Homework 3

Your full names go here

Problem 1 (Golan 124). Let F be a field and let $(K, \bullet)$ be an associative unital F-algebra. If A and B are subsets of K, we let $A \bullet B$ be the set of all elements of K of the form $a \bullet b$, with $a \in A$ and $b \in B$ (in particular, $\emptyset \bullet B = A \bullet \emptyset = \emptyset$). We know that the set V of all subsets of K is a vector space over GF(2). Is $(V, \bullet)$ a GF(2)-algebra? If so, is it associative? Is it unital?

Solution. (type your solution here)

Problem 2 (Golan 132). Let F be a field and let L be the set of all polynomials $f(X) \in F[X]$ satisfying the condition that $f(-a) = -f(a)$ for all $a \in F$. Is L a subspace of $F[X]$?

Solution. (type your solution here)

Problem 3 (Golan 133). Let F be a field and let L be the set of all polynomials $f(X) \in F[X]$ satisfying the condition that $\deg(f)$ is even. Is L a subspace of $F[X]$?

Solution. (type your solution here)

Problem 4 (Golan 142). For a field F, compare the subsets $F[X^2]$ and $F[X^2 + 1]$ of $F[X]$.

Solution. (type your solution here)

Problem 5 (Golan 154). Let F be a field and let $K = F^{\mathbb{N}}$. Define operations $+$ and $\bullet$ on K by setting $f + g : i \mapsto f(i) + g(i)$ and $f \bullet g : i \mapsto \sum_{j+k=i} f(j)g(k)$. Show that K is a an associative and commutative unital F-algebra. Is it entire?

Solution. (type your solution here)

Problem 6 (Golan 157). A *trigonometric polynomial* in $\mathbb{R}^{\mathbb{R}}$ is a function of the form $t \mapsto a_0 + \sum_{h=1}^{k}[a_h \cos(ht) + b_h \sin(ht)]$, where $a_0, \ldots, a_k, b_1, \ldots, b_k \in \mathbb{R}$. Show that the subset of $\mathbb{R}^{\mathbb{R}}$ consisting of all trigonometric polynomials is an entire $\mathbb{R}$-algebra.

Solution. (type your solution here)

Problem 7 (Golan 163). Let $F = \mathbb{Q}$. Is the subset

$$\left\{ \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} \right\}$$

of F^3 linearly independent? What happens if $F = \mathrm{GF}(5)$?

Solution. (type your solution here)

Problem 8 (Golan 177). Show that the subset

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is a linearly independent subset of $\mathrm{GF}(p)^3$ if and only if $p \neq 3$.

Solution. (type your solution here)

Clarification/amplification of Props 5.12 and 5.13. Let V be a vector space, and let $\{W_\omega : \omega \in \Omega\}$ be a collection of subspaces of V. Recall that $\sum_{\omega\in\Omega} W_\omega$ denotes the subspace of all (finite) linear combinations of vectors in $\{W_\omega : \omega \in \Omega\}$, which is equivalent to the subspace of vectors w of the form $w = \sum_{\lambda\in\Lambda} w_\lambda$, where $w_\lambda \in W_\lambda$ and Λ is a finite subset of Ω.
We call the set $\{W_\omega : \omega \in \Omega\}$ *independent* if and only if it satisfies the following condition: If Λ is a finite subset of Ω, if $w_\lambda \in W_\lambda$ for each $\lambda \in \Lambda$, and if $\sum_{\lambda\in\Lambda} w_\lambda = 0_V$, then $w_\lambda = 0_V$ for all $\lambda \in \Lambda$.

Problem 9. Let

$$W_1 = \left\{ \begin{bmatrix} a \\ 0 \end{bmatrix} : a \in \mathbb{R} \right\},\ W_2 = \left\{ \begin{bmatrix} b \\ b \end{bmatrix} : b \in \mathbb{R} \right\},\ W_3 = \left\{ \begin{bmatrix} 0 \\ c \end{bmatrix} : a \in \mathbb{R} \right\}.$$

1. Would our textbook author describe $\{W_1, W_2, W_3\}$ as "pairwise disjoint?" (explain)
2. Describe the space $W_1 + W_2 + W_3$.
3. Describe the space $W_i + W_j$ for each pair $i \neq j$ in $\{1, 2, 3\}$.
4. Is the set $\{W_1, W_2, W_3\}$ independent?

Solution. (type your solution here)

Problem 10. Below is an alleged theorem that attempts to combine Propositions 5.12 and 5.13 of the textbook. Prove it, or disprove it by providing a counterexample.

Proposition. Suppose $\{W_\omega : \omega \in \Omega\}$ is a collection of subspaces of a vector space, and suppose, for each $\omega \in \Omega$, the set B_ω is a basis for W_ω. Then the following are equivalent:

1. The set $\{W_\omega : \omega \in \Omega\}$ is independent.
2. Every $w \in \sum_{\omega\in\Omega} W_\omega$ can be written as $w = \sum_{\lambda\in\Lambda} w_\lambda$ in exactly one way.
3. For every $\lambda \in \Omega$, $W_\lambda \cap \sum_{\omega\neq\lambda} W_\omega = \{0_V\}$.
4. The set $B = \bigcup_{\omega\in\Omega} B_\omega$ is a basis for $\sum_{\omega\in\Omega} W_\omega$.

(type your proof or counterexample here)

Final remark. We use $\bigoplus_{\omega\in\Omega} W_\omega$ to denote $\sum_{\omega\in\Omega} W_\omega$ *only when* the set $\{W_\omega : \omega \in \Omega\}$ is independent. When I mentioned this in class, instead of *only when*, I used the phrase *when and only when*. This is incorrect since the expression $\sum_{\omega\in\Omega} W_\omega$ does not necessarily imply that the set $\{W_\omega : \omega \in \Omega\}$ is dependent.