AME50541: Finite Element Methods
Homework 2: Due Monday, February 18, 2019

Problem 1: (10 points) Re-write the Navier equations using indicial notation and Einstein summation convention. Replace $x \to 1$, $y \to 2$, $z \to 3$.

$$\begin{aligned}
\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} + F_x = 0 \\
\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z} + F_y = 0 \\
\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + F_z = 0
\end{aligned}$$

Problem 2: (15 points) The elasticity tensor for a St. Venant-Kirchhoff material is given by $C_{ijkl} = \lambda \delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$, where λ, μ are the Lamé parameters. Calculate the stress tensor σ_{ij}, where $\sigma_{ij} = C_{ijkl}\epsilon_{kl}$ and ϵ_{kl} is the strain tensor. Make sure to use the fact that the strain tensor is symmetric ($\epsilon_{ij} = \epsilon_{ji}$). Also, calculate the deviatoric stress $s_{ij} = \sigma_{ij} - \frac{\sigma_{kk}}{3}\delta_{ij}$. In both cases, your answer should be in terms of λ, μ, and the strain tensor ϵ.

Problem 3: (10 points) From JNR 2.1: Construct the weak form of the nonlinear equation

$$\begin{aligned}
-\frac{d}{dx}\left(u\frac{du}{dx}\right) + f = 0 \quad \text{for} \quad 0 < x < L \\
\left.\left(u\frac{du}{dx}\right)\right|_{x=0} = 0, \quad u(L) = \sqrt{2}
\end{aligned}$$

Problem 4: (20 points) Re-write the incompressible Navier-Stokes equations

$$\begin{aligned}
-\frac{\partial}{\partial x}\left(\rho\nu\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial y}\left(\rho\nu\frac{\partial u}{\partial y}\right) - \frac{\partial}{\partial z}\left(\rho\nu\frac{\partial u}{\partial z}\right) + \rho u\frac{\partial u}{\partial x} + \rho v\frac{\partial u}{\partial y} + \rho w\frac{\partial u}{\partial z} + \frac{\partial p}{\partial x} = 0 \\
-\frac{\partial}{\partial x}\left(\rho\nu\frac{\partial v}{\partial x}\right) - \frac{\partial}{\partial y}\left(\rho\nu\frac{\partial v}{\partial y}\right) - \frac{\partial}{\partial z}\left(\rho\nu\frac{\partial v}{\partial z}\right) + \rho u\frac{\partial v}{\partial x} + \rho v\frac{\partial v}{\partial y} + \rho w\frac{\partial v}{\partial z} + \frac{\partial p}{\partial y} = 0 \\
-\frac{\partial}{\partial x}\left(\rho\nu\frac{\partial w}{\partial x}\right) - \frac{\partial}{\partial y}\left(\rho\nu\frac{\partial w}{\partial y}\right) - \frac{\partial}{\partial z}\left(\rho\nu\frac{\partial w}{\partial z}\right) + \rho u\frac{\partial w}{\partial x} + \rho v\frac{\partial w}{\partial y} + \rho w\frac{\partial w}{\partial z} + \frac{\partial p}{\partial z} = 0 \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0
\end{aligned}$$

with the boundary conditions $u = \bar{u}$, $v = \bar{v}$, $w = \bar{w}$ on $\partial\Omega_1$ and

$$\begin{aligned}
\rho\nu\left(\frac{\partial u}{\partial x}n_x + \frac{\partial u}{\partial y}n_y + \frac{\partial u}{\partial z}n_z\right) - pn_x = \rho\bar{t}_x \\
\rho\nu\left(\frac{\partial v}{\partial x}n_x + \frac{\partial v}{\partial y}n_y + \frac{\partial v}{\partial z}n_z\right) - pn_y = \rho\bar{t}_y \quad \text{on } \partial\Omega_2 \\
\rho\nu\left(\frac{\partial w}{\partial x}n_x + \frac{\partial w}{\partial y}n_y + \frac{\partial w}{\partial z}n_z\right) - pn_z = \rho\bar{t}_z
\end{aligned}$$

using indicial notation and Einstein summation convention, where $\boldsymbol{u} = (u, v, w)^T$ and $\boldsymbol{n}$ is the outward normal to $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2$. Replace $x \to 1$, $y \to 2$, $z \to 3$ and $u \to u_1$, $v \to u_2$, $w \to u_3$. Then construct the weak form of the equations.

Problem 5: (20 points) Consider a system of m conservation laws in a d-dimensional space

$$\nabla \cdot \boldsymbol{F}(\boldsymbol{U}, \nabla \boldsymbol{U}) = \boldsymbol{S}(\boldsymbol{U}) \quad \text{in } \Omega,$$

where $\boldsymbol{U} \in \mathbb{R}^m$ is the state, $\boldsymbol{F}(\boldsymbol{U}, \nabla \boldsymbol{U}) \in \mathbb{R}^{m \times d}$ is the flux function, $\boldsymbol{S}(\boldsymbol{U}) \in \mathbb{R}^m$ is a source term, and $\Omega \subset \mathbb{R}^d$ is the domain. The boundary conditions are $\boldsymbol{U} = \bar{\boldsymbol{U}}$ on $\partial\Omega_1$ and $\boldsymbol{F}(\boldsymbol{U}, \nabla \boldsymbol{U})\boldsymbol{n} = \bar{\boldsymbol{f}}$ on $\partial\Omega_2$, where $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2$ and $\boldsymbol{n}$ is the outward normal.

(a) Write the conservation law in indicial notation. Drop the arguments to the flux function and source term.

(b) Construct both the weighted residual and weak formulation of the governing equations.

(c) What conditions must the approximate solution $\boldsymbol{U}_N(\boldsymbol{x}) \approx \boldsymbol{U}(\boldsymbol{x})$ satisfy if applying (a) the method of weighted residuals or (b) the Ritz method? Why is it difficult to construct a solution basis if $\boldsymbol{F}$ is nonlinear in $\boldsymbol{U}$ or $\nabla \boldsymbol{U}$ if using the method of weighted residuals?

Problem 6: (40 points) Consider the equations associated with a simply supported beam and subjected to a uniform transverse load $q = q_0$:

$$\begin{aligned} \frac{d^2}{dx^2}\left(EI\frac{d^2 w}{dx^2}\right) &= q_0 \quad \text{for} \quad 0 < x < L \\ w = EI\frac{d^2 w}{dx^2} &= 0 \quad \text{at} \quad x = 0, L. \end{aligned}$$

Take $L = 1$, $EI = 1$, and $q_0 = -1$ and approximate the solution using the following methods:

(a) the method of weighted residuals (Galerkin) using the two-term trigonometric basis $w(x) \approx w_2(x) = c_1 \sin\left(\frac{\pi x}{L}\right) + c_2 \sin\left(\frac{2\pi x}{L}\right)$,

(b) the method of weighted residuals (collocation) using the same trigonometric basis and collocation nodes $x_1 = 0.25$ and $x_2 = 0.75$,

(c) the Ritz method using the same trigonometric basis, and

(d) the Ritz method using a two-term polynomial basis ($w(x) \approx w_2(x) = c_1 x(x - L) + c_2 x^2(x - L)^2$).

For each method, verify the solution basis satisfies the appropriate conditions. Plot the approximate solution generated by each method as well as the analytical solution. In a separate figure, plot the error of each method $e(x) = |w(x) - \tilde{w}(x)|$, where $\tilde{w}$ is the approximate solution, and the residual over the domain. Finally, quantify the error of each approximation using the L^2-norm

$$e_{L^2(\Omega)} = \int_\Omega |e(x)|^2 \, dV.$$

I recommend using some symbolic mathematics software (Maple, Mathematics, MATLAB, etc) to assist with the calculations.