Gödel's Theorem

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An Informal Statement

If a set of axioms A in the language of arithmetic is consistent, "strong enough" and "reasonable", there is a sentence G such that neither G nor its negation can be proved from the axioms A.

Syntax and Semantics

- Semantics - key notion is truth. Semantics is about the meaning of the statement.
- Syntax - key notion is proof. Syntax is about the form of the statement.

Proof

A proof of a sentence S from axioms A is a sequence of sentences

$$S_1, S_2, \ldots, S_n = S$$

such that for each i, either S_i is an axiom or S_i follows from $S_1, \ldots, S_{i-1}$ by applying some logical rule of inference.

The logical rules are syntactical (based on the form of the statements not their meaning) and are independent of the meaning we give the sentences.

The language of Arithmetic

- A constant symbol: $\underline{0}$
- A unary operator: S
  $S(\underline{0}), S(S(\underline{0})), S(S(S(\underline{0}))), \ldots$ are intended to be names for the numbers $1, 2, 3, \ldots$
- Binary operators: $+ \quad \times$
- Equality: $=$
- Propositional logical operators: $\neg \quad \vee \quad \wedge \quad \implies$
- Variables (countably many): $x, y, z, x_1, y_1, z_1, \ldots$
- Quantifiers: $\forall \quad \exists$
- Punctuation: $(\quad) \quad ,$

First order logic: Variables range over numbers quantification is over numbers and not sets of numbers.

Axioms

- Consistent (doesn't prove both ϕ and $\neg\phi$ for any ϕ)
- "Reasonable" (computable)
- "Strong Enough" (Includes the following axioms Q)
  1. $\forall x(\neg(0 = S(x)))$
  2. $\forall x \forall y((S(x) = S(y) \implies (x = y))$
  3. $\forall x(\neg(x = 0) \implies \exists y(x = S(y))$
  4. $\forall x(x + 0 = x)$
  5. $\forall x \forall y(x + S(y) = S(x + y))$
  6. $\forall x(x \times 0 = 0)$
  7. $\forall x \forall y(x \times S(y) = x \times y + y)$

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Digression into semantics

- The axioms of Q are all “true.”
- In fact Q proves all the true quantifier-free facts about the natural numbers.
- The axioms of Q do not determine the structure of the natural numbers up to isomorphism (are not “categorical”). There are “non-standard” models of Q.
- No set of axioms for arithmetic that are all “true” is categorical. (There are non-standard models of arithmetic.)

Computability

This requirement for the rules and the axioms is equivalent to the requirement that it should be possible to build a finite machine, in the precise sense of a "Turing machine", which will write down all the consequences of the axioms one after the other.

Gödel (1951)

The set of theorems that can be proved from A is computably enumerable.

Gödel's Theorem

If A is a computable, consistent set of axioms in the first-order language of arithmetic and $Q \subseteq A$, there is a sentence of first-order arithmetic, G, such that neither G nor $\neg G$ can be proven from axioms A.

About G

- G is relative; it depends on A. (G_A)
- G can be produced computably from A
- If the axioms of A are all "true", then G is evidently true.

The Arithmetization of Syntax

Assign a unique number to every syntactic object, its Gödel number.
There is a formula $\mathrm{Prf}_A(x, y)$ with two free variables x and y such that

- If n is the number of proof of a sentence with number m then A proves the sentence $\mathrm{Prf}_A(\underline{n}, \underline{m})$
- IF n is not the number of a proof of a sentence with number m then A proves the sentence $\neg\mathrm{Prf}_A(\underline{n}, \underline{m})$

Diagonalization

There is a sentence of arithmetic G such that, if n is the number of G, A proves

$$G \iff \neg \exists x(\mathrm{Prf}_A(x, \underline{n}))$$

G says “I am not provable."

Gödel's Second Incompleteness Theorem

Suppose that A is a consistent, computable, "strong enough" set of axioms in the language of arithmetic. Let n be the number of the sentence $\neg(\underline{0} = \underline{0})$. Then A cannot prove

$$\neg \exists x (\mathrm{Prf}(x, \underline{n}))$$

In other words, A cannot prove its own consistency.

Extensions, Developments

Gödel's Theorem applies to other domains/languages. In particular, first-order set theory.

Are there "natural" undecidable sentences about arithmetic? (Kirby-Paris-Harrington, Friedman)

The Continuum Hypothesis?

Second order number theory (analysis) and reverse mathematics.

Gödel's Gibbs Lecture, 1951

Research in the foundations of mathematics during the past few decades has produced some results which seem to me of interest, not only in themselves, but also with regard to their implications for the traditional philosophical problems about the nature of mathematics.

Gödel on the future of foundational research

. . . after sufficient clarification of the concepts in question . . . the result will be that . . . the Platonistic view is the only one tenable. Thereby I mean the view that mathematics describes a non-sensual reality, which exists independently both of the acts and the dispositions of the human mind. This view is rather unpopular among mathematicians . . .

Gödel's Dichotomy

Either

> *The human mind (even within the realm of pure mathematics) inifinitely surpasses the powers of any finite machine*

Or

> *there exists absolutely unsolvable Diophantine problems.*

Gödel's Diophantine Problem

$$\forall x_1 \forall x_2 \ldots \forall x_n \exists y_1 \exists y_2 \ldots \exists y_m [P(x_1, \ldots, x_n, y_1, \ldots, y_m) = 0]$$

P a polynomial in $n + m$ variables (of degree at most 4)

Hilbert's 10$^{\text{th}}$ Problem is unsolvable.

There is no algorithm for determining whether an arbitrary Diophantine equation has a solution in integers.

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The Mechanistic Hypothesis

T	true arithmetic sentences	“objective” mathematics
K	knowable arithmetic sentences	“subjective” mathematics

The “mechanistic” thesis: K can be generated by a machine M

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An arugment against mechanism

1. Assume the mind is a machine, M, and K is the output of M.
2. Every knowable sentence is true.
3. Then the Gödel sentence G_M is true.
4. And, G_M is not in the output of M. (So $G_M \notin K$)
5. But we know that G_M is true. (So $G_M \in K$.)

References

- Gödel's Gibbs Lecture, https://partiallyexaminedlife.com/wp-content/uploads/Godel-Basic-Theorems-and-Their-Implications-1.pdf
- Stanford Encyclopedia of Philosophy, https://plato.stanford.edu/entries/goedel-incompleteness/