Mathematics of Gaming (COMP-10185)

Assignment 1 – Investigating Probability

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Question 1: The Monty Hall Problem

Investigate the conjecture that it is always best to switch doors by producing Monte Carlo simulations that simulate each of the following:

- *100 doors, where 1 is chosen and 98 are revealed*
- *10 doors, where 1 is chosen and 8 are revealed*
- *x doors, where 1 is chosen and (x-2) are revealed, where x is the last digit of your student number*

Run each simulation 1,000,000 times. What is the final probability for each scenario?

I remember this problem from a "Brooklyn Nine-Nine" episode years ago and it was interesting enough I looked into it and tried it out by hand with 3 doors. I never thought to look at it with more doors; the assigned video from "Numberphile" and doing this assignment was very illustrative and clear in explaining what's happening.

The simulation was designed to choose a random winning door, a random contestant's door, and to open all doors except two (one of which is the contestant's door). The other closed door depending on randomness would be the winning door or an empty door (if the contestant's door is the winning door). Since there are only two doors left, one of which is supposed to be the winning door, one is inclined to think that it's a simple 50/50 chance and switching doesn't matter.

The results of the simulations, 1 million times each, show that it is always best to switch doors regardless of the number of doors. The final probability for each scenario is as follows:

- 100 doors, where 1 is chosen and 98 are revealed: **99.0009%**
- 10 doors, where 1 is chosen and 8 are revealed: **90.0431%**
- 8 doors, where 1 is chosen and 6 are revealed: **87.5391%**

From this it is possible to estimate the chance of winning when switching for any number of doors (let's say greater than 3 so the game makes sense) with this formula, (n – 1) / n, where n is the number of doors:

- 100 doors; (100-1)/100 = 99/100 = **99%**
- 10 doors; (10-1)/10 = 9/10 = **90%**
- 8 doors; (8-1)/8 = 7/8 = **87.5%**

The above values reflect the results of the simulation. For the original problem of 3 doors, just for fun:

- 3 doors; (3-1)/3 = 2/3 = **66.67%**

And running that simulation resulted in 66.6057% which isn't as close as I'd like but that's probability right, as another simulation resulted in 66.6892%.

Having run the simulations and doing a little math it is clear that having two doors left doesn't make it a 50/50 chance. The game is designed so that the contestant's choice, from the beginning, has a smaller probability of being the winning door. The contestant's door has a 1/n chance of being the winning door whereas the other remaining door, having once been a part of a larger group of doors, will have a higher (n-1)/n chance of being the winning door. To put it succinctly, the game gives the contestant a house advantage or edge for switching doors. Therefore, I agree with the conjecture that it is always best to switch doors.

Monte Carlo Screenshots

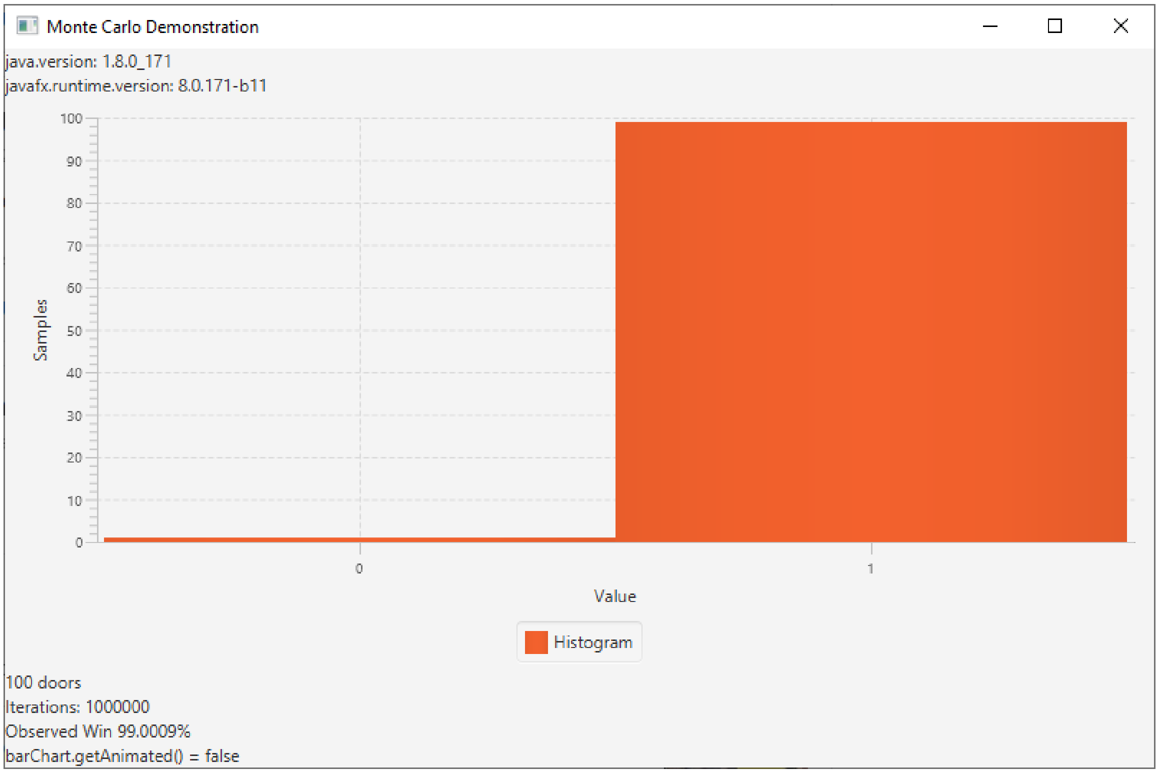

Figure 1: Monty Hall Monte Carlo for 100 doors

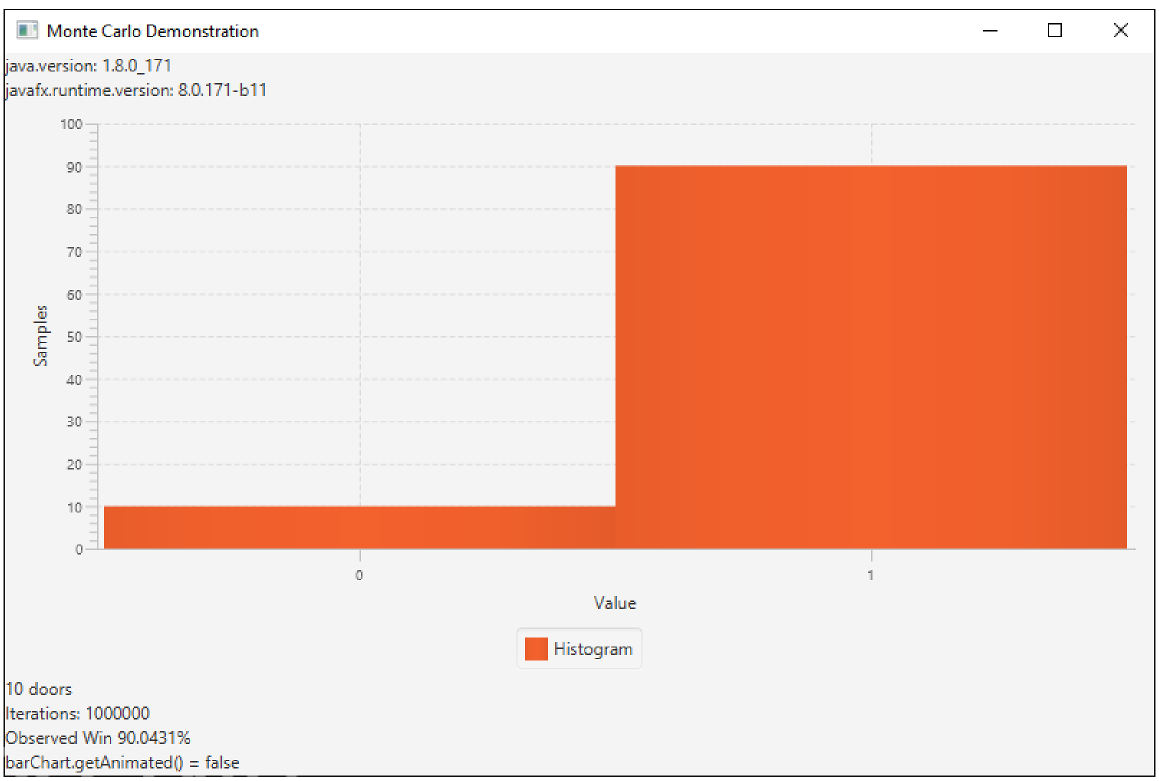

Figure 2: Monty Hall Monte Carlo for 10 doors

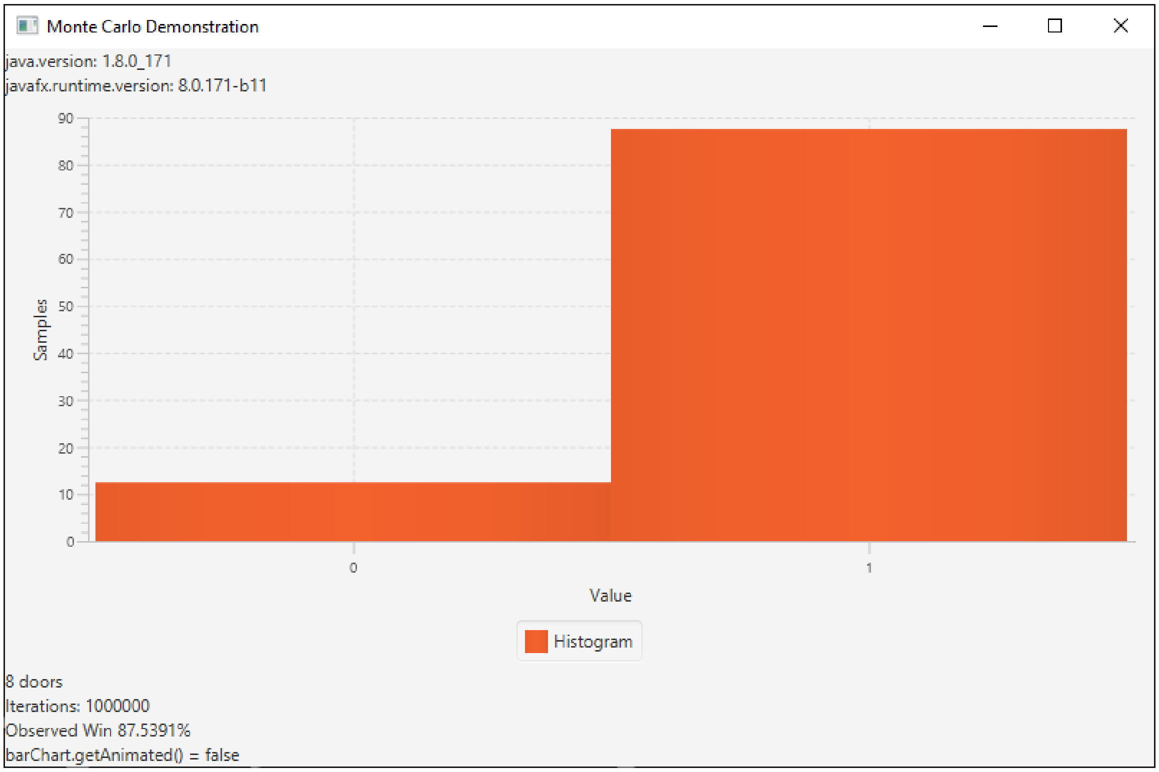

Figure 3: Monty Hall Monte Carlo for 8 doors

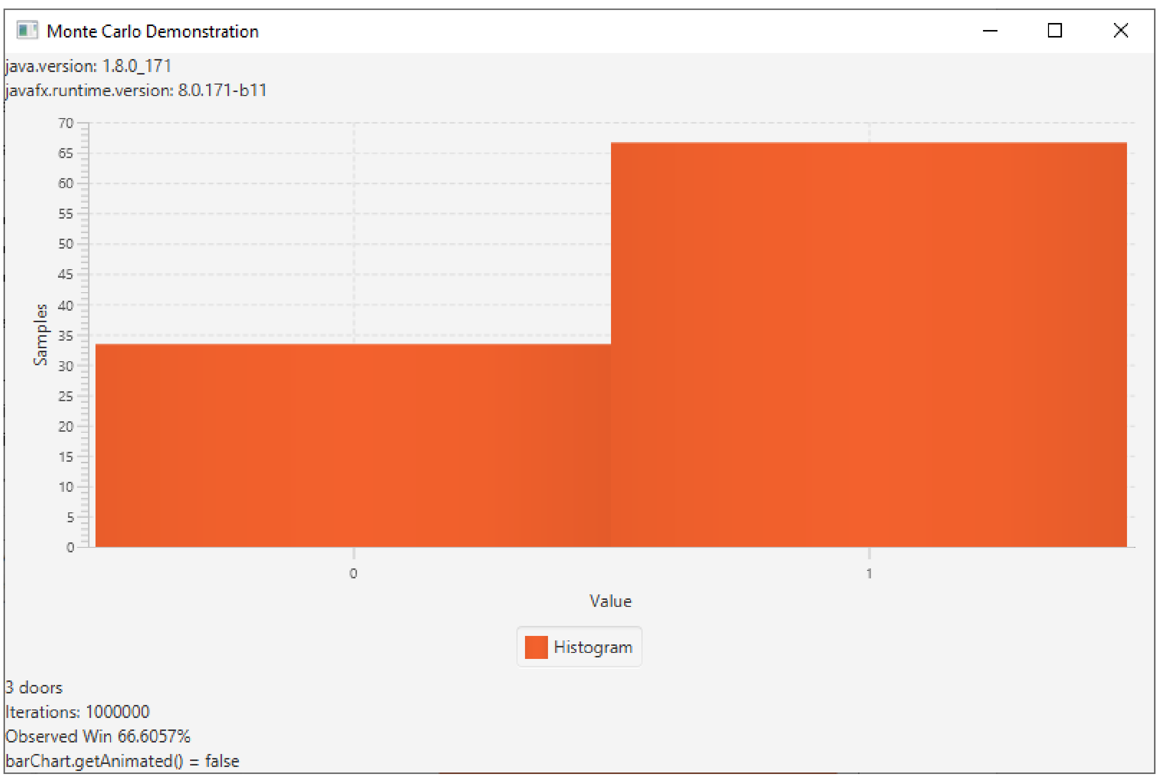

Figure 4: Monty Hall Monte Carlo for 3 doors

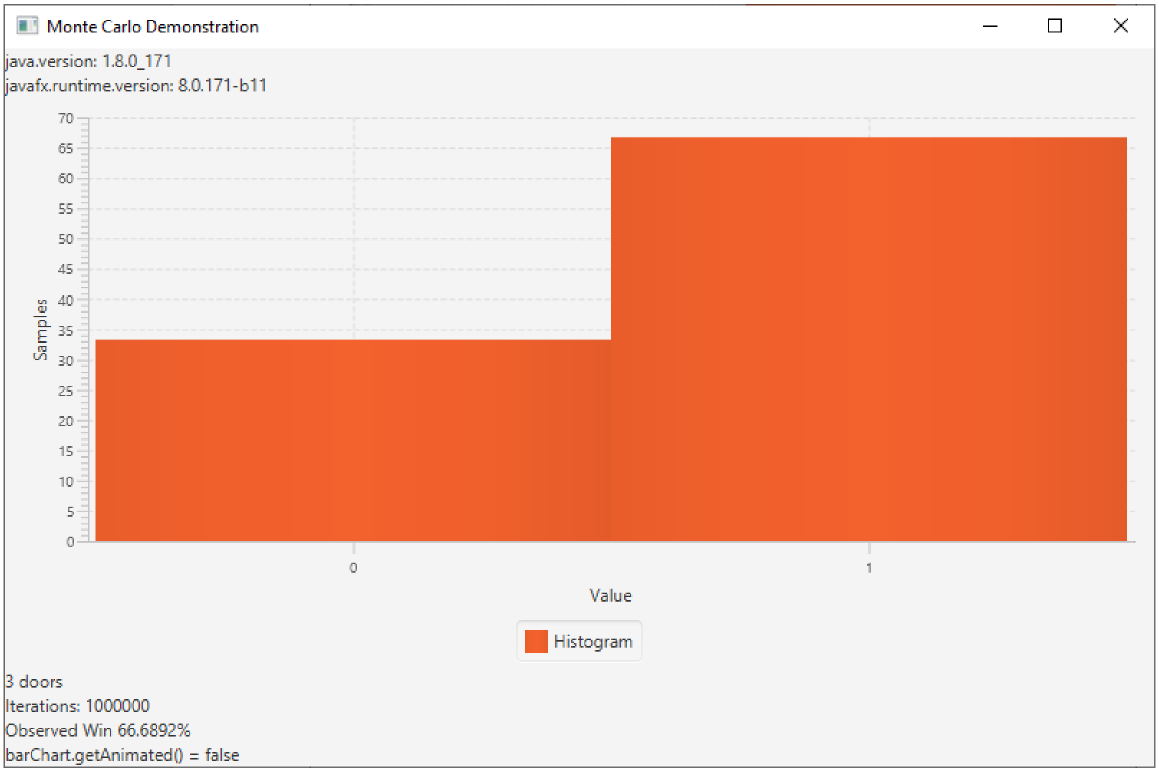

Figure 5: Monty Hall Monte Carlo for 3 doors simulation 2

Question 2: Yahtzee Strategy

Introduction to Strategy

I will explain the strategy that was implemented in the simulation code for Yahtzee using the presented course topics. To consistently earn a high score, it is necessary to understand the probability of outcomes for dice rolls and knowing which outcomes are the most beneficial depending on the current board state (the available boxes to score).

I decided that the starting point of my strategy is to prioritize the upper sections for the bonus. At this point I am not sure of the math except it sounds better: if I can consistently score 98 points in the upper section then only 52 points are required in the lower section, versus, if I can consistently score some known/unknown points in the lower section then some known/unknown points are required in the upper section.

In order to verify this idea, I first needed to estimate the unknowns by calculating the average expected scores for non-Yahtzee rolls that can be scored in either the upper section, without and with the bonus, and the 3K, 4K, and Chance boxes in the lower section. Then I filled in the average scores for the other boxes to give me a total expected score, if every box could be filled, to create this comparison table:

Box	Avg Score	Bonus Avg
Aces	3.0	8.8
Twos	6.0	11.8
Threes	9.0	14.8
Fours	12.0	17.8
Fives	15.0	20.8
Sixes	18.0	23.8
3K	17.4	17.4
4K	17.5	17.5
FH	25	25
SS	30	30
LS	40	40
Yahtzee	50	50
Chance	17.5	17.5
Total	260.4	295.4

Table 1: Potential Average Total Scores

The Bonus Avg is calculated by dividing up the 35-point bonus into each upper box (35/6 = 5.8). Supporting data for Table 1, average score calculations and individual roll scores, can be seen in Table A and Table B, respectively, in the Appendix.

Table 1 shows that with all boxes being filled with an average score, achieving the bonus results in a higher total score. That is obvious and this table alone is not valuable, but it can be expanded on. By using the first roll probabilities as a baseline and the bonus average score, a Weighted score can be calculated that allows me to sort by what box is likely to score the most points:

Box	Bonus Avg	First Roll	Prob.%	Weight
SS	30	1200/7776	15.4%	4.630
LS	40	240/7776	3.1%	1.235
FH	25	300/7776	3.9%	0.965
Sixes	23.8	200/7776	2.6%	0.613
Fives	20.8	200/7776	2.6%	0.536
Fours	17.8	200/7776	2.6%	0.459
Chance	17.5	200/7776	2.6%	0.450
3K	17.4	200/7776	2.6%	0.447
Threes	14.8	200/7776	2.6%	0.382
Twos	11.8	200/7776	2.6%	0.304
Aces	8.8	200/7776	2.6%	0.227
4K	17.5	25/7776	0.3%	0.056
Yahtzee	50.0	6/7776	0.1%	0.039
Total	295.4			

Table 2: Expected Scores and Probabilities by Weight With Bonus

First roll probabilities are from lecture slides (https://blog.plover.com/math/yahtzee.html).

Table 2 shows which boxes, ranked by Weight (Bonus Avg * Prob.%), are the most likely and worth scoring. This gives me an idea of how to prioritize the rolls. It shows that a small straight (SS) is highly weighted because it gives a relatively large amount of points for having a relatively high probability. For Yahtzee, even though the score is high, its chance of being hit is so small that the weight is overall low.

The top 6 ranks by weight are three boxes from the lower section (SS, LS, FH) and three boxes from the upper section (sixes, fives, fours). The more important point here is that the 3K, Chance, and the 4K box especially have a lower weight than the upper sections 5-6. In order to get the most out of the 3K, 4K, and Chance boxes, rolls containing 5-6s must be used, however, this table shows that these rolls are best used to score the upper sections. 4Ks are unlikely to hit and should be used for the upper sections to get the most use out of it.

The question is – is it better to prioritize the upper or lower section? My answer so far is – getting the upper section bonus is clearly better than not getting it, so it must be better to prioritize the upper section. Seems a bit circular so next question – is it possible to get the upper section bonus while prioritizing the lower section? Answer – Table 2 shows that I'm better off scoring high rolls in the upper section than lower section, and two of the lower sections are very unlikely. But again, this is based on including the bonus points, so looking at the weights without the bonus:

Box	Avg Score	First Roll	Prob.%	Weight
SS	30	1200/7776	15.4%	4.630
LS	40	240/7776	3.1%	1.235
FH	25	300/7776	3.9%	0.965
Sixes	18.0	200/7776	2.6%	0.463
Chance	17.5	200/7776	2.6%	0.450
3K	17.4	200/7776	2.6%	0.447
Fives	15.0	200/7776	2.6%	0.386
Fours	12.0	200/7776	2.6%	0.309
Threes	9.0	200/7776	2.6%	0.231
Twos	6.0	200/7776	2.6%	0.154
Aces	3.0	200/7776	2.6%	0.077
4K	17.5	25/7776	0.3%	0.056
Yahtzee	50.0	6/7776	0.1%	0.039
Total	260.4			

Table 3: Expected Scores and Probabilities by Weight Without Bonus

Table 3 shows that the priority has shifted, and the upper sections are worth less, obviously, since it doesn't have the bonus. One very important note here is that a total of the Avg Score for the first 6 boxes, SS-LS-FH-Sixes-Chance-3K, total 148 which is very close to the minimum goal of 150 – and it can be obtained with just 6 boxes out of 13 turns.

It seems better to go for the lower section to achieve the minimum score. However, the goal for us also includes getting scores over 200 as often as possible so pursuing the upper section bonus makes sense. Excluding the unlikely 4K and Yahtzee boxes, the avg score total of the 11 other boxes without the bonus is 192.9 versus 228 with the bonus.

In summary, in order to meet the requirements of the assignment, 150 points at least 90% of the time and 200 points as often as possible, I find it necessary to get the bonus from the upper section.

More Strategy, Markov Chain Diagrams

With all that said, the strategy is to prioritize filling in the upper sections for the 35-point bonus. On average, each upper section box needs a 3K, however, the focus will be to maximize rolls with 5s and 6s, filling them in the upper sections and lower sections by priority, and scratching the 1s and 2s box as required.

Early in the game and only after the first roll, rerolling a pair of 1s or 2s in favor of keeping the highest single 5 or 6, and rerolling the other four dice may be a good idea. Rerolling the low pairs is justified because there is a ~91% (7056/7776) chance of getting a pair or better on the first roll so giving up a pair in itself is not bad throughout the course of a game. The more important aspect to consider is that 1s and 2s are worth significantly less points than 5s and 6s in most of the boxes. Calculations in Table A show that on average, 3Ks of 1s and 2s are worth around half the points of 5s and 6s in all boxes except for FH. Furthermore, if the turn ends up with a worthless roll it can be scratched in the 1s or 2s and the points can be made up elsewhere.

As far as probability for getting the desired number, keeping a high single and rerolling the other four dice has a ~52% chance (671/1296) of resulting in at least one more of that high single. A third roll gives another chance of an ideal result – this is why this should only be done after first roll and not after second roll. 671/1296 is the complement of $(5/6)^4$, the chance of not rolling a single of a number, can also be derived by using the advanced formula, and was verified using the "Create the Universe" script. The hope is, as with every roll, that this turns into the most ideal roll but the more importantly, I'm taking the opportunity to go for a higher roll when the card is relatively blank and there are other boxes to score or scratch as required. By throwing out 1s and 2s early in favor of higher scoring dice, in the long run, I should score higher.

Having prioritized the upper section, I can move on to how to score the lower section. Yahtzee's will be taken as they come. The 3K, 4K, and Chance boxes will take high scoring rolls, based on the calculations table >=18), or be scratched as required. The LS will be immediately scored but a SS or FH will depend on the situation.

This is not very interesting or special but starting from pair there is a reasonable chance to go to a 3K or better (at least 35% per Figure 6 - A). With the high probability of rolling a pair, here's a reasonable chance that within a game, a good number of 3K or better can be earned. However, not all 3K are equal since 1s and 2s don't score well on their own so the strategy will consider when to use 3K to score in the upper section, when to pursue a 4K or Yahtzee, and when to score a FH vs a 3K.

Consider that a FH of 1s or 2s is worth considerably less in the upper, 3K, or Chance boxes than the FH whereas a FH of 6s or 5s is worth more in those same boxes. The important concept here is the opportunity cost of scoring a FH using 6s and 5s is high; it's better to score it elsewhere. Looking at the Markov Chain "A" in Figure 6 I can calculate, starting from a pair, the odds of rolling a FH: Pair to FH, Pair to Pair to FH, Pair to 3K to FH => (0.092 + 0.579*0.092 + 0.347*0.069) = 16.9%. By that estimation, starting from a pair I can expect to roll into a full house every six times. 18 is the magic number

A pair can be part of a SS or turn into a SS/LS, 5s and 6s will be prioritized to fill in their upper sections, instead of scoring the SS or going for a LS. As with a FH, it is not wasteful reroll a SS for other opportunities. Looking at the Markov Chain "B", for a full turn the probability of rolling a SS from a pair of 2-5: Pair to SS, Pair to Pair to SS => (0.148 + 0.579*0.148) = 23.4%, which means it can happen every 4-5 turns.

In addition to rolling from a pair, a SS or better is possible 1200/7776 or 15.4% of the time on a first roll. Again, it seems safe to reroll SS in favor of a pair of 5s or 6s since throughout a game an SS or better will turn up frequently enough. The exception to this rule is for a SS of "2345" since this open-ended SS can turn into a LS 33% of the time and is worth double the score of any potential 3K scores from it with similar odds (35% chance to roll a 3K by keeping the pair). This will be taken for the easy points.

In summary, my strategy involves prioritizing the upper sections for the bonus which requires rolling for 3Ks and keeping 5s and 6s whenever possible in favor of lower pairs, SS, and FH based on probabilities and the board state. Using this strategy, this was my best run:

```
run:
Iterations: 1000000                    Min Score: 52            Max Score: 832           Average Score: 229.76
Games>150: 92.10%        Games>200: 61.69%

BUILD SUCCESSFUL (total time: 1 minute 4 seconds)
```

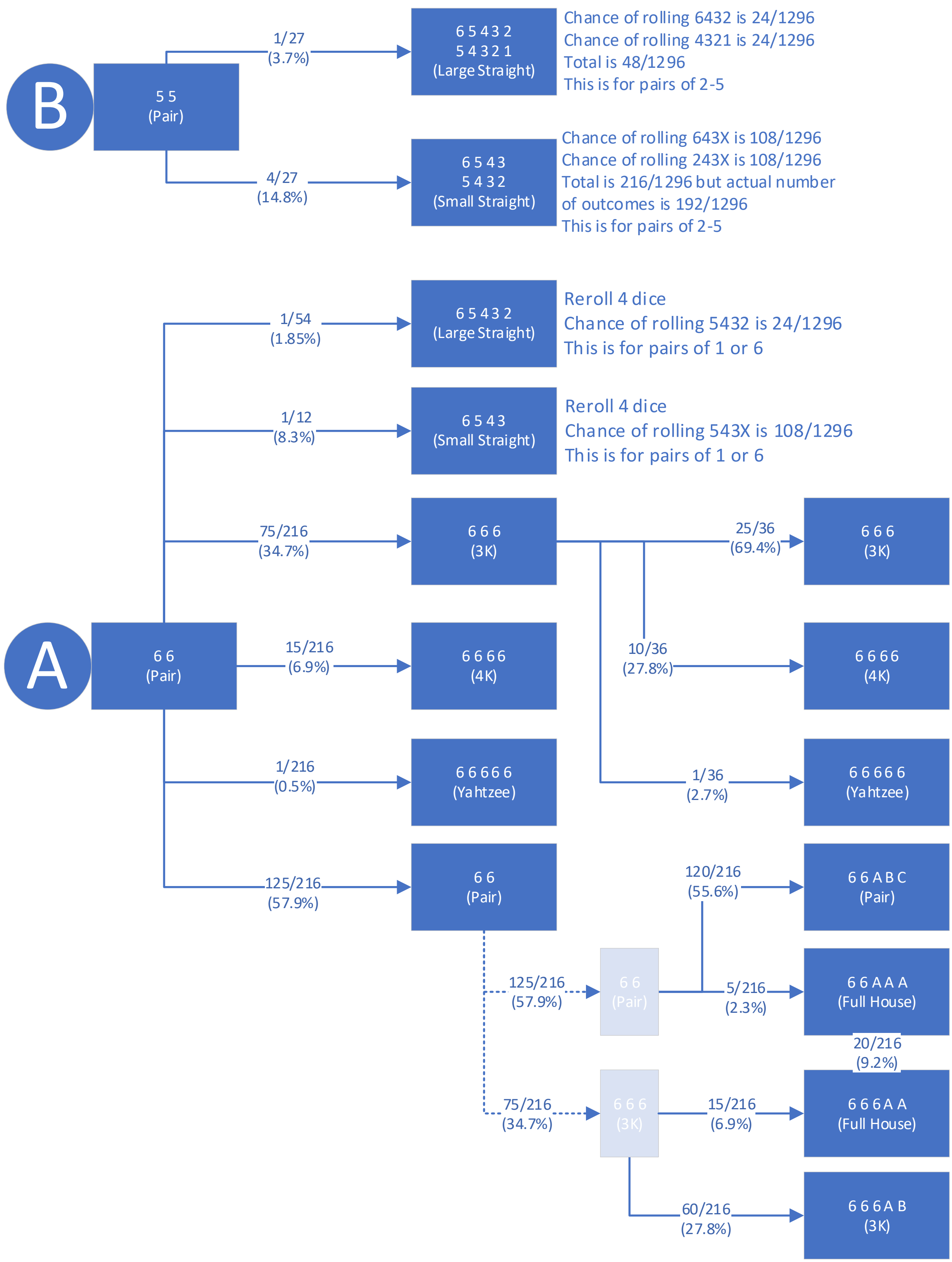

Figure 6: Markov Chain Diagrams – this is not entirely complete but there is enough to extrapolate and calculate some outcomes for all three rolls.

Appendix

Yahtzee Strategy Result Screenshots

```
run:
Iterations: 1000000                    Min Score: 52          Max Score: 832          Average Score: 229.76
Games>150: 92.10%       Games>200: 61.69%

BUILD SUCCESSFUL (total time: 1 minute 4 seconds)
```

Figure 7: Final Result

Tables

Roll	Upper	Upper (+35/6)	3K Average	4K Average	FH	Chance Average	Points Range
3K - 1	3	8.8	10.9	-	25.0	10.9	[7-15]
3K - 2	6	11.8	13.5	-	25.0	13.5	[8-18]
3K - 3	9	14.8	16.0	-	25.0	16.0	[11-21]
3K - 4	12	17.8	18.6	-	25.0	18.6	[14-24]
3K - 5	15	20.8	21.4	-	25.0	21.4	[17-27]
3K - 6	18	23.8	24.0	-	25.0	24.0	[20-28]
Avg	10.5	16.3	17.4		25.0	17.4	
4K - 1	4	9.8	8.0	8.0	-	8.0	[6-10]
4K - 2	8	13.8	11.8	11.8	-	11.8	[9-14]
4K - 3	12	17.8	15.6	15.6	-	15.6	[13-18]
4K - 4	16	21.8	19.4	19.4	-	19.4	[17-22]
4K - 5	20	25.8	23.2	23.2	-	23.2	[21-26]
4K - 6	24	29.8	27.0	27.0	-	27.0	[25-29]
Avg	14.0	19.8	17.5	17.5		17.5	

Table A: Average Scores for 3K and 4K Rolls

D1	D2	D3	D4	D5	U	3K	4K	FH	C
1	1	1	2	2	3	7		25	7
1	1	1	2	3	3	8			8
1	1	1	2	4	3	9			9
1	1	1	2	5	3	10			10
1	1	1	2	6	3	11			11
1	1	1	3	2	3	8			8
1	1	1	3	3	3	9		25	9
1	1	1	3	4	3	10			10
1	1	1	3	5	3	11			11
1	1	1	3	6	3	12			12
1	1	1	4	2	3	9			9
1	1	1	4	3	3	10			10
1	1	1	4	4	3	11		25	11
1	1	1	4	5	3	12			12
1	1	1	4	6	3	13			13
1	1	1	5	2	3	10			10
1	1	1	5	3	3	11			11
1	1	1	5	4	3	12			12
1	1	1	5	5	3	13		25	13
1	1	1	6	2	3	11			11
1	1	1	6	3	3	12			12
1	1	1	6	4	3	13			13
1	1	1	6	5	3	14			14
1	1	1	6	6	3	15		25	15
					4.7	10.9		25.0	10.9

D1	D2	D3	D4	D5	U	3K	4K	FH	C
1	1	1	1	2	4	6	6		6
1	1	1	1	3	4	7	7		7
1	1	1	1	4	4	8	8		8
1	1	1	1	5	4	9	9		9
1	1	1	1	6	4	10	10		10
					6.2	8.0	8.0		8.0

D1	D2	D3	D4	D5	U	3K	4K	FH	C
2	2	2	1	1	6	8		25	8
2	2	2	1	3	6	10			10
2	2	2	1	4	6	11			11
2	2	2	1	5	6	12			12
2	2	2	1	6	6	13			13
2	2	2	3	1	6	10			10
2	2	2	3	3	6	12		25	12
2	2	2	3	4	6	13			13
2	2	2	3	5	6	14			14
2	2	2	3	6	6	15			15
2	2	2	4	1	6	11			11
2	2	2	4	3	6	13			13
2	2	2	4	4	6	14		25	14
2	2	2	4	5	6	15			15
2	2	2	4	6	6	16			16
2	2	2	5	1	6	12			12
2	2	2	5	3	6	14			14
2	2	2	5	4	6	15			15
2	2	2	5	5	6	16		25	16
2	2	2	6	1	6	13			13
2	2	2	6	3	6	15			15
2	2	2	6	4	6	16			16
2	2	2	6	5	6	17			17
2	2	2	6	6	6	18		25	18
					9.3	13.5		25	13.5

D1	D2	D3	D4	D5	U	3K	4K	FH	C
2	2	2	2	1	8	9	9		9
2	2	2	2	3	8	11	11		11
2	2	2	2	4	8	12	12		12
2	2	2	2	5	8	13	13		13
2	2	2	2	6	8	14	14		14
					12.4	11.8	11.8		11.8

D1	D2	D3	D4	D5	U	3K	4K	FH	C
3	3	3	2	2	9	13		25	13
3	3	3	2	1	9	12			12
3	3	3	2	4	9	15			15
3	3	3	2	5	9	16			16
3	3	3	2	6	9	17			17
3	3	3	1	2	9	12			12
3	3	3	1	1	9	11		25	11
3	3	3	1	4	9	14			14
3	3	3	1	5	9	15			15
3	3	3	1	6	9	16			16
3	3	3	4	2	9	15			15
3	3	3	4	1	9	14			14
3	3	3	4	4	9	17		25	17
3	3	3	4	5	9	18			18
3	3	3	4	6	9	19			19
3	3	3	5	2	9	16			16
3	3	3	5	1	9	15			15
3	3	3	5	4	9	18			18
3	3	3	5	5	9	19		25	19
3	3	3	6	2	9	17			17
3	3	3	6	1	9	16			16
3	3	3	6	4	9	19			19
3	3	3	6	5	9	20			20
3	3	3	6	6	9	21		25	21
					14.0	16.0		25	16.0

D1	D2	D3	D4	D5	U	3K	4K	FH	C
3	3	3	3	2	12	14	14		14
3	3	3	3	1	12	13	13		13
3	3	3	3	4	12	16	16		16
3	3	3	3	5	12	17	17		17
3	3	3	3	6	12	18	18		18
					18.7	15.6	15.6		15.6

D1	D2	D3	D4	D5	U	3K	4K	FH	C
4	4	4	2	2	12	16		25	16
4	4	4	2	3	12	17			17
4	4	4	2	1	12	15			15
4	4	4	2	5	12	19			19
4	4	4	2	6	12	20			20
4	4	4	3	2	12	17			17
4	4	4	3	3	12	18		25	18
4	4	4	3	1	12	16			16
4	4	4	3	5	12	20			20
4	4	4	3	6	12	21			21
4	4	4	1	2	12	15			15
4	4	4	1	3	12	16			16
4	4	4	1	1	12	14		25	14
4	4	4	1	5	12	18			18
4	4	4	1	6	12	19			19
4	4	4	5	2	12	19			19
4	4	4	5	3	12	20			20
4	4	4	5	1	12	18			18
4	4	4	5	5	12	22		25	22
4	4	4	6	2	12	20			20
4	4	4	6	3	12	21			21
4	4	4	6	1	12	19			19
4	4	4	6	5	12	23			23
4	4	4	6	6	12	24		25	24
					18.7	18.6		25	18.6

D1	D2	D3	D4	D5	U	3K	4K	FH	C
4	4	4	4	2	16	18	18		18
4	4	4	4	3	16	19	19		19
4	4	4	4	1	16	17	17		17
4	4	4	4	5	16	21	21		21
4	4	4	4	6	16	22	22		22
					24.9	19.4	19.4		19.4

D1	D2	D3	D4	D5	U	3K	4K	FH	C
5	5	5	2	2	15	19		25	19
5	5	5	2	3	15	20			20
5	5	5	2	4	15	21			21
5	5	5	2	1	15	18			18
5	5	5	2	6	15	23			23
5	5	5	3	2	15	20			20
5	5	5	3	3	15	21		25	21
5	5	5	3	4	15	22			22
5	5	5	3	1	15	19			19
5	5	5	3	6	15	24			24
5	5	5	4	2	15	21			21
5	5	5	4	3	15	22			22
5	5	5	4	4	15	23		25	23
5	5	5	4	1	15	20			20
5	5	5	4	6	15	25			25
5	5	5	1	2	15	18			18
5	5	5	1	3	15	19			19
5	5	5	1	4	15	20			20
5	5	5	1	1	15	17		25	17
5	5	5	6	2	15	23			23
5	5	5	6	3	15	24			24
5	5	5	6	4	15	25			25
5	5	5	6	1	15	22			22
5	5	5	6	6	15	27		25	27
					23.3	21.4		25	21.4

D1	D2	D3	D4	D5	U	3K	4K	FH	C
6	6	6	2	2	18	22		25	22
6	6	6	2	3	18	23			23
6	6	6	2	4	18	24			24
6	6	6	2	5	18	25			25
6	6	6	2	1	18	21			21
6	6	6	3	2	18	23			23
6	6	6	3	3	18	24		25	24
6	6	6	3	4	18	25			25
6	6	6	3	5	18	26			26
6	6	6	3	1	18	22			22
6	6	6	4	2	18	24			24
6	6	6	4	3	18	25			25
6	6	6	4	4	18	26		25	26
6	6	6	4	5	18	27			27
6	6	6	4	1	18	23			23
6	6	6	5	2	18	25			25
6	6	6	5	3	18	26			26
6	6	6	5	4	18	27			27
6	6	6	5	5	18	28		25	28
6	6	6	1	2	18	21			21
6	6	6	1	3	18	22			22
6	6	6	1	4	18	23			23
6	6	6	1	5	18	24			24
6	6	6	1	1	18	20		25	20
					28.0	24		25	24

D1	D2	D3	D4	D5	U	3K	4K	FH	C
5	5	5	5	2	20	22	22		22
5	5	5	5	3	20	23	23		23
5	5	5	5	4	20	24	24		24
5	5	5	5	1	20	21	21		21
5	5	5	5	6	20	26	26		26
					31.1	23.2	23.2		23.2

D1	D2	D3	D4	D5	U	3K	4K	FH	C
6	6	6	6	2	24	26	26		26
6	6	6	6	3	24	27	27		27
6	6	6	6	4	24	28	28		28
6	6	6	6	5	24	29	29		29
6	6	6	6	1	24	25	25		25
					37.3	27	27		27

<u>Table B</u>: All Scores for 3K and 4K Rolls