Steady One-Dimensional Fluid Flow in a Semi-Infinite Porous Medium with Sinusoidal Surface Temperature

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Abstract. An equation is presented for computing infiltration rates from a study of temperatures observed near the land surface. The boundary conditions of heat and liquid movement assumed are (1) a sinusoidal temperature fluctuation of constant amplitude at the land surface and (2) a constant and uniform percolation rate normal to the land surface in a homogeneous medium. In natural mediums of average heat properties, percolation rates of the order of 2 cm/day or greater can be detected with ease by analyzing the temperature profile resulting from diurnal temperature fluctuations. Percolation rates of the order of 0.1 cm/day can be detected by analyzing the temperature profile resulting from annual temperature fluctuations at the land surface. With the most favorable conditions of low heat conductivity, large amplitude of temperature at the surface, and careful measurements of temperature, analysis of diurnal temperature fluctuations may yield accurate detection of velocity to a minimum of 0.3 cm/day.

Introduction. Concurrent flow of heat and fluid in saturated porous mediums has been studied in connection with a variety of engineering applications. Citing only a few examples of such work here will suffice to illustrate the scope of interest in this phenomenon [*Baumeister and Bennett*, 1958; *Morduchow et al.*, 1957; *Podolsky*, 1951; *Weinbaum and Wheeler*, 1949; *Wooding*, 1957]. Interest ranges from evaporative cooling of porous metals for high-temperature environments through heat exchange in chemical processing to monitoring the motion of fluids in vegetation.

Also within the field of hydrology, the relation between temperature and flow of water underground has been studied. *Schneider* [1961] examined the relation between air temperature and release of water by thawing in the unsaturated zone. *Lovering and Morris* (in press) estimated flow from an underground fissure by study of the temperature distribution. *Suzuki* [1960] proposed a formula relating infiltration rate to the temperature profile below the land surface. *Stallman* [1960] suggested applying finite-difference methods to the differential equation of heat and liquid flow for the computation of groundwater velocity and permeability.

The case investigated by *Suzuki* [1960] is studied further in this paper. His work relates to constant velocity of water entering through the land surface, with the temperature at the surface varying sinusoidally. He derived an algebraic equation relating temperature to groundwater velocity, depth below the land surface, and time. His equation was used for estimating the infiltration velocity from flooded rice fields. This measurement approach could be used in several situations, such as monitoring seepage from stream channels and ponds or return flow from irrigated fields.

In attempting to apply Suzuki's result (p. 2884, equations 2–4) for determining the sensitivity of the thermal measurements of infiltration velocity, I noticed that Suzuki's equation is an incomplete approximation. An exact solution of his flow problem is presented in this paper. Usefulness of the solution in field studies of groundwater flow is also discussed.

Development of analytical equation. The definitions of symbols used in this paper and the dimensional units adopted for use in the equations are given in the following list:

- a, b, constants, in cm^{-1}.
- c, specific heat of the fluid and rock in combination, in cal g^{-1} °C^{-1}.
- c_r, specific heat of the rock, in cal g^{-1} °C^{-1}.
- c_0, specific heat of the fluid, in cal g^{-1} °C^{-1}.
- k, heat conductivity of fluid and solid in combination, in cal sec^{-1} cm^{-1} °C^{-1}.
- m, a constant defining the relation between V and K, dimensionless.
- t, time, in seconds.

t_l, time lag of temperature wave at depth z (see Figure 2), in seconds.
v, gross velocity of fluid movement along z through a porous medium, taken positive downward from the land surface, in cm sec^{-1}.
z, depth below the land surface, in centimeters.
K, a constant, equals $\pi c\rho/k\tau$, in cm^{-2}.
T, temperature at any point z, t, in °C.
T_{A0}, ambient temperature at $z = 0$ and time t, in °C.
T_{AZ}, ambient temperature at point z, t due to thermal boundary conditions other than sinusoidal temperatuie fluctuation at the land surface, in °C.
T_{zm}, maximum or minimum temperature observed at depth z, in °C.
ΔT, amplitude of the temperature variation at the land surface (see Figure 2), in °C.
V, a constant, equals $vc_0\rho_0/2k$, in cm^{-1}.
ρ, density of fluid and rock in combination, in g cm^{-3}.
ρ_0, density of fluid, in g cm^{-3}.
ρ_r, density of rock, in g cm^{-3}.
τ, period of oscillation of temperature at land surface (see Figure 2), in seconds.
θ, porosity of rock as a decimal fraction of total volume, dimensionless.

One-dimensional anisothermal flow of an incompressible fluid through homogeneous porous mediums can be described by

$$k\frac{\partial^2 T}{\partial z^2} - vc_0\rho_0\frac{\partial T}{\partial z} = c\rho\frac{\partial T}{\partial t} \qquad (1)$$

where T is temperature, z is distance along the direction of flow, t is time, v is the fluid velocity in L^3 per unit gross area per unit time, c_0 and ρ_0 are the specific heat and density of the fluid, respectively, c and ρ are the specific heat and density of the fluid and medium in combination, respectively, and k is the heat conductivity of the fluid and medium in combination. Equation 1 applies only to situations wherein the following conditions are satisfied:

1. Fluid flow is parallel with the z axis. In the solution given in this paper, fluid velocity is assumed to be steady and uniform along the z axis.
2. Heat characteristics of the medium and fluid are constant in space and time.
3. All components of heat and fluid flow occur along only the z axis.
4. Temperature of the water at every point in the interstices equals the temperature of the adjoining rock at all times.

Diurnal heating and cooling of the land surface produces a temperature fluctuation on the surface which can be defined as

$$T = T_{A0} + \Delta T \sin 2\pi t/\tau \qquad (2)$$

in which T_{A0} is the average ambient temperature at the surface, ΔT is the amplitude of the temperature variation at the surface, and τ is the period of oscillation of temperature at the surface. If we take z to be positive downward and zero at the land surface, v is positive for infiltration. At $z = \infty$ the temperature is not influenced by fluctuations imposed at the surface. Thus

$$\lim_{z\to\infty} T = T_{AZ} \qquad (3)$$

where T_{AZ} is the ambient temperature at depth z. Equations 2 and 3 are the boundary conditions to be satisfied by the algebraic solution fitting equation 1 over the region $0 < z < \infty$ and $0 < t < \infty$. Furthermore, it is assumed tacitly that temperature fluctuations at all depths will be in equilibrium with the sinusoidal fluctuations of constant amplitude generated at the land surface.

Following *Suzuki* [1960], assume a solution of equation 1 in the form

$$T - T_{AZ} = \Delta T e^{-az} \sin(2\pi t/\tau - bz) \qquad (4)$$

where a and b are assumed to be constants. Taking the appropriate derivatives of $T - T_{AZ}$, substituting into (1), accounting for ambient conditions by superposition, and solving for a and b, we find

$$a = [(K^2 + V^4/4)^{1/2} + V^2/2]^{1/2} - V \qquad (5)$$

and

$$b = [(K^2 + V^4/4)^{1/2} - V^2/2]^{1/2} \qquad (6)$$

where

$$K = \pi c\rho/k\tau \qquad (7)$$

and

$$V = vc_0\rho_0/2k \qquad (8)$$

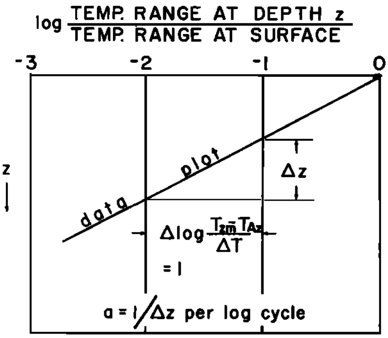

Fig. 1. Method of calculating a from temperature profile.

Substituting from (5) and (6) into (4) we get a solution which satisfies (1) and the boundary conditions. Hence this is the solution sought. Note that if V is very large in comparison with K, both a and b approach zero, and temperature at depth is nearly equal to the temperature at the surface at all times. For $v = 0$, a and b reduce to the familiar form [*Jakob,* 1949, pp. 293–294]

$$a = b = (\pi c\rho/k\tau)^{1/2} \tag{9}$$

The value of a can be observed from field data by analyzing the attenuation of the temperature wave with depth. At the maximum and minimum values of $T - T_{Az}$ for any depth z, $\sin(2\pi t/\tau - bz) = \pm 1$. Taking the logarithm of (4) we find

$$\log\left(\frac{T_{zm} - T_{Az}}{\Delta T}\right) = -az \tag{10a}$$

where T_{zm} is the maximum or minimum temperature observed at depth z. The value of a can be determined easily by observing the slope of the curve $\log[(T_{zm}-T_{Az})/\Delta T]$ versus z, as demonstrated in Figure 1, from the equation

$$a = 1/\Delta z \tag{10b}$$

where Δz is the difference in z over one log cycle.

Letting $t = 0$ when $T = T_{A0}$ at the surface, and observing that the time lag t_l between the times $T = T_{A0}$ occurs at the surface and $T = T_{Az}$ occurs at various depths z (the difference between times of occurrence of ambient temperatures), we can derive from (4) an equation for b,

$$b = 2\pi t_l/\tau z \tag{11}$$

The definition of t_l is shown in Figure 2.

Dependence of the temperature profile on velocity. The expressions for a and b given as (5) and (6) can be put in a more convenient form by letting

$$V^2/2 = mK \tag{12}$$

where m is a constant. Substituting (12) into (5) and (6) yields

$$a = [[(m^2 + 1)^{1/2} + m]^{1/2} - (2m)^{1/2}]K^{1/2} = K^{1/2}f_1(m) \tag{13}$$

and

$$b = [(m^2 + 1)^{1/2} - m]^{1/2}K^{1/2} = K^{1/2}f_2(m) \tag{14}$$

From (7), (8), and (12),

$$\frac{vc_0\rho_0}{k}\left(\frac{k\tau}{\pi c\rho}\right)^{1/2} = 2(2m)^{1/2} = \frac{vc_0\rho_0}{k}K^{-1/2} \tag{15}$$

and from (13) and (14),

$$a/b = f_1(m)/f_2(m) \tag{16}$$

Values of $aK^{-1/2}$, $bK^{-1/2}$, and a/b as functions of $(vc_0\rho_0/k)(k\tau/\pi c\rho)^{1/2}$ are given in Table 1.

Equations 1 and 4 yield the following definitions:

$$\frac{vc_0\rho_0}{k} = \frac{b^2 - a^2}{a} \tag{17}$$

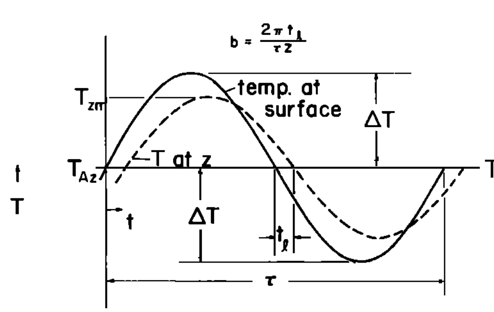

Fig. 2. Method of calculating b from time lag of temperature.

TABLE 1. Numerical Coefficients Useful for Analysis of Temperature Profile

$2(2m)^{1/2} = (vc_0\rho_0/k)(k\tau/\pi c\rho)^{1/2}$	$f_1(m) = a(k\tau/\pi c\rho)^{1/2}$	$f_2(m) = b(k\tau/\pi c\rho)^{1/2}$	$f_1(m)/f_2(m) = a/b$
3.0×10^{-3}	0.998 501	0.999 999	0.998 501
3.5	.998 251	.999 999	.998 252
4.0	.998 001	.999 999	.998 002
5.0	.997 502	.999 998	.997 503
6.0	.997 002	.999 998	.997 004
7.0	.996 503	.999 997	.996 506
8.0	.996 004	.999 996	.996 008
1.0×10^{-2}	.995 006	.999 994	.995 012
1.3	.993 511	.999 989	.993 521
1.6	.992 016	.999 984	.992 032
2.0	.990 025	.999 975	.990 050
2.5	.987 539	.999 961	.987 578
3.0	.985 056	.999 944	.985 112
3.5	.982 577	.999 923	.982 652
4.0	.980 100	.999 900	.980 198
5.0	.975 156	.999 844	.975 309
6.0	.970 225	.999 775	.970 443
7.0	.965 306	.999 694	.965 602
8.0	.960 400	.999 600	.960 784
1.0×10^{-1}	.950 625	.999 375	.951 220
1.3	.936 057	.998 944	.937 046
1.6	.921 601	.998 401	.923 077
2.0	.902 503	.997 503	.904 762
2.5×10^{-1}	.878 914	.996 101	.882 354
3.0	.855 641	.994 391	.860 467
3.5	.832 685	.992 373	.839 085
4.0	.810 049	.990 050	.818 190
5.0	.765 745	.984 499	.777 802
6.0	.722 747	.977 759	.739 188
7.0	.681 079	.969 858	.702 246
8.0	.640 766	.960 830	.666 888
1.0×10^{0}	.564 322	.939 565	.600 621
1.3	.460 549	.900 455	.511 463
1.6	.370 450	.854 373	.433 593
2.0	.272 020	.786 151	.346 014
2.5	.181 868	.698 388	.260 411
3.0	.121 789	.616 603	.197 516
3.5	.083 060 1	.545 536	.152 254
4.0	.058 171 0	.485 868	.119 726
5.0	.031 027 7	.395 096	.078 532 0
6.0	.018 239 9	.331 319	.055 052 5
7.0	.011 566 0	.284 773	.040 614 8
8.0	.007 774 66	.249 515	.031 159 1
1.0×10^{1}	.003 992 03	.199 840	.019 976 1
1.3	.001 819 39	.153 803	.011 829 4
1.6	.000 976 265	.124 985	.007 811 07
2.0	.000 499 94	.099 995 0	.004 999 6
2.5	.000 255 99	.079 998 4	.003 199 9
3.0	.000 148 15	.066 667	.002 222 3
3.5	.000 093 29	.057 143	.001 632 6
4.0	.000 062 50	.050 000	.001 250 0
5.0	.000 032 00	.040 000	.000 800 0
6.0	.000 018 52	.033 333	.000 555 6
7.0	.000 011 66	.028 57	.000 408 1
8.0	.000 007 81	.025 00	.000 312
1.0×10^{2}	.000 004 00	.020 00	.000 200

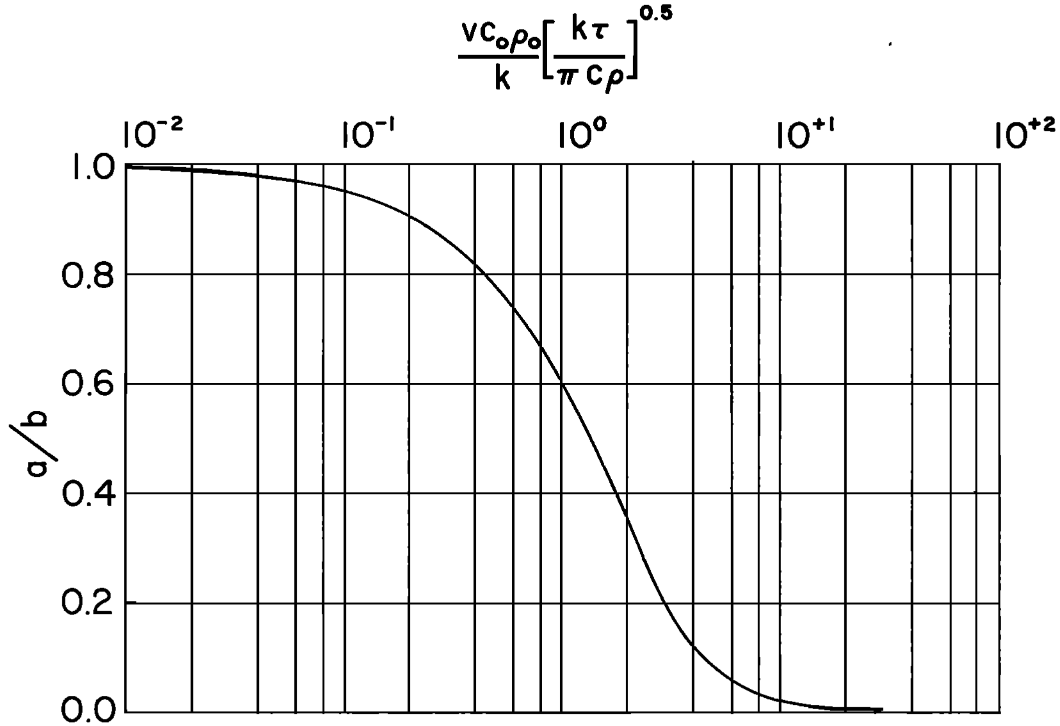

Fig. 3. Graph of a/b versus $(vc_0\rho_0/k)(k\tau/\pi c\rho)^{1/2}$.

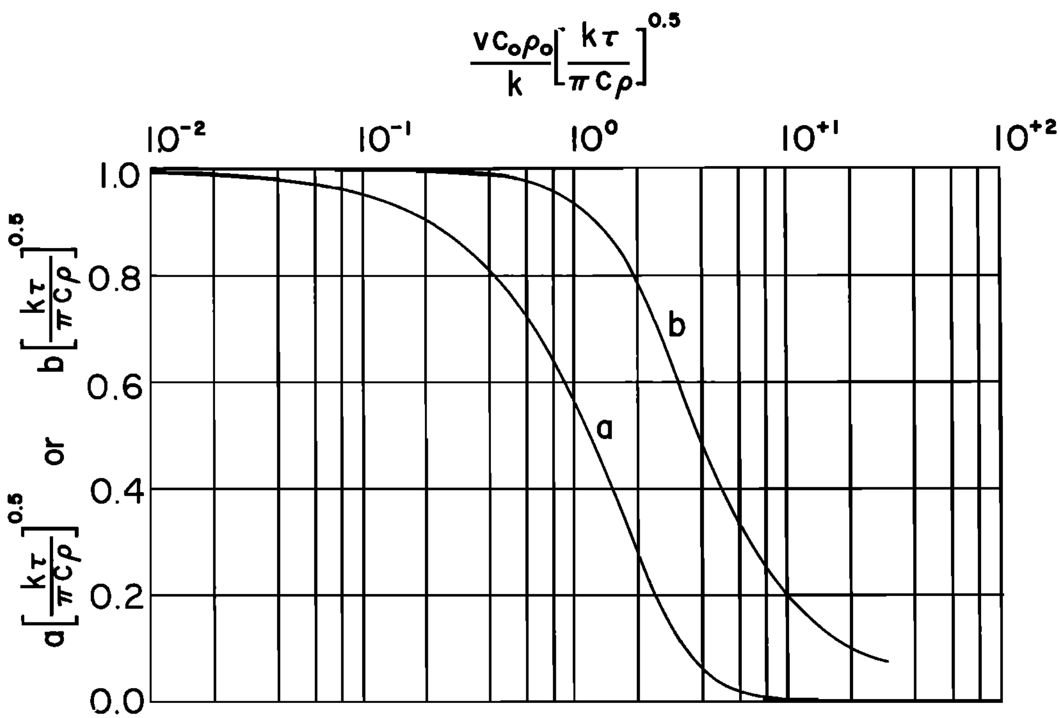

Fig. 4. Graph of $aK^{-1/2}$ and $bK^{-1/2}$ versus $(vc_0\rho_0/k)(k\tau/\pi c\rho)^{1/2}$.

The value of $vc_0\rho_0/k$ is computed from (17) after a and b have been determined from (10b) and (11). Through the value of $f_1(m)/f_2(m)$ calculated by (16), we can determine $vc_0\rho_0/k)\cdot(k\tau/\pi c\rho)^{1/2}$ from Figure 3, which has been constructed from Table 1. Knowing $vc_0\rho_0/k$ and $(vc_0\rho_0/k)$ $(k\tau/\pi c\rho)^{1/2}$ permits computation of $\pi c\rho/k\tau$ from (15). Thus all parameters are obtainable. If v is negligible, (9) and (4) constitute the complete solution, and a is obtained from (10).

According to (17), the accuracy of the determination of v from analysis of the temperature profile is dependent on the degree of accuracy to which a and b can be resolved from field data. Graphs of $aK^{-1/2}$ and $bK^{-1/2}$ versus $(vc_0\rho_0/k)$ $(k\tau/\pi c\rho)^{1/2}$ are given in Figure 4. It can be shown from equation 17 that error in $vc_0\rho_0/k$ equals twice the error in b minus twice the error in a. Errors in a and b have the most profound effect on computed v where the ratio a/b is approximately equal to one. However, in the following assessment of sensitivity between temperature and velocity, errors arising from calculated values of a and b are neglected.

Assuming that the value of a must be affected as much as, or more than, 5% by the liquid flow to ensure a sensible observation of v, we see from Figure 4 and Table 1 that $(vc_0\rho_0/k)$ $(k\tau/\pi c\rho)^{1/2}$ must be greater than about 10^{-1} for reliable detection of v. Thus, from (15), the smallest v reliably observed can be computed as

$$\frac{vc_0\rho_0}{k}\left[\frac{k\tau}{\pi c\rho}\right]^{1/2} = 10^{-1}$$

Letting $\tau = 1$ day $= 8.64 \times 10^4$ seconds, $k = 5 \times 10^{-3}$ cal/sec cm °C, $c\rho = 0.43$ cal/cm^3 °C, and $c_0\rho_0 = 1$ cal/cm^3 °C, $v = 2.8 \times 10^{-5}$ cm/sec $= 2.4$ cm/day $= 0.08$ ft/day. Thus, for rocks with heat characteristics as assumed, it seems possible to compute percolation velocities of about 2 cm/day or greater by analyzing the temperature response in the rock profile. Some rocks have a heat conductivity considerably less than 5×10^{-3} cal/sec cm °C, and in such materials velocities less than 2 cm/day could be determined by analysis of the temperature profile. Although sensitivity of temperature to velocity seems rather high, the possible error in computed v due to error in a and b is relatively large when v is small. For example, assuming $a/b = 0.95$, with v small, and a $\pm 2\%$ error in both a and b, the value of $vc_0\rho_0/k$ lies between zero and 10^{-2}, approximately. For this case, considering the effects of an assumed error of $\pm 2\%$ in a and b, one could only conclude that v is within the range 0 to 4 cm/day, even though sensitivity is 2.4 cm/day.

Use of the procedure outlined for observing fluid velocity is dependent on estimates, or measurements, of the heat conductivity, k, of the rock and water matrix. For naturally occurring rocks values of k range over more than an order of magnitude. Thus estimates of k may lead to erroneous values of v. The simplest approach to evaluating k for field observation appears to be as follows:

1. Measure the specific heat, c_r, density, ρ_r, and porosity, θ, of rock samples collected in the field.
2. Calculate $c\rho$ from

$$c\rho = c_r\rho_r(1 - \theta) + c_0\rho_0\theta \qquad (18)$$

3. Calculate k from the value of $\pi c\rho/k\tau$ which is obtained from (15) and (17).

Because $c_r\rho_r$ has a much smaller range of values than k for naturally occurring rocks, the evlauation of k through measurement, or estimation, of $c\rho$ is likely to be the most satisfactory approach.

Conclusions. Percolation rates from surface-water bodies can be determined by analyzing the attenuation of diurnal sinusoidal fluctuations of temperature with depth if the percolation rates are of the order of 2 cm/day or greater. Applying a similar analysis to the annual temperature fluctuation will permit measurement of percolation rates as small as about 0.1 cm/day. If the heat conductivity of the rocks is at the low end of the range common to porous rocks, and special care is exercised in measurement and analysis of temperatures, it may be possible to calculate percolation rates as small as 0.3 cm/day from a study of diurnal temperature.

Error in evaluation of constants in the analytical equation may exceed sensitivity of the technique if velocity is of the order of 2 cm/day or less. It may be possible to reduce such errors by using finite-difference analysis rather than the analytical equation given in this paper, particularly where heat characteristics of the medium are heterogeneous. Field trials are needed to further define the usefulness of temperature as an indicator of groundwater velocity.

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