# Chapter 1

# Direct stiffness method

## 1.1. Introduction

The direct stiffness method (DSM) is a method to solve statically determinant or indeterminant structures that is particularly well-suited for computer implementation. It is the finite element method (FEM) applied to a naturally discrete system, e.g., one modeled as a set of idealized elements connected at nodes, rather than a partial differential equation (PDE). As such, the DSM will serve as a gentle introduction to finite element concepts such as an unstructured mesh, assembly, and application of boundary conditions without the complexity of partial differential equations.

In this document, we will solely consider the DSM in the context of a truss structure, defined as a structure that consists of slender, linear elastic members joined at their endpoints by pin joints (free rotation, i.e., does not support moments) with all loads (external loads and reaction forces) applied at nodes. The members are assumed to be of negligible weight (relative to the external loads), have a constant area and stiffness along their length, and the stress on any cross section is uniform. The assumption that the structure consists of members of negligible weight connected by pins and is only loaded at its nodes implies the force in each member is purely axial (pure compression or tension, no transverse force) and constant along its length. The assumption that the members have constant area and stiffness implies the strain in each element is constant, which in turn implies the axial displacement varies linearly along the length of the member. In the remainder of the document, we will consider the truss in Figure 1.1 for concreteness.

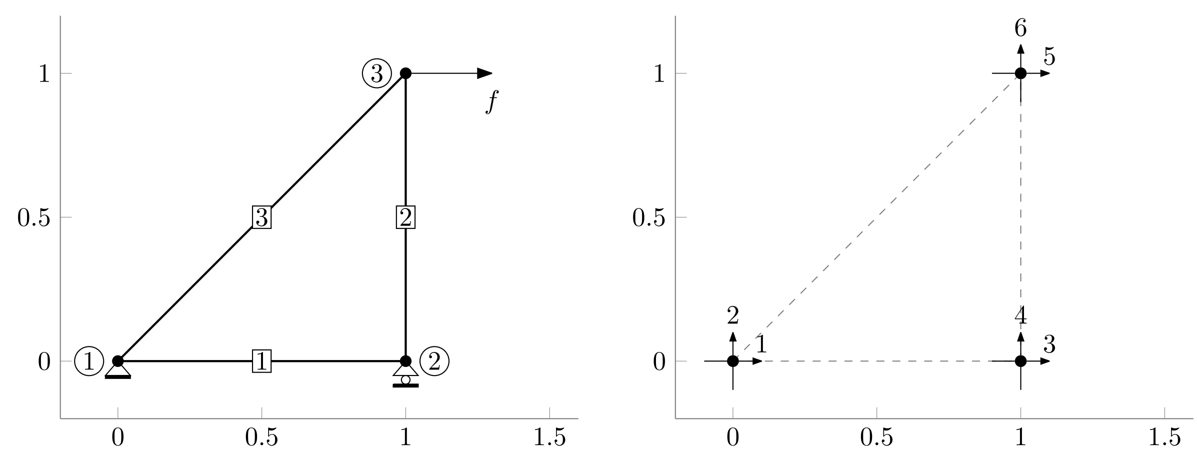


Figure 1.1: *Left*: Truss structure with three nodes and elements, an $x$-directed load at node 3, a pinned (fixed $x$ and $y$ displacement) boundary condition at node 1, and a vertical roller (fixed $y$ displacement) at node 2. The node and element numbers are shown in the figure; the node numbers are contained in circles and the element numbers in rectangles. *Right*: Numbering of global degrees of freedom.

## 1.2. Element contribution to global equations

The goal of this section is to derive a relationship between the force in each element and the displacement of its nodes in the coordinate system of the structure ($x$-$y$). However, the force-displacement relationship is most readily derived in a coordinate system aligned with the element. Therefore we consider an arbitrary element $e$ from the truss and introduce a coordinate system ($\bar{x}^e$-$\bar{y}^e$) such that the first coordinate direction ($\bar{x}^e$) is aligned with the axis of the bar (Figure 1.2). Each element consists of two local nodes whose numbering is independent of the global node numbering in Figure 1.1 and taken as $\{1,2\}$ for convenience. Let $\theta_e$ denote the angle (counterclockwise) from the horizontal to the element axis, i.e., the angle between the $\bar{x}^e$- and $x$-axis. Let $(\bar{u}_1^e, \bar{u}_2^e)$ and $(\bar{u}_3^e, \bar{u}_4^e)$ denote the displacement in the $(\bar{x}, \bar{y})$ direction of local node 1 and 2, respectively. Similarly, let $\bar{F}_1^e$ and $\bar{F}_3^e$ be the force at local node 1 and 2, respectively, in the $\bar{x}^e$-direction. The force in the $\bar{y}^e$-direction has been intentionally excluded because, from the assumptions stated in Section 1.1, the force in the members is purely axial. Finally, we denote the displacement and forces at local node 1 of element $e$ in the global coordinate system ($x$-$y$) as $(u_1^e, u_2^e)$ and $(F_1^e, F_2^e)$, respectively. Similarly, the displacements and forces at local node 2 of element $e$ in the global coordinate system are $(u_3^e, u_4^e)$ and $(F_3^e, F_4^e)$, respectively.

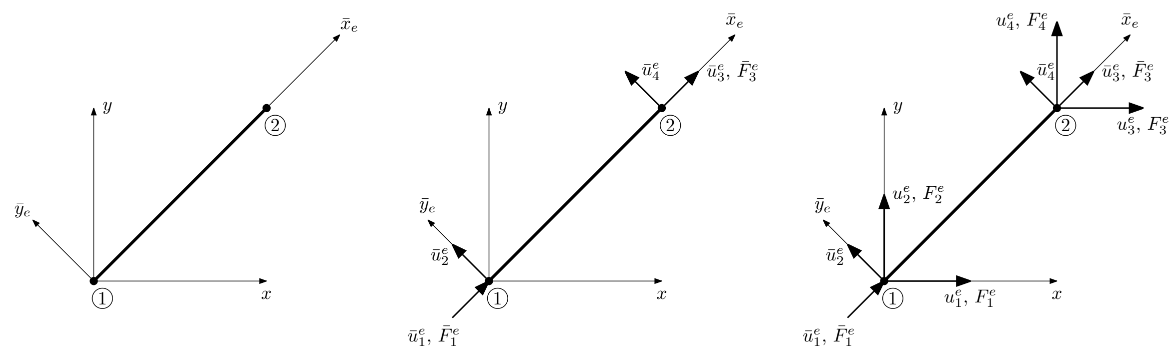


Figure 1.2: Local and global coordinate system for bar 2 in the truss from Figure 1.1 (*left*), the forces and displacements in the element coordinate system (*center*), and the forces and displacements in the global coordinate system (*right*).

With this notation and the assumptions in Section 1.1, the (constant) strain in the element, defined as the change in length of a member relative to its original length, is

$$\bar{\epsilon}_e = \frac{\bar{u}_3^e - \bar{u}_1^e}{h_e}. \tag{1.1}$$

Then, from Hooke's law (linear elasticity) that linearly relates stress and strain ($\sigma = E\epsilon$), and the definition of stress as force per unit area ($F = \sigma A$), we have

$$\begin{aligned} \bar{F}_1^e &= -\bar{\sigma}_e A_e = -E_e A_e \bar{\epsilon}_e = -\frac{E_e A_e}{h_e}(\bar{u}_3^e - \bar{u}_1^e) \\ \bar{F}_3^e &= \phantom{-}\bar{\sigma}_e A_e = \phantom{-}E_e A_e \bar{\epsilon}_e = \phantom{-}\frac{E_e A_e}{h_e}(\bar{u}_3^e - \bar{u}_1^e), \end{aligned} \tag{1.2}$$

where $A_e$, $E_e$, $\bar{\epsilon}_e$, and $\bar{\sigma}_e$ are the cross-sectional area, stiffness (Young's modulus), strain, and stress of member $e$, respectively, all of which are constant along its length, or equivalently in matrix form

$$\begin{bmatrix} \bar{F}_1^e \\ \bar{F}_3^e \end{bmatrix} = \frac{E_e A_e}{L_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \bar{u}_1^e \\ \bar{u}_3^e \end{bmatrix}. \tag{1.3}$$

It is easy to verify the signs have been chosen correctly by considering the case where $\bar{u}_3^e > \bar{u}_1^e$, i.e., the bar is in tension therefore $\bar{F}_3^e$ is correctly oriented and thus has a positive sign, while $\bar{F}_1^e$ should be reversed and thus has a negative sign.

Before closing this section, we relate the forces and displacements in the element coordinate system to the global coordinate system for global assembly of the elements in the next section. The following rotation matrix will rotate a vector, $\boldsymbol{v} \in \mathbb{R}^2$, *clockwise* by $\theta$

$$\boldsymbol{T}(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}. \tag{1.4}$$

A convenient property of rotation matrices is they are orthogonal, i.e., $\boldsymbol{T}(\theta)^T = \boldsymbol{T}(\theta)^{-1}$. Let $\boldsymbol{e}_i$ and $\bar{\boldsymbol{e}}_i^e$ be unit vectors aligned with the $x$-$y$ and $\bar{x}^e$-$\bar{y}^e$ coordinate axes, respectively. From the configuration of the $x$-$y$ coordinate system, $\boldsymbol{e}_1 = (1,0)^T$ and $\boldsymbol{e}_2 = (0,1)^T$. Then, we have the relationship

$$\boldsymbol{e}_i = \boldsymbol{T}(\theta_e)\bar{\boldsymbol{e}}_i^e \tag{1.5}$$

from the definition of the $\bar{x}^e$-$\bar{y}^e$ coordinate system. Expansion of any vector $\boldsymbol{v} \in \mathbb{R}^2$ in these coordinate systems gives

$$\boldsymbol{v} = v_1\boldsymbol{e}_1 + v_2\boldsymbol{e}_2 = \bar{v}_1\bar{\boldsymbol{e}}_1^e + \bar{v}_2\bar{\boldsymbol{e}}_2^e, \tag{1.6}$$

where $(v_1, v_2)$ are the coordinates of $\boldsymbol{v}$ in the $x$-$y$ coordinates system and $(\bar{v}_1, \bar{v}_2)$ are the coordinates of $\boldsymbol{v}$ in the $\bar{x}^e$-$\bar{y}^e$ coordinate system. From the above equivalence between coordinate systems and (1.5), we have

$$\bar{v}_1\boldsymbol{e}_1 + \bar{v}_2\boldsymbol{e}_2 = \bar{v}_1\boldsymbol{T}(\theta_e)\bar{\boldsymbol{e}}_1^e + \bar{v}_2\boldsymbol{T}(\theta_e)\bar{\boldsymbol{e}}_2^e = \boldsymbol{T}(\theta_e)(\bar{v}_1\bar{\boldsymbol{e}}_1^e + \bar{v}_2\bar{\boldsymbol{e}}_2^e) = \boldsymbol{T}(\theta_e)\boldsymbol{v}. \tag{1.7}$$

Thus, the displacements are transferred between coordinate systems as

$$\begin{bmatrix} \bar{u}_1^e \\ \bar{u}_2^e \end{bmatrix} = \begin{bmatrix} \cos\theta_e & \sin\theta_e \\ -\sin\theta_e & \cos\theta_e \end{bmatrix} \begin{bmatrix} u_1^e \\ u_2^e \end{bmatrix}, \qquad \begin{bmatrix} \bar{u}_3^e \\ \bar{u}_4^e \end{bmatrix} = \begin{bmatrix} \cos\theta_e & \sin\theta_e \\ -\sin\theta_e & \cos\theta_e \end{bmatrix} \begin{bmatrix} u_3^e \\ u_4^e \end{bmatrix}, \tag{1.8}$$

which we combine into a single matrix equation for convenience

$$\begin{bmatrix} \bar{u}_1^e \\ \bar{u}_2^e \\ \bar{u}_3^e \\ \bar{u}_4^e \end{bmatrix} = \begin{bmatrix} \cos\theta_e & \sin\theta_e & 0 & 0 \\ -\sin\theta_e & \cos\theta_e & 0 & 0 \\ 0 & 0 & \cos\theta_e & \sin\theta_e \\ 0 & 0 & -\sin\theta_e & \cos\theta_e \end{bmatrix} \begin{bmatrix} u_1^e \\ u_2^e \\ u_3^e \\ u_4^e \end{bmatrix}. \tag{1.9}$$

Similarly the forces are transferred as

$$\begin{bmatrix} \bar{F}_1^e \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta_e & \sin\theta_e \\ -\sin\theta_e & \cos\theta_e \end{bmatrix} \begin{bmatrix} F_1^e \\ F_2^e \end{bmatrix}, \qquad \begin{bmatrix} \bar{F}_3^e \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta_e & \sin\theta_e \\ -\sin\theta_e & \cos\theta_e \end{bmatrix} \begin{bmatrix} F_3^e \\ F_4^e \end{bmatrix}, \tag{1.10}$$

which we combine into a single matrix equation

$$\begin{bmatrix} \bar{F}_1^e \\ 0 \\ \bar{F}_3^e \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta_e & \sin\theta_e & 0 & 0 \\ -\sin\theta_e & \cos\theta_e & 0 & 0 \\ 0 & 0 & \cos\theta_e & \sin\theta_e \\ 0 & 0 & -\sin\theta_e & \cos\theta_e \end{bmatrix} \begin{bmatrix} F_1^e \\ F_2^e \\ F_3^e \\ F_4^e \end{bmatrix}. \tag{1.11}$$

Padding the element equations in (1.3) with zeros

$$\begin{bmatrix} \bar{F}_1^e \\ 0 \\ \bar{F}_3^e \\ 0 \end{bmatrix} = \frac{E_e A_e}{L_e} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{u}_1^e \\ \bar{u}_2^e \\ \bar{u}_3^e \\ \bar{u}_4^e \end{bmatrix}. \tag{1.12}$$

and combining with the transformation between coordinate systems (1.9), (1.11) leads to the desired relationship between the elemental displacements and forces in the global coordinate system

$$\boldsymbol{F}^e = \boldsymbol{K}^e\boldsymbol{u}^e, \tag{1.13}$$

where $\boldsymbol{F}^e \in \mathbb{R}^4$ is the element force vector, $\boldsymbol{u}^e \in \mathbb{R}^4$ is the element displacement vector

$$\boldsymbol{F}^e = \begin{bmatrix} F_1^e \\ F_2^e \\ F_3^e \\ F_4^e \end{bmatrix}, \quad \boldsymbol{u}^e = \begin{bmatrix} u_1^e \\ u_2^e \\ u_3^e \\ u_4^e \end{bmatrix}, \tag{1.14}$$

and the element stiffness matrix is $\boldsymbol{K}^e \in \mathbb{R}^{4\times 4}$

$$\boldsymbol{K}^e = \frac{E_e A_e}{h_e} \begin{bmatrix} \cos\theta_e & \sin\theta_e & 0 & 0 \\ -\sin\theta_e & \cos\theta_e & 0 & 0 \\ 0 & 0 & \cos\theta_e & \sin\theta_e \\ 0 & 0 & -\sin\theta_e & \cos\theta_e \end{bmatrix}^T \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\theta_e & \sin\theta_e & 0 & 0 \\ -\sin\theta_e & \cos\theta_e & 0 & 0 \\ 0 & 0 & \cos\theta_e & \sin\theta_e \\ 0 & 0 & -\sin\theta_e & \cos\theta_e \end{bmatrix}. \tag{1.15}$$

## 1.3. Assembly of global equations: equilibrium

With the relationship between the nodal displacements and forces for each element in the global coordinate system established in (1.13), we are ready to derive the governing equations for the global system. Since this is a static structure (not a mechanism), each node must be in equilibrium, that is, the sum of forces acting of the node from the elements, external loads, and reactions, should be zero. Equilibrium of each node in the truss in Figure 1.1 leads to

$$\begin{aligned}
R_1 &= F_1^1 + F_1^3 - r_1 = 0 \\
R_2 &= F_2^1 + F_2^3 - r_2 = 0 \\
R_3 &= F_3^1 + F_1^2 = 0 \\
R_4 &= F_4^1 + F_2^2 - r_4 = 0 \\
R_5 &= F_3^2 + F_3^3 - f = 0 \\
R_6 &= F_4^2 + F_4^3 = 0
\end{aligned} \tag{1.16}$$

where $r_1$, $r_2$ are the reaction forces at node 1, $r_4$ is the reaction force at node 2, and all elements have been assigned local node numbers such that local node 1 corresponds to the smaller global node number (Figure 1.3). The equations are written in this expanded form to highlight that the global equations are merely a summation over the appropriate element equations (1.13), a procedure usually referred to as *assembly*. Substituting the element contributions (1.13) into the equilibrium equations (1.16) leads to the system that relates displacements and forces for the entire structure

$$\begin{aligned}
R_1 &= K_{11}^1 u_1^1 + K_{12}^1 u_2^1 + K_{13}^1 u_3^1 + K_{14}^1 u_4^1 + K_{11}^3 u_1^3 + K_{12}^3 u_2^3 + K_{13}^3 u_3^3 + K_{14}^3 u_4^3 - r_1 \\
R_2 &= K_{21}^1 u_1^1 + K_{22}^1 u_2^1 + K_{23}^1 u_3^1 + K_{24}^1 u_4^1 + K_{21}^3 u_1^3 + K_{22}^3 u_2^3 + K_{23}^3 u_3^3 + K_{24}^3 u_4^3 - r_2 \\
R_3 &= K_{31}^1 u_1^1 + K_{32}^1 u_2^1 + K_{33}^1 u_3^1 + K_{34}^1 u_4^1 + K_{11}^2 u_1^2 + K_{12}^2 u_2^2 + K_{13}^2 u_3^2 + K_{14}^2 u_4^2 \\
R_4 &= K_{41}^1 u_1^1 + K_{42}^1 u_2^1 + K_{43}^1 u_3^1 + K_{44}^1 u_4^1 + K_{21}^2 u_1^2 + K_{22}^2 u_2^2 + K_{23}^2 u_3^2 + K_{24}^2 u_4^2 - r_4 \\
R_5 &= K_{31}^2 u_1^2 + K_{32}^2 u_2^2 + K_{33}^2 u_3^2 + K_{34}^2 u_4^2 + K_{31}^3 u_1^3 + K_{32}^3 u_2^3 + K_{33}^3 u_3^3 + K_{34}^2 u_4^3 - f \\
R_6 &= K_{41}^2 u_1^2 + K_{42}^2 u_2^2 + K_{43}^2 u_3^2 + K_{44}^2 u_4^2 + K_{41}^3 u_1^3 + K_{42}^3 u_2^3 + K_{43}^3 u_3^3 + K_{44}^2 u_4^3.
\end{aligned} \tag{1.17}$$

Next, we enforce compatibility of displacements at nodes by relating element displacements (displacement of the nodes of each element) to global nodal displacements. Let $(u_{2(i-1)+1}, u_{2i})$ be the displacement of node $i$ in the global truss structure; see numbering of global degrees of freedom in Figure 1.1. Then, the displacement of the end of every element that meet at node $i$ must be equal to $(u_{2(i-1)+1}, u_{2i})$ due to the pin connection

$$\begin{aligned}
u_1 &= u_1^1 = u_1^3 \qquad & u_3 &= u_3^1 = u_1^2 \qquad & u_5 &= u_3^2 = u_3^3 \\
u_2 &= u_2^1 = u_2^3 \qquad & u_4 &= u_4^1 = u_2^2 \qquad & u_6 &= u_4^2 = u_4^3.
\end{aligned} \tag{1.18}$$

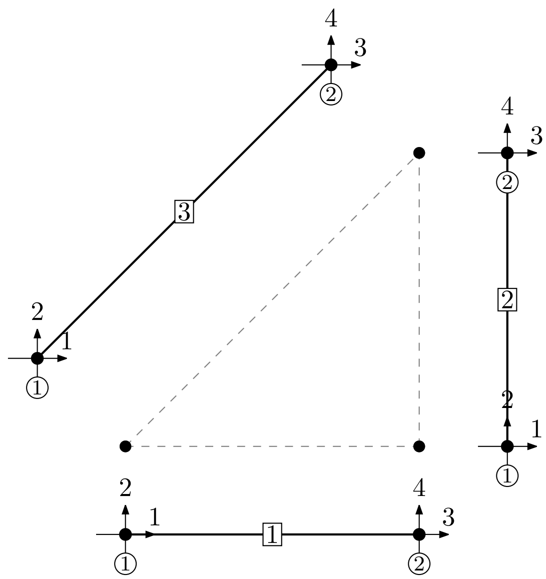


Figure 1.3: Truss structure with local nodes and degrees of freedom labeled. The local node numbers are contained in circles, the global element numbers are in rectangles, and numbered arrows identify the local degrees of freedom for each element.

Substitution of these compatibility conditions into (1.25) leads to the final form of the assembled equations that enforces equilibrium, compatibility of displacements at nodes, and the element equations

$$\begin{aligned}
R_1 &= (K^1_{11} + K^3_{11})u_1 + (K^1_{12} + K^3_{12})u_2 + K^1_{13}u_3 + K^1_{14}u_4 + K^3_{13}u_5 + K^3_{14}u_6 - r_1 \\
R_2 &= (K^1_{21} + K^3_{21})u_1 + (K^1_{22} + K^3_{22})u_2 + K^1_{23}u_3 + K^1_{24}u_4 + K^3_{23}u_5 + K^3_{24}u_6 - r_2 \\
R_3 &= K^1_{31}u_1 + K^1_{32}u_2 + (K^1_{33} + K^2_{11})u_3 + (K^1_{34} + K^2_{12})u_4 + K^2_{13}u_5 + K^2_{14}u_6 \\
R_4 &= K^1_{41}u_1 + K^1_{42}u_2 + (K^1_{43} + K^2_{21})u_3 + (K^1_{44} + K^2_{22})u_4 + K^2_{23}u_5 + K^2_{24}u_6 - r_4 \\
R_5 &= K^3_{31}u_1 + K^3_{32}u_2 + K^2_{31}u_3 + K^2_{32}u_4 + (K^2_{33} + K^3_{33})u_5 + (K^3_{34} + K^2_{34})u_6 - f \\
R_6 &= K^2_{41}u_1 + K^2_{42}u_2 + K^2_{41}u_3 + K^2_{42}u_4 + (K^2_{43} + K^3_{43})u_5 + (K^3_{44} + K^2_{44})u_6.
\end{aligned} \tag{1.19}$$

This can be compactly written as

$$\boldsymbol{R}(\boldsymbol{u}, \boldsymbol{f}) = \boldsymbol{K}\boldsymbol{u} - \boldsymbol{f} = \boldsymbol{0}, \tag{1.20}$$

where the residual ($\boldsymbol{R}$), vector of nodal displacements ($\boldsymbol{u}$), and vector of external forces ($\boldsymbol{f}$) are

$$\boldsymbol{R} = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \\ R_6 \end{bmatrix}, \qquad \boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix}, \qquad \boldsymbol{f} = \begin{bmatrix} r_1 \\ r_2 \\ 0 \\ r_4 \\ f \\ 0 \end{bmatrix} \tag{1.21}$$

and the stiffness matrix ($\boldsymbol{K}$) is

$$\boldsymbol{K} = \begin{bmatrix}
K^1_{11} + K^3_{11} & K^1_{12} + K^3_{12} & K^1_{13} & K^1_{14} & K^3_{13} & K^3_{14} \\
K^1_{21} + K^3_{21} & K^1_{22} + K^3_{22} & K^1_{23} & K^1_{24} & K^3_{23} & K^3_{24} \\
K^1_{31} & K^1_{32} & K^1_{33} + K^2_{11} & K^1_{34} + K^2_{12} & K^2_{13} & K^2_{14} \\
K^1_{41} & K^1_{42} & K^1_{43} + K^2_{21} & K^1_{44} + K^2_{22} & K^2_{23} & K^2_{24} \\
K^3_{31} & K^3_{32} & K^2_{31} & K^2_{32} & K^2_{33} + K^3_{33} & K^3_{34} + K^2_{34} \\
K^3_{41} & K^3_{42} & K^2_{41} & K^2_{42} & K^2_{43} + K^3_{43} & K^3_{44} + K^2_{44}
\end{bmatrix}. \tag{1.22}$$

The stiffness matrix is exactly the derivative of the residual with respect to the nodal coordinates, i.e.,

$$\boldsymbol{K} = \frac{\partial \boldsymbol{R}}{\partial \boldsymbol{u}}, \tag{1.23}$$

and therefore will also be referred to as the *Jacobian* matrix (the matrix of partial derivatives) of the residual.

## 1.4. Assembly of global equations: connectivity

In this section, we generalize the assembly procedure introduced in the previous section, which illuminates a convenient shortcut for assembling the stiffness matrix. Consider a truss with $N_v$ vertices and $N_e$ elements in $d$-dimensions. Let the $j$th column of $\boldsymbol{x} \in \mathbb{R}^{d\times N_v}$ be the coordinates of the $j$th node and $\boldsymbol{\Theta} \in \mathbb{N}^{2\times N_e}$ be the connectivity of the truss, i.e., $\Theta_{je}$ is the global node corresponding to local node $j$ of element $e$. Finally define $\boldsymbol{\Xi} \in \mathbb{N}^{dN_v \times N_e}$ as the mapping from local to global degrees of freedom, i.e., $\Xi_{je}$ is the global degree of freedom corresponding to local degree of freedom $j$ of element $e$.

**Example: Truss from Figure 1.1**

For the truss in Figure 1.1, the matrices defining the truss are

$$\boldsymbol{x} = \begin{bmatrix} 0.0 & 1.0 & 1.0 \\ 0.0 & 0.0 & 1.0 \end{bmatrix}, \qquad \boldsymbol{\Theta} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 3 \end{bmatrix}, \qquad \boldsymbol{\Xi} = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 4 & 2 \\ 3 & 5 & 5 \\ 4 & 6 & 6 \end{bmatrix}. \tag{1.24}$$

From these definitions, it is easy to see that the equilibrium conditions can be written compactly as

$$R_i = \sum_{e=1}^{N_e} \sum_{j=1}^{2d} F_j^e \delta_{i\Xi_{je}} - f_i, \tag{1.25}$$

for $i = 1, \ldots, dN_v$, where $R_i$ is the equilibrium residual and $f_i$ is the external force corresponding to global degree of freedom $i$, $F_j^e$ is the force acting on local degree for freedom $j$ ($j = 1, \ldots, 2d$) of element $e$ ($e = 1, \ldots, N_e$), and the Kronecker delta function (additional detail in Chapter 2) is defined as

$$\delta_{ab} = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{if } a \neq b. \end{cases} \tag{1.26}$$

The equilibrium system in (1.16) is recovered using the truss definition in (1.24) in the general system (1.25).

Compatibility of the truss structure requires all degrees of freedom coincident at a node must be equal due to the pin connection. That is, the $j$th local degree of freedom of element $e$ must be equal to the corresponding global degree of freedom $u_{\Xi_{je}}$:

$$u_j^e = u_{\Xi_{je}} = \sum_{s=1}^{dN_v} u_s \delta_{s\Xi_{je}}, \tag{1.27}$$

for local degrees of freedom $j = 1, \ldots, 2d$ and elements $e = 1, \ldots, N_e$, where $u_j^e$ is the local degree of freedom, $u_s$ ($s = 1, \ldots, dN_v$) are the global degrees of freedom, and the last equality follows from a simple identity. The compatibility conditions in (1.18) is recovered using the truss definition in (1.24) with the general conditions in (1.27).

The element equations (1.13) take the form of a matrix-vector product

$$F_j^e = \sum_{k=1}^{2d} K_{jk}^e u_k^e \tag{1.28}$$

for local degrees of freedom $j = 1, \ldots, 2d$, where $K_{jk}^e$ ($j, k = 1, \ldots, 2d$) are the entries of the local stiffness matrix for element $e$. Combining the element equations (1.28) with global compatibility (1.25)

$$R_i = \sum_{e=1}^{N_e} \sum_{j=1}^{2d} \sum_{k=1}^{2d} K_{jk}^e u_k^e \delta_{i\Xi_{je}} - f_i. \tag{1.29}$$

for global degrees of freedom $i = 1, \ldots, dN_v$. Compatibility is enforced by replacing the local degrees of freedom with the appropriate global degree of freedom from (1.27)

$$R_i = \sum_{e=1}^{N_e} \sum_{j=1}^{2d} \sum_{k=1}^{2d} \sum_{s=1}^{dN_v} K_{jk}^e u_s \delta_{i\Xi_{je}} \delta_{s\Xi_{ke}} - f_i. \tag{1.30}$$

After swapping the order of the summation, the governing equations reduce to a matrix-vector product as in (1.20)

$$R_i = \sum_{s=1}^{dN_v} \left[ \sum_{e=1}^{N_e} \sum_{j=1}^{2d} \sum_{k=1}^{2d} K_{jk}^e \delta_{i\Xi_{je}} \delta_{s\Xi_{ke}} \right] u_s - f_i, \tag{1.31}$$

and therefore, we identify the entries of the stiffness matrix explicitly as

$$K_{is} = \sum_{e=1}^{N_e} \sum_{j=1}^{2d} \sum_{k=1}^{2d} K_{jk}^e \delta_{i\Xi_{je}} \delta_{s\Xi_{ke}} \tag{1.32}$$

for $i, s = 1, \ldots, dN_v$. This equation suggests a simple procedure for constructing the global (assembled) stiffness matrix: 1) evaluate the element stiffness matrix using (1.14) and 2) fill entries of the global stiffness matrix using (1.32).

**Example: Entries of global stiffness matrix, truss from Figure 1.1**

We use the expression in (1.32) to compute $K_{11}$, $K_{21}$, $K_{43}$, $K_{64}$:

$$K_{11} = \sum_{e=1}^{3} \sum_{j=1}^{4} \sum_{k=1}^{4} K_{jk}^e \delta_{1\Xi_{je}} \delta_{1\Xi_{ke}} = K_{11}^1 + K_{11}^3$$

$$K_{21} = \sum_{e=1}^{3} \sum_{j=1}^{4} \sum_{k=1}^{4} K_{jk}^e \delta_{2\Xi_{je}} \delta_{1\Xi_{ke}} = K_{12}^1 + K_{12}^3$$

$$K_{43} = \sum_{e=1}^{3} \sum_{j=1}^{4} \sum_{k=1}^{4} K_{jk}^e \delta_{4\Xi_{je}} \delta_{3\Xi_{ke}} = K_{43}^1 + K_{21}^2$$

$$K_{64} = \sum_{e=1}^{3} \sum_{j=1}^{4} \sum_{k=1}^{4} K_{jk}^e \delta_{6\Xi_{je}} \delta_{4\Xi_{ke}} = K_{42}^2.$$

## 1.5. Displacement boundary conditions

The final task before we can solve for the nodal displacements of the truss structure is to apply the displacement boundary conditions. These are also called essential or Dirichlet boundary conditions. From the truss in Figure 1.1, we know the $x$ and $y$ displacement of node 1 are zero and the $y$ displacement of node 2 is zero, i.e., $u_1 = u_2 = u_4 = 0$. Since these displacements are *known*, we do not need to solve for them and will eliminate the corresponding equations from the system of equations in (1.20).

Consider a partition of the global degrees of freedom into those that are constrained (displacement known) and unconstrained (displacement unknown). Let $\mathcal{I}_c$ and $\mathcal{I}_u$ be sets of indices that partition the global degrees of freedom into constrained and unconstrained degrees of freedom, respectively. Then we apply this partition to the nodal displacements to yield the vector of unknown displacements as $\boldsymbol{u}_u = \boldsymbol{u}|_{\mathcal{I}_u}$ and known displacements as $\boldsymbol{u}_c = \boldsymbol{u}|_{\mathcal{I}_c}$, where $\boldsymbol{u}_{\mathcal{I}}$ is the restriction of the vector $\boldsymbol{u}$ to the indices in $\mathcal{I}$. We define $\boldsymbol{f}_c$, $\boldsymbol{f}_u$, $\boldsymbol{K}_{cc}$, $\boldsymbol{K}_{uc}$, $\boldsymbol{K}_{cu}$, $\boldsymbol{K}_{uu}$ similarly, where, e.g., $\boldsymbol{K}_{uc}$ results from restricting the rows of $\boldsymbol{K}$ to the indices in $\mathcal{I}_u$ and the columns to the indices in $\mathcal{I}_c$. In the context of the truss in Figure 1.1, these quantities are defined as: $\mathcal{I}_u = \{3, 5, 6\}$, $\mathcal{I}_c = \{1, 2, 4\}$,

$$\boldsymbol{u}_u = \begin{bmatrix} u_3 \\ u_5 \\ u_6 \end{bmatrix}, \quad \boldsymbol{u}_c = \begin{bmatrix} u_1 \\ u_2 \\ u_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{f}_u = \begin{bmatrix} 0 \\ f \\ 0 \end{bmatrix}, \quad \boldsymbol{f}_c = \begin{bmatrix} r_1 \\ r_2 \\ r_4 \end{bmatrix}, \tag{1.33}$$

and

$$\begin{aligned} \boldsymbol{K}_{uu} &= \begin{bmatrix} K_{33} & K_{35} & K_{36} \\ K_{53} & K_{55} & K_{56} \\ K_{63} & K_{65} & K_{66} \end{bmatrix} \qquad \boldsymbol{K}_{uc} = \begin{bmatrix} K_{31} & K_{32} & K_{34} \\ K_{51} & K_{52} & K_{54} \\ K_{61} & K_{62} & K_{64} \end{bmatrix} \\ \boldsymbol{K}_{cu} &= \begin{bmatrix} K_{13} & K_{15} & K_{16} \\ K_{23} & K_{25} & K_{26} \\ K_{43} & K_{45} & K_{46} \end{bmatrix} \qquad \boldsymbol{K}_{cc} = \begin{bmatrix} K_{11} & K_{12} & K_{14} \\ K_{21} & K_{22} & K_{24} \\ K_{41} & K_{42} & K_{44} \end{bmatrix}. \end{aligned} \tag{1.34}$$

Observe that both $\boldsymbol{u}_u$ (unknown nodal displacements) and $\boldsymbol{f}_c$ (reaction forces) are *unknown*, while $\boldsymbol{u}_c$ (prescribed displacements) and $\boldsymbol{f}_u$ (external load) are known. This will always be the case because wherever the displacement is known, there will be an unknown reaction force from the boundary condition and whenever the displacement is unknown (i.e., without a displacement boundary condition), there will be a known force. For more complex boundary conditions such as an elastic foundation, the displacement is unknown and the force is given as a function of the unknown displacement.

After re-arranging the ordering of the equations and variables, we can write

$$\boldsymbol{u} = \begin{bmatrix} \boldsymbol{u}_u \\ \boldsymbol{u}_c \end{bmatrix}, \qquad \boldsymbol{f} = \begin{bmatrix} \boldsymbol{f}_u \\ \boldsymbol{f}_c \end{bmatrix}, \qquad \boldsymbol{K} = \begin{bmatrix} \boldsymbol{K}_{uu} & \boldsymbol{K}_{uc} \\ \boldsymbol{K}_{cu} & \boldsymbol{K}_{cc} \end{bmatrix} \tag{1.35}$$

and the governing equation in (1.20) becomes

$$\boldsymbol{R}(\boldsymbol{u}) = \boldsymbol{K}\boldsymbol{u} - \boldsymbol{f} = \begin{bmatrix} \boldsymbol{K}_{uu} & \boldsymbol{K}_{uc} \\ \boldsymbol{K}_{cu} & \boldsymbol{K}_{cc} \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_u \\ \boldsymbol{u}_c \end{bmatrix} - \begin{bmatrix} \boldsymbol{f}_u \\ \boldsymbol{f}_c \end{bmatrix} \tag{1.36}$$

Expansion of this system of equations leads to two distinct systems: one for the nodal displacements $\boldsymbol{R}_u(\boldsymbol{u}_u; \boldsymbol{u}_c, \boldsymbol{f}_u) = 0$ and the other for the reaction forces $\boldsymbol{R}_c(\boldsymbol{f}_c; \boldsymbol{u}_u, \boldsymbol{u}_c) = 0$:

$$\begin{aligned} \boldsymbol{R}_u(\boldsymbol{u}_u; \boldsymbol{u}_c, \boldsymbol{f}_u) &= \boldsymbol{K}_{uu}\boldsymbol{u}_u + \boldsymbol{K}_{uc}\boldsymbol{u}_c - \boldsymbol{f}_u = \boldsymbol{0} \\ \boldsymbol{R}_c(\boldsymbol{f}_c; \boldsymbol{u}_u, \boldsymbol{u}_c) &= \boldsymbol{K}_{cu}\boldsymbol{u}_u + \boldsymbol{K}_{cc}\boldsymbol{u}_c - \boldsymbol{f}_c = \boldsymbol{0}. \end{aligned} \tag{1.37}$$

The semicolon notation is used to distinguish the primary variable (unknown) in the system of equations (left of semicolon) from the *data* or known quantities.

## 1.6. Solution of the global system

Finally, the solution of the truss problem reduces to the solution of the systems of equations in (1.37). Since the equations are linear, the unknown nodal displacement are defined as

$$\boldsymbol{u}_u = \boldsymbol{K}_{uu}^{-1}(\boldsymbol{f}_u - \boldsymbol{K}_{uc}\boldsymbol{u}_c). \tag{1.38}$$

The combination of this solution with the prescribed displacements in $\boldsymbol{u}_c$ gives the displacement of the entire truss structure and completes the analysis. From this information, the force, stress, strain, or any other quantity of interest can be computed. If the reaction forces are required, substitute $\boldsymbol{u}_u$ into $\boldsymbol{R}_c$ in (1.37) to yield

$$\boldsymbol{f}_c = \boldsymbol{K}_{cu}\boldsymbol{K}_{uu}^{-1}(\boldsymbol{f}_u - \boldsymbol{K}_{uc}\boldsymbol{u}_c) + \boldsymbol{K}_{cc}\boldsymbol{u}_c. \tag{1.39}$$

This process of applying boundary conditions to the global system in (1.20) and solving the resulting system in (1.38) and (1.39) sequentially is referred to as *static condensation*.

As a final note, the reader should always interpret $\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{b}$ as the solution of the linear system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ rather than explicit inversion of the matrix $\boldsymbol{A}$, which is unstable, time- and resource-intensive, and destroys sparsity. Either direct solvers such as LU factorizations or iterative solvers such as Conjugate Gradient can be used to solve the linear systems that arise in the DSM (or FEM). In this class, we will use MATLAB's backslash function which uses a direct method.

## 1.7. Connection to the finite element method

As mentioned in Section 1.1, the direct stiffness method is the finite element method applied to a naturally discrete system derived by physical laws. As such the DSM and FEM share many common features such as the assembly of global equations from element contributions, enforcement of compatibility based on the connectivity of the mesh, application of Dirichlet boundary conditions through static condensation, and solution of the resulting system of equations using direct or iterative solvers. By beginning with the direct stiffness method, we were able to avoid the complication of partial differential equations and their reformulation as a weak form while introducing the aforementioned critical steps of the FEM. As we will see, application of the FEM to PDEs will simply lead us to different element equations, but the remaining steps (assembly, compatibility, boundary conditions, solve) will be the same.

## 1.8. Summary

We summarize the key points from this chapter:

1) The direct stiffness method is introduced as the analog to the finite element method for naturally discrete problems such as the deformation of a truss structure.

2) The DSM contains many of the same ingredients of the FEM: element contribution to global system, compatibility, assembly of global system, enforcement of essential boundary conditions.

3) The element contribution for a truss element was derived by transforming to a local coordinate system aligned with each element.

4) Compatibility between the displacements of connected elements is enforced by introducing global degrees of freedom and requiring all coincident local degrees of freedom be equal.

5) Equilibrium at each node is used to derive a global system that combines all element contributions.

6) Displacement boundary conditions are enforced by partitioning the degrees of freedom into constrained and unconstrained and using static condensation to derive the governing equations for the unconstrained degrees of freedom.

7) The concepts of nodes, connectivity, and local-to-global degree of freedom mapping were introduced and used to derive an explicit expression for the entries of the assembled stiffness matrix in terms of the element stiffness matrix.