Available online at www.sciencedirect.com

ScienceDirect

Journal of Approximation Theory 175 (2013) 1–18

JOURNAL OF Approximation Theory

www.elsevier.com/locate/jat

Full length article

Fractal interpolation functions with variable parameters and their analytical properties☆

Hong-Yong Wang*, Jia-Shan Yu

School of Applied Mathematics, Nanjing University of Finance and Economics, Nanjing 210023, PR China

Received 21 August 2012; received in revised form 9 May 2013; accepted 21 July 2013
Available online 30 July 2013

Communicated by Ulrich Reif

Abstract

Based on a widely used class of iterated function systems (IFSs), a class of IFSs with variable parameters is introduced, which generates the fractal interpolation functions (FIFs) with more flexibility. Some analytical properties of these FIFs are investigated in the present paper. Their smoothness is first considered and the related results are presented in three different cases. The stability is then studied in the case of the interpolation points having small perturbations. Finally, the sensitivity analysis is carried out by providing an upper estimate of the errors caused by the slight perturbations of the IFSs generating these FIFs.
© 2013 Elsevier Inc. All rights reserved.

Keywords: Iterated function systems; Fractal interpolation functions; Smoothness; Stability; Sensitivity

1. Introduction

The concept of fractal interpolation functions (FIFs) was introduced by Barnsley [1,2] on the basis of the theory of iterated function systems (IFSs). As a new type of interpolants, FIFs have more advantages than the classical interpolants like polynomials and splines in fitting and approximation of naturally occurring functions that display some kind of self-similarity in nature. For more than two decades, FIFs have been extensively applied to structural mechanics, signal

☆ This work was supported by the National Natural Science Foundation of China (No. 11071152) and the Scientific Research Foundation of Nanjing University of Finance and Economics (No. A2011019).
* Corresponding author.
E-mail addresses: 98wanghy@163.com, 98wanghy@njue.edu.cn (H.-Y. Wang).

0021-9045/$ - see front matter © 2013 Elsevier Inc. All rights reserved.
http://dx.doi.org/10.1016/j.jat.2013.07.008

processing, approximation of functions, and computer graphics, etc. (see, e.g., [13,18,19,26]) and have become powerful analytical tools in many areas of applied sciences and engineering.

In the developments of theory of FIFs, many researchers have generalized the notion of FIFs in different ways. They have constructed various types of FIFs, including the multivariable FIFs (see, e.g., [6,12,15,17,25,27]) generated by using higher-dimensional or recurrent IFSs, the hidden variable FIFs (see, e.g., [3,5,7]) produced by projecting the attractors of vector valued IFSs to a lower-dimensional space, and the Hermite or spline FIFs (see, e.g., [8,20]) constructed by using Hermite or spline functions. Some significant properties of such FIFs, including calculus, dimensions, smoothness, stability, and perturbation errors, etc., were widely studied (see, e.g., [4,9–11,14,16,21,23,24]). As we know, an FIF is essentially generated by an IFS that consists of a finite set of iterated mappings on a complete metric space. An important set of free parameters in the iterated mappings, the set of vertical scaling factors, has a decisive influence on the properties and shape of the corresponding FIF. Especially, in the case of affine FIFs, the vertical scaling factors uniquely determine the corresponding affine FIF provided that the interpolation points are prescribed in advance. It is worth noting that almost all researches on the various types of FIFs mentioned above are limited within the cases of constant vertical scaling factors (constant parameters). Using constant parameters in an IFS, one can make each iterated mapping possess the same vertical compression ratios on an identical subinterval that belongs to a partition of some closed interval defining the FIF. This means that the FIFs generated by those IFSs with constant parameters usually have obvious self-similarity character, which could lead to the loss of flexibility, and might cause obvious errors in fitting and approximation of some complicated curves and non-stationary data that show less self-similarity. In [22] this problem was examined in a particular case where the IFS consists of affine mappings. The authors introduced a class of FIFs with variable vertical scaling factors based on the affine IFSs and analyzed the characteristics of the class of FIFs.

In the present paper, to get the FIFs with more flexibility and diversity in a more general sense, we will first generalize the construction of a widely used class of IFSs of fractal interpolation from the case of using constant parameters to the case of using variable parameters and then investigate the analytical properties of the FIFs generated by the class of IFSs with variable parameters, including the smoothness, stability and sensitivity. We believe that the FIFs presented in this paper would have more flexibility than those classical FIFs in fitting and approximation of many complicated phenomena and patterns. The research results could provide a necessary theoretical basis for the practical applications of such FIFs.

The paper is organized as follows. In Section 2, we introduce the class of IFSs with variable parameters on the basis of a class of IFSs with extensive applications and verify the existence of the corresponding FIFs. In Section 3, we first give two auxiliary lemmas and then discuss the smoothness of the class of FIFs with variable parameters. The related results about the Hölder continuity are given in three different cases. In Section 4, the stability of the FIFs with a special structure is studied by using the results obtained in Section 3. Section 5 is devoted to the sensitivity analysis for the above FIFs. The errors caused by the perturbations of the IFSs generating the FIFs are discussed and an upper estimate of the errors is given. As a consequence, the sensitivity result of such FIFs is obtained. Finally, conclusions are drawn in Section 6.

2. The FIFs with variable parameters

Let a set of interpolation points $\{(x_i, y_i) \in I \times \mathbb{R} : \ i = 0, 1, \dots, N\}$ be given, where $x_0 < x_1 < \cdots < x_N$ is a partition of the closed interval $I = [x_0, x_N]$, $y_i \in [h_1, h_2] \subset \mathbb{R}$, $i =$

$0, 1, \ldots, N$. Set $I_i = [x_{i-1}, x_i]$ and $K = I \times [h_1, h_2]$. Let $L_i : I \to I_i,\ i = 1, 2, \ldots, N$, be contractive homeomorphisms such that

$$L_i(x_0) = x_{i-1}, \qquad L_i(x_N) = x_i, \tag{2.1}$$

$$|L_i(c_1) - L_i(c_2)| \le l|c_1 - c_2|, \quad \forall c_1, c_2 \in I, \tag{2.2}$$

for some $0 \le l < 1$. Furthermore, let N continuous functions $F_i : K \to \mathbb{R}$ be given such that

$$F_i(x_0, y_0) = y_{i-1}, \qquad F_i(x_N, y_N) = y_i, \tag{2.3}$$

$$|F_i(x, \xi) - F_i(x, \xi')| \le |\alpha_i||\xi - \xi'|, \quad \forall x \in I,\ \xi_1, \xi_2 \in [h_1, h_2], \tag{2.4}$$

for some $\alpha_i \in (-1, 1),\ i = 1, 2, \ldots, N$. Now define mappings $W_i : K \to I_i \times \mathbb{R},\ i = 1, 2, \ldots, N$ by

$$W_i(x, y) = (L_i(x), F_i(x, y)), \quad \forall (x, y) \in K.$$

Barnsley presented the following famous result.

Proposition 2.1 ([1]). *The IFS $\{K, W_i : i = 1, 2, \ldots, N\}$ defined above has a unique attractor G. G is the graph of a continuous function $f : I \to \mathbb{R}$ which obeys $f(x_i) = y_i$ for $i = 0, 1, \ldots, N$.*

The previous function f is called a fractal interpolation function (FIF) and is the unique function satisfying the following functional equation (a fixed point equation):

$$f(x) = F_i\left(L_i^{-1}(x), f(L_i^{-1}(x))\right), \quad \forall x \in I_i,\ i = 1, 2, \ldots, N. \tag{2.5}$$

The most widely studied FIFs in theory and applications so far are defined by the iterated mappings

$$\begin{cases} L_i(x) = a_i x + b_i, \\ F_i(x, y) = \alpha_i y + q_i(x), \end{cases} \quad i = 1, 2, \ldots, N, \tag{2.6}$$

where the real constants a_i and b_i are determined by conditions (2.1), $q_i(x)$ are some continuous functions such that conditions (2.3) and (2.4) hold, α_i are free parameters obeying $\alpha_i \in (-1, 1)$, and are called vertical scaling factors of the transformations W_i. If $q_i(x)$ are linear, i.e. $q_i(x) = d_i x + e_i,\ i = 1, 2, \ldots, N$, in (2.6), then the corresponding FIF is referred to as an affine FIF.

Let us now introduce an IFS with variable parameters based on iterated mappings (2.6). For convenience, in what follows we always take the interval $I = [0, 1]$ associated with a partition $0 = x_0 < x_1 < \cdots < x_N = 1$ and the set of interpolation data $\Delta = \{(x_i, y_i) \in I \times [h_1, h_2] : i = 0, 1, \ldots, N\}$. Let us consider

$$\begin{cases} L_i(x) = a_i x + x_{i-1}, \\ F_i(x, y) = \alpha_i(x) y + q_i(x), \end{cases} \quad i = 1, 2, \ldots, N, \tag{2.7}$$

where $a_i = x_i - x_{i-1},\ \alpha_i(x)$ are Lipschitz functions defined on I satisfying for $i = 1, 2, \ldots, N$, $\|\alpha_i\|_\infty = \sup\{|\alpha_i(x)| : x \in I\} < 1,\ q_i(x)$ are also Lipschitz functions defined on I such that

$$q_i(0) = y_{i-1} - \alpha_i(0) y_0, \qquad q_i(1) = y_i - \alpha_i(1) y_N. \tag{2.8}$$

It is easy to verify that $L_i(x)$ and $F_i(x, y)$ in (2.7) satisfy conditions (2.1)–(2.4). So, we get an IFS with variable parameters

$$\{K, (L_i(x), F_i(x, y)) : i = 1, 2, \ldots, N\}, \tag{2.9}$$

where $L_i(x)$ and $F_i(x, y)$ are defined in (2.7). According to Proposition 2.1, IFS (2.9) defines an FIF $f : I \to \mathbb{R}$ with variable parameters. From (2.5) and (2.7), we obtain the fixed point equation

$$f(x) = \alpha_i(L_i^{-1}(x)) f(L_i^{-1}(x)) + q_i(L_i^{-1}(x)), \quad \forall x \in I_i. \tag{2.10}$$

3. Smoothness of FIFs with variable parameters

In this section we will discuss the smoothness of the FIFs with variable parameters introduced in Section 2. To this end, we first give two lemmas that will be used in our subsequent discussion.

For any $x \in I = [0, 1]$, let $L_{i_1 i_2 \cdots i_n}(x) = L_{i_1} \circ L_{i_2} \circ \cdots \circ L_{i_n}(x)$ and $L_{i_1 i_2 \cdots i_n}(I) = L_{i_1} \circ L_{i_2} \circ \cdots \circ L_{i_n}(I)$, where $i_j \in \{1, 2, \ldots, N\}$, $j = 1, 2, \ldots, n$. Define a shift operator σ by $\sigma(i_1 i_2 \cdots i_n) = (i_2 i_3 \cdots i_n)$. Let σ^k denote the k-fold composition of σ with itself such that $L_{\sigma^k(i_1 i_2 \cdots i_n)}(x) = L_{i_{k+1} \cdots i_n}(x)$ for $1 \le k \le n-1$, while $L_{\sigma^n(i_1 i_2 \cdots i_n)}(x) = x$.

Using the successive iteration and induction, we are able to prove the following Lemma 3.1 (a similar lemma can be found in [23]).

Lemma 3.1. *Let $f(x)$ be the FIF defined by IFS (2.9). For any $x \in I$, $\forall i_j \in \{1, 2, \ldots, N\}$, $j = 1, 2, \ldots, n$, set $\prod_{k=1}^{0} \alpha_{i_k}(x) = 1$ and $\prod_{j=1}^{0} a_{i_j} = 1$. Then we have*

$$L_{i_1 i_2 \cdots i_n}(x) = \left(\prod_{j=1}^{n} a_{i_j}\right) x + \sum_{k=1}^{n} \left(\prod_{j=1}^{k-1} a_{i_j}\right) x_{i_k - 1}, \tag{3.1}$$

$$\begin{aligned} f(L_{i_1 i_2 \cdots i_n}(x)) = {} & \left[\prod_{k=1}^{n} \alpha_{i_k}(L_{\sigma^k(i_1 i_2 \cdots i_n)}(x))\right] f(x) \\ & + \sum_{r=1}^{n} \left[\prod_{k=1}^{r-1} \alpha_{i_k}(L_{\sigma^k(i_1 i_2 \cdots i_n)}(x))\right] q_{i_r}(L_{\sigma^r(i_1 i_2 \cdots i_n)}(x)), \end{aligned} \tag{3.2}$$

where

$$L_{\sigma^k(i_1 i_2 \cdots i_n)}(x) = \left(\prod_{j=1}^{n-k} a_{i_{k+j}}\right) x + \sum_{l=1}^{n-k} \left(\prod_{j=1}^{l-1} a_{i_{k+j}}\right) x_{i_{k+l} - 1}. \tag{3.3}$$

Lemma 3.2. *Let s_i, t_i, $i = 1, 2, \ldots, n$, be arbitrarily given real numbers. Then*

$$\prod_{i=1}^{n} s_i - \prod_{i=1}^{n} t_i = \sum_{i=1}^{n} \left(\prod_{k=1}^{n-1} c_k^{(i)}\right) (s_i - t_i), \tag{3.4}$$

where, for all $k = 1, 2, \ldots, n-1$, each $c_k^{(i)}$, $i = 1, 2, \ldots, n$, is a real number from the set $\{s_1, s_2, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n, t_1, t_2, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n\}$. We put $\prod_{k=1}^{n-1} c_k^{(i)} = 1$ when $n = 1$.

Moreover, we also have

$$\left|\prod_{i=1}^{n}(s_i+t_i)-\prod_{i=1}^{n}t_i\right| \le (s+t)^n - t^n, \tag{3.5}$$

where $s = \max_{1\le i\le n}\{|s_i|\},\ t = \max_{1\le i\le n}\{|t_i|\}$.

Proof. We will proceed by mathematical induction with respect to n. First, we observe that the equality (3.4) is clearly true for $n = 1$. And then when $n = 2$, we have

$$\prod_{i=1}^{2}s_i-\prod_{i=1}^{2}t_i = s_1s_2 - t_1t_2 = s_2(s_1-t_1)+t_1(s_2-t_2) = \sum_{i=1}^{2}\left(\prod_{k=1}^{1}c_k^{(i)}\right)(s_i-t_i),$$

where $c_1^{(1)} = s_2$ and $c_1^{(2)} = t_1$. We now suppose that (3.4) holds for $n = m$. Hence, using the induction hypothesis, we obtain

$$\begin{aligned}
\prod_{i=1}^{m+1}s_i-\prod_{i=1}^{m+1}t_i &= s_{m+1}\prod_{i=1}^{m}s_i - t_{m+1}\prod_{i=1}^{m}t_i\\
&= (s_{m+1}-t_{m+1})\prod_{i=1}^{m}s_i + t_{m+1}\left(\prod_{i=1}^{m}s_i-\prod_{i=1}^{m}t_i\right)\\
&= (s_{m+1}-t_{m+1})\prod_{i=1}^{m}s_i + t_{m+1}\sum_{i=1}^{m}\left(\prod_{k=1}^{m-1}\tilde{c}_k^{(i)}\right)(s_i-t_i)\\
&= \sum_{i=1}^{m}\left(\prod_{k=1}^{m}\tilde{c}_k^{(i)}\right)(s_i-t_i)+\left(\prod_{k=1}^{m}s_k\right)(s_{m+1}-t_{m+1}),
\end{aligned}$$

where, for $1 \le k \le m-1$, each $\tilde{c}_k^{(i)} \in \{s_1,\ldots,s_{i-1},s_{i+1},\ldots,s_m,t_1,\ldots,t_{i-1},t_{i+1},\ldots,t_m\}$ and $\tilde{c}_m^{(i)} = t_{m+1}$ for $i = 1,2,\ldots,m$. So, we can write

$$\prod_{i=1}^{m+1}s_i-\prod_{i=1}^{m+1}t_i = \sum_{i=1}^{m+1}\left(\prod_{k=1}^{m}c_k^{(i)}\right)(s_i-t_i),$$

where, for $1 \le k \le m$, each $c_k^{(i)} \in \{s_1,\ldots,s_{i-1},s_{i+1},\ldots,s_{m+1},t_1,\ldots,t_{i-1},t_{i+1},\ldots,t_{m+1}\}$ for $i = 1,2,\ldots,m+1$. This means that (3.4) is available for $n = m+1$. By induction, the equality (3.4) holds for all $n \ge 1$.

Finally, we prove the inequality (3.5). Obviously, (3.5) is true for $n = 1$. Suppose that (3.5) holds for $n = m$. Then, we have

$$\begin{aligned}
\left|\prod_{i=1}^{m+1}(s_i+t_i)-\prod_{i=1}^{m+1}t_i\right| &= \left|(s_{m+1}+t_{m+1})\prod_{i=1}^{m}(s_i+t_i)-t_{m+1}\prod_{i=1}^{m}t_i\right|\\
&= \left|s_{m+1}\prod_{i=1}^{m}(s_i+t_i)+t_{m+1}\left(\prod_{i=1}^{m}(s_i+t_i)-\prod_{i=1}^{m}t_i\right)\right|\\
&\le s(s+t)^m + t[(s+t)^m - t^m] = (s+t)^{m+1}-t^{m+1}.
\end{aligned}$$

Thus, the inequality (3.5) holds for all $n \ge 1$ by induction. $\square$

Definition 3.1. Let $0 < \lambda \leq 1$. A function $f(x)$ defined on interval I is called λ-Hölder continuous if there exists a positive constant C such that

$$|f(x) - f(x')| \leq C|x - x'|^{\lambda}, \quad \forall x, x' \in I.$$

The class Lipλ consists of all λ-Hölder continuous functions f on I for any positive constant C.

We now establish the smoothness results of the FIFs with variable parameters by using Lemmas 3.1 and 3.2.

Theorem 3.1. *Let $f(x)$ be the FIF with variable parameters defined by IFS (2.9). Set $a = \min_{1\leq i\leq N}\{a_i\}$, $A = \max_{1\leq i\leq N}\{a_i\}$, $\alpha = \max_{1\leq i\leq N}\{\|\alpha_i\|_\infty\}$, and $\delta = \alpha/a$. We have for all $x \in I$,*

(i) *If $\delta < 1$, then $f \in$ Lip1.*
(ii) *If $\delta = 1$, then $f \in$ Lip$(1-\mu)$, where μ is a constant obeying $0 < \mu < 1$.*
(iii) *If $\delta > 1$, then $f \in$ Lipν, where $0 < \nu \leq (\ln \alpha)/\ln a < 1$.*

Proof. For any $x, x' \in I = [0, 1]$, it is possible to find $m \geq 0$ such that $x \in L_{i_1 i_2 \cdots i_m}(I)$ and

$$\prod_{j=1}^{m+1} a_{i_j} \leq |x - x'| \leq \prod_{j=1}^{m} a_{i_j}. \tag{3.6}$$

If $m = 0$, we then prescribe $L_{i_1 i_2 \cdots i_m}(I) = I$ and $\prod_{j=1}^{0} a_{i_j} = 1$.

Let us first consider the case when $x, x' \in L_{i_1 i_2 \cdots i_m}(I)$. Since $x \in L_{i_1 i_2 \cdots i_m}(I)$, there exists an $\bar{x} \in I$ from (3.1) and (3.2) such that

$$x = L_{i_1 i_2 \cdots i_m}(\bar{x}) = \left(\prod_{j=1}^{m} a_{i_j}\right)\bar{x} + \sum_{h=1}^{m}\left(\prod_{j=1}^{h-1} a_{i_j}\right) x_{i_h - 1}, \tag{3.7}$$

$$\begin{aligned} f(x) \;=\; f(L_{i_1 i_2 \cdots i_m}(\bar{x})) &= \left[\prod_{k=1}^{m} \alpha_{i_k}(L_{\sigma^k(i_1 i_2 \cdots i_m)}(\bar{x}))\right] f(\bar{x}) \\ &\quad + \sum_{r=1}^{m}\left[\prod_{k=1}^{r-1} \alpha_{i_k}(L_{\sigma^k(i_1 i_2 \cdots i_m)}(\bar{x}))\right] q_{i_r}(L_{\sigma^r(i_1 i_2 \cdots i_m)}(\bar{x})). \end{aligned} \tag{3.8}$$

From (3.7), $\bar{x}$ then can be written as

$$\bar{x} = \left(\prod_{j=1}^{m} a_{i_j}^{-1}\right)\left[x - \sum_{h=1}^{m}\left(\prod_{j=1}^{h-1} a_{i_j}\right) x_{i_h - 1}\right]. \tag{3.9}$$

Using (3.3) and (3.9), one gets

$$\begin{aligned} L_{\sigma^k(i_1 i_2 \cdots i_m)}(\bar{x}) &= \left(\prod_{j=1}^{k} a_{i_j}^{-1}\right)\left[x - \sum_{h=1}^{m}\left(\prod_{j=1}^{h-1} a_{i_j}\right) x_{i_h - 1}\right] \\ &\quad + \sum_{l=1}^{m-k}\left(\prod_{j=1}^{l-1} a_{i_{k+j}}\right) x_{i_{k+l} - 1}. \end{aligned} \tag{3.10}$$

Similarly, since $x' \in L_{i_1 i_2 \cdots i_m}(I)$, there is an $\bar{x}' \in I$ such that x' and $f(x')$ have the expressions similar to (3.7) and (3.8), respectively. So, using (3.8), we have

$$\begin{aligned}
|f(x) - f(x')| \leq \sum_{r=1}^{m} &\left| \left[\prod_{k=1}^{r-1} \alpha_{i_k}(L_{\sigma^k(i_1 \cdots i_m)}(\bar{x})) \right] q_{i_r}(L_{\sigma^r(i_1 \cdots i_m)}(\bar{x})) \right. \\
&\left. - \left[\prod_{k=1}^{r-1} \alpha_{i_k}(L_{\sigma^k(i_1 \cdots i_m)}(\bar{x}')) \right] q_{i_r}(L_{\sigma^r(i_1 \cdots i_m)}(\bar{x}')) \right| \\
&+ \left| \left[\prod_{k=1}^{m} \alpha_{i_k}(L_{\sigma^k(i_1 \cdots i_m)}(\bar{x})) \right] f(\bar{x}) \right. \\
&\left. - \left[\prod_{k=1}^{m} \alpha_{i_k}(L_{\sigma^k(i_1 \cdots i_m)}(\bar{x}')) \right] f(\bar{x}') \right| \\
\leq \sum_{r=1}^{m} &\left[\left| \prod_{k=1}^{r-1} \alpha_{i_k}(L_{\sigma^k(i_1 \cdots i_m)}(\bar{x})) \right| \left| q_{i_r}(L_{\sigma^r(i_1 \cdots i_m)}(\bar{x})) \right. \right. \\
&\left. - q_{i_r}(L_{\sigma^r(i_1 \cdots i_m)}(\bar{x}')) \right| + \left| \prod_{k=1}^{r-1} \alpha_{i_k}(L_{\sigma^k(i_1 \cdots i_m)}(\bar{x})) \right. \\
&\left. - \prod_{k=1}^{r-1} \alpha_{i_k}(L_{\sigma^k(i_1 \cdots i_m)}(\bar{x}')) \right| \\
&\left. \times \left| q_{i_r}(L_{\sigma^r(i_1 \cdots i_m)}(\bar{x}')) \right| \right] + 2\|f\|_\infty \alpha^m.
\end{aligned} \tag{3.11}$$

Taking into account that $q_i(x)$, $i = 1, 2, \ldots, N$, are Lipschitz functions defined on I, we denote by Q_i the Lipschitz constant of $q_i(x)$ and set $Q = \max_{1 \leq i \leq N}\{Q_i\}$. This gives that for $i = 1, 2, \ldots, N$,

$$|q_i(x) - q_i(x')| \leq Q|x - x'|, \quad \forall x, x' \in I. \tag{3.12}$$

Noting (3.10), we have

$$\left| L_{\sigma^r(i_1 \cdots i_m)}(\bar{x}) - L_{\sigma^r(i_1 \cdots i_m)}(\bar{x}') \right| = \left(\prod_{j=1}^{r} a_{i_j}^{-1} \right) |x - x'|. \tag{3.13}$$

Using (3.12) and (3.13), we can write

$$\left| q_{i_r}(L_{\sigma^r(i_1 \cdots i_m)}(\bar{x})) - q_{i_r}(L_{\sigma^r(i_1 \cdots i_m)}(\bar{x}')) \right| \leq Q \left(\prod_{j=1}^{r} a_{i_j}^{-1} \right) |x - x'|. \tag{3.14}$$

Let P_i denote the Lipschitz constant of $\alpha_i(x)$ and $P = \max_{1 \leq i \leq N}\{P_i\}$. Using (3.4) with $n = r-1$ and (3.13), we have

$$\left| \prod_{k=1}^{r-1} \alpha_{i_k}(L_{\sigma^k(i_1 \cdots i_m)}(\bar{x})) - \prod_{k=1}^{r-1} \alpha_{i_k}(L_{\sigma^k(i_1 \cdots i_m)}(\bar{x}')) \right|$$

$$\leq \sum_{k=1}^{r-1} \alpha^{r-2} P \left| L_{\sigma^k(i_1\cdots i_m)}(\bar{x}) - L_{\sigma^k(i_1\cdots i_m)}(\bar{x}') \right| \leq \sum_{k=1}^{r-1} \alpha^{r-2} P \left(\prod_{j=1}^{k} a_{i_j}^{-1} \right) |x - x'|$$

$$\leq P\alpha^{r-2} |x - x'| \sum_{k=1}^{r-1} a^{-k} \leq \frac{P|x - x'|\alpha^{r-2}}{1 - a} a^{-r+1} = \frac{P|x - x'|}{a(1 - a)} \delta^{r-2}, \tag{3.15}$$

where $\delta = \alpha/a$. Let $M_0 = \max_{1\leq i\leq N}\{\|q_i\|_\infty\}$. Noting that the notation $\prod_{k=1}^{r-1} \alpha_{i_k}(x) = 1$ when $r = 1$, we can deduce from (3.11), (3.14) and (3.15) that

$$\begin{aligned}
|f(x) - f(x')| &\leq \sum_{r=1}^{m} \alpha^{r-1} Q \left(\prod_{j=1}^{r} a_{i_j}^{-1} \right) |x - x'| \\
&\quad + \sum_{r=2}^{m} \frac{M_0 P}{a(1 - a)} \delta^{r-2} |x - x'| + 2\|f\|_\infty \alpha^m \\
&\leq Q|x - x'| \sum_{r=1}^{m} \alpha^{r-1} a^{-r} + \frac{M_0 P|x - x'|}{a(1 - a)} \sum_{r=2}^{m} \delta^{r-2} + 2\|f\|_\infty \delta^m a^m \\
&= \frac{Q|x - x'|}{a} \sum_{r=1}^{m} \delta^{r-1} + \frac{M_0 P|x - x'|}{a(1 - a)} \sum_{r=2}^{m} \delta^{r-2} + \frac{2\|f\|_\infty}{a} \delta^m a^{m+1}.
\end{aligned}$$

From (3.6), we obtain $a^{m+1} \leq \prod_{j=1}^{m+1} a_{i_j} \leq |x - x'|$. Therefore,

$$\begin{aligned}
|f(x) - f(x')| &\leq \frac{Q(1 - a) + M_0 P}{a(1 - a)} |x - x'| \sum_{r=1}^{m} \delta^{r-1} + \frac{2\|f\|_\infty}{a} \delta^m |x - x'| \\
&\leq M_1 |x - x'| \sum_{r=1}^{m+1} \delta^{r-1},
\end{aligned} \tag{3.16}$$

where $M_1 = \max\{\frac{Q(1-a)+M_0P}{a(1-a)}, \frac{2\|f\|_\infty}{a}\}$.

(i) If $\delta < 1$, then it follows from (3.16) that $|f(x) - f(x')| \leq C_1|x - x'|$, where the positive constant $C_1 = M_1/(1 - \delta) = M_1 a/(a - \alpha)$. This means that $f \in \mathrm{Lip}1$.

(ii) If $\delta = 1$, then we have $|f(x) - f(x')| \leq M_1(m + 1)|x - x'|$. From (3.6), we obtain $|x - x'| \leq \prod_{j=1}^{m} a_{i_j} \leq A^m < 1$, which means that $m \leq (\ln|x - x'|)/\ln A$. Applying a known inequality $0 < -x^\mu \ln x \leq 1/(\mu\mathrm{e})$ for $0 < x \leq 1$ and $0 < \mu < 1$, let us take some $0 < \mu < 1$ so that

$$\begin{aligned}
(m + 1)|x - x'| &\leq \left(1 + \frac{\ln|x - x'|}{\ln A}\right) |x - x'| \\
&= |x - x'| + \frac{-|x - x'|^\mu \ln|x - x'|}{|\ln A|} |x - x'|^{1-\mu} \\
&\leq |x - x'| + \frac{1}{\mu\mathrm{e}|\ln A|} |x - x'|^{1-\mu} \leq \left(1 + \frac{1}{\mu\mathrm{e}|\ln A|}\right) |x - x'|^{1-\mu}.
\end{aligned}$$

Thus, $|f(x) - f(x')| \leq C_2|x - x'|^{1-\mu}$, where the positive constant $C_2 = M_1(1 + 1/(\mu\mathrm{e}|\ln A|))$. This shows $f \in \mathrm{Lip}(1 - \mu)$.

(iii) If $\delta > 1$, then $|f(x) - f(x')| \le \frac{M_1}{\delta-1}\delta^{m+1}|x - x'|$. Let us choose a positive number ν with $0 < \nu < 1$ such that $\delta^{m+1}|x - x'| \le |x - x'|^{\nu}$. This implies that $\nu \le 1 + \frac{(m+1)\ln\delta}{\ln|x-x'|}$. Since $a^{m+1} \le |x - x'|$, the inequality $\frac{1}{\ln|x-x'|} \le \frac{1}{(m+1)\ln a}$ holds. This gives $\nu \le 1 + \frac{\ln\delta}{\ln a} = \frac{\ln\alpha}{\ln a} < 1$. Therefore, we obtain $|f(x) - f(x')| \le C_3|x - x'|^{\nu}$, where the positive constant $C_3 = M_1/(\delta - 1) = M_1 a/(\alpha - a)$. Thus, it follows that $f \in \mathrm{Lip}\nu$.

Next let us consider the case when x and x' do not belong to the same subinterval $L_{i_1 i_2 \cdots i_m}(I)$, but condition (3.6) holds. In this case, x and x' must belong to the adjacent two subintervals, respectively. Let $\tilde{x}$ be the common boundary point of the adjacent subintervals. Then we have $|f(x)-f(x')| \le |f(x)-f(\tilde{x})|+|f(\tilde{x})-f(x')|$. Thus, corresponding to the three subcases $\delta < 1$, $\delta = 1$ and $\delta > 1$, we obtain $|f(x)-f(x')| \le 2C_1|x-x'|$, $|f(x)-f(x')| \le 2C_2|x-x'|^{1-\mu}$ and $|f(x) - f(x')| \le 2C_3|x - x'|^{\nu}$, respectively. Therefore, the results of Theorem 3.1 follow from above arguments. $\square$

From Theorem 3.1 we can obtain the following corollary that will be used in our discussion on stability for a special class of FIFs with variable parameters.

Corollary 3.1. *Let $f(x)$ be the FIF defined by IFS (2.9) associated with the interpolation points $\{(x_i, y_i) \in [0, 1]\times[h_1, h_2] : i = 0, 1, \dots, N\}$. Then there exist constants M and τ $(0 < \tau \le 1)$, independent of y_i, $i = 0, 1, \dots, N$, such that*

$$|f(x) - f(x')| \le M|x - x'|^{\tau}, \quad \forall x, x' \in [0, 1].$$

Remark 3.1. In the case where the vertical scaling factor parameters are constants, Chen [11], Chand and Kapoor [9,10] investigated respectively the smoothness of a class of FIFs and the smoothness of coalescence hidden variable FIFs by using the techniques of operator approximation. Here, we study the smoothness of a class of FIFs with variable parameters, and the techniques used in Theorem 3.1 are completely different from those used in [9–11]. Our smoothness results are obtained by directly evaluating $|f(x) - f(x')|$ based on the multiresolution expressions (3.2) and (3.8) for the FIF f.

4. Stability of FIFs with variable parameters

In this section we focus on a class of FIFs with a special structure. We will investigate the stability of the class of FIFs by means of the smoothness results obtained in the previous section.

Let us take $q_i(x)$, $i = 1, 2, \dots, N$, in iterated mappings (2.7) to be a special class of Lipschitz functions. To avoid confusion, we write

$$\tilde{q}_i(x) = g(L_i(x)) - \alpha_i(x)b(x) \tag{4.1}$$

for the $q_i(x)$ in (2.7), where $g(x)$ is the piecewise linear interpolation function through the interpolation points $\Delta = \{(x_i, y_i) \in I \times [h_1, h_2] : i = 0, 1, \dots, N\}$ with $I = [0, 1]$, $b(x)$ is the linear function through the points (x_0, y_0) and (x_N, y_N), and $\alpha_i(x)$ are the Lipschitz functions as defined in (2.7).

It is easy to check that the $\tilde{q}_i(x)$, $i = 1, 2, \dots, N$, defined in (4.1) are Lipschitz functions satisfying conditions (2.8). So, the mappings

$$\begin{cases} L_i(x) = a_i x + x_{i-1}, \\ \tilde{F}_i(x, y) = \alpha_i(x)y + g(L_i(x)) - \alpha_i(x)b(x), \end{cases} \quad i = 1, 2, \dots, N, \tag{4.2}$$

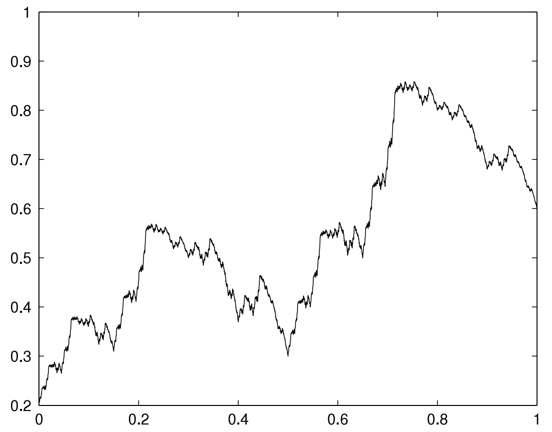

Fig. 1. The graph of the FIF with constant parameters.

as a particular type of iterated mappings (2.7), constitute a particular IFS with variable parameters

$$\{K, (L_i(x), \tilde{F}_i(x, y)) : i = 1, 2, \ldots, N\}. \tag{4.3}$$

According to (2.10), the FIF $f : I \to \mathbb{R}$ generated by IFS (4.3) satisfies the fixed point equation

$$f(x) = \alpha_i(L_i^{-1}(x))[f(L_i^{-1}(x)) - b(L_i^{-1}(x))] + g(x), \quad \forall x \in I_i. \tag{4.4}$$

Note that if all $\alpha_i(x)$, $i = 1, 2, \ldots, N$, in (4.2) are chosen to be constants, then the corresponding FIF is an affine FIF introduced by Barnsley in [1]. In general, if we choose at least an $\alpha_i(x)$ such that $\alpha_i(x)$ is a Lipschitz function obeying $\|\alpha_i\|_\infty < 1$ instead of a constant α_i obeying $|\alpha_i| < 1$, then the resulting FIF will not be an affine (linear) FIF. The motivation of selecting variable parameters is originated from the facts that the graphs of affine FIFs show obvious self-similarity, which might lose the flexibility to a certain extent in some practical applications, while the graphs of FIFs with variable parameters could display a weaker self-similarity, which can enlarge the flexibility and diversity of FIFs. Two examples of FIFs are given in the following, one is an affine FIF, and the other is an FIF with variable parameters.

Examples. Let interpolation points $\{(0, 0.2), (0.3, 0.5), (0.5, 0.3), (0.8, 0.8), (1, 0.6)\}$ be given. Let $g(x)$ be the piecewise linear function passing through the interpolation points given above, and $b(x)$ the linear function through the points $(0, 0.2)$ and $(1, 0.6)$.

(i) Let us consider a set of constant parameters, that is, $\alpha_1 = 0.4, \alpha_2 = 0.3, \alpha_3 = 0.5$ and $\alpha_4 = 0.2$. Then the graph of the FIF generated by IFS (4.3) with $\alpha_i(x) = \alpha_i$ for $i = 1, 2, 3, 4$, is plotted in Fig. 1.

(ii) Take a set of variable parameters, $\alpha_1(x) = 0.4\sin(5x) + 0.2$, $\alpha_2(x) = 0.3\cos(10x)$, $\alpha_3(x) = 0.5\exp(-2x) + 0.2$ and $\alpha_4(x) = 0.2\exp(x)\sin(x) + 0.1$. The graph of the FIF with the above variable parameters is displayed in Fig. 2.

Comparing Figs. 1 and 2, we can find that the self-similarity of the fractal interpolation curve shown in Fig. 2 is weaker than that of the affine fractal interpolation curve in Fig. 1. Hence, we can say that the FIFs with variable parameters may have more flexibility and applicability.

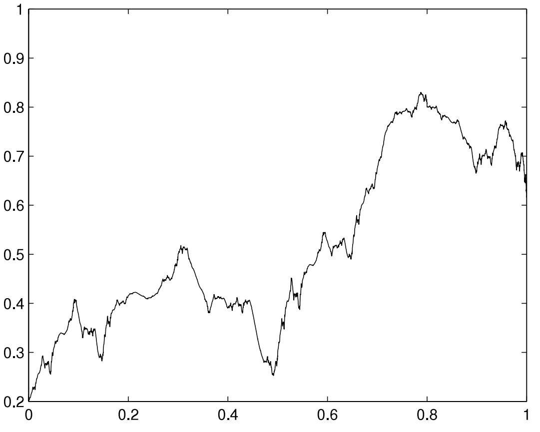

Fig. 2. The graph of the FIF with variable parameters.

Let us now proceed to study the stability of the FIF with variable parameters generated by IFS (4.3).

Let $\hat{\Delta} = \{(x_i, \bar{y}_i) : i = 0, 1, \ldots, N\}$ be another set of interpolation points in $K = I \times [h_1, h_2] \subset \mathbb{R}^2$, which is generated by the perturbations of ordinates of the points in $\Delta = \{(x_i, y_i) : i = 0, 1, \ldots, N\}$. Associated with interpolation data set $\hat{\Delta}$, we can define an IFS

$$\{K, (L_i(x), \hat{F}_i(x, y)) : i = 1, 2, \ldots, N\}, \tag{4.5}$$

where $L_i(x)$, $i = 1, 2, \ldots, N$, are the mappings defined in (4.2), and $\hat{F}_i(x, y) = \alpha_i(x)y + \hat{g}(L_i(x)) - \alpha_i(x)\hat{b}(x)$, $i = 1, 2, \ldots, N$, where $\hat{g}(x)$ is the piecewise linear function that interpolates the data set $\hat{\Delta}$, $\hat{b}(x)$ is the straight line through the points $(x_0, \bar{y}_0)$ and $(x_N, \bar{y}_N)$ and $\alpha_i(x)$, $i = 1, 2, \ldots, N$, are chosen to be the same as those in (4.2).

According to the theory of fractal interpolation, IFS (4.5) defines an FIF $\hat{f} : I \to \mathbb{R}$, whose graph passes through the data set $\hat{\Delta}$. The following Theorem 4.1 reveals the relationship between $f(x)$ and $\hat{f}(x)$.

Theorem 4.1. *Let $\Delta = \{(x_i, y_i) : i = 0, 1, \ldots, N\}$ and $\hat{\Delta} = \{(x_i, \bar{y}_i) : i = 0, 1, \ldots, N\}$ be two sets of interpolation points in a compact set $K \subset \mathbb{R}^2$. Let $f(x)$ be the FIF generated by IFS (4.3) associated with the interpolation data set Δ, and $\hat{f}(x)$ be the FIF determined by IFS (4.5) associated with the interpolation data set $\hat{\Delta}$. Set $\alpha = \max_{1\le i\le N}\{\|\alpha_i\|_\infty\}$. Then we have*

$$\|f - \hat{f}\|_\infty \le \frac{1+\alpha}{1-\alpha} \max_{0\le i\le N}\{|y_i - \bar{y}_i|\}. \tag{4.6}$$

Proof. Using the fixed point equation of FIF, we obtain that for any $x \in I$,

$$\begin{aligned} |f(L_i(x)) - \hat{f}(L_i(x))| &= \left|\tilde{F}_i(x, f(x)) - \hat{F}_i(x, \hat{f}(x))\right| \\ &= \Big|g(L_i(x)) + \alpha_i(x)(f(x) - b(x)) - \hat{g}(L_i(x)) \end{aligned}$$

$$\begin{aligned}&\quad - \alpha_i(x)(\hat{f}(x) - \hat{b}(x))\Big| \\ &\le \left|g(L_i(x)) - \hat{g}(L_i(x))\right| + |\alpha_i(x)| \left|f(x) - \hat{f}(x)\right| \\ &\quad + |\alpha_i(x)| \left|b(x) - \hat{b}(x)\right|.\end{aligned}$$

Since $g(x)$ and $\hat{g}(x)$ are two piecewise linear functions through the interpolation data sets Δ and $\hat{\Delta}$, respectively, and $L_i(x) \in I_i$ for any $x \in I$, it follows that

$$|g(L_i(x)) - \hat{g}(L_i(x))| \le \max_{0\le i\le N}\{|y_i - \bar{y}_i|\}.$$

Similarly, we have from the definitions of $b(x)$ and $\hat{b}(x)$ that

$$|b(x) - \hat{b}(x)| \le \max_{0\le i\le N}\{|y_i - \bar{y}_i|\}.$$

Consequently, it follows that

$$\begin{aligned}|f(L_i(x)) - \hat{f}(L_i(x))| &\le \max_{0\le i\le N}\{|y_i - \bar{y}_i|\} + \alpha|f(x) - \hat{f}(x)| + \alpha \max_{0\le i\le N}\{|y_i - \bar{y}_i|\} \\ &\le (1+\alpha)\max_{0\le i\le N}\{|y_i - \bar{y}_i|\} + \alpha\|f - \hat{f}\|_\infty,\end{aligned}$$

from which the inequality (4.6) is deduced. □

We now consider that the perturbations of abscissas of interpolation points are how to affect the values of the FIFs associated with the interpolation points.

Let $\bar{\Delta} = \{(\bar{x}_i, y_i) : i = 0, 1, \ldots, N\} \subset K$ be a set of interpolation points, which is obtained by the perturbations of abscissas of points in $\Delta = \{(x_i, y_i) : i = 0, 1, \ldots, N\}$, satisfying $0 = \bar{x}_0 < \bar{x}_1 < \cdots < \bar{x}_N = 1$. Set $\bar{I}_i = [\bar{x}_{i-1}, \bar{x}_i], \bar{a}_i = \bar{x}_i - \bar{x}_{i-1},\ i = 1, 2, \ldots, N$. Let $\bar{g}(x)$ be the piecewise linear interpolation function through the data set $\bar{\Delta}$, and $\bar{b}(x)$ be the linear function obeying $\bar{b}(\bar{x}_0) = y_0$ and $\bar{b}(\bar{x}_N) = y_N$. Then an IFS associated with $\bar{\Delta}$, $\bar{g}(x)$ and $\bar{b}(x)$ is defined as

$$\{K, (\bar{L}_i(x), \bar{F}_i(x, y)) : i = 1, 2, \ldots, N\}, \tag{4.7}$$

where $\bar{L}_i(x) = \bar{a}_i x + \bar{x}_{i-1}$ and $\bar{F}_i(x, y) = \alpha_i(x)y + \bar{g}(\bar{L}_i(x)) - \alpha_i(x)\bar{b}(x)$ with the same choice of variable parameters $\alpha_i(x)$ as in (4.2).

It is easy to see that $\bar{b}(x) = b(x)$ for $x \in I$ due to the fact that $\bar{x}_0 = x_0 = 0$ and $\bar{x}_N = x_N = 1$. The FIF generated by IFS (4.7) is denoted as $\bar{f}(x)$, we then have the following result.

Theorem 4.2. *Let $\Delta = \{(x_i, y_i) : i = 0, 1, \ldots, N\}$ and $\bar{\Delta} = \{(\bar{x}_i, y_i) : i = 0, 1, \ldots, N\}$ be two sets of interpolation points in a compact set $K \subset \mathbb{R}^2$ such that $\bar{x}_0 = x_0 = 0$ and $\bar{x}_N = x_N = 1$. Let $\bar{A} = \max_{1\le i\le N}\{\bar{a}_i\}$, $\beta = \max_{1\le i\le N}\{|a_i - \bar{a}_i|\}$ and $\bar{A} + \beta < 1$. Let $f(x)$ and $\bar{f}(x)$ are the FIFs generated by IFSs (4.3) and (4.7), respectively. Then there exist constants $\tilde{M}$ and τ $(0 < \tau \le 1)$ independent of y_i, $i = 0, 1, \ldots, N$, such that*

$$\|f - \bar{f}\|_\infty \le \tilde{M} \max_{0\le i\le N}\{|x_i - \bar{x}_i|^\tau\}. \tag{4.8}$$

Proof. Since $L_i(x),\ i = 1, 2, \ldots, N$, are contractive on the closed interval I, the sequence of sets $\{L_{i_1 i_2 \cdots i_n}(I)\}$ is monotonically decreasing for $i_j \in \{1, 2, \ldots, N\},\ j = 1, 2, \ldots, n$. Hence,

$\bigcap_{n=1}^{\infty} L_{i_1 i_2 \cdots i_n}(I)$ consists of a single point in I. Obviously, for any given $x \in I$, there exists a sequence of integers $\{i_j\}$, $i_j \in \{1, 2, \ldots, N\}$, such that

$$\{x\} = \bigcap_{n=1}^{\infty} L_{i_1 i_2 \cdots i_n}(I) = \lim_{n\to\infty} L_{i_1 i_2 \cdots i_n}(I).$$

Noting that each a_{i_j} in (3.1) obeys $0 < a_{i_j} < 1$, thus using (3.1), x can be expressed as

$$x = \lim_{n\to\infty} L_{i_1 i_2 \cdots i_n}(\hat{x}) = \sum_{k=1}^{\infty} \left(\prod_{j=1}^{k-1} a_{i_j} \right) x_{i_k - 1}, \tag{4.9}$$

where $\hat{x}$ may be arbitrarily chosen in I.

Similarly, for the same sequence of integers $\{i_j\}$ mentioned above, we write

$$x' = \lim_{n\to\infty} \bar{L}_{i_1 i_2 \cdots i_n}(\tilde{x}) = \sum_{k=1}^{\infty} \left(\prod_{j=1}^{k-1} \bar{a}_{i_j} \right) \bar{x}_{i_k - 1}, \quad \tilde{x} \in I, \tag{4.10}$$

because $\bar{L}_i(x) = \bar{a}_i x + \bar{x}_{i-1}$, $i = 1, 2, \ldots, N$, are also contractive on I. It is easy to see that $x' \in I$. From (4.9) and (4.10), it follows that

$$\begin{aligned} |x - x'| &\le \sum_{k=1}^{\infty} \left| \left(\prod_{j=1}^{k-1} a_{i_j} \right) x_{i_k - 1} - \left(\prod_{j=1}^{k-1} \bar{a}_{i_j} \right) \bar{x}_{i_k - 1} \right| \\ &\le \sum_{k=1}^{\infty} \left(\prod_{j=1}^{k-1} a_{i_j} \right) |x_{i_k - 1} - \bar{x}_{i_k - 1}| + \sum_{k=1}^{\infty} \left| \prod_{j=1}^{k-1} a_{i_j} - \prod_{j=1}^{k-1} \bar{a}_{i_j} \right| \bar{x}_{i_k - 1}. \end{aligned} \tag{4.11}$$

Set $\beta_{i_j} = a_{i_j} - \bar{a}_{i_j}$, $j = 1, 2, \ldots, k-1$. By the notation $\beta = \max_{1\le i\le N}\{|a_i - \bar{a}_i|\}$, we have $|\beta_{i_j}| \le \beta$, and $\beta = \max_{1\le i\le N}\{|x_i - x_{i-1} - (\bar{x}_i - \bar{x}_{i-1})|\} \le 2\max_{0\le i\le N}\{|x_i - \bar{x}_i|\}$. Using (3.5), it follows that

$$\left| \prod_{j=1}^{k-1} a_{i_j} - \prod_{j=1}^{k-1} \bar{a}_{i_j} \right| = \left| \prod_{j=1}^{k-1} \left(\bar{a}_{i_j} + \beta_{i_j} \right) - \prod_{j=1}^{k-1} \bar{a}_{i_j} \right| \le (\bar{A} + \beta)^{k-1} - \bar{A}^{k-1}. \tag{4.12}$$

From (4.11) and (4.12), we obtain

$$\begin{aligned} |x - x'| &\le \sum_{k=1}^{\infty} A^{k-1} |x_{i_k - 1} - \bar{x}_{i_k - 1}| + \sum_{k=1}^{\infty} \left[(\bar{A} + \beta)^{k-1} - \bar{A}^{k-1} \right] \\ &\le \frac{1}{1 - A} \max_{0\le i\le N} \{|x_i - \bar{x}_i|\} + \frac{\beta}{(1 - \bar{A} - \beta)(1 - \bar{A})} \\ &\le \frac{\max_{0\le i\le N}\{|x_i - \bar{x}_i|\}}{1 - A} + \frac{2\max_{0\le i\le N}\{|x_i - \bar{x}_i|\}}{(1 - \bar{A} - \beta)(1 - \bar{A})} \\ &= M_2 \max_{0\le i\le N} \{|x_i - \bar{x}_i|\}, \end{aligned} \tag{4.13}$$

where the constant $M_2 = 1/(1 - A) + 2/[(1 - \bar{A} - \beta)(1 - \bar{A})]$.

Substituting $q_i(x)$ with $\tilde{q}_i(x) = g(L_i(x)) - \alpha_i(x)b(x)$ in (3.2) and using (3.2), (3.3), (4.9) and (4.10), we obtain

$$|f(x) - \bar{f}(x')| = \left| \lim_{n\to\infty} f(L_{i_1 i_2 \cdots i_n}(\hat{x})) - \lim_{n\to\infty} \bar{f}(\bar{L}_{i_1 i_2 \cdots i_n}(\tilde{x})) \right|$$

$$= \left| \sum_{r=1}^{\infty} \left[\left(\prod_{k=1}^{r-1} \alpha_{i_k}(z_k) \right) \tilde{q}_{i_r}(z_r) - \left(\prod_{k=1}^{r-1} \alpha_{i_k}(\bar{z}_k) \right) \tilde{q}_{i_r}(\bar{z}_r) \right] \right|$$
$$\le \sum_{r=1}^{\infty} \left| \prod_{k=1}^{r-1} \alpha_{i_k}(z_k) \right| |\tilde{q}_{i_r}(z_r) - \tilde{q}_{i_r}(\bar{z}_r)|$$
$$+ \sum_{r=1}^{\infty} \left| \prod_{k=1}^{r-1} \alpha_{i_k}(z_k) - \prod_{k=1}^{r-1} \alpha_{i_k}(\bar{z}_k) \right| |\tilde{q}_{i_r}(\bar{z}_r)|, \tag{4.14}$$

where the meanings of the notations z_k, z_r, $\bar{z}_k$ and $\bar{z}_r$ are given as follows:

$$z_k = \sum_{l=1}^{\infty} \left(\prod_{j=1}^{l-1} a_{i_{k+j}} \right) x_{i_{k+l}-1}, \qquad z_r = \sum_{l=1}^{\infty} \left(\prod_{j=1}^{l-1} a_{i_{r+j}} \right) x_{i_{r+l}-1},$$
$$\bar{z}_k = \sum_{l=1}^{\infty} \left(\prod_{j=1}^{l-1} \bar{a}_{i_{k+j}} \right) \bar{x}_{i_{k+l}-1}, \qquad \bar{z}_r = \sum_{l=1}^{\infty} \left(\prod_{j=1}^{l-1} \bar{a}_{i_{r+j}} \right) \bar{x}_{i_{r+l}-1}.$$

It is easy to see from (3.3) that z_k, z_r, $\bar{z}_k$ and $\bar{z}_r$ belong to I. Taking into account that

$$|z_k - \bar{z}_k| \le \sum_{l=1}^{\infty} \left| \left(\prod_{j=1}^{l-1} a_{i_{k+j}} \right) x_{i_{k+l}-1} - \left(\prod_{j=1}^{l-1} \bar{a}_{i_{k+j}} \right) \bar{x}_{i_{k+l}-1} \right|$$
$$\le \sum_{l=1}^{\infty} \left(\prod_{j=1}^{l-1} a_{i_{k+j}} \right) |x_{i_{k+l}-1} - \bar{x}_{i_{k+l}-1}| + \sum_{l=1}^{\infty} \left| \prod_{j=1}^{l-1} a_{i_{k+j}} - \prod_{j=1}^{l-1} \bar{a}_{i_{k+j}} \right| \bar{x}_{i_{k+l}-1},$$

similar to the proofs of (4.11) and (4.13), we can obtain $|z_k - \bar{z}_k| \le M_2 \max_{0 \le i \le N}\{|x_i - \bar{x}_i|\}$. Similarly, we also have $|z_r - \bar{z}_r| \le M_2 \max_{0 \le i \le N}\{|x_i - \bar{x}_i|\}$.

Noting that $\alpha_i(x)$, $i = 1, 2, \ldots, N$, are Lipschitz functions on I and using (3.4) with $n = r - 1$, we obtain

$$\left| \prod_{k=1}^{r-1} \alpha_{i_k}(z_k) - \prod_{k=1}^{r-1} \alpha_{i_k}(\bar{z}_k) \right| \le \sum_{k=1}^{r-1} \alpha^{r-2} P |z_k - \bar{z}_k|$$
$$\le P M_2 \max_{0 \le i \le N} \{|x_i - \bar{x}_i|\} \cdot (r-1)\alpha^{r-2}. \tag{4.15}$$

Since all $\tilde{q}_i(x)$, $i = 1, 2, \ldots, N$, are Lipschitz functions, we can write

$$|\tilde{q}_i(x) - \tilde{q}_i(x')| \le \tilde{Q}|x - x'|, \quad i = 1, 2, \ldots, N, \ \forall x, x' \in I, \tag{4.16}$$

where $\tilde{Q}$ is a positive constant. From (4.16), we have

$$|\tilde{q}_{i_r}(z_r) - \tilde{q}_{i_r}(\bar{z}_r)| \le \tilde{Q}|z_r - \bar{z}_r| \le \tilde{Q} M_2 \max_{0 \le i \le N} \{|x_i - \bar{x}_i|\}. \tag{4.17}$$

By (4.14)–(4.17), it follows that

$$|f(x) - \bar{f}(x')| \le \tilde{Q} M_2 \max_{0 \le i \le N} \{|x_i - \bar{x}_i|\} \sum_{r=1}^{\infty} \alpha^{r-1} + P M_2 \max_{0 \le i \le N} \{|x_i - \bar{x}_i|\}$$
$$\times \sum_{r=1}^{\infty} (r-1)\alpha^{r-2} \|\tilde{q}_{i_r}\|_{\infty}$$

$$\begin{aligned} &\le \frac{\tilde{Q}M_2}{1-\alpha}\max_{0\le i\le N}\{|x_i-\bar{x}_i|\} + \frac{PM_2(\|g\|_\infty+\alpha\|b\|_\infty)}{(1-\alpha)^2}\max_{0\le i\le N}\{|x_i-\bar{x}_i|\} \\ &= M_3\max_{0\le i\le N}\{|x_i-\bar{x}_i|\}, \end{aligned} \tag{4.18}$$

where the constant $M_3 = [\tilde{Q}M_2(1-\alpha) + PM_2(\|g\|_\infty + \alpha\|b\|_\infty)]/(1-\alpha)^2$.

Taking into account that $\bar{f}(x)$ is the FIF associated with interpolation data set $\bar{\Delta}$, and according to Corollary 3.1, we conclude that there exist constants $\bar{M}$ and τ, independent of y_i, $i = 0, 1, \ldots, N$, such that $|\bar{f}(x) - \bar{f}(x')| \le \bar{M}|x - x'|^\tau$. Using (4.13), we obtain

$$|\bar{f}(x) - \bar{f}(x')| \le \bar{M}M_2^\tau \max_{0\le i\le N}\{|x_i-\bar{x}_i|^\tau\}. \tag{4.19}$$

Thus, we deduce from (4.18) and (4.19) that for any $x \in I$,

$$\begin{aligned} |f(x) - \bar{f}(x)| &\le |f(x) - \bar{f}(x')| + |\bar{f}(x) - \bar{f}(x')| \\ &\le \tilde{M}\max_{0\le i\le N}\{|x_i-\bar{x}_i|^\tau\}, \end{aligned} \tag{4.20}$$

where the constant $\tilde{M} = M_3 + \bar{M}M_2^\tau$. This means that (4.8) of Theorem 4.2 holds. □

Theorem 4.3. *Let $\Delta = \{(x_i, y_i) : i = 0, 1, \ldots, N\}$ and $\tilde{\Delta} = \{(\bar{x}_i, \bar{y}_i) : i = 0, 1, \ldots, N\}$ be two sets of interpolation points in a compact set $K \subset \mathbb{R}^2$ such that $\bar{x}_0 = x_0 = 0$ and $\bar{x}_N = x_N = 1$. Let $\bar{A} = \max_{1\le i\le N}\{\bar{a}_i\}$, $\beta = \max_{1\le i\le N}\{|a_i-\bar{a}_i|\}$, $a_i = x_i - x_{i-1}$, $\bar{a}_i = \bar{x}_i - \bar{x}_{i-1}$, and $\bar{A} + \beta < 1$. Suppose that $f(x)$ and $\tilde{f}(x)$ are the FIFs associated with the interpolation data sets Δ and $\tilde{\Delta}$, respectively. Then we have*

$$\|f - \tilde{f}\|_\infty \le \frac{1+\alpha}{1-\alpha}\max_{0\le i\le N}\{|y_i-\bar{y}_i|\} + \tilde{M}\max_{0\le i\le N}\{|x_i-\bar{x}_i|^\tau\}, \tag{4.21}$$

where $\alpha = \max_{1\le i\le N}\{\|\alpha_i\|_\infty\} < 1$, and $\tilde{M}$, τ $(0 < \tau \le 1)$ are two constants independent of the ordinates of interpolation points.

Proof. Let $\hat{\Delta} = \{(x_i, \bar{y}_i) : i = 0, 1, \ldots, N\}$ be another set of interpolation points. The FIF associated with $\hat{\Delta}$ is represented as $\hat{f}$. Then

$$\|f - \tilde{f}\|_\infty \le \|f - \hat{f}\|_\infty + \|\hat{f} - \tilde{f}\|_\infty.$$

Thus, (4.21) follows by applying Theorem 4.1 and (4.2). □

5. Sensitivity of FIFs with variable parameters

In this section we will discuss the sensitivity of the FIF defined by IFS (4.3) and give the sensitivity result.

Let $T_i : K \to \mathbb{R}$, $i = 1, 2, \ldots, N$, be the continuous functions defined on $K = I \times [h_1, h_2]$ such that for all $(x, y) \in K$,

$$T_i(x, y) = [\alpha_i(x) + \varepsilon_i\varphi_i(x)]y + g(L_i(x)) - [\alpha_i(x) + \varepsilon_i\varphi_i(x)]b(x) + \eta_i\psi_i(x),$$

where $\alpha_i(x)$, $L_i(x)$, $g(x)$ and $b(x)$ are the same functions as those defined in (4.2), ε_i and η_i are perturbation parameters obeying $0 < |\varepsilon_i| < 1$ and $0 < |\eta_i| < 1$, $\varphi_i(x)$ and $\psi_i(x)$ are Lipschitz functions defined on $I = [0, 1]$ satisfying $\max_{1\le i\le N}\{\|\alpha_i + \varepsilon_i\varphi_i\|_\infty\} < 1$ and $\psi_i(0) = \psi_i(1) = 0$.

Obviously, for each $i = 1, 2, \ldots, N$, the function $T_i(x, y)$ is a perturbation of the function $\tilde{F}_i(x, y)$ in (4.2). Consequently, the IFS

$$\{K, (L_i(x), T_i(x, y)) : \ i = 1, 2, \ldots, N\} \tag{5.1}$$

associated with the interpolation data set $\Delta = \{(x_i, y_i) \in K : \ i = 0, 1, \ldots, N\}$ constitutes a perturbation IFS of the IFS (4.3). It is easy to verify that the continuous functions $T_i(x, y)$ are contractive in the second variable y and satisfy $T_i(0, y_0) = y_{i-1}$ and $T_i(1, y_N) = y_i$ for all $i = 1, 2, \ldots, N$. Thus, IFS (5.1) determines a unique FIF, denoted by $f_{\varepsilon,\eta}(x)$, whose graph is the attractor of IFS (5.1) passing through the data set Δ. The following theorem presents the sensitivity result of the FIF defined by IFS (4.3).

Theorem 5.1. *Let $f(x)$ and $f_{\varepsilon,\eta}(x)$ be the FIFs generated by the IFS (4.3) and its perturbed IFS (5.1), respectively. Set $\alpha = \max_{1\le i\le N}\{\|\alpha_i\|_\infty\}$, $\varphi = \max_{1\le i\le N}\{\|\varphi_i\|_\infty\}$, $\psi = \max_{1\le i\le N}\{\|\psi_i\|_\infty\}$, $\varepsilon = \max_{1\le i\le N}\{|\varepsilon_i|\}$ and $\eta = \max_{1\le i\le N}\{|\eta_i|\}$. If $0 < \alpha + \varepsilon\varphi < 1$, then*

$$\|f_{\varepsilon,\eta} - f\|_\infty \le \frac{(\|g - b\|_\infty)\varphi}{(1-\alpha)(1-\alpha-\varepsilon\varphi)}\varepsilon + \frac{\psi}{1-\alpha-\varepsilon\varphi}\eta. \tag{5.2}$$

Proof. Since $f_{\varepsilon,\eta}(x)$ is the FIF generated by IFS (5.1), it satisfies the fixed point equation $f_{\varepsilon,\eta}(x) = T_i(L_i^{-1}(x), f_{\varepsilon,\eta}(L_i^{-1}(x)))$ for all $x \in I_i$ and $i = 1, 2, \ldots, N$, that is,

$$\begin{aligned} f_{\varepsilon,\eta}(x) = &\left[\alpha_i(L_i^{-1}(x)) + \varepsilon_i\varphi_i(L_i^{-1}(x))\right]\cdot\left[f_{\varepsilon,\eta}(L_i^{-1}(x)) - b(L_i^{-1}(x))\right] \\ &+ g(x) + \eta_i\psi_i(L_i^{-1}(x)). \end{aligned} \tag{5.3}$$

From (5.3) and the fixed point equation (4.4) satisfied by $f(x)$, we obtain for $x \in I_i$,

$$\begin{aligned} f_{\varepsilon,\eta}(x) - f(x) = \alpha_i(L_i^{-1}(x))&\left[f_{\varepsilon,\eta}(L_i^{-1}(x)) - f(L_i^{-1}(x))\right] + \varepsilon_i\varphi_i(L_i^{-1}(x)) \\ &\times\left[f_{\varepsilon,\eta}(L_i^{-1}(x)) - b(L_i^{-1}(x))\right] + \eta_i\psi_i(L_i^{-1}(x)). \end{aligned}$$

Therefore,

$$|f_{\varepsilon,\eta}(x) - f(x)| \le \alpha\|f_{\varepsilon,\eta} - f\|_\infty + \varepsilon\varphi\|f_{\varepsilon,\eta} - b\|_\infty + \eta\psi,$$

which means that

$$\begin{aligned} \|f_{\varepsilon,\eta} - f\|_\infty &\le \frac{\varepsilon\varphi}{1-\alpha}\|f_{\varepsilon,\eta} - b\|_\infty + \frac{\eta\psi}{1-\alpha} \\ &\le \frac{\varepsilon\varphi}{1-\alpha}\|f_{\varepsilon,\eta} - g\|_\infty + \frac{\varepsilon\varphi}{1-\alpha}\|g - b\|_\infty + \frac{\eta\psi}{1-\alpha}. \end{aligned} \tag{5.4}$$

Using (5.3), we have

$$\begin{aligned} \|f_{\varepsilon,\eta} - g\|_\infty &\le (\alpha + \varepsilon\varphi)\|f_{\varepsilon,\eta} - b\|_\infty + \eta\psi \\ &\le (\alpha + \varepsilon\varphi)\|f_{\varepsilon,\eta} - g\|_\infty + (\alpha + \varepsilon\varphi)\|g - b\|_\infty + \eta\psi. \end{aligned}$$

This implies

$$\|f_{\varepsilon,\eta} - g\|_\infty \le \frac{\alpha + \varepsilon\varphi}{1-\alpha-\varepsilon\varphi}\|g - b\|_\infty + \frac{\eta\psi}{1-\alpha-\varepsilon\varphi}. \tag{5.5}$$

Substituting (5.5) into (5.4) gives the inequality (5.2). □

6. Conclusions

In the present work, a class of FIFs with variable parameters is introduced, which generalizes the classical FIFs with constant parameters considered by Barnsley [1,2]. Compared with many types of FIFs (see, e.g., [1,4,22]), the FIFs considered in this paper have more flexibility and diversity and are more suitable to the fitting and approximation of many complicated curves and non-stationary experimental data that display less self-similarity. Their smoothness is discussed in three cases by means of the derived multiresolution expression and some auxiliary results. The stability of the FIFs with a special structure is studied in a general setup (equidistant or non-equidistant interpolation data). Our stability results generalize those of the self-affine FIFs generated by using constant parameters and equidistant interpolation data given by Feng and Xie [14]. Finally, the sensitivity analysis of the FIFs mentioned above is made. Their perturbation errors caused by the perturbations of the IFSs generating these FIFs are investigated, and an upper bound of the errors is given. From the stability results we can see that the FIFs with variable parameters introduced in the paper are stable to the small perturbations of the interpolation points. On the other hand, from the sensitivity result we know that the slight perturbations in the iterated mappings will lead to small changes in the corresponding FIFs. This means that such FIFs are not sensitive to the small perturbations of IFSs generating them. Our results on stability and sensitivity of the FIFs with variable parameters lay a theoretical foundation for their practical applications including data fitting, approximation of functions, signal processing and computer graphics, etc.

Acknowledgments

The authors would like to thank the reviewers for many valuable suggestions and useful comments.

References

[1] M.F. Barnsley, Fractal functions and interpolation, Constr. Approx. 2 (1986) 303–329.
[2] M.F. Barnsley, Fractals Everywhere, Academic Press, New York, 1988.
[3] M.F. Barnsley, J. Elton, D. Hardin, P.R. Massopust, Hidden variable fractal interpolation functions, SIAM J. Math. Anal. 20 (5) (1989) 1218–1242.
[4] M.F. Barnsley, A.N. Harrington, The calculus of fractal interpolation functions, J. Approx. Theory 57 (1989) 14–34.
[5] P. Bouboulis, L. Dalla, Hidden variable vector valued fractal interpolation functions, Fractals 13 (3) (2005) 227–232.
[6] P. Bouboulis, L. Dalla, A general construction of fractal interpolation functions on grids of $\mathbb{R}^n$, European J. Appl. Math. 18 (2007) 449–476.
[7] A.K.B. Chand, G.P. Kapoor, Hidden variable bivariate fractal interpolation surfaces, Fractals 11 (3) (2003) 277–288.
[8] A.K.B. Chand, G.P. Kapoor, Generalized cubic spline fractal interpolation functions, SIAM J. Numer. Anal. 44 (2) (2006) 655–676.
[9] A.K.B. Chand, G.P. Kapoor, Smoothness analysis of coalescence hidden variable fractal interpolation functions, Int. J. Nonlinear Sci. 3 (1) (2007) 15–26.
[10] A.K.B. Chand, G.P. Kapoor, Stability of affine coalescence hidden variable fractal interpolation functions, Nonlinear Anal. 68 (2008) 3757–3770.
[11] G. Chen, The smoothness and dimension of fractal interpolation function, Appl. Math. J. Chinese Univ. Ser. B 11 (4) (1996) 409–418.
[12] L. Dalla, Bivariate fractal interpolation functions on grids, Fractals 10 (1) (2002) 53–58.
[13] V. Drakopoulos, P. Bouboulis, S. Theodoridis, Image compression using affine fractal interpolation on rectangular lattices, Fractals 14 (4) (2006) 259–269.
[14] Z. Feng, H. Xie, On stability of fractal interpolation, Fractals 6 (3) (1998) 269–273.
[15] J.S. Geronimo, D. Hardin, Fractal interpolation surfaces and a related 2-D multiresolution analysis, J. Math. Anal. Appl. 176 (1993) 561–586.

[16] R. Malysz, The Minkowski dimension of the bivariate fractal interpolation surfaces, Chaos Solitons Fractals 27 (2006) 1147–1156.
[17] P.R. Massopust, Fractal surfaces, J. Math. Anal. Appl. 151 (1990) 275–290.
[18] D.S. Mazel, M.H. Hayes, Using iterated function system to model discrete sequences, IEEE Trans. Signal Process. 40 (1992) 1724–1734.
[19] M.A. Navascués, Fractal trigonometric approximation, Electron. Trans. Numer. Anal. 20 (2005) 64–74.
[20] M.A. Navascués, M.V. Sebastián, Generalization of Hermite functions by fractal interpolation, J. Approx. Theory 131 (2004) 19–29.
[21] H.-Y. Wang, On smoothness for a class of fractal interpolation surfaces, Fractals 14 (3) (2006) 223–230.
[22] H.-Y. Wang, Z.-L. Fan, Analytical characteristics of fractal interpolation functions with function vertical scaling factors, Acta Math. Sinica (Chin. Ser.) 54 (1) (2011) 147–158 (in Chinese).
[23] H.-Y. Wang, X.-J. Li, Perturbation error analysis for fractal interpolation functions and their moments, Appl. Math. Lett. 21 (2008) 441–446.
[24] H.-Y. Wang, S.-Z. Yang, X.-J. Li, Error analysis for bivariate fractal interpolation functions generated by 3D perturbed iterated function systems, Comput. Math. Appl. 56 (2008) 1684–1692.
[25] M. Wolfgang, H.Y. Chol, Construction of fractal interpolation surfaces on rectangular grids, Int. J. Bifur. Chaos 20 (12) (2010) 4079–4086.
[26] H. Xie, H. Sun, Y. Ju, Z. Feng, Study on generation of rock fracture surfaces by using fractal interpolation, Int. J. Solids Struct. 38 (2001) 5765–5787.
[27] N. Zhao, Construction and application of fractal interpolation surfaces, Vis. Comput. 12 (1996) 132–146.