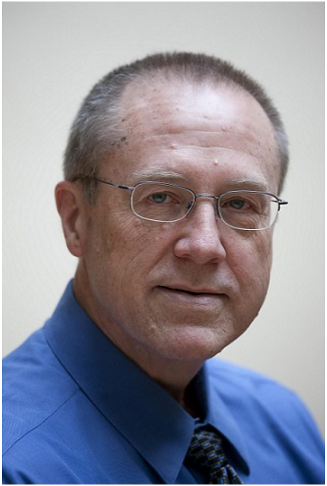
Gödel's Incompleteness Theorems

Acknowledgement



Mic Detlefsen (1948-2019)

An Informal Statement

If a set of axioms A in the language of arithmetic is consistent, “strong enough” and “reasonable”, there is a sentence G such that neither G nor its negation can be proved from the axioms A.

Syntax and Semantics

- Semantics - key notion is truth. Semantics is about the meaning of the statement.
- Syntax - key notion is proof. Syntax is about the form of the statement.

Proof

A proof of a sentence S from axioms A is a sequence of sentences

$$S_1, S_2, \ldots, S_n = S$$

such that for each i, either S_i is an axiom or S_i follows from $S_1, \ldots, S_{i-1}$ by applying some logical rule of inference.

The logical rules are syntactical (based on the form of the statements not their meaning) and are independent of the meaning we give the sentences.

But real proofs have semantic and intuitive content.

Computability

> *This requirement for the rules and the axioms is equivalent to the requirement that it should be possible to build a finite machine, in the precise sense of a "Turing machine", which will write down all the consequences of the axioms one after the other.*
>
> Gödel (1951)

The set of theorems that can be proved from A is computably enumerable.

The language of Arithmetic

- A constant symbol: $\underline{0}$
- A unary operator: S
  $S(\underline{0}), S(S(\underline{0})), S(S(S(\underline{0}))), \ldots$ are intended to be names for the numbers $1, 2, 3, \ldots$
- Binary operators: $+ \quad \times$
- Equality: $=$
- Propositional logical operators: $\neg \quad \vee \quad \wedge \quad \rightarrow$
- Variables (countably many): $x, y, z, x_1, y_1, z_1, \ldots$
- Quantifiers: $\forall \quad \exists$
- Punctuation: ( ) ,

First order logic: Variables range over numbers quantification is over numbers and not sets of numbers.

But often real proofs are conducted in second-order mathematics (e.g., mathematical induction, completeness property for the reals)

Robinson's Q

1. $\forall x(\neg(\underline{0} = S(x)))$
2. $\forall x \forall y((S(x) = S(y) \rightarrow (x = y))$
3. $\forall x(\neg(x = \underline{0}) \rightarrow \exists y(x = S(y))$
4. $\forall x(x + \underline{0} = x)$
5. $\forall x \forall y(x + S(y) = S(x + y))$
6. $\forall x(x \times \underline{0} = \underline{0})$
7. $\forall x \forall y(x \times S(y) = x \times y + y)$

From the axioms Q, we can prove all quantifier-free sentences that are "true" about arithmetic. ("Third-grade arithmetic")

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Gödel's Theorem

If A is a Turing-computable, consistent set of axioms in the first-order language of arithmetic and $Q \subseteq A$, there is a sentence of first-order arithmetic, G, such that neither G nor its negation can be proven from axioms A.

Gödel's original hypotheses on the axioms A were stronger.

About G

- G is relative; it depends on A. (G_A)
- G can be produced computably (and uniformly) from A
- If the axioms of A are all “true”, then G is evidently true.

Gödel's sentence G

G says "G is not provable from axioms A"

If G is provable from A, then both these sentences are provable (meaning A is inconsistent):

- G is provable from axioms A
- G is not provable from axioms A

Therefore G is not provable and therefore true.

But G is not a statement about arithmetic. Also, this argument is semantic.

The Arithmetization of Syntax

Assign a unique number to every syntactic object, its Gödel number.
There is a formula $\mathrm{Prf}_A(x, y)$ with two free variables x and y such that

- If n is the number of proof of a sentence with number m then the sentence $\mathrm{Prf}_A(\underline{n}, \underline{m})$ can be proved from axioms A. $A \vdash \mathrm{Prf}_A(\underline{n}, \underline{m})$.
- IF n is not the number of a proof of a sentence with number m then $A \vdash \neg\mathrm{Prf}_A(\underline{n}, \underline{m})$

The Fixed-Point Theorem

There is a sentence of arithmetic G such that, if n is the number of G, A proves

$$G \leftrightarrow \neg \exists x (\mathrm{Prf}_A(x, \underline{n}))$$

G says “I am not provable."

Gödel's Second Incompleteness Theorem

Suppose that A is a consistent, computable, "strong enough" set of axioms in the language of arithmetic. Let n be the number of the sentence $\neg(\underline{0} = \underline{0})$. Then A cannot prove

$$\neg \exists x (\mathrm{Prf}(x, \underline{n}))$$

In other words, A cannot prove its own consistency.

Philosophical/Meta-mathematical Implications

1. Incompletability (inexhaustability) of mathematics
2. Mechanism (Lucas, Penrose)
3. Consistency
4. Platonism, Formalism, etc.

Gödel's Gibbs Lecture, 1951

Some basic theorems on the foundations of mathematics and their philosophical implications.

> *Research in the foundations of mathematics during the past few decades has produced some results which seem to me of interest, not only in themselves, but also with regard to their implications for the traditional philosophical problems about the nature of mathematics.*

Undecidable Sentences

Undedicable Sentence

True by proof of GT

Not obviously true by proof of GT

Decidable in an evident extension

Not decidable in an evident extension

Undecidable Sentences - Arithmetic

- First-order Peano Arithmetic (PA):

  The following theorem is true but unprovable in PA (Paris-Harrington (1976)):

  For all n, k, r there is an m such that $m \underset{*}{\rightarrow} (n)^k_r$

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Undecidable Sentences - Set Theory

- ZFC:

  The Continuum Hypothesis is independent of ZFC (Gödel, Cohen)

Gödel's Dichotomy

Either

> *The human mind (even within the realm of pure mathematics) infinitely surpasses the powers of any finite machine*

Or

> *there exists absolutely unsolvable Diophantine problems.*

Gödel's Diophantine Problem

Gödel's sentence is equivalent to one of the following form:

$$\forall x_1 \ldots \forall x_n \exists y_1 \ldots \exists y_m [P(x_1, \ldots, x_n, y_1, \ldots, y_m) = Q(x_1, \ldots, x_n, y_1, \ldots, y_m)]$$

P and Q are polynomials in $n + m$ variables over the natural numbers (of degree at most 4)

MRPD Theorem

Can get rid of the existential quantifiers. (Solution to Hilbert's 10th Problem.)

$$\forall x_1 \forall x_2 \ldots \forall x_n [P(x_1, \ldots, x_n) \neq Q(x_1, \ldots, x_n)]$$

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Gödel on the future of foundational research

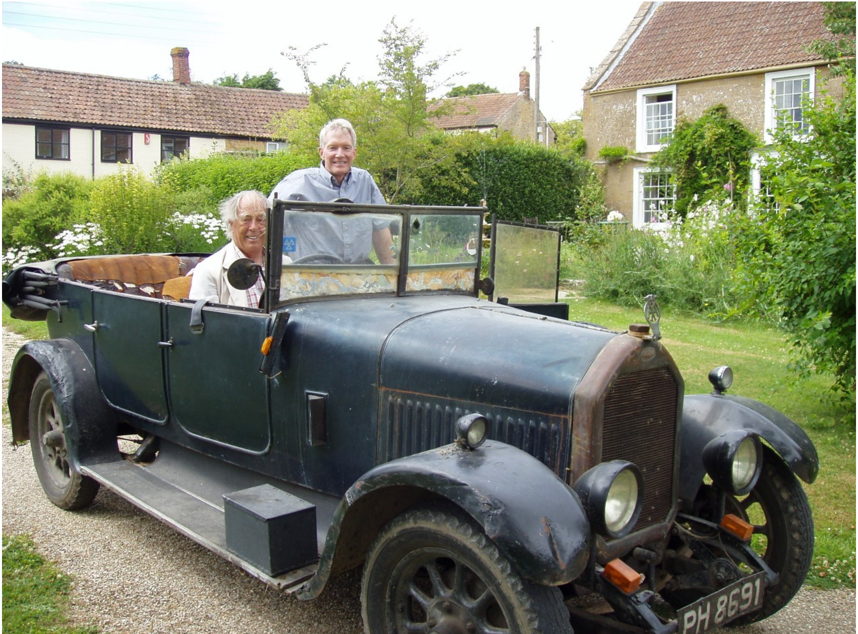
... after sufficient clarification of the concepts in question ... the result will be that ... the Platonistic view is the only one tenable. Thereby I mean the view that mathematics describes a non-sensual reality, which exists independently both of the acts and the dispositions of the human mind. This view is rather unpopular among mathematicians ...

The Mechanistic Hypothesis

T	true arithmetic sentences	“objective” mathematics
K	knowable arithmetic sentences	“subjective” mathematics

The “mechanistic” thesis: K can be generated by a machine M

Lucas



An arugment against mechanism (Lucas, Penrose)

1. Assume the mind is a machine, M, and K is the output of M.
2. Every knowable sentence is true.
3. Then the Gödel sentence G_M is true.
4. And, G_M is not in the output of M. (So $G_M \notin K$)
5. But we know that G_M is true. (So $G_M \in K$.)

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How do we know arithmetic (say PA) is consistent?

Semantic Proof: The axioms have a model - the “standard” model of arithmetic.

Syntactic Proof: Any syntactic proof requires something besides PA by the second incompleteness theorem.

The first consistency proof generally accepted was Gentzen (1936). It required transfinite induction up to a certain transfinite (but countable) ordinal number ϵ_0.

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Hilbert's Program

- Formalize mathematics in axiomatic form
- Prove that the formalization is consistent
- The consistency proof should only use "finitary" methods.

Second-Order Logic

Allows variables that range over relations and functions defined on individuals.

Advantage: Expressive power - categoricity

The following second order axiom together with the first order axioms for the successor (S) function characterize the natural numbers with the successor operation.

$$(\forall X)[X(0) \wedge (\forall y(X(y) \rightarrow X(Sy))) \rightarrow (\forall y)X(y)]$$

Disadvantage: No complete proof theory

Gödel Proposition VI

To every ω-consistent recursive class c of formulae, there corresponds a recursive class-signs r, such that neither v Gen r nor Neg v Gen r belongs to Flg(c) (where v is the free variable of r).

Gödel's First (Unpublished)Proof

- The set of true arithmetic statements is not definable in arithmetic. (Tarski's Theorem and the Diagonalization Lemma)
- The set of provable arithmetic statements is definable in arithmetic. (Arithmetization of Syntax)
- Therefore, the set of provable statements is not the same as the set of true statements.

References

- Gödel's Gibbs Lecture
- John Lucas Against Mechanism, issue of Ethics and Politics, https://www2.units.it/etica/2003_1/index.html
- Stanford Encyclopedia of Philosophy, https://plato.stanford.edu/entries/goedel-incompleteness/
- The Popular Impact of Gödel's Theorem, Torkel Franzen, Notices AMS, 2006

Topics

- Incompletability of Mathematics
  - “Absolutely” Undecidable Sentences?
  - Gödel Gibbs Lecture
  - Gödel’s Dichotomy
  - Natural Undecidable Sentences
- The argument against mechanism
  - Mechanism
  - Lucas Argument
- Consistency proofs
  - Consistency of arithmetic
  - Hilbert’s Program
- Platonism, Formalism, etc.
  - Gödel on Platonism
- Miscellaneous
  - Second-order Logic
  - Gödel’s Original Statement (Proposition VI)
  - Gödel’s Original (Unpublished) Proof
  - References

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