On the Spectral Bias of Neural Networks

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Introduction

- Observe characteristics of neural networks which show bias against frequencies in various operations.
- Transform input manifold to facilitate performance against different frequencies

Experiment 1: LF Learned First

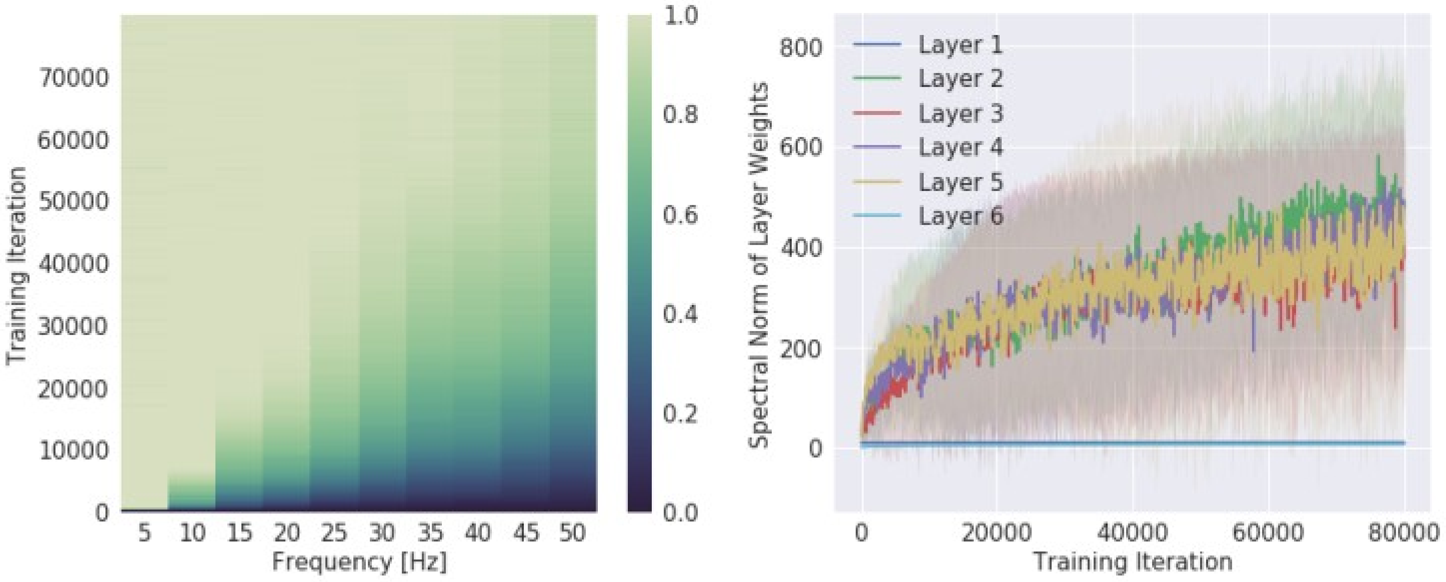
- We want to approximate mappin $\lambda : [0,1] \to \mathbb{R}$ given by:

$$\lambda(z) = \sum_i A_i \sin(2\pi k_i z + \varphi_i).$$

via a neural network (6-layer deep 256-unit wide ReLU network).

where $\kappa = (5, 10, ..., 45, 50)$ with 200 equally-spaced samples over [0, 1].

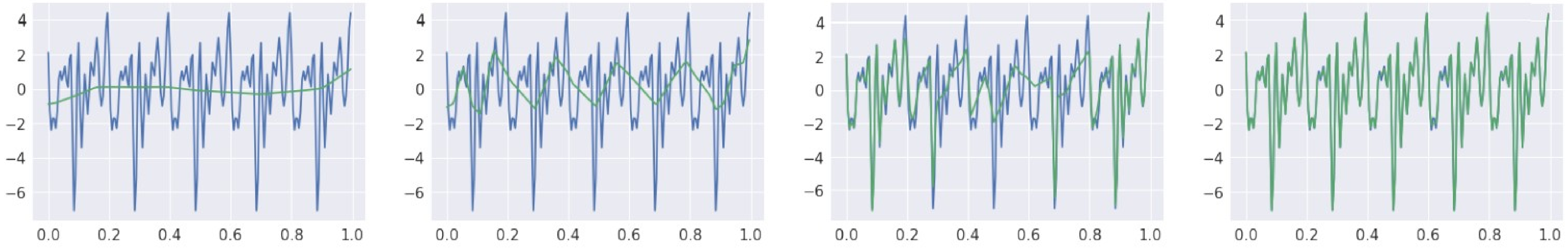
Experiment 1: LF Learned First



(a) Equal Amplitudes

- Left: color represents normalized magnitu $|\tilde{f}_\theta(k_i)|/A_i$ (f: DFT)
- Observe that lower frequencies are learned first
- Different amplitudes have minimal effect

Experiment 1: LF Learned First



(a) Iteration 100 (b) Iteration 1000 (c) Iteration 10000 (d) Iteration 80000

- Green: learnt function
- Blue: target functior $\lambda(z) = \sum_i A_i \sin(2\pi k_i z + \varphi_i).$)

Experiment 2: LF Robust to Parameter Perturbations

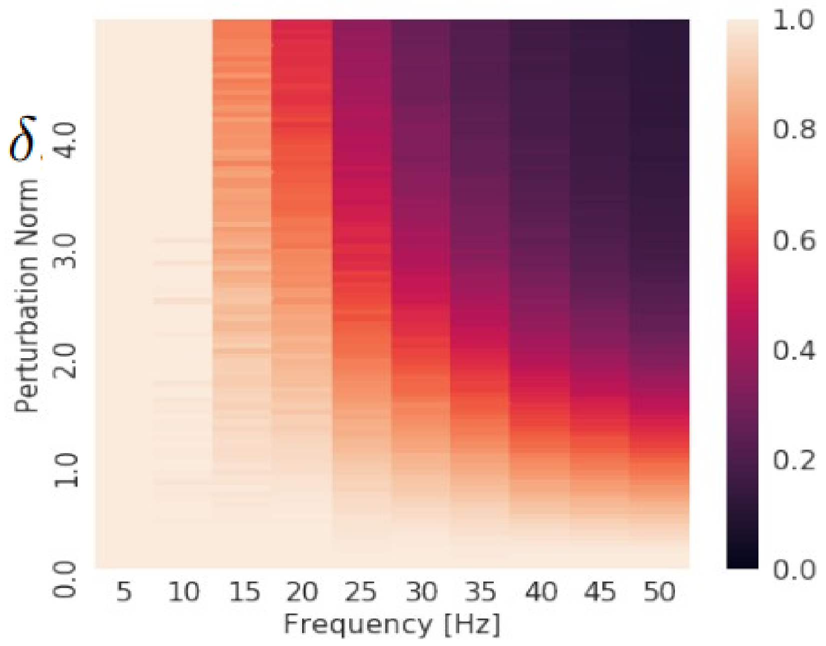
- Same target function $\lambda : [0,1] \to \mathbb{R}$, $\lambda(z) = \sum_i A_i \sin(2\pi k_i z + \varphi_i)$. $\kappa = (5, 10, ..., 45, 50)$ $A_i = 1 \, \forall i$.

- After convergence to θ^* we consider random isotropic perturbations:

$$\theta = \theta^* + \delta\hat{\theta}$$

- We average over 100 samples of $\hat{\theta}$ to obtain $|\tilde{f}_{\mathbb{E}\theta}(k_i)|$ (DFT), normalize by $|\tilde{f}_{\theta*}(k_i)|$ then average over phases ϕ.

Experiment 2: LF Robust to Parameter Perturbations



- Values close to 0: HF components not observed in average?
- *Higher frequencies are significantly less robust than the lower ones, guiding the intuition that expressing higher frequencies requires the parameters to be finely-tuned to work together, i.e. parameters that contribute towards expressing high-frequency components occupy a small volume in the parameter space.*

Experiment 3 & 4: LF Robust to Parameter Perturbations (Real Data)

- Test on MNIST dataset; target binary functi $\tau_0 : X \to \{0,1\}$

defined on input space $X = [0,1]^{784}$

- Can only evaluate by loss on validation set with LF/HF noise and their amplitudes (Experiment 3) because of large input dimension: 28x28=784.

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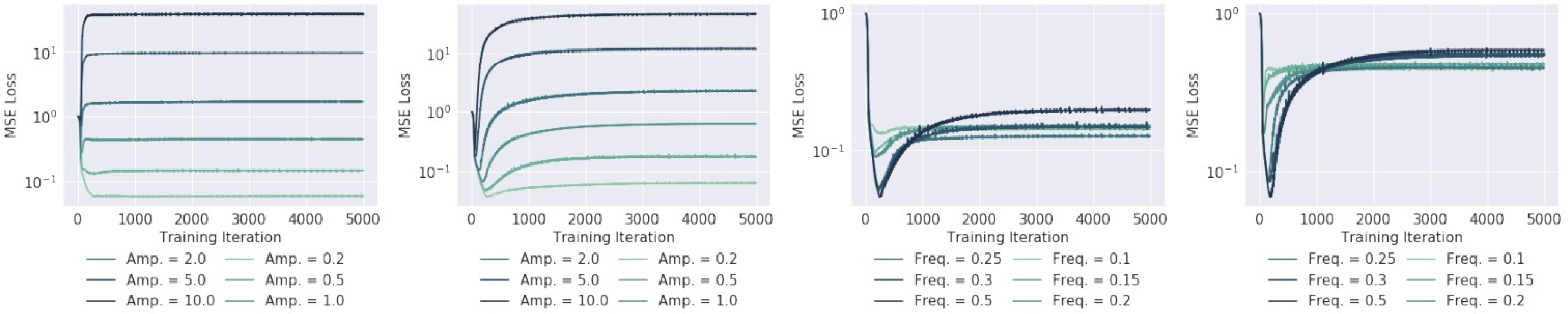


(a) $k = 0.1$ (b) $k = 1$ (c) $\beta = 0.5$ (d) $\beta = 1.$

- *Validation performance is adversely affected by the amplitude of the LF noise, whereas Figure 4b shows that the amplitude of HF noise does not significantly affect the best validation score.*
- This is because HF signal is only fit later in the training.

Experiment 3 & 4: LF Robust to Parameter Perturbations (Real Data)

- We project the network function to the space spanned by orthonormal eigenfunctions φ_n of Gaussian RBF kernel (Braun et al., 2006),

sorted by decreasing eigenvalues which resemble sinusoids, index n being thought of as "frequency".

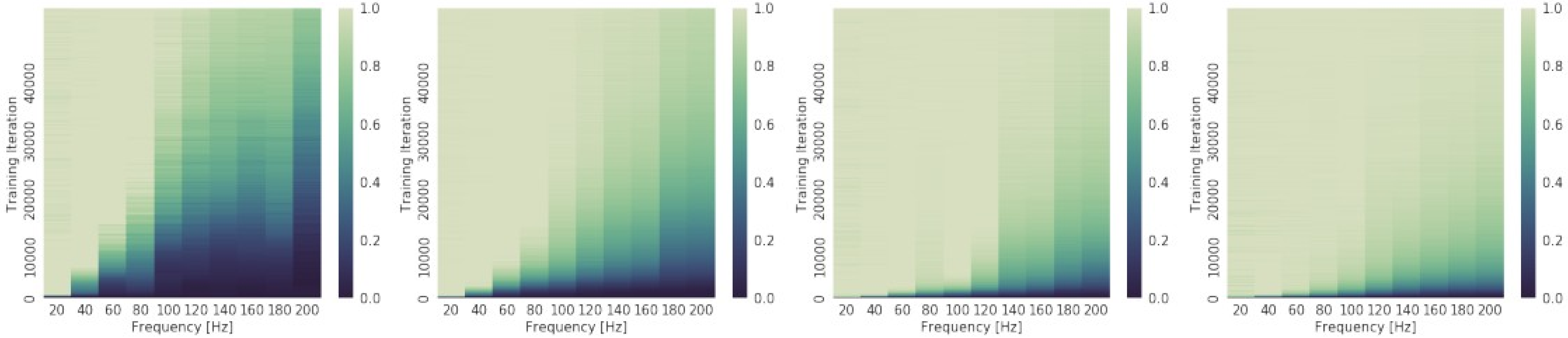
Experiment 5: Manifold hypothesis

- Identify functio $\lambda = \tau \circ \gamma$ where $\gamma : [0,1]^m \to \mathbb{R}^d$ and $\tau : \mathbb{R}^d \to \mathbb{R}$
- We approximate this wit $f : \mathbb{R}^d \to \mathbb{R}$ via a neural network.
- Specifically, we desi $\gamma_L : [0,1] \to \mathbb{R}^2$:

$$\gamma_L(z) = R_L(z)(\cos(2\pi z), \sin(2\pi z))$$
$$\text{where } R_L(z) = 1 + \frac{1}{2}\sin(2\pi L z)$$

- We use same function as in the previous experiments ($\kappa = (20, 40, ..., 180, 200)$

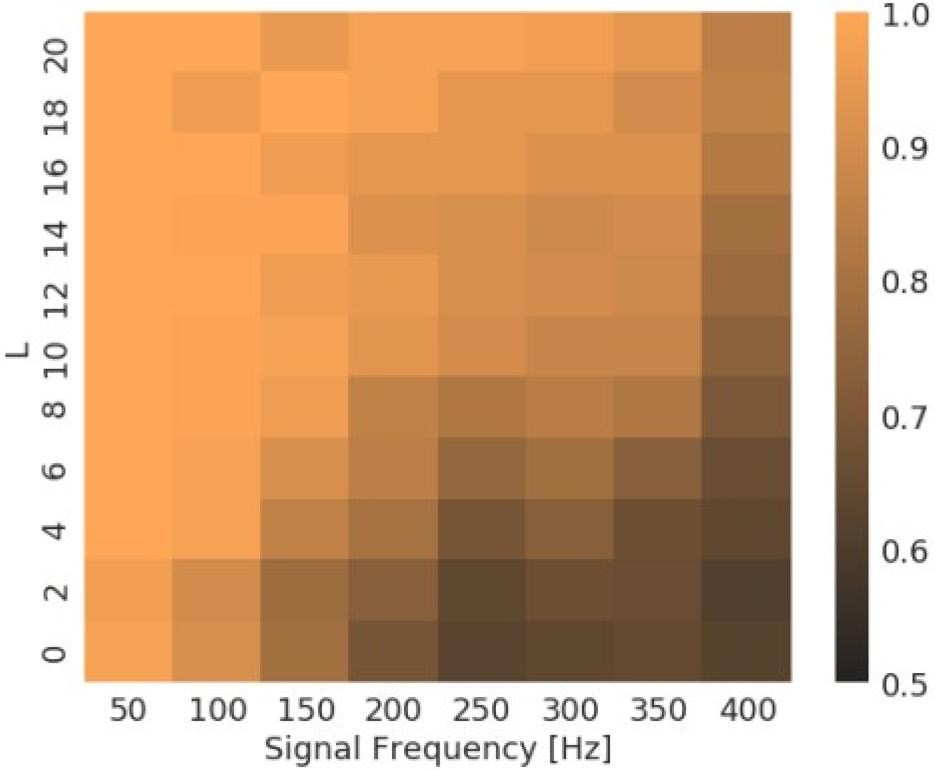
Experiment 5: Manifold hypothesis



(a) $L = 0$ (b) $L = 4$ (c) $L = 10$ (d) $L = 16$

- *Some manifolds (here with larger L) make it easier for the network to learn higher frequencies than others.*

Experiment 6: Manifold hypothesis II



- adapt the setting of Experiment 5 to binary classification by simply thresholding the function λ at 0.5 to obtain a binary

$$\lambda(z) = \sin(2\pi k z + \varphi)$$

$k \in \{50, 100, ..., 350, 400\}$ and $L \in \{0, 2, ..., 18, 20\}$.

Discussion

- Mathematical relationships can be derived btw. k and L
- E6 is also tested on MNIST dataset (See Experiment 8 in supp. mat.)

- To apply to our own training, identify what HF/LF components mean for our data.
- Can run frequency analyses on our own model.