A General Theory for Inertial Navigator Error Modeling

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Abstract - This paper presents a general theory for the development of error equations for an inertial navigator. A large space of possibilities is presented, based on how one defines the error parameters. For attitude, rotation vector parameterizations of the errors are introduced that are based on the difference between true can computed versions of each coordinate frame. Since these definitions are valid for large error angles, one can obtain general nonlinear equations for the psi- and phi-angle differential equations. These definitions are applicable to any mechanization and to either strapdown or platform navigators. This allows the development of general nonlinear formulas for the error dynamics, as well as new linear formulas using a different parameterization of the errors.

There are two commonly used choices for the errors in position and velocity [1]; the more commonly used choice is to form the error as that of the true variable in the true frame less the computed variable in the computed frame. The alternate approach is one in which the error is the true variable less the computed variable, with both errors coordinatized in the computed frame. This latter approach gives new error equations not previously considered in the literature. Transformations between the new error parameterizations and the traditional ones are presented.

I. INTRODUCTION

The error equations of an inertial navigator are chosen to provide a computationally efficient means of representing the navigator error dynamics in a Kalman filter. The original formulation was developed for a local level platform navigator by Pinson and Kachikas at Autonetics (now Boeing-Anaheim) [2],[3]. Here, the term 'local-level' denotes any navigator that uses Schuler tuning to maintain one axis along the local vertical. This was followed by formulations for other mechanizations, most notably for the Earth-Centered-Inertial (ECI) platform navigator [4], and for a strapdown local-level navigator [5]. Alternative parameterizations were also developed for the local-level platform navigator, most notably the Britting equations [6] and the RAIDES equations (named for the Radio Astro-Inertial Development & Evaluation Simulation written during the 1960s at Litton, now Northrop-Grumman; this parameterization consists of the phi and delta-theta angles, as well as the velocity and altitude) [1]. An error model based on the quaternion parameterization was originally developed by Friedland [7].

Subsequent efforts saw attempts at a general formulation of the linear error equations of inertial navigation. A clever and original approach was presented by Arshal [8], with the most systematic attempt by Drora Goshen-Meskin and Itzhack Bar-Itzhack at Technion [9], [10]. There they presented a taxonomy of coordinate frames and methods for parameterizing the navigation error states. A key feature of this taxonomy was the designation of certain coordinate frames as 'known' to the computer so that it was unnecessary to take the perturbation between such frames in the formulation of error angles.

This paper takes a different approach, with three key features: The first is an extension of the concept of 'known' frames (referred to herein as 'computed' frames) with the recognition that, regardless of the mechanization, an inertial navigator can in principle implement computed counterparts to the body, inertial, Earth, geodetic, and wander azimuth frames. The second feature is that attitude errors are not treated as representing perturbations in attitude [5], [11]-[12], but as extensions of the Pinson-Kachikas notion of errors as difference between the computed or platform implementation of a coordinate frame and the corresponding true frame. Indeed, this formulation allows one to reconcile the perturbation definitions with the Pinson-Kachikas definitions, which at first sight seem non-intuitive. The third feature is that position and velocity errors are formulated in the computed version of the coordinate frame. By contrast, previous practice has been to formulate these errors as the true quantity in the true frame less the computed quantity in the computed frame. This convention matches that used in the indirect Kalman filter implementation [13], in that the error state is the difference between the true and computed states.

This approach results in a new set of linear error equations for each mechanization. These equations are obtained from the traditional equations by a coordinate transformation.

The most recent efforts in error modeling have been in the extension of the error equations to handle large-amplitude attitude errors. Both Yu et Al. [14] and Kong et Al. [15] use nonlinear differential equations for the psi angle. This paper will present two alternate methods for this approach. More notable are the approaches by Scherzinger [16], and Rogers [17] [18], both of which try to approximate large amplitude heading errors using linear models. By treating attitude errors as between true and computed versions of a coordinate frame, one can address large amplitude errors using the quaternion differential equations for this attitude error. This allows one to model large amplitude angular errors using linear error equations, which reduce to the familiar error equations for small angles.

II. A SURVEY OF ATTITUDE ERROR DEFINITIONS

The traditional method of defining attitude errors, as described by Pinson & Kachikas was intended to apply to local-level platform navigators (Figure 1) [19].

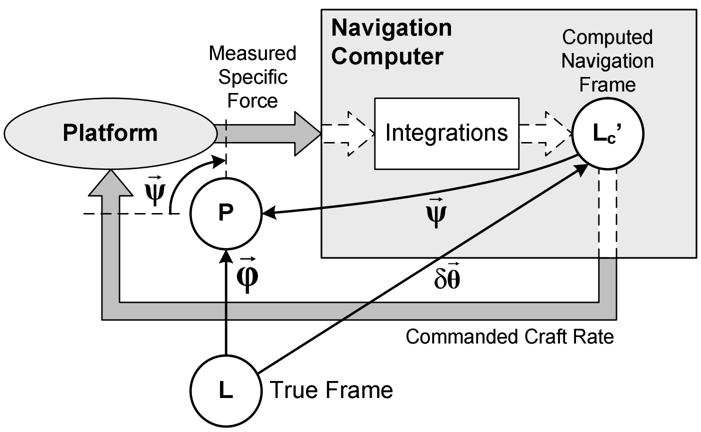

Fig. 1: The original attitude error relationships for a local-level platform navigator [2],[3].

In the original definition, there are three versions of the local level navigation frame: First, there is the actual local-level frame that the navigator attempts to model, designated as the true navigation frame, L. The computed position and in the case of a wander azimuth navigator, wander angle, are used to determine a computed version of the navigation frame L_c'. The inertial platform is commanded to track this frame, but errors in the gyro cause the platform orientation to differ from the computed version, giving rise to a third frame, denoted P.

In this mechanization, the outputs from accelerometers mounted to the platform are provided to the navigation computer, which removes gravity from the inputs and integrates the result twice to obtain velocity and position. In doing so, errors from the accelerometers enter the picture, causing the L_c' frame to be in error relative to the platform frame. The angular velocity of the computed navigation frame relative to inertial (known as the craft rate or transport rate) is sent to the inertial platform, closing the loop. This angular velocity will be different from the true craft rate due to accelerometer errors.

For an inertial platform operating in local-level mode, the original definition of the psi angle is the rotation vector [20] that relates the orientation in the computed version of the navigation frame, as determined by the position and wander angle, relative to the inertial platform. This error arises from the fact that the platform will not track the commands based on the computed navigation frame perfectly because of gyro errors.

The phi angle was defined as the rotation vector of the inertial platform relative to the true navigation frame attitude relative to the inertial platform. This is the actual 'navigation frame' for the platform navigator in that this is the frame in which the integrations of acceleration and velocity are carried out and in which vectors are coordinatized in. Since the inertial platform is commanded at the angular velocity of the true frame, the phi angle will include the effects of both gyro and accelerometer errors.

The delta-theta angle was defined as the rotation vector of the computed nav frame L_c' relative to the true frame L. This error was shown to be a parameterization of the error of true position relative to computed [2].

The question arises of how to apply these definitions to other mechanizations and to strapdown navigators. A popular approach in the strapdown literature [11] [12] is to define the error angles as perturbations in the direction cosine matrices from their true values. The convention taken by these authors is based on additive perturbations to the direction cosine matrices, assuming small angles in the rotation vectors. In this convention, the psi angle represents the perturbation in the orientation of the Earth (E) frame relative to the vehicle body (B) frame:

$$\delta C_B^E = C_B^E|_c - C_B^E = -[\vec{\psi}^E \times]C_B^E \tag{2.1}$$

where the subscript 'c' denotes the computed version and no subscript denotes the true version.

The phi angle accounts for the perturbation in the direction cosine matrix of the local-level (L) frame relative to the body frame:

$$\delta C_B^L = C_B^L|_c - C_B^L = -[\vec{\varphi}^L \times]C_B^N \tag{2.2}$$

Finally, the delta-theta angle represents the perturbation of the direction cosine matrix of the Earth relative to the local-level frame :

$$\delta C_L^E = C_L^E|_c - C_L^E = -[\delta\vec{\theta}^E \times]C_L^E \tag{2.3}$$

Although they give error equations similar to those used for platform navigators, the connection between these definitions and the original ones by Pinson and Kachikas is unclear.

A different approach, pursued by Arshal [9] and Goshen-Meskin & Bar Itzhack [10] [11] is to define the error rotation vectors from the version of the coordinate frame maintained in the navigation computer to the true version. Doing so allows resulting error equations to be easily extended to nonlinear angular formulations. These definitions correspond to the original Pinson-Kachikas approach in that they relate different versions of the same coordinate frame, differing only due to instrument errors.

At this point, two terms deserve clarification: First is the term 'navigation frame' will be taken to mean that coordinate frame that in which the acceleration and velocity integrals are computed. The error equations will be derived in this frame for each mechanization. The second is what is meant by a 'computed frame.' Here, the term 'computed' will apply to any coordinate frame maintained by the navigation system, including that of the inertial platform, since it is in this coordinate frame that acceleration and velocity are integrated. Hence the original error angle formulation is viewed as having two computed navigation frames: One is based on the navigation frame and the other the frame that one would obtain directly from the position. The coordinate frame N'_c is an example of the latter.

A consequence of this approach is that with the exception of the delta-theta angle, perturbations in a direction cosine matrix are now treated as multiplicative as opposed to additive. Letting A and F be two true coordinate frames and A_c and F_c representing their versions maintained in the navigation computer, the true and computed versions of the direction cosine matrix are related by:

$$C_A^F = C_{F_c}^F C_{A_c}^{F_c} C_A^{A_c} \tag{2.4}$$

One can write equation 2.4 as in terms of the individual error rotation vectors as:

$$C_A^F = C(\vec{\Phi}_{F_cF}^{F_c}) C_{A_c}^{F_c} C(-\vec{\Phi}_{A_cA}^{A_c}) \tag{2.5}$$

where C( ) denotes the direction cosine matrix corresponding to the error rotation vector. The rotation vector of the true F-frame relative to the computed F-frame in computed F-frame coordinates is denoted by $\vec{\Phi}_{F_cF}^{F_c}$.

The error direction cosine matrix is related to the error rotation vector for large angles by [20] :

$$C(\vec{\Phi}_{F_cF}^{F_c}) = I_3 - \frac{\sin\phi_{F_cF}}{\phi_{F_cF}}[\vec{\Phi}_{F_cF}^{F_c}\times] + \frac{1-\cos\phi_{F_cF}}{\phi_{F_cF}^2}[\vec{\Phi}_{F_cF}^{F_c}\times]^2 \tag{2.6}$$

where ϕ_{F_cF} is the magnitude of the error rotation vector and $[\vec{\Phi}_{F_cF}^{F_c}\times]$ is a cross-product matrix, defined as follows:

$$\left[\begin{bmatrix} x \\ y \\ z \end{bmatrix}\times\right] = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} \tag{2.7}$$

The rotation vector and formulas for the F-frame are defined similarly. The multiplicative perturbation approach that will be expanded upon in Sections V-VII, but conventions for modeling must be established first.

III. MODELING CONVENTIONS FOR THE INERTIAL SENSORS

The modeling convention for sensors is the traditional approach, where the instrument error is defined as the difference between the measured and true values. The gyro and accelerometer errors are defined by the equations:

$$\delta\vec{\omega}_{gyro}^{M_c} = \vec{\omega}_{IM_m}^{M_c} - \vec{\omega}_{IM}^{M} \tag{3.1}$$

$$\delta\vec{f}_{acc}^{M_c} = \vec{f}_m^{M_c} - \vec{f}^{M} \tag{3.2}$$

where M and M_c denote the true and computed frames in which the inertial instrument measurements are taken.

This definition can be appreciated as that between the sensor output and its input. The error terms can be measured on a test table, where one knows the input and measures the output.

IV. ERROR MODELS FOR THE NATURAL ENVIRONMENT

For errors in the natural environment models, one must take into account how the models were obtained and how they will be used. Both the Earth rotation rate and gravitational field models are derived with a very precise knowledge of the Earth and inertial frames [21]. In drawing up these models, the modeling errors will have the form:

$$\delta\vec{\omega}_{IE_{model}}^{E_c} = \vec{\omega}_{IE}^{E_c} - \vec{\omega}_{IE_{model}}^{E_c} \tag{4.1}$$

$$\delta\vec{G}_{model}^{E_c} = \vec{G}^{E_c}(\vec{R}_{O_cP_c}^{E_c}) - \vec{G}_c^{E_c}(\vec{R}_{O_cP_c}^{E_c}) \tag{4.2}$$

where E_c the Earth frame computed by the navigator. These errors will be in the common computed Earth frame in that this is the frame in which the geodetic measurements are interpreted.

The error in the gravitation in the computed Earth frame will be dependent on the true and computed versions of the position (next page):

$$\delta\vec{\mathbf{G}}^{E_c}(\vec{\mathbf{R}}^{E_c}_{O_cN_c}) = \vec{\mathbf{G}}^{E_c}(\vec{\mathbf{R}}^{E}_{ON}) - \vec{\mathbf{G}}^{E_c}_{c}(\vec{\mathbf{R}}^{E_c}_{O_cN_c}) \tag{4.3}$$

Two error effects are manifested in this equation. The first is the modeling error in the computed gravitation. The other is the error in the gravitation due to an error in position.

The true position can be related to the computed position by:

$$\vec{\mathbf{R}}_{ON} = \vec{\mathbf{R}}_{OO_c} + \vec{\mathbf{R}}_{O_cN_c} + \vec{\mathbf{R}}_{N_cN} \tag{4.4}$$

where point O is the true Earth center, point O_c is the computed version of the Earth center, N is the reference point in the navigator, and N_c its computed version.

The first term on the right hand side can usually be considered negligible, since the position of the Earth center in the International Terrestrial Reference System is known to an accuracy of approximately 1 cm [21]. Coordinatizing equation 4.4 in the computed Earth frame gives:

$$\vec{\mathbf{R}}^{E}_{ON} = \vec{\mathbf{R}}^{E_c}_{O_cN_c} + \vec{\mathbf{R}}^{E_c}_{N_cN} \tag{4.5}$$

Expansion of the true gravitation to first order results in:

$$\boxed{\begin{aligned}\vec{\mathbf{G}}^{E_c}(\vec{\mathbf{R}}^{E}_{ON}) - \vec{\mathbf{G}}^{E_c}_{c}(\vec{\mathbf{R}}^{E_c}_{O_cN_c}) &= \nabla\vec{\mathbf{G}}^{E_c}_{c}(\vec{\mathbf{R}}^{E_c}_{O_cN_c})\,\vec{\mathbf{R}}^{E_c}_{N_cN} \\ &+ \delta\vec{\mathbf{G}}^{E_c}_{residual}(\vec{\mathbf{R}}^{E_c}_{O_cN_c})\end{aligned}} \tag{4.6}$$

where the last term on the right hand side is the combined effect of modeling error and any truncation error when using a spherical harmonic series expansion for the computed gravitation. One can likewise develop a similar expression for the gravity, $\vec{\mathbf{g}}$:

$$\boxed{\begin{aligned}\vec{\mathbf{g}}^{E_c}(\vec{\mathbf{R}}^{E}_{ON}) - \vec{\mathbf{g}}^{E_c}_{c}(\vec{\mathbf{R}}^{E_c}_{O_cN_c}) &= \nabla\vec{\mathbf{g}}^{E_c}_{c}(\vec{\mathbf{R}}^{E_c}_{O_cN_c})\,\vec{\mathbf{R}}^{E_c}_{N_cN} \\ &+ \delta\vec{\mathbf{g}}^{E_c}_{residual}(\vec{\mathbf{R}}^{E_c}_{O_cN_c})\end{aligned}} \tag{4.7}$$

V. ATTITUDE ERROR FORMULATION FOR THE EARTH-CENTERED-INERTIAL MECHANIZATIONS

In the Earth-Centered-Inertial (ECI) platform mechanization, the inertial platform is only commanded to remove the effects of gyro errors, so that the platform tracks the true inertial frame. Here there 'computed' navigation frame is that which is fixed to the platform since it is the one in which the acceleration is integrated to obtain velocity and again to yield position. In this step, accelerometer errors will contribute to the velocity and position errors so that if one were to compute a local level frame based on position, this coordinate frame would be in error due not only to platform attitude errors, but also due to accelerometer errors.

Using the Pinson/Kachikas formulation, the error angle relationships for the platform ECI navigator become those depicted in Figure 2. The delta-theta angle still represents the error in the knowledge of the local-level frame L'_c. A difference with this concept from the original one by Pinson and Kachikas is that the psi angle is replaced by $\vec{\boldsymbol{\psi}}_I$, which is the error in attitude due to gyro errors alone; This is now the error between true and platform versions of the inertial frame. In contrast with the sign convention of rotation vector errors being that of the true frame relative to computed, the sign of the delta-theta angle will be same as that used in the literature.

In this formulation, the true local level frame, L, does not possess an error relative to true inertial frame, I. For the ECI navigator, one could define a second local level frame L_c by computing the inertial craft rate (the angular velocity of local-level frame relative to inertial) to drive a solver for the kinematic differential equation of the attitude of the L_c frame relative to platform. To actually do so would require that the navigation calculations be performed in a manner like that for a strapdown navigator, where the space-stable platform acts as the 'body' frame in which the inertial sensors are fixed.

This raises an important point: Although one speaks of a 'navigation' frame, there are in fact a number of different computed coordinate frames that can be determined in a navigator. In the case of the local-level frame, there can also be more than one version of the computed frame, each with different errors relative to the true frame. One can define error rotation vectors that relate these coordinate frames, which for the ECI platform navigator, will be related as shown in Figure 3. Here again the subscript 'c' denotes the computed version of a coordinate frame and no subscript denotes the true version.

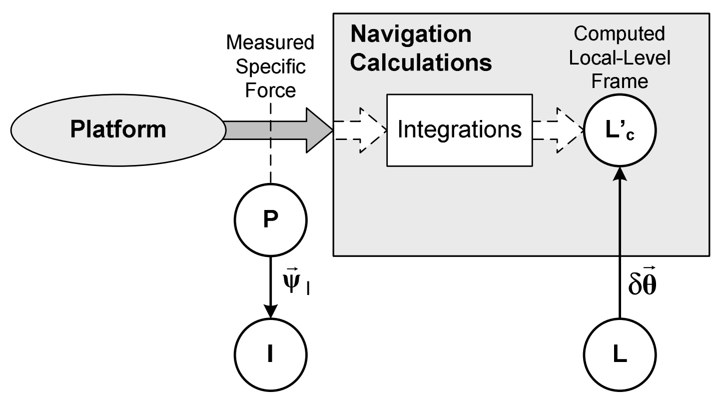

Figure 2: The role of the error angles in an Earth-Centered-Inertial platform navigator

In this figure, one can see the relation of the body frame errors to inertial frame. Here, the error in the determination of the body frame for the platform navigator can be seen to be due to a combination of gyro and resolver errors. Note also that the computed inertial frame is the best known of all the frames, with other frames being less known due to the effect of the additional error sources.

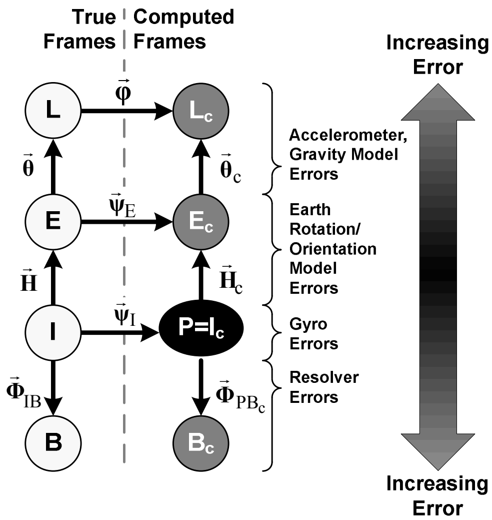

Fig. 3: Error angle/coordinate frame relationships for the ECI platform navigator.

In a strapdown navigator, the inertial platform is replaced by an 'analytic platform inside the navigation computer. This set of calculations is still distinct from the 'Navigation Calculations,' the formulas of which would be essentially the same regardless of whether the navigator is a platform or a strapdown system (Figure 4).

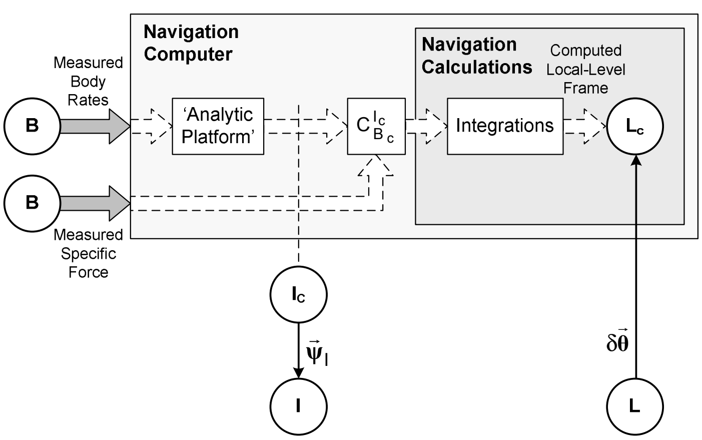

Fig. 4: The role of the error angles in an Earth-Centered-Inertial strapdown navigator.

One can also construct a chart similar to that in Figure 3 for the remaining coordinate frames (Figure 5). The first essential difference with Figure 3 is the relationship of the body frame angular errors to the inertial errors. Also note that now it is the computed body frame that is the best known, with the error increasing as one moves up the diagram.

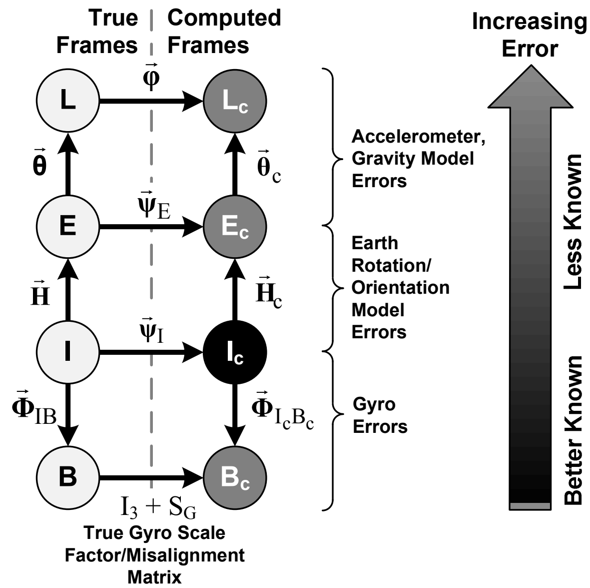

Figure 5: Error angle/coordinate frame relationships for the ECI strapdown navigator.

The relationship between the true angular velocity of the body frame relative to inertial and its value as perceived by the navigator will be given by a combination of the misalignment of the inertial measurement unit with respect to the vehicle and the individual gyro errors. The approach adopted herein will be to treat the true gyro scale factor/misalignment matrix S_G as zero, with these errors lumped into the gyro instrument errors.

To obtain the ECI attitude error equation, the true angular velocity is first expanded in terms of the computed angular velocity as follows:

$$\vec{\omega}_{IM} = \vec{\omega}_{II_c} + \vec{\omega}_{I_cB_c} + \vec{\omega}_{B_cB} \tag{5.1}$$

It was established earlier that the measured and computed frames are the same so that the angular velocity of the true body frame relative to computed is zero, and the last term of the above equation vanishes.

Now coordinatize the particular components as follows:

$$\vec{\omega}_{IB}^{B_c} = \vec{\omega}_{I_cB_c}^{B_c} - C_{I_c}^{B_c}\vec{\omega}_{I_cI}^{I_c} \tag{5.2}$$

For large angles, the nonlinear equation for the rotation vector is [20]:

$$\dot{\vec{\psi}}_I^{I_c} = \{ I_3 + \tfrac{1}{2}[\vec{\psi}_I^{I_c}\times] + \frac{1}{\psi_I^2}\left(1 - \frac{\psi_I \sin\psi_I}{2(1-\cos\psi_I)}\right)[\vec{\psi}_I^{I_c}\times]^2\}\vec{\omega}_{I_cI}^{I_c} \tag{5.3}$$

For a platform navigator

$$\boxed{\vec{\omega}_{I_cI}^{I_c} = \delta\vec{\omega}_{gyro}^{I_c} + \delta\vec{\omega}_{cal}^{I_c}} \tag{5.4}$$

while for the strapdown navigator the relationship is:

$$\boxed{\vec{\omega}^{I_c}_{I_cI} = C^{I_c}_{B_c}\,\delta\vec{\omega}^{B_c}_{gyro}} \tag{5.5}$$

Here the term $\delta\vec{\omega}_{gyro}$ represents the gyro instrument errors including scale factors and misalignments and $\delta\vec{\omega}^{I_c}_{cal}$ is the platform calibration command from the integration filter in the navigator. The latter has the same role as the integration filter corrections for a strapdown navigator.

Substitution of eq. 5.4 into equation gives the nonlinear error equation for the inertial attitude error for the platform navigator:

$$\boxed{\begin{aligned}\dot{\vec{\psi}}^{I_c}_{I} &= \{I_3 + \tfrac{1}{2}[\vec{\psi}^{I_c}_{I}\times] \\ &+ \frac{1}{\psi_I^2}\left(1 - \frac{\psi_I \sin\psi_I}{2(1-\cos\psi_I)}\right)[\vec{\psi}^{I_c}_{I}\times]^2\}(\delta\vec{\omega}^{I_c}_{gyro} + \delta\vec{\omega}^{I_c}_{cal})\end{aligned}} \tag{5.6}$$

The strapdown version is :

$$\boxed{\begin{aligned}\dot{\vec{\psi}}^{I_c}_{I} &= \{I_3 + \tfrac{1}{2}[\vec{\psi}^{I_c}_{I}\times] \\ &+ \frac{1}{\psi_I^2}\left(1 - \frac{\psi_I \sin\psi_I}{2(1-\cos\psi_I)}\right)[\vec{\psi}^{I_c}_{I}\times]^2\}C^{I_c}_{B_c}\,\delta\vec{\omega}^{B_c}_{gyro}\end{aligned}} \tag{5.7}$$

For small error angles, these reduce to the linear versions:

Platform:

$$\boxed{\dot{\vec{\psi}}^{I_c}_{I} = \delta\vec{\omega}^{I_c}_{gyro} + \delta\vec{\omega}^{I_c}_{cal}} \tag{5.8}$$

Strapdown:

$$\boxed{\dot{\vec{\psi}}^{I_c}_{I} = C^{I_c}_{B_c}\,\delta\vec{\omega}^{B_c}_{IB}} \tag{5.9}$$

For large amplitude errors, the quaternion parameterization gives a set of linear equations. The psi-angle is related to the scalar and vector parts of the error quaternion using the following sign convention [22]:

$$a_{I_c,I} = \cos\frac{\psi_I}{2} \tag{5.10}$$

$$\vec{\mathbf{p}}^{I_c}_{I_c,I} = \sin\frac{\psi_I}{2}\,\vec{\mathbf{u}}^{I_c}_{\psi_I} \tag{5.11}$$

From the quaternion differential [22], one has:

$$\boxed{(2\dot{\mathbf{q}}^{I_c}_{I_c,I}) = A(\vec{\omega}^{I_c}_{I_cI})(2\mathbf{q}^{I_c}_{I_c,I})} \tag{5.12}$$

where the matrix A is given by:

$$A(\vec{\omega}) = \frac{1}{2}\left[\begin{array}{c|c} -[\vec{\omega}\times] & -\vec{\omega} \\ \hline \vec{\omega}^T & 0 \end{array}\right] \tag{5.13}$$

In a navigation integration filter for coarse and fine alignment, one could use equation 5.12, which is linear instead of equations 5.6 and 5.7. As the attitude error becomes small during coarse alignment, the vector part of twice the quaternion will approach the psi angle and the differential equation for the scalar part could be ignored beyond this point.

Equations 5.6-5.9 and 5.12 represent the psi-angle error state equations for the ECI platform and strapdown navigators.

VI. ATTITUDE ERROR FORMULATION FOR THE EARTH-CENTERED-EARTH-FIXED MECHANIZATIONS

For the Earth-Centered-Earth-Fixed (ECEF) mechanization, a model of the Earth rate is used to drive the inertial platform so that it tracks the Earth orientation (Figure 6).

The model of the Earth frame will be in error compared to the true Earth frame due to errors in the Earth orientation/rate model itself. Here the Earth frame plays a role analogous to that of the 'computed' navigation frame in Figure 1, L'_c, in that it commands the platform. Unlike Figures 1 and 2, the modeled Earth frame stands alone, not affected by the other navigation algorithms in the navigation computer.

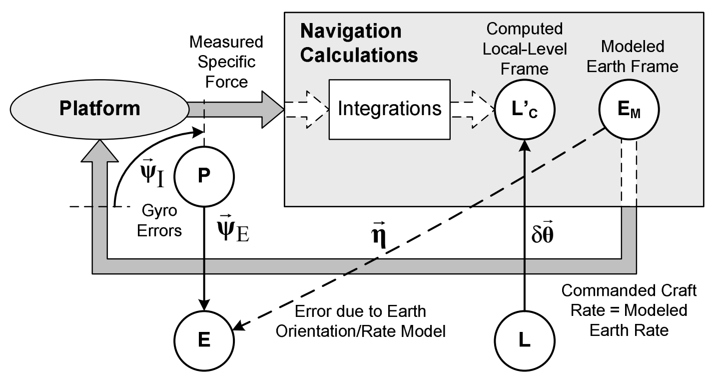

Fig. 6: The role of error angles in an ECEF platform navigator.

For the Earth-Centered-Earth-Fixed (ECEF) navigator, the error in the platform frame relative to the true Earth frame is now represented by $\vec{\psi}_E$. This is because the platform command will be in error by the error in the modeled Earth rate through the platform command as well as the gyro instrument errors. The error angle of the platform frame relative to the modeled Earth frame is once again $\vec{\psi}_I$. This is because this measures the ability of the platform to track the commanded frame, which in this case is the modeled Earth frame. The difference in the psi-angles $\vec{\psi}_I$ and $\vec{\psi}_E$ is solely caused by modeling errors in Earth orientation, $\vec{\boldsymbol{\eta}}$. This effect is negligible for navigators near the Earth, although it may be significant due to unmodeled lunar physical libration for space vehicles operating in the vicinity of the moon [23].

The relationships between the other coordinate frames for the ECEF platform navigator are described in Figure 7. The main differences with Figure 3 are the relationships of the modeled Earth frame to computed inertial frame and from the body frame to the platform frame P, which now serves as the computed Earth frame, E_c.

The 'computed' body frame B_c is obtained from the platform frame from resolver measurements, so this frame is in error relative to the platform due to errors in these sensors. The modeled Earth frame E_m is used to generate the platform command and will be in error relative to the true Earth frame by angle $\vec{\boldsymbol{\eta}}$. Were one to construct the computed inertial frame in the navigator, its orientation error would be described by the angle $\vec{\boldsymbol{\psi}}_I$ because this frame would be obtained by backing out the modeled Earth orientation.

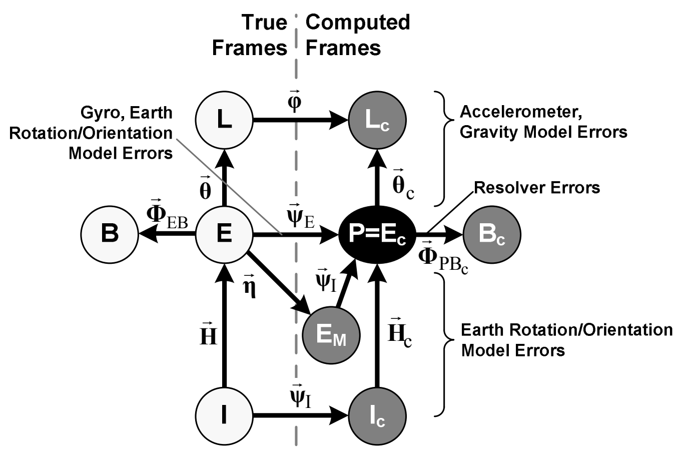

Fig. 7: Error angle/coordinate frame relationships for the ECEF platform navigator.

The mechanization for the ECEF strapdown navigator is similar to the platform (Figure 8), with the inertial platform replaced by the 'analytic platform,' which consists of the attitude equations from computed body to computed Earth frame.

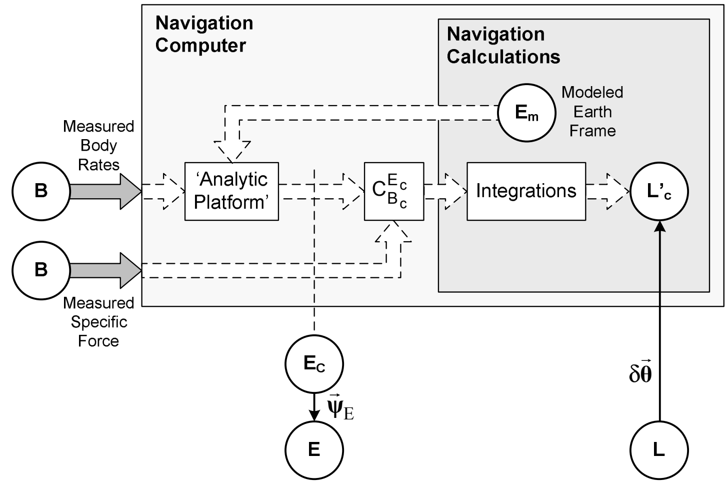

Fig. 8: The role of error angles in an ECEF strapdown navigator.

The relationships between coordinate frames for the strapdown navigator are depicted Figure 9. These relationships are the same as for the platform navigator except for the relationship of the body frame to inertial which are the same as for the ECI strapdown navigator.

The error angle relationships between true, computed, and modeled Earth frames follow by the same reasoning as for the platform case. However, for a strapdown system, the computed inertial frame could be solved for directly from the gyro measurements. Whether it is obtained by backing out the Earth orientation error or computed directly, the error formulation will be the same.

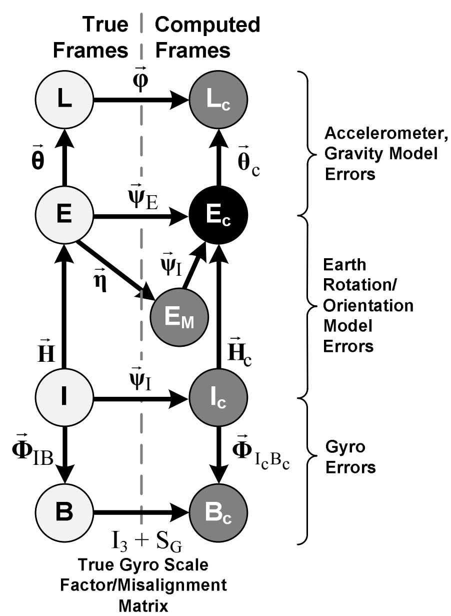

Fig. 9: Error angle/coordinate frame relationships for the ECEF strapdown navigator.

These relationships can be expressed mathematically as follows: For the ECEF platform navigator, the actual angular velocity achieved will be the sum of the modeled Earth rate and the gyro error:

$$\vec{\omega}^{E_c}_{IE_c} = \vec{\omega}^{E_c}_{IE_{model}} + \delta\vec{\omega}^{E_c}_{gyro} + \delta\vec{\omega}^{E_c}_{cal} \qquad (6.1)$$

An important observation is that the modeled Earth rate will be expressed in the computed Earth frame E_c, which is the platform-fixed frame. The error angular velocity of the true frame E relative to computed E_c will be:

$$\vec{\omega}^{E_c}_{E_cE} = C^{E_c}_{E}\vec{\omega}^{E}_{IE} - \vec{\omega}^{E_c}_{IE_{model}} + \delta\vec{\omega}^{E_c}_{gyro} + \delta\vec{\omega}^{E_c}_{cal} \qquad (6.2)$$

Substitution of equation 4.1 and 6.1 into the above gives the error angular velocity in terms of the Earth rate model and gyro errors:

$$\vec{\omega}^{E_c}_{E_cE} = (C^{E_c}_{E} - I_3)\vec{\omega}^{E_c}_{IE_{model}} + \delta\vec{\omega}^{E_c}_{gyro} + \delta\vec{\omega}^{E_c}_{cal} + C^{E_c}_{E}\,\delta\vec{\omega}^{E_c}_{IE_{model}} \qquad (6.3)$$

In this paper, we will define a psi-angle, $\vec{\psi}_E$, as the error between true and computed Earth frames. The nonlinear formula for the direction cosine matrix in terms of this angle is [20]:

$$C^{E}_{E_c} = I_3 - \frac{\sin\psi_E}{\psi_E}[\vec{\psi}^{E_c}_{E}\times] + \frac{1-\cos\psi_E}{\psi_E^2}[\vec{\psi}^{E_c}_{E}\times]^2 \qquad (6.4)$$

This psi-angle corresponds to the one normally used for inertial navigation error modeling, both in the platform and strapdown literature. For small angles, this reduces to:

$$C^{E}_{E_c} \cong I_3 - [\vec{\psi}^{E_c}_{E}\times] \qquad (6.5)$$

The nonlinear differential equation for the psi-angle error is [20]:

$$\begin{aligned}\dot{\vec{\psi}}^{E_c}_{E} = \{ & I_3 + \tfrac{1}{2}[\vec{\psi}^{E_c}_{E}\times] \\ & + \frac{1}{\psi_E}\left(1 - \frac{\psi_E\sin\psi_E}{2(1-\cos\psi_E)}\right)[\vec{\psi}^{E_c}_{E}\times]^2\}\vec{\omega}^{E_c}_{E_cE}\end{aligned} \qquad (6.6)$$

For small angles, this reduces to, upon substitution of equation 6.5 into 6.3 and the result into the above (top):

$$\boxed{\begin{aligned}\dot{\vec{\psi}}^{E_c}_{E} &= \vec{\omega}^{E_c}_{E_cE} \\ &= -[\vec{\omega}^{E_c}_{IE_{model}}\times]\vec{\psi}^{E_c}_{E} + \delta\vec{\omega}^{E_c}_{gyro} + \delta\vec{\omega}^{E_c}_{cal} + \delta\vec{\omega}^{E_c}_{IE_{model}}\end{aligned}} \qquad (6.7)$$

For the strapdown case, the 'analytic platform' is driven at the modeled Earth rate:

$$\vec{\omega}^{E_c}_{I_cE_c} = \vec{\omega}^{E_c}_{IE_{model}} \qquad (6.8)$$

The error angular velocity is solved for using:

$$\vec{\omega}^{E_c}_{E_cE} = -\vec{\omega}^{E_c}_{I_cE_c} + \vec{\omega}^{E_c}_{I_cI} + \vec{\omega}^{E_c}_{IE} \qquad (6.9)$$

Substitution of equations 5.5 and 6.8 into the above gives the nonlinear equation for the Earth frame error angular velocity:

$$\vec{\omega}^{E_c}_{E_cE} = (C^{E_c}_{E} - I_3)\vec{\omega}^{E_c}_{IE_{model}} + C^{E_c}_{B_c}\delta\vec{\omega}^{B_c}_{gyro} + C^{E_c}_{E}\delta\vec{\omega}^{E_c}_{IE_{model}} \qquad (6.10)$$

or, for small angles, from eq. 6.5 :

$$\boxed{\begin{aligned}\dot{\vec{\psi}}^{E_c}_{E} = \vec{\omega}^{E_c}_{E_cE} &= -[\vec{\omega}^{E_c}_{IE_{model}}\times]\vec{\psi}^{E_c}_{E} \\ &\quad + C^{E_c}_{B_c}\delta\vec{\omega}^{B_c}_{gyro} + \delta\vec{\omega}^{E_c}_{IE_{model}}\end{aligned}} \qquad (6.11)$$

It should be remarked that in the absence of Earth rate modeling error, by coordinatizing equations 6.7 and 6.11 in the inertial frame gives equations 5.8 and 5.9. In this case, both the psi-I and psi-E angles will represent the same rotation vector, but coordinatized in different frames.

VII. ATTITUDE ERROR FORMULATIONS FOR THE LOCAL-LEVEL MECHANIZATIONS

The error angle relationships for local-level platform navigator are those introduced by Pinson and Kachikas (Figure 1). The key difference between a local level navigator and ECI and ECEF navigators is that the craft rate closes a feedback loop so that the past error history in the velocity contributes to both errors in platform and computed frame attitudes relative to true.

The relationships between the coordinate frames for the local-level platform navigator are depicted in Figure 10. The primary difference in this diagram with those for the ECI and ECEF navigators is that the relationships of the body frames to the local level frames. The phi angle will the result of the effects of gyro, accelerometer, and Earth orientation model errors.

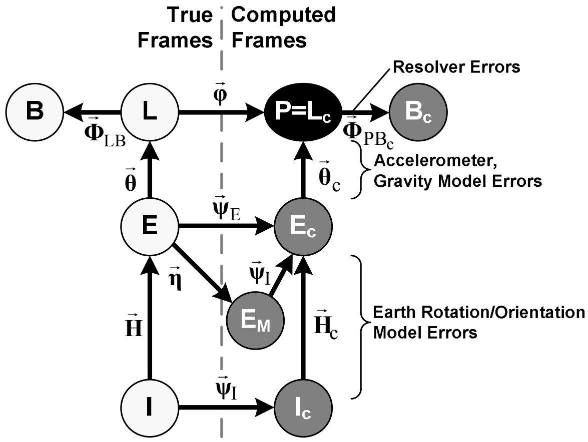

Figure 10: Error angle/coordinate frame relationships for the local level platform navigator

One can use the psi-angle to parameterize the Earth orientation error in the local level navigator, but a change of coordinates will be necessary. The change in coordinates is accomplished by:

$$\vec{\psi}_E^{E_c} = C_{L_c}^{E_c}\vec{\psi}_E^{L_c} \tag{7.1}$$

The derivative of the psi-angle is obtained using Coriolis' law:

$$\dot{\vec{\psi}}_E^{E_c} = C_{L_c}^{E_c}\{\dot{\vec{\psi}}_E^{L_c} + [\vec{\omega}_{E_cL_c}^{L_c}\times]\vec{\psi}_E^{L_c}\} \tag{7.2}$$

The term in the cross product matrix can be seen to be the craft rate. Substitution of equations 7.1 and 7.2 into equation 6.7 gives the psi-angle equation for the platform navigator:

$$\boxed{\begin{aligned}\dot{\vec{\psi}}_E^{L_c} &= -[(\vec{\omega}_{E_cL_c}^{L_c} + \vec{\omega}_{IE_{model}}^{L_c})\times]\vec{\psi}_E^{L_c} \\ &\quad + \delta\vec{\omega}_{gyro}^{L_c} + \delta\vec{\omega}_{cal}^{L_c} + C_{E_c}^{L_c}\delta\vec{\omega}_{IE_{model}}^{E_c}\end{aligned}} \tag{7.3}$$

The strapdown counterpart to Figure 1 is depicted in Figure 11. Note that the L'_c frame has taken the place of the modeled Earth frame as the source of commands to the 'analytic platform.' The relationships between the coordinate frames are exactly the same as that in Figure 9, except that the computed local level frame is now the navigation frame.

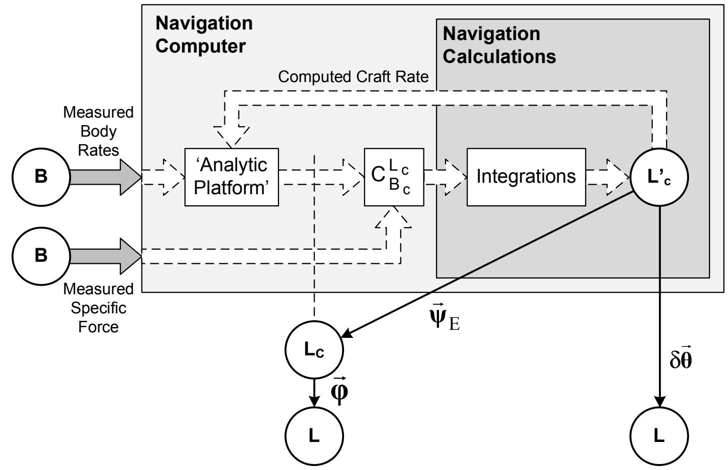

Figure 11: The role of error angles in the local level strapdown navigator.

Following the same steps as for the platform navigator, one obtains the psi-angle equation for the local-level strapdown navigator:

$$\boxed{\begin{aligned}\dot{\vec{\psi}}_E^{L_c} &= -[(\vec{\omega}_{E_cL_c}^{L_c} + \vec{\omega}_{IE_{model}}^{L_c})\times]\vec{\psi}_E^{L_c} \\ &\quad + C_{B_c}^{L_c}\delta\vec{\omega}_{gyro}^{B_c} + C_{E_c}^{L_c}\delta\vec{\omega}_{IE_{model}}^{E_c}\end{aligned}} \tag{7.4}$$

VIII. VELOCITY AND POSITION ERROR MODELING

A. Error Definitions

There are two general approaches to position and velocity error modeling for inertial navigators. The conventional approach is to treat position and velocity errors as perturbations in the parameters used to represent both. For example, the perturbation in position between the true and computed versions of the navigation point N relative to Earth center O in the F frame is:

$$\delta\vec{R}_{ON}^{F_c} = \vec{R}_{ON}^{F} - \vec{R}_{O_cN_c}^{F_c} \tag{8.1}$$

This approach has been developed extensively in the literature [11][12][25].

An alternate approach is to define position and velocity as relative between true and computed positions of the navigation point. Recalling the above example, the 'common frame' version of the error is:

$$\vec{R}_{N_cN}^{F_c} = \vec{R}_{ON}^{F_c} - \vec{R}_{O_cN_c}^{F_c} \tag{8.2}$$

where it was assumed earlier that the difference between true and computed positions of the Earth center was negligible.

Notice that equation 8.2 is simply the position vector from the computed navigation point N_c to the true value N, which happens to be coordinatized in F_c-frame; The definition of this error does not depend on the specification of a coordinate frame. By contrast, the 'perturbation' position error is a construct based on mathematical vectors and is consequently tied to the F and F_c frames.

The position and velocity equations, since they are performed as part of the Navigation Calculations are not dependent on whether the navigator is a strapdown or platform type, except when one includes terms in the Earth orientation error, the psi-E angle.

B. ECI Common-Frame Error Equations

The remainder of this section will focus on development of the 'common frame' error equations. The ECI position and velocity 'common-frame' equations are developed by differentiation of the position of the navigation point relative to Earth center in the inertial frame and relating to the computed inertial frame using Coriolis' law:

$$\vec{\mathbf{V}}_{I;ON} = \vec{\mathbf{V}}_{I_c;ON} + \vec{\boldsymbol{\omega}}_{I,I_c} \times \vec{\mathbf{R}}_{ON} \tag{8.3}$$

Next, equation (8.2) is differentiated in the computed inertial frame:

$$\vec{\mathbf{V}}_{I_c;ON} = \vec{\mathbf{V}}_{I_c;O_cN_c} + \vec{\mathbf{V}}_{I_c;N_cN} \tag{8.4}$$

Substitution of this result into equation (8.3), followed by substitution of equation (8.2) with reversal of the cross-product gives, to first order:

$$\vec{\mathbf{V}}_{I;ON} = \vec{\mathbf{V}}_{I_c;O_cN_c} + \vec{\mathbf{V}}_{I_c;N_cN} + \vec{\mathbf{R}}_{O_cN_c} \times \vec{\boldsymbol{\omega}}_{I_cI} \tag{8.5}$$

Differentiating equation 8.5 with respect to inertial frame and applying Coriolis' law a second time results in:

$$\begin{aligned} \vec{\mathbf{a}}_{I;ON} &= \vec{\mathbf{a}}_{I_c;O_cN_c} + D_{I_c}(\vec{\mathbf{V}}_{I_c;N_cN}) \\ &+ 2\vec{\mathbf{V}}_{I_c;O_cN_c} \times \vec{\boldsymbol{\omega}}_{I_cI} + \vec{\mathbf{R}}_{O_cN_c} \times D_{I_c}(\vec{\boldsymbol{\omega}}_{I_cI}) \end{aligned} \tag{8.6}$$

where the symbol $D_{Ic}(\ )$ denotes differentiation with respect to time in the I_c-frame.

The true and computed versions of the ECI velocity mechanization equations are:

$$\vec{\mathbf{a}}_{I;ON} = \vec{\mathbf{f}} + \vec{\mathbf{G}} \tag{8.7}$$

$$\vec{\mathbf{a}}_{I_c;O_cN_c} = \vec{\mathbf{f}}_m + \vec{\mathbf{G}}_c \tag{8.8}$$

Substitute equations 8.7 and 8.8 into equation 8.6, followed by a re-arrangement of terms (top):

$$\begin{aligned} D_{I_c}(\vec{\mathbf{V}}_{I_c;N_cN}) &= -2\vec{\mathbf{V}}_{I_c;O_cN_c} \times \vec{\boldsymbol{\omega}}_{I_cI} - \vec{\mathbf{R}}_{O_cN_c} \times D_{I_c}(\vec{\boldsymbol{\omega}}_{I_cI}) \\ &+ (\vec{\mathbf{f}} - \vec{\mathbf{f}}_m) + (\vec{\mathbf{G}} - \vec{\mathbf{G}}_c) \end{aligned} \tag{8.9}$$

Now take coordinates in the computed inertial frame:

$$\begin{aligned} \dot{\vec{\mathbf{V}}}^{I_c}_{I_c;N_cN} &= -2[\vec{\mathbf{V}}^{I_c}_{I_c;O_cN_c} \times]\,\vec{\boldsymbol{\omega}}^{I_c}_{I_cI} - [\vec{\mathbf{R}}^{I_c}_{O_cN_c} \times]\,\dot{\vec{\boldsymbol{\omega}}}^{I_c}_{I_cI} \\ &+ (\vec{\mathbf{f}}^{I_c} - \vec{\mathbf{f}}^{I_c}_m) + C^{I_c}_{E_c}(\vec{\mathbf{G}}^{E_c} - \vec{\mathbf{G}}^{E_c}_c) \end{aligned} \tag{8.10}$$

Substitute in equation 4.6 for the error in gravitation:

$$\begin{aligned} \dot{\vec{\mathbf{V}}}^{I_c}_{I_c;N_cN} &= \nabla\vec{\mathbf{G}}^{I_c}_c \vec{\mathbf{R}}^{I_c}_{N_cN} \\ &- 2[\vec{\mathbf{V}}^{I_c}_{I_c;O_cN_c} \times]\,\vec{\boldsymbol{\omega}}^{I_c}_{I_cI} - [\vec{\mathbf{R}}^{I_c}_{O_cN_c} \times]\,\dot{\vec{\boldsymbol{\omega}}}^{I_c}_{I_cI} \\ &+ (\vec{\mathbf{f}}^{I_c} - \vec{\mathbf{f}}^{I_c}_m) + C^{I_c}_{E_c}\,\delta\vec{\mathbf{G}}^{E_c}_{residual} \end{aligned} \tag{8.11}$$

At this point, the derivations of the platform and strapdown ECI error equations diverge. For the platform navigator, the specific force error is in the common frame:

$$\vec{\mathbf{f}}^{I_c} - \vec{\mathbf{f}}^{I_c}_m = -\delta\vec{\mathbf{f}}^{I_c}_{acc} \tag{8.12}$$

Substitution of the above, along with equation 5.4, gives the velocity error equation for the platform ECI navigator :

$$\boxed{\begin{aligned} \dot{\vec{\mathbf{V}}}^{I_c}_{I_c;N_cN} &= \nabla\vec{\mathbf{G}}^{I_c}_c \vec{\mathbf{R}}^{I_c}_{N_cN} - 2[\vec{\mathbf{V}}^{I_c}_{I_c;O_cN_c} \times]\,(\delta\vec{\boldsymbol{\omega}}^{I_c}_{gyro} + \delta\vec{\boldsymbol{\omega}}^{I_c}_{cal}) \\ &- [\vec{\mathbf{R}}^{I_c}_{O_cN_c} \times]\,(\delta\dot{\vec{\boldsymbol{\omega}}}^{I_c}_{gyro} + \delta\dot{\vec{\boldsymbol{\omega}}}^{I_c}_{cal}) \\ &- \delta\vec{\mathbf{f}}^{I_c}_{acc} + C^{I_c}_{E_c}\,\delta\vec{\mathbf{G}}^{E_c}_{residual} \end{aligned}} \tag{8.13}$$

For the strapdown navigator, one starts with obtaining the derivative of the angular velocity of the true inertial frame relative to computed, by differentiation of equation 5.5:

$$\dot{\vec{\boldsymbol{\omega}}}^{I_c}_{I_cI} = \dot{C}^{I_c}_{B_c}\,\delta\vec{\boldsymbol{\omega}}^{B_c}_{gyro} + C^{I_c}_{B_c}\,\delta\dot{\vec{\boldsymbol{\omega}}}^{B_c}_{gyro} \tag{8.14}$$

Since,

$$\dot{C}^{I_c}_{B_c} = C^{I_c}_{B_c}[\vec{\boldsymbol{\omega}}^{B_c}_{I_cB_c} \times] \tag{8.15}$$

equation 8.14 becomes:

$$\dot{\vec{\boldsymbol{\omega}}}^{I_c}_{I_cI} = C^{I_c}_{B_c}\{[\vec{\boldsymbol{\omega}}^{B_c}_{I_cB_c} \times]\,\delta\vec{\boldsymbol{\omega}}^{B_c}_{gyro} + \delta\dot{\vec{\boldsymbol{\omega}}}^{B_c}_{gyro}\} \tag{8.16}$$

The true specific force in body frame is again in the common frame:

$$\vec{\mathbf{f}}^{I_c} - \vec{\mathbf{f}}^{I_c}_m = -C^{I_c}_{B_c}\,\delta\vec{\mathbf{f}}^{B_c}_{acc} \tag{8.17}$$

where the scale factor/misalignment errors are incorporated into the accelerometer instrument error term.

Substitution of equations 5.5, 8.16 and 8.17 into equation 8.11 gives the velocity error equation for the ECI strapdown navigator:

$$\boxed{\begin{aligned}\dot{\vec{\mathbf{V}}}^{I_c}_{I_c;N_cN} &= \nabla\vec{\mathbf{G}}^{I_c}_c \vec{\mathbf{R}}^{I_c}_{N_cN} \\ &- \left\{\begin{aligned} &2[\vec{\mathbf{V}}^{I_c}_{I_c;O_cN_c}\times]\,C^{I_c}_{B_c} \\ &+ [\vec{\mathbf{R}}^{I_c}_{O_cN_c}\times]\,C^{I_c}_{B_c}[\vec{\boldsymbol{\omega}}^{B_c}_{I_cB_c}\times]\end{aligned}\right\}\delta\vec{\boldsymbol{\omega}}^{B_c}_{gyro} \\ &- [\vec{\mathbf{R}}^{I_c}_{O_cN_c}\times]\,C^{I_c}_{B_c}\,\delta\dot{\vec{\boldsymbol{\omega}}}^{B_c}_{gyro} - C^{I_c}_{B_c}\,\delta\vec{\mathbf{f}}^{B_c}_{acc} + C^{I_c}_{E_c}\,\delta\vec{\mathbf{G}}^{E_c}_{residual}\end{aligned}} \tag{8.18}$$

For both platform and strapdown navigators, the position error equation will be the same for both:

$$\boxed{\dot{\vec{\mathbf{R}}}^{I_c}_{N_cN} = \vec{\mathbf{V}}^{I_c}_{I_c;N_cN}} \tag{8.19}$$

C. ECEF Common-Frame Error Equations

The ECEF version of the velocity equations is derived in a manner similar to the ECI error equations. By analogy to equations 8.5 and 8.6, one has:

$$\vec{\mathbf{V}}_{E;ON} = \vec{\mathbf{V}}_{E_c;O_cN_c} + \vec{\mathbf{V}}_{E_c;N_cN} + \vec{\mathbf{R}}_{O_cN_c}\times\vec{\boldsymbol{\omega}}_{E_cE} \tag{8.20}$$

$$\begin{aligned}\vec{\mathbf{a}}_{E;ON} &= \vec{\mathbf{a}}_{E_c;O_cN_c} + D_{E_c}(\vec{\mathbf{V}}_{E_c;N_cN}) \\ &+ 2\vec{\mathbf{V}}_{E_c;O_cN_c}\times\vec{\boldsymbol{\omega}}_{E_cE} + \vec{\mathbf{R}}_{O_cN_c}\times D_{E_c}(\vec{\boldsymbol{\omega}}_{E_cE})\end{aligned} \tag{8.21}$$

The true and computed versions of the ECEF velocity mechanization equations are:

$$\vec{\mathbf{a}}_{E;ON} = 2\vec{\mathbf{V}}_{E;ON}\times\vec{\boldsymbol{\omega}}_{IE} + \vec{\mathbf{f}} + \vec{\mathbf{g}} \tag{8.22}$$

$$\vec{\mathbf{a}}_{E_c;O_cN_c} = 2\vec{\mathbf{V}}_{E_c;O_cN_c}\times\vec{\boldsymbol{\omega}}_{I_cE_c} + \vec{\mathbf{f}}_m + \vec{\mathbf{g}}_c \tag{8.23}$$

Substitution of equations 8.22 and 8.23 into equation 8.21, followed by equation 8.20 results in the following:

$$\begin{aligned}D_{E_c}(\vec{\mathbf{V}}_{E_c;N_cN}) &= 2\vec{\mathbf{V}}_{E_c;O_cN_c}\times(\vec{\boldsymbol{\omega}}_{IE}-\vec{\boldsymbol{\omega}}_{I_cE_c}) \\ &- \vec{\boldsymbol{\omega}}_{IE}\times\vec{\mathbf{V}}_{E_c;N_cN} - \vec{\boldsymbol{\omega}}_{IE}\times(\vec{\mathbf{R}}_{O_cN_c}\times\vec{\boldsymbol{\omega}}_{E_cE}) \\ &- 2\vec{\mathbf{V}}_{E_c;O_cN_c}\times\vec{\boldsymbol{\omega}}_{E_cE} - \vec{\mathbf{R}}_{O_cN_c}\times D_{E_c}(\vec{\boldsymbol{\omega}}_{E_cE}) \\ &+ (\vec{\mathbf{f}}-\vec{\mathbf{f}}_m) + (\vec{\mathbf{g}}-\vec{\mathbf{g}}_c)\end{aligned} \tag{8.24}$$

Coordinatizing the above in computed Earth frame, followed by insertion of equations 4.1 and 4.7 gives, to first-order:

$$\begin{aligned}\dot{\vec{\mathbf{V}}}^{E_c}_{E_c;N_cN} &= \nabla\vec{\mathbf{g}}^{E_c}_c(\vec{\mathbf{R}}^{E_c}_{O_cN_c})\,\vec{\mathbf{R}}^{E_c}_{N_cN} - [\vec{\boldsymbol{\omega}}^{E_c}_{IE_{model}}\times]\vec{\mathbf{V}}^{E_c}_{E_c;N_cN} \\ &- \{2[\vec{\mathbf{V}}^{E_c}_{E_c;O_cN_c}\times] + [\vec{\boldsymbol{\omega}}^{E_c}_{IE_{model}}\times][\vec{\mathbf{R}}^{E_c}_{O_cN_c}\times]\}\vec{\boldsymbol{\omega}}^{E_c}_{E_cE} \\ &- [\vec{\mathbf{R}}^{E_c}_{O_cN_c}\times]\dot{\vec{\boldsymbol{\omega}}}^{E_c}_{E_cE} + (\vec{\mathbf{f}}^{E_c}-\vec{\mathbf{f}}^{E_c}_m) \\ &+ \delta\vec{\mathbf{g}}^{E_c}_{residual} + 2[\vec{\mathbf{V}}^{E_c}_{E_c;O_cN_c}\times]\delta\vec{\boldsymbol{\omega}}^{E_c}_{IE_{model}}\end{aligned} \tag{8.25}$$

Beyond this point, the derivations of the platform and strapdown versions of the velocity error equation differ. For the ECEF platform system, differentiation of equation 6.7 followed by substitution of the same equation gives the angular velocity derivative term:

$$\begin{aligned}\dot{\vec{\boldsymbol{\omega}}}^{E_c}_{E_cE} &= [\vec{\boldsymbol{\omega}}^{E_c}_{IE_{model}}\times]^2\,\vec{\boldsymbol{\psi}}^{E_c}_E \\ &- [\vec{\boldsymbol{\omega}}^{E_c}_{IE_{model}}\times](\delta\vec{\boldsymbol{\omega}}^{E_c}_{gyro}+\delta\vec{\boldsymbol{\omega}}^{E_c}_{cal}) + \delta\dot{\vec{\boldsymbol{\omega}}}^{E_c}_{gyro} + \delta\dot{\vec{\boldsymbol{\omega}}}^{E_c}_{cal} \\ &- [\vec{\boldsymbol{\omega}}^{E_c}_{IE_{model}}\times]\delta\vec{\boldsymbol{\omega}}^{E_c}_{IE_{model}}\end{aligned} \tag{8.26}$$

where the time derivative of the modeled Earth rate is assumed to be zero.

Substitution of equations 6.7, 8.26, and the Earth frame counterpart of equation 8.12 gives the velocity error equation for the ECEF platform navigator :

$$\boxed{\begin{aligned}\dot{\vec{\mathbf{V}}}^{E_c}_{E_c;N_cN} &= \nabla\vec{\mathbf{g}}^{E_c}_c(\vec{\mathbf{R}}^{E_c}_{O_cN_c})\,\vec{\mathbf{R}}^{E_c}_{N_cN} - [\vec{\boldsymbol{\omega}}^{E_c}_{IE_{model}}\times]\vec{\mathbf{V}}^{E_c}_{E_c;N_cN} \\ &+ \left\{\begin{aligned} &2[\vec{\mathbf{V}}^{E_c}_{E_c;O_cN_c}\times][\vec{\boldsymbol{\omega}}^{E_c}_{IE_{model}}\times] \\ &+ [(\vec{\boldsymbol{\omega}}^{E_c}_{IE_{model}}\times\vec{\mathbf{R}}^{E_c}_{O_cN_c})\times][\vec{\boldsymbol{\omega}}^{E_c}_{IE_{model}}\times]\end{aligned}\right\}\vec{\boldsymbol{\psi}}^{E_c}_E \\ &- \{2[\vec{\mathbf{V}}^{E_c}_{E_c;O_cN_c}\times] + [(\vec{\boldsymbol{\omega}}^{E_c}_{IE_{model}}\times\vec{\mathbf{R}}^{E_c}_{O_cN_c})\times]\}(\delta\vec{\boldsymbol{\omega}}^{E_c}_{gyro}+\delta\vec{\boldsymbol{\omega}}^{E_c}_{cal}) \\ &- [\vec{\mathbf{R}}^{E_c}_{O_cN_c}\times]\{[\vec{\boldsymbol{\omega}}^{E_c}_{IE_{model}}\times](\delta\dot{\vec{\boldsymbol{\omega}}}^{E_c}_{gyro}+\delta\dot{\vec{\boldsymbol{\omega}}}^{E_c}_{cal}) - \delta\vec{\mathbf{f}}^{E_c}_{acc} \\ &+ \delta\vec{\mathbf{g}}^{E_c}_{residual} - [(\vec{\boldsymbol{\omega}}^{E_c}_{IE_{model}}\times\vec{\mathbf{R}}^{E_c}_{O_cN_c})\times]\delta\vec{\boldsymbol{\omega}}^{E_c}_{IE_{model}}\end{aligned}} \tag{8.27}$$

For the strapdown navigator, differentiation of equation 6.11 followed by substitution of this equation results in the following (next page):

$$\begin{aligned}\dot{\vec{\boldsymbol{\omega}}}^{E_c}_{E_cE} &= [\vec{\boldsymbol{\omega}}^{E_c}_{IE_{model}} \times]^2\, \vec{\boldsymbol{\psi}}^{E_c}_{E} \\ &\quad - [\vec{\boldsymbol{\omega}}^{E_c}_{IE_{model}} \times] C^{E_c}_{B_c} \delta\vec{\boldsymbol{\omega}}^{B_c}_{gyro} \\ &\quad + \dot{C}^{E_c}_{B_c} \delta\vec{\boldsymbol{\omega}}^{B_c}_{gyro} + C^{E_c}_{B_c} \delta\dot{\vec{\boldsymbol{\omega}}}^{B_c}_{gyro} - [\vec{\boldsymbol{\omega}}^{E_c}_{IE_{model}} \times] \delta\vec{\boldsymbol{\omega}}^{E_c}_{IE_{model}}\end{aligned} \tag{8.28}$$

But since,

$$\dot{C}^{E_c}_{B_c} = C^{E_c}_{B_c} [\vec{\boldsymbol{\omega}}^{B_c}_{E_cB_c} \times] \tag{8.29}$$

equation 8.28 becomes:

$$\begin{aligned}\dot{\vec{\boldsymbol{\omega}}}^{E_c}_{E_cE} &= [\vec{\boldsymbol{\omega}}^{E_c}_{IE_{model}} \times]^2\, \vec{\boldsymbol{\psi}}^{E_c}_{E} \\ &\quad + \{C^{E_c}_{B_c} [\vec{\boldsymbol{\omega}}^{B_c}_{E_cB_c} \times] - [\vec{\boldsymbol{\omega}}^{E_c}_{IE_{model}} \times] C^{E_c}_{B_c}\} \delta\vec{\boldsymbol{\omega}}^{B_c}_{gyro} \\ &\quad + C^{E_c}_{B_c} \delta\dot{\vec{\boldsymbol{\omega}}}^{B_c}_{gyro} - [\vec{\boldsymbol{\omega}}^{E_c}_{IE_{model}} \times] \delta\vec{\boldsymbol{\omega}}^{E_c}_{IE_{model}}\end{aligned} \tag{8.30}$$

The velocity error equation for the ECEF strapdown navigator is obtained by substitution of equations 6.11, 8.30, and the Earth frame counterpart of eq. 8.17 into equation 8.25:

$$\boxed{\begin{aligned}\dot{\vec{\mathbf{V}}}^{E_c}_{E_c;N_cN} &= \nabla\vec{\mathbf{g}}^{E_c}_{c}(\vec{\mathbf{R}}^{E_c}_{O_cN_c})\, \vec{\mathbf{R}}^{E_c}_{N_cN} - [\vec{\boldsymbol{\omega}}^{E_c}_{IE_{model}} \times] \vec{\mathbf{V}}^{E_c}_{E_c;N_cN} \\ &\quad + \left\{\begin{aligned}&2[\vec{\mathbf{V}}^{E_c}_{E_c;O_cN_c} \times][\vec{\boldsymbol{\omega}}^{E_c}_{IE_{model}} \times] \\ &+ [(\vec{\boldsymbol{\omega}}^{E_c}_{IE_{model}} \times \vec{\mathbf{R}}^{E_c}_{O_cN_c})\times][\vec{\boldsymbol{\omega}}^{E_c}_{IE_{model}} \times]\end{aligned}\right\} \vec{\boldsymbol{\psi}}^{E_c}_{E} \\ &\quad - \left\{\begin{aligned}&2[\vec{\mathbf{V}}^{E_c}_{E_c;O_cN_c} \times] + [(\vec{\boldsymbol{\omega}}^{E_c}_{IE_{model}} \times \vec{\mathbf{R}}^{E_c}_{O_cN_c})\times] \\ &+ [\vec{\mathbf{R}}^{E_c}_{O_cN_c} \times][C^{E_c}_{B_c} \vec{\boldsymbol{\omega}}^{B_c}_{E_cB_c} \times]\end{aligned}\right\} C^{E_c}_{B_c} \delta\vec{\boldsymbol{\omega}}^{B_c}_{gyro} \\ &\quad - [\vec{\mathbf{R}}^{E_c}_{O_cN_c} \times] C^{E_c}_{B_c} \delta\dot{\vec{\boldsymbol{\omega}}}^{B_c}_{gyro} - C^{E_c}_{B_c} \delta\vec{\mathbf{f}}^{B_c}_{acc} + \delta\vec{\mathbf{g}}^{E_c}_{residual} \\ &\quad - [(\vec{\boldsymbol{\omega}}^{E_c}_{IE_{model}} \times \vec{\mathbf{R}}^{E_c}_{O_cN_c})\times] \delta\vec{\boldsymbol{\omega}}^{E_c}_{IE_{model}}\end{aligned}} \tag{8.31}$$

The accompanying position equation is the same for both the platform and strapdown ECEF navigators:

$$\boxed{\dot{\vec{\mathbf{R}}}^{E_c}_{N_cN} = \vec{\mathbf{V}}^{E_c}_{E_c;N_cN}} \tag{8.32}$$

D. Local-Level Common-Frame Error Equations

What remains is to obtain the velocity and position equations in a local level frame, L. Because the Earth-referenced velocity is used for both local level navigators, one can obtain the position and velocity equations from equations 8.27, 8.31 and 8.32 by transformations from ECEF to local level frame. The velocity state in computed local level frame is (top):

$$\vec{\mathbf{V}}^{L_c}_{E_c;N_cN} = C^{L_c}_{E_c} \vec{\mathbf{V}}^{E_c}_{E_c;N_cN} \tag{8.33}$$

Differentiate the above and noting that:

$$\dot{C}^{L_c}_{E_c} = [-\vec{\boldsymbol{\omega}}^{L_c}_{E_cL_c} \times] C^{L_c}_{E_c} \tag{8.34}$$

gives:

$$\dot{\vec{\mathbf{V}}}^{L_c}_{E_c;N_cN} = C^{L_c}_{E_c} \dot{\vec{\mathbf{V}}}^{E_c}_{E_c;N_cN} - [\vec{\boldsymbol{\omega}}^{L_c}_{E_cL_c} \times] \vec{\mathbf{V}}^{L_c}_{E_c;N_cN} \tag{8.35}$$

The position error parameter for both the platform and strapdown local level navigator is related to that for the ECEF equations by:

$$\vec{\mathbf{R}}^{L_c}_{N_cN} = C^{L_c}_{E_c} \vec{\mathbf{R}}^{E_c}_{N_cN} \tag{8.36}$$

Differentiation of this equation, followed by substitution of equation 8.35 gives the position equation for a local-level navigator:

$$\boxed{\dot{\vec{\mathbf{R}}}^{L_c}_{N_cN} = -[\omega^{L_c}_{E_cL_c} \times] \vec{\mathbf{R}}^{L_c}_{N_cN} + \vec{\mathbf{V}}^{L_c}_{E_c;N_cN}} \tag{8.37}$$

For the platform local-level navigator, substitution of equation 8.27 into equation 8.35 results in the velocity error equation for the local-level platform navigator:

$$\boxed{\begin{aligned}\dot{\vec{\mathbf{V}}}^{L_c}_{E_c;N_cN} &= \nabla\vec{\mathbf{g}}^{L_c}_{c}\, \vec{\mathbf{R}}^{L_c}_{N_cN} - [(\vec{\boldsymbol{\omega}}^{L_c}_{E_cL_c} + C^{L_c}_{E_c} \vec{\boldsymbol{\omega}}^{E_c}_{IE_{model}})\times] \vec{\mathbf{V}}^{L_c}_{E_c;N_cN} \\ &\quad + \left\{\begin{aligned}&2[\vec{\mathbf{V}}^{L_c}_{E_c;O_cN_c} \times][C^{L_c}_{E_c} \vec{\boldsymbol{\omega}}^{E_c}_{IE_{model}} \times] \\ &+ [C^{L_c}_{E_c}(\vec{\boldsymbol{\omega}}^{E_c}_{IE_{model}} \times \vec{\mathbf{R}}^{E_c}_{O_cN_c})\times][C^{L_c}_{E_c} \vec{\boldsymbol{\omega}}^{E_c}_{IE_{model}} \times]\end{aligned}\right\} \vec{\boldsymbol{\psi}}^{L_c}_{E} \\ &\quad - \{2[\vec{\mathbf{V}}^{L_c}_{E_c;O_cN_c} \times] + [C^{L_c}_{E_c}(\vec{\boldsymbol{\omega}}^{E_c}_{IE_{model}} \times \vec{\mathbf{R}}^{E_c}_{O_cN_c})\times]\}(\delta\vec{\boldsymbol{\omega}}^{L_c}_{gyro} + \delta\vec{\boldsymbol{\omega}}^{L_c}_{cal}) \\ &\quad - [C^{L_c}_{E_c} \vec{\mathbf{R}}^{E_c}_{O_cN_c} \times]\{[C^{L_c}_{E_c} \vec{\boldsymbol{\omega}}^{E_c}_{IE_{model}} \times](\delta\dot{\vec{\boldsymbol{\omega}}}^{L_c}_{gyro} + \delta\dot{\vec{\boldsymbol{\omega}}}^{L_c}_{cal}) - \delta\vec{\mathbf{f}}^{L_c}_{acc} \\ &\quad + C^{L_c}_{E_c} \delta\vec{\mathbf{g}}^{E_c}_{residual} - C^{L_c}_{E_c}[(\vec{\boldsymbol{\omega}}^{E_c}_{IE_{model}} \times \vec{\mathbf{R}}^{E_c}_{O_cN_c})\times] \delta\vec{\boldsymbol{\omega}}^{E_c}_{IE_{model}}\end{aligned}} \tag{8.38}$$

For the strapdown one equation 8.31 is inserted into 8.35 to get:

$$\boxed{\begin{aligned}\dot{\vec{\mathbf{V}}}^{L_c}_{E_c;N_cN} &= \nabla\vec{\mathbf{g}}^{L_c}_{c}\, \vec{\mathbf{R}}^{L_c}_{N_cN} - [(\vec{\boldsymbol{\omega}}^{L_c}_{E_cL_c} + C^{L_c}_{E_c} \vec{\boldsymbol{\omega}}^{E_c}_{IE_{model}})\times] \vec{\mathbf{V}}^{L_c}_{E_c;N_cN} \\ &\quad + \left\{\begin{aligned}&2[\vec{\mathbf{V}}^{L_c}_{E_c;O_cN_c} \times][C^{L_c}_{E_c} \vec{\boldsymbol{\omega}}^{E_c}_{IE_{model}} \times] \\ &+ [C^{L_c}_{E_c}(\vec{\boldsymbol{\omega}}^{E_c}_{IE_{model}} \times \vec{\mathbf{R}}^{E_c}_{O_cN_c})\times][C^{L_c}_{E_c} \vec{\boldsymbol{\omega}}^{E_c}_{IE_{model}} \times]\end{aligned}\right\} \vec{\boldsymbol{\psi}}^{L_c}_{E} \\ &\quad - \left\{\begin{aligned}&2[\vec{\mathbf{V}}^{L_c}_{E_c;O_cN_c} \times] + [C^{L_c}_{E_c}(\vec{\boldsymbol{\omega}}^{E_c}_{IE_{model}} \times \vec{\mathbf{R}}^{E_c}_{O_cN_c})\times] \\ &+ [C^{L_c}_{E_c} \vec{\mathbf{R}}^{E_c}_{O_cN_c} \times][C^{L_c}_{B_c} \vec{\boldsymbol{\omega}}^{B_c}_{E_cB_c} \times]\end{aligned}\right\} C^{L_c}_{B_c} \delta\vec{\boldsymbol{\omega}}^{B_c}_{gyro} \\ &\quad - [C^{L_c}_{E_c} \vec{\mathbf{R}}^{E_c}_{O_cN_c} \times] C^{L_c}_{B_c} \delta\dot{\vec{\boldsymbol{\omega}}}^{B_c}_{gyro} - C^{L_c}_{B_c} \delta\vec{\mathbf{f}}^{B_c}_{acc} + C^{L_c}_{E_c} \delta\vec{\mathbf{g}}^{E_c}_{residual} \\ &\quad - C^{L_c}_{E_c}[(\vec{\boldsymbol{\omega}}^{E_c}_{IE_{model}} \times \vec{\mathbf{R}}^{E_c}_{O_cN_c})\times] \delta\vec{\boldsymbol{\omega}}^{E_c}_{IE_{model}}\end{aligned}} \tag{8.39}$$

Note that the error equations are independent of the actual form of the mechanization equations that are implemented in the navigator.

Using the common frame parameterization of the error, the resulting velocity error equations will possess terms in the position of the computed navigation point N_c relative to computed Earth center O_c. However, in all cases, there are no cross product terms in the specific force. These features distinguish the common frame velocity error equations from the perturbation form of these equations found in the literature. The reconciliation of these forms of the error equations is accomplished using the transformation formulae in the following section:

IX. TRANSFORMATION FORMULAS BETWEEN THE 'PERTURBATION' AND 'COMMON-FRAME' PARAMETERIZATIONS

The new parameterizations, based on relative errors between true and computed position and velocity in the computed frame give terms not previously encountered, while familiar terms, like those in the specific force, are absent. These effects are accounted for by the transformation equations between the perturbation formulation [11], [12], [24] and the new parameterization.

In the perturbation approach, the perturbation in position is the difference of true less computed position, where the true position is coordinatized in the true frame (F) and the computed position in the computed frame (F_c):

$$\delta\vec{\mathbf{R}}^{F_c}_{ON} = \vec{\mathbf{R}}^{F}_{ON} - \vec{\mathbf{R}}^{F_c}_{O_cN_c} \tag{9.1}$$

Relating true frame to computed and by substitution of eq. 6.7 gives :

$$\delta\vec{\mathbf{R}}^{F_c}_{ON} = C^{F}_{F_c}(\vec{\mathbf{R}}^{F_c}_{O_cN_c} + \vec{\mathbf{R}}^{F_c}_{N_cN}) - \vec{\mathbf{R}}^{F_c}_{O_cN_c} \tag{9.2}$$

Now, the true F-frame is related to the computed frame to first-order by:

$$C^{F}_{F_c} \approx I_3 - [\vec{\boldsymbol{\gamma}}^{F_c}\times] \tag{9.3}$$

where $\vec{\boldsymbol{\gamma}}$ represents the appropriate error angle (psi-E in the case of Earth frame error). Substitution into eq. 9.2 and simplifying gives the position error transformation formula:

$$\boxed{\delta\vec{\mathbf{R}}^{F_c}_{ON} = \vec{\mathbf{R}}^{F_c}_{N_cN} + [\vec{\mathbf{R}}^{F_c}_{O_cN_c}\times]\vec{\boldsymbol{\gamma}}^{F_c}} \tag{9.4}$$

For velocity, the perturbation definition of error is the difference:

$$\delta\vec{\mathbf{V}}^{F_c}_{F;ON} = \vec{\mathbf{V}}^{F}_{F;ON} - \vec{\mathbf{V}}^{F_c}_{F_c;O_cN_c} \tag{9.5}$$

Again, the true frame term is related to computed frame (top):

$$\delta\vec{\mathbf{V}}^{F_c}_{F;ON} = C^{F}_{F_c}\vec{\mathbf{V}}^{F_c}_{F;ON} - \vec{\mathbf{V}}^{F_c}_{F_c;O_cN_c} \tag{9.6}$$

From Coriolis' law,

$$\vec{\mathbf{V}}^{F_c}_{F;ON} = \vec{\mathbf{V}}^{F_c}_{F_c;ON} - [\vec{\boldsymbol{\omega}}^{F_c}_{F_cF}\times]\vec{\mathbf{R}}^{F_c}_{ON} \tag{9.7}$$

Taking to first-order and reversing the cross product:

$$\vec{\mathbf{V}}^{F_c}_{F;ON} = \vec{\mathbf{V}}^{F_c}_{F_c;ON} + [\vec{\mathbf{R}}^{F_c}_{O_cN_c}\times]\vec{\boldsymbol{\omega}}^{F_c}_{F_cF} \tag{9.8}$$

Since the velocity of true Earth center relative to computed is negligible,

$$\vec{\mathbf{V}}^{F_c}_{F_c;ON} = \vec{\mathbf{V}}^{F_c}_{F_c;O_cN_c} + \vec{\mathbf{V}}^{F_c}_{F_c;N_cN} \tag{9.9}$$

Substitution of the above into equation 9.8 and the result along with equation 9.3 into equation 9.6 gives the velocity error transformation formula:

$$\boxed{\delta\vec{\mathbf{V}}^{F_c}_{F;ON} = \vec{\mathbf{V}}^{F_c}_{F_c;N_cN} + [\vec{\mathbf{V}}^{F_c}_{F_c;O_cN_c}\times]\vec{\boldsymbol{\gamma}}^{F_c} + [\vec{\mathbf{R}}^{F_c}_{O_cN_c}\times]\vec{\boldsymbol{\omega}}^{F_c}_{F_cF}} \tag{9.10}$$

For attitude, the perturbation in attitude will be taken here to be computed less true:

$$\delta C^{B}_{A} = C^{B_c}_{A_c} - C^{B}_{A} \tag{9.11}$$

Letting $\vec{\boldsymbol{\alpha}}$ denote the error in A frame and $\vec{\boldsymbol{\beta}}$ the error in B-frame, one has, from equation 2.5:

$$C^{B}_{A} = C(\vec{\boldsymbol{\beta}}^{B_c})C^{B_c}_{A_c}C(-\vec{\boldsymbol{\alpha}}^{A_c}) \tag{9.12}$$

To first-order, this becomes:

$$\boxed{C^{B}_{A} = \{I_3 - [\vec{\boldsymbol{\beta}}^{B_c}\times]\}C^{B_c}_{A_c}\{I_3 + [\vec{\boldsymbol{\alpha}}^{A_c}\times]\}} \tag{9.13}$$

Expanding terms, followed by substitution into equation 9.11 and simplifying gives the additive correction:

$$\boxed{\delta C^{B}_{A} = C^{B_c}_{A_c}[\vec{\boldsymbol{\alpha}}^{A_c}\times] - [\vec{\boldsymbol{\beta}}^{B_c}\times]C^{B_c}_{A_c}} \tag{9.14}$$

Letting $\vec{\boldsymbol{\gamma}}$ denote the error rotation vector corresponding to the perturbation, following the convention in equations 2.1-2.3, the error angles are related by:

$$\boxed{\vec{\gamma} \;=\; \vec{\beta} - \vec{\alpha}} \tag{9.15}$$

which holds in any coordinate frame.

To obtain the transformation formula for acceleration, one follows the same steps used to derive equation 9.10. The acceleration error transformation formula has the same form:

$$\boxed{\delta\vec{a}^{F_c}_{F;ON} \;=\; \vec{a}^{F_c}_{F_c;N_cN} + [\vec{a}^{F_c}_{F_c;O_cN_c}\times]\vec{\gamma}^{F_c} + [\vec{V}^{F_c}_{F_c;O_cN_c}\times]\vec{\omega}^{F_c}_{F_cF}} \tag{9.16}$$

The transformation equation for specific force is obtained simply from definition of the perturbation:

$$\delta\vec{f}^{\,F_c} \;=\; \vec{f}^{\,F} \;-\; \vec{f}^{\,F_c}_{m} \tag{9.17}$$

Substitution of equations 3.2 and 9.3 into the above gives the transformation equation for specific force:

$$\boxed{\delta\vec{f}^{\,F_c} \;=\; [\vec{f}^{\,F_c}_{m}\times]\vec{\gamma}^{F_c} \;+\; \delta\vec{f}^{\,F_c}_{m}} \tag{9.18}$$

where $\vec{\gamma}$ once again is interpreted as the rotation vector error from computed to true F-frame.

X. A COMMENT ON MODES OF MOTION

What is generally missing from the discussion of the error equations in the literature is the role of modes of motion. Despite the position and velocity parameterization used, the dominant effect is the gravity gradient matrix (equations 8.13, 8.18, 8.27, 8.31, 8.38, and 8.39). For the parameterizations that use both position and velocity errors, the position error equations are essentially integrators. The combined system gives rise to four modes of motion. There are two Schuler modes at approximately the same frequency, the altitude divergence mode, and an altitude convergence mode. The latter has a time constant equal to that of the altitude divergence mode, but of opposite sign. These modes follow from the geometry of the isopotentials and rhumb line, which is described by the Marussi tensor [25]. The Schuler modes depend primarily on the curvatures of the gravitational isopotential surface and on the curvature of the rhumb line, while the altitude modes have eigenvectors approximately in the direction across isopotentials. Note that the psi angles are not responsible for these modes.

The remaining three modes are inferred from the psi-angle equation. For the ECI navigators, the error angular rates are simply integrated to get the psi-I error, so in this case they act as integrators. The psi-E equation for the ECEF navigator will have an eigenvalue at zero, as well as a complex pair of eigenvalues having a magnitude equal to the Earth rate, that is, a Foucault mode. For the local-level navigator, these eigenvalues get shifted in magnitude by the craft rate.

Assuming that the Earth orientation/rate modeling errors are zero, the psi-E error reduces to the psi-I error and one can obtain the differential equation of the psi-angle error in a given coordinate frame from that in another frame. One can see that in going from the inertial to the Earth to the local-level frames, that the eigenvalues of a pair of modes acts as a complex pair in all cases, differing only in the angular velocity of each frame relative to inertial. This behavior follows from Floquet-Lyapunov theory, discussed in reference [26].

XI. WHOLE-VALUE UPDATE FORMULAS

In an indirect implementation of an inertial navigator [13] [27], the error equations are used in an integration filter to correct the ‘whole-valued’ states, those obtained by integration of the nonlinear navigator mechanization equations. For the common-frame error parameterization developed in this paper, the whole-valued corrections for position and velocity are simply equations 9.4 and 9.10, because the whole-valued corrections are applied in the ‘true’ frame. Equations 9.13 or 9.14 are used to correct the direction cosine matrix.

XII. CONCLUSIONS

A general theory for inertial navigator error modeling must define as large a space of models and the options that one can choose from that lead to these models. While there is insufficient room here to present all possible error equations, a rigorous framework for deriving them is also presented (the phi and delta-theta error equations will be presented later). To achieve this rigor, it is necessary to be precise as to the coordinate frames involved, their roles, and their interrelationships. In the process, much has been said about Earth rate/orientation modeling errors, where in fact these errors are usually negligible in an inertial navigator. But its presence makes it clear that different error angles will be necessary to relate computed and true versions of the coordinate frames.

In the process of precisely defining the coordinate frames, the traditional Pinson-Kachikas definition of the error angles have been extended and related to the definitions used for strapdown by Savage [11] and Rogers [12] to form a consistent picture. Because this new framework treats error angles as between coordinate frames, this makes it easier to develop nonlinear error equations. A few examples are presented, although one can mitigate the nonlinearities using quaternions, confining the nonlinearities to the error angular velocity expressions.

It has been long recognized that there are two possible approaches for representing position and velocity errors [1]. The conventional approach is what is termed the ‘perturbation’ method. Error equations derived using this approach are common in the literature. This paper develops the other approach, where the position and velocity errors are formulated as true less computed, regardless of the coordinate frame in which they are expressed.

No presentation of error equations for the ECEF platform and strapdown navigators is known in the literature, and scant reference is made to the ECI error equations in the open literature [4] [28].

It should be clear from the derivations herein that the utmost care should be taken in the derivation of the error equations for a particular navigator. It is how to do this which puts the theory of inertial navigator error modeling on a firm foundation.

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