

A geometric computation of cohomotopy groups in co-degree one

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joint work with

Thomas Rot

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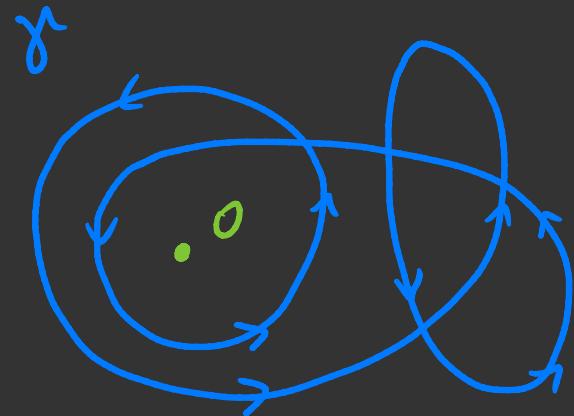
| Baby example



Q: Curves $\gamma : s^1 \rightarrow \mathbb{R}^2 - \{0\}$ up to homotopy?

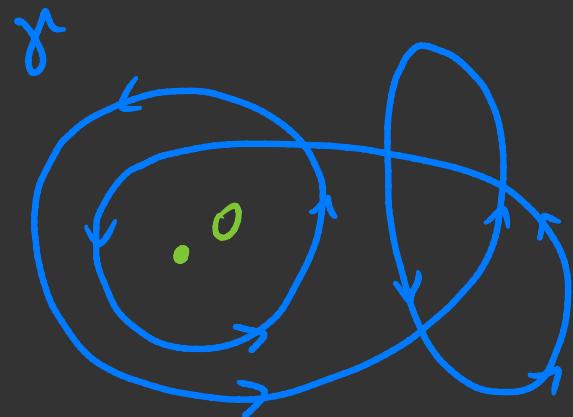
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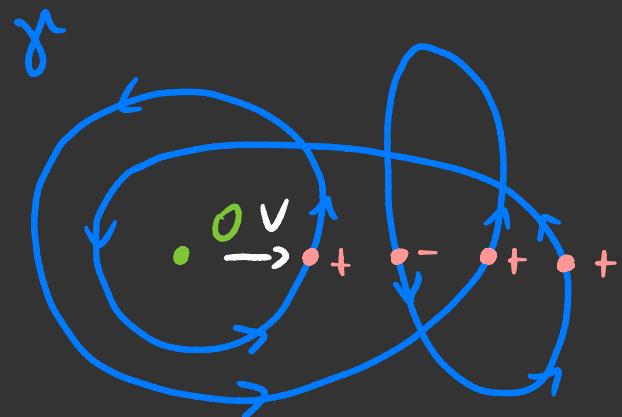
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γ has winding number
 $w(\gamma) = 2$.

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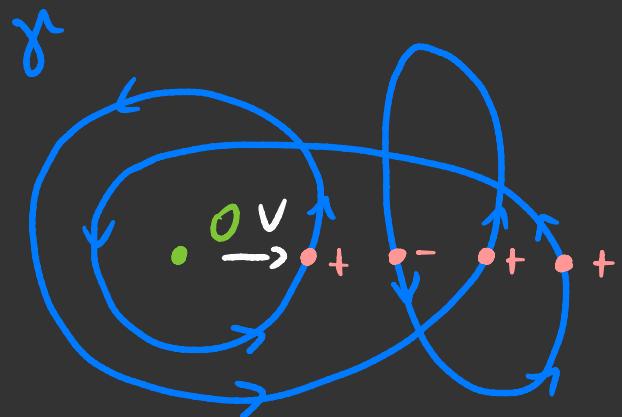
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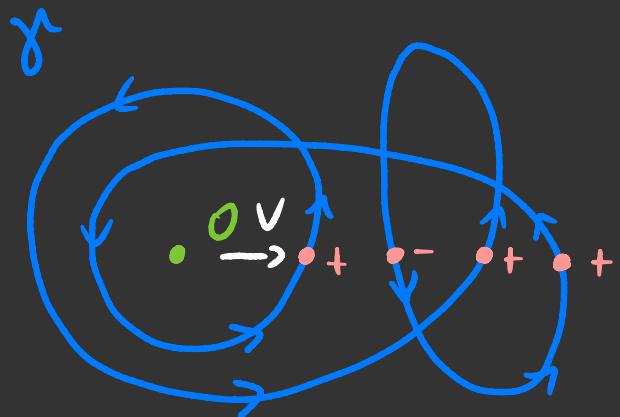
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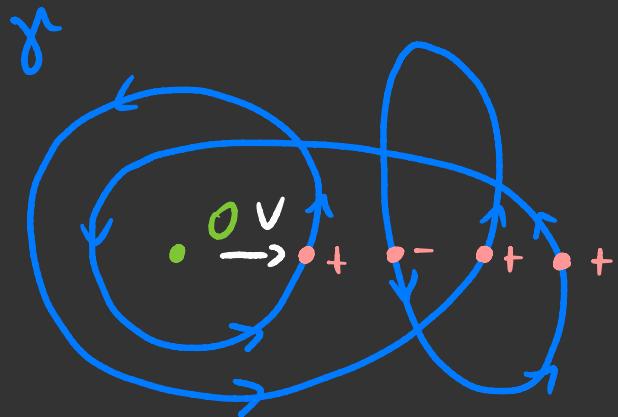
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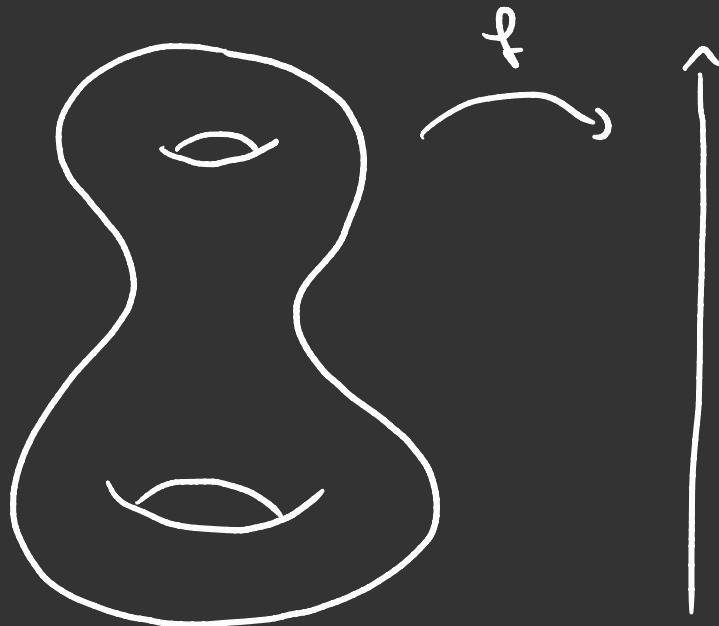


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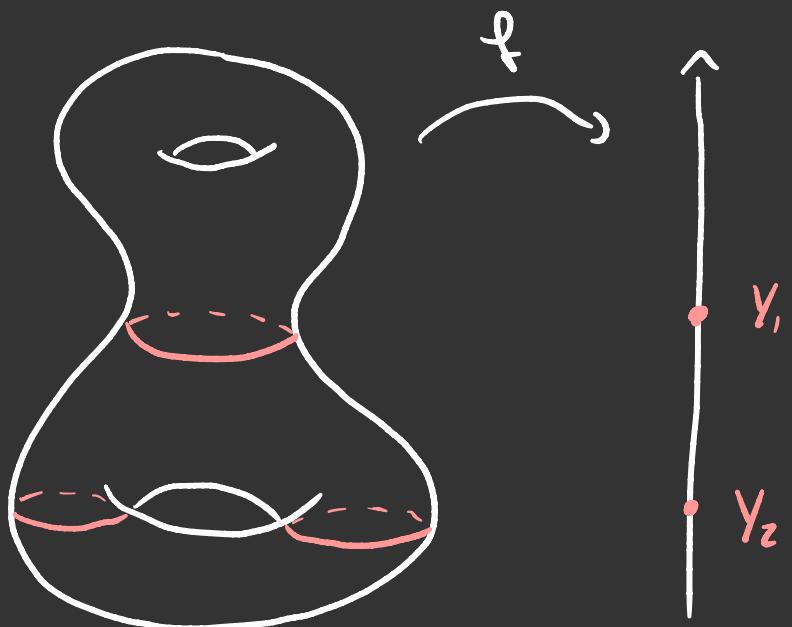
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 $\therefore \deg(f; v)$
mapping degree

I Regular values



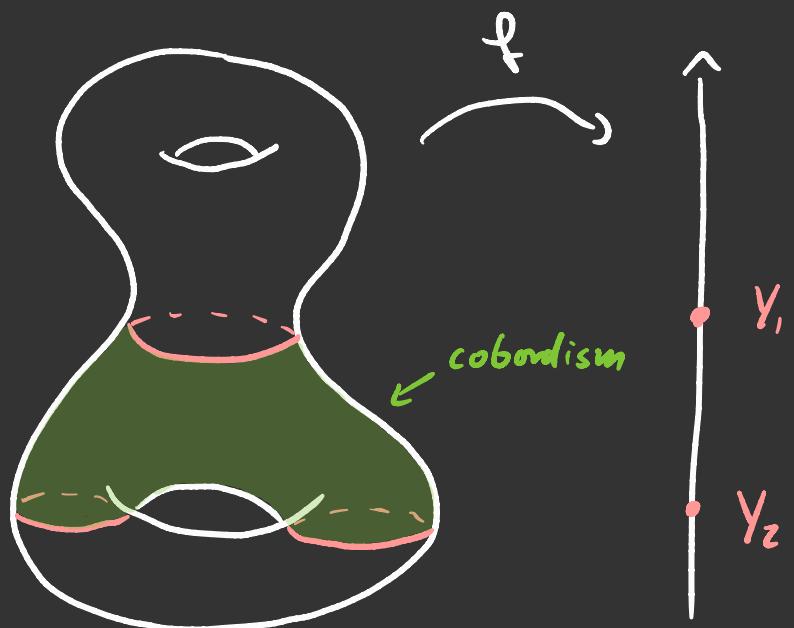
Regular values



$f: X \rightarrow M$ smooth
 $\Rightarrow f^{-1}(y) \subset X$ submfld.
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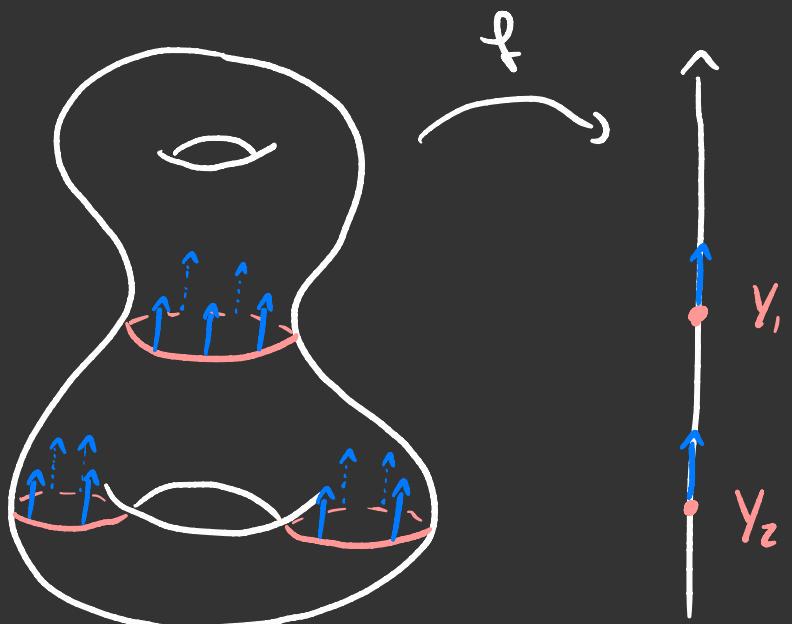
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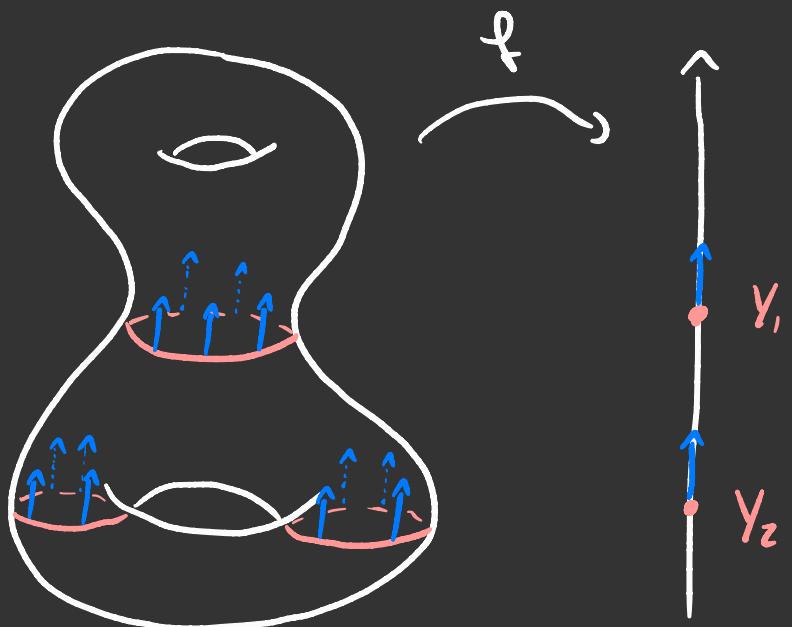


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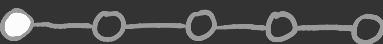
Fact: Reg. values are "generic".

| Back to the baby example

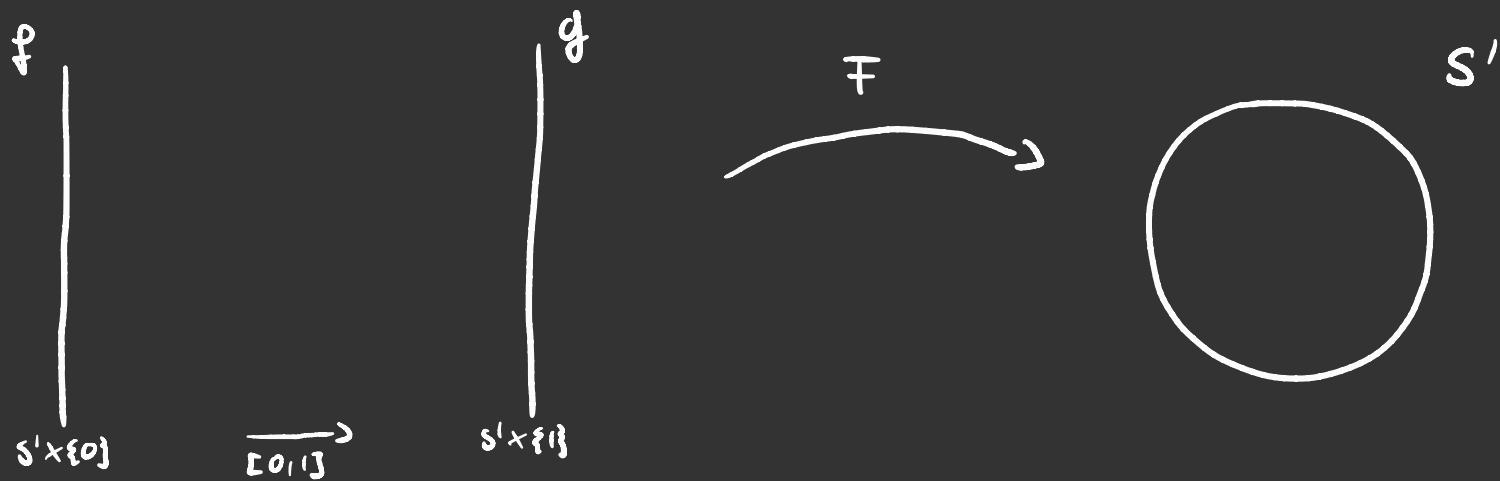


Q: Is mapping degree a homotopy invariant?

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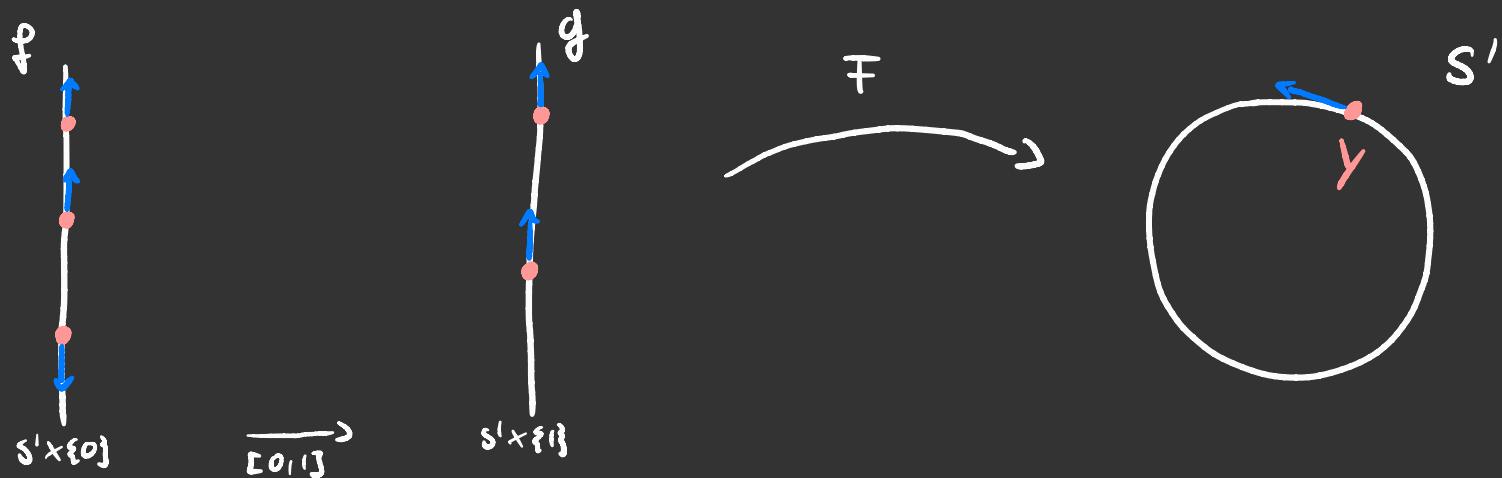
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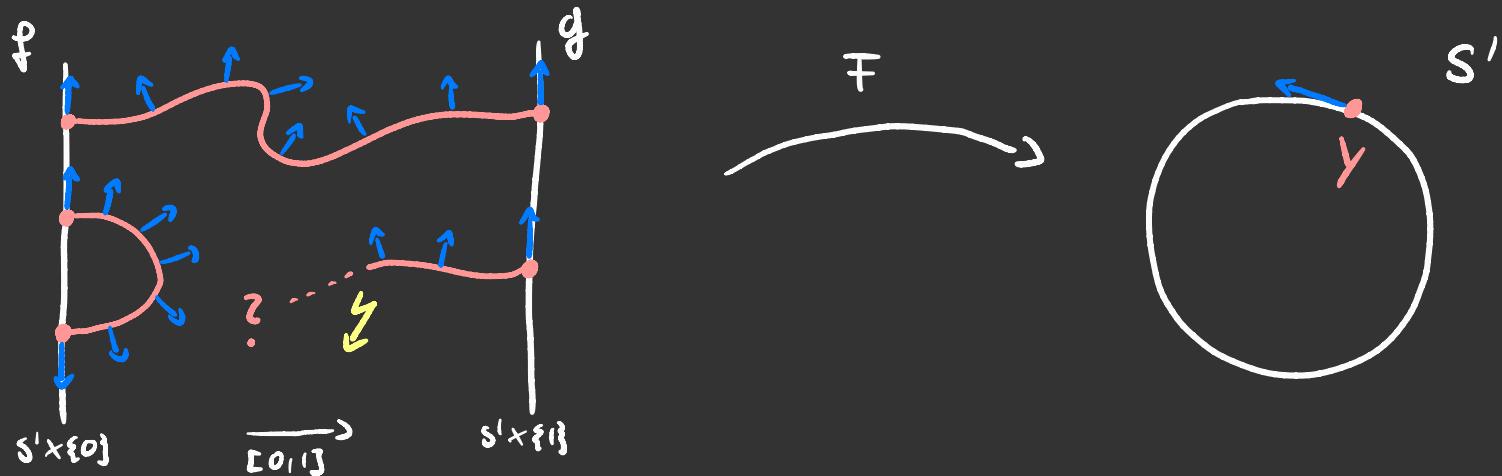


Assume $\deg(f; y) \neq \deg(g; y)$.

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Pontryagin-Thom construction



Q: Homotopy invariant for $f: X \rightarrow S^n$?

Pontryagin-Thom construction



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Remember: $f'(y) \subset X$ submfld.

Pontryagin-Thom construction



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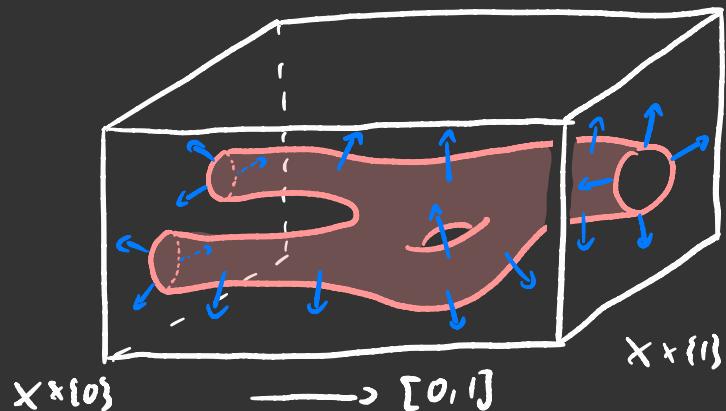
Remember: $f^{-1}(y) \subset X$ submfld. \leadsto cobordisms?

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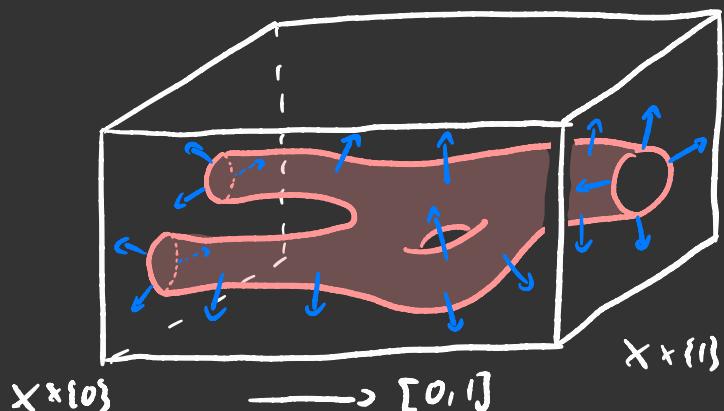


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Define:

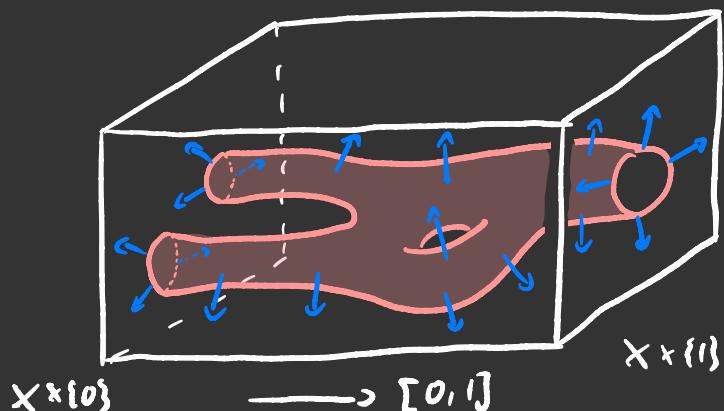
$$IF_n(X) := \frac{\{n\text{-dim closed } M \subset X \text{ w/}\}}{\text{normally framed cobord.}}$$

| Pontryagin-Thom construction



Q: Homotopy invariant for $f: X \rightarrow S^n$?

Remember: $f^{-1}(y) \subset X$ submfld. \leadsto cobordisms?



Define:

$$\text{IF}_k(X) := \frac{\{\text{k-dim closed } M \subset X \text{ w/ framing of } \Sigma_M\}}{\text{normally framed cobord.}}$$

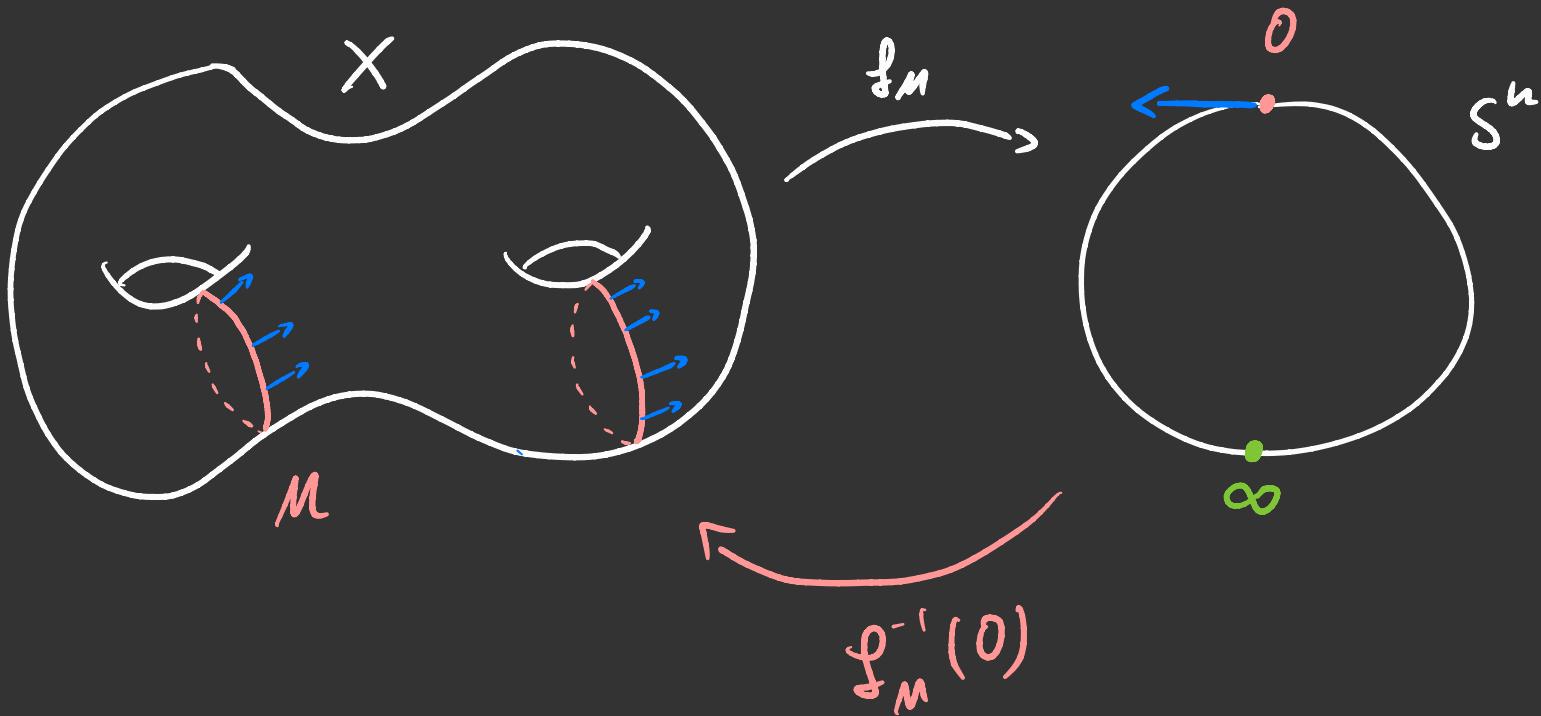
Note: $\text{IF}_k(X)$ is group if $\dim X \gg k$:

- i) $[M_1, e] + [M_2, e] = [M_1 \cup M_2, e]$
- ii) $-[M, e] = [M, -e]$ (reverse orientation)

Pontryagin-Thom construction



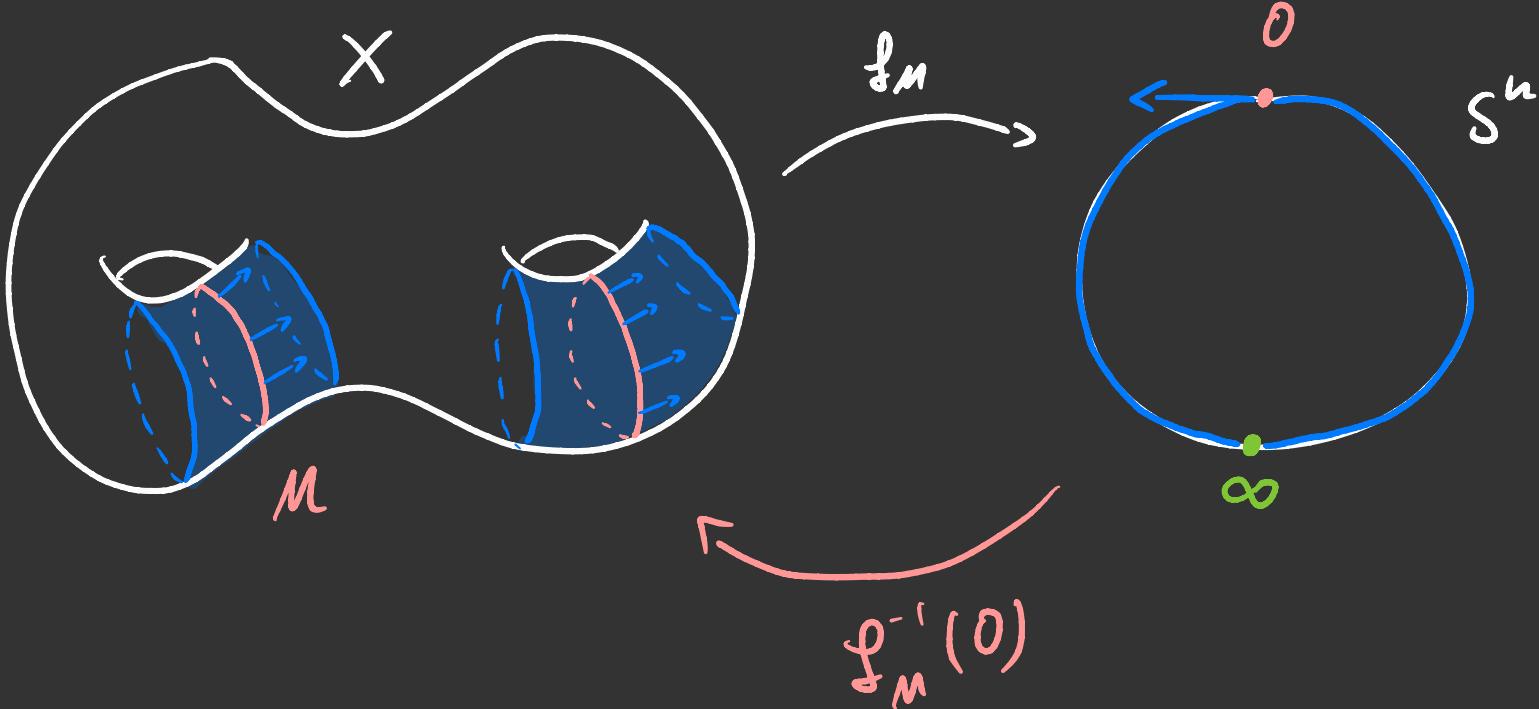
$$\underline{A}: [x^{n+k}, S^n] \cong \mathbb{F}_k(x)$$



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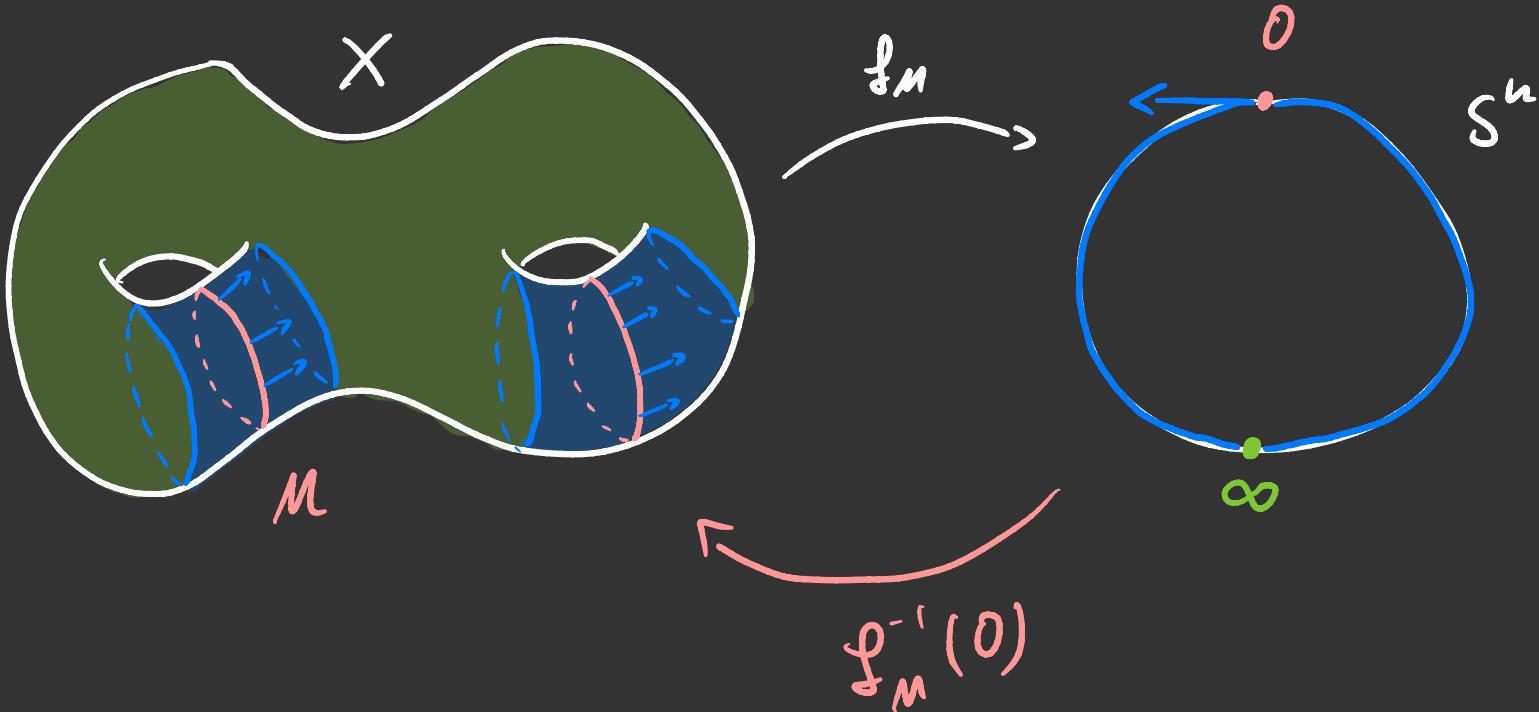
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- Enumerated by Steenrod '47

What is $[x^{n+1}, S^n]$ (aka $\text{IF}_1(x)$)?



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- Group str. by Taylor '09 but purely algebraically

What is $[X^{n+1}, S^n]$ (aka $\pi_{n+1}(X)$)?



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Q: Geometric proof for $n \geq 3$ and
 X^{n+1} (non-orientable) manifold?

Twisted homology



$$H_1(X; \mathbb{Z}_\omega) := \frac{\{\text{links } L \subset X \text{ w/ orientation of } \mathcal{V}_L\}}{\text{normally oriented bordisms}}$$

| Twisted homology



$$| L_1(X; \mathbb{Z}_\omega) := \frac{\{ \text{links } L \subset X \text{ w/ orientation of } \mathcal{V}_L \}}{\text{normally oriented bordisms}}$$

"Algebraically":

$$| L_1(X; \mathbb{Z}_\omega) \cong H_1(X; \mathbb{Z}_\omega) = \text{first homology w/ "twisted" coefficients}$$

(Thom '54, Atiyah '60)

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↳ obvious map $h: F_1(X) \longrightarrow H_1(X; \mathbb{Z}_\omega)$

forgets framing, remembers orientation

| The short exact sequence



$$F_1(x) \xrightarrow{h} H_1(x; \mathbb{Z}_\omega)$$

| The short exact sequence



$$| F_1(x) \xrightarrow{h} H_1(X; \mathbb{Z}_\omega) \rightarrow 0$$

v.b. over 1-dim CW-complex
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$$0 \rightarrow \text{ker } h \rightarrow F_1(x) \xrightarrow{h} H_1(x; \mathbb{Z}_\omega) \rightarrow 0$$

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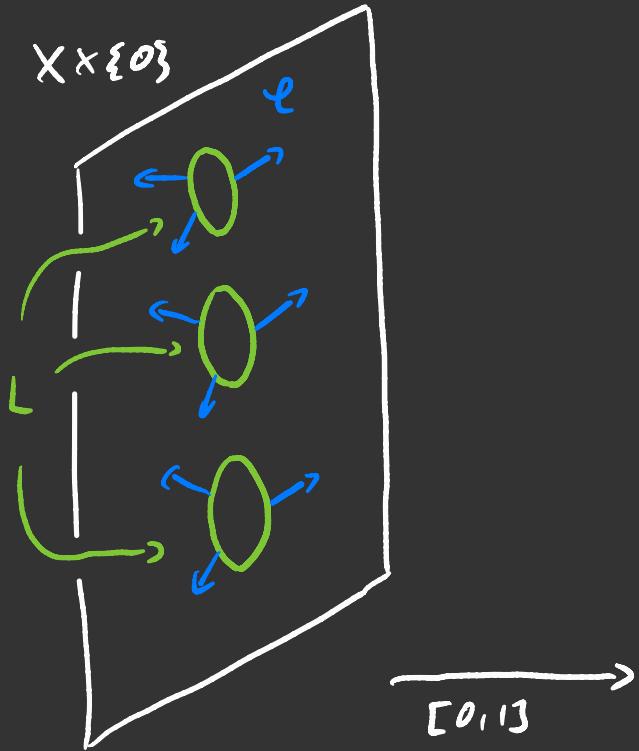
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Also: what is the extension?

Determine $\text{Ker } h$



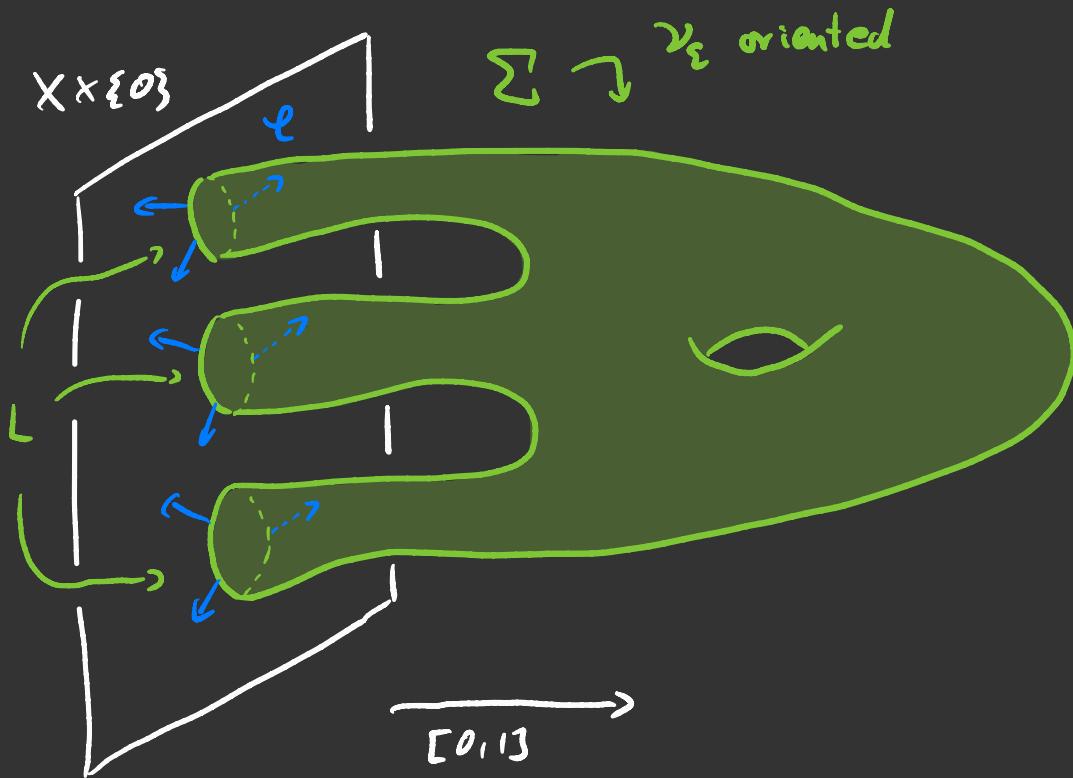
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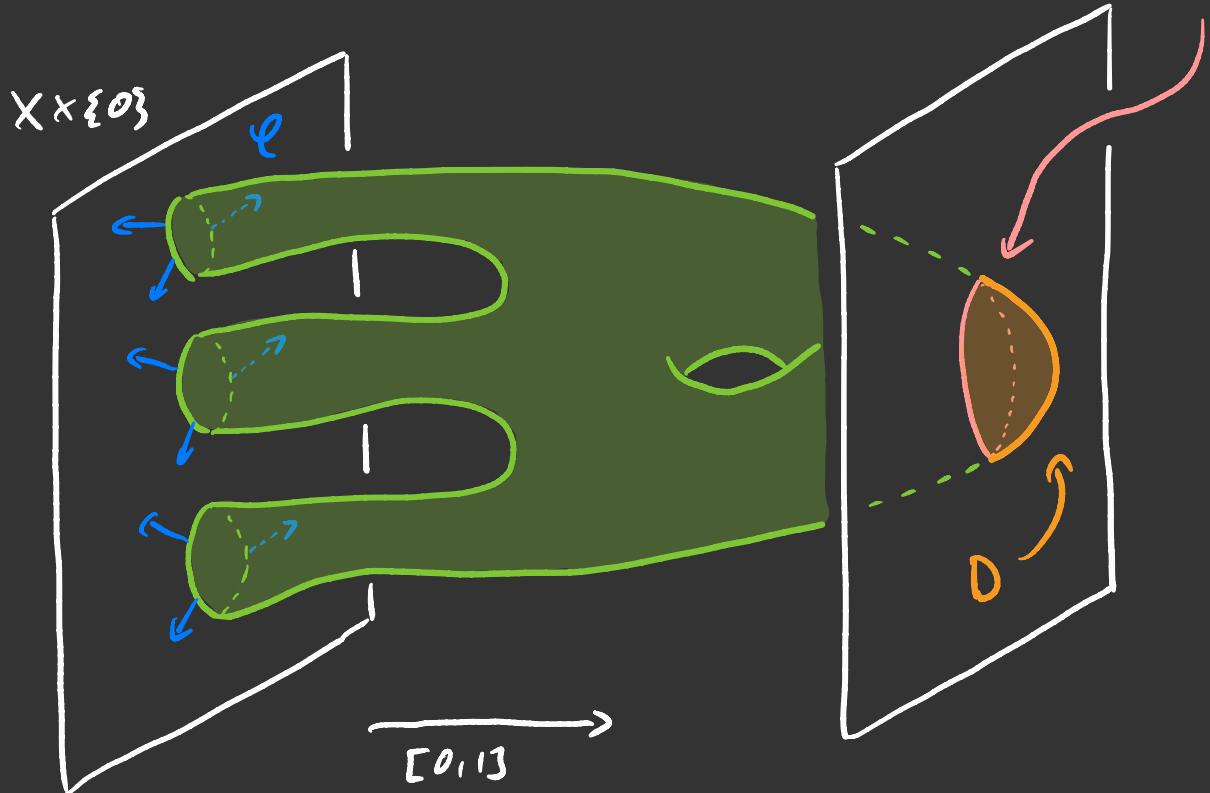


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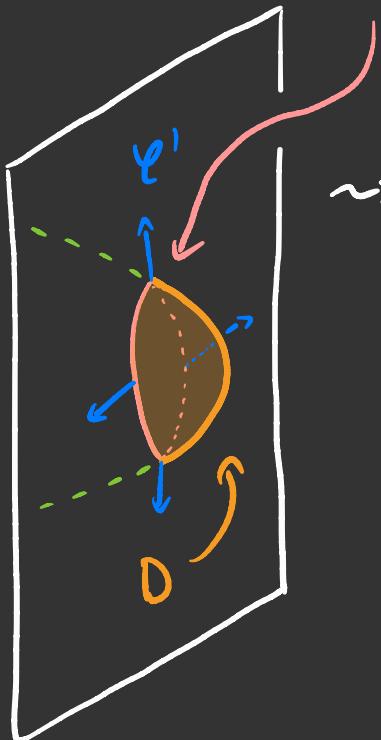
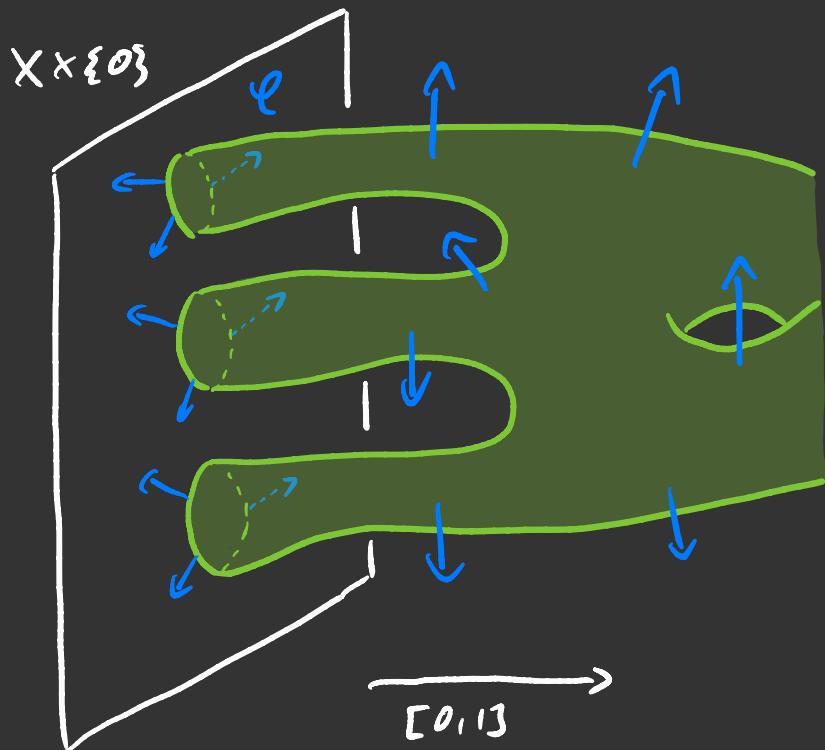
u contractible



Determine Kerh



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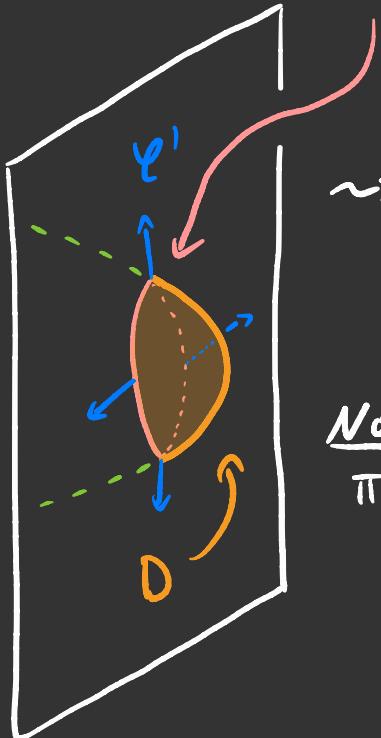
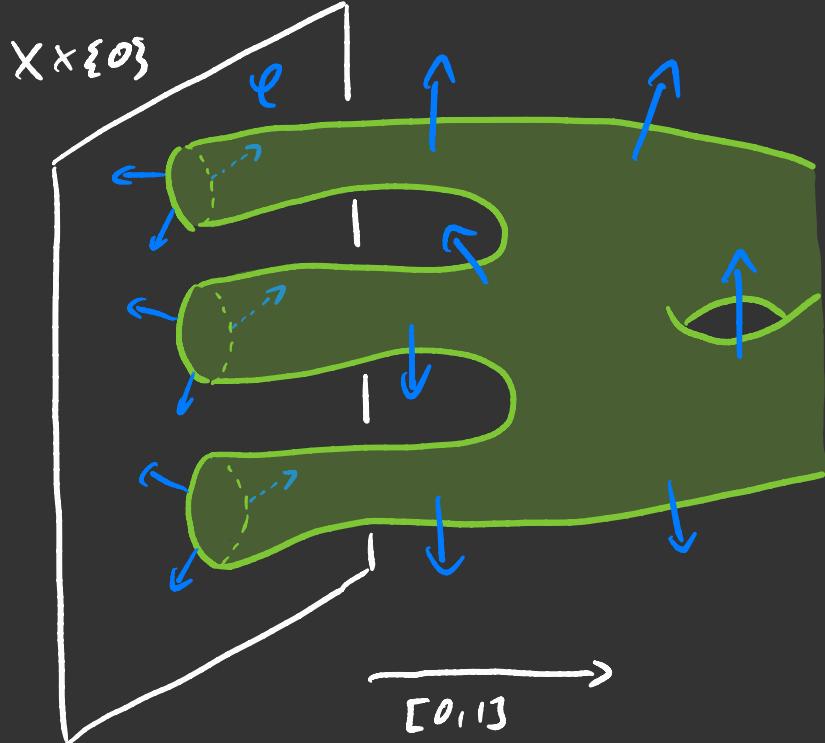


~ induces (unique)
framing φ' over u
s.t. $[L, \varphi] = [u, \varphi']$.

Determine Kerh



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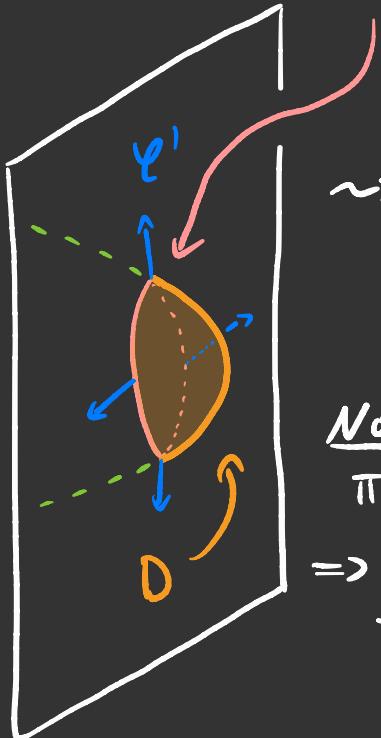
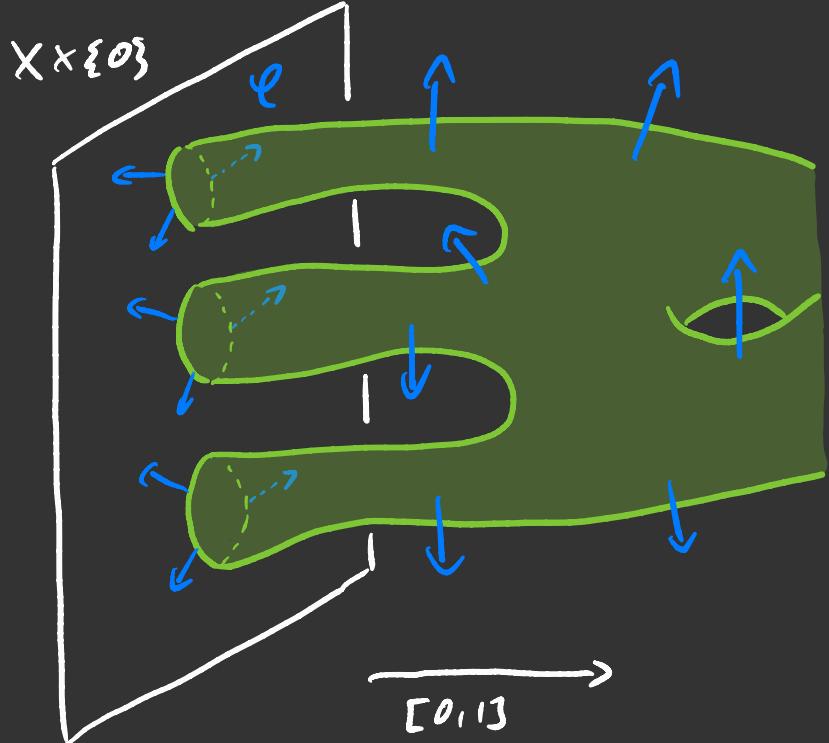
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Now:
 $\pi_1(\text{SO}(u)) = \mathbb{Z}_2$

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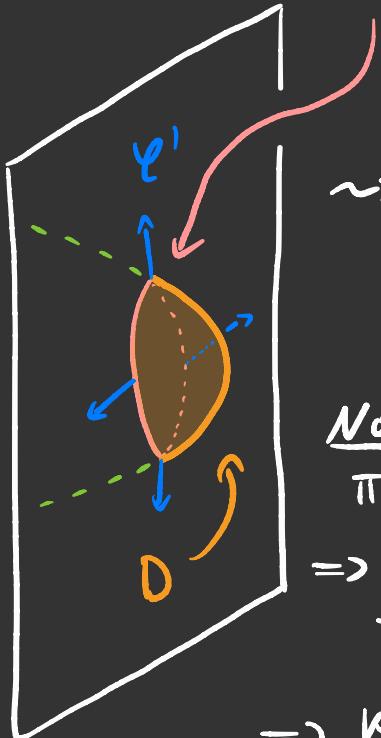
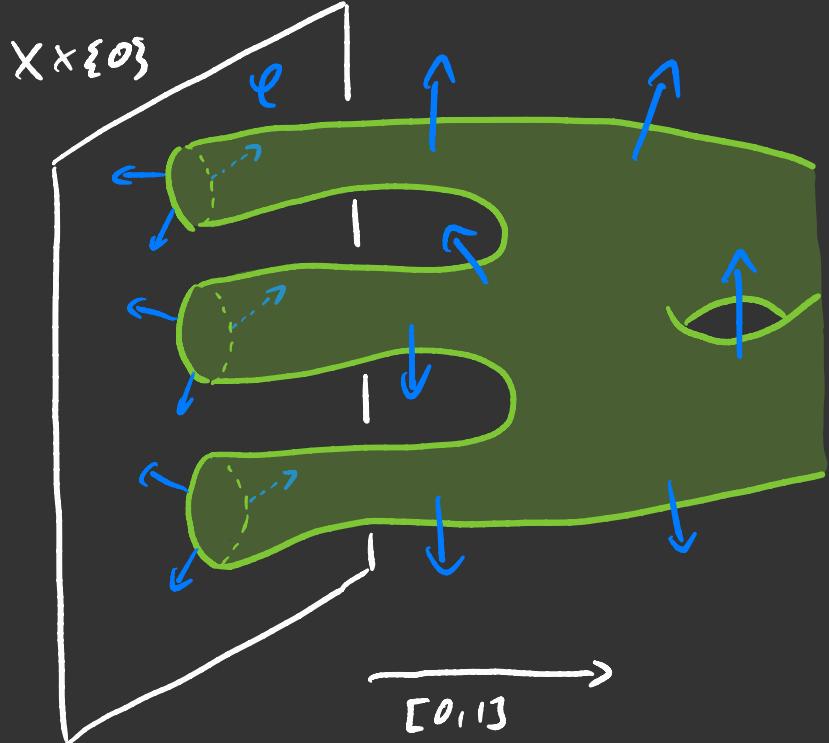
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$\Rightarrow u$ has two possible
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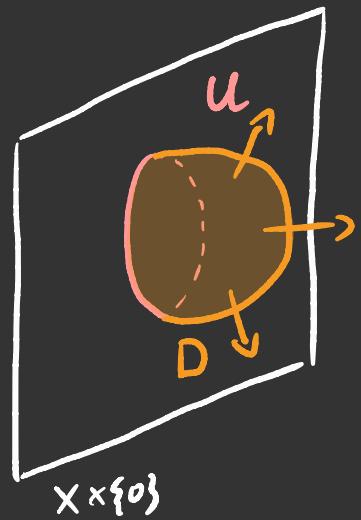
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$\Rightarrow \text{Kerh}$ is at most \mathbb{Z}_2 !

Determine $\ker h$

Assume $\ker h = 0$.

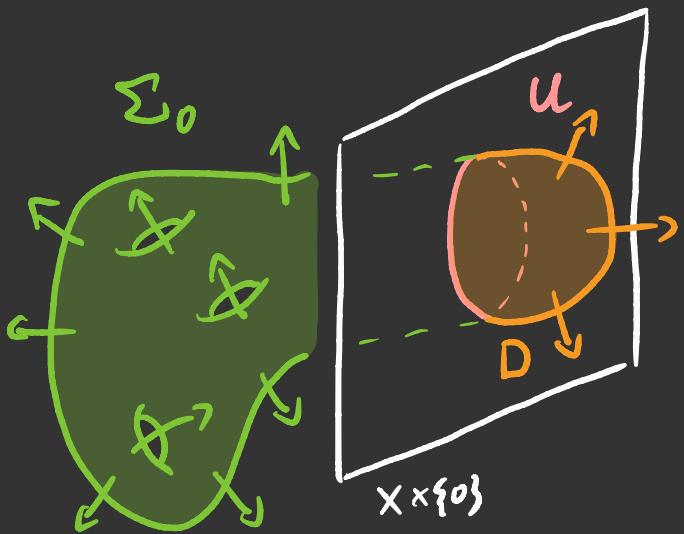


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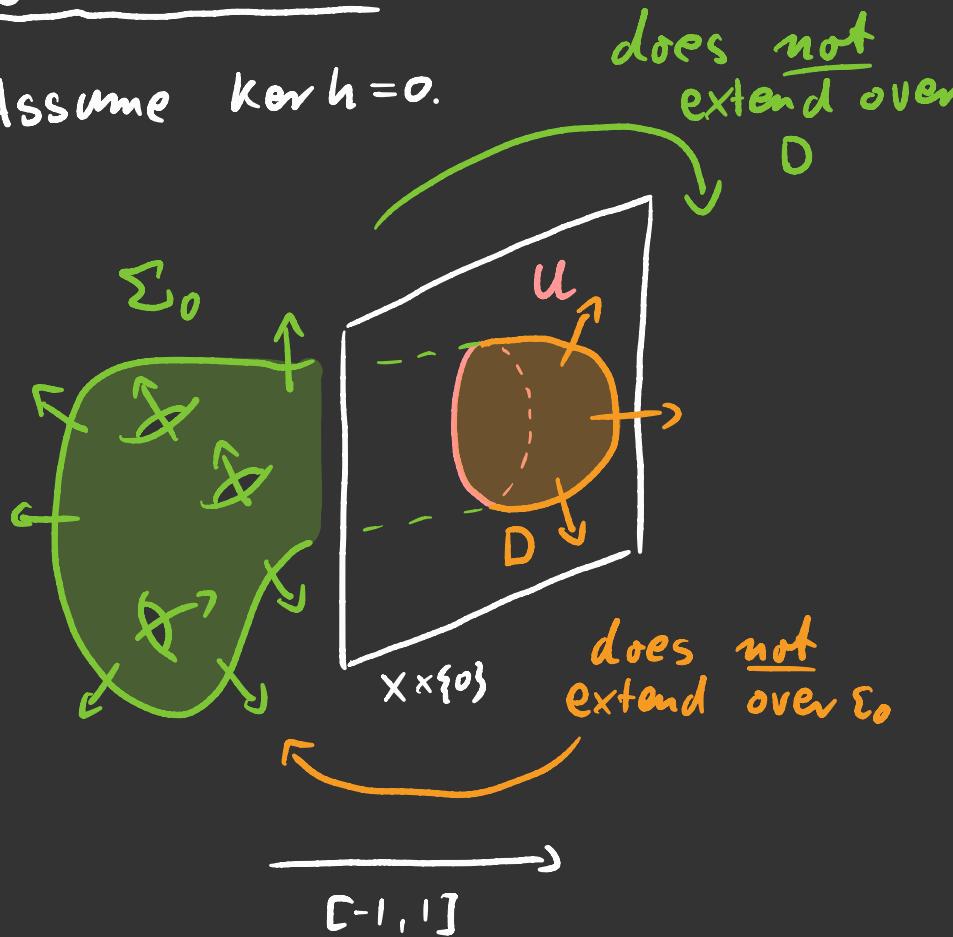


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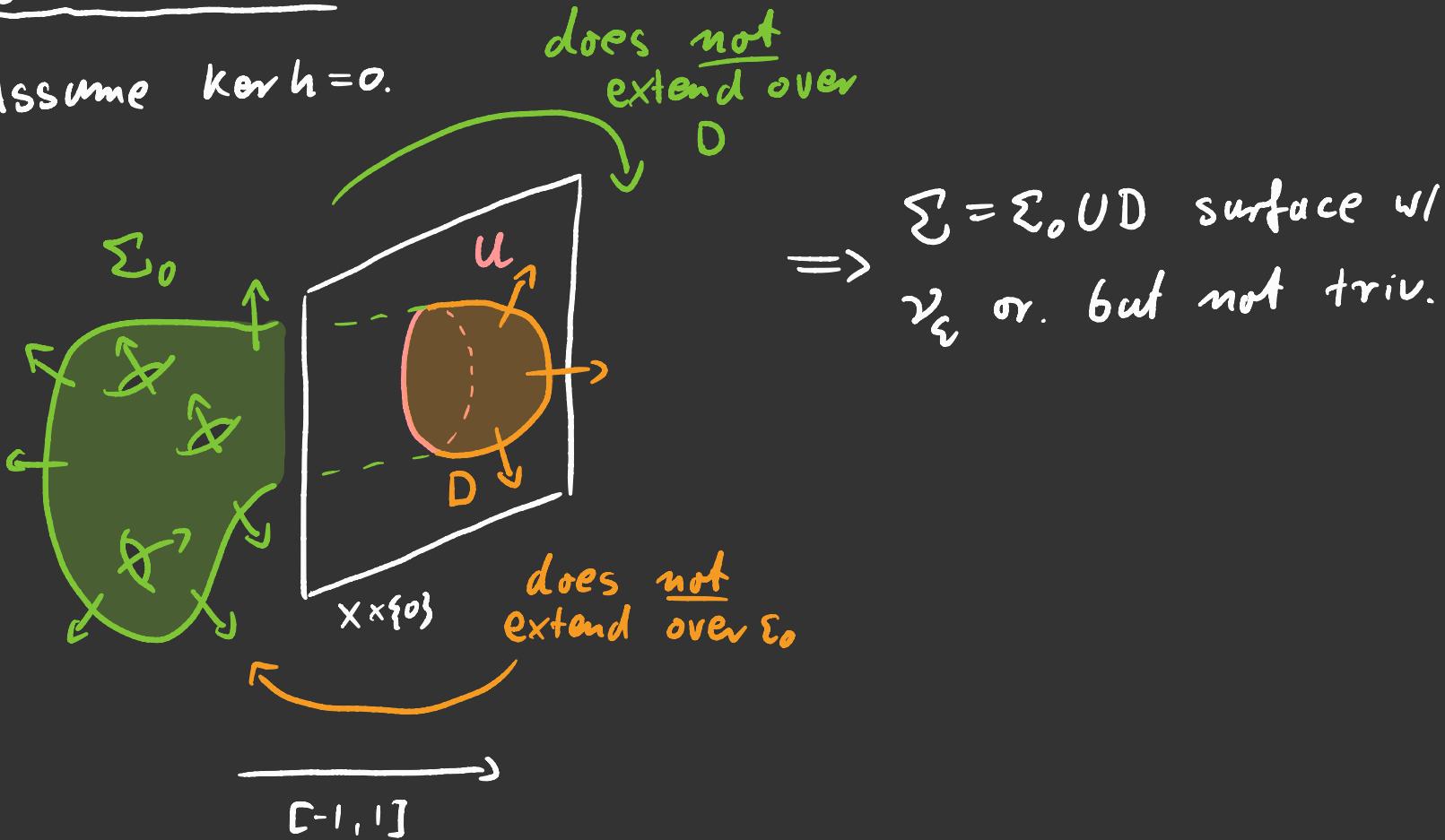


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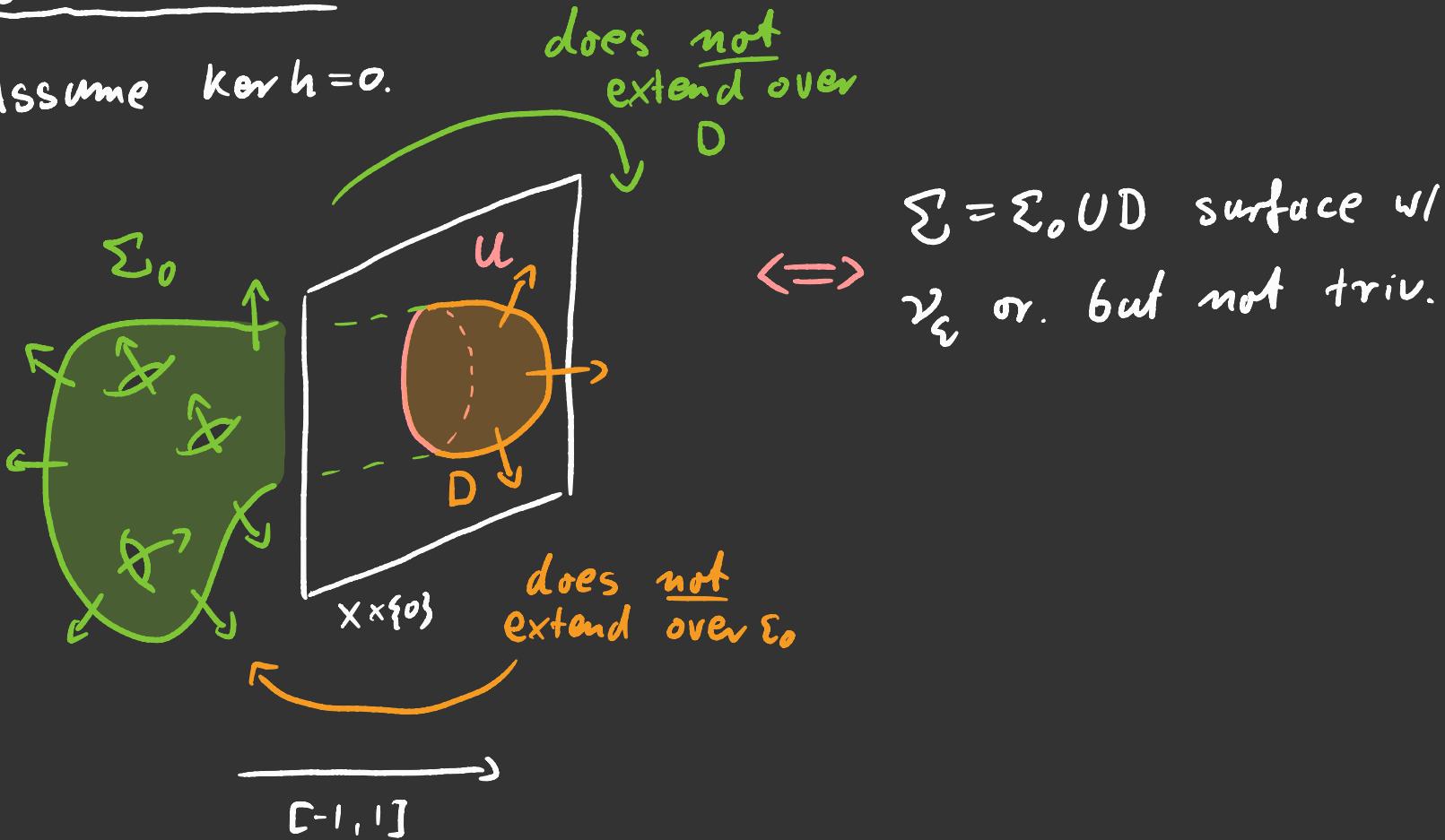
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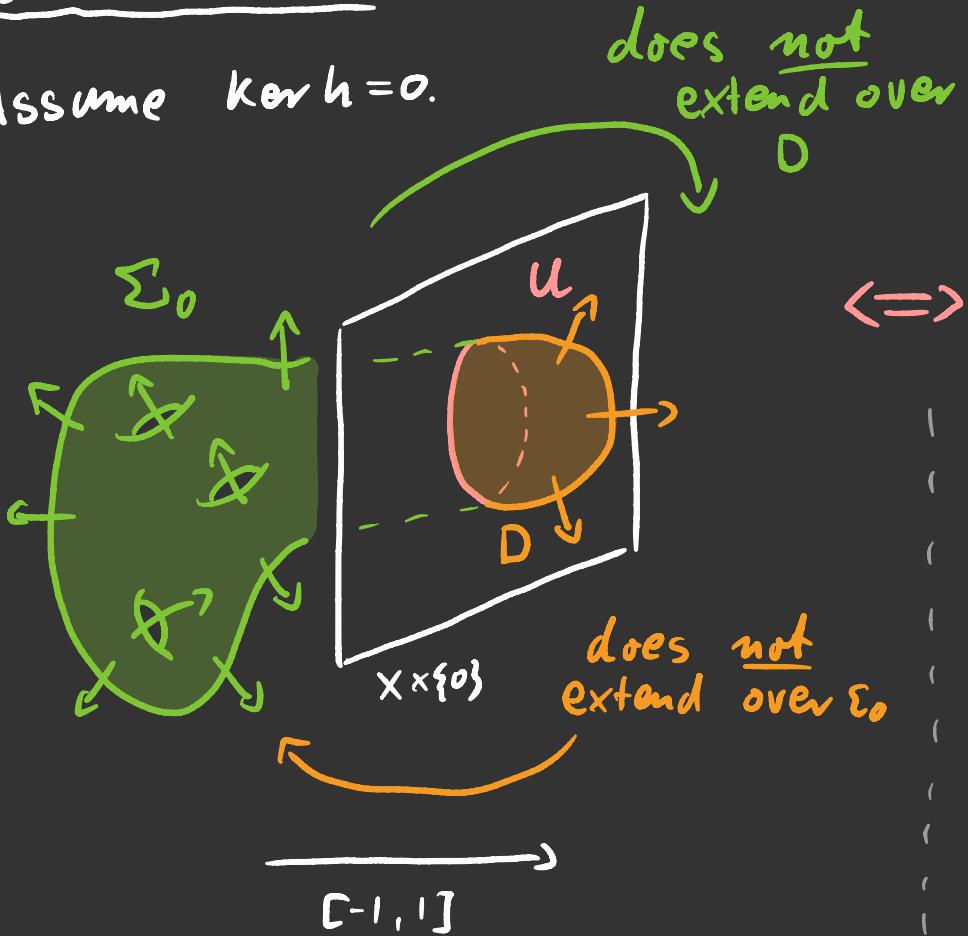
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Determine $\ker h$



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\Leftrightarrow

$\Sigma = \Sigma_0 \cup D$ surface w/
 ν_Σ or. but not triv.

say X is type I
iff $\exists \Sigma \subset X$ s.t.
 ν_Σ or. but not triv.

First result



Theorem: X is type I iff $h : \text{IF}_1(X) \rightarrow H_1(X; \mathbb{Z}_\omega)$ is an isomorphism.

First result



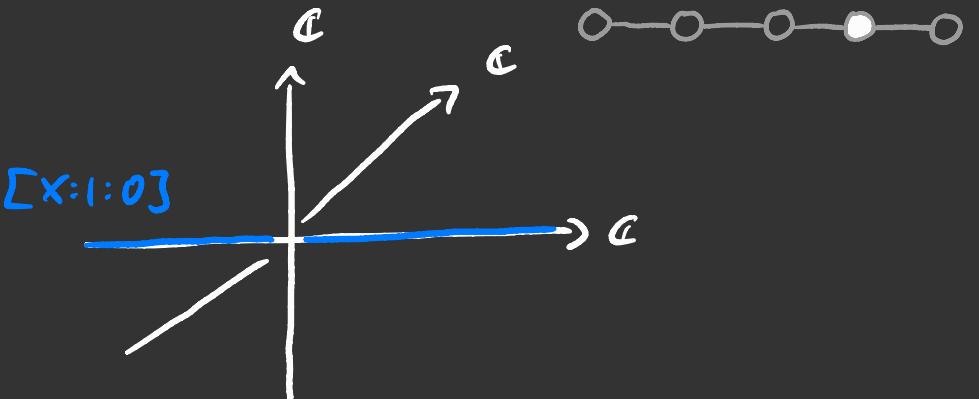
Theorem: X is type I iff $h: \mathrm{HF}_1(X) \rightarrow H_1(X; \mathbb{Z}_\omega)$ is an isomorphism.

And define:

X is type II iff $\forall \Sigma \subset X$ surface
 Σ or. $\Rightarrow \Sigma$ triv.

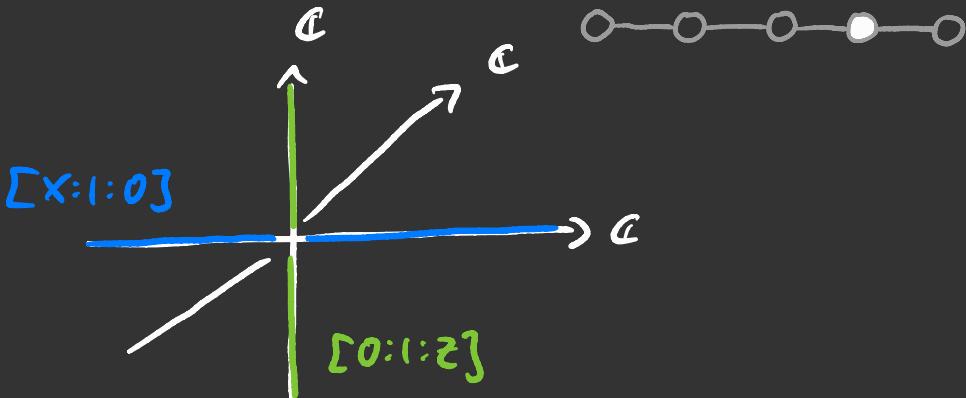
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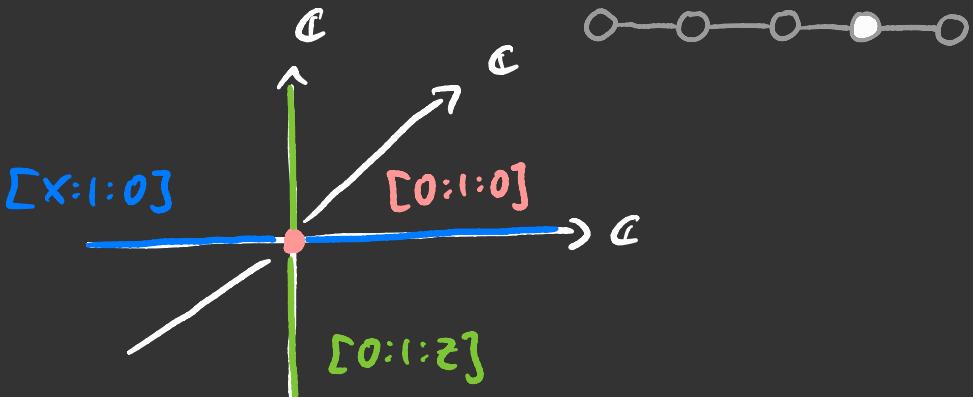


Examples

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$\leadsto \mathbb{C}P^1$ has odd self-intersection

$\Rightarrow \mathbb{C}P^2$ is of type I



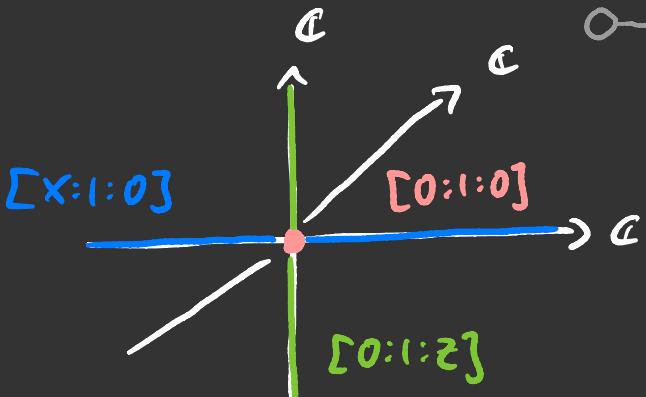
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Examples

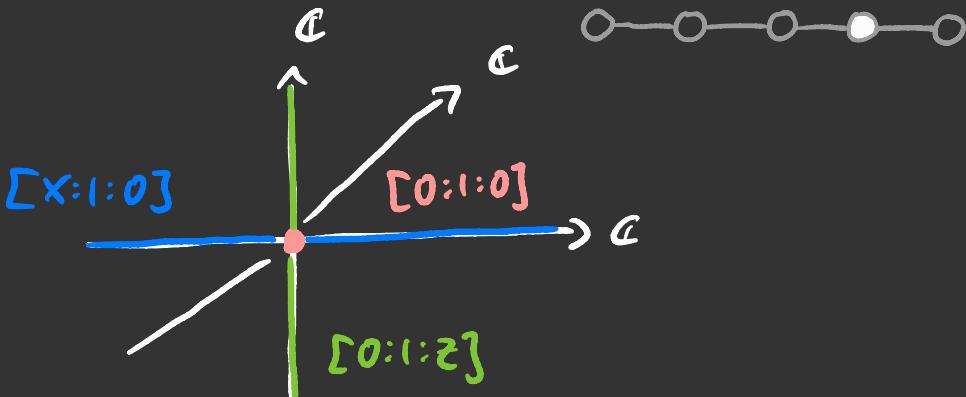
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2) $\mathbb{R}P^2 \subset \mathbb{R}P^{4k}$



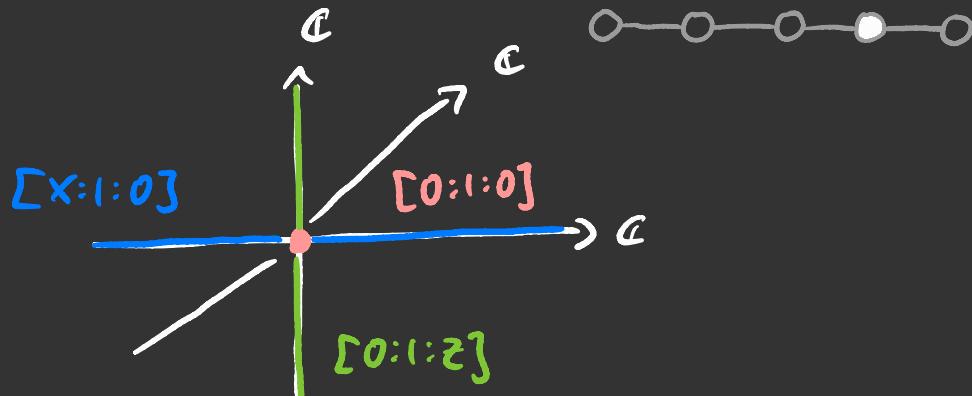
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2) $\mathbb{R}P^2 \subset \mathbb{R}P^{4k}$ $\xrightarrow{\text{char. classes}}$ $\mathcal{V}_{\mathbb{R}P^2}$ or. but non-triv.

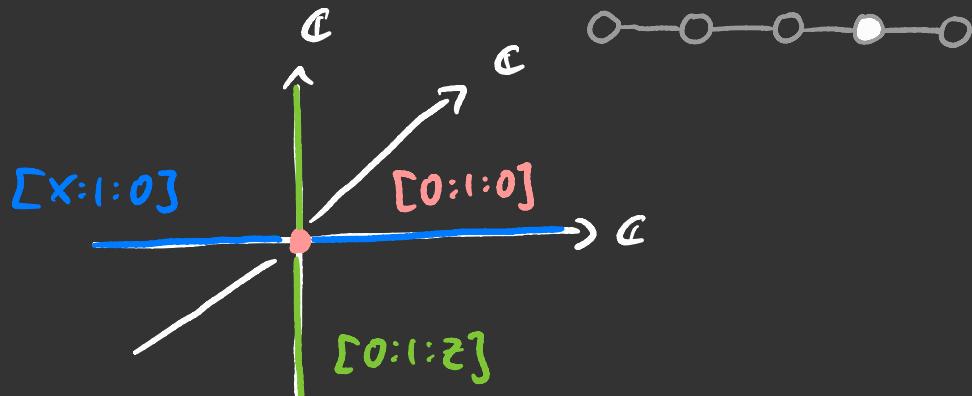
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$\leadsto [F_1(\mathbb{R}P^{4k})] \cong H_1(\mathbb{R}P^{4k}; \mathbb{Z}_2) \cong \overset{\curvearrowleft}{0}$ all circles \$c\$ with orientable \$\nu_c\$ are contractible

Result for type II



Theorem: If X is type II, then $F_1(x)$ fits into

$$0 \rightarrow Z_2 \longrightarrow F_1(x) \rightarrow H_1(x; Z_2) \rightarrow 0.$$

The extension is uniquely determined by $\omega_1^2(x) + \omega_2(x)$.

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$$0 \rightarrow Z_2 \longrightarrow F_1(x) \rightarrow H_1(x; \mathbb{Z}_2) \rightarrow 0.$$

The extension is uniquely determined by $\underbrace{\omega_1^2(x) + \omega_2(x)}_{\text{Pin}^- \text{-obstruction class!}}$.

Result for type II



Theorem: If X is type II, then $F_1(x)$ fits into

$$0 \rightarrow \mathbb{Z}_2 \longrightarrow F_1(x) \rightarrow H_1(X; \mathbb{Z}_2) \rightarrow 0.$$

The extension is uniquely determined by $\underbrace{\omega_1^2(x) + \omega_2(x)}_{\text{Pin}^- \text{-obstruction class!}}$.

In particular:

$$F_1(x) \cong H_1(X; \mathbb{Z}_2) \oplus \mathbb{Z}_2 \text{ iff } X \text{ is Pin}^-.$$



I Proof idea

Fact: Extension of

$$0 \rightarrow \mathbb{Z}_2 \rightarrow F_i(x) \rightarrow H_i(x; \mathbb{Z}_2) \rightarrow 0$$

is (uniquely) determined by hom.

$$\epsilon_x: \underbrace{\text{Tor}_2(H_i(x; \mathbb{Z}_2))}_{= \{ \sum c_j \in H_i(x; \mathbb{Z}_2) \mid 2c_j = 0 \}} \rightarrow \mathbb{Z}_2.$$



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Fact: Extension of

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Link to geom. of X :

- 1) $[c] \in \text{Tor}_2(H_1(x; \mathbb{Z}_2))$ circle.



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Link to geom. of X :

- 1) $[c] \in \text{Tor}_2(H_1(x; \mathbb{Z}_2))$ circle.
- 2) Endow c with normal framing \mathcal{L} .

I Proof idea



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- 1) $[c] \in \text{Tor}_2(H_1(x; \mathbb{Z}_2))$ circle.
- 2) Endow c with normal framing ℓ .
- 3) Then $z[c, \ell] = 0$ iff $\epsilon_X([c]) = 0$.



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Link to geom. of X :

- 1) $[c] \in \text{Tor}_2(H_1(x; \mathbb{Z}_2))$ circle.
- 2) Endow c with normal framing ℓ .
- 3) Then $z[c, \ell] = 0$ iff $\epsilon_X([c]) = 0$.
- 4) Build up extension via generators.



| Proof idea

$[c]_{t \in \text{Tor}_2(H_1(x_i z_j))}$ generator

| Proof idea



$[c]_{+v} \in \text{Tor}_2(H_1(x; \mathbb{Z}_v))$ generator

$\exists \beta : H_2(x; \mathbb{Z}_v) \longrightarrow \text{Tor}_2(H_1(x; \mathbb{Z}_v))$ surjective s.t.

$\beta([\Sigma]_v) = [c]_{+v}$ and c represents $w_1(\gamma_\Sigma)$

| Proof idea



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Prop: $z[c, v] = 0$ iff $w_1(\gamma_v) = 0$

| Proof idea



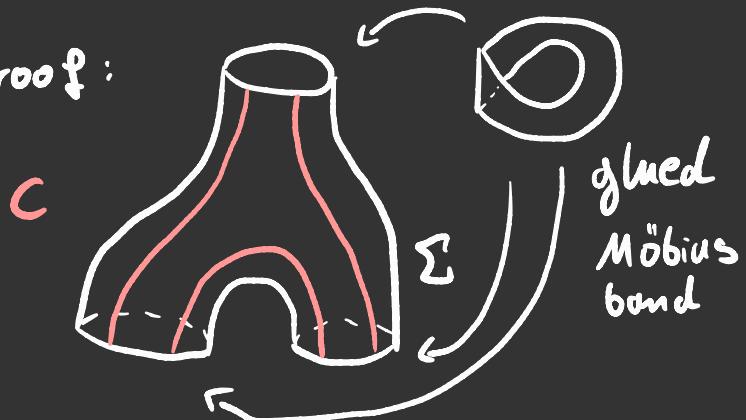
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Prop: $\exists [c, \rho] = 0$ iff $w_1(\gamma_\Sigma) = 0$

Proof:



| Proof idea



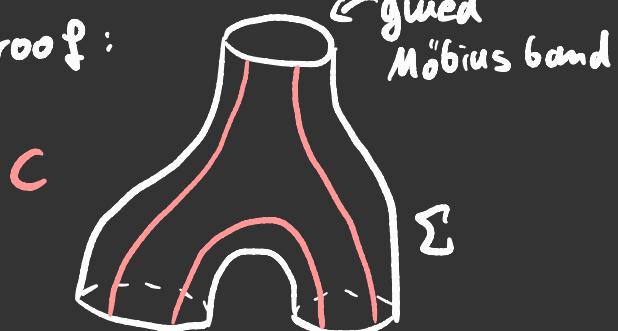
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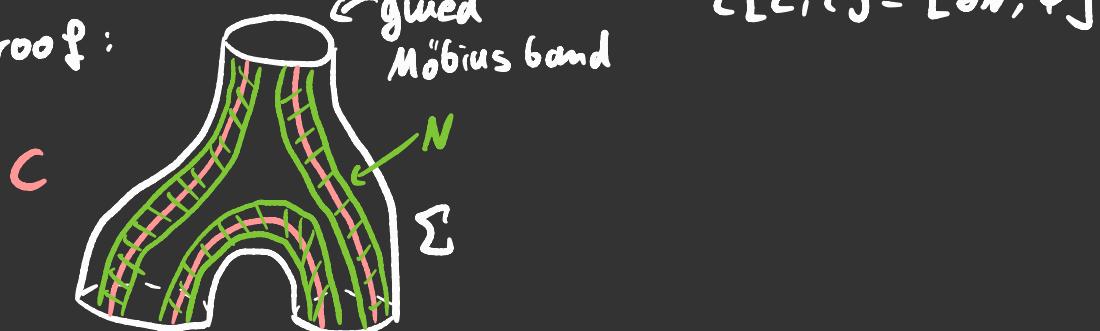
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Proof:



| Proof idea



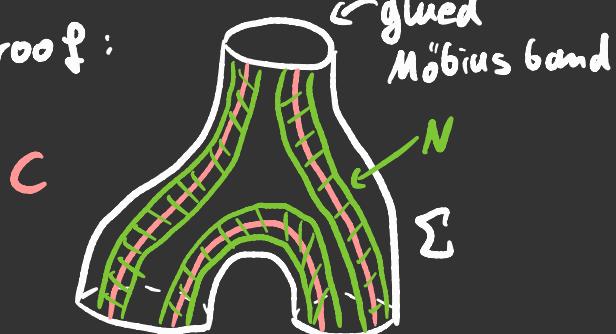
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$\beta([\Sigma]_2) = [C]_{+w}$ and C represents $w_1(\gamma_\Sigma)$

Prop: $\gamma [C, \psi] = 0$ iff $w_2(\gamma_\Sigma) = 0$

Proof:



$$\gamma [C, \psi] = [\partial N, \psi]$$

Now: $[\partial N, \psi] = 0$ iff ψ extends over $\Sigma - N$

| Proof idea



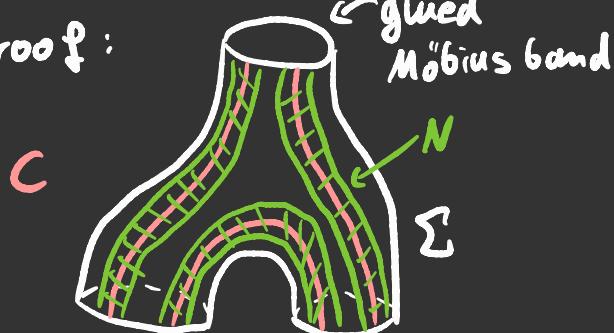
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Prop: $z[C, \varphi] = 0$ iff $w_2(\gamma_\Sigma) = 0$

Proof:



$$z[C, \varphi] = [\partial N, \psi]$$

Now: $[\partial N, \psi] = 0$ iff ψ extends over $\Sigma - N$

Obstruction theory:

ψ extends over $\Sigma - N$
 $w_2(\gamma_C) = 0$



| Example



Consider $X = \mathbb{R}P^{n+1}$ and $\mathbb{R}P^2 \subset \mathbb{R}P^{n+1}$

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$[\mathbb{R}P^2]$ generates $H_2(X; \mathbb{Z}_2)$ and $\text{Tor}_2(H_1(X; \mathbb{Z}_2)) = H_1(X; \mathbb{Z}_2)$.

$(n+1) \bmod 4$	0	1	2	3
$w_1(\gamma_{\mathbb{R}P^2})$	0	1	0	1
$w_2(\gamma_{\mathbb{R}P^2})$	1	1	0	0
type	I	II, not Pin^-	II, Pin^-	II, Pin^-
$\pi^n(\mathbb{R}P^{n+1})$	0	\mathbb{Z}_4	\mathbb{Z}_2	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$

Pin manifolds



X has Pin-structure, then

$$0 \rightarrow \mathcal{S}^{\text{Pin}}_1 \xrightarrow{\quad} \mathbb{F}_1(x) \rightarrow H_1(X; \mathbb{Z}_2) \rightarrow 0$$

| Pin manifolds



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$$0 \rightarrow \mathcal{S}^{\text{Pin}}_1 \longrightarrow \mathbb{F}_1(x) \longrightarrow H_1(X; \mathbb{Z}_2) \rightarrow 0$$

$\mathbb{Z}_2 \xleftarrow{\text{induced splitting map}}$

| Pin manifolds



X has Pin-structure, then

$$0 \rightarrow \mathcal{L}_i^{\text{Pin}} \longrightarrow \mathbb{F}_i(x) \longrightarrow H_i(x; \mathbb{Z}_2) \rightarrow 0$$

\mathbb{Z}_2 induced
splitting map

$$\text{Pin}(x) \times H^i(x; \mathbb{Z}_2) \xrightarrow{\text{act}} \text{Pin}(x)$$

Then:

$$\text{split}(x) \times \frac{H^i(x; \mathbb{Z}_2)}{\langle \omega_i(x) \rangle} \xrightarrow{\text{act}} \text{split}(x)$$

| Applications to vector bundles



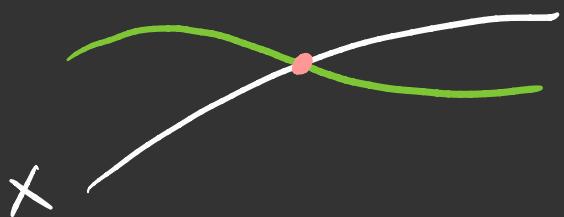
$\mathbb{R}^n \hookrightarrow E \longrightarrow X^{n+1}$ w/ orientation & spin structure

| Applications to vector bundles



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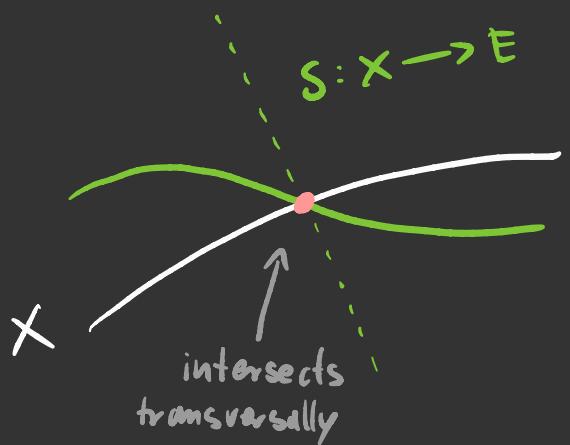
$s: X \rightarrow E$



| Applications to vector bundles



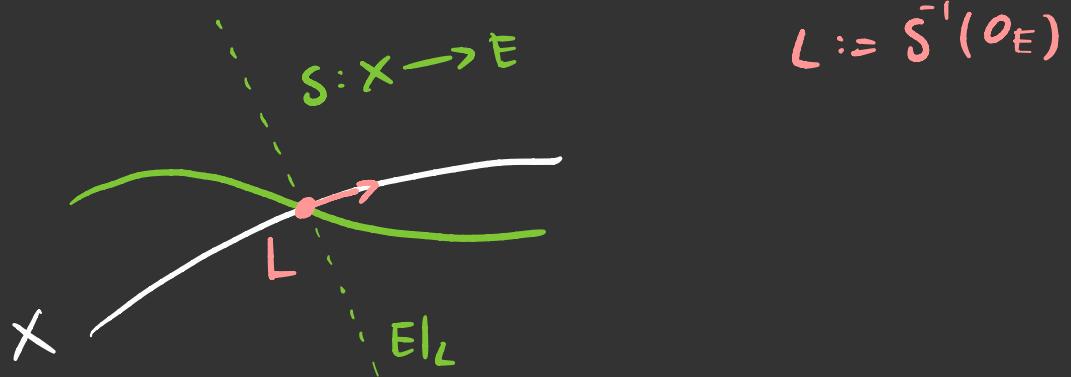
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| Applications to vector bundles



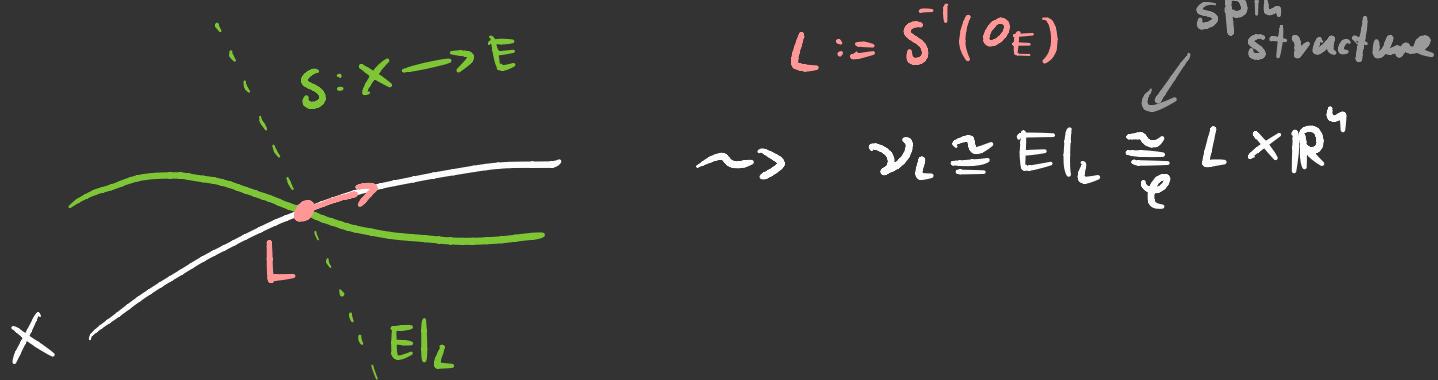
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Applications to vector bundles



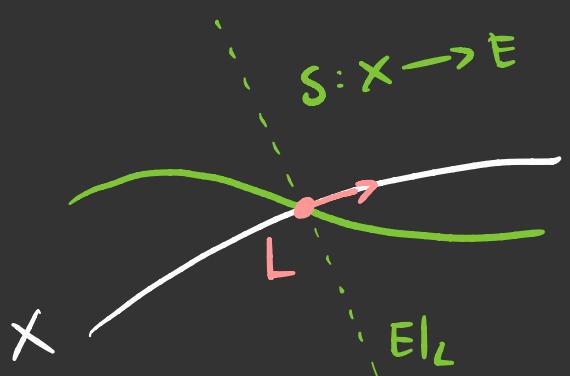
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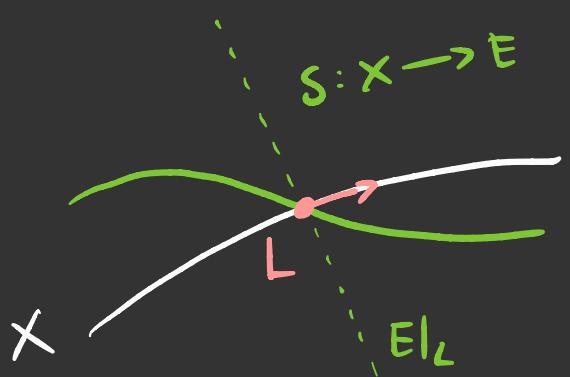


$$\begin{aligned} L &:= S^{-1}(0_E) && \text{spin structure} \\ \rightsquigarrow & \quad \mathcal{V}_L \cong E|_L \cong \varphi_L L \times \mathbb{R}^4 \\ \rightsquigarrow & \quad [L, \varphi] \in \mathrm{IF}_+(X) \end{aligned}$$

Applications to vector bundles



$\mathbb{R}^n \hookrightarrow E \longrightarrow X^{n+1}$ w/ orientation & spin structure



$$L := S^1(O_E)$$

$$\rightsquigarrow \mathcal{V}_L \cong E|_L \xrightarrow{\varphi} L \times \mathbb{R}^4$$

$$\rightsquigarrow [L, \varphi] \in \text{IF}_+(X)$$

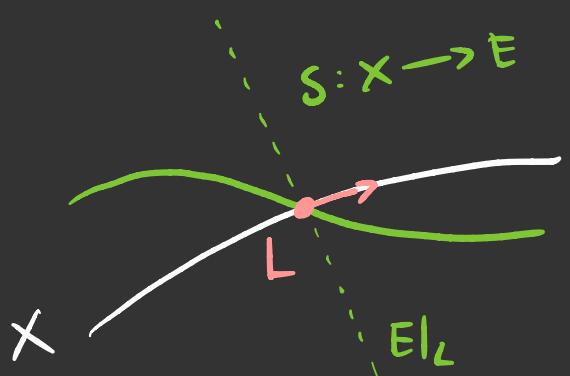
$$\hookrightarrow h([L, \varphi]) = \text{PD of Euler class } e(E)$$

spin
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| Applications to vector bundles



$\mathbb{R}^n \hookrightarrow E \longrightarrow X^{n+1}$ w/ orientation & spin structure



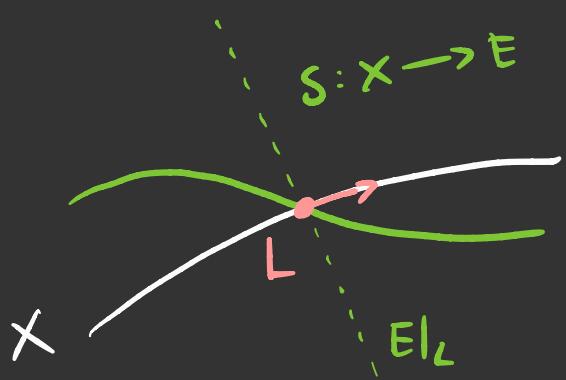
$$\begin{aligned} L &:= S^{-1}(0_E) && \text{spin structure} \\ \rightsquigarrow & \quad \nu_L \cong E|_L \cong L \times \mathbb{R}^4 \\ \rightsquigarrow & [L, \varphi] \in \mathrm{IF}_+(X) \\ \hookrightarrow & h([L, \varphi]) = \text{PD of Euler class } e(E) \end{aligned}$$

Theorem: E admits a non-vanishing section if and only if $[L, \varphi] = 0$.

Applications to vector bundles



$\mathbb{R}^n \hookrightarrow E \longrightarrow X^{n+1}$ w/ orientation & spin structure



$$L := S^1(0_E) \quad \text{spin structure}$$

$$\rightsquigarrow \mathcal{V}_L \cong E|_L \xrightarrow{\varphi} L \times \mathbb{R}^4$$

$$\rightsquigarrow [L, \varphi] \in \text{IF}_1(X)$$

$$\hookrightarrow h([L, \varphi]) = \text{PD of Euler class } e(E)$$

Theorem: E admits a non-vanishing section if and only if $[L, \varphi] = 0$.

Idea: Use null-bordism to "push" s away from zero (similar to Whitney trick).

| Applications to vector bundles



Corollary 1: If X is of type I, then
 E admits a non-vanishing section
iff $e(E) = 0$.

| Applications to vector bundles



Corollary 1: If X is of type I, then

E admits a non-vanishing section
iff $e(E) = 0$.

Corollary 2: If X is Pin⁻, then E admits
a non-vanishing section iff
 $e(E) = 0$ and $\partial\Sigma(\Sigma L, E) = 0$.

↑ splitting map
 $\pi: \mathbb{F}_1(x) \rightarrow \mathbb{Z}_2$

THANK YOU
FOR YOUR
ATTENTION!

