Homework 2

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Problem 1 (Golan 56). Is it possible to define on $\mathbb{Z}/(4)$ the structure of a vector space over GF(2) in such a way that the vector addition is the usual addition in $\mathbb{Z}/(4)$?

[*Hints*: Recall that (n) denotes the set $\{\dots, -2n, -n, 0, n, 2n, 3n, \dots\}$, which we denoted in lecture by $n\mathbb{Z}$. This is the ideal generated by n in the ring $\mathbb{Z}$, but don't worry about that for now. Just take $\mathbb{Z}/(n)$ to be the abelian group of integers $\{0, 1, 2, \dots, n-1\}$ with addition modulo n. In lecture, we used $\mathbb{Z}/n\mathbb{Z}$ to denote $\mathbb{Z}/(n)$. Use whichever notation you prefer.]

Solution. Assume toward a contradiction that $\mathbb{Z}/(4)$ is a vector space over GF(2), with vector addition defined as the usual modular addition. Then for any $v \in \mathbb{Z}/(4)$,

$$\begin{aligned} 0 &= (0)v \\ &= (1+1)v \\ &= 1 \cdot v + 1 \cdot v \\ &= v + v \end{aligned}$$

Now, let $v = 3$ to see that $3 + 3 = 2 \mod 4 \neq 0$. Thus we have a contradiction. The answer is no, it is not possible with the usual modular vector addition.

Problem 2 (Golan 60). Let $V = C(0,1)$. Define an operation $\boxplus$ on V by setting $f \boxplus g : x \mapsto \max\{f(x), g(x)\}$. Does this operation of vector addition, together with the usual scalar multiplication make V into a vector space over $\mathbb{R}$?

Solution. We must show that the $\boxplus$ operation is closed in V, that is, $f \boxplus g \in C(0,1)$. We will prove that $f \boxplus g$ is continuous by the "delta-epsilon" method from real analysis.

Let $\epsilon > 0$. As f,g continuous on $(0,1)$, there exist $\delta_f > 0$ and $\delta_g > 0$ such that

$$|x - x_0| < \delta_f \implies |f(x) - f(x_0)| < \frac{\epsilon}{2}$$
$$|x - x_0| < \delta_g \implies |g(x) - g(x_0)| < \frac{\epsilon}{2}$$

Let $h(x) = max\{f(x), g(x)\}$. Let $\delta_h = \min\{\delta_f, \delta_g\}$. Then

$$\begin{aligned}
|x - x_0| < \delta_h \quad \implies & \quad |h(x) - h(x_0)| \\
= & \quad \left| \frac{f(x) + g(x) + |f(x) - g(x)| - f(x_0) - g(x_0) - |f(x_0) - g(x_0)|}{2} \right| \\
\leq & \quad \left| \frac{f(x) - f(x_0)}{2} \right| + \left| \frac{g(x) - g(x_0)}{2} \right| + \left| \frac{f(x) - f(x_0) - (g(x) - g(x_0))}{2} \right| \\
< & \quad \frac{\epsilon}{2} + \left| \frac{f(x) - f(x_0) - (g(x) - g(x_0))}{2} \right| \\
< & \quad \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\end{aligned}$$

Therefore $f \boxplus g$ is continuous for $x_0 \in (0, 1)$, so vector addition is closed in V. With the usual scalar multiplication, V is a vector space over $\mathbb{R}$.

Problem 3 (Golan 63). Let $V = \{i \in \mathbb{Z} \mid 0 \leq i < 2^n\}$ for some fixed positive integer n. Define operations of vector addition and scalar multiplication on V in such a way as to turn it into a vector space over GF(2).

[*Hints*: Recall that GF(2) denotes the Galois field with two elements, $\{0, 1\}$, with addition mod 2 and the usual multiplication. Other than this field, the only restriction given in the problem is that V must have 2^n elements. Do you know of any sets of this size?]

Solution. Let us define vector addition in a "bitwise xor" fashion such that $v + v = 0$ and $v + w = 1$ for all $v, w \in V, w \neq v$. Furthermore, let us define scalar multiplication in the natural way such that $1 \cdot v = v$ and $0 \cdot v = 0$. Then we can see that vector addition is closed as $0, 1 \in V$, as well as being associative and commutative. And every v has an additive inverse, namely v. Scalar multiplication is also closed in V, as the product is always $0 \in V$ or $v \in V$. So V is a vector space over GF(2).

Problem 4 (Golan 70). Show that $\mathbb{Z}$ is not a vector space over any field.

Solution. Let $\mathbb{F} = \mathrm{GF}(2)$. Assume $\mathbb{Z}$ is a vector space over $\mathbb{F}$. Then

$$\begin{aligned} 0 &= 1_Z(1_F + 1_F) \\ &= 1_Z(1_F) +_Z 1_Z(1_F) \\ &= 2 \end{aligned}$$

But $0 \neq 2$, so $\mathbb{Z}$ is not a vector space over GF(2). Now let $\mathbb{F}$ be a field with characteristic greater than 2. Assume $\mathbb{Z}$ a vector space over $\mathbb{F}$. Then we know

$$1_F + 1_F = 2_F \implies 1_F = \frac{1}{2_F} + \frac{1}{2_F}$$

Therefore

$$\begin{aligned} 1_Z &= 1_F(1_Z) \\ &= (\frac{1}{2_F} + \frac{1}{2_F})1_Z \\ &= \frac{1}{2_F}(1_Z) +_Z \frac{1}{2_F}(1_Z) \end{aligned}$$

Now let $\frac{1}{2_F}(1_Z) = n \in \mathbb{Z}$. But there is no element in $n \in \mathbb{Z}$ that satisfies $n + n = 1$. So $\mathbb{Z}$ is not a vector space over any field with characteristic greater than 2. Thus $\mathbb{Z}$ is not a vector space over any field.

Problem 5 (Golan 76). Let $V = \mathbb{R}^{\mathbb{R}}$ and let W be the subset of V containing the constant function $x \mapsto 0$ and all of those functions $f \in V$ satisfying the following condition: $f(a) = 0$ for at most finitely many real numbers a. Is W a subspace of V.

[*Hint:* It's easy.]

Solution. Let us assume toward a contradiction that W is a subspace of V. Let $p(x), g(x), h(x)$ be distinct functions in W, such that $p(x) = g(x)+h(x)$ and $g(x) = 0$ for $x = b_0, \ldots, b_l \in \mathbb{R}$ and $h(x) = 0$ for $x = c_0, \ldots, c_m \in \mathbb{R}$. Then $p(x) = 0$ when $g(x) = h(x) = 0$ or when $g(x) = -h(x)$. The former happens when $b_i = c_j$ for $1 \leq i \leq l, 1 \leq j \leq m$. We can see this is a finite set of points. The latter, however, could happen for an infinite number of points (e.g. define $g(x) = -h(x)$ for $x > max(b_l, c_m) \in \mathbb{R}$). In that case, $p(a) = 0$ for an infinitely many real numbers a, but it is not the constant function $x \mapsto 0$. So vector addition is not closed and therefore W is not a subspace in V.

Problem 6 (Golan 79). A function $f \in \mathbb{R}^{\mathbb{R}}$ is *piecewise constant* if and only if it is a constant function $x \mapsto c$ or there exist $a_1 < a_2 < \cdots < a_n$ and $c_0 < c_1 < \cdots < c_n$ in $\mathbb{R}$ such that

$$f : x \mapsto \begin{cases} c_0 & \text{if } x < a_1, \\ c_i & \text{if } a_i \leq x < a_i \text{ for } 1 \leq i < n, \\ c_n & \text{if } a_n \leq x. \end{cases}$$

Does the set of all piecewise constant functions form a subspace of the vector space $\mathbb{R}^{\mathbb{R}}$ over $\mathbb{R}$?

Solution. (type your solution here)

Problem 7 (Golan 81). Let W be the subspace of $V = \mathrm{GF}(2)^5$ consisting of all vectors $(a_1, \ldots, a_5)$ satisfying $\sum_{i=1}^{5} a_i = 0$. Is W a subspace of V?

Solution. (type your solution here)

Problem 8 (Golan 85). Let $V = \mathbb{R}^{\mathbb{R}}$ and let W be the subset of V consisting of all functions f satisfying the following condition: there exists $c \in \mathbb{R}$ (that depends on f) such that $|f(a)| \leq c|a|$ for all $a \in \mathbb{R}$. Is W a subspace of V?

Solution. (type your solution here)

Problem 9 (Golan 93). Let V be a vector space over a field F and let P be the collection of all subsets of V, which we know is a vector space over $\mathrm{GF}(2)$. Is the collection of all subpaces of V a subspace of P?

Solution. (type your solution here)

Problem 10 (Golan 105). Let V be a vector space over a field F and let $0_V \neq w \in V$. Given a vector $v \in V \setminus Fw$, find the set G of all scalars $a \in F$ satisfying $F\{v, w\} = F\{v, aw\}$.

Solution. (type your solution here)