Homework 1 hints

Problem (Golan 16). Let z_1, z_2, and z_3 be complex numbers satisfying $|z_i| = 1$ for $i = 1, 2, 3$. Show that $|z_1 z_2 + z_1 z_3 + z_2 z_3| = |z_1 + z_2 + z_3|$.

Hint. Let $w = z_1 z_2 z_3$. Note that the modulus of w is $|w| = 1$, since each z_i has modulus 1. The left hand side of the equality we want to establish is $|w/z_3 + w/z_2 + w/z_1|$. Factoring out $|w|$, we have $|w||1/z_3 + 1/z_2 + 1/z_1|$. Note that since $|z_i| = 1$, the conjugate of z_i is $1/z_i$.

Problem (Golan 22 *Abel's inequality*). Let $z_1, \ldots, z_n$ be a list of complex numbers and, for each $1 \le k \le n$, let $s_k = \sum_{i=1}^k z_i$. For real numbers $a_1, \ldots, a_n$ satisfying $a_1 \ge a_2 \ge \cdots \ge a_n \ge 0$, show that

$$\left| \sum_{i=1}^n a_i z_i \right| \le a_1 \left(\max_{1 \le k \le n} |s_k| \right). \tag{1}$$

Hint. The kth term of $\sum a_i z_i$ is

$$a_k z_k = a_k(z_1 + \cdots + z_k) - a_k(z_1 + \cdots + z_{k-1}) = a_k s_k - a_k s_{k-1},$$

so

$$\begin{aligned} \left| \sum_{i=1}^n a_i z_i \right| &= |a_n s_n - a_n s_{n-1} + a_{n-1} s_{n-1} - a_{n-1} s_{n-2} + \cdots + a_2 s_2 - a_2 s_1 + a_1 s_1| \\ &= |s_1(a_1 - a_2) + s_2(a_2 - a_3) + \cdots + s_{n-1}(a_{n-1} - a_n) + s_n a_n| \,. \end{aligned}$$

Now use the triangle inequality to bound this expression, and then factor a certain maximum out of the sum. If you recognize what remains of the sum as a telescoping series, the desired inequality will reveal itself. (You should try to write down the proof, either following the suggestions outlined above and filling in the missing details, or by finding an alternative solution of your own.)