A geometric computation of cohomotopy groups in co-degree one

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joint work with Thomas Rot

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1 Baby example

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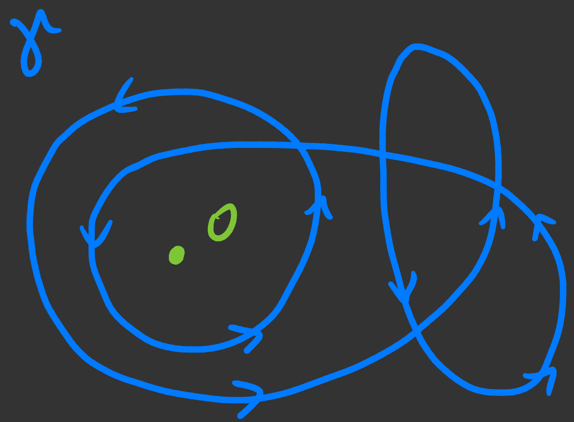

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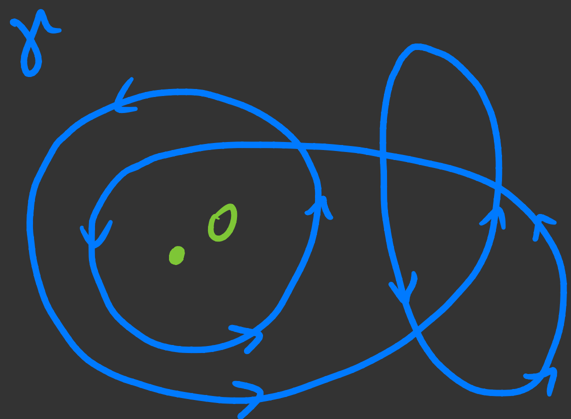

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$w(\gamma) = 2$.

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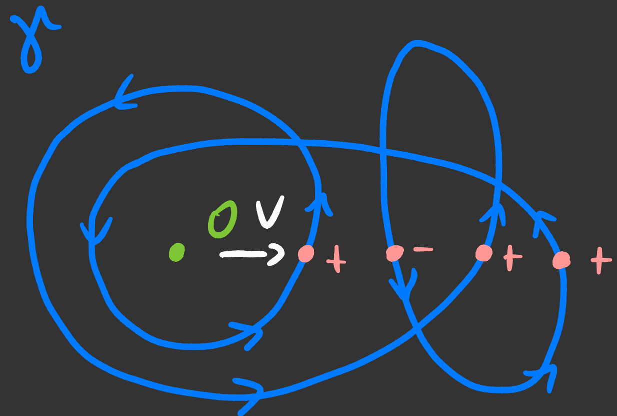

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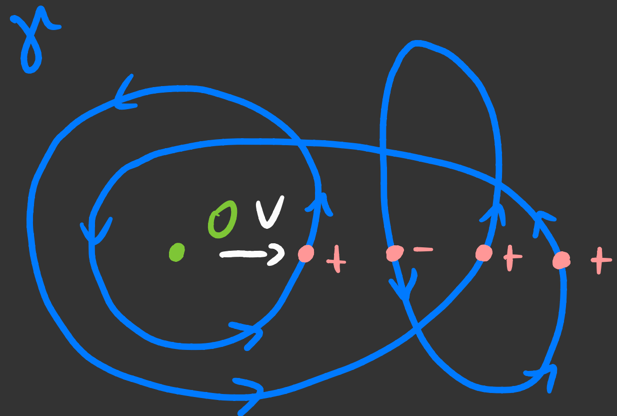

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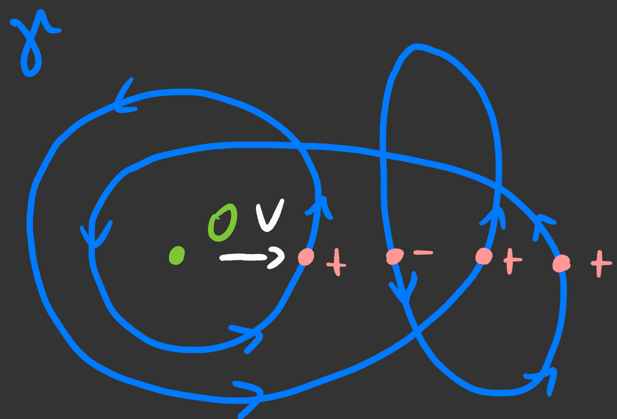

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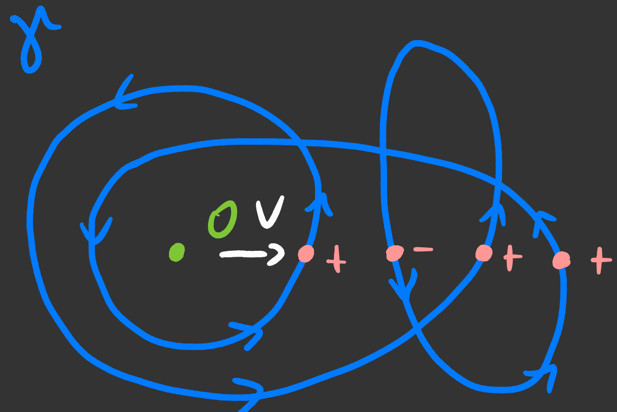

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$=: \deg(f; v)$

mapping degree

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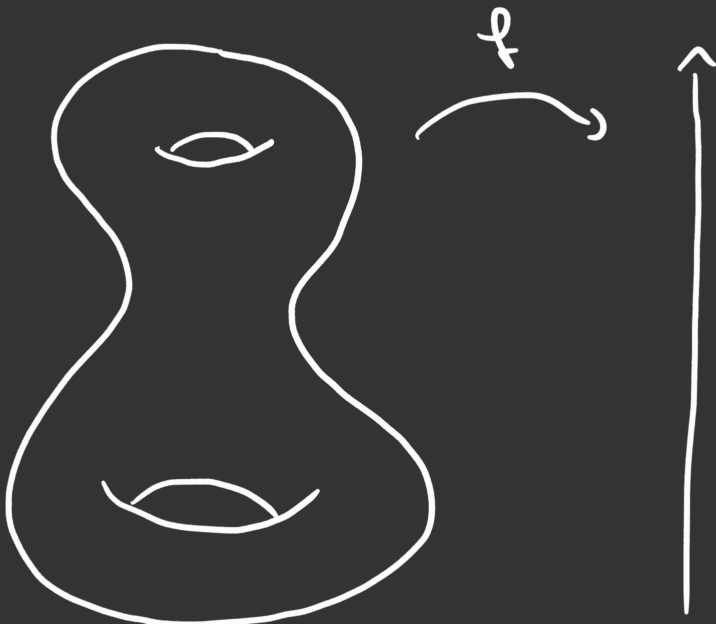

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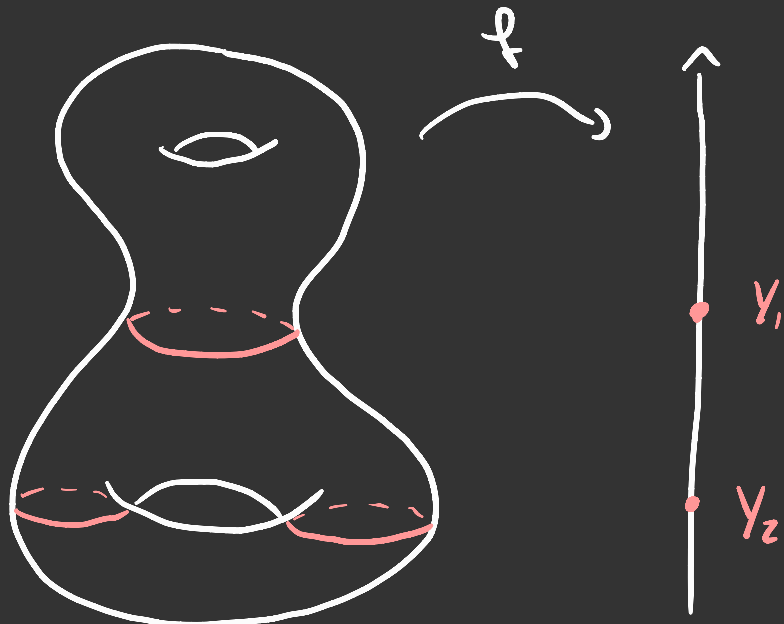

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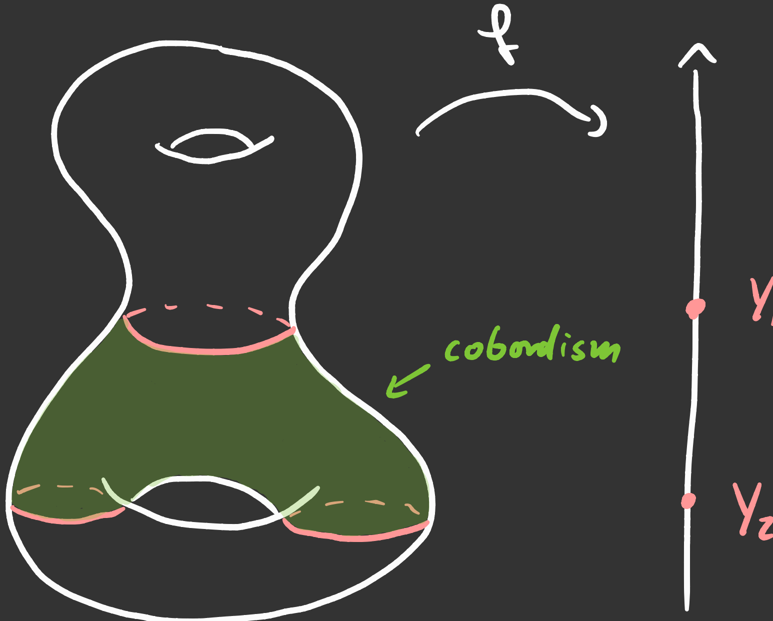

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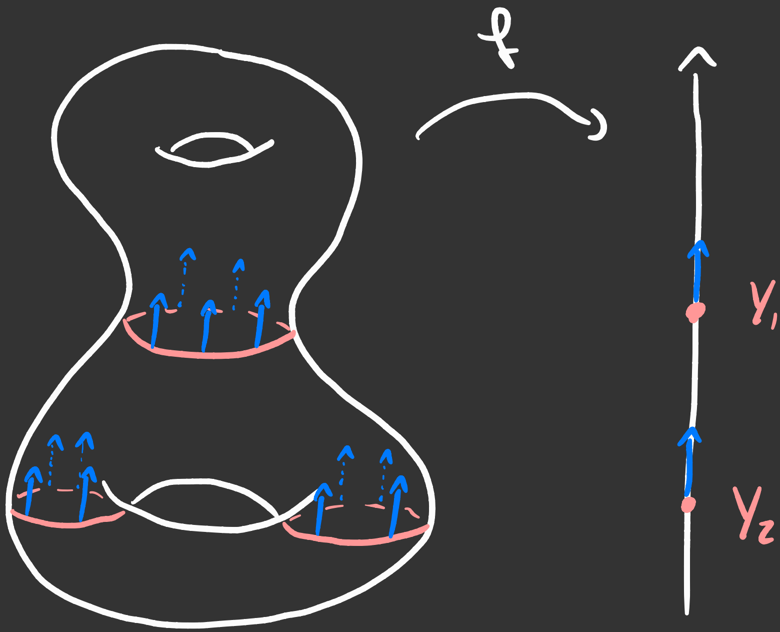

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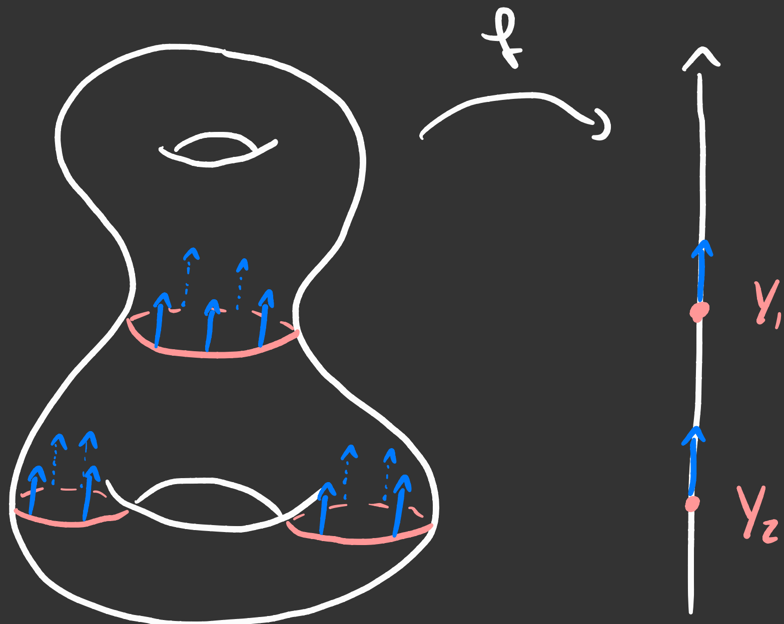

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Fact: Reg. values are "generic".

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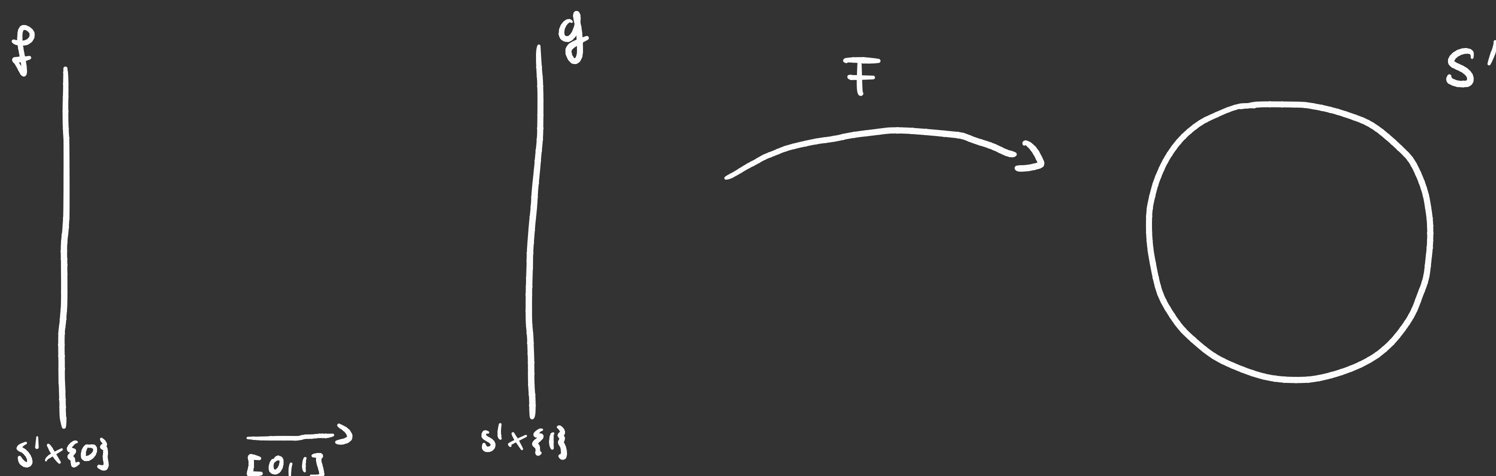

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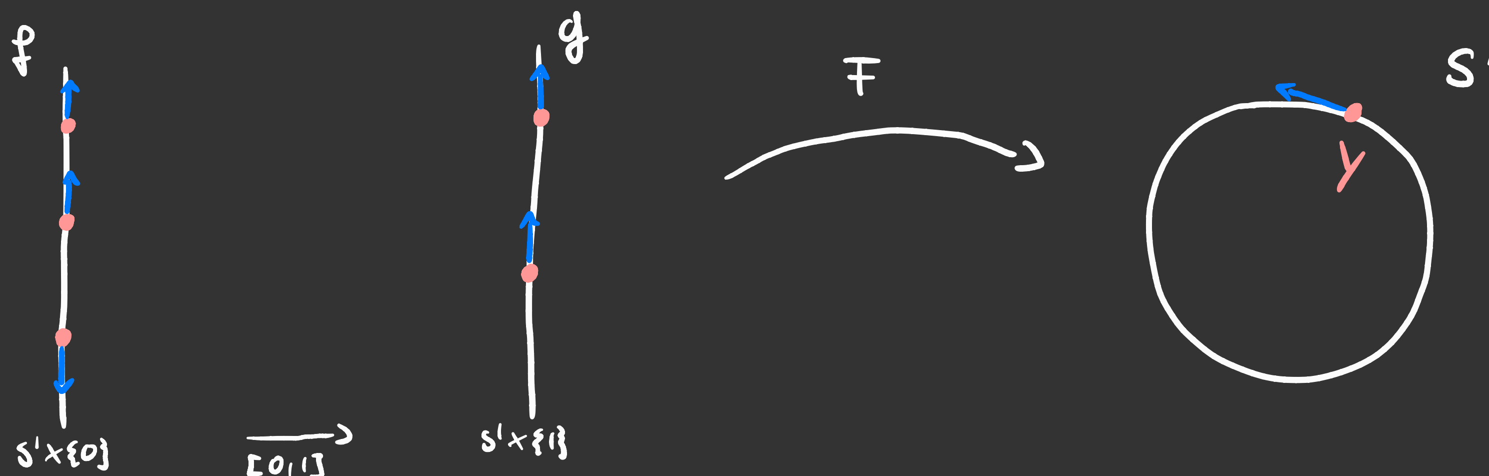

Assume $\deg(f;y)\neq\deg(g;y)$.

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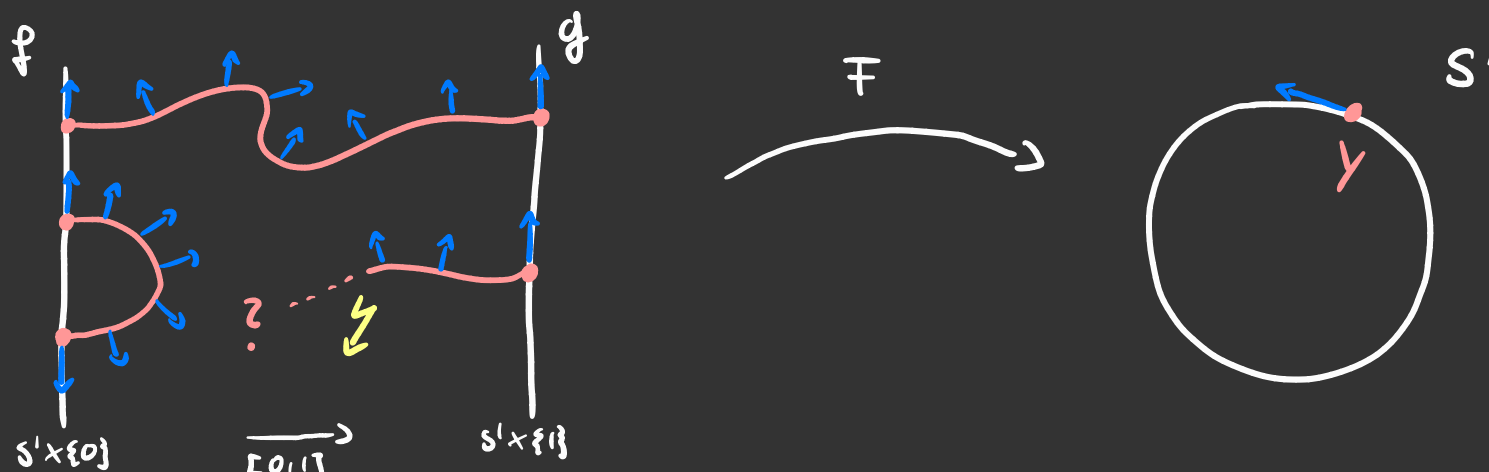

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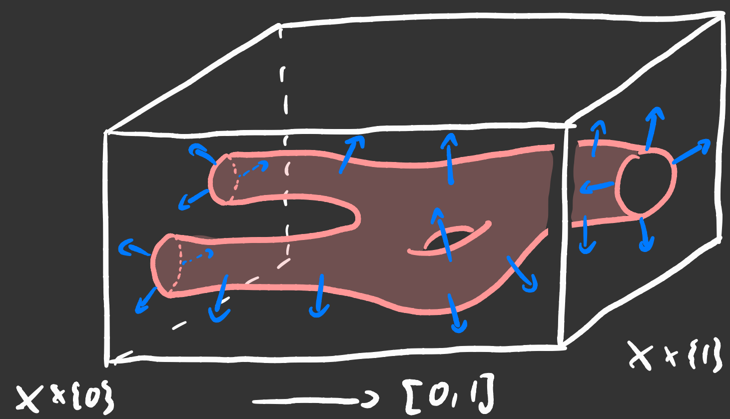

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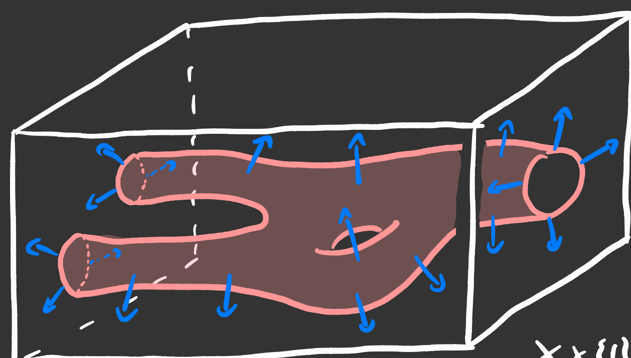

$X \times \{0\}$ $\longrightarrow [0,1]$ $X \times \{1\}$

Define:

$$\mathbb{F}_k(X) := \frac{\left\{ \begin{array}{l} k\text{-dim closed } M \subset X \text{ w/} \\ \text{framing of } \nu_M \end{array} \right\}}{\text{normally framed cobord.}}$$

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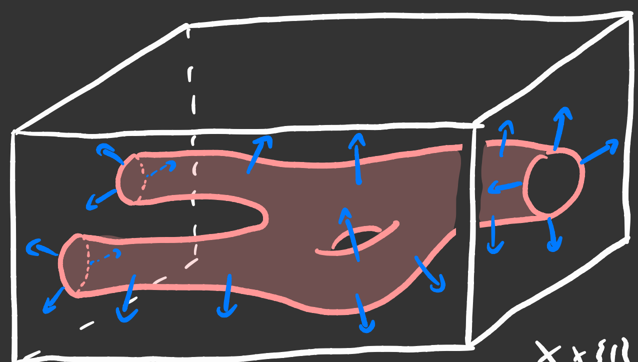

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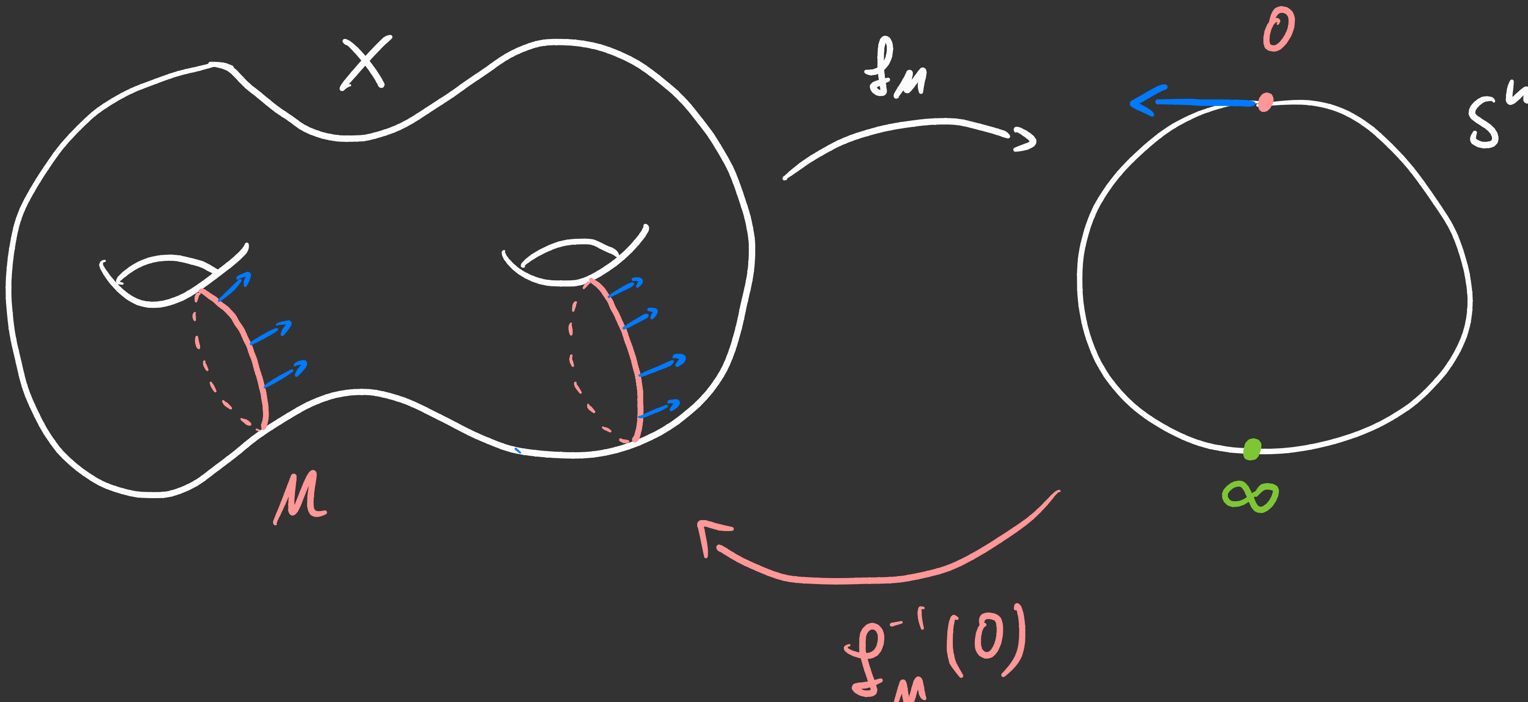
Note: $\mathbb{F}_k(X)$ is group if $\dim X \gg k$:

i) $[M_1, \varphi_1] + [M_2, \varphi_2] = [M_1 \sqcup M_2, \varphi_1 \sqcup \varphi_2]$

ii) $-[M, \varphi] = [M, -\varphi]$ (reverse orientation)

1 Pontryagin-Thom construction

A: $[X^{n+k}, S^n] \cong \mathbb{F}_k(X)$



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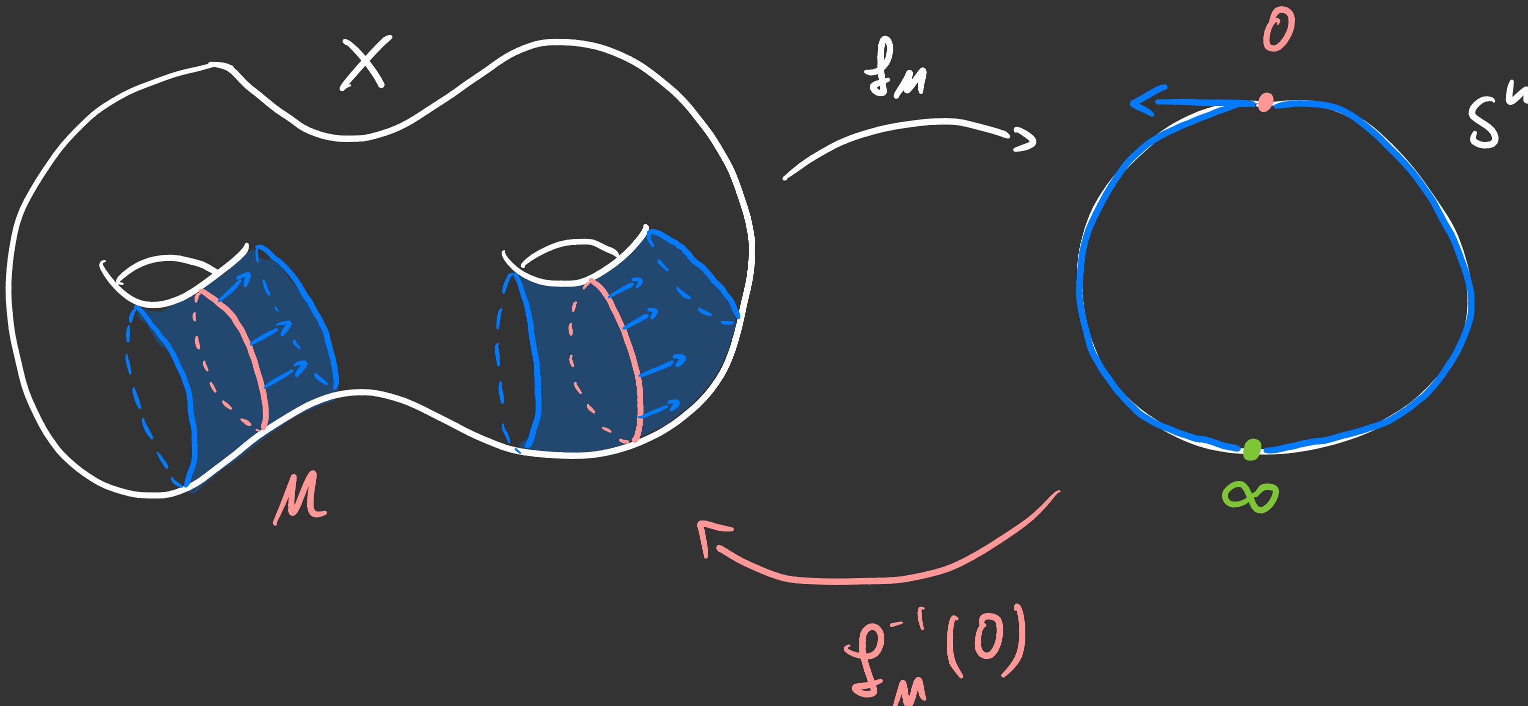

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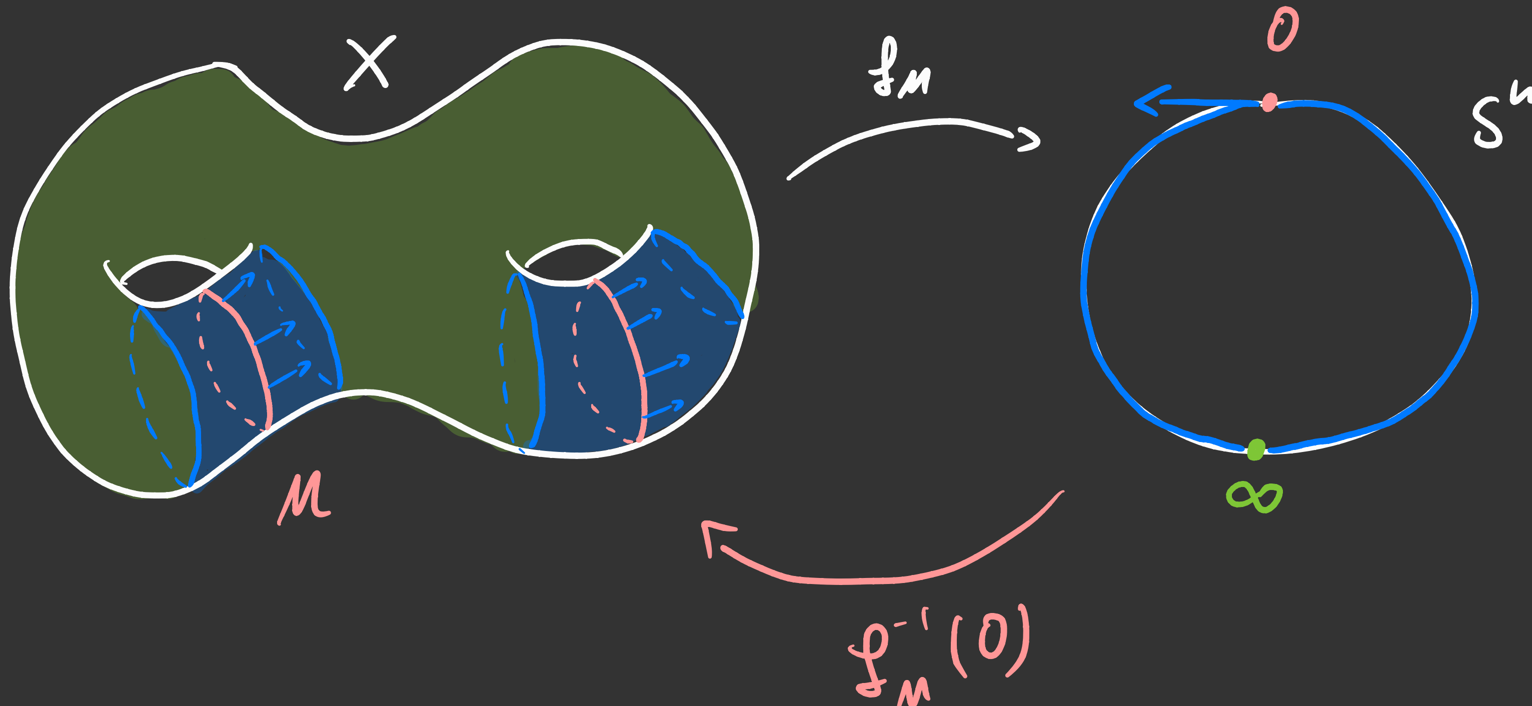

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Q: Geometric proof for $n \geq 3$ and X^{n+1} (non-orientable) manifold?

1 Twisted homology

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$\mathbb{L}_1(X;\mathbb{Z}w) \cong H_1(X;\mathbb{Z}w)$ = first homology w/ "twisted" coefficients

(Thom '54, Atiyah '60)

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↳ obvious map $h : \mathbb{F}_1(X) \longrightarrow H_1(X;\mathbb{Z}w)$

forgets framing, remembers orientation

1 The short exact sequence

$$\mathbb{F}_1(X) \xrightarrow{h} H_1(X; \mathbb{Z}_w)$$

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$$\mathbb{F}_1(X) \xrightarrow{h} H_1(X; \mathbb{Z}_w) \longrightarrow 0$$

v.b. over 1-dim CW-complex
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$$0 \longrightarrow \overset{?}{\ker h} \longrightarrow \mathbb{F}_1(X) \xrightarrow{h} H_1(X; \mathbb{Z}_w) \longrightarrow 0$$

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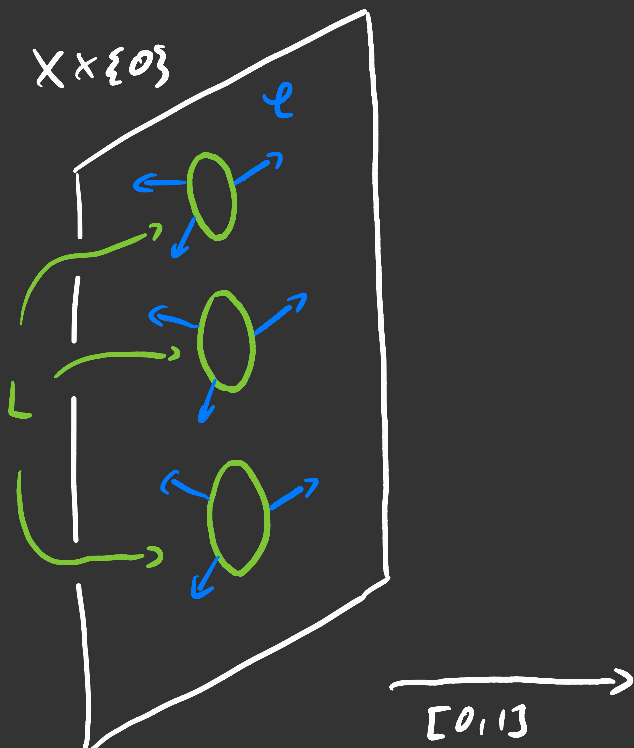
Also: what is the extension?

1 Determine Ker h

Suppose $[L, e] \in \operatorname{Ker} h$.



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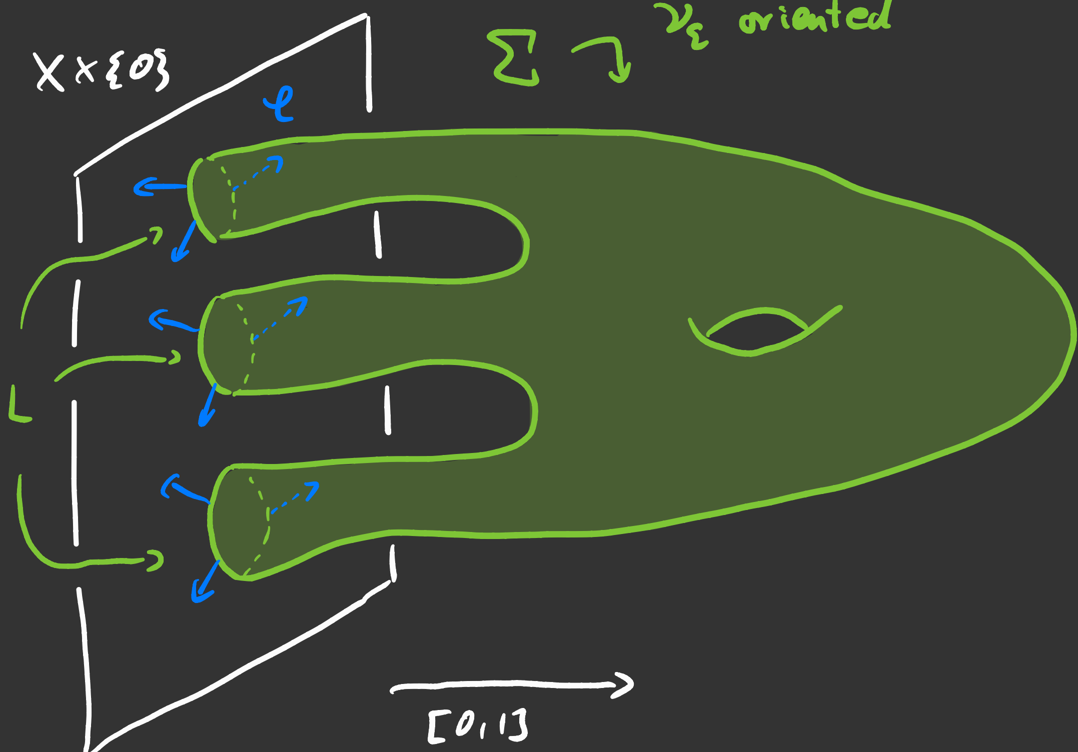

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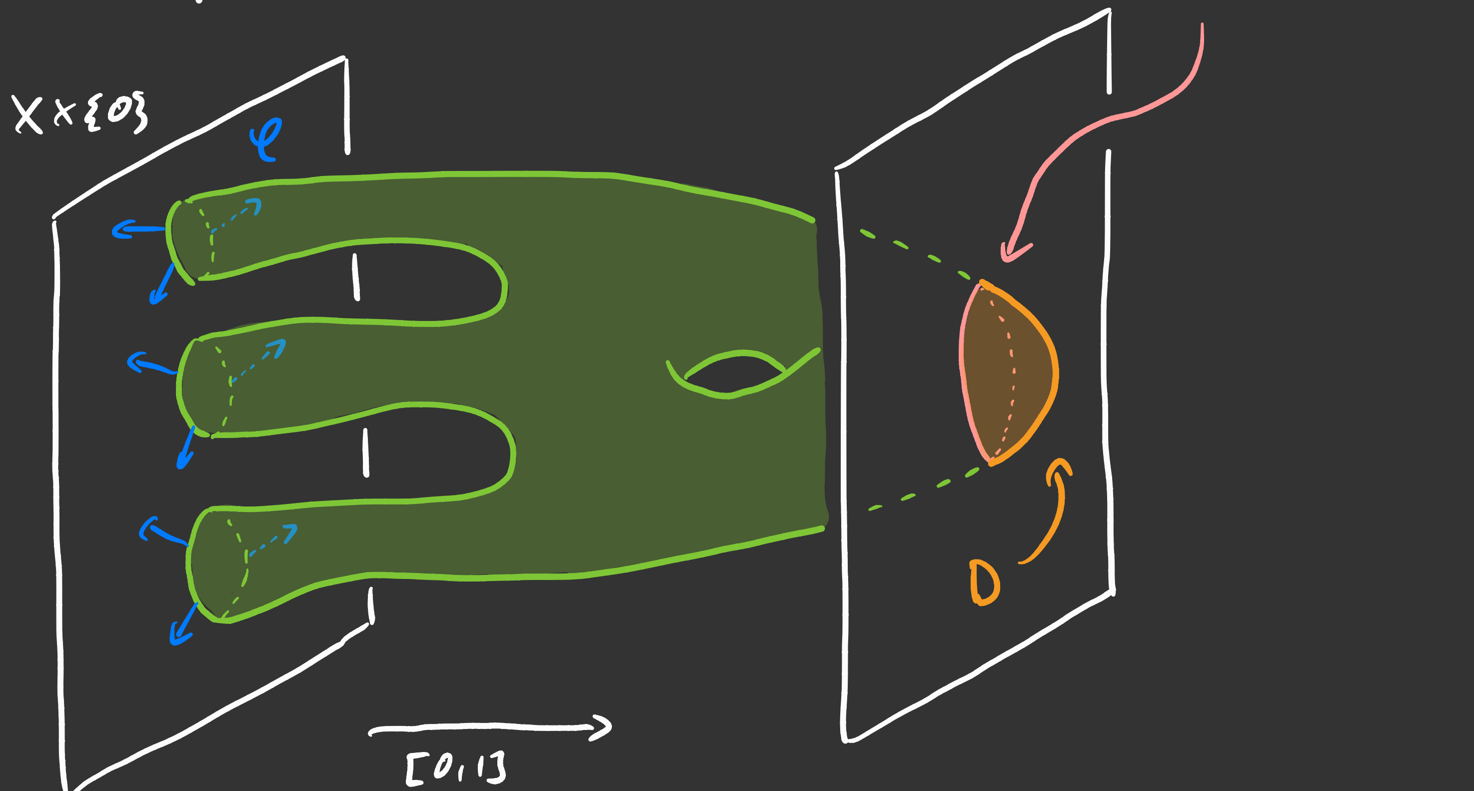

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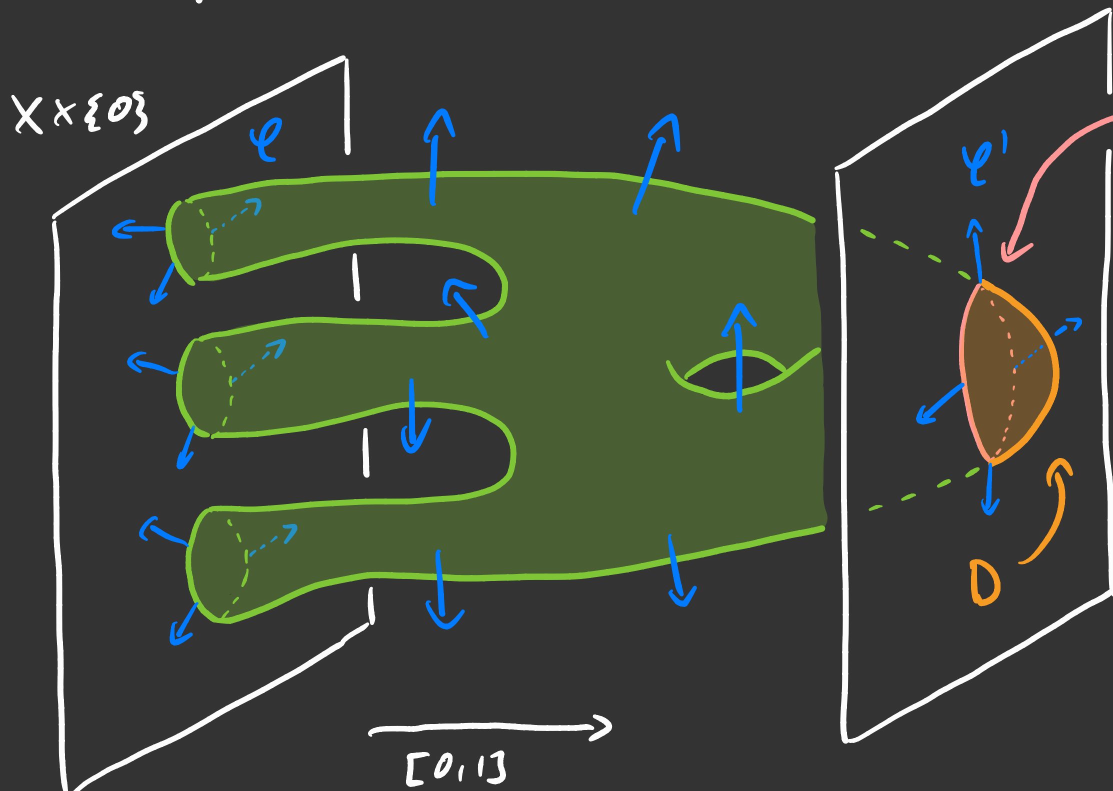

$\rightsquigarrow$ induces (unique) framing φ' over $\mathcal{U}$ s.t. $[L, \varphi] = [\mathcal{U}, \varphi']$.

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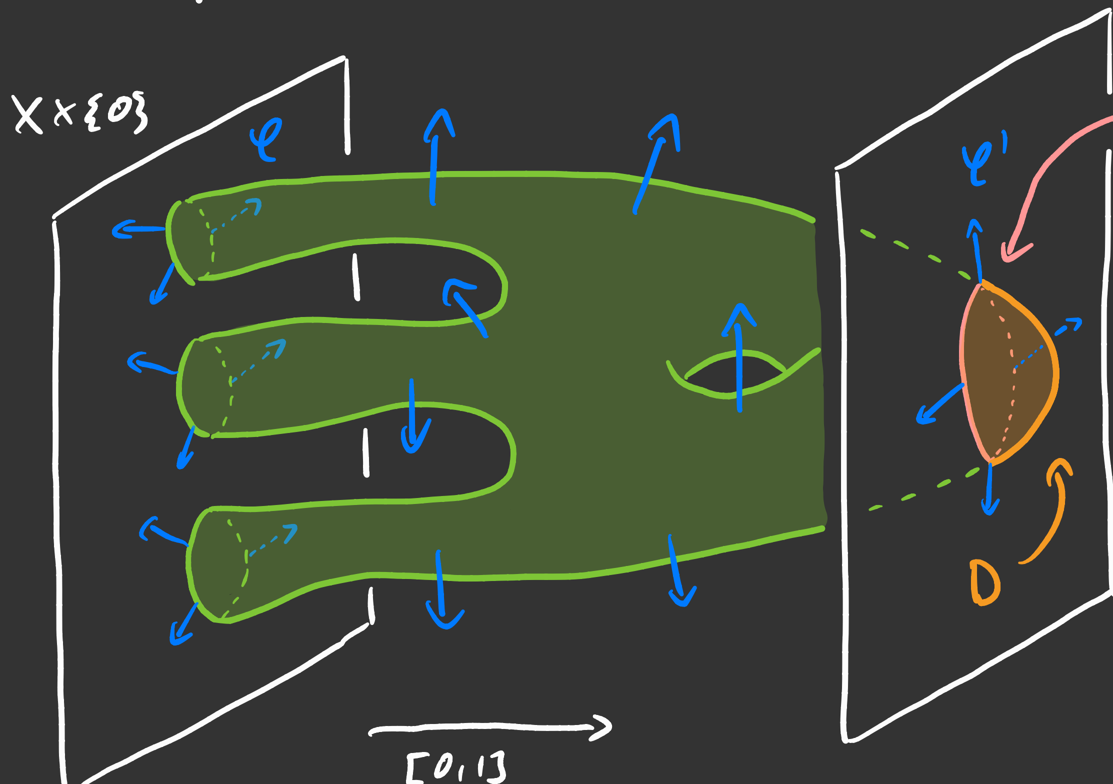

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<u>Now:</u>

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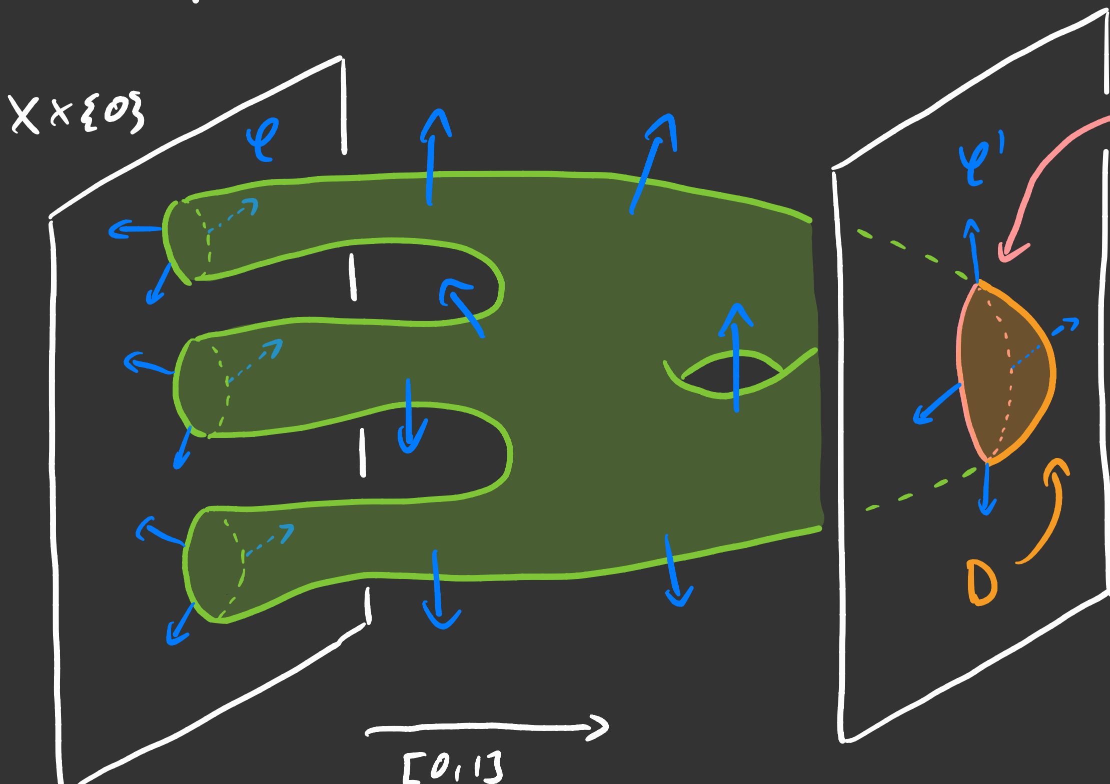

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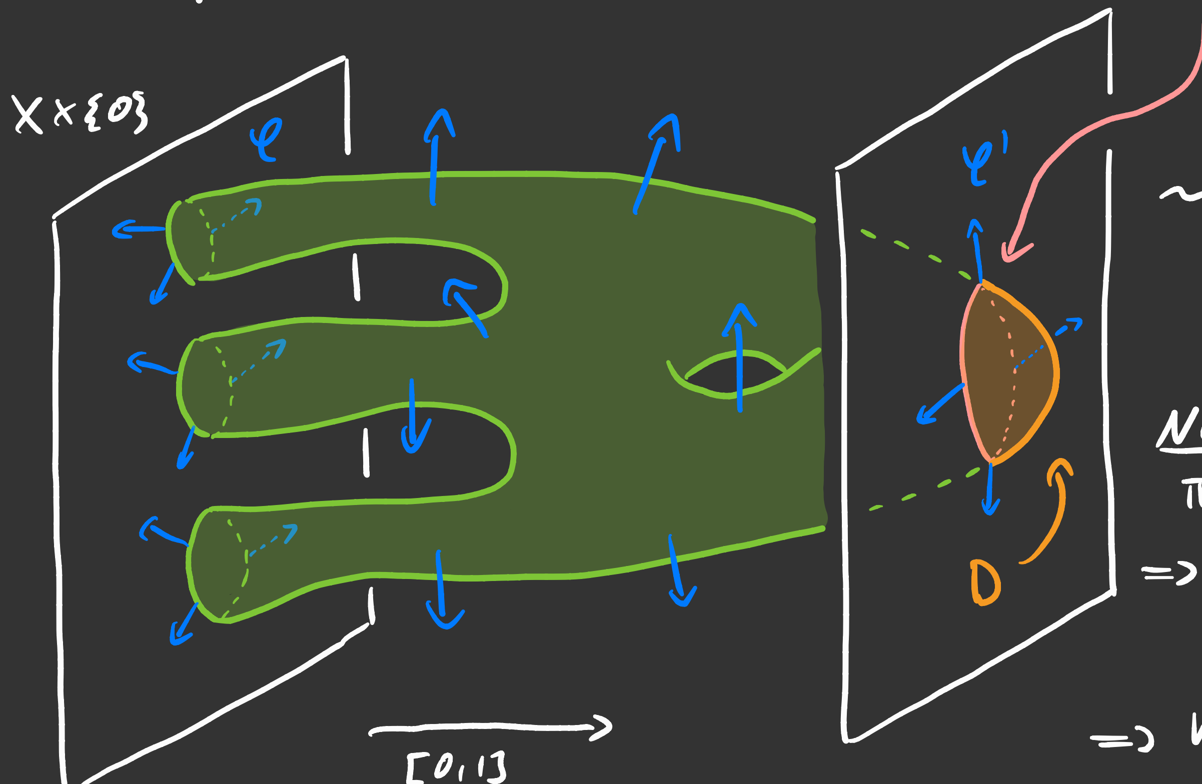

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$\Rightarrow$ $\operatorname{Ker} h$ is <u>at most</u> $\mathbb{Z}_2$!

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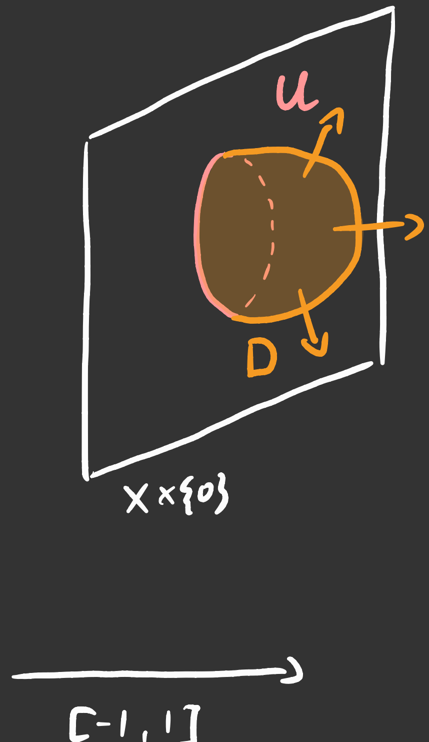
Assume $\ker h = 0$.



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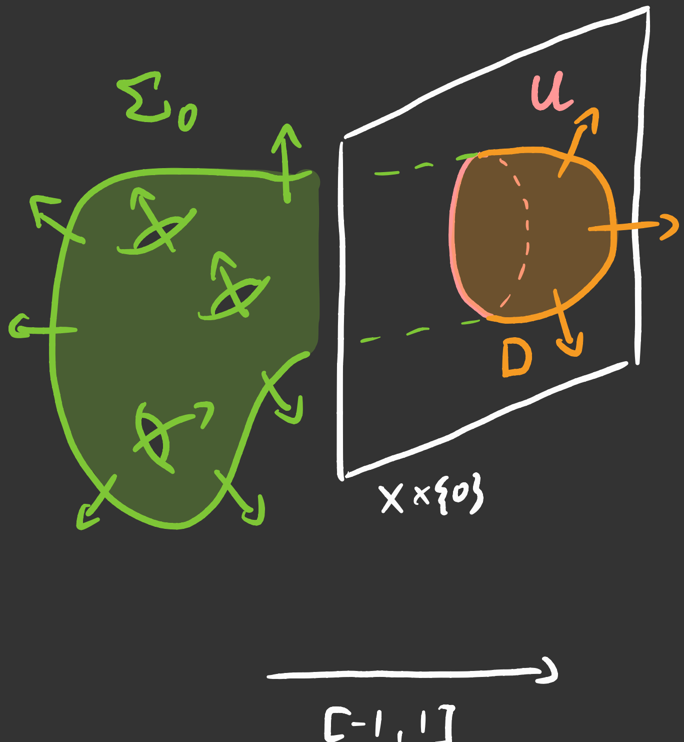

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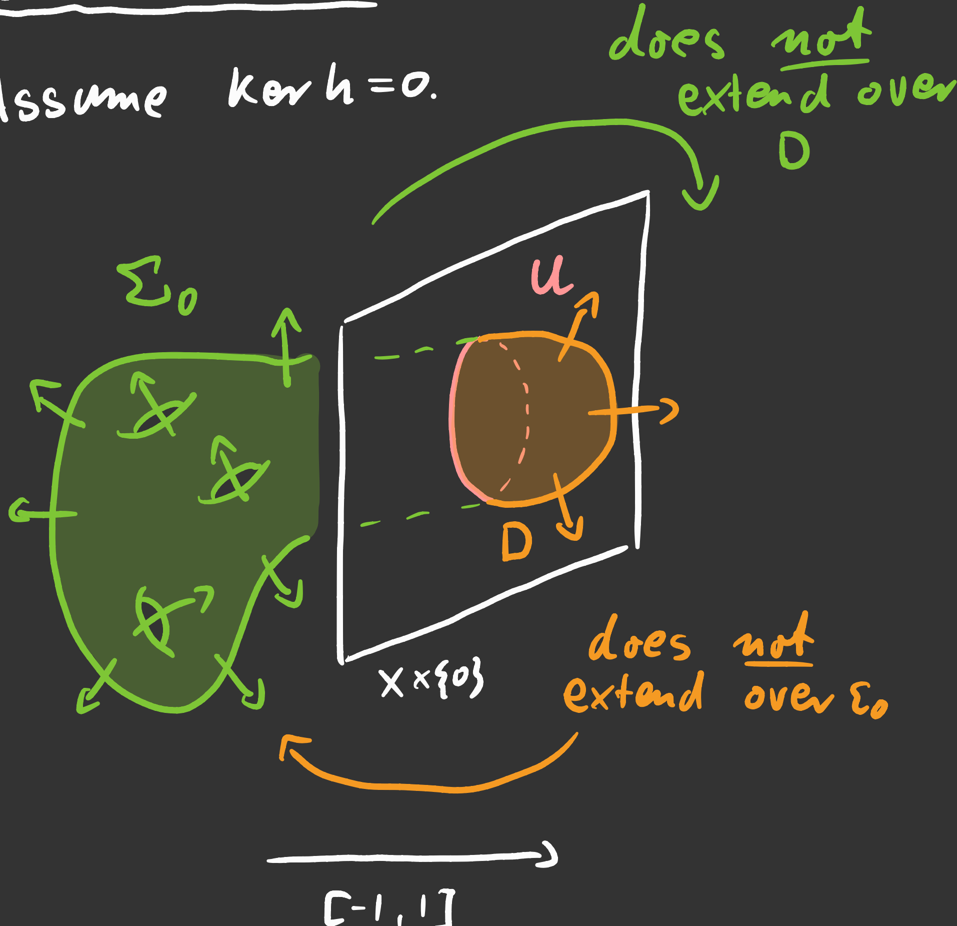

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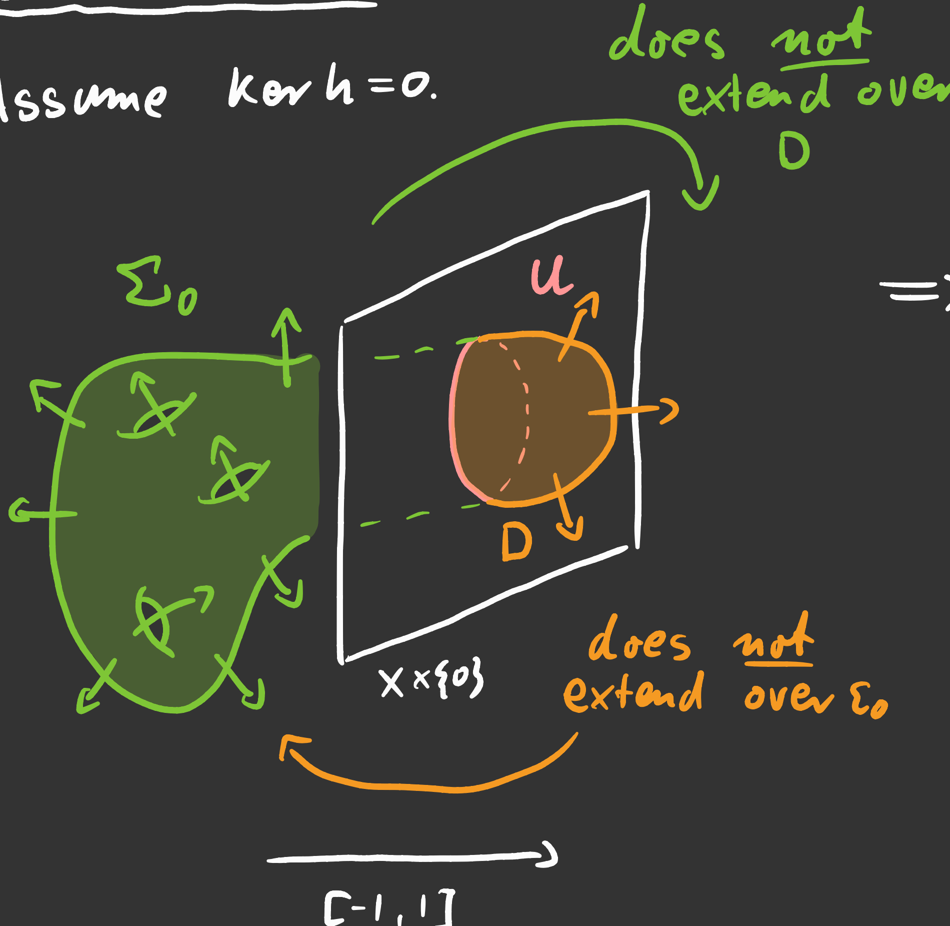

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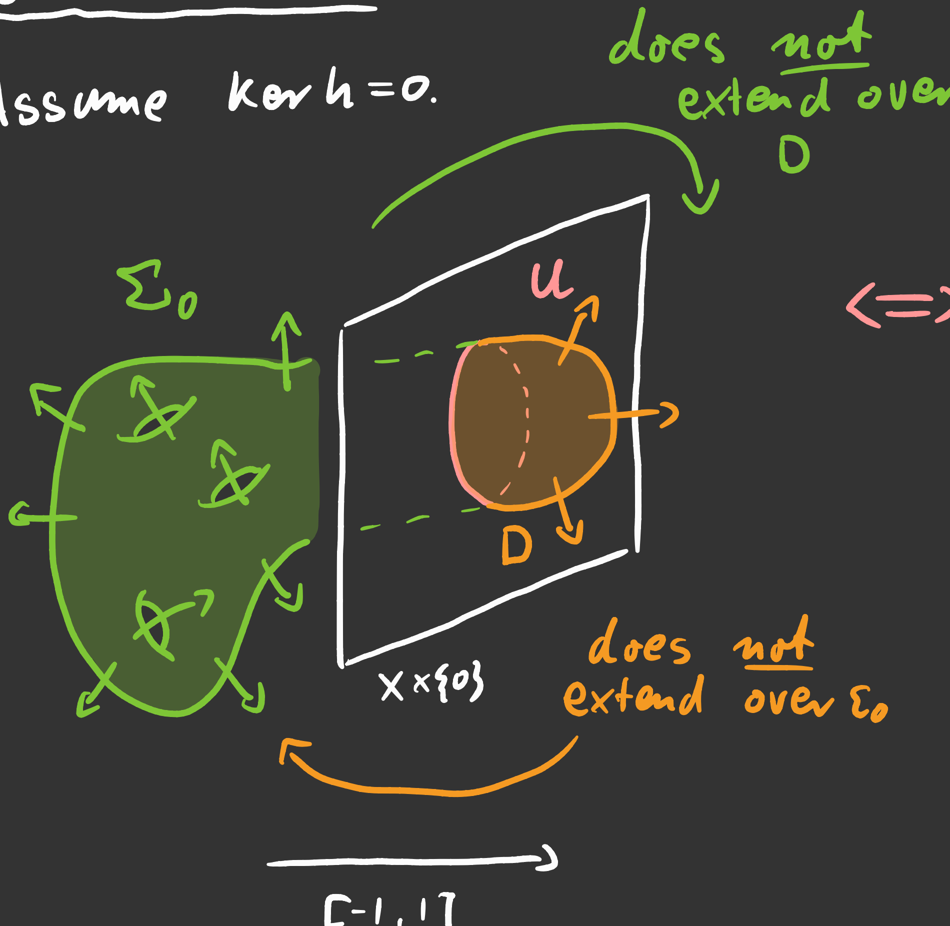

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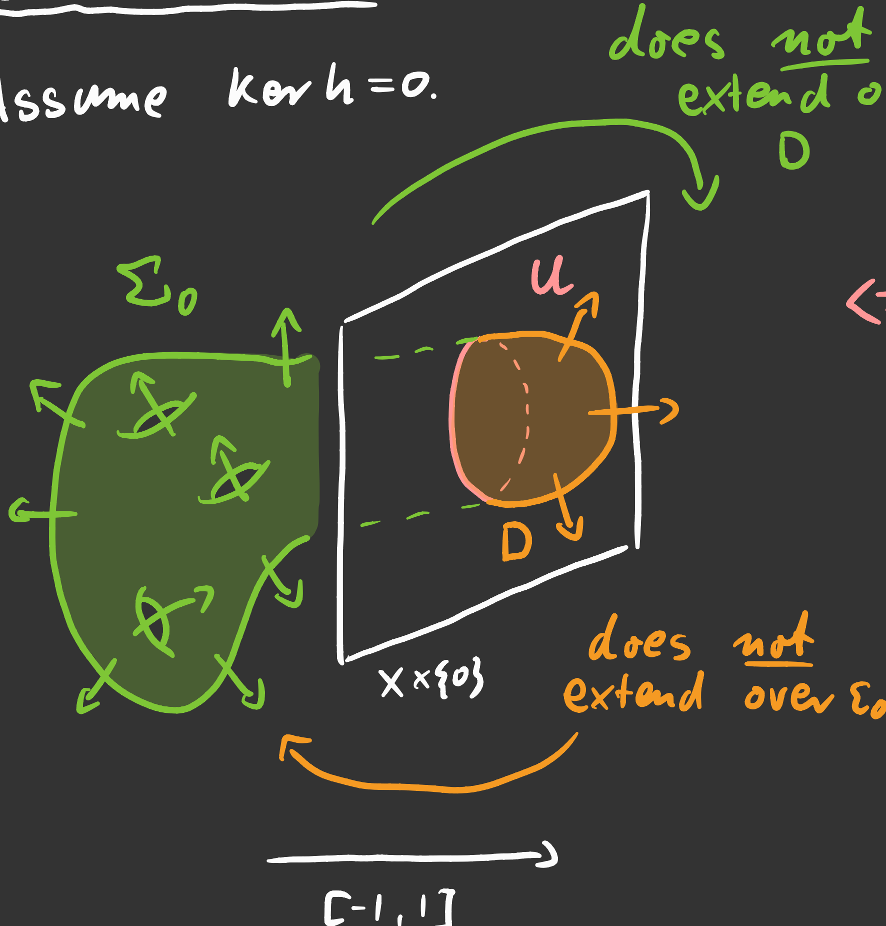

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$\Downarrow$

say X is type I iff $\exists\, \Sigma \subset X$ s.t. ν_Σ or. but not triv.

First result

Theorem: X is type I iff $h : \Pi_1(X) \longrightarrow H_1(X; \mathbb{Z}w)$ is an isomorphism.

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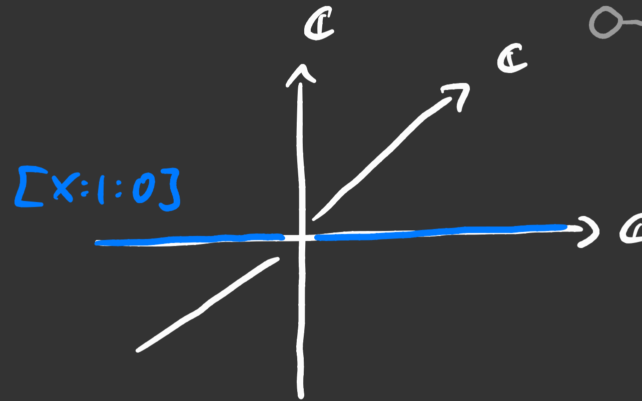
Theorem: X is type I iff $h: \mathbb{F}_1(X) \longrightarrow H_1(X; \mathbb{Z}w)$ is an isomorphism.

And *define*:

X is type II iff $\forall \Sigma \subset X$ surface ν_Σ or. $\Rightarrow \nu_\Sigma$ triv.

Examples

1) $\mathbb{C}P^1 \subset \mathbb{C}P^2$



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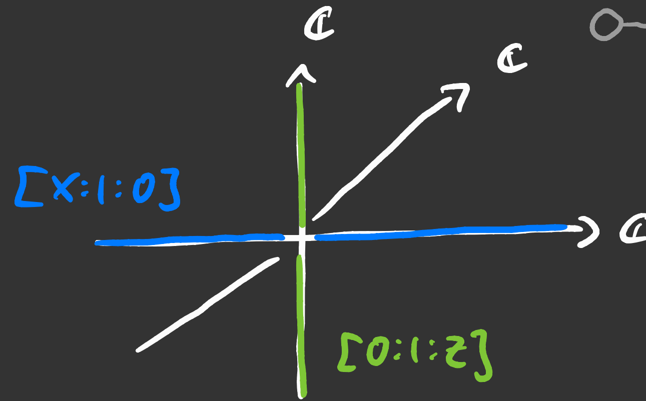

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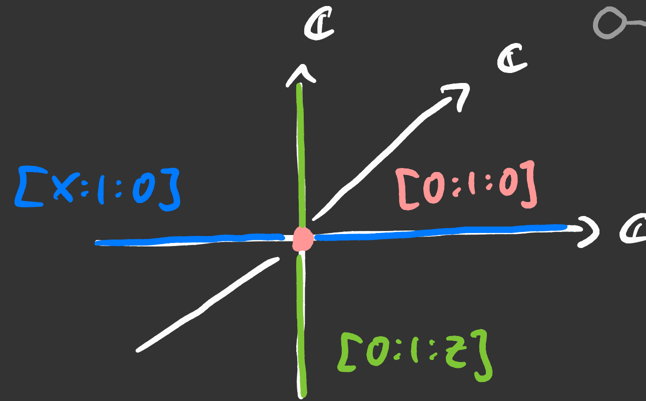

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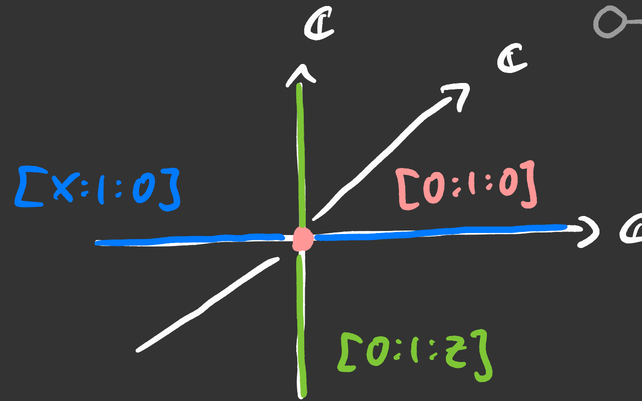

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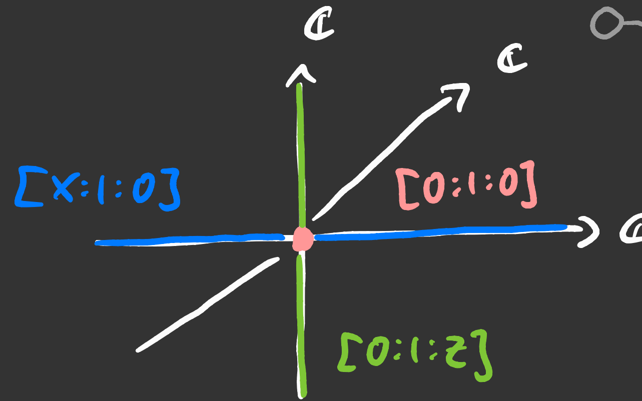

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2) $\mathbb{RP}^2 \subset \mathbb{RP}^{4k}$

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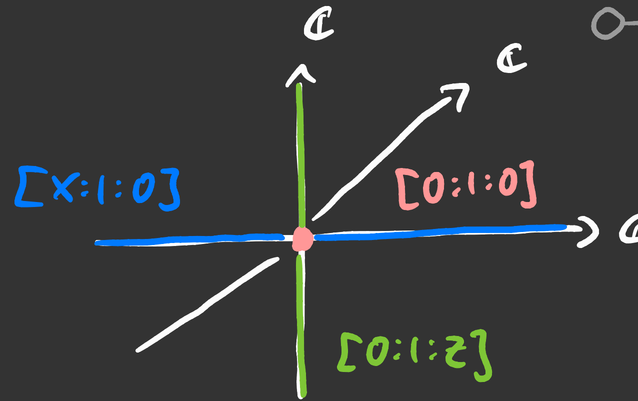

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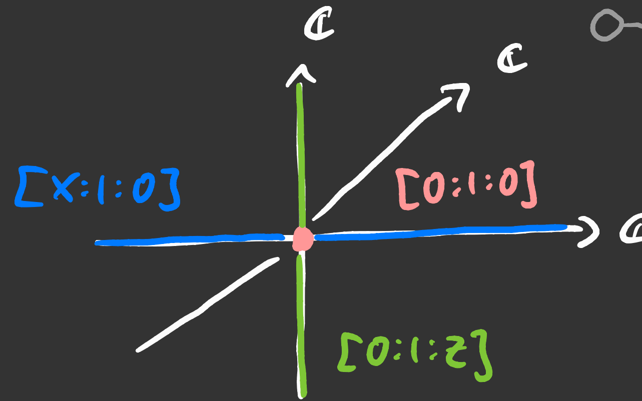
2) $\mathbb{R}P^2 \subset \mathbb{R}P^{4k} \xrightarrow{\text{char. classes}} \nu_{\mathbb{R}P^2}$ or. but non-triv.

Examples

1) $\mathbb{C}P^1 \subset \mathbb{C}P^2$



$\rightsquigarrow$ $\mathbb{C}P^1$ has odd self-intersection

$\Rightarrow$ $\mathbb{C}P^2$ is of type I

$\rightsquigarrow$ $\mathbb{F}_1(\mathbb{C}P^2) \cong H_1(\mathbb{C}P^2;\mathbb{Z}_w) \cong H_1(\mathbb{C}P^2;\mathbb{Z}) = 0$

2) $\mathbb{R}P^2 \subset \mathbb{R}P^{4k}$ $\xrightarrow{\text{char. classes}}$ $\nu_{\mathbb{R}P^2}$ or. but non-triv.

$\rightsquigarrow$ $\mathbb{F}_1(\mathbb{R}P^{4k}) \cong H_1(\mathbb{R}P^{4k};\mathbb{Z}_w) \cong 0$

all circles c with orientable ν_c are contractible

Result for type II

Theorem: If X is type II, then $\mathbb{F}_1(X)$ fits into

$$0 \to \mathbb{Z}_2 \longrightarrow \mathbb{F}_1(X) \to H_1(X; \mathbb{Z}_w) \to 0.$$

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In particular:

$$\mathbb{F}_1(X) \cong H_1(X;\mathbb{Z}_w)\oplus\mathbb{Z}_2 \text{ iff } X \text{ is Pin}^-.$$

Proof idea

Fact: Extension of

$$0 \to \mathbb{Z}_2 \to \mathbb{F}_1(X) \to H_1(X;\mathbb{Z}_w) \to 0$$

is (uniquely) determined by hom.

$$\varepsilon_X : \underbrace{\mathrm{Tor}_2(H_1(X;\mathbb{Z}_w))}_{=: \{[c] \in H_1(X;\mathbb{Z}_w) \mid 2[c]=0\}} \to \mathbb{Z}_2.$$

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<u>Link to geom. of X:</u>

1) $[c] \in \mathrm{Tor}_2(H_1(X;\mathbb{Z}_w))$ circle.

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Link to geom. of X:

1) $[C] \in \mathrm{Tor}_2(H_1(X;\mathbb{Z}_w))$ circle.

2) Endow C with normal framing ℓ.

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4) Build up extension via generators.

Proof idea

$$[C]_{tv} \in \mathrm{Tor}_2(H_1(X;\mathbb{Z}_v))$$ generator

Proof idea

$[C]_{tw} \in \mathrm{Tor}_2(H_1(X;\mathbb{Z}_w))$ generator

$\exists\, \beta : H_2(X;\mathbb{Z}_2) \longrightarrow \mathrm{Tor}_2(H_1(X;\mathbb{Z}_w))$ surjective s.t.

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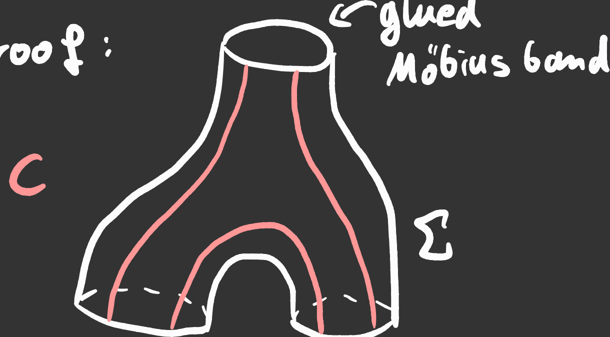

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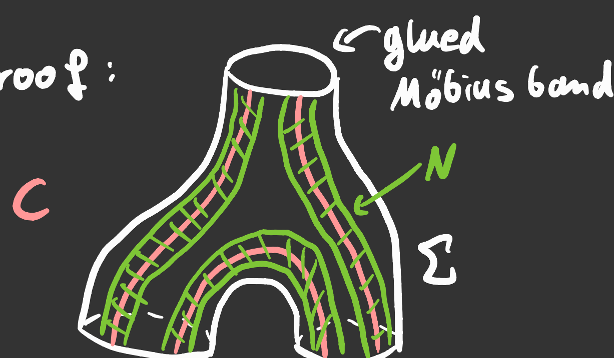

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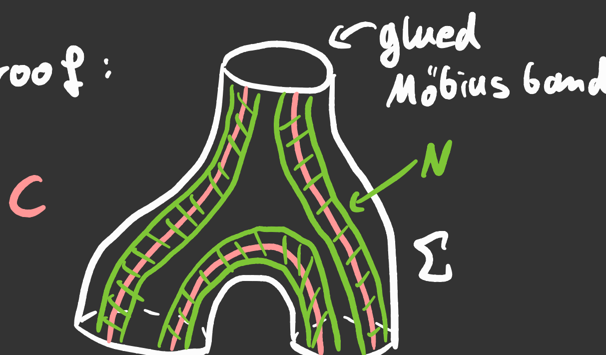

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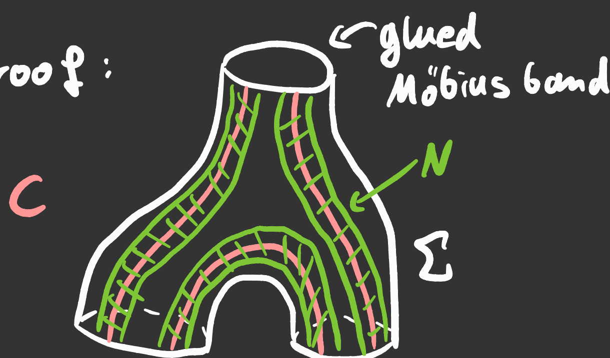

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Obstruction theory: ψ extends over D iff $w_2(\nu_\Sigma) = 0$ $\square$

Example

Consider $X = \mathbb{R}P^{4+1}$ and $\mathbb{R}P^2 \subset \mathbb{R}P^{4+1}$

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$[\mathbb{RP}^2]$ generates $H_2(X; \mathbb{Z}_2)$ and $\mathrm{Tor}_2(H_1(X;\mathbb{Z}_\omega)) = H_1(X;\mathbb{Z}_\omega)$.

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$[\mathbb{R}P^2]$ generates $H_2(X;\mathbb{Z}_2)$ and $\mathrm{Tor}_2(H_1(X;\mathbb{Z}_w)) = H_1(X;\mathbb{Z}_w)$.

$(4+1) \bmod 4$	0	1	2	3
$w_1(\nu_{\mathbb{R}P^2})$	0	1	0	1
$w_2(\nu_{\mathbb{R}P^2})$	1	1	0	0
type	I	II, not Pin^-	II, Pin^-	II, Pin^-
$\pi^4(\mathbb{R}P^{4+1})$	0	$\mathbb{Z}_4$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$

1 Pin$^-$ manifolds

X has Pin$^-$-structure, then

$$0 \longrightarrow \underset{\mathbb{Z}_2}{\overset{\|\wr}{\Omega_1^{\mathrm{Pin}^-}}} \longrightarrow \mathbb{F}_1(X) \longrightarrow H_1(X;\mathbb{Z}_w) \longrightarrow 0$$

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Then:

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\mathrm{Pin}^-(X) \times H^1(X;\mathbb{Z}_2) \arrow[r, "\mathrm{act}"] \arrow[d] & \mathrm{Pin}^-(X) \arrow[d] \\
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Applications to vector bundles

$\mathbb{R}^n \hookrightarrow E \longrightarrow X^{n+1}$ w/ orientation & spin structure

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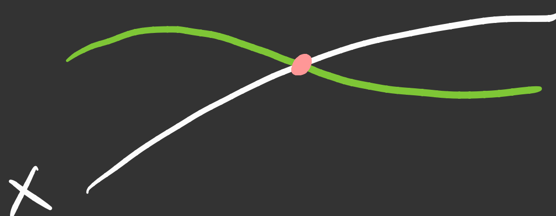
$s : X \to E$



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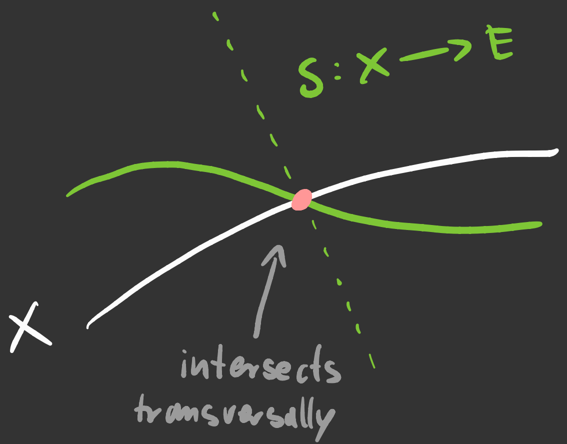

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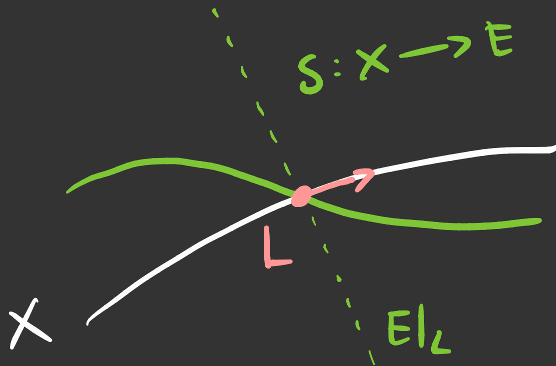
$L := s^{-1}(0_E)$



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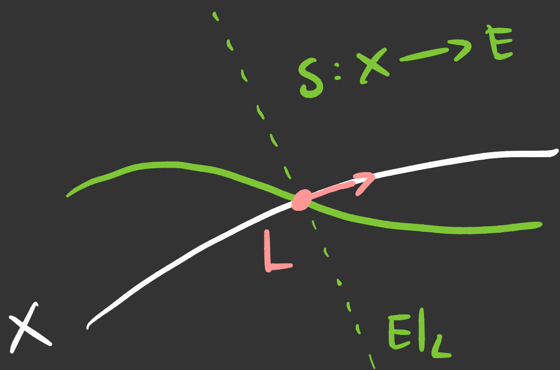

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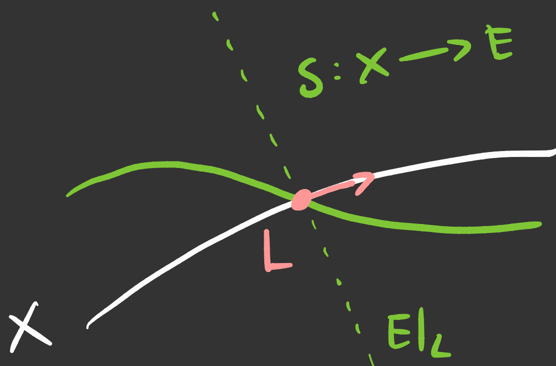

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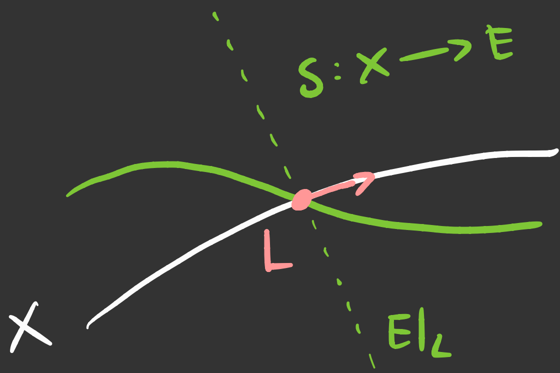

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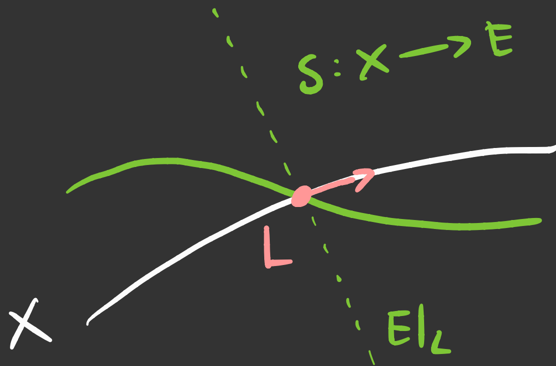

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<u>Theorem:</u> E admits a non-vanishing section if and only if $[L, \varphi] = 0$.

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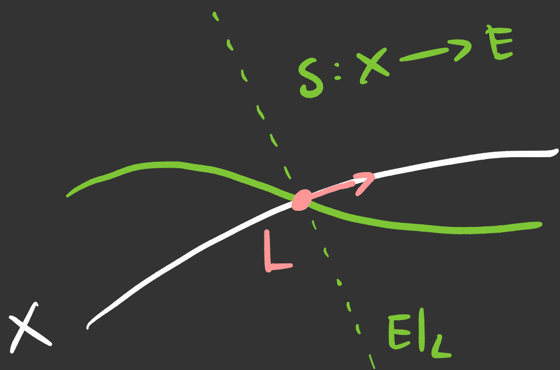

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Idea: Use null-bordism to "push" s away from zero (similar to Whitney trick).

Applications to vector bundles

<u>Corollary 1:</u> If X is of type I, then E admits a non-vanishing section iff $e(E) = 0$.

Applications to vector bundles

Corollary 1: If X is of type I, then E admits a non-vanishing section iff $e(E) = 0$.

Corollary 2: If X is Pin^-, then E admits a non-vanishing section iff $e(E) = 0$ and $\mathfrak{X}_2([L, e]) = 0$.

splitting map

$\mathfrak{X}_2 : \mathbb{F}_1(X) \longrightarrow \mathbb{Z}_2$

THANK YOU FOR YOUR ATTENTION!