Dorota Leszczyńska-Jasion*, Yaroslav Petrukhin**, Vasilyi Shangin**, Marcin Jukiewicz*

Functional completeness in CPL *via* correspondence analysis

* Department of Logic and Cognitive Science, Adam Mickiewicz University, Poznań, Poland
Dorota.Leszczynska[at]amu.edu.pl
Marcin.Jukiewicz[at]amu.edu.pl
** Department of Logic, Faculty of Philosophy, Lomonosov Moscow State University, Moscow, Russia
petrukhin[at]philos.msu.ru
shangin[at]philos.msu.ru

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Lemma 2. Let $x, y, z \in \{1, 0\}$, $A, B \in \mathscr{F}^{\circ}_{\neg}$, v_x, v_y, and v_z be valuations such that $v_x(A) = x$, $v_y(B) = y$, and $v_z(A \circ B) = z$. Then if $f_\circ(x, y) = z$, then $A^{v_x}, B^{v_y} \Rightarrow (A \circ B)^{v_z}$ is provable in the respective $\mathscr{C}$ for $\circ$.

Proof. Consider the case $f_\circ(0, 0) = 0$. We need to show that $\neg A, \neg B \Rightarrow \neg(A \circ B)$ is provable in the respective $\mathscr{C}$ for $\circ \in \{\circ_\perp, \wedge, \not\to, \circ_1, \not\leftarrow, \circ_2, \veebar, \vee\}$. If $\circ = \circ_1$, then we have $R^{(01)}_\circ$ in the calculus, if $\circ \in \{\circ_\perp, \not\to, \not\leftarrow\}$, then we have $R^{(02)}_\circ$, if $\circ = \circ_2$, then there is $R^{(03)}_\circ$, and if $\circ = \veebar$, then one of the three rules is present. In these cases we follow the appropriate scheme of derivation:

$$\cfrac{\cfrac{\cfrac{\cfrac{A \Rightarrow B, A}{\neg A, A \Rightarrow B}\,(\neg\Rightarrow)}{\neg A, A \circ B \Rightarrow B}\,R^{(01)}_\circ\downarrow}{\neg A, \neg B, A \circ B \Rightarrow}\,(\neg\Rightarrow)}{\neg A, \neg B \Rightarrow \neg(A \circ B)}\,(\Rightarrow\neg) \qquad \cfrac{\cfrac{\cfrac{\cfrac{B \Rightarrow \neg(A \circ B), B}{\neg B, B \Rightarrow \neg(A \circ B)}\,(\neg\Rightarrow)}{\neg B, A \circ B \Rightarrow A}\,R^{(02)}_\circ\downarrow}{\neg A, \neg B, A \circ B \Rightarrow}\,(\neg\Rightarrow)}{\neg A, \neg B \Rightarrow \neg(A \circ B)}\,(\Rightarrow\neg) \qquad \cfrac{\cfrac{\cfrac{\cfrac{B \Rightarrow A, B}{\neg B, B \Rightarrow A}\,(\neg\Rightarrow)}{\neg B, A \circ B \Rightarrow A}\,R^{(03)}_\circ\downarrow}{\neg A, \neg B, A \circ B \Rightarrow}\,(\neg\Rightarrow)}{\neg A, \neg B \Rightarrow \neg(A \circ B)}\,(\Rightarrow\neg)$$

In the case of $\circ = \{\wedge, \vee\}$, we cannot apply any of the above rules. If $\mathscr{C}_P = \mathscr{C}^A_P$, then we have the axiom $A^{(\mathrm{I})}_{\circ\downarrow}$, and the following shows that $\vdash_{\mathscr{C}^A_P} \neg A, \neg B \Rightarrow \neg(A \circ B)$:

$$\cfrac{\cfrac{A \circ B \Rightarrow A, B}{\neg A, \neg B, A \circ B \Rightarrow}\,(\neg\Rightarrow)\times 2}{\neg A, \neg B \Rightarrow \neg(A \circ B)}\,(\Rightarrow\neg)$$

If $\mathscr{C}_P = \mathscr{C}^R_P$, then we have the rule $R^{(\mathrm{I})}_\circ$:

$$\cfrac{\cfrac{\cfrac{\cfrac{A, B \Rightarrow A, A \circ B}{A \circ B \Rightarrow A, A, B}\,R^{(\mathrm{I})}_\circ\downarrow}{\neg A, \neg B, A \circ B \Rightarrow A}\,(\neg\Rightarrow)\times 2}{\neg A, \neg B \Rightarrow \neg(A \circ B), A}\,(\Rightarrow\neg) \quad \cfrac{A, \neg B \Rightarrow A, \neg(A \circ B)}{A, \neg A, \neg B \Rightarrow \neg(A \circ B)}\,(\neg\Rightarrow)}{\neg A, \neg B \Rightarrow \neg(A \circ B)}\,(\mathrm{cut})$$

Consider the case $f_\circ(0, 0) = 1$. We need to show that $\vdash_{\mathscr{C}} \neg A, \neg B \Rightarrow A \circ B$ is provable in the respective $\mathscr{C}$ for $\circ \in \{\downarrow, \equiv, \circ_{\neg 2}, \leftarrow, \circ_{\neg 1}, \rightarrow, \uparrow, \circ_\top\}$. If $\circ = \circ_{\neg 1}$, then we have the rule $R^{(04)}_\circ$, and then we follow:

$$\cfrac{\cfrac{A \Rightarrow B, A}{\neg A, A \Rightarrow B}\,(\neg\Rightarrow)}{\neg A, \neg B \Rightarrow A \circ B}\,R^{(04)}_\circ\downarrow$$

If $\circ \in \{\leftarrow, \rightarrow, \circ_{\top}\}$, then we have the rule $R^{(05)}_{\circ}$, and we follow:

$$\dfrac{\dfrac{B \Rightarrow A \circ B, B}{B, \neg B \Rightarrow A \circ B}\,(\neg \Rightarrow)}{\neg A, \neg B \Rightarrow A \circ B}\,R^{(05)}_{\circ}\downarrow$$

If $\circ = \circ_{\neg 2}$, then we have $R^{(06)}_{\circ}$, and we follow:

$$\dfrac{\dfrac{A \circ B \Rightarrow A, A \circ B}{\neg A \Rightarrow B, A \circ B}\,R^{(06)}_{\circ}\downarrow}{\neg A, \neg B \Rightarrow A \circ B}\,(\neg \Rightarrow)$$

If $\circ$ is $\equiv$, then one of $R^{(04)}_{\circ}$, $R^{(05)}_{\circ}$, $R^{(06)}_{\circ}$ is present. Finally, assume that $\circ \in \{\downarrow, \uparrow\}$. If $\mathscr{C} = \mathscr{C}^{A}_{P}$, then the required sequent is the axiom $A^{(\mathrm{II})}_{\circ\uparrow}$. Let $\mathscr{C} = \mathscr{C}^{R}_{P}$. Then sequent $\neg A, \neg B \Rightarrow A \circ B$ is proved as follows:

$$\dfrac{\neg A, \neg B \Rightarrow A \circ B, \neg A \qquad \dfrac{\neg A, A \circ B \Rightarrow \neg A, \neg B}{\neg A, \neg A, \neg B \Rightarrow A \circ B}\,R^{(\mathrm{II})}_{\circ}\uparrow}{\neg A, \neg B \Rightarrow A \circ B}\,(\mathrm{cut})$$

Consider the case $f_{\circ}(0,1) = 0$. We need to show that $\vdash_{\mathscr{C}} \neg A, B \Rightarrow \neg(A \circ B)$ is provable in the respective $\mathscr{C}$ for $\circ \in \{\circ_{\bot}, \wedge, \nrightarrow, \circ_{1}, \downarrow, \equiv, \circ_{\neg 2}, \leftarrow\}$. If $\circ = \circ_{\neg 2}$, then we have $R^{(06)}_{\circ}$, and we follow:

$$\dfrac{\dfrac{\dfrac{\neg A, B \Rightarrow B}{A \circ B, B \Rightarrow A}\,R^{(06)}_{\circ}\uparrow}{A \circ B, \neg A, B \Rightarrow}\,(\neg \Rightarrow)}{\neg A, B \Rightarrow \neg(A \circ B)}\,(\neg \Rightarrow)$$

If $\circ = \circ_{1}$, then we have $R^{(07)}_{\circ}$, and then we follow:

$$\dfrac{\dfrac{\dfrac{A \circ B, B \Rightarrow A \circ B}{B \Rightarrow \neg(A \circ B), A \circ B}\,(\Rightarrow \neg)}{B \Rightarrow \neg(A \circ B), A}\,R^{(07)}_{\circ}\downarrow}{\neg A, B \Rightarrow \neg(A \circ B)}\,(\neg \Rightarrow)$$

If $\circ \in \{\circ_{\bot}, \wedge, \downarrow\}$, then $R^{(08)}_{\circ}$ is present, and then:

$$\dfrac{\dfrac{\dfrac{A \circ B, B \Rightarrow B}{A \circ B, B \Rightarrow A}\,R^{(08)}_{\circ}\downarrow}{A \circ B, \neg A, B \Rightarrow}\,(\neg \Rightarrow)}{\neg A, B \Rightarrow \neg(A \circ B)}\,(\Rightarrow \neg)$$

In the case of $\equiv$, one of the three rules is present in the calculus. Suppose that $\circ \in \{\nrightarrow, \leftarrow\}$, and that $\mathscr{C} = \mathscr{C}^{A}_{P}$. Then the following:

$$\dfrac{\dfrac{\dfrac{\neg A, B \Rightarrow \neg(A \circ B), B}{\neg B, \neg A, B \Rightarrow \neg(A \circ B)}\,(\neg \Rightarrow)}{\neg A, B \Rightarrow \neg(A \circ B), \neg\neg B}\,(\Rightarrow \neg) \qquad \dfrac{\dfrac{\dfrac{A \circ B, B \Rightarrow A, \neg B}{\neg A, A \circ B, B \Rightarrow \neg B}\,(\neg \Rightarrow)}{\neg\neg B, \neg A, A \circ B, B \Rightarrow}\,(\neg \Rightarrow)}{\neg\neg B, \neg A, B \Rightarrow \neg(A \circ B)}\,(\Rightarrow \neg)}{\neg A, B \Rightarrow \neg(A \circ B)}\,(\mathrm{cut})$$

is a proof of the required sequent, since the left leaf follows under the scheme of (Ax), and the right one – under the scheme of $A^{(\mathrm{III})}_{\circ\downarrow}$. If $\mathscr{C} = \mathscr{C}^{R}_{P}$, then we have the rule $R^{(\mathrm{III})}_{\circ}$, and we follow:

$$
\begin{prooftree}
\AxiomC{$\neg A, B \Rightarrow B, \neg(A \circ B)$}
\RightLabel{$(\neg \Rightarrow)$}
\UnaryInfC{$\neg A, B, \neg B \Rightarrow \neg(A \circ B)$}
\RightLabel{$(\Rightarrow \neg)$}
\UnaryInfC{$\neg A, B \Rightarrow \neg(A \circ B), \neg\neg B$}
\AxiomC{$B, A \Rightarrow A \circ B, B$}
\RightLabel{$(\neg \Rightarrow)$}
\UnaryInfC{$B, A, \neg B \Rightarrow A \circ B$}
\RightLabel{$R_\circ^{(\mathrm{III})} \downarrow$}
\UnaryInfC{$B, A \circ B \Rightarrow A, \neg B$}
\RightLabel{$(\neg \Rightarrow)$}
\UnaryInfC{$\neg A, B, A \circ B \Rightarrow \neg B$}
\RightLabel{$(\neg \Rightarrow)$}
\UnaryInfC{$\neg\neg B, \neg A, B, A \circ B \Rightarrow$}
\RightLabel{$(\Rightarrow \neg)$}
\UnaryInfC{$\neg\neg B, \neg A, B \Rightarrow \neg(A \circ B)$}
\RightLabel{(cut)}
\BinaryInfC{$\neg A, B \Rightarrow \neg(A \circ B)$}
\end{prooftree}
$$

Consider the case $f_\circ(0,1) = 1$. We need to show that $\vdash_{\mathscr{C}} \neg A, B \Rightarrow A \circ B$ is provable in the respective $\mathscr{C}$ for $\circ \in \{\not\leftarrow, \circ_2, \veebar, \vee, \circ_{\neg 1}, \rightarrow, \uparrow, \circ_\top\}$. If $\circ = \circ_2$, then $R_\circ^{(03)}$ is present in the calculus and we have:

$$
\begin{prooftree}
\AxiomC{$A \circ B \Rightarrow A \circ B, A$}
\RightLabel{$R_\circ^{(03)} \uparrow$}
\UnaryInfC{$B \Rightarrow A \circ B, A$}
\RightLabel{$(\neg \Rightarrow)$}
\UnaryInfC{$\neg A, B \Rightarrow A \circ B$}
\end{prooftree}
$$

If $\circ = \circ_{\neg 1}$, then $R_\circ^{(09)}$ is present, and we derive:

$$
\begin{prooftree}
\AxiomC{$A \circ B, B \Rightarrow A \circ B$}
\RightLabel{$(\Rightarrow \neg)$}
\UnaryInfC{$B \Rightarrow A \circ B, \neg(A \circ B)$}
\RightLabel{$R_\circ^{(09)} \downarrow$}
\UnaryInfC{$B \Rightarrow A \circ B, A$}
\RightLabel{$(\neg \Rightarrow)$}
\UnaryInfC{$\neg A, B \Rightarrow A \circ B$}
\end{prooftree}
$$

If $\circ \in \{\vee, \uparrow, \circ_\top\}$, then we have $R_\circ^{(10)}$.

$$
\begin{prooftree}
\AxiomC{$B \Rightarrow A \circ B, B$}
\RightLabel{$(\neg \Rightarrow)$}
\UnaryInfC{$\neg B, B \Rightarrow A \circ B$}
\RightLabel{$R_\circ^{(10)} \downarrow$}
\UnaryInfC{$\neg A, B \Rightarrow A \circ B$}
\end{prooftree}
$$

In the case of $\veebar$, one of the three rules is in the calculus. Finally, suppose that $\circ \in \{\not\leftarrow, \rightarrow\}$. If $\mathscr{C} = \mathscr{C}_P^A$, then the desired sequent is of the form of the axiom scheme $A_{\circ\uparrow}^{(\mathrm{IV})}$. If $\mathscr{C} = \mathscr{C}_P^R$, then we apply:

$$
\begin{prooftree}
\AxiomC{$\neg A, B \Rightarrow A \circ B, \neg A$}
\AxiomC{$\neg A, A \circ B \Rightarrow \neg A, B$}
\RightLabel{$R_\circ^{(\mathrm{IV})} \uparrow$}
\UnaryInfC{$\neg A, \neg A, B \Rightarrow A \circ B$}
\RightLabel{(cut)}
\BinaryInfC{$\neg A, B \Rightarrow A \circ B$}
\end{prooftree}
$$

Consider the case $f_\circ(1,0) = 0$. We need to show that $\vdash_{\mathscr{C}} A, \neg B \Rightarrow \neg(A \circ B)$ is provable in the respective $\mathscr{C}$ for $\circ \in \{\circ_\bot, \wedge, \not\leftarrow, \circ_2, \downarrow, \equiv, \circ_{\neg 1}, \rightarrow\}$. If $\circ = \circ_{\neg 1}$, then we have: $R_\circ^{(04)}$, and:

$$
\begin{prooftree}
\AxiomC{$A \circ B, \neg B \Rightarrow A \circ B$}
\RightLabel{$(\Rightarrow \neg)$}
\UnaryInfC{$\neg B \Rightarrow \neg(A \circ B), A \circ B$}
\RightLabel{$R_\circ^{(04)} \uparrow$}
\UnaryInfC{$A \Rightarrow \neg(A \circ B), B$}
\RightLabel{$(\neg \Rightarrow)$}
\UnaryInfC{$A, \neg B \Rightarrow \neg(A \circ B)$}
\end{prooftree}
$$

If $\circ \in \{\circ_\bot, \wedge, \downarrow\}$, then we have $R_\circ^{(08)}$, and if $\circ$ is $\circ_2$ – $R_\circ^{(11)}$. We go as follows:

$$
\begin{prooftree}
\AxiomC{$A \circ B, A \Rightarrow A$}
\RightLabel{$R_\circ^{(08)} \uparrow$}
\UnaryInfC{$A \circ B, A \Rightarrow B$}
\RightLabel{$(\Rightarrow \neg)$}
\UnaryInfC{$A \Rightarrow \neg(A \circ B), B$}
\RightLabel{$(\neg \Rightarrow)$}
\UnaryInfC{$A, \neg B \Rightarrow \neg(A \circ B)$}
\end{prooftree}
$$

$$
\begin{prooftree}
\AxiomC{$A \circ B, A \Rightarrow A \circ B$}
\RightLabel{$(\Rightarrow \neg)$}
\UnaryInfC{$A \Rightarrow \neg(A \circ B), A \circ B$}
\RightLabel{$R_\circ^{(11)} \downarrow$}
\UnaryInfC{$A \Rightarrow \neg(A \circ B), B$}
\RightLabel{$(\neg \Rightarrow)$}
\UnaryInfC{$A, \neg B \Rightarrow \neg(A \circ B)$}
\end{prooftree}
$$

If $\circ$ is $\equiv$, then we must have one of the three rules. Let $\circ \in \{\not\leftarrow, \rightarrow\}$ and $\mathscr{C} = \mathscr{C}_P^A$. Then we go as follows (the right leaf is the appropriate axiom):

$$\dfrac{\dfrac{\dfrac{A, \neg B \Rightarrow A, \neg(A \circ B)}{\neg A, A, \neg B \Rightarrow \neg(A \circ B)}\,(\neg\Rightarrow)}{A, \neg B \Rightarrow \neg(A \circ B), \neg\neg A}\,(\Rightarrow\neg) \qquad \dfrac{\dfrac{\dfrac{A, A \circ B \Rightarrow \neg A, B}{A, \neg B, A \circ B \Rightarrow \neg A}\,(\neg\Rightarrow)}{\neg\neg A, A, \neg B, A \circ B \Rightarrow}\,(\neg\Rightarrow)}{\neg\neg A, A, \neg B \Rightarrow \neg(A \circ B)}\,(\Rightarrow\neg)}{A, \neg B \Rightarrow \neg(A \circ B)}\,(\mathrm{cut})$$

whereas in the case of $\mathscr{C} = \mathscr{C}_P^R$, as follows:

$$\dfrac{\dfrac{\dfrac{A, \neg B \Rightarrow A, \neg(A \circ B)}{\neg A, A, \neg B \Rightarrow \neg(A \circ B)}\,(\neg\Rightarrow)}{A, \neg B \Rightarrow \neg(A \circ B), \neg\neg A}\,(\Rightarrow\neg) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A, B \Rightarrow A \circ B, A}{A, \neg A, B \Rightarrow A \circ B}\,(\neg\Rightarrow)}{A, A \circ B \Rightarrow \neg A, B}\,R_\circ^{(\mathrm{IV})}\downarrow}{A, \neg B, A \circ B \Rightarrow \neg A}\,(\neg\Rightarrow)}{\neg\neg A, A, \neg B, A \circ B \Rightarrow}\,(\neg\Rightarrow)}{\neg\neg A, A, \neg B \Rightarrow \neg(A \circ B)}\,(\Rightarrow\neg)}{A, \neg B \Rightarrow \neg(A \circ B)}\,(\mathrm{cut})$$

Consider the case $f_\circ(1,0) = 1$. We need to show that $\vdash_{\mathscr{C}} A, \neg B \Rightarrow A \circ B$ is provable in the respective $\mathscr{C}$ for $\circ \in \{\not\rightarrow, \circ_1, \veebar, \vee, \circ_{\neg 2}, \leftarrow, \uparrow, \circ_\top\}$. If $\circ$ is $\circ_1$, then the calculus contains $R_\circ^{(01)}$; if $\circ \in \{\vee, \uparrow, \circ_\top\}$, then it contains $R_\circ^{(10)}$, if $\circ = \circ_{\neg 2}$, then $R_\circ^{(12)}$ is present, and if $\circ$ is $\veebar$, then it is one of the three rules. In these cases we apply:

$$\dfrac{\dfrac{A \circ B \Rightarrow A \circ B, B}{A \Rightarrow A \circ B, B}\,R_\circ^{(01)}\uparrow}{A, \neg B \Rightarrow \neg(A \circ B)}\,(\neg\Rightarrow)$$

or:

$$\dfrac{\dfrac{A \Rightarrow A \circ B, A}{A, \neg A \Rightarrow A \circ B}\,(\neg\Rightarrow)}{A, \neg B \Rightarrow A \circ B}\,R_\circ^{(10)}\uparrow$$

or:

$$\dfrac{\dfrac{\dfrac{A \circ B, A \Rightarrow A \circ B}{A \Rightarrow A \circ B, \neg(A \circ B)}\,(\Rightarrow\neg)}{A \Rightarrow A \circ B, B}\,R_\circ^{(12)}\downarrow}{A, \neg B \Rightarrow A \circ B}\,(\neg\Rightarrow)$$

If $\circ \in \{\not\rightarrow, \leftarrow\}$, and $\mathscr{C}$ contains $A_{\circ\uparrow}^{(\mathrm{III})}$, then the desired sequent is an instance of the axiom scheme. Suppose $\mathscr{C} = \mathscr{C}_P^R$. Then we apply:

$$\dfrac{A, \neg B \Rightarrow A \circ B, \neg B \qquad \dfrac{\neg B, A \circ B \Rightarrow A, \neg B}{\neg B, A, \neg B \Rightarrow A \circ B}\,R_\circ^{(\mathrm{III})}\uparrow}{A, \neg B \Rightarrow A \circ B}\,(\mathrm{cut})$$

Consider the case $f_\circ(1,1) = 0$. We need to show that $\vdash_{\mathscr{C}} A, B \Rightarrow \neg(A \circ B)$ is provable in the respective $\mathscr{C}$ for $\circ \in \{\circ_\bot, \not\rightarrow, \not\leftarrow, \veebar, \downarrow, \circ_{\neg 2}, \circ_{\neg 1}, \uparrow\}$. If $\circ \in \{\circ_\bot, \not\rightarrow, \not\leftarrow\}$, then the rule $R_\circ^{(02)}$ is in the calculus; if $\circ = \circ_{\neg 1}$, then it is $R_\circ^{(09)}$; if $\circ = \circ_{\neg 2}$, then it is $R_\circ^{(12)}$; and if $\circ$ is $\veebar$, then it is one of the three. In these cases we follow:

$$\dfrac{A, A \circ B \Rightarrow A}{A, B \Rightarrow \neg(A \circ B)}\,R_\circ^{(02)}\uparrow$$

or:

$$\dfrac{A, B \Rightarrow A}{A, B \Rightarrow \neg(A \circ B)}\,R_\circ^{(09)}\uparrow$$

or:

$$\dfrac{A, B \Rightarrow B}{A, B \Rightarrow \neg(A \circ B)}\ R^{(12)}_{\circ}\uparrow$$

If $\circ \in \{\downarrow, \uparrow\}$, then either $A^{(\mathrm{II})}_{\circ\downarrow}$ or $R^{(\mathrm{II})}_{\circ\downarrow}$ is used as follows:

$$\dfrac{\dfrac{\dfrac{A, B \Rightarrow \neg(A \circ B), B}{\neg B, A, B \Rightarrow \neg(A \circ B)}\,(\neg \Rightarrow)}{A, B \Rightarrow \neg(A \circ B), \neg\neg B}\,(\Rightarrow \neg) \qquad \dfrac{\dfrac{\dfrac{\dfrac{A, B \Rightarrow \neg(A \circ B), \neg B, A}{\neg A, A, B \Rightarrow \neg(A \circ B), \neg B}\,(\neg \Rightarrow)}{A, B \Rightarrow \neg(A \circ B), \neg B, \neg\neg A}\,(\Rightarrow \neg) \qquad \dfrac{\dfrac{A, B, A \circ B \Rightarrow \neg B, \neg A}{A, B \Rightarrow \neg(A \circ B), \neg B, \neg A}\,(\Rightarrow \neg)}{\neg\neg A, A, B \Rightarrow \neg(A \circ B), \neg B}\,(\Rightarrow \neg)}{A, B \Rightarrow \neg(A \circ B), \neg B}\,(\mathrm{cut})}{\neg\neg B, A, B \Rightarrow \neg(A \circ B)}\,(\neg \Rightarrow)}{A, B \Rightarrow \neg(A \circ B)}\,(\mathrm{cut})$$

For the case of $\mathscr{C}^R_P$ the rightmost branch is extended as follows:

$$\dfrac{\dfrac{A, B, \neg B \Rightarrow A \circ B, A}{A, B, \neg A, \neg B \Rightarrow A \circ B}\,(\neg \Rightarrow)}{A, B, A \circ B \Rightarrow \neg A, \neg B}\ R^{(\mathrm{II})}_{\circ}\downarrow$$

Finally, consider the case $f_\circ(1,1) = 1$. We need to show that $\vdash_{\mathscr{C}} A, B \Rightarrow A \circ B$ is provable in the respective $\mathscr{C}$ for $\circ \in \{\wedge, \circ_1, \circ_2, \vee, \equiv, \leftarrow, \rightarrow, \circ_\top\}$. For $\circ \in \{\leftarrow, \rightarrow, \circ_\top\}$ we have $R^{(05)}_\circ$; for $\circ_1$ we have $R^{(07)}_\circ$; for $\circ_2$ we have $R^{(11)}_\circ$; and for $\equiv$ we have one of the three rules. Then we follow:

$$\dfrac{\dfrac{A \Rightarrow A \circ B, A}{A, \neg A \Rightarrow A \circ B}\,(\neg \Rightarrow)}{A, B \Rightarrow A \circ B}\ R^{(05)}_{\circ}\uparrow$$

or:

$$\dfrac{A, B \Rightarrow A}{A, B \Rightarrow A \circ B}\ R^{(07)}_{\circ}\uparrow$$

or:

$$\dfrac{A, B \Rightarrow B}{A, B \Rightarrow A \circ B}\ R^{(11)}_{\circ}\uparrow$$

If $\circ \in \{\wedge, \vee\}$ and $\mathscr{C} = \mathscr{C}^A_P$, then the respective sequent is an instance of $A^{(\mathrm{I})}_{\circ\uparrow}$. If $\mathscr{C} = \mathscr{C}^R_P$, then the sequent is derived as follows:

$$\dfrac{\dfrac{\dfrac{A, B \Rightarrow A \circ B, A}{\neg A, A, B \Rightarrow A \circ B}\,(\neg \Rightarrow)}{A, B \Rightarrow A \circ B, \neg\neg B}\,(\Rightarrow \neg) \qquad \dfrac{\dfrac{\dfrac{B, A \circ B \Rightarrow A, B}{A \circ B \Rightarrow A, B, \neg B}\,(\Rightarrow \neg)}{A, B \Rightarrow A \circ B, \neg B}\ R^{(\mathrm{I})}_{\circ}\uparrow}{\neg\neg B, A, B \Rightarrow A \circ B}\,(\neg \Rightarrow)}{A, B \Rightarrow A \circ B}\,(\mathrm{cut})$$

$\square$