Lab 3

● Graded

1 Day, 21 Hours Late

Student

MJ Wu

Total Points

30 / 30 pts

Question 1

Q2: Matrix-Vector DFT

10 / 10 pts

✔ **– 0 pts** Correct

– 2.5 pts a Partially Correct

– 5 pts a Incorrect

– 2.5 pts b Partially Correct

– 5 pts b Incorrect

Question 2

Q3: The Fast Fourier Transform (FFT)

10 / 10 pts

✔ **– 0 pts** Correct

– 2.5 pts a Partially Correct

– 5 pts a Incorrect

– 2.5 pts b Partially Correct

– 5 pts b Incorrect

Question 3

Q4: Efficient Oscilloscope Calibration

10 / 10 pts

✔ **– 0 pts** Correct

– 2 pts a Incorrect

– 2 pts b Incorrect

– 2 pts c Incorect

– 2 pts d Incorrect

– 2 pts e Incorrect

No questions assigned to the following page.

EE 120 Lab 3: Practical Fourier Analysis

Signals and Systems at UC Berkeley

Acknowledgements:

- **Fall 2018** (Beta): Dominic Carrano
- **Spring 2019** (v1.0): Dominic Carrano, Babak Ayazifar
- **Fall 2019** (v2.0): Dominic Carrano
- **Spring 2020** (v3.0): Dominic Carrano
- **Fall 2020** (v3.1): Anmol Parande

In [2]:
```
from scipy import signal
from scipy.io.wavfile import read
from IPython.display import Audio
from math import ceil, floor
import lab3_helper
import numpy as np
import timeit, time
import matplotlib.pyplot as plt
%matplotlib inline
```

Background

Recap: Why Fourier Analysis Matters

You've already seen that complex exponentials are a key player in signals and systems for two reasons. In particular, a complex exponential at frequency ω:

1. (**The signals perspective**) Corresponds to a wave or oscillator with frequency ω.
2. (**The systems perspective**) Is an eigenfunction of any LTI system, with the (typically, complex) eigenvalue given by the system's frequency response evaluated at ω.

Perhaps unsurprisingly, then, *Fourier analysis*, which is a set of techniques for decomposing signals as linear combinations of complex exponentials, is incredibly useful. We can make use of it to study the frequency composition of a signal as suggested by (1), or analyze and even implement LTI systems as suggested by (2). In this lab, we'll work through an example of how both are done in practice.

No questions assigned to the following page.

Fourier Analysis on a Computer

We have nearly endless computational power at our fingertips today, so it's only natural that we seek a way to add Fourier analysis to our ever-expanding Python signals and systems toolkit. You've already seen in the first couple labs that all signals we work with on computers are discrete and have finite duration N, enabling a convenient representation as a length N numpy array. Outside these N samples, we assume the signal is zero. **The question, then, is how do we digitally take a Fourier Transform of the signal, given this array representation?**

To answer this question, let's take a look at the four main flavors available to us—summarized in the table below—and see if we can adapt any to fit our needs. As a reminder, FS is an abbreviation for Fourier Series, FT for Fourier Transform.

	Discrete time (DT)	Continuous Time (CT)
Periodic	DTFS	CTFS
Not necessarily periodic	DTFT	CTFT

Since we operate in DT on a computer, working with sets of bits, the CTFT and CTFS are both ruled out. Our signals are typically aperiodic, so the most natutral choice seems like the DTFT; let's try computing it. The analysis equation would reduce from a sum over $n = -\infty$ to ∞ to a sum over $n = 0$ to $N - 1$ due to our signal being of finite duration, giving

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-i\omega n} = \sum_{n=0}^{N-1} x(n)e^{-i\omega n}$$

but this still presents an issue: ω can be any value in the interval $[0, 2\pi)$, requiring us to compute an uncountably infinite number of values. Instead, we use the *Discrete Fourier Transform (DFT)*

$$X[k] = \sum_{k=0}^{N-1} x(n)e^{-i\frac{2\pi}{N}kn}$$

which corresponds to *sampling* the DTFT at N evenly spaced frequencies in the range $[0, 2\pi)$. This idea that the DFT corresponds to sampling the DTFT is clear from the formulae: the DTFT analysis equation (for a duration N signal) evaluated at the frequency $\omega = 2\pi k/N$ yields the expression for the kth DFT coefficient.

In total, to compute the signal's N*-point DFT*, we calculate N samples of the DTFT

No questions assigned to the following page.

$$\{X[k]\}_{k=0,1,2,\ldots,N-1} = X(\omega)|_{\omega=\frac{2\pi}{N}k,\ k=0,1,2,\ldots,N-1}$$
$$= \left\{X(0), X\left(\frac{2\pi}{N}\right), X\left(2\cdot\frac{2\pi}{N}\right), \ldots, X\left((N-1)\cdot\frac{2\pi}{N}\right)\right\}$$

and use this as our computer-ready Fourier Transform. We can't get the entire DTFT, unfortunately, but we can increase N by *zero-padding*, or appending some of the implicit zeros to the end of our signal, until the number of samples we have of the DTFT is sufficient. **This is how we can use the Fourier Transform in practice!**

We're done, right?

It seems like our problem is solved: if we want to take the Fourier Transform of a signal in Python, we can just compute its N-point DFT through the analysis equation. As you'll soon see, however, directly using the analysis equation is limitingly slow, yielding an $O(N^2)$ (https://en.wikipedia.org/wiki/Big_O_notation) algorithm.

In this lab, we'll see how clever manipulation of the DFT analysis equation to eliminate redundant computations can bring that down to $O(N \log N)$, and just how much of a difference that makes. The latter algorithm is known as the *Fast Fourier Transform (FFT)*. It is **not** a different transform, simply a particular algorithm for implementing the DFT.

Q1: The Naive DFT [Optional]

To motivate the need for a *fast* Fourier Transform, we'll compute the DFT of a clip of music, a bottleneck operation in Shazam (https://en.wikipedia.org/wiki/Shazam_(application)), which calculates and compares DFTs of songs to identify them.

Before we can do that, though, we need to actually implement the DFT itself.

No questions assigned to the following page.

Q1a: DFT Code [Optional]

Your job is to fill in the function `dft` below according to the docstring.

We can compute an N-point DFT using the DFT analysis equation

$$X[k] = \sum_{n=0}^{N-1} x(n) e^{-i \frac{2\pi}{N} nk}$$

for $k \in \{0, 1, 2, \ldots, N-1\}$. We collect the coefficients $X[0], X[1], \ldots, X[N-1]$ in a numpy array and call this sequence the N-point DFT of x.

Do not use any vectorization, we're saving that for the next question. Implement this in the simplest way possible, using for loops.

In [3]:
```
def dft(x):
    """
    Compute the N-point Discrete Fourier Transform of x based on the ana
    the length of the numpy array x.

    In your array X that you output, X[k], the kth element (k = 0, 1, ..
    should be the value of the kth DFT coefficient.

    **Hint 1:** The function call `np.exp(z)` returns $e^z$, where $z$ i
    **Hint 2:** In Python, `1j` is used to represent the complex number
                typically denoted as $i$ or $j$ depeding on the notation
    """
    N = len(x)
    X = np.zeros(N, dtype=np.complex128) # output array - have to declar
    ## TODO your code here ##



    ## TODO your code here ##
    return X
```

Let's do a couple quick sanity checks on `dft` to make sure it returns the correct results. Run the cells below to check your work. Make sure all tests pass before you move on.

No questions assigned to the following page.

In [4]:
```
lab3_helper.run_fft_tests(dft)
```

```
Test 1 passed: False
Test 2 passed: False
Test 3 passed: True
Test 4 passed: False
Test 5 passed: False
```

Q1b: DFT of Music Clip [Optional]

Now that we have a way to compute the DFT, let's put our method to the test! Run the cell below to load in a snippet of *Mirrors* by Justin Timberlake, which we'll use for benchmarking.

In [5]:
```
fs, song = read("mirrors_intro.wav")
song = np.swapaxes(song, 0, 1)
Audio(song, rate=fs)
```

Out[5]:

When we listen to the song, we hear both the left and right audio channels superimposed. The variable `song` is actually a 2 row matrix, with each row a distinct signal for each audio channel. We'll average these together to obtain a single 1D signal for analysis.

In [6]:
```
# Note: The "10*fs:" crop grabs samples starting 10 seconds in
left = song[0, 10*fs:]
right = song[1, 10*fs:]
mirrors = (left + right) / 2
```

Now to put our DFT function to the test! Typically, Shazam would split the signal into blocks, compute FFTs of those blocks, and compare against a database to match the song. Let's see how long it takes to compute the DFT for the first:

1. 512 samples.
2. 1024 samples.
3. 2048 samples.
4. 4096 samples.

Will our DFT be fast enough to handle Shazam's real-time computation needs? Run the cell to find out! (**Note: This will probably take a few minutes to run**)

No questions assigned to the following page.

In [7]:
```
mirrors_512 = mirrors[:512]
mirrors_1024 = mirrors[:1024]
mirrors_2048 = mirrors[:2048]
mirrors_4096 = mirrors[:4096]

t0 = time.time() # initial time
dft(mirrors_512)
t1 = time.time()
print("Took {0} sec for 512-point DFT".format(round(t1 - t0, 5)))

dft(mirrors_1024)
t2 = time.time()
print("Took {0} sec for 1024-point DFT".format(round(t2 - t1, 5)))

dft(mirrors_2048)
t3 = time.time()
print("Took {0} sec for 2048-point DFT".format(round(t3 - t2, 5)))

dft(mirrors_4096)
t4 = time.time()
print("Took {0} sec for 4096-point DFT".format(round(t4 - t3, 5)))
```

```
Took 0.00011 sec for 512-point DFT
Took 0.00104 sec for 1024-point DFT
Took 0.00045 sec for 2048-point DFT
Took 0.0004 sec for 4096-point DFT
```

Is it practical? [Optional]

We'll use the 4096-point DFT as the standard for benchmarking. Truthfully, we couldn't find any online resources indicating what DFT block size the production version of Shazam uses. However, we dug through the source code of Dejavu (https://github.com/worldveil/dejavu), an open source version, and found that they use 4096 (see the `DEFAULT_WINDOW_SIZE` definition in this (https://github.com/worldveil/dejavu/blob/master/dejavu/fingerprint.py) file), so this seems like a good size to test with.

Q: Shazam usually needs 3-5 seconds of data before confidently making a decision about what song is being played. Our data was sampled at 44.1 kHz, so we'd have to separately compute around fifty 4096-point DFTs before being done (since $4096 \times 50 \approx 44100 \times 5$).

Extrapolating from the data you got in the cell above for 4096, roughly how long it take to compute those fifty 4096-point DFTs, so Shazam can identify our song?

A: (TODO)

No questions assigned to the following page.

Q: If Shazam took as long to classify songs as you found in the question immediately preceding this one, would you still use it?

A: (TODO)

Can't we just use 512?

It's tempting to think we can just use a 512-point DFT instead of 4096-point one, since it's faster. After all, why would Dejavu or Shazam bother to split songs into 4096-sample chunks instead of 512-sample chunks?

In many applications that rely on DFTs, Shazam included, the DFT length has a serious impact on functionality. EE 123 gives a more in-depth treatment of this problem, but the upshot is that shorter DFTs mean less time-domain information per block of samples, and past a certain point, you don't have enough data to do anything meaningful.

As a brief example, let's consider music, since that's what we care about from a Shazam perspective. Typically, songs are sampled at 44.1 kHz or 48 kHz (for the song we have here, it was 44.1). 512 datapoints per DFT represents only $512/44100 \approx .012$ seconds of data per block, whereas 4096 gives us $4096/44100 \approx .1$ seconds of data per block, which is a much better tradeoff.

Q1c: DFT Computational Analysis [Optional]

If you'd like to fully understand why the DFT runtime is $O(N^2)$, take a crack at answering the questions below. This part can be done entirely by inspecting the DFT analysis equation, and does not require a working implementation of the previous parts. You should answer these with respect to the analysis equation, not your specific implementation, although the two should match.

Q: How many complex multipications are performed by `dft` in computing a *single* DFT coefficient for a size N input?

Note: Multiplying two complex numbers counts as a single multiplication. For example, $x(1) \cdot e^{-i\frac{2\pi}{N}}$ counts as one complex multiplication, as each element of $x(n)$ is a complex number, in general.

A: (TODO)

No questions assigned to the following page.

Q: How many complex additions (adding two complex numbers) are performed by `dft` in computing a **single** DFT coefficient for a size N input?

A: (TODO)

Q: Adding up your previous two answers, how many total complex number operations (complex additions and complex multiplications together) are performed in calculating a single DFT coefficient for an input of size N?

A: (TODO)

Q: Multiplying your previous by the number of coefficients, N, what is the total number of complex number operations performed in calculating the entire N-point DFT?

A: (TODO)

Q: Expressing your previous answer in Big-O notation, what is the asymptotic runtime of the function `dft` for calculation of the DFT via the analysis equation?

A: (TODO)

Question assigned to the following page: 1

Q2: Matrix-Vector DFT

Directly implementing the DFT as a set of for loops much too slow for practical applications. Fortunately, our for loops are just a series of dot products, and we can implement the DFT as a matrix-vector multiplication.

The DFT analysis equation can be written as the matrix-vector equation $\vec{X} = M\vec{x}$, where:

$$\underbrace{\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ \vdots \\ X[N-1] \end{bmatrix}}_{\vec{X}} = \underbrace{\begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & e^{-i\frac{2\pi}{N}} & e^{-i\frac{2\pi}{N}2} & \dots & e^{-i\frac{2\pi}{N}(N-1)} \\ 1 & e^{-i\frac{2\pi}{N}2} & e^{-i\frac{2\pi}{N}4} & \dots & e^{-i\frac{2\pi}{N}2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & e^{-i\frac{2\pi}{N}(N-1)} & e^{-i\frac{2\pi}{N}2(N-1)} & \dots & e^{-i\frac{2\pi}{N}(N-1)(N-1)} \end{bmatrix}}_{M} \underbrace{\begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ \vdots \\ x(N-1) \end{bmatrix}}_{\vec{x}}$$

Note that the entry of M at row n, column k is $e^{-i\frac{2\pi}{N}nk}$, where $n, k \in \{0, 1, 2, \dots, N-1\}$.

Q2a: Matrix-Vector DFT Code

Implement `dft_mtx_vec` below, which constructs the $N \times N$ DFT matrix M (where N is the length of the input vector) and returns the matrix-vector product $M\vec{x}$, which yields the result of applying the DFT analysis equation.

Hint: After determining N, the code `n = np.arange(N); k = n.reshape((N, 1)); idx = n * k` will generate an $N \times N$ matrix `idx` such that `idx[m][l] = m * l`. See how you can use this matrix to then easily compute the DFT matrix.

In [16]:
```
def dft_matrix_vector(x):
    ### TODO - Your code here
    N = len(x)
    n = np.arange(N)
    k = n.reshape((N, 1))
    idx = n * k
    M = np.exp(idx * (2 * np.pi / N)*-1j)
    DFT = np.dot(x,M)
    return DFT
```

Let's run the same sanity checks.

Question assigned to the following page: 1

In [17]:
```
lab3_helper.run_fft_tests(dft_matrix_vector)
```

```
Test 1 passed: True
Test 2 passed: True
Test 3 passed: True
Test 4 passed: True
Test 5 passed: True
```

Q2b: Benchmarking the Matrix-Vector DFT

Same setup: we want to see how long it takes to compute 512, 1024, 2048, and 4096-point DFTs of our song, now using our improved method. Run the cell below to benchmark `dft_mtx_vec` .

In [19]:
```
mirrors_512 = mirrors[:512]
mirrors_1024 = mirrors[:1024]
mirrors_2048 = mirrors[:2048]
mirrors_4096 = mirrors[:4096]

t0 = time.time() # initial time
dft_matrix_vector(mirrors_512)
t1 = time.time()
print("Took {0} sec for 512-point DFT".format(round(t1 - t0, 5)))

dft_matrix_vector(mirrors_1024)
t2 = time.time()
print("Took {0} sec for 1024-point DFT".format(round(t2 - t1, 5)))

dft_matrix_vector(mirrors_2048)
t3 = time.time()
print("Took {0} sec for 2048-point DFT".format(round(t3 - t2, 5)))

dft_matrix_vector(mirrors_4096)
t4 = time.time()
print("Took {0} sec for 4096-point DFT".format(round(t4 - t3, 5)))
```

```
Took 0.01886 sec for 512-point DFT
Took 0.06333 sec for 1024-point DFT
Took 0.21407 sec for 2048-point DFT
Took 1.17536 sec for 4096-point DFT
```

Is it practical?

Questions assigned to the following page: 1 and 2

Q: Extrapolating from the data you got in the cell above for 4096, roughly how long it take to compute the fifty 4096-point DFTs needed for Shazam to identify our song?

A: It took about 1.1 sec for 4096-point DFT and hence it would take between 50 sec and 50 sec to compute the fifty 4096-point DFTs needed for Shazam to identify the song.

```
We're doing much better now, but can still improve some more.
```

Q3: The Fast Fourier Transform (FFT)

Matrix-vector DFT does pretty well, but we can still do better.

The strategy in developing a faster method, which is formally called the *Decimation in Time Fast Fourier Transform* algorithm, will be to split up the signal $x(n)$ into successively smaller signals, take DFTs of those smaller signals, and build up the full result from these smaller parts. If you've taken CS 170, you've heard this (fittingly) referred to as a *divide-and-conquer* algorithm.

Powers of 2

Throughout this question, we'll assume that the signal duration, N, is a power of two so that we can keep splitting the signal in half until we have N separate duration 1 signals. If not, we can just pad on implicit zeros until the length is a power of two (which is what signal processing practitioners often do anyways), so this is a fine assumption.

Dividing

Our goal is to divide the task of computing the N-point DFT of $x(n)$ into computing two $N/2$-point DFTs:

- One over the even-indexed entries $x(0), x(2), x(4), \ldots, x(N-2)$.
- One over the odd-indexed entries $x(1), x(3), x(5), \ldots, x(N-1)$.

Remember, N is a power of two, so both it and $N-2$ are even. This means $N-1$, the index of the last signal value (since we start counting at zero), is odd. We can use this division to split up the analysis equation as

$$X[k] = \sum_{n=0}^{N-1} x(n) e^{-i\frac{2\pi}{N}nk} = \sum_{n \text{ even}} x(n) e^{-i\frac{2\pi}{N}nk} + \sum_{n \text{ odd}} x(n) e^{-i\frac{2\pi}{N}nk}$$

which will make things a bit easier to work with.

Question assigned to the following page: 2

Conquering

Writing n even and n odd as our summation indices is a bit annoying. Let's reindex things so we have contiguous indices rather than ones with gaps (e.g., $0, 2, 4, \ldots$) in them.

To generate a contiguous set of $N/2$ integers to sum over for the evens, note that

$$n = 0, 2, 4, \ldots, N-2 = 2 \cdot \left(0, 1, 2, \ldots, \frac{N-2}{2}\right) = 2 \cdot \underbrace{\left(0, 1, 2, \ldots, \frac{N}{2} - 1\right)}_{m}$$

and $N/2$ is an integer since N is a power of two.

Similarly, to generate the odds, we have

$$n = 1, 3, 5, \ldots, N-1 = (0, 2, 4, \ldots, N-2) + 1 = 2 \cdot \underbrace{\left(0, 1, 2, \ldots, \frac{N}{2} - 1\right)}_{m} + 1$$

This provides a key insight: in both cases, if we have a summation running over $m = 0, 1, 2, \ldots, \frac{N}{2} - 1$, which would correspond to an $N/2$-point DFT, to generate even values of n, we can use $n = 2m$. Similarly, to generate the odds, we use $n = 2m + 1$.

Equipped with this, we can write

$$\begin{aligned} X[k] &= \sum_{m=0}^{N/2-1} x(2m) e^{-i\frac{2\pi}{N}(2m)k} + \sum_{m=0}^{N/2-1} x(2m+1) e^{-i\frac{2\pi}{N}(2m+1)k} \\ &= \left[\sum_{m=0}^{N/2-1} x(2m) e^{-i\frac{2\pi}{N/2}mk}\right] + e^{-i\frac{2\pi}{N}k}\left[\sum_{m=0}^{N/2-1} x(2m+1) e^{-i\frac{2\pi}{N/2}mk}\right] \end{aligned}$$

Defining

$$a(n) = [x(0), x(2), x(4), \ldots, x(N-2)], \quad b(n) = [x(1), x(3), x(5), \ldots, x(N-1)]$$

and $A[k]$, $B[k]$ as their $N/2$-point DFTs, respectively, we see that

$$\boxed{X[k] = A[k] + e^{-i\frac{2\pi}{N}k} B[k], \quad k = 0, 1, 2, \ldots, N-1}$$

We can now compute the N-point DFT X from the two $N/2$-point DFTs A, B using this formula!

This provides a recursive method for computing the DFT:

Question assigned to the following page: 2

- Construct a, b.
- Recursively compute their $N/2$-point DFTs A, B.
- For the kth DFT coefficient, $k = 0$ to $k = N - 1$, compute $X[k] = A[k] + e^{-i\frac{2\pi}{N}k} B[k]$.

Important implementation detail: When indexing into A, B, the index k must be interpreted $\mod N/2$ since these are $N/2$-point DFTs, not N-point DFTs.

Q3a: FFT Code

Implement the function `my_fft` according to the docstring. **If the length of the input array `x` is not a power of two, just call `dft_matrix_vector` on it.** You may, but are not required to, use the provided `is_power_of_two` function.

The FFT algorithm often can often look daunting to students when they learn about it the first time, this lab hopes to show that the its implementation is relatively simple. For reference, the staff solution has only 10 lines of code between the `TODO` markers.

In [92]:
```
def is_power_of_two(n):
    """
    Return true if n is a power of 2, else false, assuming n is a positi
    Adapted from https://stackoverflow.com/a/57027610.
    """
    return n & (n - 1) == 0
```

Question assigned to the following page: 2

In [95]:
```
def my_fft(x):
    """
    Uses the decimation in time FFT algorithm to compute an N-point DFT
    where N is the length of x.

    Hint 1: The FFT is a recursive algorithm, and the base case is N=1,
            when you only have one signal value and have to compute one

            What does the analysis equation tell you that this one DFT c
            X[0], is in terms of the one signal value, x[0]?

    Hint 2: Python's slicing features - check out https://stackoverflow.
            be very useful for grabbing the even and odd-indexed compone
    """
    N = len(x)
    X = np.zeros(N, dtype=np.complex128)

    ## TODO - your FFT implementation here ##
    if is_power_of_two(N) == False:
        return dft_matrix_vector(x)
    elif N == 1:
        return x
    even, odd = x[::2], x[1::2]
    a = my_fft(even)
    b = my_fft(odd)
    for i in range(N//2):
        X[i] = b[i] * (np.exp(-1j*i*(2*np.pi/N))) + a[i]
        X[i + N//2] = a[i] - b[i] * (np.exp(-1j*i*(2*np.pi/N)))

    ## TODO ##

    return X
```

Now for some sanity tests.

If you're failing any tests: remember, $A[k]$ and $B[k]$ only have $N/2$ elements, but we use them to compute a total of N DFT coefficients. We already discussed above how to deal with this issue.

In [96]:
```
lab3_helper.run_fft_tests(my_fft)
```

```
Test 1 passed: True
Test 2 passed: True
Test 3 passed: True
Test 4 passed: True
Test 5 passed: True
```

Question assigned to the following page: 2

Q3b: Benchmarking our Homemade FFT

Let's see how our from-scratch FFT does. For full credit, it should blow your Q1 implementation out of the water (the runtimes won't be close), and be faster than your Q2 implementation.

In [42]:
```
mirrors_512 = mirrors[:512]
mirrors_1024 = mirrors[:1024]
mirrors_2048 = mirrors[:2048]
mirrors_4096 = mirrors[:4096]

t0 = time.time() # initial time
my_fft(mirrors_512)
t1 = time.time()
print("Took {0} sec for 512-point DFT".format(round(t1 - t0, 5)))

my_fft(mirrors_1024)
t2 = time.time()
print("Took {0} sec for 1024-point DFT".format(round(t2 - t1, 5)))

my_fft(mirrors_2048)
t3 = time.time()
print("Took {0} sec for 2048-point DFT".format(round(t3 - t2, 5)))

my_fft(mirrors_4096)
t4 = time.time()
print("Took {0} sec for 4096-point DFT".format(round(t4 - t3, 5)))
```

```
Took 0.02385 sec for 512-point DFT
Took 0.04749 sec for 1024-point DFT
Took 0.0788 sec for 2048-point DFT
Took 0.19009 sec for 4096-point DFT
```

Q: Extrapolating from the data you got in the cell above for 4096, roughly how long it take to compute the fifty 4096-point DFTs needed for Shazam to identify our song?

A: It took about 0.2 sec for 4096-point DFT, and hence it would roughly take 10 sec to compute the fifty 4096-DFTs needed for Shazam to indentify our song.

Let's see how NumPy's FFT function does, since in practice, this is what we'd want to use.

Question assigned to the following page: 2

In [77]:
```
t0 = time.time()
np.fft.fft(mirrors_512)
t1 = time.time()
print("Took {0} sec for 512-point DFT".format(round(t1 - t0, 10)))

np.fft.fft(mirrors_1024)
t2 = time.time()
print("Took {0} sec for 1024-point DFT".format(round(t2 - t1, 10)))

np.fft.fft(mirrors_2048)
t3 = time.time()
print("Took {0} sec for 2048-point DFT".format(round(t3 - t2, 10)))

np.fft.fft(mirrors_4096)
t4 = time.time()
print("Took {0} sec for 4096-point DFT".format(round(t4 - t3, 10)))
```

```
Took 0.0002121925 sec for 512-point DFT
Took 0.0004308224 sec for 1024-point DFT
Took 0.0003499985 sec for 2048-point DFT
Took 0.0014281273 sec for 4096-point DFT
```

Q: Roughly how long it take to compute the fifty 4096-point DFTs needed for Shazam to identify our song, using NumPy's library grade FFT implementation?

A: It took about 0.0014 sec for 4096-point DFT, and hence it would roughly take 0.07 sec to compute the fifty 4096-DFTs needed for Shazam to indentify our song.

(Optional) Q3c: FFT Computational Analysis

We've experimentally seen that the FFT is much faster than the DFT. If you'd like a step by step walkthrough of how to formally prove the algorithm's $O(N \log N)$ runtime, have a go at this question. This type of algorithmic analysis is taught in CS 61B and used throughout CS 170, so it should be accessible with CS 61AB under your belt. It is optional, however, so don't feel the need to fully understand it for EE 120 purposes.

Q: The FFT is a recursive algorithm, dividing its input size in half with each recursive call. When we compute an N-point FFT, assuming N is a power of two, how many *layers* of recursion are there? (Equivalently, how many times can N be divided by two before it becomes 1, the base case?) **Your answer should be a function of N.**

A: (TODO)

Question assigned to the following page: 2

Q: We know each recursive call to the FFT produces two more recursive calls (assuming we haven't hit the base case). At the kth layer of the recursion, $k = 0, 1, \ldots, ?$, where $?$ is your answer to the previous question, what is the total number of recursive calls? To clarify, $k = 0$ corresponds to the initial call, where we have only one recursive call. At $k = 1$ we have two recursive calls, spawned by the initial call, and so on. **Your answer should be a function of k.**

A: (TODO)

Q: Consider a specific call to FFT at layer k of the recursion. What is the size of the input to FFT at this layer? **Your answer should be a function of k and N.** (Hint: We keep halving our input size each time, going from N to $N/2$ to $N/4$ and so on - how does this generalize for an arbitrary $k = 0, 1, \ldots ?$)

A: (TODO)

Q: Note that at any given layer, the total amount of work we do (other than the recursion) is a linear function of our input size, since the only other work done is combining the recursion results and computing M DFT coefficients, where M is the input size. So, the total work done (as a function of N), obtained by summing over the work done by all recursive calls, can be computed as:

$$W(N) = \sum_{k=0}^{\text{Number of layers} - 1} (\text{Number of recursive calls of size } k) \cdot (\text{Work done by a recursive call of size } k)$$

which simplifies to

$$W(N) = \sum_{k=0}^{a-1} b(k)c(N, k)$$

where a is your answer to the first question, $b(k)$ is your answer to the second question, and $c(N, k)$ is your answer to the third question. Plug in your answers and obtain a simplified expression for $W(N)$.

(Hint: $b(k)c(N, k)$ should cancel nicely to be independent of k.)

A: (TODO)

Questions assigned to the following page: 2 and 3

Q: Express your previous answer in Big-O notation, providing the asymptotic runtime of the FFT as a function of its input size N. Does this runtime grow more slowly than that of the naive DFT?

A: (TODO)

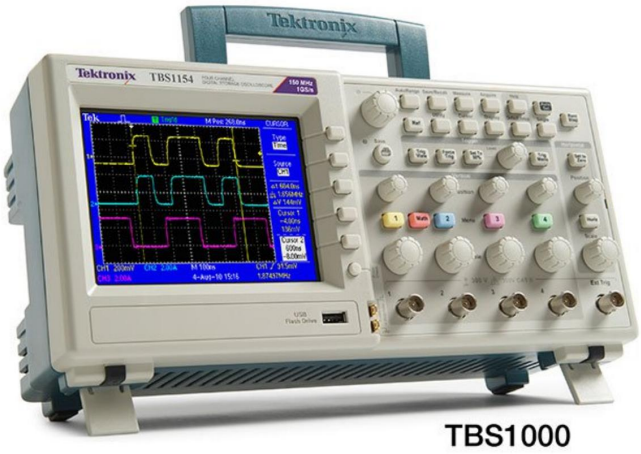
Q4: Efficient Oscilloscope Calibration

In Q1-Q3, you saw the Fourier Transform's use in analyzing signal frequency content, and the necessity of a fast algorithm for doing so. Now, we'll turn our attention to implementing LTI systems (i.e., filters) with it. Specifically, we'll use the FFT to devise an efficient algorithm for cross-correlation (https://en.wikipedia.org/wiki/Cross-correlation), and make use of it for calibrating measured signals.

Question assigned to the following page: 3

Motivation

An *oscilloscope*, shown below (image credit: Tektronix (https://www.tek.com/oscilloscope/tbs1000-digital-storage-oscilloscope)), is the tool of choice for measuring electronic signals and testing equipment. You might have worked with one in courses such as EE 16AB or Physics 111A.



On the bottom right of the image, there is a row of five metallic ports. The first four are inputs for a separate measurement *channel* of the scope, which are juxtaposed on the scope's display (here, in yellow, blue, and purple).

In many cases, we measure the same signal on multiple channels (say, at different points in a circuit) and want to see what differences, if any, exist. However, this poses a challenge: the signals may be temporally offset from each other, so we need to align them before we can do any meaningful comparision. For example, this often happens because the two oscilloscope cables have different lengths—one signal has to travel through a physically longer path to the scope, and thus shows up later in time.

In this question, we'll develop an efficient algorithm to:
1) Compute the number of samples one signal lags behind another by, and
2) Align the signals by shifting by the offset.

The second item is pretty easy to accomplish—NumPy has functions for doing this—so most of our work will be in handling the first item.

Question assigned to the following page: 3

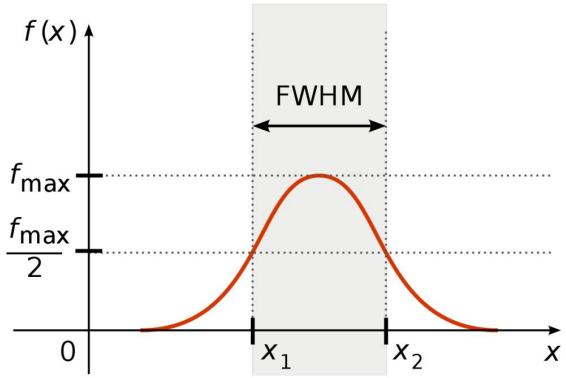
Q4a: The Gaussian Pulse

The *Gaussian pulse*, a bell-curve shaped signal, is one of the most interesting and important entities in both theoretical and practical signal processing setups, so we'll be using it throughout this question as the input to our virtual oscilloscope.

One of the many things it's used for is measuring the impulse response of a CT-LTI system. Since the Dirac delta is a mathematical abstraction, and not a physically realizable signal, we can't just generate one and send it into a CT-LTI system to measure its impulse response. Instead, it's common practice to send in a very narrow, unit area Gaussian pulse (since we recover the Dirac delta in the limit of the width approaching zero), and treat the system's output as a measured impulse response. The approximation is good as long as the Gaussian used is sufficiently narrow. For example, according to reference 8 (page 26), Lawrence Livermore National Laboratory's National Ignition Facility generates impulses as Gaussian pulses with a full width at half maximum (explained below) of 88 picoseconds, which is about as close to an ideal impulse as modern hardware can generate.

Full Width at Half Maximum

In signal processing, we typically characterize Gaussians by their *Full Width at Half Maximum* (FWHM), shown below (image credit: Wikipedia). It is what it sounds like: the width of the pulse at half its max amplitude.



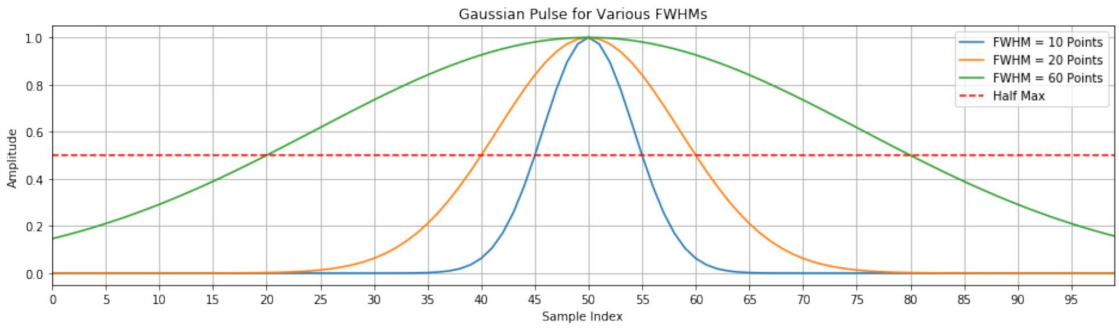
We can express the Gaussian, which is assumed to be centered at $t = 0$, in terms of its FWHM as $f(t) = 2^{-(2t \,/\, \mathrm{FWHM})^2}$. Using this formula, implement `gaussian_pulse` below according to the docstring.

Question assigned to the following page: 3

In [111]:
```
def gaussian_pulse(L, fwhm):
    """
    Parameters:
    L    - The number of samples to generate the pulse over.
    fwhm - The pulse's full width at half maximum.

    Returns:
    Gaussian pulse with FWHM of "fwhm" defined over the sample points t
    """
    # TODO your code here
    h = list()
    for i in range (-L // 2, L // 2):
        h.append(2**(-(2 * i / fwhm) ** 2))
    return h
```

Now, check your code by running the cell below. Think about the meaning of FWHM, and use that as a sanity check on whether your results are correct or not. All Gaussians should attain a maximum value of 1 at the middle of the plot.

In [112]:
```
L = 100
FWHMs = [10, 20, 60]
plt.figure(figsize=(16, 4))
for fwhm in FWHMs:
    plt.plot(gaussian_pulse(L, fwhm))
plt.xlim([0, L-1])
plt.plot(.5 * np.ones(L), 'r--')
plt.legend(["FWHM = {0} Points".format(fwhm) for fwhm in FWHMs] + ["Half
plt.title("Gaussian Pulse for Various FWHMs")
plt.ylabel("Amplitude")
plt.xlabel("Sample Index")
plt.xticks(np.arange(0, 100, 5))
plt.grid()
plt.show()
```



Svstem Model

Question assigned to the following page: 3

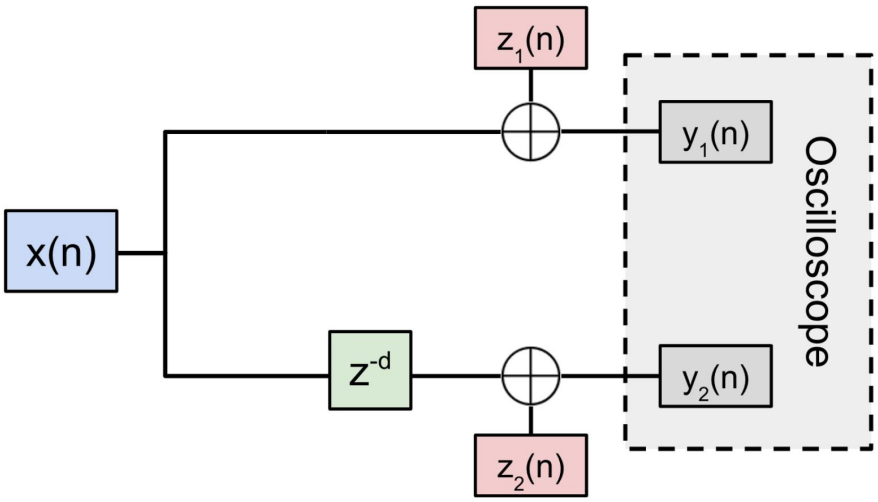
We'll assume each oscilloscope channel measures the signal fed to it by:

1. Delaying it by some amount (different for each channel), and then
2. Adding some amount of random noise.

For simplicity, we'll consider the case of two channels, although the same principles apply if you had more channels and wanted to align all of their measurements. **Crucially, we only care about the *relative* delay between channel 1 and channel 2 because we want to align them with each other, not shift them to some absolute point in time.** This allows us to think of channel 1 as having zero delay, acting as our *reference channel*.

Block Diagram

Below is a delay-adder-gain block diagram of our model, where $y_1(n)$ and $y_2(n)$ are the signals measured on channels 1 and 2, respectively. In reality, the delays shown would be in continuous time, as the signals involved are only discretized once they reach the oscilloscope, but we'll think of everything as one DT system for simplicity here.



Note:

- The z^{-d} delay block says that $y_2(n)$ arrives d samples after $y_1(n)$. **If $d < 0$, channel 2's signal arrives before channel 1's.**
- $z_1(n)$, $z_2(n)$ are additive noise corrupting our signals.
  - Each has a different strength, depending on the channel.
  - We'll assume *White Gaussian Noise* (WGN). We're not just doing this for academic purposes—there's ample empirical evidence that this is a good model for oscilloscope noise (see reference 9).
- Channel 1 measures $y_1(n) = x(n) + z_1(n)$.

Question assigned to the following page: 3

- Channel 2 measures $y_2(n) = x(n-d) + z_2(n)$.

Our goal: Find d, and shift $y_2(n)$ to the left by d samples.

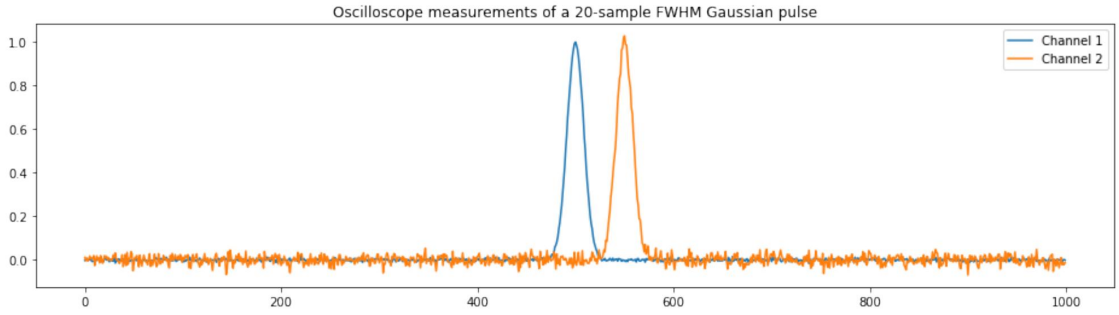
We've provided the class `Oscilloscope` for you below, which models this system. You are not responsible for understanding it, just run the next two cells to get a feel for what it does.

In [113]:
```
class Oscilloscope:
    """
    Oscilloscope modeled as a set of channels, each of which has its own
    All noise is modeled as Additive White Gaussian Noise (AWGN), and de
    """
    def __init__(self, num_channels, channel_noise_levels, channel_delay
        """
        Parameters:
        num_channels         - The number of oscilloscope channels.
        channel_noise_levels - The standard deviation of the AWGN on eac
        channel_delays       - The delays, in samples, for each channel.

        A Value_Error is raised if num_channels does not match the numbe
        channel_noise_levels and channel_delays, or if all entries of ch
        not integers.
        """
        channel_delays = np.array(channel_delays)
        if not num_channels == len(channel_noise_levels):
            raise ValueError("Number of channels must match number of no
        if not num_channels == len(channel_delays):
            raise ValueError("Number of channels must match number of de
        if not np.array_equal(channel_delays, channel_delays.astype("int
            raise ValueError("All delays must be integer-valued.")
        self.num_channels = num_channels
        self.channel_noise_levels = channel_noise_levels
        self.channel_delays = channel_delays

    def measure(self, signal):
        measurements = []
        for i in range(self.num_channels):
            curr_noise = np.random.normal(scale=self.channel_noise_level
            curr_channel_measurement = np.roll(signal, self.channel_dela
            measurements.append(curr_channel_measurement)
        return measurements
```

Let's simulate measurement of a Gaussian pulse using our Oscilloscope. We'll assume a 50-sample delay, and a modest amount of noise on each channel.

Question assigned to the following page: 3

In [114]:
```
# Measure a 20-sample FWHM gaussian pulse
noise_levels = [.005, .02]
delays = [0, 50]
num_channels = 2
scope = Oscilloscope(num_channels, noise_levels, delays)
signal = gaussian_pulse(1000, 20)
measured_signals = scope.measure(signal)

# Display results
plt.figure(figsize=(16, 4))
plt.title("Oscilloscope measurements of a 20-sample FWHM Gaussian pulse")
plt.plot(measured_signals[0], label="Channel 1")
plt.plot(measured_signals[1], label="Channel 2")
plt.legend()
plt.show()
```



You should see two Gaussian pulses, each noised, with a 50-sample offset between them. This is what our raw data would look like in a real world setup!

Now, we'll align these signals and compensate for the 50-sample delay using cross-correlation.

Question assigned to the following page: 3

Q4b: Cross-Correlation (Conceptual)

The *normalized* cross-correlation of two deterministic DT signals x and y, which is another DT signal R_{xy}, is defined as

$$R_{xy}(n) = \frac{\sum_{k=-\infty}^{\infty} x(k)y(k+n)}{\left(\sqrt{\sum_{k=-\infty}^{\infty} |x(k)|^2}\right)\left(\sqrt{\sum_{k=-\infty}^{\infty} |y(k)|^2}\right)}.$$

The question remains: We want to know what shift n makes $x(k)$ and $y(k+n)$ the most "aligned" in some sense, and then align them by shifting by that amount. How does cross-correlation tell us this?

Cosine Similarity Interpretation

Recall that in $\mathbb{R}^N$ (N-dimensional Euclidean space), if the angle between two vectors $\vec{x}, \vec{y}$ is θ, then

$$\cos(\theta) = \frac{\vec{x} \cdot \vec{y}}{||\vec{x}|| \cdot ||\vec{y}||},$$

where $||\vec{x}|| = \sqrt{\vec{x} \cdot \vec{x}}$ is the length, or *norm*, of the vector $\vec{x}$. The expression on the right hand side of the equation above is often referred to as the *cosine similarity* between $\vec{x}$ and $\vec{y}$.

The reason why we call this a "similarity" pops out when we look at what it assigns for different values of θ:

- If $\theta = 0$, the vectors are parallel, and their cosine similarity is 1.
- If $\theta = \pi/2$, the vectors are orthogonal, and their cosine similarity is 0.
- If $\theta = \pi$, the vectors are antiparallel, and their cosine similarity is -1.
- The cosine similarity is bounded between -1 and 1 for any θ due to the normalization (this can be proved via the Cauchy-Schwarz inequality). **This means the cosine similarity is maximized when the vectors are perfectly aligned!**

If we think of our duration N signals (remember, we're on a computer, so everything is finite-duration) as vectors in $\mathbb{R}^N$, then $R_{xy}(n)$ is precisely the cosine similarity between a shifted (left by n samples) version of y and an unshifted version of x.

We can find the offset between the signals, then, by determining what shift produces the largest cosine similarity (i.e., makes the signals most "aligned" in Euclidean space) between them! Equivalently, we just find where $R_{xy}(n)$ is maximized, and call that the relative delay between them. That is, the number of samples *after* x that y appears.

Question assigned to the following page: 3

An Example

To get a feel for why a bigger cross-correlation means the signals are more aligned, let's do an example. We'll manually take the dot products ourselves so it's clear which dot product corresponds to what offset.

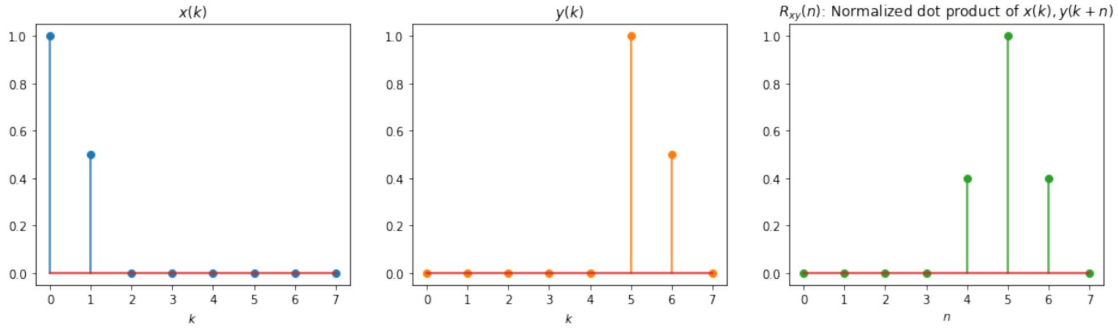
Run the cell below and answer the questions that come after it.

In [132]:
```
x = np.array([1, .5, 0, 0,  0,  0, 0, 0])
y = np.array([0,  0,  0, 0, 0, 1, .5, 0])
N = len(x)

# np.roll(y, -n) shifts x to the left by n, i.e. produces y(k+n)
R_xy = np.array([np.dot(x, np.roll(y, -n)) for n in range(N)])
R_xy = R_xy / (np.linalg.norm(x) * np.linalg.norm(y)) # normalize

plt.figure(figsize=(16, 4))
plt.subplot(1, 3, 1)
plt.stem(x)
plt.title("$x(k)$")
plt.xlabel("$k$")

plt.subplot(1, 3, 2)
plt.stem(y, linefmt="C1", markerfmt="C1o")
plt.title("$y(k)$")
plt.xlabel("$k$")

plt.subplot(1, 3, 3)
plt.stem(R_xy, linefmt="C2", markerfmt="C2o")
plt.title("$R_{xy}(n)$: Normalized dot product of $x(k), y(k+n)$")
plt.xlabel("$n$")
plt.show()
```

```
/opt/anaconda3/lib/python3.7/site-packages/ipykernel_launcher.py:11: U
serWarning: In Matplotlib 3.3 individual lines on a stem plot will be
added as a LineCollection instead of individual lines. This significan
tly improves the performance of a stem plot. To remove this warning an
d switch to the new behaviour, set the "use_line_collection" keyword a
rgument to True.
  # This is added back by InteractiveShellApp.init_path()
/opt/anaconda3/lib/python3.7/site-packages/ipykernel_launcher.py:16: U
serWarning: In Matplotlib 3.3 individual lines on a stem plot will be
added as a LineCollection instead of individual lines. This significan
tly improves the performance of a stem plot. To remove this warning an
d switch to the new behaviour, set the "use_line_collection" keyword a
rgument to True.
  app.launch_new_instance()
/opt/anaconda3/lib/python3.7/site-packages/ipykernel_launcher.py:21: U
```

Question assigned to the following page: 3

```
serWarning: In Matplotlib 3.3 individual lines on a stem plot will be
added as a LineCollection instead of individual lines. This significan
tly improves the performance of a stem plot. To remove this warning an
d switch to the new behaviour, set the "use_line_collection" keyword a
rgument to True.
```



For all questions, only consider the eight shift values of n that are shown in the cross-correlation output, 0 through 7.

Q: For what shift values n do $x(k)$ and $y(k+n)$ *not* overlap? What is $R_{xy}(n)$ there? Considered as vectors in $\mathbb{R}^8$, what is the angle θ between them for those shift values?

A: There is no value overlap between $x(k)$ and $y(k+n)$ and $R_{xy}(n)$ is 0. The angle θ between them is 90 degree or pi/2.

Q: For what shift value n_* are $x(k)$ and $y(k+n_*)$ identical? What is $R_{xy}(n_*)$? Considered as vectors in $\mathbb{R}^8$, what is the angle θ between them for this shift value?

A: There is overlap between $x(k)$ and $y(k+n)$ when n_* is 5. $R_{xy}(n)$ is 1 and the angle θ between them is 0.

Question assigned to the following page: 3

Q4c: Cross-Correlation (Implementation)

Now that we've seen how cross-correlation can solve our signal alignment problem, we need an efficient way to implement it.

Filtering Interpretation

The numerator in our expression for cross-correlation is the main troublemaker, since the denominator is just a product of two norms which we know how to calculate with `np.linalg.norm`. However, the numerator looks suspiciously similar to a convolution. By making the change of variables $m = -k$, so that $k = -m$, we see that

$$R_{xy}(n) = \frac{\sum_{k=-\infty}^{\infty} x(k)y(k+n)}{||x|| \cdot ||y||} = \frac{\sum_{m=-\infty}^{\infty} x(-m)y(n-m)}{||x|| \cdot ||y||},$$

and it's clear that R_{xy} is the convolution of y with a time-reversed version of x, all divided by the product of their norms. We've rewritten the denominator in terms of the norms of x and y here for brevity.

Question assigned to the following page: 3

Implementation Details

As we saw above, cross-correlation reduces to a convolution. What a relief! Convolution can be implemented efficiently by taking the product of the FFTs (convolution in time is multiplication in frequency). We just need to figure out how to handle the time-reversal.

Time-Reversal

Since we're working with finite-duration signals here, we can't just use `x[::-1]` and declare victory. If x holds values corresponding to $n = 0, \ldots, N-1$, then reversing the array `x` in-place puts the value for $n = N-1$ at the place for $n = 0$, not $n = -N+1$. We could solve this issue by carefully doing some zero-padding, but this is a bit annoying. This is where the Fourier Transform properties come to the rescue.

If $x(n)$ and $X(\omega)$ form a time-frequency pair (i.e., $X(\omega)$ is the DTFT of $x(n)$), then $x(-n)$ and $X(-\omega)$ also form a time-frequency pair. This doesn't appear to help much, though, as there's still a reversal involved.

However, we know that when x is real-valued (as is the case here), its spectrum is *conjugate-symmetric*: $X(-\omega) = X^*(\omega)$. So, instead of dealing with any reversal going on in either domain, we can just conjugate the spectrum! **This is much simpler and more efficient, as complex conjugation is an element-wise operation.** Contrast this with array reversal, which typically requires you to swap a bunch of data (or pointers).

Zero-Padding

We know that the DT convolution (or cross-correlation) of a duration M signal and duration N signal produces a duration $M + N - 1$ signal. In the most general case, all $M + N - 1$ points are nonzero and must be stored in our output. Since we're implementing a modified form of convolution here, we need to account for this, and make sure our output is at least of size $M + N - 1$. Essentially, we're doing a "full" mode cross-correlation.

For added efficiency, we'll just use the next power of two after or including $M + N - 1$. This won't affect our results, since the zeros contribute nothing to the dot products.

Question assigned to the following page: 3

Your Job

Implement the function `xcorr` below which returns the **normalized** cross-correlation of two signals x and y through the algorithm we've developed in this question:

1. Zero pad x and y to `next_power_of_2(len(x) + len(y) - 1)`, obtaining $x_{\text{pad}}, y_{\text{pad}}$.
2. Multiply the FFT of y_{pad} with the complex conjugate of the FFT of x_{pad}.
3. Take the inverse FFT of the product computed in step 2.
4. Divide by $||x|| \cdot ||y||$.
5. *Center* the cross-correlation using np.fft.fftshift (https://docs.scipy.org/doc/numpy/reference/generated/numpy.fft.fftshift.html). This places the dot product corresponding to a delay of zero in the middle of the output.
6. Return the real part. This eliminates the small, but fake, imaginary part caused by numerical imprecision. The cross-correlation of two real signals is always real-valued.

You are welcome to use np.fft.fft (https://docs.scipy.org/doc/numpy/reference/generated/numpy.fft.fft.html), np.fft.ifft (https://docs.scipy.org/doc/numpy/reference/generated/numpy.fft.ifft.html), np.linalg.norm (https://docs.scipy.org/doc/numpy/reference/generated/numpy.linalg.norm.html), np.concatenate (https://docs.scipy.org/doc/numpy/reference/generated/numpy.concatenate.html), np.zeros (https://docs.scipy.org/doc/numpy/reference/generated/numpy.zeros.html), np.conj (https://docs.scipy.org/doc/numpy/reference/generated/numpy.conj.html), and the provided `next_power_of_2` function in your implementation. You may **NOT** use `np.convolve`, `np.correlate`, or any other library functions that directly compute convolution or cross-correlation.

Question assigned to the following page: 3

In [139]:
```
def next_power_of_2(n):
    """Returns the next power of 2 after and including n."""
    return 2 ** int(np.ceil(np.log2(n)))

def xcorr(x, y):
    """
    Returns the cross-correlation of x and y, padded to the next power o
    Assumes that x, y are both real signals.
    """
    # TODO your code here
    l = next_power_of_2(len(x) + len(y) - 1)
    l_x = np.zeros(l-len(x))
    l_y = np.zeros(l-len(y))
    x_new = np.concatenate((x, l_x))
    y_new = np.concatenate((y, l_y))
    x_fft = np.fft.fft(x_new)
    y_fft = np.fft.fft(y_new)
    fft_inv = np.fft.ifft(np.conjugate(x_fft)*y_fft)
    norminv = fft_inv / (np.linalg.norm(x) * np.linalg.norm(y))
    num = np.fft.fftshift(norminv)
    real = num.real
    return real
```

Run the cell below to test your `xcorr` function. These tests aren't comprehensive, but should be good enough for our purposes. This will be used for grading purposes, so make sure you're passing all tests before moving on.

In [140]:
```
lab3_helper.run_xcorr_tests(xcorr, gaussian_pulse) # one test uses gauss
```

```
Test 1 (same input lengths) passed: True
Test 2 (same input lengths) passed: True
Test 3 (same input lengths) passed: True
Test 4 (diff input lengths) passed: True
Test 5 (same input lengths) passed: True
5 of 5 tests passed
```

Q4d: Optimal Offset

Implement `opt_offset`, which takes in two signals, x and y, and returns the "optimal offset" between them, defined as the index (relative to zero) at which their cross-correlation is maximized. This corresponds to the number of samples that x is *ahead* (i.e., to the left of) y by.

Hint: Our cross-correlation is zero-centered. To find the actual delay, we need to extract the index at which the cross-correlation is maximized, and subtract the index corresponding to a zero delay (i.e., the center) from it.

Question assigned to the following page: 3

In [141]:
```
def opt_offset(x, y):
    """Returns the most likely number of samples that x lags behind y vi
    # TODO your code here
    c = xcorr(x, y)
    i, max = 0, 0
    for n in range(len(c)):
        if c[n]>max:
            i, max = n, c[n]
    o_s = i - (len(c) // 2)
    return o_s
```

Here a few sanity checks on `opt_offset` you can run. If any of them `AssertionError`, that's saying that the test failed. If not, you're in good shape.

In [142]:
```
assert(opt_offset([0, 0, 1, 0], [0, 1, 0, 0]) == -1)      # 1st lags behi
assert(opt_offset([0, 0, 1, 0], [1, 0, 0, 0]) == -2)      # 1st lags behi
assert(opt_offset([1, 0, 0, 0, 0], [0, 0, 0, 0, 1]) == 4) # 1st is ahead
```

Q4e: Realignment

To complete the lab, and provide us with a nice abstraction to solve our original problem, implement `align`. This function takes in two signals, finds the optimal offset between them, and shifts the first to the right by that amount so they're aligned.

This should be very simple (two lines of code, at most) once you have `opt_offset` working.

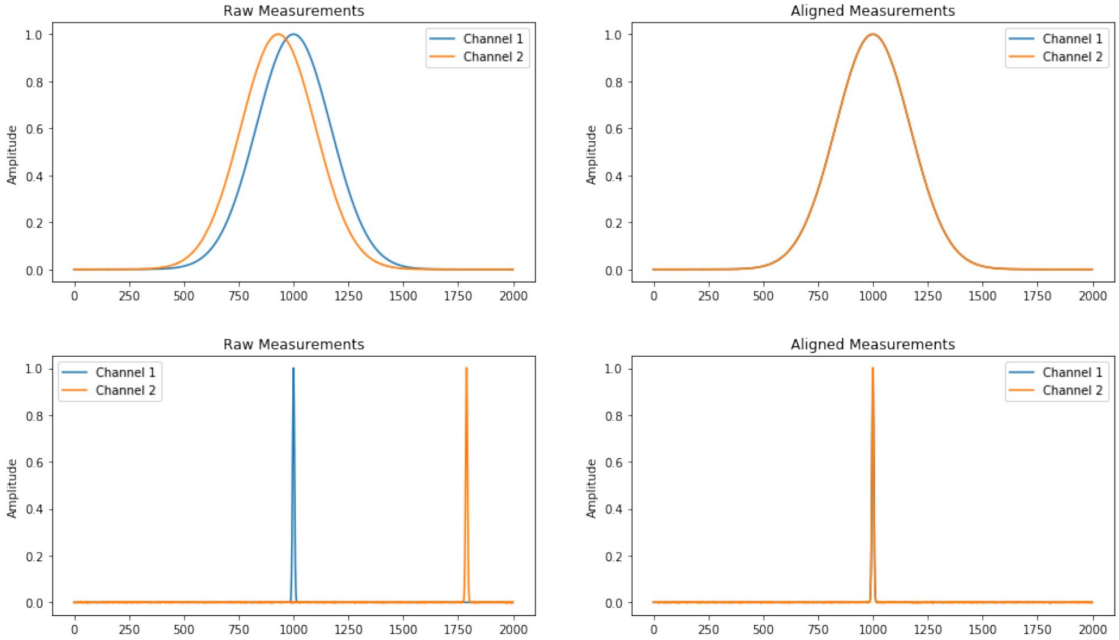
In [143]:
```
def align(x, x_ref):
    """Assuming x is a delayed copy of x_ref, finds the timing offset an
    # TODO your code here
    o_s = opt_offset(x, x_ref)
    return np.roll(x, o_s)
```

And now, the moment we've been building to: let's align some oscilloscope measurements! Run the cell below.

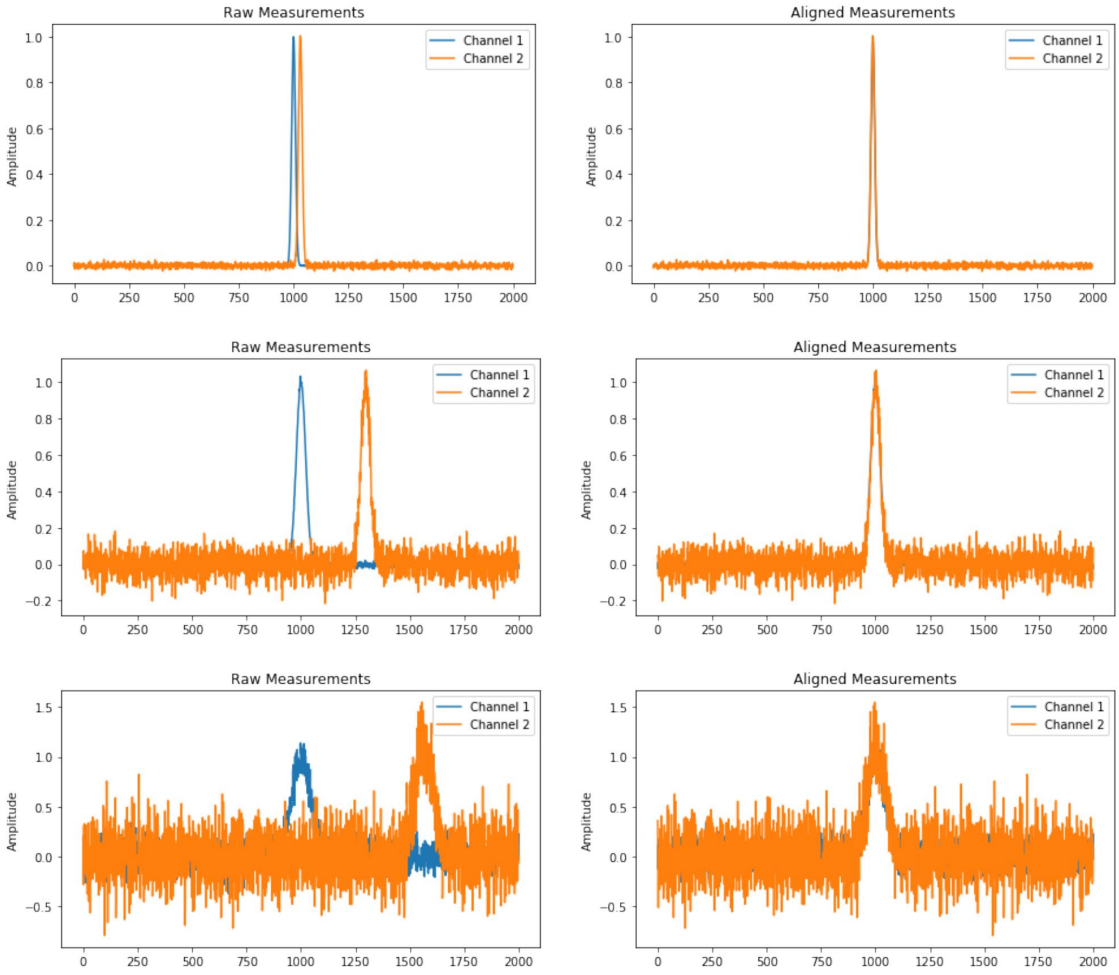
In [144]:
```
# Mock setup to test out the calibration
num_channels = 2
record_length = 2000
fwhms = [400, 10, 20, 50, 100]
ch1_noise_levels = [0,     .0001, .001, .01, .1]
ch2_noise_levels = [.0003, .0009, .008, .06, .23]
ch1_delays = [0, 0, 0,  0, 0]
ch2_delays = [-70, 788, 31, 297, 563]
```

Question assigned to the following page: 3

```
for i in range(len(fwhms)):
    # Construct oscilloscope
    curr_noise_levels = [ch1_noise_levels[i], ch2_noise_levels[i]]
    curr_delays = [ch1_delays[i], ch2_delays[i]]
    scope = Oscilloscope(num_channels, curr_noise_levels, curr_delays)

    # Measure
    curr_signal = gaussian_pulse(record_length, fwhms[i])
    measured_signals = scope.measure(curr_signal)
    ch1_meas = measured_signals[0]
    ch2_meas = measured_signals[1]

    # Align and display results
    ch2_corrected = align(ch2_meas, ch1_meas)
    plt.figure(figsize=(16, 4))
    plt.subplot(1, 2, 1)
    plt.title("Raw Measurements")
    plt.plot(ch1_meas, label="Channel 1")
    plt.plot(ch2_meas, label="Channel 2")
    plt.ylabel("Amplitude")
    plt.legend()

    plt.subplot(1, 2, 2)
    plt.title("Aligned Measurements")
    plt.plot(ch1_meas, label="Channel 1")
    plt.plot(ch2_corrected, label="Channel 2")
    plt.ylabel("Amplitude")
    plt.legend()
plt.show()
```



Question assigned to the following page: 3



You should see two columns and five rows. The first column contains the "raw", unaligned measurements, and the second is the result of calling your `align` function to compensate for the delays. If everything is correct, you should see nicely aligned signals in the second column.

Take a look at how well your alignment works, even on the bottom row, where the noise is incredibly strong. Cross-correlation's offset detection capability in the presence of additive random noise is remarkably robust, which is why it's the standard solution used in practice for this problem.

No questions assigned to the following page.

Optional Extension

See how high you can crank up the noise levels before our realignment technique fails. Since the noise is random, you can even re-run the experiment multiple times and empirically find the probability of the procedure failing as a function of the white Gaussian noise's standard deviation.

References

[1] *Cooley-Tukey FFT algorithm (Wikipedia).* Link (https://en.wikipedia.org/wiki/Cooley%E2%80%93Tukey_FFT_algorithm)
[2] *EE 123 - Spring 2018 - Lecture 3C, Fast Convolutions and the FFT.* Link (https://inst.eecs.berkeley.edu/~ee123/sp18/Notes/Lecture3C.pdf)
[3] *EE 123 - Spring 2018 - Lecture 4A, The FFT.* Link (https://inst.eecs.berkeley.edu/~ee123/sp18/Notes/Lecture4A.pdf)
[3] *Full width at half maximum (Wikipedia).* Link (https://en.wikipedia.org/wiki/Full_width_at_half_maximum)
[4] *Matched filter (Wikipedia).* Link (https://en.wikipedia.org/wiki/Matched_filter)
[5] *An Industrial Strength Audio Search Algorithm.* Link (https://www.ee.columbia.edu/~dpwe/papers/Wang03-shazam.pdf)
[6] *EE 120 - Fall 2019 - Lecture 10 Notes.* Link (https://inst.eecs.berkeley.edu/~ee120/fa19/LectureNotes/Lecture10.pdf)
[7] *EE 123 - Spring 2016 - Lab 3, Time Frequency Part I.* Link (https://inst.eecs.berkeley.edu/~ee123/sp16/lab/lab3/lab3-Part_I_Time-Frequency-Spectrogram.html)
[8] *NIF Laser Capabilities.* Link (https://lasers.llnl.gov/workshops/user_group_2012/docs/5.2_haynam.pdf)
[9] *Evaluating Oscilloscope Vertical Noise Characteristics.* Link (https://www.testunlimited.com/pdf/an/5989-3020EN.pdf)

Additionally, these materials, while not used in developing this lab, may be of interest:

- *Explained: The Discrete Fourier Transform.* Link (http://news.mit.edu/2009/explained-fourier)
- *Rensselaer Polytechnic Institute. ECSE-4530, Digital Signal Processing - Lecture 11: Radix 2 Fast Fourier Transforms.* Link (https://www.youtube.com/watch?v=1mVbZLHLaf0)