Homework 2

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Problem 1 (Golan 56). Is it possible to define on $\mathbb{Z}/(4)$ the structure of a vector space over GF(2) in such a way that the vector addition is the usual addition in $\mathbb{Z}/(4)$?

[*Hints*: Recall that (n) denotes the set $\{\ldots, -2n, -n, 0, n, 2n, 3n, \ldots\}$, which we denoted in lecture by $n\mathbb{Z}$. This is the ideal generated by n in the ring $\mathbb{Z}$, but don't worry about that for now. Just take $\mathbb{Z}/(n)$ to be the abelian group of integers $\{0, 1, 2, \ldots, n-1\}$ with addition modulo n. In lecture, we used $\mathbb{Z}/n\mathbb{Z}$ to denote $\mathbb{Z}/(n)$. Use whichever notation you prefer.]

Solution. (type your solution here)

Problem 2 (Golan 60). Let $V = C(0,1)$. Define an operation $\boxplus$ on V by setting $f \boxplus g : x \mapsto \max\{f(x), g(x)\}$. Does this operation of vector addition, together with the usual scalar multiplication make V into a vector space over $\mathbb{R}$?

Solution. (type your solution here)

Problem 3 (Golan 63). Let $V = \{i \in \mathbb{Z} \mid 0 \leq i < 2^n\}$ for some fixed positive integer n. Define operations of vector addition and scalar multiplication on V in such a way as to turn it into a vector space over GF(2).

[*Hints*: Recall that GF(2) denotes the Galois field with two elements, $\{0, 1\}$, with addition mod 2 and the usual multiplication. Other than this field, the only restriction given in the problem is that V must have 2^n elements. Do you know of any sets of this size?]

Solution. (type your solution here)

Problem 4 (Golan 70). Show that $\mathbb{Z}$ is not a vector space over any field.

Solution. (type your solution here)

Problem 5 (Golan 76). Let $V = \mathbb{R}^{\mathbb{R}}$ and let W be the subset of V containing the constant function $x \mapsto 0$ and all of those functions $f \in V$ satisfying the following condition: $f(a) = 0$ for at most finitely many real numbers a. Is W a subspace of V.

[*Hint:* It's easy.]

Solution. (type your solution here)

Problem 6 (Golan 79). A function $f \in \mathbb{R}^{\mathbb{R}}$ is *piecewise constant* if and only if it is a constant function $x \mapsto c$ or there exist $a_1 < a_2 < \cdots < a_n$ and $c_0 < c_1 < \cdots < c_n$ in $\mathbb{R}$ such that

$$f : x \mapsto \begin{cases} c_0 & \text{if } x < a_1, \\ c_i & \text{if } a_i \leq x < a_i \text{ for } 1 \leq i < n, \\ c_n & \text{if } a_n \leq x. \end{cases}$$

Does the set of all piecewise constant functions form a subspace of the vector space $\mathbb{R}^{\mathbb{R}}$ over $\mathbb{R}$?

Solution. (type your solution here)

Problem 7 (Golan 81). Let W be the subspace of $V = \mathrm{GF}(2)^5$ consisting of all vectors $(a_1, \ldots, a_5)$ satisfying $\sum_{i=1}^{5} a_i = 0$. Is W a subspace of V?

Solution. (type your solution here)

Problem 8 (Golan 85). Let $V = \mathbb{R}^{\mathbb{R}}$ and let W be the subset of V consisting of all functions f satisfying the following condition: there exists $c \in \mathbb{R}$ (that depends on f) such that $|f(a)| \leq c|a|$ for all $a \in \mathbb{R}$. Is W a subspace of V?

Solution. (type your solution here)

Problem 9 (Golan 93). Let V be a vector space over a field F and let P be the collection of all subsets of V, which we know is a vector space over $\mathrm{GF}(2)$. Is the collection of all subpaces of V a subspace of P?

Solution. (type your solution here)

Problem 10 (Golan 105). Let V be a vector space over a field F and let $0_V \neq w \in V$. Given a vector $v \in V \setminus Fw$, find the set G of all scalars $a \in F$ satisfying $F\{v, w\} = F\{v, aw\}$.

Solution. (type your solution here)