AME50541: Finite Element Methods
Note 1: The Direct Stiffness Method

1 Introduction

The direct stiffness method (DSM) is a method to solve statically determinant or indeterminant structures that is particularly well-suited for computer implementation. It is the finite element method (FEM) applied to a naturally discrete system, e.g., one modeled as a set of idealized elements connected at nodes, rather than a partial differential equation (PDE). As such, the DSM will serve as a gentle introduction to finite element concepts such as an unstructured mesh, assembly, and application of boundary conditions without the complexity of partial differential equations.

In this document, we will solely consider the DSM in the context of a truss structure, defined as a structure that consists of slender, linear elastic members joined at their endpoints by pin joints (free rotation, i.e., does not support moments) with all loads (external loads and reaction forces) applied at nodes. The members are assumed to be of negligible weight (relative to the external loads), have a constant area and stiffness along their length, and the stress on any cross section is uniform. The assumption that the structure consists of members of negligible weight connected by pins and is only loaded at its nodes implies the force in each member is purely axial (pure compression or tension, no transverse force) and constant along its length. The assumption that the members have constant area and stiffness implies the strain in each element is constant, which in turn implies the axial displacement varies linearly along the length of the member. In the remainder of the document, we will consider the truss in Figure 1 for concreteness.

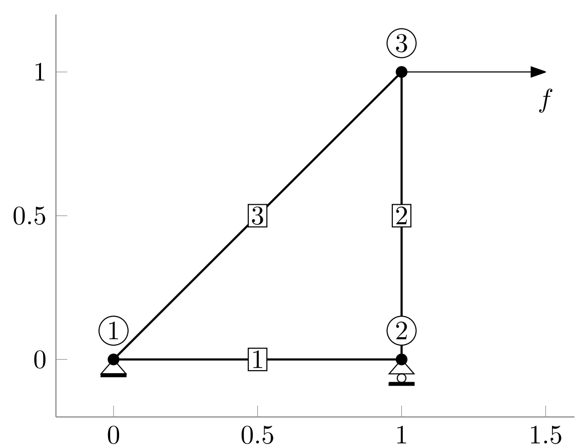

Figure 1: Truss structure with three nodes and elements, an x-directed load at node 3, a pinned (fixed x and y displacement) boundary condition at node 1, and a vertical roller (fixed y displacement) at node 2. The node and element numbers are shown in the figure; the node numbers are contained in circles and the element numbers in rectangles.

The direct stiffness method, like the finite element method, consists of four core tasks: (1) compute **element contributions** to global equations, (2) **assemble** the global equations from element contributions, (3) apply of **boundary conditions**, and (4) **solve** the global system. The remainder of this document details each of these tasks using the truss in Figure 1 as a guiding example.

2 Element contribution to global equations

The goal of this section is to derive a relationship between the force in each element and the displacement of its nodes in the coordinate system of the structure (x-y). However, the force-displacement relationship is most readily derived in a coordinate system aligned with the element. To this end, we consider an arbitrary element e from the truss and introduce a coordinate system ($\bar{x}^e$-$\bar{y}^e$) such that the first coordinate direction

($\bar{x}^e$) is aligned with the axis of the bar (Figure 2). Each element consists of two local nodes whose numbering is independent of the global node numbering in Figure 1 and taken as $\{1, 2\}$ for convenience. Let θ_e denote the angle (counterclockwise) from the horizontal to the element axis, i.e., the angle between the $\bar{x}^e$- and x-axis. Let $(\bar{u}_i^e, \bar{v}_i^e)$ denote the displacement in the $(\bar{x}, \bar{y})$ direction of local node $i \in \{1, 2\}$ and $\bar{F}_i^e$ be the force at local node i in the $\bar{x}^e$-direction. The force in the $\bar{y}^e$-direction has been intentionally excluded because, from the assumptions stated in Section 1, the force is the members is purely axial. Finally, we denote the displacement and forces at local node i of element e in the global coordinate system (x-y) as (u_i^e, v_i^e) and (F_{ix}^e, F_{iy}), respectively.

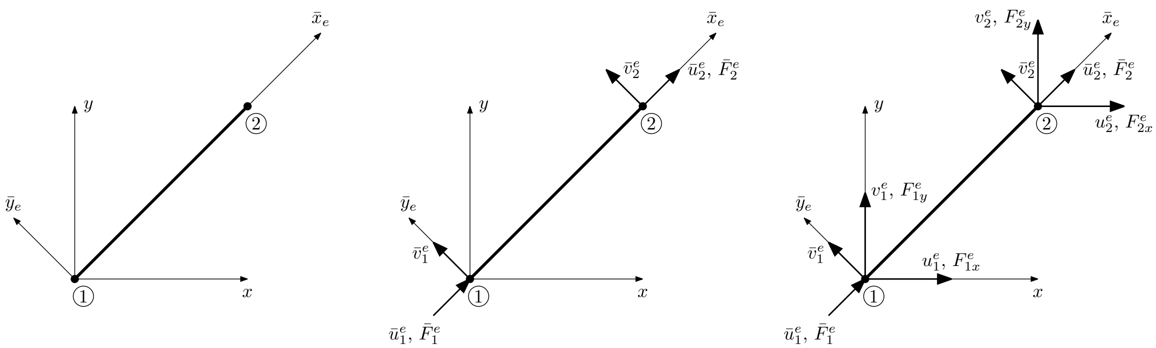

Figure 2: Local and global coordinate system for bar 2 in the truss from Figure 1 (*left*), the forces and displacements in the element coordinate system (*center*), and the forces and displacements in the global coordinate system (*right*).

With this notation and the assumptions in Section 1, the (constant) strain in the element, defined as the change in length of a member relative to its original length, is

$$\bar{\epsilon}_e = \frac{\bar{u}_2^e - \bar{u}_1^e}{h_e}. \tag{1}$$

Then, from Hooke's law (linear elasticity) that linearly relates stress and strain ($\sigma = E\epsilon$), and the defintion of stress as force per unit area ($F = \sigma A$), we have

$$\begin{aligned} \bar{F}_1^e &= -\bar{\sigma}_e A_e = -E_e A_e \bar{\epsilon}_e = -\frac{E_e A_e}{h_e}(\bar{u}_2^e - \bar{u}_1^e) \\ \bar{F}_2^e &= \quad \bar{\sigma}_e A_e = \quad E_e A_e \bar{\epsilon}_e = \quad \frac{E_e A_e}{h_e}(\bar{u}_2^e - \bar{u}_1^e), \end{aligned} \tag{2}$$

where A_e, E_e, $\bar{\epsilon}_e$, and $\bar{\sigma}_e$ are the cross-sectional area, stiffness (Young's modulus), strain, and stress of member e, respectively, all of which are constant along its length. The sign of the force-stress relationship is chosen such that the direction of the force agrees with the sign of the stress, e.g., a compressive (negative) stress must correspond to a force directed into the bar.

Before closing this section, we relate the forces and displacements in the element coordinate system to the global coordinate system for global assembly of the elements in the next section. The following rotation matrix will rotate a vector, $\boldsymbol{v} \in \mathbb{R}^2$, *clockwise* by θ

$$\boldsymbol{T}(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}. \tag{3}$$

A convenient property of rotation matricies is they are orthogonal, i.e., $\boldsymbol{T}(\theta)^T = \boldsymbol{T}(\theta)^{-1}$. Let $\boldsymbol{e}_i$ and $\bar{\boldsymbol{e}}_i^e$ be unit vectors aligned with the x-y and $\bar{x}^e$-$\bar{y}^e$ coordinate axes, respectively. From the configuration of the x-y coordinate system, $\boldsymbol{e}_1 = (1, 0)^T$ and $\boldsymbol{e}_2 = (0, 1)^T$. Then, we have the relationship

$$\boldsymbol{e}_i = \boldsymbol{T}(\theta_e)\bar{\boldsymbol{e}}_i^e \tag{4}$$

from the defintion of the $\bar{x}^e$-$\bar{y}^e$ coordinate system. Expansion of any vector $\boldsymbol{v} \in \mathbb{R}^2$ in these coordinate systems gives

$$\boldsymbol{v} = v_1\boldsymbol{e}_1 + v_2\boldsymbol{e}_2 = \bar{v}_1\bar{\boldsymbol{e}}_1^e + \bar{v}_2\bar{\boldsymbol{e}}_2^e, \tag{5}$$

where (v_1, v_2) are the coordinates of $\boldsymbol{v}$ in the x-y coordinates system and $(\bar{v}_1, \bar{v}_2)$ are the coordinates of $\boldsymbol{v}$ in the $\bar{x}^e$-$\bar{y}^e$ coordinate system. From the above equivalence between coordinate systems and (4), we have

$$\begin{bmatrix}\bar{v}_1\\ \bar{v}_2\end{bmatrix} = \bar{v}_1\boldsymbol{e}_1 + \bar{v}_2\boldsymbol{e}_2 = \bar{v}_1\boldsymbol{T}(\theta_e)\bar{\boldsymbol{e}}_1^e + \bar{v}_2\boldsymbol{T}(\theta_e)\bar{\boldsymbol{e}}_2^e = \boldsymbol{T}(\theta_e)(\bar{v}_1\bar{\boldsymbol{e}}_1^e + \bar{v}_2\bar{\boldsymbol{e}}_2^e) = \boldsymbol{T}(\theta_e)\boldsymbol{v} = \boldsymbol{T}(\theta_e)\begin{bmatrix}v_1\\ v_2\end{bmatrix}. \tag{6}$$

Thus, the displacements and forces are transferred between coordinate systems as

$$\begin{bmatrix}\bar{u}_i^e\\ \bar{v}_i^e\end{bmatrix} = \boldsymbol{T}(\theta_e)\begin{bmatrix}u_i^e\\ v_i^e\end{bmatrix}, \qquad \begin{bmatrix}\bar{F}_i^e\\ 0\end{bmatrix} = \boldsymbol{T}(\theta_e)\begin{bmatrix}F_{ix}^e\\ F_{iy}^e\end{bmatrix}. \tag{7}$$

Combining these expressions with the element equations in (2) leads to the desired relationship between the elemental displacements and forces in the global coordinate system

$$\begin{bmatrix}F_{1x}^e\\ F_{1y}^e\\ F_{2x}^e\\ F_{2y}^e\end{bmatrix} = \frac{E_eA_e}{h_e}\begin{bmatrix}\boldsymbol{T}(\theta_e) & \\ & \boldsymbol{T}(\theta_e)\end{bmatrix}^T\begin{bmatrix}1 & 0 & -1 & 0\\ 0 & 0 & 0 & 0\\ -1 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\boldsymbol{T}(\theta_e) & \\ & \boldsymbol{T}(\theta_e)\end{bmatrix}\begin{bmatrix}u_1^e\\ v_1^e\\ u_2^e\\ v_2^e\end{bmatrix}. \tag{8}$$

By introducing the element force, $\boldsymbol{F}^e$, displacement, $\boldsymbol{u}^e$, and stiffness matrix, $\boldsymbol{K}^e$, as

$$\boldsymbol{F}^e = \begin{bmatrix}F_{1x}^e\\ F_{1y}^e\\ F_{2x}^e\\ F_{2y}^e\end{bmatrix}, \quad \boldsymbol{u}^e = \begin{bmatrix}u_1^e\\ v_1^e\\ u_2^e\\ v_2^e\end{bmatrix}, \quad \boldsymbol{K}^e = \frac{E_eA_e}{h_e}\begin{bmatrix}\boldsymbol{T}(\theta_e) & \\ & \boldsymbol{T}(\theta_e)\end{bmatrix}^T\begin{bmatrix}1 & 0 & -1 & 0\\ 0 & 0 & 0 & 0\\ -1 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\boldsymbol{T}(\theta_e) & \\ & \boldsymbol{T}(\theta_e)\end{bmatrix}, \tag{9}$$

the preceeding equations simplify to

$$\boldsymbol{F}^e = \boldsymbol{K}^e\boldsymbol{u}^e. \tag{10}$$

3 Assembly of global equations

With the relationship between the nodal displacements and forces for each element in the global coordinate system established in (8), we are ready to derive the governing equations for the global system. Since this is a static structure (not a mechanism), each node must be in equilibrium, that is, the sum of forces acting of the node from the elements, external loads, and reactions, should be zero. Equilibrium of each node in the truss in Figure 1 leads to

$$\begin{aligned}
R_{1x} &= F_{1x}^1 + F_{1x}^3 - r_{1x} = 0\\
R_{1y} &= F_{1y}^1 + F_{1y}^3 - r_{1y} = 0\\
R_{2x} &= F_{2x}^1 + F_{1x}^2 = 0\\
R_{2y} &= F_{2y}^1 + F_{1y}^2 - r_{2y} = 0\\
R_{3x} &= F_{2x}^2 + F_{2x}^3 - f = 0\\
R_{3y} &= F_{2y}^2 + F_{2y}^3 = 0
\end{aligned} \tag{11}$$

where r_{1x}, r_{1y} are the reaction forces at node 1, r_{2y} is the reaction force at node 2, and all elements have been assigned local node numbers such that local node 1 corresponds to the smaller global node number. The equations are written in this expanded form to highlight that the global equations are merely a summation over the appropriate element equations (8), a procedure usually referred to as *assembly*. Substituting the element contributions (8) into the equilibrium equations (11) leads to the system that relates displacements and forces for the entire structure

$$\begin{aligned}
R_{1x} &= K^1_{11}u^1_1 + K^1_{12}v^1_1 + K^1_{13}u^1_2 + K^1_{14}v^1_2 + K^3_{11}u^3_1 + K^3_{12}v^3_1 + K^3_{13}u^3_2 + K^3_{14}v^3_2 - r_{1x}\\
R_{1y} &= K^1_{21}u^1_1 + K^1_{22}v^1_1 + K^1_{23}u^1_2 + K^1_{24}v^1_2 + K^3_{21}u^3_1 + K^3_{22}v^3_1 + K^3_{23}u^3_2 + K^3_{24}v^3_2 - r_{1y}\\
R_{2x} &= K^1_{31}u^1_1 + K^1_{32}v^1_1 + K^1_{33}u^1_2 + K^1_{34}v^1_2 + K^2_{11}u^2_1 + K^2_{12}v^2_1 + K^2_{13}u^2_2 + K^2_{14}v^2_2\\
R_{2y} &= K^1_{41}u^1_1 + K^1_{42}v^1_1 + K^1_{43}u^1_2 + K^1_{44}v^1_2 + K^2_{21}u^2_1 + K^2_{22}v^2_1 + K^2_{23}u^2_2 + K^2_{24}v^2_2 - r_{2y}\\
R_{3x} &= K^2_{31}u^2_1 + K^2_{32}v^2_1 + K^2_{33}u^2_2 + K^2_{34}v^2_2 + K^3_{31}u^3_1 + K^3_{32}v^3_1 + K^3_{33}u^3_2 + K^2_{34}v^3_2 - f\\
R_{3y} &= K^2_{41}u^2_1 + K^2_{42}v^2_1 + K^2_{43}u^2_2 + K^2_{44}v^2_2 + K^3_{41}u^3_1 + K^3_{42}v^3_1 + K^3_{43}u^3_2 + K^2_{44}v^3_2.
\end{aligned} \tag{12}$$

Next, we enforce compatibility of displacements at nodes by relating element displacements (displacement of the nodes of each element) to global nodal displacements. Let (u_i, v_i) be the displacement of node i in the global truss structure. Then, the displacement of the end of every element that meet at node i must be equal to (u_i, v_i) due to the pin connection

$$\begin{aligned}
u_1 &= u^1_1 = u^3_1 \qquad & u_2 &= u^1_2 = u^2_1 \qquad & u_3 &= u^2_2 = u^3_2\\
v_1 &= v^1_1 = v^3_1 \qquad & v_2 &= v^1_2 = v^2_1 \qquad & v_3 &= v^2_2 = v^3_2.
\end{aligned} \tag{13}$$

Substition of these compatibility conditions into (12) leads to the final form of the assembled equations that enforces equilibrium, compatibility of displacements at nodes, and the element equations

$$\begin{aligned}
R_{1x} &= (K^1_{11} + K^3_{11})u_1 + (K^1_{12} + K^3_{12})v_1 + K^1_{13}u_2 + K^1_{14}v_2 + K^3_{13}u_3 + K^3_{14}v_3 - r_{1x}\\
R_{1y} &= (K^1_{21} + K^3_{21})u_1 + (K^1_{22} + K^3_{22})v_1 + K^1_{23}u_2 + K^1_{24}v_2 + K^3_{23}u_3 + K^3_{24}v_3 - r_{1y}\\
R_{2x} &= K^1_{31}u_1 + K^1_{32}v_1 + (K^1_{33} + K^2_{11})u_2 + (K^1_{34} + K^2_{12})v_2 + K^2_{13}u_3 + K^2_{14}v_3\\
R_{2y} &= K^1_{41}u_1 + K^1_{42}v_1 + (K^1_{43} + K^2_{21})u_2 + (K^1_{44} + K^2_{22})v_2 + K^2_{23}u_3 + K^2_{24}v_3 - r_{2y}\\
R_{3x} &= K^3_{31}u_1 + K^3_{32}v_1 + K^2_{31}u_2 + K^2_{32}v_2 + (K^2_{33} + K^3_{33})u_3 + (K^3_{34} + K^2_{34})v_3 - f\\
R_{3y} &= K^2_{41}u_1 + K^2_{42}v_1 + K^2_{41}u_2 + K^2_{42}v_2 + (K^2_{43} + K^3_{43})u_3 + (K^3_{44} + K^2_{44})v_3.
\end{aligned} \tag{14}$$

This can be compactly written as

$$\boldsymbol{R}(\boldsymbol{u}, \boldsymbol{f}) = \boldsymbol{K}\boldsymbol{u} - \boldsymbol{f} = \boldsymbol{0}, \tag{15}$$

where the residual ($\boldsymbol{R}$), vector of nodal displacements ($\boldsymbol{u}$), and vector of external forces ($\boldsymbol{f}$) are

$$\boldsymbol{R} = \begin{bmatrix} R_{1x}\\ R_{1y}\\ R_{2x}\\ R_{2y}\\ R_{3x}\\ R_{3y} \end{bmatrix}, \qquad \boldsymbol{u} = \begin{bmatrix} u_1\\ v_1\\ u_2\\ v_2\\ u_3\\ v_3 \end{bmatrix}, \qquad \boldsymbol{f} = \begin{bmatrix} r_{1x}\\ r_{1y}\\ 0\\ r_{2y}\\ f\\ 0 \end{bmatrix} \tag{16}$$

and the stiffness matrix ($\boldsymbol{K}$) is

$$\boldsymbol{K} = \begin{bmatrix}
K^1_{11} + K^3_{11} & K^1_{12} + K^3_{12} & K^1_{13} & K^1_{14} & K^3_{13} & K^3_{14}\\
K^1_{21} + K^3_{21} & K^1_{22} + K^3_{22} & K^1_{23} & K^1_{24} & K^3_{23} & K^3_{24}\\
K^1_{31} & K^1_{32} & K^1_{33} + K^2_{11} & K^1_{34} + K^2_{12} & K^2_{13} & K^2_{14}\\
K^1_{41} & K^1_{42} & K^1_{43} + K^2_{21} & K^1_{44} + K^2_{22} & K^2_{23} & K^2_{24}\\
K^3_{31} & K^3_{32} & K^2_{31} & K^2_{32} & K^2_{33} + K^3_{33} & K^3_{34} + K^2_{34}\\
K^3_{41} & K^3_{42} & K^2_{41} & K^2_{42} & K^2_{43} + K^3_{43} & K^3_{44} + K^2_{44}
\end{bmatrix}. \tag{17}$$

The stiffness matrix is exactly the derivative of the residual with respect to the nodal coordinates, i.e.,

$$\boldsymbol{K} = \frac{\partial \boldsymbol{R}}{\partial \boldsymbol{u}}, \tag{18}$$

and therefore will also be referred to as the *Jacobian* matrix (the matrix of partial derivatives) of the residual.

4 Boundary conditions

The final task before we can solve for the nodal displacements of the truss structure is to apply the displacement boundary conditions. These are also called essential or Dirichlet boundary conditions. From the truss in Figure 1, we know the x and y displacement of node 1 are zero and the y displacement of node 2 is zero, i.e., $u_1 = v_1 = v_2 = 0$. Since these displacements are *known*, we do not need to solve for them and will eliminate the corresponding equations from the system of equations in (15).

Consider a partition of the global degrees of freedom into those with prescribed displacements (displacement known) and forces (displacement unknown). Let $\mathcal{I}_d$ and $\mathcal{I}_f$ be sets of indices that partition the global degrees of freedom into those with prescribed displacements and forces, respectively. Then we apply this partition to the nodal displacements to yield the vector of unknown displacements as $\boldsymbol{u}_f = \boldsymbol{u}|_{\mathcal{I}_f}$ and known displacements as $\boldsymbol{u}_d = \boldsymbol{u}|_{\mathcal{I}_d}$, where $\boldsymbol{u}_{\mathcal{I}}$ is the restriction of the vector $\boldsymbol{u}$ to the indices in $\mathcal{I}$. We define $\boldsymbol{f}_d$, $\boldsymbol{f}_d$, $\boldsymbol{K}_{dd}$, $\boldsymbol{K}_{fu}$, $\boldsymbol{K}_{uf}$, $\boldsymbol{K}_{ff}$ similarly, where, e.g., $\boldsymbol{K}_{uf}$ results from restricting the rows of $\boldsymbol{K}$ to the indices in $\mathcal{I}_f$ and the columns to the indices in $\mathcal{I}_d$. In the context of the truss in Figure 1, these quantities are defined as: $\mathcal{I}_f = \{3, 5, 6\}$, $\mathcal{I}_d = \{1, 2, 4\}$,

$$\boldsymbol{u}_f = \begin{bmatrix} u_2 \\ u_3 \\ v_3 \end{bmatrix}, \quad \boldsymbol{u}_d = \begin{bmatrix} u_1 \\ v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{f}_f = \begin{bmatrix} 0 \\ f \\ 0 \end{bmatrix}, \quad \boldsymbol{f}_d = \begin{bmatrix} r_{1x} \\ r_{1y} \\ r_{2y} \end{bmatrix}, \tag{19}$$

and

$$\begin{aligned} \boldsymbol{K}_{ff} &= \begin{bmatrix} K_{33} & K_{35} & K_{36} \\ K_{53} & K_{55} & K_{56} \\ K_{63} & K_{65} & K_{66} \end{bmatrix} \quad & \boldsymbol{K}_{fd} &= \begin{bmatrix} K_{31} & K_{32} & K_{34} \\ K_{51} & K_{52} & K_{54} \\ K_{61} & K_{62} & K_{64} \end{bmatrix} \\ \boldsymbol{K}_{df} &= \begin{bmatrix} K_{13} & K_{15} & K_{16} \\ K_{23} & K_{25} & K_{26} \\ K_{43} & K_{45} & K_{46} \end{bmatrix} \quad & \boldsymbol{K}_{dd} &= \begin{bmatrix} K_{11} & K_{12} & K_{14} \\ K_{21} & K_{22} & K_{24} \\ K_{41} & K_{42} & K_{44} \end{bmatrix}. \end{aligned} \tag{20}$$

Observe that both $\boldsymbol{u}_f$ (unknown nodal displacements) and $\boldsymbol{f}_d$ (reaction forces) are *unknown*, while $\boldsymbol{u}_d$ (prescribed displacements) and $\boldsymbol{f}_f$ (external load) are known. This will always be the case because wherever the displacement is known, there will be an unknown reaction force from the boundary condition and whenever the displacement is unknown (i.e., without a displacement boundary condition), there will be a known force. For more complex boundary conditions such as an elastic foundation, the displacement is unknown and the force is given as a function of the unknown displacement.

After re-arranging the ordering of the equations and variables, we can write

$$\boldsymbol{u} = \begin{bmatrix} \boldsymbol{u}_f \\ \boldsymbol{u}_d \end{bmatrix}, \qquad \boldsymbol{f} = \begin{bmatrix} \boldsymbol{f}_f \\ \boldsymbol{f}_d \end{bmatrix}, \qquad \boldsymbol{K} = \begin{bmatrix} \boldsymbol{K}_{ff} & \boldsymbol{K}_{fd} \\ \boldsymbol{K}_{df} & \boldsymbol{K}_{dd} \end{bmatrix} \tag{21}$$

and the governing equation in (15) becomes

$$\boldsymbol{R}(\boldsymbol{u}) = \boldsymbol{K}\boldsymbol{u} - \boldsymbol{f} = \begin{bmatrix} \boldsymbol{K}_{ff} & \boldsymbol{K}_{fd} \\ \boldsymbol{K}_{df} & \boldsymbol{K}_{dd} \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_f \\ \boldsymbol{u}_d \end{bmatrix} - \begin{bmatrix} \boldsymbol{f}_f \\ \boldsymbol{f}_d \end{bmatrix} \tag{22}$$

Expansion of this system of equations leads to two distinct systems: one for the nodal displacements $\boldsymbol{R}_f(\boldsymbol{u}_f; \boldsymbol{u}_d, \boldsymbol{f}_f) = 0$ and the other for the reaction forces $\boldsymbol{R}_d(\boldsymbol{f}_d; \boldsymbol{u}_f, \boldsymbol{u}_d) = 0$:

$$\begin{aligned} \boldsymbol{R}_f(\boldsymbol{u}_f; \boldsymbol{u}_d, \boldsymbol{f}_f) &= \boldsymbol{K}_{ff}\boldsymbol{u}_f + \boldsymbol{K}_{fd}\boldsymbol{u}_d - \boldsymbol{f}_f = \boldsymbol{0} \\ \boldsymbol{R}_d(\boldsymbol{f}_d; \boldsymbol{u}_f, \boldsymbol{u}_d) &= \boldsymbol{K}_{df}\boldsymbol{u}_f + \boldsymbol{K}_{dd}\boldsymbol{u}_d - \boldsymbol{f}_d = \boldsymbol{0}. \end{aligned} \tag{23}$$

The semicolon notation is used to distinguish the primary variable (unknown) in the system of equations (left of semicolon) from the *data* or known quantities.

5 Solution of the global system

Finally, the solution of the truss problem reduces to the solution of the systems of equations in (23). Since the equations are linear, the uknown nodal displacement are defined as

$$\boldsymbol{u}_f = \boldsymbol{K}_{ff}^{-1}(\boldsymbol{f}_f - \boldsymbol{K}_{fd}\boldsymbol{u}_d). \tag{24}$$

The combination of this solution with the prescribed displacements in $\boldsymbol{u}_d$ gives the displacement of the entire truss structure and completes the analysis. From this information, the force, stress, strain, or any other quantity of interest can be computed. If the reaction forces are required, substitue $\boldsymbol{u}_f$ into $\boldsymbol{R}_d$ in (23) to yield

$$\boldsymbol{f}_d = \boldsymbol{K}_{df}\boldsymbol{K}_{ff}^{-1}(\boldsymbol{f}_f - \boldsymbol{K}_{fd}\boldsymbol{u}_d) + \boldsymbol{K}_{dd}\boldsymbol{u}_d. \tag{25}$$

This process of applying boundary conditions to the global system in (15) and solving the resulting system in (24) and (25) sequentially is referred to as *static condensation*.

As a final note, the reader should always interpret $\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{b}$ as the solution of the linear system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ rather than explicit inversion of the matrix $\boldsymbol{A}$, which is unstable, time- and resource-intensive, and destroys sparsity. Either direct solvers such as LU factorizations or iterative solvers such as Conjugate Gradient can be used to solve the linear systems that arise in the DSM (or FEM). In this class, we will use MATLAB's backslash function which uses a direct method.

6 Connection to the finite element method

As mentioned in Section 1, the direct stiffness method is the finite element method applied to a naturally discrete system derived by physical laws. As such the DSM and FEM share many common features such as the assembly of global equations from element contributions, enforcement of compatibility based on the connectivity of the mesh, application of Dirichlet boundary conditions through static condensation, and solution of the resulting system of equations using direct or iterative solvers. By beginning with the direct stiffness method, we were able to avoid the complication of partial differential equations and their reformulation as a weak form while introducing the aforementioned critical steps of the FEM. As we will see, application of the FEM to PDEs will simply lead us to different element equations, but the remaining steps (assembly, compatibility, boundary conditions, solve) will be the same.