Homework 1 hints

Problem (Golan 16). Let z_1, z_2, and z_3 be complex numbers satisfying $|z_i| = 1$ for $i = 1, 2, 3$. Show that $|z_1 z_2 + z_1 z_3 + z_2 z_3| = |z_1 + z_2 + z_3|$.

Hint. Let $w = z_1 z_2 z_3$. Note that the modulus of w is $|w| = 1$, since each z_i has modulus 1. The left hand side of the equality we want to establish is $|w/z_3 + w/z_2 + w/z_1|$. Factoring out $|w|$, we have $|w||1/z_3 + 1/z_2 + 1/z_1|$. Note that since $|z_i| = 1$, the conjugate of z_i is $1/z_i$.

Problem (Golan 22 *Abel's inequality*). Let $z_1, \ldots, z_n$ be a list of complex numbers and, for each $1 \le k \le n$, let $s_k = \sum_{i=1}^{k} z_i$. For real numbers $a_1, \ldots, a_n$ satisfying $a_1 \ge a_2 \ge \cdots \ge a_n \ge 0$, show that

$$\left| \sum_{i=1}^{n} a_i z_i \right| \le a_1 \left(\max_{1 \le k \le n} |s_k| \right). \tag{1}$$

Hint. The kth term of $\sum_{i=1}^{n} a_i z_i$ is

$$a_k z_k = a_k(z_1 + \cdots + z_k) - a_k(z_1 + \cdots + z_{k-1}),$$

so

$$\begin{aligned}
\left| \sum_{i=1}^{n} a_i z_i \right| &= |a_n(z_1 + \cdots + z_n) - a_n(z_1 + \cdots + z_{n-1}) \\
&\quad + a_{n-1}(z_1 + \cdots + z_{n-1}) - a_{n-1}(z_1 + \cdots + z_{n-2}) \\
&\quad \vdots \\
&\quad + a_2(z_1 + z_2) - a_2 z_1 + a_1 z_1| .
\end{aligned}$$

Collecting like terms in the expression on the right, we have

$$|z_1(a_1 - a_2) + (z_1 + z_2)(a_2 - a_3) + \cdots + (z_1 + \cdots + z_{n-1})(a_{n-1} - a_n) + (z_1 + \cdots + z_n)a_n| .$$

Consider bounding this expression by a term like $(a_j - a_{j+1})|z_1 + \cdots + z_k|$ (take the maximum such term as j and k vary). Finally, note that $a_j - a_{j+1} < a_1$. Use all of these facts together to establish the claim.