Chapter 6

Finite element method: d-dimensions

6.1. Introduction

In this chapter we generalize the formulation, construction, implementation, and error analysis of the finite element method introduced in Chapter 4 for one-dimensional problems to PDEs over domains in $\Omega \subset \mathbb{R}^d$ ($d \geqslant 1$) using the variational formalism introduced in Chapter 5. For simplicity, we restrict attention to linear, second-order, scalar-valued PDEs. The extension to nonlinear, systems of PDEs will be addressed in Chapters 7-8; we will not have time to consider higher order PDEs. As we will see, most of the details of the construction are unchanged in the more complex setting (nonlinear, systems of PDEs), which makes this a reasonable and highly relevant starting point.

6.2. Finite element formulation

Consider a domain $\Omega \subset \mathbb{R}^d$ with boundary $\partial\Omega$ partitioned as $\partial\Omega = \overline{\partial\Omega_D} \cup \overline{\partial\Omega_N}$ and the following abstract variational problem:

$$\text{find } u \in \mathcal{V} \text{ such that } B(w, u) = \ell(w) \text{ for all } w \in \mathcal{V}^0, \tag{6.1}$$

where $B : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ is a bilinear form, $\ell : H^1(\Omega) \to \mathbb{R}$ is a linear functional, and $\mathcal{V}, \mathcal{V}^0 \subset H^1(\Omega)$ are the following subsets

$$\mathcal{V} := \left\{ f \in H^1(\Omega) \;\middle|\; f|_{\partial\Omega_D} = g \right\}, \qquad \mathcal{V}^0 := \left\{ f \in H^1(\Omega) \;\middle|\; f|_{\partial\Omega_D} = 0 \right\}. \tag{6.2}$$

The function $g \in L^2(\partial\Omega)$ is the prescribed essential boundary condition, $\mathcal{V}$ is an affine subspace, and $\mathcal{V}^0$ is a linear subspace. Since $\mathcal{V}$ is affine, it can be written as $\mathcal{V} = \varphi + \mathcal{V}^0$ for any $\varphi \in \mathcal{V}$. Following the procedure in Chapter **??**, the variational problem can be converted to a bilinear form with the same test and trial space:

$$\text{find } \bar{u} \in \mathcal{V}^0 \text{ such that } B(w, \bar{u}) = \bar{\ell}(w) := \ell(w) - B(w, \varphi) \text{ for all } w \in \mathcal{V}^0. \tag{6.3}$$

We assume the bilinear form B is continuous and coercive on $\mathcal{V}^0$ and the linear functional $\bar{\ell}$ is continuous on $\mathcal{V}^0$. Then the Lax-Milgram theorem guarantees (6.3) (and therefore (6.1)) possesses a unique solution.

The finite element method introduces a finite-dimensional linear subspace $\mathcal{V}_h^0 \subset \mathcal{V}^0$, which leads to the variational Galerkin formulation:

$$\text{find } u_h \in \mathcal{V}_h \text{ such that } B(w_h, u_h) = \ell(w_h) \text{ for all } w_h \in \mathcal{V}_h^0, \tag{6.4}$$

where $\mathcal{V}_h := \varphi + \mathcal{V}_h^0$ for any $\varphi \in \mathcal{V}$. The FE variational problem possesses a unique solution (Lax-Milgram theorem) since the properties of the bilinear form (continuity, coercivity) and linear functional (continuity) are inherited by virtue of $\mathcal{V}_h^0$ being a linear subspace of $\mathcal{V}^0$. Similar to Chapter 4, the finite element subspace is defined by partitioning the domain into elements and introducing a local, finite-dimensional function space over each element.

Consider a partition $\mathcal{E}_h$ of Ω, e.g., a collection of non-overlapping elements (open sets) $\Omega_1, \dots, \Omega_{N_{\text{el}}} \subset \Omega$ that cover the domain, that is,

$$\mathcal{E}_h \coloneqq \{\Omega_e\}_{e=1}^{N_{\text{el}}}, \qquad \Omega = \bigcup_{e=1}^{N_{\text{el}}} \overline{\Omega}_e, \qquad \Omega_e \cap \Omega_{e'} = \varnothing \ \ (e \neq e'). \tag{6.5}$$

We associate a local, finite-dimensional function space $\mathcal{Y}(\Omega_e) \subset \mathcal{C}^\infty(\Omega_e)$ to each element $\Omega_e \in \mathcal{E}_h$ and define the finite element subspace as

$$\mathcal{V}_h^0 \coloneqq \left\{ f \in H^1(\Omega) \;\middle|\; f|_{\Omega_e} \in \mathcal{Y}(\Omega_e) \ \forall \Omega_e \in \mathcal{E}_h, \ f|_{\partial\Omega_D} = 0 \right\}. \tag{6.6}$$

It can be shown (Example 5.3) that any $f \in \mathcal{V}_h^0$ is *continuous*, i.e., $\mathcal{V}_h^0 \subset \mathcal{C}^0(\Omega)$; in the following sections, when we construct the finite element space, special attention will be given to enforcing this continuity requirement. Furthermore, $\mathcal{V}_h^0 \subset \mathcal{V}^0$ is a finite-dimensional linear subspace ($\bar{N}_{\text{dof}} \coloneqq \dim \mathcal{V}_h^0$).

Let $\{\Phi_1, \dots, \Phi_{\bar{N}_{\text{dof}}}\}$ be a basis for $\mathcal{V}_h^0$, then any $u \in \mathcal{V}_h$ can be written as

$$u = \varphi + \sum_{I=1}^{\bar{N}_{\text{dof}}} \hat{u}_I^{\text{u}} \Phi_I. \tag{6.7}$$

In finite-dimensional setting, the variational problem (6.1) is equivalent to

$$\text{find } u_h \in \mathcal{V}_h \text{ such that } B(\Phi_I, u_h) = \ell(\Phi_I) \text{ for } I = 1, \dots, \bar{N}_{\text{dof}} \tag{6.8}$$

by Proposition 3.1, which reduces to

$$\sum_{J=1}^{\bar{N}_{\text{dof}}} B(\Phi_I, \Phi_J)\hat{u}_J = \ell(\Phi_I) - B(\Phi_I, \varphi). \tag{6.9}$$

This can be written compactly as a linear system of equations

$$\hat{\boldsymbol{K}}^{\text{uu}} \hat{\boldsymbol{u}}^{\text{u}} = \hat{\boldsymbol{b}}^{\text{u}}, \tag{6.10}$$

where we defined $\hat{\boldsymbol{K}}^{\text{uu}} \in M_{\bar{N}_{\text{dof}}, \bar{N}_{\text{dof}}}(\mathbb{R})$ and $\hat{\boldsymbol{b}}^{\text{u}} \in \mathbb{R}^{\bar{N}_{\text{dof}}}$ as

$$\hat{K}_{IJ}^{\text{uu}} \coloneqq B(\Phi_I, \Phi_J), \qquad \hat{f}_I^{\text{u}} \coloneqq \ell(\Phi_I), \qquad \hat{b}_I^{\text{u}} \coloneqq \hat{f}_I^{\text{u}} - B(\Phi_I, \varphi) \tag{6.11}$$

for $I, J = 1, \dots, \bar{N}_{\text{dof}}$. In the following section, we introduce a concrete definition of a mesh and local function space $\mathcal{Y}(\Omega_e)$ to reduce this expression to a concrete, *computable* form.

6.3. H^1-conforming finite elements

In this section we provide a formal definition of a finite element and construct the most widely used finite elements: H^1 conforming (nodal) elements.

Definition 6.3.1 (Finite element)**.** Let

(i) $K \subset \mathbb{R}^d$ be a compact set with non-empty interior and piecewise smooth boundary (element domain),
(ii) $\mathcal{Y}$ be a finite-dimensional function space on K ($M \coloneqq \dim \mathcal{Y}$) (space of shape functions), and
(iii) $\mathcal{D} \coloneqq \{D_1, \dots, D_M\}$ is a basis for $\mathcal{Y}'$ (dual space of $\mathcal{Y}$) (degrees of freedom).

Then $(K, \mathcal{Y}, \mathcal{D})$ is called a *finite element.*

In this course we will only consider *nodal* elements. Let $\mathcal{N} \subset K$ be a set of $N_{\text{nd}}^{\text{el}}$ ordered *nodes* denoted $\hat{\boldsymbol{\xi}}_1, \dots, \hat{\boldsymbol{\xi}}_{N_{\text{nd}}^{\text{el}}}$ with the jth coordinate ($j = 1, \dots, d$) of node i ($i = 1, \dots, N_{\text{nd}}^{\text{el}}$) denoted $\hat{\xi}_{ji}$. For now, we only allow for a single degree of freedom per node ($M = N_{\text{nd}}^{\text{el}}$) because we are considering scalar-valued variational problems, i.e., u in (6.1) is a scalar-valued function; this will be generalized in Chapter 8 when considering vector-valued variational problems. In this setting, let $(K, \mathcal{Y}, \mathcal{D})$ be a nodal finite element with associated node set $\mathcal{N}_h^K$, then the degrees of freedom of any $v \in \mathcal{Y}$ are

$$D_i(v) := v(\hat{\boldsymbol{\xi}}_i). \tag{6.12}$$

In the remainder, we use the notation $\hat{v}_i := v(\hat{\boldsymbol{\xi}}_i)$. Furthermore, let $\{\psi_1, \dots, \psi_{N_{\mathrm{nd}}^{\mathrm{el}}}\}$ be a basis of $\mathcal{Y}$ that possesses the *nodal* property, i.e.,

$$\psi_i(\hat{\boldsymbol{\xi}}_j) = \delta_{ij}. \tag{6.13}$$

Recall this property guarantees the basis functions are linearly independent. In this setting, any $v \in \mathcal{Y}$ can be expressed in terms of the degrees of freedom and basis functions as

$$v = \sum_{i=1}^{M} D_i(v)\psi_i = \sum_{i=1}^{M} \hat{v}_i \psi_i, \tag{6.14}$$

i.e., the coefficients that expand v in the basis $\{\psi_1, \dots, \psi_{N_{\mathrm{nd}}^{\mathrm{el}}}\}$ are equal to the *value* of the function at the nodes.

Example 6.1: Linear finite element in one dimension

The linear, one-dimensional finite element used in Chapter 4 is a finite element $(K, \mathcal{Y}, \mathcal{D})$. The element domain is simply a closed interval $K := [a, b]$ for $a, b \in \mathbb{R}$ and the local function space is $\mathcal{Y} := \mathcal{P}^1(K)$. The nodes of the element are located at the endpoints of K, i.e., $\mathcal{N} := \{\hat{\xi}_1, \hat{\xi}_2\}$ where $\hat{\xi}_1 = a$ and $\hat{\xi}_2 = b$, and $\mathcal{D} = \{D_1, D_2\}$ are the nodal degrees of freedom associated with these nodes, i.e., $D_i(v) := \hat{v}_i := v(\hat{\xi}_i)$ for $i = 1, 2$ and any $v \in \mathcal{Y}$. The finite element $(K, \mathcal{Y}, \mathcal{D})$ is summarized as

$$K := [a, b], \quad \mathcal{Y} := \mathcal{P}^1(K), \quad \mathcal{N} := \{a, b\}, \quad \mathcal{D} := \{v(a), v(b)\}. \tag{6.15}$$

The (unique) nodal basis $\{\psi_1, \psi_2\}$ of $\mathcal{Y}$ associated with the node set $\mathcal{N}$ is

$$\psi_1(\xi) := \frac{b - \xi}{b - a}, \qquad \psi_2(\xi) := \frac{\xi - a}{b - a}. \tag{6.16}$$

An illustration of this finite element and its nodal basis functions in the special case where $a = -1$, $b = 1$ is provided in Figure 6.2.

Example 6.2: Bilinear quadrilateral finite element in two dimensions

As another example we construct the bilinear, one-dimensional finite element used in Homework 2. The element domain is simply a closed quadrilateral $K := [a_1, b_1] \times [a_2, b_2]$ for $a_1, a_2, b_1, b_2 \in \mathbb{R}$ and the local function space is $\mathcal{Y} := \mathcal{Q}^1(K)$. The nodes of the element are located at the corners of K, i.e., $\mathcal{N}_h = \{\hat{\boldsymbol{\xi}}_1, \dots, \hat{\boldsymbol{\xi}}_4\}$, where

$$\hat{\boldsymbol{\xi}}_1 = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \quad \hat{\boldsymbol{\xi}}_2 = \begin{bmatrix} b_1 \\ a_2 \end{bmatrix}, \quad \hat{\boldsymbol{\xi}}_3 = \begin{bmatrix} a_1 \\ b_2 \end{bmatrix}, \quad \hat{\boldsymbol{\xi}}_4 = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \tag{6.17}$$

and $\mathcal{D} = \{D_1, D_2, D_3, D_4\}$ are the nodal degrees of freedom associated with these nodes, i.e., $D_i(v) := \hat{v}_i := v(\hat{\boldsymbol{\xi}}_i)$ for $i = 1, 2, 3, 4$ and any $v \in \mathcal{Y}$. The bilinear quadrilateral finite element $(K, \mathcal{Y}, \mathcal{D})$ is summarized as

$$\begin{aligned} K := [a_1, b_1] \times [a_2, b_2], \quad \mathcal{Y} := \mathcal{Q}^1(K), \quad \mathcal{N} := \left\{ \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \begin{bmatrix} b_1 \\ a_2 \end{bmatrix}, \begin{bmatrix} a_1 \\ b_2 \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \right\}, \\ \mathcal{D} := \left\{ v\left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \right), v\left(\begin{bmatrix} b_1 \\ a_2 \end{bmatrix} \right), v\left(\begin{bmatrix} a_1 \\ b_2 \end{bmatrix} \right), v\left(\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \right) \right\}. \end{aligned} \tag{6.18}$$

The (unique) nodal basis $\{\psi_1, \psi_2, \psi_3, \psi_4\}$ of $\mathcal{Y}$ associated with the node set $\mathcal{N}$ is

$$\begin{aligned} \psi_1(r, s) &:= \frac{b_1 - r}{b_1 - a_1}\frac{b_2 - s}{b_2 - a_2}, \qquad & \psi_2(r, s) &:= \frac{r - a_1}{b_1 - a_1}\frac{b_2 - s}{b_2 - a_2}, \\ \psi_3(r, s) &:= \frac{b_1 - r}{b_1 - a_1}\frac{s - a_2}{b_2 - a_2}, \qquad & \psi_4(r, s) &:= \frac{r - a_1}{b_1 - a_1}\frac{s - a_2}{b_2 - a_2}. \end{aligned} \tag{6.19}$$

An illustration of this finite element and its nodal basis functions in the special case where $a_1 = a_2 = -1$, $b_1 = b_2 = 1$ are provided in Figure 6.3, 6.4.

In the remainder of this section we introduce various classes of H^1-conforming finite elements that we will use in this class, namely, polynomial simplex and hypercube elements. For now, we will not worry about the configuration of the element in the domain. Rather we will define them on a idealized, *reference* domain, denoted $\Omega_\square \subset \mathbb{R}^d$. In the next section, we will introduce a mapping to push them to their appropriate configuration/orientation in the domain Ω. Since these ideal elements will be used to generate all of the physical elements in a mesh, we call them *master* elements. Since we have committed to nodal elements, we no longer need to discuss the degrees of freedom $\mathcal{D}$ since they will be given uniquely from the nodes. Therefore, a *master finite element* is completely defined by $(\Omega_\square, \mathcal{Y}_\square, \mathcal{N}_\square)$, where $\Omega_\square \subset \mathbb{R}^d$ is the master element geometry, $\mathcal{Y}_\square$ is the function space associated with the master element, and $\mathcal{N}_\square \subset \Omega_\square$ is the collection of $N_{\mathrm{nd}}^{\mathrm{el}}$ nodes that define the element degrees of freedom. In the remainder of this section, we define a number of useful master finite elements in $d = 1, 2$ and higher dimensions. In particular, we define the element geometry $\Omega_\square$, the distribution of nodes and their numbering $\mathcal{N}_\square$, and the associated (usually polynomial) function space $\mathcal{Y}_\square$.

6.3.1 Polynomial spaces

In Chapter 2 we introduced the polynomial space $\mathcal{P}^p(\Omega)$ for $\Omega \subset \mathbb{R}$. In higher dimensions $\Omega \subset \mathbb{R}^d$ $(d > 1)$ there many relevant polynomial spaces; however, we will consider the two most common ones $\mathcal{P}^k(\Omega)$ and $\mathcal{Q}^k(\Omega)$ defined as

$$\begin{aligned}
\mathcal{P}^k(\Omega) &:= \left\{ p \in \mathcal{F}_{\Omega\to\mathbb{R}} \;\middle|\; p(\boldsymbol{\xi}) = \sum_{\substack{\boldsymbol{\alpha}\in\mathbb{N}_0^d \\ \|\boldsymbol{\alpha}\|_1 \leqslant k}} a_{\boldsymbol{\alpha}} \boldsymbol{\xi}^{\boldsymbol{\alpha}},\ \boldsymbol{\xi} \in \Omega,\ a_{\boldsymbol{\alpha}} \in \mathbb{R} \right\} \\
\mathcal{Q}^k(\Omega) &:= \left\{ p \in \mathcal{F}_{\Omega\to\mathbb{R}} \;\middle|\; p(\boldsymbol{\xi}) = \sum_{\substack{\boldsymbol{\alpha}\in\mathbb{N}_0^d \\ \|\boldsymbol{\alpha}\|_\infty \leqslant k}} a_{\boldsymbol{\alpha}} \boldsymbol{\xi}^{\boldsymbol{\alpha}},\ \boldsymbol{\xi} \in \Omega,\ a_{\boldsymbol{\alpha}} \in \mathbb{R} \right\}.
\end{aligned} \tag{6.20}$$

It can be shown that these are linear spaces of dimension

$$\dim \mathcal{P}^k(\Omega) = \binom{k+d}{d}, \qquad \dim \mathcal{Q}^k(\Omega) = (k+1)^d. \tag{6.21}$$

Example 6.3: Polynomial spaces in $d = 1$ dimension

In $d = 1$ dimension, both the $\mathcal{P}^k(\Omega)$ and $\mathcal{Q}^k(\Omega)$ polynomial spaces are identical and equal to

$$\mathcal{P}^k(\Omega) = \mathcal{Q}^k(\Omega) = \left\{ p \in \mathcal{F}_{\Omega\to\mathbb{R}} \;\middle|\; p(\xi) = \sum_{n=0}^{k} a_n \xi^n, \xi \in \Omega, a_n \in \mathbb{R} \right\} \tag{6.22}$$

and have dimension $\dim \mathcal{P}^k(\Omega) = \dim \mathcal{Q}^k(\Omega) = k + 1$.

Example 6.4: Polynomial spaces in $d = 2$ dimension

In $d = 2$ dimensions, the polynomial spaces are

$$\begin{aligned}
\mathcal{P}^k(\Omega) &= \left\{ p \in \mathcal{F}_{\Omega\to\mathbb{R}} \;\middle|\; p(\boldsymbol{\xi}) = \sum_{\alpha_1+\alpha_2 \leqslant k} a_{\alpha_1\alpha_2} \xi_1^{\alpha_1} \xi_2^{\alpha_2}, \boldsymbol{\xi} \in \Omega, a_{\alpha_1\alpha_2} \in \mathbb{R} \right\} \\
\mathcal{Q}^k(\Omega) &= \left\{ p \in \mathcal{F}_{\Omega\to\mathbb{R}} \;\middle|\; p(\boldsymbol{\xi}) = \sum_{1 \leqslant \alpha_1,\alpha_2 \leqslant k} a_{\alpha_1\alpha_2} \xi_1^{\alpha_1} \xi_2^{\alpha_2}, \boldsymbol{\xi} \in \Omega, a_{\alpha_1\alpha_2} \in \mathbb{R} \right\}
\end{aligned} \tag{6.23}$$

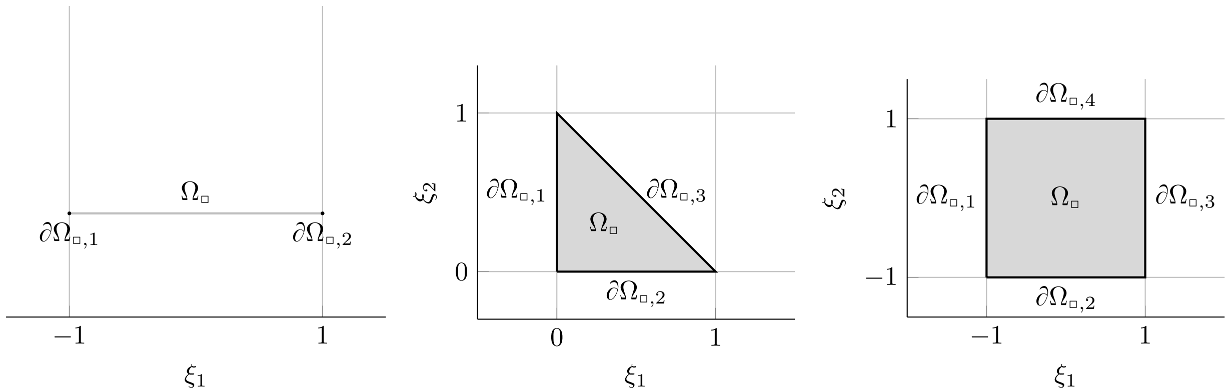

Figure 6.1: Master element geometry and boundary numbering for line ($d = 1$ hypercube) (*left*), triangle ($d = 2$ simplex) (*middle*), and quadrilateral ($d = 2$ hypercube) (*right*).

and have dimension $\dim \mathcal{P}^k(\Omega) = (k+1)(k+2)/2$ and $\dim \mathcal{Q}^k(\Omega) = (k+1)^2$. Now we state the monomial basis of $\mathcal{P}^k(\Omega)$ and $\mathcal{Q}^k(\Omega)$ for the special case of $k = 0, 1, 2$

$$\begin{aligned}
\mathcal{P}^0(\Omega) &= \operatorname{span}\{1\}, & \mathcal{Q}^0(\Omega) &= \operatorname{span}\{1\} \\
\mathcal{P}^1(\Omega) &= \operatorname{span}\{1, \xi_1, \xi_2\}, & \mathcal{Q}^1(\Omega) &= \operatorname{span}\{1, \xi_1, \xi_2, \xi_1\xi_2\} \\
\mathcal{P}^2(\Omega) &= \operatorname{span}\{1, \xi_1, \xi_2, \xi_1^2, \xi_1\xi_2, \xi_2^2\}, & \mathcal{Q}^2(\Omega) &= \operatorname{span}\{1, \xi_1, \xi_2, \xi_1^2, \xi_1\xi_2, \xi_2^2, \xi_1\xi_2^2, \xi_1^2\xi_2, \xi_1^2\xi_2^2\}
\end{aligned} \tag{6.24}$$

from which we can see the dimensions are

$$\begin{aligned}
\dim \mathcal{P}^0(\Omega) &= 1, & \dim \mathcal{Q}^0(\Omega) &= 1 \\
\dim \mathcal{P}^1(\Omega) &= 3, & \dim \mathcal{Q}^1(\Omega) &= 4 \\
\dim \mathcal{P}^2(\Omega) &= 6, & \dim \mathcal{Q}^2(\Omega) &= 9
\end{aligned} \tag{6.25}$$

in agreement with the preceeding general formula.

6.3.2 $d = 1$ dimension: $\mathcal{P}^p$ master line element

Element domain

In $d = 1$ dimension the only possible element geometry is a line segment. For convenience, we take the master element domain to be the bi-unit interval centered at zero

$$\Omega_\square := [-1, 1]. \tag{6.26}$$

Then the boundary of the master element is $\partial\Omega_\square = \{-1, 1\}$. We separate these into an ordered set of *faces* $\mathcal{F}_\square = \{\partial\Omega_{\square,1}, \partial\Omega_{\square,2}\}$, where $\partial\Omega_{\square,1} = \{-1\}$ with associated unit outward normal $N_{\square,1} = \{-1\}$ and $\partial\Omega_{\square,2} = \{1\}$ with associated unit outward normal $N_{\square,2} = \{1\}$. The complete geometry of the master line element is illustrated in Figure 6.1.

Local function space

We take the local function space to be the space of polynomials up to (and including) degree p, i.e., $\mathcal{Y}_\square := \mathcal{P}^p(\Omega_\square)$. Therefore, the local function space has dimension $\dim \mathcal{Y}_\square = p + 1$.

Distribution and numbering of nodes

Before we can construct a nodal basis of $\mathcal{Y}_\square$, we must distribute $N_{\text{nd}}^{\text{el}} = p + 1$ nodes throughout the element geometry. To ensure the nodal basis functions are linearly independent, the nodes must not overlap (and

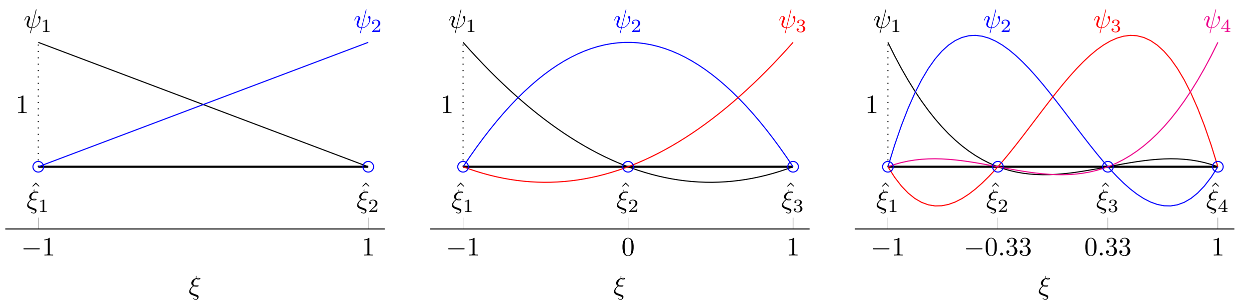

Figure 6.2: Master $\mathcal{P}^p$ line element including nodal positions/numbering and nodal basis functions for $p = 1, 2, 3$ (*left-to-right*).

should not be too close to prevent ill-conditioning). Furthermore we insist that a node lies on each face as this make enforcing global continuity straightforward. For simplicity, we uniformly distribute nodes through $\Omega_\square$, i.e., $\mathcal{N}_h = \{\hat{\xi}_1, \dots, \hat{\xi}_{p+1}\}$ where

$$\hat{\xi}_i = -1 + 2\frac{i-1}{p} \tag{6.27}$$

for $i = 1, \dots, p+1$ (Figure 6.2). It is well-known that uniform placement of nodes can lead to ill-conditioned systems for high p; this can be remedied using non-uniform points such the Chebyshev or Gauss-Legendre-Lobotto nodes.

Construction of nodal element basis functions

Finally, we turn to a construction of basis functions of $\mathcal{Y}_\square := \mathcal{P}^p(\Omega_\square)$ that satisfy the nodal property: $\psi_i(\hat{\xi}_j) = \delta_{ij}$ for $i, j = 1, \dots, p+1$. Notice that the nodal property constrains the value of each basis function at $p+1$ (unique) locations; since the basis functions are polynomials of degree p, these constraints uniquely define them. From this observation, we set out to construct the basis functions *by inspection*.

To begin, we consider the special case of $p = 1$ and observe that ψ_1 must go to zero at $\hat{\xi}_2$ from which we postulate it takes the form $\psi_1(\xi) = \alpha(\xi - \hat{\xi}_2)$ for some $\alpha \in \mathbb{R}$. Similarly, we postulate $\psi_2(\xi) = \beta(\xi - \hat{\xi}_1)$ for some $\beta \in \mathbb{R}$. It can readily be seen that these are linear functions that satisfy $\psi_1(\hat{\xi}_2) = \psi_2(\hat{\xi}_1) = 0$, which makes them valid candidates for a nodal basis of $\mathcal{P}^1(\Omega_\square)$. The only conditions that remain are the normalization conditions $\psi_1(\hat{\xi}_1) = \psi_2(\hat{\xi}_2) = 1$, which leads to the following expressions for the coefficients

$$\alpha = \frac{1}{\hat{\xi}_1 - \hat{\xi}_2}, \qquad \beta = \frac{1}{\hat{\xi}_2 - \hat{\xi}_1} \tag{6.28}$$

to yield the nodal basis

$$\psi_1(\xi) = \frac{\hat{\xi}_2 - \xi}{\hat{\xi}_2 - \hat{\xi}_1} = \frac{1-\xi}{2}, \qquad \psi_2(\xi) = \frac{\xi - \hat{\xi}_1}{\hat{\xi}_2 - \hat{\xi}_1} = \frac{\xi+1}{2}, \tag{6.29}$$

where we have used that $\hat{\xi}_1 = -1$ and $\hat{\xi}_2 = 1$.

We follow a similar procedure for the $p = 2$ case and postulate that

$$\psi_1(\xi) = \alpha(\xi - \hat{\xi}_2)(\xi - \hat{\xi}_3), \quad \psi_2(\xi) = \beta(\xi - \hat{\xi}_1)(\xi - \hat{\xi}_3), \quad \psi_3(\xi) = \gamma(\xi - \hat{\xi}_1)(\xi - \hat{\xi}_2), \tag{6.30}$$

where $\alpha, \beta, \gamma \in \mathbb{R}$ are constants that must be determined. It is easy to verify that $\psi_i(\hat{\xi}_j) = 0$ for $i \neq j$. The constants are determined to be

$$\alpha = \frac{1}{(\hat{\xi}_1 - \hat{\xi}_2)(\hat{\xi}_1 - \hat{\xi}_3)}, \quad \beta = \frac{1}{(\hat{\xi}_2 - \hat{\xi}_1)(\hat{\xi}_2 - \hat{\xi}_3)}, \quad \gamma = \frac{1}{(\hat{\xi}_3 - \hat{\xi}_1)(\hat{\xi}_3 - \hat{\xi}_2)}, \tag{6.31}$$

from the normalization conditions $\psi_i(\hat{\xi}_i) = 1$ (no summation on i), which leads to the nodal basis functions

$$\begin{aligned}
\psi_1(\xi) &= \frac{(\xi - \hat{\xi}_2)(\xi - \hat{\xi}_3)}{(\hat{\xi}_1 - \hat{\xi}_2)(\hat{\xi}_1 - \hat{\xi}_3)} &&= \frac{\xi(\xi - 1)}{2} \\
\psi_2(\xi) &= \frac{(\xi - \hat{\xi}_1)(\xi - \hat{\xi}_3)}{(\hat{\xi}_2 - \hat{\xi}_1)(\hat{\xi}_2 - \hat{\xi}_3)} &&= (\xi + 1)(1 - \xi) \\
\psi_3(\xi) &= \frac{(\xi - \hat{\xi}_1)(\xi - \hat{\xi}_2)}{(\hat{\xi}_3 - \hat{\xi}_1)(\hat{\xi}_3 - \hat{\xi}_2)} &&= \frac{\xi(\xi + 1)}{2}
\end{aligned} \tag{6.32}$$

because the nodes are $\hat{\xi}_1 = -1$, $\hat{\xi}_2 = 0$, and $\hat{\xi}_3 = 1$ from (6.27). We follow the same procedure precisely to obtain an expression for the nodal basis functions of $\mathcal{P}^p(\Omega_\square)$ associated with the nodes $\mathcal{N}_\square = \{\hat{\xi}_1, \dots, \hat{\xi}_{p+1}\}$

$$\psi_i(\xi) = \prod_{\substack{j=1 \\ j \neq i}}^{p+1} \frac{\xi - \hat{\xi}_j}{\hat{\xi}_i - \hat{\xi}_j} = \frac{\xi - \hat{\xi}_1}{\hat{\xi}_i - \hat{\xi}_1} \cdots \frac{\xi - \hat{\xi}_{i-1}}{\hat{\xi}_i - \hat{\xi}_{i-1}} \frac{\xi - \hat{\xi}_{i+1}}{\hat{\xi}_i - \hat{\xi}_{i+1}} \cdots \frac{\xi - \hat{\xi}_{p+1}}{\hat{\xi}_i - \hat{\xi}_{p+1}}, \tag{6.33}$$

see Figure 6.2 for an illustration of the basis functions up to $p = 3$. It is a simple exercise to verify that this collection of functions satisfies the nodal property.

6.3.3 $d = 2$ dimensions: $\mathcal{Q}^p$ master quadrilateral element

Unlike in $d = 1$ dimension, there are infinitely many possible element geometries in $d > 1$ dimensions; we will only consider a small, but extremely useful subset of these possibilities. We begin with the $\mathcal{Q}^p$ quadrilateral element.

Element domain

The reference domain of the master quadrilateral element is taken to be the bi-unit interval centered at zero for consistency with the line element

$$\Omega_\square := [-1, 1] \times [-1, 1]. \tag{6.34}$$

Notice that the master quadrilateral element is a Cartesian product of the master line element with itself. The boundary of the master element is $\partial\Omega_\square = \bigcup_{i=1}^4 \partial\Omega_{\square,i}$, where

$$\partial\Omega_{\square,1} := \{-1\} \times [-1, 1], \quad \partial\Omega_{\square,2} := [-1, 1] \times \{-1\}, \quad \partial\Omega_{\square,3} := \{1\} \times [-1, 1], \quad \partial\Omega_{\square,4} := [-1, 1] \times \{1\} \tag{6.35}$$

and the corresponding unit outward normals are

$$\boldsymbol{N}_{\square,1} := \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \quad \boldsymbol{N}_{\square,2} := \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \quad \boldsymbol{N}_{\square,3} := \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \boldsymbol{N}_{\square,4} := \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{6.36}$$

The complete geometry of the master quadilateral element is illustrated in Figure 6.1.

Local function space

We take the local function space to be the space $\mathcal{Y}_\square := \mathcal{Q}^p(\Omega_\square)$, i.e., polynomial functions where the largest exponent is p. The dimension of the local function space is $\dim \mathcal{Y}_\square = (p + 1)^2$. An important property of this function space is that any function $v \in \mathcal{Q}^p(\Omega_\square)$ is a one-dimensional polynomial of degree p when restricted to any face $\partial\Omega_{\square,k}$, $k = 1, \dots, 4$. To see this we introduce a parametrization of boundary 2 (chosen arbitrarily), $\gamma : [-1, 1] \to \partial\Omega_{\square,2}$ defined as $\gamma(s) := (s, -1) \in \partial\Omega_{\square,2}$. Then, for any $v \in \mathcal{Q}^p(\Omega_\square)$, we expand it as

$$v(\xi) = \sum_{\alpha_1 \leqslant p, \alpha_2 \leqslant p} a_{\alpha_1\alpha_2} \xi_1^{\alpha_1} \xi_2^{\alpha_2}, \tag{6.37}$$

where $a_{ij} \in \mathbb{R}$ for $i, j = 1, \dots, p + 1$. We then restrict v to $\partial\Omega_{\square,2}$ by composing with the face parametrization γ

$$f(s) := v(\gamma(s)) = \sum_{\alpha_1 \leqslant p, \alpha_2 \leqslant p} a_{\alpha_1\alpha_2} \gamma_1(s)^{\alpha_1} \gamma_2(s)^{\alpha_2} = \sum_{\alpha_2 \leqslant p} \left(\sum_{\alpha_1 \leqslant p} (-1)^{\alpha_1} a_{\alpha_1\alpha_2} \right) s^{\alpha_2}, \tag{6.38}$$

which is clearly a polynomial of degree at most p, i.e., $f \in \mathcal{P}^p([-1, 1])$.

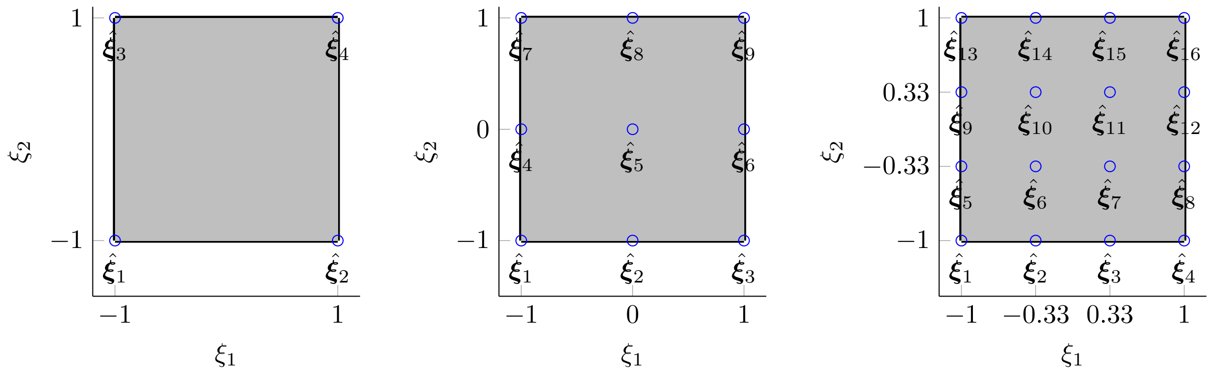

Figure 6.3: Master $\mathcal{Q}^p$ quadrilateral element including nodal positions and numbering for $p = 1, 2, 3$ (*left-to-right*).

Distribution and numbering of nodes

Before we construct a nodal basis of $\mathcal{Y}_\square$, we must distribute $N_{\text{nd}}^{\text{el}} = (p+1)^2$ nodes throughout the element domain $\Omega_\square$. To ensure all basis functions are linearly independent, the nodes must not overlap (or be too close to prevent ill-conditioning). We also require that $p+1$ nodes lie on each of the four faces of the quadrilateral $\Omega_\square$. Recall from the previous section that functions of $\mathcal{Q}^p(\Omega_\square)$ are one-dimensional polynomials of degree p when restricted to faces $\partial\Omega_\square$ and are therefore uniquely determined by $p+1$ nodal values. This gives a convenient way to enforce global continuity between elements: if the nodal values of two abutting elements match at the $p+1$ nodes on their common face, then the functions will match everywhere on that face since they will define the same one-dimensional polynomial.

To satisfy these requirements, we define the nodes of the $\mathcal{Q}^p$ master quadrilateral to be the Cartesian product of the nodes of the $\mathcal{Q}^p$ master line element, numbered first in the ξ_1-direction then in the ξ_2-direction (Figure 6.3 for $p = 1, 2, 3$). To make this precise, let $\{\hat{s}_1, \dots, \hat{s}_{p+1}\} \subset [-1, 1]$ be the nodes of the master line element and introduce to mappings

$$\mathcal{I} : \{1, \dots, (p+1)^2\} \to \{1, \dots, p+1\}, \quad \mathcal{J} : \{1, \dots, (p+1)^2\} \to \{1, \dots, p+1\}, \tag{6.39}$$

where $\mathcal{I}$ maps the quadrilateral node number to the node number along the ξ_1-axis and $\mathcal{J}$ maps to the node number along the ξ_2-axis, i.e., the ith quadrilateral node is the $\mathcal{I}(i)$th node in the ξ_1-direction and the $\mathcal{J}(i)$th node in the ξ_2-direction. To agree with the node numbering in Figure 6.3, we have

$$\mathcal{I}(k) \coloneqq 1 + [(k-1)\%(p+1)], \qquad \mathcal{J}(k) \coloneqq 1 + \left\lfloor \frac{k-1}{p+1} \right\rfloor \tag{6.40}$$

for $k = 1, \dots, (p+1)^2$, where % is the modulus operator (remainder after division) and $\lfloor \cdot \rfloor$ is the floor operator. With this notation, the kth quadrilateral node is defined in terms of the line element nodes as

$$\hat{\boldsymbol{\xi}}_k \coloneqq \begin{bmatrix} \hat{\xi}_{1k} \\ \hat{\xi}_{2k} \end{bmatrix} \coloneqq \begin{bmatrix} \hat{s}_{\mathcal{I}(k)} \\ \hat{s}_{\mathcal{J}(k)} \end{bmatrix}. \tag{6.41}$$

Example 6.5: Nodes of bilinear $\mathcal{Q}^1$ quadrilateral

To define the nodes of $\mathcal{Q}^1$ master quadrilateral element, recall the nodes of the $\mathcal{Q}^1$ master line element $\hat{s}_1 = -1$, $\hat{s}_2 = 1$ and, in the special case of $p = 1$, the quadrilateral-to-line mappings (6.39)-(6.40) are

$$\mathcal{I} = \{1, 2, 1, 2\}, \quad \mathcal{J} = \{1, 1, 2, 2\}. \tag{6.42}$$

From this and (6.41), the nodes of the master $\mathcal{Q}^1$ quadrilateral element are

$$\begin{aligned}
\hat{\boldsymbol{\xi}}_1 &= \begin{bmatrix} \hat{\xi}_{11} \\ \hat{\xi}_{21} \end{bmatrix} = \begin{bmatrix} \hat{s}_{\mathcal{I}(1)} \\ \hat{s}_{\mathcal{J}(1)} \end{bmatrix} = \begin{bmatrix} \hat{s}_1 \\ \hat{s}_1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \qquad & \hat{\boldsymbol{\xi}}_2 &= \begin{bmatrix} \hat{\xi}_{12} \\ \hat{\xi}_{22} \end{bmatrix} = \begin{bmatrix} \hat{s}_{\mathcal{I}(2)} \\ \hat{s}_{\mathcal{J}(2)} \end{bmatrix} = \begin{bmatrix} \hat{s}_2 \\ \hat{s}_1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \\
\hat{\boldsymbol{\xi}}_3 &= \begin{bmatrix} \hat{\xi}_{13} \\ \hat{\xi}_{23} \end{bmatrix} = \begin{bmatrix} \hat{s}_{\mathcal{I}(3)} \\ \hat{s}_{\mathcal{J}(3)} \end{bmatrix} = \begin{bmatrix} \hat{s}_1 \\ \hat{s}_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \qquad & \hat{\boldsymbol{\xi}}_4 &= \begin{bmatrix} \hat{\xi}_{14} \\ \hat{\xi}_{24} \end{bmatrix} = \begin{bmatrix} \hat{s}_{\mathcal{I}(4)} \\ \hat{s}_{\mathcal{J}(4)} \end{bmatrix} = \begin{bmatrix} \hat{s}_2 \\ \hat{s}_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix},
\end{aligned} \tag{6.43}$$

which clearly agrees with Figure 6.3.

Example 6.6: Nodes of biquadratic $\mathcal{Q}^2$ quadrilateral

To define the nodes of $\mathcal{Q}^2$ master quadrilateral element, recall the nodes of the $\mathcal{Q}^2$ master line element $\hat{s}_1 = -1$, $\hat{s}_2 = 0$, $\hat{s}_3 = 1$ and, in the special case of $p = 2$, the quadrilateral-to-line mappings (6.39)-(6.40) are

$$\mathcal{I} = \{1, 2, 3, 1, 2, 3, 1, 2, 3\}, \quad \mathcal{J} = \{1, 1, 1, 2, 2, 2, 3, 3, 3\}. \tag{6.44}$$

From this and (6.41), the nodes of the master $\mathcal{Q}^2$ quadrilateral element are

$$\begin{aligned}
\hat{\boldsymbol{\xi}}_1 &= \begin{bmatrix} \hat{\xi}_{11} \\ \hat{\xi}_{21} \end{bmatrix} = \begin{bmatrix} \hat{s}_{\mathcal{I}(1)} \\ \hat{s}_{\mathcal{J}(1)} \end{bmatrix} = \begin{bmatrix} \hat{s}_1 \\ \hat{s}_1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \qquad & \hat{\boldsymbol{\xi}}_2 &= \begin{bmatrix} \hat{\xi}_{12} \\ \hat{\xi}_{22} \end{bmatrix} = \begin{bmatrix} \hat{s}_{\mathcal{I}(2)} \\ \hat{s}_{\mathcal{J}(2)} \end{bmatrix} = \begin{bmatrix} \hat{s}_2 \\ \hat{s}_1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \\
\hat{\boldsymbol{\xi}}_3 &= \begin{bmatrix} \hat{\xi}_{13} \\ \hat{\xi}_{23} \end{bmatrix} = \begin{bmatrix} \hat{s}_{\mathcal{I}(3)} \\ \hat{s}_{\mathcal{J}(3)} \end{bmatrix} = \begin{bmatrix} \hat{s}_3 \\ \hat{s}_1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \qquad & \hat{\boldsymbol{\xi}}_4 &= \begin{bmatrix} \hat{\xi}_{14} \\ \hat{\xi}_{24} \end{bmatrix} = \begin{bmatrix} \hat{s}_{\mathcal{I}(4)} \\ \hat{s}_{\mathcal{J}(4)} \end{bmatrix} = \begin{bmatrix} \hat{s}_1 \\ \hat{s}_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \\
\hat{\boldsymbol{\xi}}_5 &= \begin{bmatrix} \hat{\xi}_{15} \\ \hat{\xi}_{25} \end{bmatrix} = \begin{bmatrix} \hat{s}_{\mathcal{I}(5)} \\ \hat{s}_{\mathcal{J}(5)} \end{bmatrix} = \begin{bmatrix} \hat{s}_2 \\ \hat{s}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad & \hat{\boldsymbol{\xi}}_6 &= \begin{bmatrix} \hat{\xi}_{16} \\ \hat{\xi}_{26} \end{bmatrix} = \begin{bmatrix} \hat{s}_{\mathcal{I}(6)} \\ \hat{s}_{\mathcal{J}(6)} \end{bmatrix} = \begin{bmatrix} \hat{s}_3 \\ \hat{s}_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\
\hat{\boldsymbol{\xi}}_7 &= \begin{bmatrix} \hat{\xi}_{17} \\ \hat{\xi}_{27} \end{bmatrix} = \begin{bmatrix} \hat{s}_{\mathcal{I}(7)} \\ \hat{s}_{\mathcal{J}(7)} \end{bmatrix} = \begin{bmatrix} \hat{s}_1 \\ \hat{s}_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \qquad & \hat{\boldsymbol{\xi}}_8 &= \begin{bmatrix} \hat{\xi}_{18} \\ \hat{\xi}_{28} \end{bmatrix} = \begin{bmatrix} \hat{s}_{\mathcal{I}(8)} \\ \hat{s}_{\mathcal{J}(8)} \end{bmatrix} = \begin{bmatrix} \hat{s}_2 \\ \hat{s}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \\
\hat{\boldsymbol{\xi}}_9 &= \begin{bmatrix} \hat{\xi}_{19} \\ \hat{\xi}_{29} \end{bmatrix} = \begin{bmatrix} \hat{s}_{\mathcal{I}(9)} \\ \hat{s}_{\mathcal{J}(9)} \end{bmatrix} = \begin{bmatrix} \hat{s}_3 \\ \hat{s}_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, & &
\end{aligned} \tag{6.45}$$

which clearly agrees with Figure 6.3.

Construction of element basis functions

Lastly we turn to a construction of the nodal basis of $\mathcal{Y}_{\square} := \mathcal{Q}^p(\Omega_{\square})$. Given the *tensor product structure* of the master quadrilateral domain and its nodes, we construct the nodal basis via a Cartesian product of the nodal basis functions of the master line element. Again, let $\{\hat{s}_1, \dots, \hat{s}_{p+1}\}$ be the nodes of the $\mathcal{P}^p$ master line element, $\mathcal{I}$ and $\mathcal{J}$ be the quadrilateral-to-line nodal mapping from the previous section, and $\{\tilde{\psi}_1, \dots, \tilde{\psi}_{p+1}\}$ be the corresponding nodal basis, i.e., $\tilde{\psi}_i(\hat{s}_j) = \delta_{ij}$ for $i, j = 1, \dots, p+1$. Then we define the nodal basis functions $\{\psi_1, \dots, \psi_{N_{\mathrm{nd}}^{\mathrm{el}}}\}$ of the $\mathcal{Q}^p$ quadrilateral to be

$$\psi_i(\boldsymbol{\xi}) := \tilde{\psi}_{\mathcal{I}(i)}(\xi_1)\tilde{\psi}_{\mathcal{J}(i)}(\xi_2). \tag{6.46}$$

Using the expression in (6.33) for the one-dimensional nodal basis, this becomes

$$\psi_i(\boldsymbol{\xi}) = \left(\prod_{\substack{j=1 \\ j \neq \mathcal{I}(i)}}^{p+1} \frac{\xi_1 - \hat{s}_j}{\hat{s}_{\mathcal{I}(i)} - \hat{s}_j} \right) \left(\prod_{\substack{k=1 \\ k \neq \mathcal{J}(i)}}^{p+1} \frac{\xi_2 - \hat{s}_k}{\hat{s}_{\mathcal{J}(i)} - \hat{s}_k} \right) \tag{6.47}$$

To verify this choice of basis has the nodal property, we evaluate ψ_i at node $\hat{\boldsymbol{\xi}}_j$

$$\psi_i(\hat{\boldsymbol{\xi}}_j) = \tilde{\psi}_{\mathcal{I}(i)}(\hat{\xi}_{1j})\tilde{\psi}_{\mathcal{J}(i)}(\hat{\xi}_{2j}) = \tilde{\psi}_{\mathcal{I}(i)}(\hat{s}_{\mathcal{I}(j)})\tilde{\psi}_{\mathcal{J}(i)}(\hat{s}_{\mathcal{J}(j)}) = \delta_{\mathcal{I}(i)\mathcal{I}(j)}\delta_{\mathcal{J}(i)\mathcal{J}(j)} = \delta_{ij}, \tag{6.48}$$

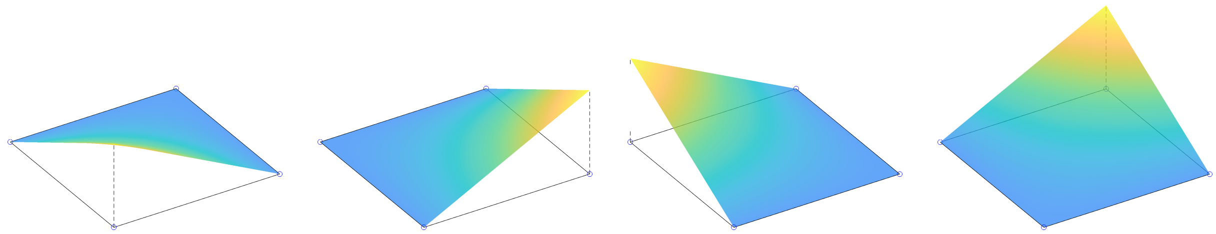

Figure 6.4: Nodal basis functions of $\mathcal{Q}^1$ master quadrilateral element.

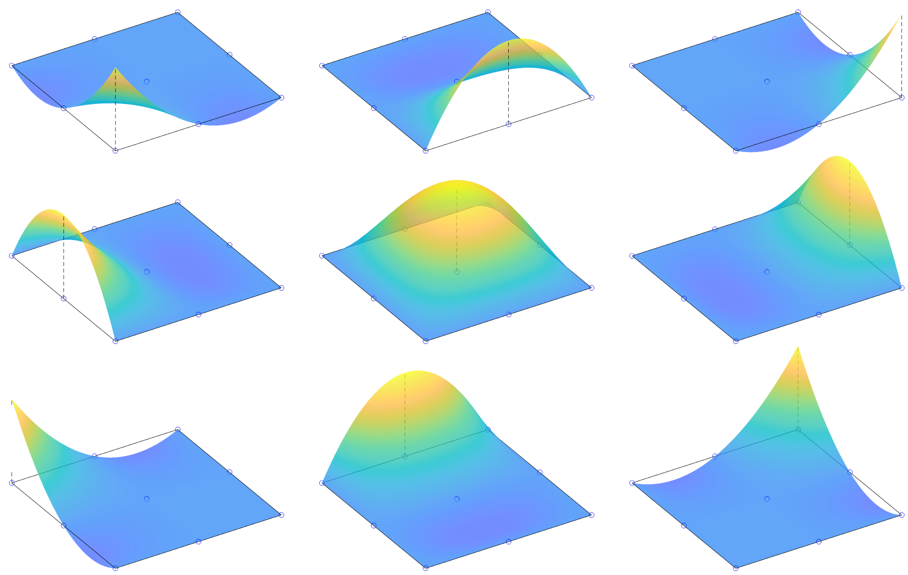

Figure 6.5: Nodal basis functions of $\mathcal{Q}^2$ master quadrilateral element.

where the last equality follows because the product $\delta_{\mathcal{I}(i)\mathcal{I}(j)}\delta_{\mathcal{J}(i)\mathcal{J}(j)}$ only survives if $\mathcal{I}(i) = \mathcal{I}(j)$ (the ξ_1 index of nodes i and j must match) and $\mathcal{J}(i) = \mathcal{J}(j)$ (the ξ_2 index of nodes i and j must match), which can only happen if $i = j$. The nodal basis functions for the $\mathcal{Q}^1$, $\mathcal{Q}^2$, and $\mathcal{Q}^3$ master quadrilateral are shown in Figures 6.4-6.6.

Example 6.7: Bilinear $\mathcal{Q}^1$ quadrilateral nodal basis

Recall the quadrilateral-to-line mappings for the bilinear quadrilateral (6.39)-(6.40). Then the nodal basis functions of the $\mathcal{Q}^1$ quadrilateral (Figure 6.4) using the tensor product formula in (6.47) are

$$\begin{aligned}
\psi_1(\xi) = \tilde{\psi}_1(\xi_1)\tilde{\psi}_1(\xi_2) \quad &= \frac{\xi_1 - \hat{s}_2}{\hat{s}_1 - \hat{s}_2}\frac{\xi_2 - \hat{s}_2}{\hat{s}_2 - \hat{s}_1} \quad &&= \frac{1}{4}(1-\xi_1)(1-\xi_2) \\
\psi_2(\xi) = \tilde{\psi}_2(\xi_1)\tilde{\psi}_1(\xi_2) \quad &= \frac{\xi_1 - \hat{s}_1}{\hat{s}_2 - \hat{s}_1}\frac{\xi_2 - \hat{s}_2}{\hat{s}_1 - \hat{s}_2} \quad &&= \frac{1}{4}(\xi_1+1)(1-\xi_2) \\
\psi_3(\xi) = \tilde{\psi}_1(\xi_1)\tilde{\psi}_2(\xi_2) \quad &= \frac{\xi_1 - \hat{s}_1}{\hat{s}_2 - \hat{s}_1}\frac{\xi_2 - \hat{s}_2}{\hat{s}_1 - \hat{s}_2} \quad &&= \frac{1}{4}(1-\xi_1)(\xi_2+1) \\
\psi_4(\xi) = \tilde{\psi}_2(\xi_1)\tilde{\psi}_2(\xi_2) \quad &= \frac{\xi_1 - \hat{s}_1}{\hat{s}_2 - \hat{s}_1}\frac{\xi_2 - \hat{s}_1}{\hat{s}_2 - \hat{s}_1} \quad &&= \frac{1}{4}(\xi_1+1)(\xi_2+1),
\end{aligned} \tag{6.49}$$

where the last equality used that the nodes of the $\mathcal{P}^1$ master element are $\hat{s}_1 = -1$ and $\hat{s}_2 = 1$.

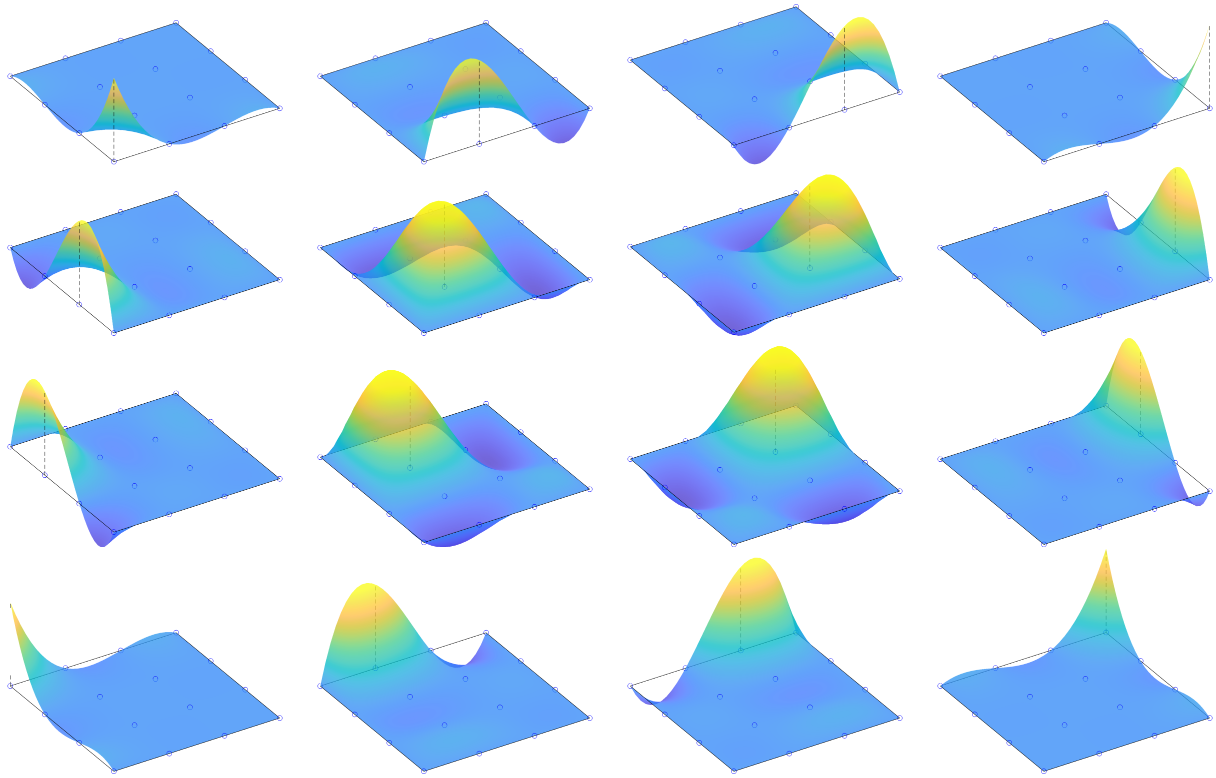

Figure 6.6: Nodal basis functions of $\mathcal{Q}^3$ master quadrilateral element.

Example 6.8: Biquadratic $\mathcal{Q}^2$ quadrilateral nodal basis

Recall the quadrilateral-to-line mappings for the biquadratic quadrilateral (6.39)-(6.40). Then the nodal basis functions of the $\mathcal{Q}^2$ quadrilateral (Figure 6.5) using the tensor product formula in (6.47) are

$$
\begin{aligned}
\psi_1(\xi) = \tilde\psi_1(\xi_1)\tilde\psi_1(\xi_2) &= \frac{\xi_1-\hat s_2}{\hat s_1-\hat s_2}\frac{\xi_1-\hat s_3}{\hat s_1-\hat s_3}\frac{\xi_2-\hat s_2}{\hat s_1-\hat s_2}\frac{\xi_2-\hat s_3}{\hat s_1-\hat s_3} &&= \frac{\xi_1(\xi_1-1)\xi_2(\xi_2-1)}{4}\\
\psi_2(\xi) = \tilde\psi_2(\xi_1)\tilde\psi_1(\xi_2) &= \frac{\xi_1-\hat s_1}{\hat s_2-\hat s_1}\frac{\xi_1-\hat s_3}{\hat s_2-\hat s_3}\frac{\xi_2-\hat s_2}{\hat s_1-\hat s_2}\frac{\xi_2-\hat s_3}{\hat s_1-\hat s_3} &&= \frac{(\xi_1+1)(1-\xi_1)\xi_2(\xi_2-1)}{2}\\
\psi_3(\xi) = \tilde\psi_3(\xi_1)\tilde\psi_1(\xi_2) &= \frac{\xi_1-\hat s_1}{\hat s_3-\hat s_1}\frac{\xi_1-\hat s_2}{\hat s_3-\hat s_2}\frac{\xi_2-\hat s_2}{\hat s_1-\hat s_2}\frac{\xi_2-\hat s_3}{\hat s_1-\hat s_3} &&= \frac{\xi_1(\xi_1+1)\xi_2(\xi_2-1)}{4}\\
\psi_4(\xi) = \tilde\psi_1(\xi_1)\tilde\psi_2(\xi_2) &= \frac{\xi_1-\hat s_2}{\hat s_1-\hat s_2}\frac{\xi_1-\hat s_3}{\hat s_1-\hat s_3}\frac{\xi_2-\hat s_1}{\hat s_2-\hat s_1}\frac{\xi_2-\hat s_3}{\hat s_2-\hat s_3} &&= \frac{\xi_1(\xi_1-1)(\xi_2+1)(1-\xi_2)}{2}\\
\psi_5(\xi) = \tilde\psi_2(\xi_1)\tilde\psi_2(\xi_2) &= \frac{\xi_1-\hat s_1}{\hat s_2-\hat s_1}\frac{\xi_1-\hat s_3}{\hat s_2-\hat s_3}\frac{\xi_2-\hat s_1}{\hat s_2-\hat s_1}\frac{\xi_2-\hat s_3}{\hat s_2-\hat s_3} &&= (\xi_1+1)(1-\xi_1)(\xi_2+1)(1-\xi_2)\\
\psi_6(\xi) = \tilde\psi_3(\xi_1)\tilde\psi_2(\xi_2) &= \frac{\xi_1-\hat s_1}{\hat s_3-\hat s_1}\frac{\xi_1-\hat s_2}{\hat s_3-\hat s_2}\frac{\xi_2-\hat s_1}{\hat s_2-\hat s_1}\frac{\xi_2-\hat s_3}{\hat s_2-\hat s_3} &&= \frac{\xi_1(\xi_1+1)(\xi_2+1)(1-\xi_2)}{2}\\
\psi_7(\xi) = \tilde\psi_1(\xi_1)\tilde\psi_3(\xi_2) &= \frac{\xi_1-\hat s_2}{\hat s_1-\hat s_2}\frac{\xi_1-\hat s_3}{\hat s_1-\hat s_3}\frac{\xi_2-\hat s_1}{\hat s_3-\hat s_1}\frac{\xi_2-\hat s_2}{\hat s_3-\hat s_2} &&= \frac{\xi_1(\xi_1-1)\xi_2(\xi_2+1)}{4}\\
\psi_8(\xi) = \tilde\psi_2(\xi_1)\tilde\psi_3(\xi_2) &= \frac{\xi_1-\hat s_1}{\hat s_2-\hat s_1}\frac{\xi_1-\hat s_3}{\hat s_2-\hat s_3}\frac{\xi_2-\hat s_1}{\hat s_3-\hat s_1}\frac{\xi_2-\hat s_2}{\hat s_3-\hat s_2} &&= \frac{(\xi_1+1)(1-\xi_1)\xi_2(\xi_2+1)}{2}\\
\psi_9(\xi) = \tilde\psi_3(\xi_1)\tilde\psi_3(\xi_2) &= \frac{\xi_1-\hat s_1}{\hat s_3-\hat s_1}\frac{\xi_1-\hat s_2}{\hat s_3-\hat s_2}\frac{\xi_2-\hat s_1}{\hat s_3-\hat s_1}\frac{\xi_2-\hat s_2}{\hat s_3-\hat s_2} &&= \frac{\xi_1(\xi_1+1)\xi_2(\xi_2+1)}{4}
\end{aligned}
\tag{6.50}
$$

where the last equality used that the nodes of the $\mathcal{P}^2$ master element are $\hat s_1 = -1$, $\hat s_2 = 0$, and $\hat s_3 = 1$.

6.3.4 $d = 2$ dimensions: $\mathcal{P}^p$ triangle elements

Next we introduce the most common, versatile two-dimensional finite element, the triangle.

Element domain

The reference domain of the master triangle element is taken to be the unit right triangle

$$\Omega_\square := \left\{\boldsymbol{\xi} \in \mathbb{R}^d \mid \xi_1 \geqslant 0, \xi_2 \geqslant 0, \xi_1 + \xi_2 \leqslant 1\right\}. \tag{6.51}$$

The boundary of the master triangle consists of three faces $\partial\Omega_\square = \bigcup_{i=1}^3 \partial\Omega_{\square,i}$, where

$$\partial\Omega_{\square,1} := \{0\} \times [0,1], \quad \partial\Omega_{\square,2} := [0,1] \times \{0\}, \quad \partial\Omega_{\square,3} := \{(s, 1-s) \mid s \in [0,1]\} \tag{6.52}$$

and the corresponding unit outward normals are

$$\boldsymbol{N}_{\square,1} := \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \quad \boldsymbol{N}_{\square,2} := \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \quad \boldsymbol{N}_{\square,3} := \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{6.53}$$

The complete geometry of the master triangle element is illustrated in Figure 6.1.

Local function space

We take the local function space to be $\mathcal{Y}_\square := \mathcal{P}^p(\Omega_\square)$. The dimension of the local function space is

$$\dim \mathcal{Y}_\square = \frac{(p+1)(p+2)}{2}. \tag{6.54}$$

An important property of this function space is every $v \in \mathcal{P}^p(\Omega_\square)$ is a one-dimensional polynomial of degree p when restricted to any line $\Gamma = \{(a_1 + b_1 s, a_2 + b_2 s) \mid s \in \mathbb{R}\}$ for any $a_1, a_2, b_1, b_2 \in \mathbb{R}^2$. To see this, let $\boldsymbol{\gamma} : \mathbb{R} \to \Gamma$ be a parametrization of Γ defined as $\boldsymbol{\gamma}(s) := (a_1 + b_1 s, a_2 + b_2 s)$. Then, for any $v \in \mathcal{P}^p(\Omega_\square)$, we expand in a monomial basis as

$$v(\boldsymbol{\xi}) = \sum_{\alpha_1 + \alpha_2 \leqslant p} a_{\alpha_1 \alpha_2} \xi_1^{\alpha_1} \xi_2^{\alpha_2}. \tag{6.55}$$

We restrict v to Γ by composing with the parametrization $\boldsymbol{\gamma}$ to obtain

$$f(s) := v(\boldsymbol{\gamma}(s)) = \sum_{\alpha_1 + \alpha_2 \leqslant p} a_{\alpha_1 \alpha_2} \gamma_1(s)^{\alpha_1} \gamma_2^{\alpha_2} = \sum_{\alpha_1 + \alpha_2 \leqslant p} a_{\alpha_1 \alpha_2} (a_1 + b_1 s)^{\alpha_1} (a_2 + b_2 s)^{\alpha_2}, \tag{6.56}$$

where clearly the largest monomial term is $s^{\alpha_1 + \alpha_2}$ and since $\alpha_1 + \alpha_2 \leqslant p$, this is a polynomial of degree at most p. Since all boundaries of the master triangular element are striaght lines, this implies that functions of $\mathcal{P}^p(\Omega_\square)$ restricted to the faces of the master element are one-dimensional polynomials of degree at most p.

Distribution and numbering of nodes

Before we construct a nodal basis of $\mathcal{Y}$, we must distribute $N_{\text{nd}}^{\text{el}} = (p+1)(p+2)/2$ nodes throughout the element domain $\Omega_\square$. To ensure all basis functions are linearly independent, the nodes must not overlap (or be too close to prevent ill-conditioning). Similar to the quadrilateral element, we require that $p+1$ nodes lie on each of the three faces of the triangle $\Omega_\square$. Again, this comes from the fact that functions of $\mathcal{P}^p(\Omega_\square)$ are one-dimensional polynomials of degree p when restricted to faces $\partial\Omega_\square$ and are therefore uniquely determined by $p+1$ nodal values, which gives a convenient way to enforce global continuity (ensure the nodal values of abutting elements agree at the $p+1$ nodes). A convenient and systematic way to populate the master triangle with nodes is to:

(1) uniformly distribute $p+1$ nodes $\{\hat{s}_1, \dots, \hat{s}_{p+1}\}$ throughout the unit interval $[0,1]$, i.e., $\hat{s}_i = (i-1)/p$,

(2) form their tensor product following the procedure in Section 6.3.3 to yield $(p+1)^2$ nodes $\{\boldsymbol{\zeta}_1, \dots, \boldsymbol{\zeta}_{(p+1)^2}\}$ in the unit square $[0,1]^2$, and

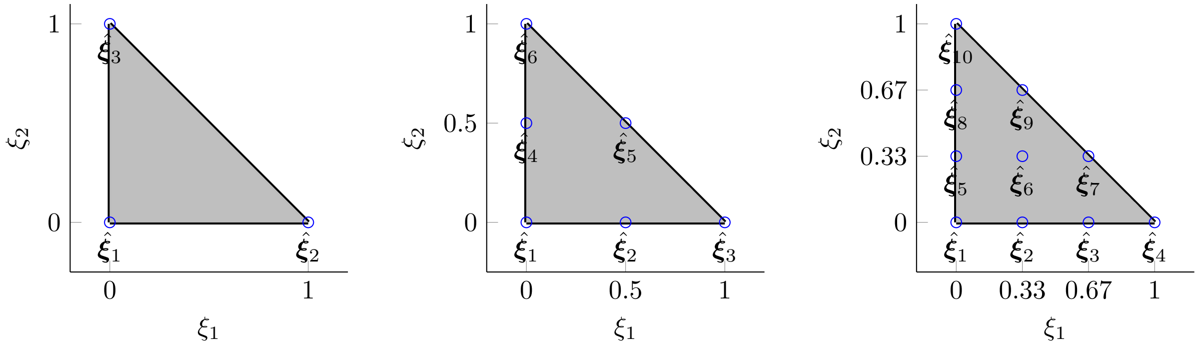

Figure 6.7: Master $\mathcal{P}^p$ triangle element including nodal positions and numbering for $p = 1, 2, 3$ (*left-to-right*).

(3) retain only the nodes that lie in the master triangle domain $\Omega_\square$ and re-number sequentially (preserve ordering) to obtain the nodes $\{\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_{N^{\text{el}}_{\text{nd}}}\}$.

This procedure will generate nodes in the master triangle that are uniformly spaced with $p+1$ nodes lying on each boundary (Figure 6.7 for $p = 1, 2, 3$).

Example 6.9: Nodes of linear $\mathcal{P}^1$ triangle

From (6.54) with $p = 1$, there are 3 nodes associated with the $\mathcal{P}^1$ master triangle. The only locations we can place these nodes to ensure each face has 2 nodes is at the triangle vertices

$$\hat{\boldsymbol{\xi}}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \hat{\boldsymbol{\xi}}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \hat{\boldsymbol{\xi}}_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{6.57}$$

which clearly agrees with Figure 6.7.

Example 6.10: Nodes of quadratic $\mathcal{P}^2$ triangle

From (6.54) with $p = 2$, there are 6 nodes associated with the $\mathcal{P}^2$ master triangle and each face must contain $p+1 = 3$ nodes. Following the procedure outlined in this section, we define equally spaced nodes in the unit interval $\hat{s}_1 = 0$, $\hat{s}_2 = 0.5$, $\hat{s}_3 = 1$, which leads to the following nodes in the unit square (following the procedure in Section 6.3.3 to construct nodes in $\mathbb{R}^2$ as tensor products of those in $\mathbb{R}$):

$$\begin{aligned} \boldsymbol{\zeta}_1 &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}, & \boldsymbol{\zeta}_2 &= \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}, & \boldsymbol{\zeta}_3 &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\ \boldsymbol{\zeta}_4 &= \begin{bmatrix} 0 \\ 0.5 \end{bmatrix}, & \boldsymbol{\zeta}_5 &= \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}, & \boldsymbol{\zeta}_6 &= \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}, \\ \boldsymbol{\zeta}_7 &= \begin{bmatrix} 0 \\ 1 \end{bmatrix}, & \boldsymbol{\zeta}_8 &= \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}, & \boldsymbol{\zeta}_9 &= \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \end{aligned} \tag{6.58}$$

The only nodes that lie in the master triangle domain are $\boldsymbol{\zeta}_i$ for $i \in \{1, 2, 3, 4, 5, 7\}$, so we re-number these nodes sequentially (retaining their original ordering) as the nodes of the master triangle

$$\begin{aligned} \boldsymbol{\xi}_1 &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}, & \boldsymbol{\xi}_2 &= \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}, & \boldsymbol{\xi}_3 &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\ \boldsymbol{\xi}_4 &= \begin{bmatrix} 0 \\ 0.5 \end{bmatrix}, & \boldsymbol{\xi}_5 &= \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}, & \boldsymbol{\xi}_6 &= \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \end{aligned} \tag{6.59}$$

which clearly agrees with Figure 6.7.

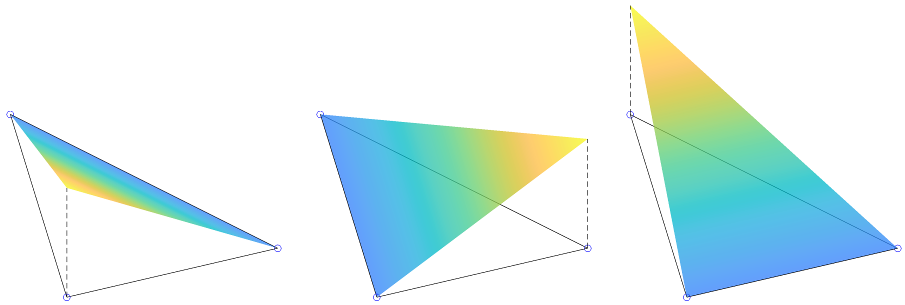

Figure 6.8: Nodal basis functions of $\mathcal{P}^1$ master triangle element.

Construction of element basis functions

Unfortunately the triangle does not posses the Cartesian product structure that we utilized to build up the nodal basis functions of the $\mathcal{Q}^p$ quadrilateral from the nodal basis of the $\mathcal{P}^p$ line elements. Instead, we introduce a systematic procedure, known as Vandermonde's method, to construct the nodal basis. Let $\{\psi_1, \dots, \psi_{N_{\text{nd}}^{\text{el}}}\}$ denote the nodal basis of the local function space $\mathcal{Y}_\square$ of the master triangle $\Omega_\square$. Since each $\psi_i \in \mathcal{P}^p(\Omega_\square)$, it can can be expanded in a monomial basis that includes all terms up to those with exponents that sum to p, i.e., $\{\xi_1^\alpha \xi_2^\beta \mid \alpha + \beta \leqslant p\}$, so we can write our $N_{\text{nd}}^{\text{el}}$ basis functions as

$$\psi_i(\boldsymbol{\xi}) = \sum_{k=1}^{N_{\text{nd}}^{\text{el}}} \hat{C}_{ik} \xi_1^{\alpha_k} \xi_2^{\beta_k} \tag{6.60}$$

where $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{N}_0^{N_{\text{nd}}^{\text{el}}}$ are vectors of natural numbers such that $\alpha_i + \beta_i \leqslant p$ for $i = 1, \dots, N_{\text{nd}}^{\text{el}}$ that are used to sweep over all $N_{\text{nd}}^{\text{el}}$ permissible exponents.

Denote the $N_{\text{nd}}^{\text{el}}$ nodes of the pth order simplex element as $\{\hat{\boldsymbol{\xi}}_i\}_{i=1}^{N_{\text{nd}}^{\text{el}}}$, where $\hat{\boldsymbol{\xi}}_i = (\hat{\xi}_{1i}, \hat{\xi}_{2i})^T$. The nodal property is

$$\psi_i(\hat{\boldsymbol{\xi}}_j) = \delta_{ij},$$

for $i, j = 1, \dots, N_{\text{nd}}^{\text{el}}$, which leads to

$$\sum_{k=1}^{N_{\text{nd}}^{\text{el}}} \hat{C}_{ik} \hat{\xi}_{1j}^{\alpha_k} \hat{\xi}_{2j}^{\beta_k} = \delta_{ij}$$

once the expression for $\psi_i(\boldsymbol{\xi})$ is used from (6.92). Let $\hat{V}_{ij} = \xi_{1i}^{\alpha_j} \xi_{2i}^{\beta_j}$ be the Vandermonde matrix corresponding to the d-dimensional, pth order simplex evaluated at $\{\hat{\boldsymbol{\xi}}_i\}_{i=1}^{N_{\text{nd}}^{\text{el}}}$, then the above constraints can be written in matrix form as $\hat{\boldsymbol{V}} \hat{\boldsymbol{C}}^T = \boldsymbol{I}_{N_{\text{nd}}^{\text{el}}}$, where $\hat{\boldsymbol{V}}$, $\hat{\boldsymbol{C}}$ are the matrices with indices $\hat{V}_{ij}$, $\hat{C}_{ij}$, respectively, and $\boldsymbol{I}_{N_{\text{nd}}^{\text{el}}}$ is the $N_{\text{nd}}^{\text{el}} \times N_{\text{nd}}^{\text{el}}$ identity matrix. Once we compute the coefficients, $\hat{\boldsymbol{C}} = \hat{\boldsymbol{V}}^{-T}$, we substitute this expression into (6.60) to give the final expression for

$$\psi_i(\boldsymbol{\xi}) = \sum_{k=1}^{N_{\text{nd}}^{\text{el}}} \hat{C}_{ik} \xi_1^{\alpha_k} \xi_2^{\beta_k} = \sum_{k=1}^{N_{\text{nd}}^{\text{el}}} \left(\hat{V}^{-1}\right)_{ki} \xi_1^{\alpha_k} \xi_2^{\beta_k}. \tag{6.61}$$

The nodal basis functions for the $\mathcal{P}^1$, $\mathcal{P}^2$, and $\mathcal{P}^3$ master triangle are shown in Figures 6.8-6.10.

Example 6.11: Linear $\mathcal{P}^1$ triangle nodal basis

To provide a concrete example, we consider the $\mathcal{P}^1$ master triangle. The vectors used to sweep over the admissible monomials are

$$\boldsymbol{\alpha} = (0, 1, 0), \quad \boldsymbol{\beta} = (0, 0, 1), \tag{6.62}$$

which leads to the following monomial expansion of the basis functions

$$\psi_i(\xi) = \hat{C}_{i1} \xi_1^{\alpha_1} \xi_2^{\beta_1} + \hat{C}_{i2} \xi_1^{\alpha_2} \xi_2^{\beta_2} + \hat{C}_{i3} \xi_1^{\alpha_3} \xi_2^{\beta_3} = \hat{C}_{i1} \xi_1^0 \xi_2^0 + \hat{C}_{i2} \xi_1^1 \xi_2^0 + \hat{C}_{i3} \xi_1^0 \xi_2^1 = \hat{C}_{i1} + \hat{C}_{i2} \xi_1 + \hat{C}_{i3} \xi_2, \tag{6.63}$$

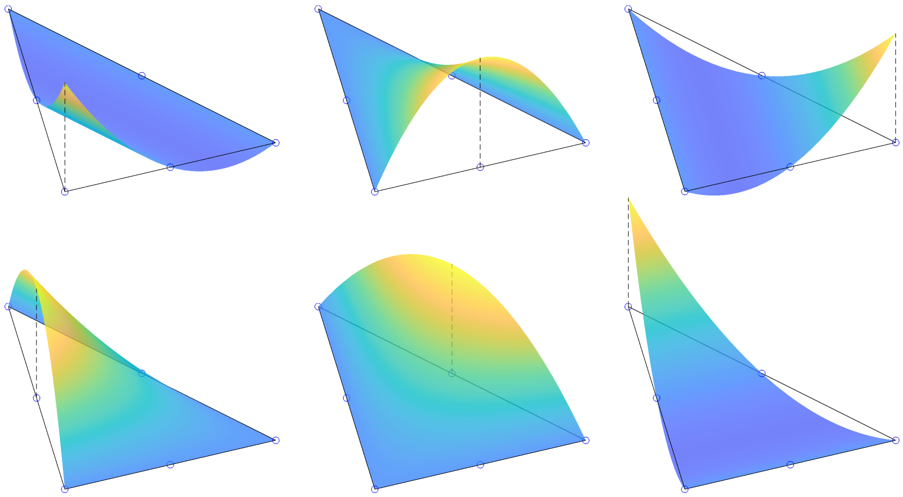

Figure 6.9: Nodal basis functions of $\mathcal{P}^2$ master triangle element.

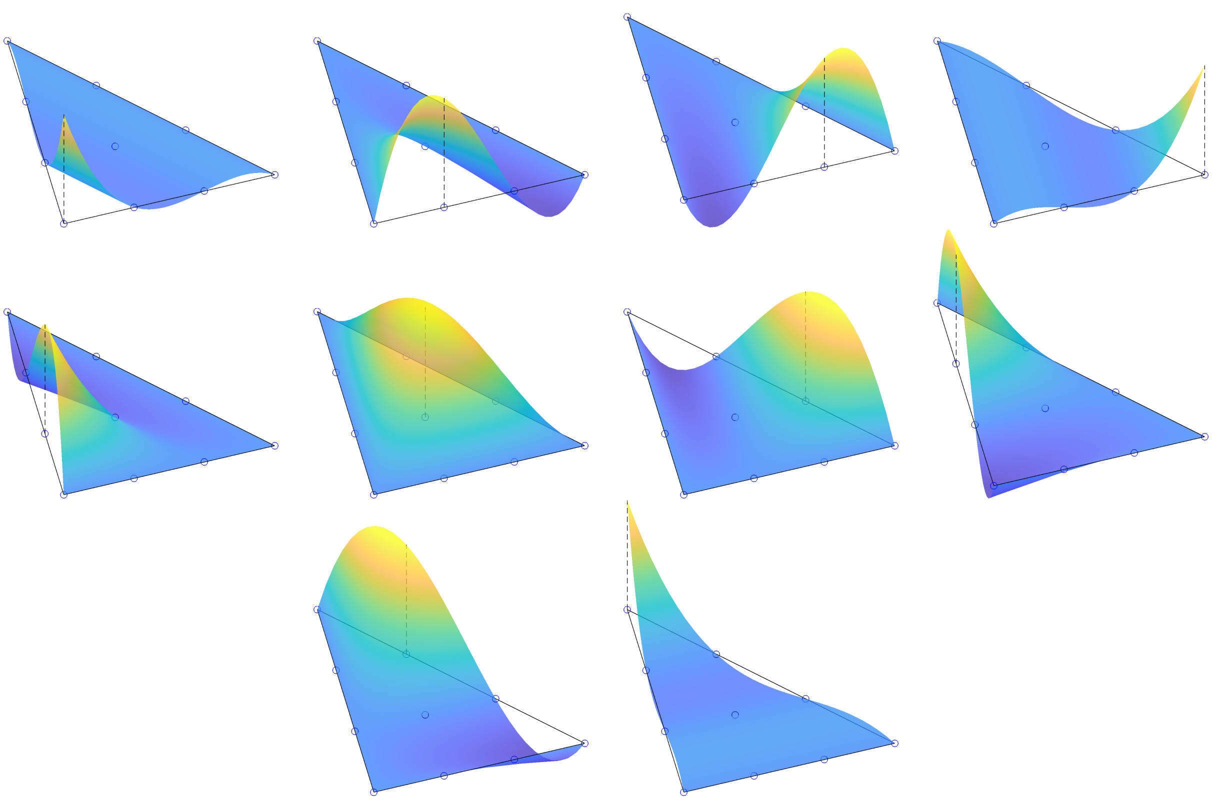

Figure 6.10: Nodal basis functions of $\mathcal{P}^3$ master triangle element.

which clearly lies in $\mathcal{P}^1(\Omega_\square)$. The corresponding Vandermonde matrix is

$$\boldsymbol{V} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \tag{6.64}$$

which leads to the following matrix of coefficients

$$\hat{\boldsymbol{C}} = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{6.65}$$

Combining these coefficients with the expansion in (6.60) we have

$$\begin{aligned} \psi_1(\xi) &= 1 - \xi_1 - \xi_2 \\ \psi_2(\xi) &= \xi_1 \\ \psi_3(\xi) &= \xi_2. \end{aligned} \tag{6.66}$$

It is a simple exercise to show these possess the nodal property with respect to the nodes of the $\mathcal{P}^1$ master triangle defined in Example 6.3.4. These nodal basis functions are shown in Figure 6.8.

Example 6.12: Quadratic $\mathcal{P}^2$ triangle nodal basis

Next we consider the $\mathcal{P}^2$ master triangle. The vectors used to sweep over the admissible monomials are

$$\boldsymbol{\alpha} = (0, 1, 0, 2, 1, 0), \quad \boldsymbol{\beta} = (0, 0, 1, 0, 1, 2), \tag{6.67}$$

which leads to the following monomial expansion of the basis functions

$$\begin{aligned} \psi_i(\xi) &= \hat{C}_{i1}\xi_1^{\alpha_1}\xi_2^{\beta_1} + \hat{C}_{i2}\xi_1^{\alpha_2}\xi_2^{\beta_2} + \hat{C}_{i3}\xi_1^{\alpha_3}\xi_2^{\beta_3} + \hat{C}_{i3}\xi_1^{\alpha_4}\xi_2^{\beta_4} + \hat{C}_{i3}\xi_1^{\alpha_5}\xi_2^{\beta_5} + \hat{C}_{i3}\xi_1^{\alpha_6}\xi_2^{\beta_6} \\ &= \hat{C}_{i1}\xi_1^0\xi_2^0 + \hat{C}_{i2}\xi_1^1\xi_2^0 + \hat{C}_{i3}\xi_1^0\xi_2^1 + \hat{C}_{i4}\xi_1^2\xi_2^0 + \hat{C}_{i5}\xi_1^1\xi_2^1 + \hat{C}_{i6}\xi_1^0\xi_2^2 \\ &= \hat{C}_{i1} + \hat{C}_{i2}\xi_1 + \hat{C}_{i3}\xi_2 + \hat{C}_{i4}\xi_1^2 + \hat{C}_{i5}\xi_1\xi_2 + \hat{C}_{i6}\xi_2^2, \end{aligned} \tag{6.68}$$

which clearly lies in $\mathcal{P}^1(\Omega_\square)$. The corresponding Vandermonde matrix is

$$\boldsymbol{V} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0.5 & 0 & 0.25 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0.5 & 0 & 0 & 0.25 \\ 1 & 0.5 & 0.5 & 0.25 & 0.25 & 0.25 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}, \tag{6.69}$$

which leads to the following matrix of coefficients

$$\hat{\boldsymbol{C}} = \begin{bmatrix} 1 & -3 & -3 & 2 & 4 & 2 \\ 0 & 4 & 0 & -4 & -4 & 0 \\ 0 & -1 & 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 & -4 & -4 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & -1 & 0 & 0 & 2 \end{bmatrix}. \tag{6.70}$$

Combining these coefficients with the expansion in (6.60) we have

$$\begin{aligned}\psi_1(\xi) &= 1 - 3(\xi_1 + \xi_2) + 2(\xi_1 + \xi_2)^2 \\ \psi_2(\xi) &= 4\xi_1(1 - \xi_1 - \xi_2) \\ \psi_3(\xi) &= \xi_1(-1 + 2\xi_1) \\ \psi_4(\xi) &= 4\xi_2(1 - \xi_1 - \xi_2) \\ \psi_5(\xi) &= 4\xi_1\xi_2 \\ \psi_6(\xi) &= \xi_2(-1 + 2\xi_2).\end{aligned} \tag{6.71}$$

It is a simple exercise to show these possess the nodal property with respect to the nodes of the $\mathcal{P}^2$ master triangle defined in Example 6.3.4. These nodal basis functions are shown in Figure 6.9.

6.3.5 d dimensions: $\mathcal{Q}^p$ hypercube elements

In $d > 2$ dimensions, the number of possible geometries explodes, i.e., in $d = 3$ could have tetrahedra, cubes, prisms, pyramids. Unfortunately we do not have time to develop all these elements; instead, we focus on elements that generalize to any number of dimensions. We introduce a systematic procedure to define the element domain, local function space, nodes, and construct nodal basis functions. We being with the hypercube element, the d-dimensional generalization of a quadrilateral.

Element domain

The reference domain of the master hypercube element is taken to be the bi-unit interval centered at zero

$$\Omega_\square := \left\{ \boldsymbol{\xi} \in \mathbb{R}^d \;\middle|\; -1 \leqslant \xi_i \leqslant 1, i = 1, \dots, d \right\}. \tag{6.72}$$

The master hypercube element is the Cartesian product of the master line element with itself d times. The boundary of the master element is $\partial\Omega_\square = \bigcup_{i=1}^{2d} \partial\Omega_{\square,i}$, where

$$\begin{aligned}\partial\Omega_{\square,i} &:= \left\{ \boldsymbol{\xi} \in \mathbb{R}^d \;\middle|\; \xi_i = -1, \;\; -1 \leqslant \xi_j \leqslant 1, \;\; j \neq i \right\} \\ \partial\Omega_{\square,d+i} &:= \left\{ \boldsymbol{\xi} \in \mathbb{R}^d \;\middle|\; \xi_i = 1, \;\; -1 \leqslant \xi_j \leqslant 1, \;\; j \neq i \right\}\end{aligned} \tag{6.73}$$

and the corresponding unit outward normals are

$$\boldsymbol{N}_{\square,i} := -\boldsymbol{e}_i, \quad \boldsymbol{N}_{\square,d+i} := \boldsymbol{e}_i. \tag{6.74}$$

for $i = 1, \dots, d$. Notice that this definition coincides with the master line element for $d = 1$ and quadrilateral element for $d = 2$. The complete geometry of the master hypercube element is illustrated in Figure 6.1 ($d = 1, 2$) and Figure 6.11 ($d = 3$).

Local function space

We take the local function space to be the space $\mathcal{Y}_\square := \mathcal{Q}^p(\Omega_\square)$, i.e., polynomial functions where the largest exponent is p. The dimension of the local function space is $\dim \mathcal{Y}_\square = (p+1)^d$. Similar to the $d = 2$ case, functions that belong to $\mathcal{Q}^p(\Omega_\square)$ are polynomials in $d-1$ dimension where the largest exponent is p when restricted to any face of $\Omega_\square$. To see this we introduce a parametrization of boundary 1 (chosen arbitrarily), $\boldsymbol{\gamma} : [-1, 1]^{d-1} \to \partial\Omega_{\square,1}$ defined as $\boldsymbol{\gamma}(s_1, \dots, s_{d-1}) := (-1, s_1, \dots, s_{d-1}) \in \partial\Omega_{\square,1}$. Then, for any $v \in \mathcal{Q}^p(\Omega_\square)$, we expand it in a monomial basis as

$$v(\boldsymbol{\xi}) = \sum_{\substack{\boldsymbol{\alpha} \in \mathbb{N}^d \\ \max \boldsymbol{\alpha} \leqslant p}} a_\alpha \xi_1^{\alpha_1} \cdots \xi_d^{\alpha_d}, \tag{6.75}$$

where $a_\alpha \in \mathbb{R}$ for $\boldsymbol{\alpha} \in \mathbb{N}^d$, $\max \boldsymbol{\alpha} \leqslant p$. We restrict v to $\partial\Omega_{\square,1}$ by composing with the face parametrization $\boldsymbol{\gamma}$

$$f(s) = v(\boldsymbol{\gamma}(s)) = \sum_{\substack{\boldsymbol{\alpha} \in \mathbb{N}^d \\ \max \boldsymbol{\alpha} \leqslant p}} = a_\alpha \gamma_1(s)^{\alpha_1} \cdots \gamma_d(s)^{\alpha_d} = \sum_{\alpha_2, \dots, \alpha_d \leqslant p} \left(\sum_{\alpha_1 \leqslant p} a_\alpha (-1)^{\alpha_1} \right) s_1^{\alpha_2} \cdots s_{d-1}^{\alpha_{d-1}}, \tag{6.76}$$

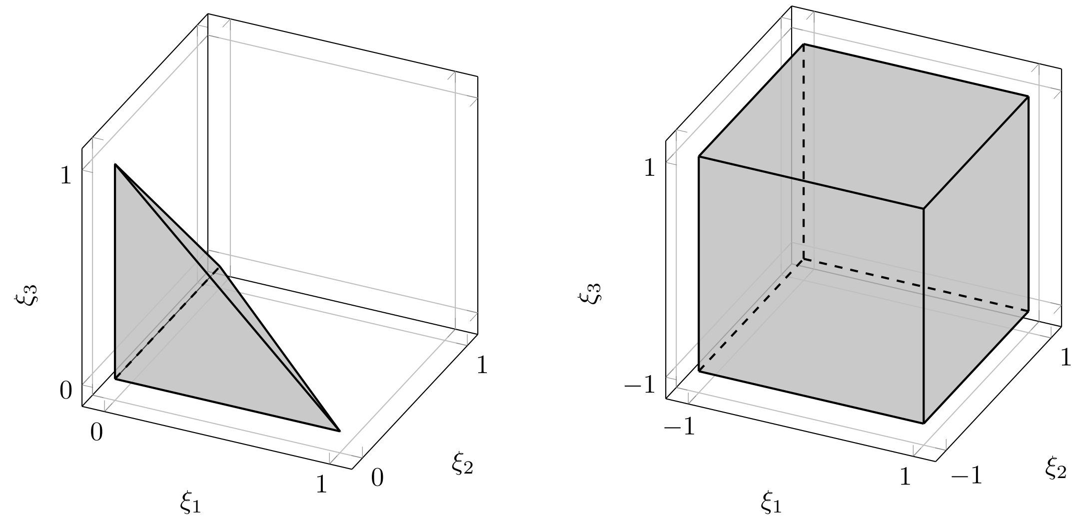

Figure 6.11: Master element geometry for $d = 3$ simplex (tetrahedra) (*left*) and $d = 3$ hypercube (hexahedron) (*right*).

which is clearly a polynomial of degree at most p in $d-1$ dimensions, i.e., $f \in \mathcal{Q}^p([-1,1]^{d-1})$.

Distribution and numbering of nodes

Before we construct a nodal basis of $\mathcal{Y}_\square$, we must distribute $N_{\text{nd}}^{\text{el}} = (p+1)^d$ nodes throughout the element domain $\Omega_\square$. To ensure all basis functions are linearly independent, the nodes must not overlap (or be too close to prevent ill-conditioning). We also require that $(p+1)^{d-1}$ nodes lie on each of the $2d$ faces of the hypercube $\Omega_\square$. Again, this is because any element of $\mathcal{Q}^p(\Omega_\square)$ restricted to a face will be uniquely determined by its value at $(p+1)^{d-1}$ nodes, which gives a straightforward way to enforce global continuity.

Mimicing the construction of the nodes for the $\mathcal{Q}^p$ quadrilateral, we define the nodes of the $\mathcal{Q}^p$ master hypercube as the Cartesian product of the nodes of the $\mathcal{Q}^p$ master line element with itself d times, numbered first in the ξ_1-direction, then ξ_2, etc (Figure 6.13). To make this precise, let $\{\hat{s}_1, \dots, \hat{s}_{p+1}\} \subset [-1,1]$ be the nodes of the master line element and introduce

$$\mathcal{I}_i : \{1, \dots, (p+1)^d\} \to \{1, \dots, p+1\}, \quad i = 1, \dots, d, \tag{6.77}$$

where $\mathcal{I}_i$ maps the hypercube node number to the node number along the ξ_i-axis, i.e., the ith hypercube node is the $\mathcal{I}_j(i)$th node in the ξ_j-direction (in the $d = 2$ case, $\mathcal{I}_1$ and $\mathcal{I}_2$ are precisely $\mathcal{I}$ and $\mathcal{J}$, respectively, in (6.39)). For our node numbering that varies first in the ξ_1-direction, then ξ_2, etc., these mappings are

$$\mathcal{I}_1 := 1 + \left[(k-1)\%(p+1)^{d-1}\right], \qquad \mathcal{I}_i(k) := 1 + \left\lfloor \frac{k-1}{(p+1)^{d-1}} \right\rfloor \tag{6.78}$$

$i = 2, \dots, d$. With this notation, the kth hypercube node is defined in terms of the line element nodes as

$$\hat{\boldsymbol{\xi}}_k = \begin{bmatrix} \hat{\xi}_{1k} \\ \vdots \\ \hat{\xi}_{dk} \end{bmatrix} = \begin{bmatrix} \hat{s}_{\mathcal{I}_1(k)} \\ \vdots \\ \hat{s}_{\mathcal{I}_d(k)} \end{bmatrix}. \tag{6.79}$$

Example 6.13: Nodes of trilinear $\mathcal{Q}^1$ hexahedral

To define the nodes of $\mathcal{Q}^1$ master hexahedral element, recall the nodes of the $\mathcal{Q}^1$ master line element $\hat{s}_1 = -1$, $\hat{s}_2 = 1$ and, in the special case of $p = 1$, the hypercube-to-line mappings (6.77)-(6.78) are

$$\mathcal{I}_1 = \{1,2,1,2,1,2,1,2\}, \quad \mathcal{I}_2 = \{1,1,2,2,1,1,2,2\}, \quad \mathcal{I}_3 = \{1,1,1,1,2,2,2,2\}. \tag{6.80}$$

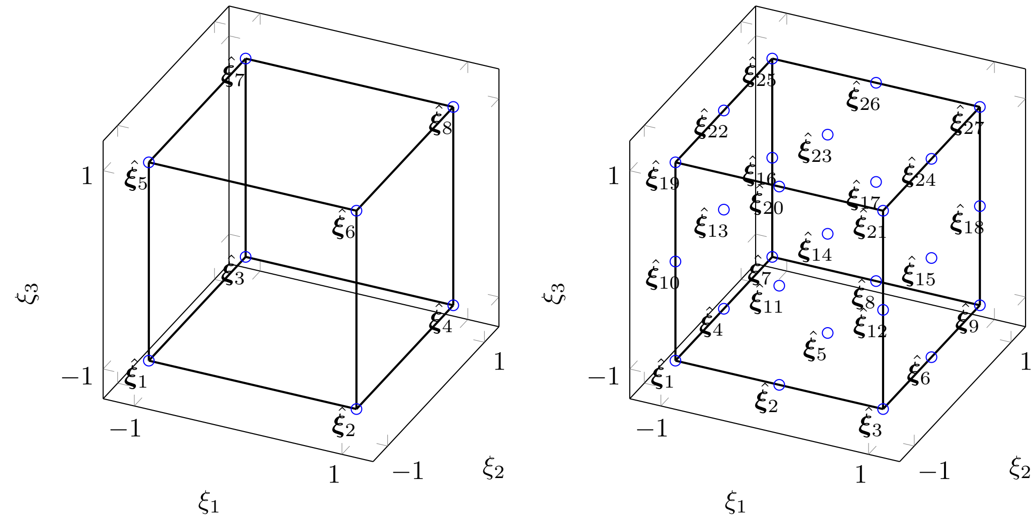

Figure 6.12: Master $\mathcal{Q}^p$ hexahedra element including nodal positions and numbering for $p = 1, 2, 3$ (*left-to-right*).

From this and (6.79), the nodes of the master $\mathcal{Q}^1$ hexahedral element are

$$
\begin{aligned}
\hat{\boldsymbol{\xi}}_1 &= \begin{bmatrix} \hat{\xi}_{11} \\ \hat{\xi}_{21} \\ \hat{\xi}_{31} \end{bmatrix} = \begin{bmatrix} \hat{s}_{\mathcal{I}_1(1)} \\ \hat{s}_{\mathcal{I}_2(1)} \\ \hat{s}_{\mathcal{I}_3(1)} \end{bmatrix} = \begin{bmatrix} \hat{s}_1 \\ \hat{s}_1 \\ \hat{s}_1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}, &\quad \hat{\boldsymbol{\xi}}_2 &= \begin{bmatrix} \hat{\xi}_{12} \\ \hat{\xi}_{22} \\ \hat{\xi}_{32} \end{bmatrix} = \begin{bmatrix} \hat{s}_{\mathcal{I}_1(2)} \\ \hat{s}_{\mathcal{I}_3(2)} \\ \hat{s}_{\mathcal{I}_3(2)} \end{bmatrix} = \begin{bmatrix} \hat{s}_2 \\ \hat{s}_1 \\ \hat{s}_1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \\
\hat{\boldsymbol{\xi}}_3 &= \begin{bmatrix} \hat{\xi}_{13} \\ \hat{\xi}_{23} \\ \hat{\xi}_{33} \end{bmatrix} = \begin{bmatrix} \hat{s}_{\mathcal{I}_1(3)} \\ \hat{s}_{\mathcal{I}_2(3)} \\ \hat{s}_{\mathcal{I}_3(3)} \end{bmatrix} = \begin{bmatrix} \hat{s}_1 \\ \hat{s}_2 \\ \hat{s}_1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}, &\quad \hat{\boldsymbol{\xi}}_4 &= \begin{bmatrix} \hat{\xi}_{14} \\ \hat{\xi}_{24} \\ \hat{\xi}_{34} \end{bmatrix} = \begin{bmatrix} \hat{s}_{\mathcal{I}_1(4)} \\ \hat{s}_{\mathcal{I}_2(4)} \\ \hat{s}_{\mathcal{I}_3(4)} \end{bmatrix} = \begin{bmatrix} \hat{s}_2 \\ \hat{s}_2 \\ \hat{s}_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \\
\hat{\boldsymbol{\xi}}_5 &= \begin{bmatrix} \hat{\xi}_{15} \\ \hat{\xi}_{25} \\ \hat{\xi}_{35} \end{bmatrix} = \begin{bmatrix} \hat{s}_{\mathcal{I}_1(5)} \\ \hat{s}_{\mathcal{I}_2(5)} \\ \hat{s}_{\mathcal{I}_3(5)} \end{bmatrix} = \begin{bmatrix} \hat{s}_1 \\ \hat{s}_1 \\ \hat{s}_2 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, &\quad \hat{\boldsymbol{\xi}}_6 &= \begin{bmatrix} \hat{\xi}_{16} \\ \hat{\xi}_{26} \\ \hat{\xi}_{36} \end{bmatrix} = \begin{bmatrix} \hat{s}_{\mathcal{I}_1(6)} \\ \hat{s}_{\mathcal{I}_3(6)} \\ \hat{s}_{\mathcal{I}_3(6)} \end{bmatrix} = \begin{bmatrix} \hat{s}_2 \\ \hat{s}_1 \\ \hat{s}_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \\
\hat{\boldsymbol{\xi}}_7 &= \begin{bmatrix} \hat{\xi}_{17} \\ \hat{\xi}_{27} \\ \hat{\xi}_{37} \end{bmatrix} = \begin{bmatrix} \hat{s}_{\mathcal{I}_1(7)} \\ \hat{s}_{\mathcal{I}_2(7)} \\ \hat{s}_{\mathcal{I}_3(7)} \end{bmatrix} = \begin{bmatrix} \hat{s}_1 \\ \hat{s}_2 \\ \hat{s}_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, &\quad \hat{\boldsymbol{\xi}}_8 &= \begin{bmatrix} \hat{\xi}_{18} \\ \hat{\xi}_{28} \\ \hat{\xi}_{38} \end{bmatrix} = \begin{bmatrix} \hat{s}_{\mathcal{I}_1(8)} \\ \hat{s}_{\mathcal{I}_2(8)} \\ \hat{s}_{\mathcal{I}_3(8)} \end{bmatrix} = \begin{bmatrix} \hat{s}_2 \\ \hat{s}_2 \\ \hat{s}_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},
\end{aligned}
\tag{6.81}
$$

which clearly agrees with Figure 6.12.

Construction of element basis functions

Finally we construct the nodal basis of $\mathcal{Y}_\square = \mathcal{Q}^p(\Omega_\square)$ using the tensor product structure of the master hypercube element. Let $\{\hat{s}_1, \dots, \hat{s}_{p+1}\}$ be the nodes of the $\mathcal{P}^p$ master line element, $\mathcal{I}_1, \dots, \mathcal{I}_d$ be the hypercube-to-line nodal mapping from the previous section, and $\{\tilde{\psi}_1, \dots, \tilde{\psi}_d\}$ be the corresponding nodal basis, i.e., $\tilde{\psi}_i(\hat{s}_j) = \delta_{ij}$ for $i, j = 1, \dots, p+1$. Then we define the nodal basis functions $\{\psi_1, \dots, \psi_{N_{\text{nd}}^{\text{el}}}\}$ of the $\mathcal{Q}^p$ hypercube element to be

$$
\psi_i(\boldsymbol{\xi}) \coloneqq \prod_{j=1}^{d} \tilde{\psi}_{\mathcal{I}_j(i)}(\xi_j). \tag{6.82}
$$

Using the expression in (6.33) for the one-dimensional nodal basis, this becomes

$$
\psi_i(\xi) = \prod_{k=1}^{d} \left(\prod_{\substack{j=1 \\ j \neq \mathcal{I}_k(i)}}^{p+1} \frac{\xi_k - \hat{s}_j}{\hat{s}_{\mathcal{I}_k(i)} - \hat{s}_j} \right). \tag{6.83}
$$

To verify this choice of basis has the nodal property, we evaluate ψ_i at node $\hat{\boldsymbol{\xi}}_j$

$$\psi_i(\hat{\boldsymbol{\xi}}_j) = \prod_{k=1}^{d} \tilde{\psi}_{\mathcal{I}_k(i)}(\xi_{kj}) = \prod_{k=1}^{d} \tilde{\psi}_{\mathcal{I}_k(i)}(\hat{s}_{\mathcal{I}_k(j)}) = \prod_{k=1}^{d} \delta_{\mathcal{I}_k(i)\mathcal{I}_k(j)} = \delta_{ij}, \tag{6.84}$$

where the last equality follows because the product $\prod_{k=1}^{d} \delta_{\mathcal{I}_k(i)\mathcal{I}_k(j)}$ only survives if $\mathcal{I}_i(i) = \mathcal{I}_i(j)$ for $i = 1, \dots, d$, which can only happen if $i = j$.

Example 6.14: Trilinear $\mathcal{Q}^1$ hexahedral nodal basis

Recall the hypercube-to-line mappings for the trilinear hexahedral element (6.77)-(6.78). Then the nodal basis functions of the $\mathcal{Q}^1$ hexahedral element using the tensor product formula in (6.84) are

$$\begin{aligned}
\psi_1(\boldsymbol{\xi}) &= \tilde{\psi}_1(\xi_1)\tilde{\psi}_1(\xi_2)\tilde{\psi}_1(\xi_3) &&= \frac{\xi_1 - \hat{s}_2}{\hat{s}_1 - \hat{s}_2}\frac{\xi_2 - \hat{s}_2}{\hat{s}_1 - \hat{s}_2}\frac{\xi_3 - \hat{s}_2}{\hat{s}_1 - \hat{s}_2} &&= \frac{1}{8}(1-\xi_1)(1-\xi_2)(1-\xi_3) \\
\psi_2(\boldsymbol{\xi}) &= \tilde{\psi}_2(\xi_1)\tilde{\psi}_1(\xi_2)\tilde{\psi}_1(\xi_3) &&= \frac{\xi_1 - \hat{s}_1}{\hat{s}_2 - \hat{s}_1}\frac{\xi_2 - \hat{s}_2}{\hat{s}_1 - \hat{s}_2}\frac{\xi_3 - \hat{s}_2}{\hat{s}_1 - \hat{s}_2} &&= \frac{1}{8}(1+\xi_1)(1-\xi_2)(1-\xi_3) \\
\psi_3(\boldsymbol{\xi}) &= \tilde{\psi}_1(\xi_1)\tilde{\psi}_2(\xi_2)\tilde{\psi}_1(\xi_3) &&= \frac{\xi_1 - \hat{s}_2}{\hat{s}_1 - \hat{s}_2}\frac{\xi_2 - \hat{s}_1}{\hat{s}_2 - \hat{s}_1}\frac{\xi_3 - \hat{s}_2}{\hat{s}_1 - \hat{s}_2} &&= \frac{1}{8}(1-\xi_1)(1+\xi_2)(1-\xi_3) \\
\psi_4(\boldsymbol{\xi}) &= \tilde{\psi}_2(\xi_1)\tilde{\psi}_2(\xi_2)\tilde{\psi}_1(\xi_3) &&= \frac{\xi_1 - \hat{s}_1}{\hat{s}_2 - \hat{s}_1}\frac{\xi_2 - \hat{s}_1}{\hat{s}_2 - \hat{s}_1}\frac{\xi_3 - \hat{s}_2}{\hat{s}_1 - \hat{s}_2} &&= \frac{1}{8}(1+\xi_1)(1+\xi_2)(1-\xi_3) \\
\psi_5(\boldsymbol{\xi}) &= \tilde{\psi}_1(\xi_1)\tilde{\psi}_1(\xi_2)\tilde{\psi}_2(\xi_3) &&= \frac{\xi_1 - \hat{s}_2}{\hat{s}_1 - \hat{s}_2}\frac{\xi_2 - \hat{s}_2}{\hat{s}_1 - \hat{s}_2}\frac{\xi_3 - \hat{s}_1}{\hat{s}_2 - \hat{s}_1} &&= \frac{1}{8}(1-\xi_1)(1-\xi_2)(1+\xi_3) \\
\psi_6(\boldsymbol{\xi}) &= \tilde{\psi}_2(\xi_1)\tilde{\psi}_1(\xi_2)\tilde{\psi}_2(\xi_3) &&= \frac{\xi_1 - \hat{s}_1}{\hat{s}_2 - \hat{s}_1}\frac{\xi_2 - \hat{s}_2}{\hat{s}_1 - \hat{s}_2}\frac{\xi_3 - \hat{s}_1}{\hat{s}_2 - \hat{s}_1} &&= \frac{1}{8}(1+\xi_1)(1-\xi_2)(1+\xi_3) \\
\psi_7(\boldsymbol{\xi}) &= \tilde{\psi}_1(\xi_1)\tilde{\psi}_2(\xi_2)\tilde{\psi}_2(\xi_3) &&= \frac{\xi_1 - \hat{s}_2}{\hat{s}_1 - \hat{s}_2}\frac{\xi_2 - \hat{s}_1}{\hat{s}_2 - \hat{s}_1}\frac{\xi_3 - \hat{s}_1}{\hat{s}_2 - \hat{s}_1} &&= \frac{1}{8}(1-\xi_1)(1+\xi_2)(1+\xi_3) \\
\psi_8(\boldsymbol{\xi}) &= \tilde{\psi}_2(\xi_1)\tilde{\psi}_2(\xi_2)\tilde{\psi}_2(\xi_3) &&= \frac{\xi_1 - \hat{s}_1}{\hat{s}_2 - \hat{s}_1}\frac{\xi_2 - \hat{s}_1}{\hat{s}_2 - \hat{s}_1}\frac{\xi_3 - \hat{s}_1}{\hat{s}_2 - \hat{s}_1} &&= \frac{1}{8}(1+\xi_1)(1+\xi_2)(1+\xi_3).
\end{aligned} \tag{6.85}$$

where the last equality used that the nodes of the $\mathcal{P}^1$ master element are $\hat{s}_1 = -1$ and $\hat{s}_2 = 1$.

6.3.6 d dimensions: $\mathcal{P}^p$ simplex elements

Finally, we turn to the most versatile class of elements, simplices, the d-dimensional generalization of a triangle.

Element domain

The reference domain of the master simplex element is taken to be

$$\Omega_\square := \left\{ \boldsymbol{\xi} \in \mathbb{R}^d \;\middle|\; \sum_{i=1}^{d} \xi_i \leqslant 1, \xi_i \geqslant 0, i = 1, \dots, d \right\}. \tag{6.86}$$

The boundary of the master element is $\partial\Omega_\square = \bigcup_{i=1}^{d+1} \partial\Omega_{\square,i}$, where

$$\begin{aligned}
\partial\Omega_{\square,i} &:= \left\{ \boldsymbol{\xi} \in \mathbb{R}^d \;\middle|\; \xi_i = 0,\ 0 \leqslant \xi_j \leqslant 1,\ j \neq i \right\} \\
\partial\Omega_{\square,d+1} &:= \left\{ \boldsymbol{\xi} \in \mathbb{R}^d \;\middle|\; \sum_{j=1}^{d} \xi_j = 1,\ 0 \leqslant \xi_j \leqslant 1,\ j = 1, \dots, d \right\}.
\end{aligned} \tag{6.87}$$

and the corresponding unit outward normals are

$$\boldsymbol{N}_{\square,i} := -\boldsymbol{e}^{(i)}, \quad \boldsymbol{N}_{\square,d+1} := \sum_{i=1}^{d} \frac{1}{\sqrt{d}} \boldsymbol{e}^{(i)} \tag{6.88}$$

for $i = 1, \dots, d$. Notice that even though a $d = 1$ dimensional simplex is a line ($\Omega_\square = [0, 1]$), it does not coincide with the master line element introduced in Section 6.3.2 . The complete geometry of the master simplex element is illustrated in Figure 6.1 ($d = 2$ triangle) and Figure 6.11 ($d = 3$ tetrahedra).

Local function space

We take the local function space to be $\mathcal{Y}_\square := \mathcal{P}^p(\Omega_\square)$. The dimension of the local function space is

$$\dim \mathcal{Y}_\square = \binom{p+d}{d}. \tag{6.89}$$

Similar to the $d = 2$ case, functions that belong to $\mathcal{P}^p(\Omega_\square)$ are polynomials of degree p in $d-1$ dimensions when restricted to any plane.

Distribution and numbering of nodes

Before we construct a nodal basis of $\mathcal{Y}_\square$, we must distribute

$$N_{\text{nd}}^{\text{el}} = \binom{p+d}{d} \tag{6.90}$$

nodes throughout the element domain $\Omega_\square$. To ensure all basis functions are linearly independent, the nodes must not overlap (or be too close to prevent ill-conditioning). We also require that $\binom{p+d-1}{d-1}$ nodes lie on each of the faces of the simplex $\Omega_\square$. Again, this is because any element of $\mathcal{P}^p(\Omega_\square)$ restricted to a face will be uniquely determined by its value at $\binom{p+d-1}{d-1}$ nodes, which gives a straightforward way to enforce global continuity. A systematic procedure to populate the master simplex with nodes is the straightforward generalization of the procedure in Section 6.3.4 to populate the master triangle with nodes:

(1) uniformly distribute $p+1$ nodes $\{\hat{s}_1, \dots, \hat{s}_{p+1}\}$ throughout the unit interval $[0, 1]$,

(2) form their tensor product following the procedure in Section 6.3.5 to yield $(p+1)^d$ nodes $\{\boldsymbol{\zeta}_1, \dots, \boldsymbol{\zeta}_{(p+1)^d}\}$ nodes in the unit hypercube $[0, 1]^d$, and

(3) retain only the nodes that lie in the master simplex domain $\Omega_\square$ and re-number sequentially (preserving order) to obtain the nodes $\{\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_{N_{\text{nd}}^{\text{el}}}\}$.

This procedure will generate nodes in the master simplex that are uniformly spaced with $\binom{p+d-1}{d-1}$ on each boundary (Figure 6.13 for $p = 1, 2, 3$).

Example 6.15: Nodes of linear $\mathcal{P}^1$ tetrahedra

From (6.89) with $p = 1$, there are $\binom{p+d}{d} = \binom{4}{1} = 4$ nodes associated with the $\mathcal{P}^1$ master tetrahedra. The only locations we can place them to ensure each face has $\binom{p+d-1}{d-1} = \binom{3}{2} = 3$ nodes is at the tetrahedra vertices

$$\hat{\boldsymbol{\xi}}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \hat{\boldsymbol{\xi}}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \hat{\boldsymbol{\xi}}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \hat{\boldsymbol{\xi}}_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \tag{6.91}$$

which clearly agrees with Figure 6.13.

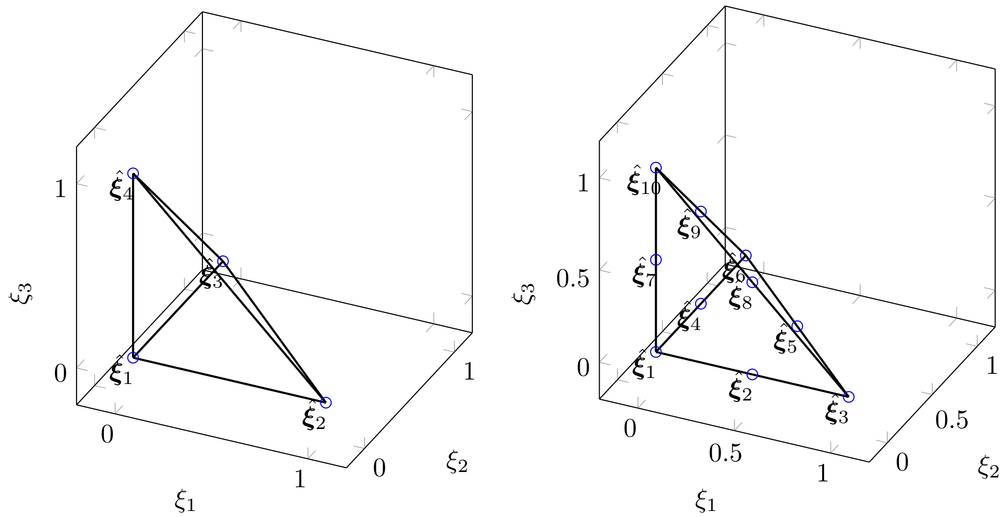

Figure 6.13: Master $\mathcal{P}^p$ tetrahedra element including nodal positions and numbering for $p = 1, 2, 3$ (*left-to-right*).

Construction of element basis functions

Finally we generalize Vandermonde's method introduced in Section 6.3.4 to define a nodal basis of $\mathcal{P}^p(\Omega_\square)$ ($\Omega_\square \subset \mathbb{R}^d$), which we denote $\{\psi_1, \dots, \psi_{N_{\text{nd}}^{\text{el}}}\}$. Since each $\psi_i \in \mathcal{P}^p(\Omega_\square)$, it can be expanded in a monomial basis that includes all terms up to those with exponents that sum to p, i.e., $\{\xi_1^{\alpha_1} \cdots \xi_d^{\alpha_d} \mid \sum_{i=1}^d \alpha_i \leqslant p\}$, so we can write our $N_{\text{nd}}^{\text{el}}$ basis functions as

$$\psi_i(\boldsymbol{\xi}) = \sum_{k=1}^{N_{\text{nd}}^{\text{el}}} \hat{C}_{ik} \prod_{j=1}^{d} \xi_j^{\Upsilon_{jk}} \tag{6.92}$$

where $\boldsymbol{\Upsilon} \in \mathbb{N}_0^{d \times N_{\text{nd}}^{\text{el}}}$ is a matrix of natural numbers such that $\sum_{i=1}^d \Upsilon_{ij} \leqslant p$ for each $j = 1, \dots, N_{\text{nd}}^{\text{el}}$ that is used to sweep over all $N_{\text{nd}}^{\text{el}}$ permissible exponents. For convenience, we introduce the function $\omega_i(\boldsymbol{\xi})$, $i = 1, \dots, N_{\text{nd}}^{\text{el}}$

$$\omega_i(\boldsymbol{\xi}) = \prod_{s=1}^{d} \xi_s^{\Upsilon_{si}},$$

so the basis functions can conveniently be expressed as $\psi_i(\boldsymbol{\xi}) = \sum_{k=1}^{N_{\text{nd}}^{\text{el}}} \hat{C}_{ik}\omega_k(\boldsymbol{\xi})$.

Example 6.16: Monomial expansion in $\mathcal{P}^1(\mathbb{R}^2)$

To demostrate the general monomial expansion in (6.92) agrees with known special cases we consider $p = 1$, $d = 2$ ($\mathcal{P}^1$ triangle). In this case, we take

$$\boldsymbol{\Upsilon} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{6.93}$$

which leads to the following monomial expansion of the basis functions

$$\psi_i(\boldsymbol{\xi}) = \hat{C}_{i1}\xi_1^{\Upsilon_{11}}\xi_2^{\Upsilon_{21}} + \hat{C}_{i2}\xi_1^{\Upsilon_{12}}\xi_2^{\Upsilon_{22}} + \hat{C}_{i3}\xi_1^{\Upsilon_{13}}\xi_2^{\Upsilon_{23}} = \hat{C}_{i1}\xi_1^0\xi_2^0 + \hat{C}_{i2}\xi_1^1\xi_2^0 + \hat{C}_{i3}\xi_1^0\xi_2^1 = \hat{C}_{i1} + \hat{C}_{i2}\xi_1 + \hat{C}_{i3}\xi_2, \tag{6.94}$$

where the monomial terms are

$$\omega_1(\boldsymbol{\xi}) = 1, \quad \omega_2(\boldsymbol{\xi}) = \xi_1, \quad \omega_3(\boldsymbol{\xi}) = \xi_2. \tag{6.95}$$

Example 6.17: Monomial expansion in $\mathcal{P}^2(\mathbb{R}^2)$

Next we consider the special case $p = 2$, $d = 2$ ($\mathcal{P}^2$ triangle). In this case, we take

$$\boldsymbol{\Upsilon} = \begin{bmatrix} 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 2 \end{bmatrix}, \tag{6.96}$$

which leads to the following monomial expansion of the basis functions

$$\begin{aligned} \psi_i(\boldsymbol{\xi}) &= \hat{C}_{i1}\xi_1^{\Upsilon_{11}}\xi_2^{\Upsilon_{21}} + \hat{C}_{i2}\xi_1^{\Upsilon_{12}}\xi_2^{\Upsilon_{22}} + \hat{C}_{i3}\xi_1^{\Upsilon_{13}}\xi_2^{\Upsilon_{23}} + \hat{C}_{i4}\xi_1^{\Upsilon_{14}}\xi_2^{\Upsilon_{24}} + \hat{C}_{i5}\xi_1^{\Upsilon_{15}}\xi_2^{\Upsilon_{25}} + \hat{C}_{i6}\xi_1^{\Upsilon_{16}}\xi_2^{\Upsilon_{26}} \\ &= \hat{C}_{i1}\xi_1^0\xi_2^0 + \hat{C}_{i2}\xi_1^1\xi_2^0 + \hat{C}_{i3}\xi_1^0\xi_2^1 + \hat{C}_{i4}\xi_1^2\xi_2^0 + \hat{C}_{i5}\xi_1^1\xi_2^1 + \hat{C}_{i6}\xi_1^0\xi_2^2 \\ &= \hat{C}_{i1} + \hat{C}_{i2}\xi_1 + \hat{C}_{i3}\xi_2 + \hat{C}_{i4}\xi_1^2 + \hat{C}_{i5}\xi_1\xi_2 + \hat{C}_{i6}\xi_2^2, \end{aligned} \tag{6.97}$$

where the monomial terms are

$$\omega_1(\boldsymbol{\xi}) = 1, \quad \omega_2(\boldsymbol{\xi}) = \xi_1, \quad \omega_3(\boldsymbol{\xi}) = \xi_2, \quad \omega_4(\boldsymbol{\xi}) = \xi_1^2, \quad \omega_5(\boldsymbol{\xi}) = \xi_1\xi_2, \quad \omega_6(\boldsymbol{\xi}) = \xi_2^2. \tag{6.98}$$

Example 6.18: Monomial expansion in $\mathcal{P}^1(\mathbb{R}^3)$

Finally we consider the special case $p = 1$, $d = 3$ ($\mathcal{P}^1$ tetrahedra). In this case, we take

$$\boldsymbol{\Upsilon} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{6.99}$$

which leads to the following monomial expansion of the basis functions

$$\begin{aligned} \psi_i(\boldsymbol{\xi}) &= \hat{C}_{i1}\xi_1^{\Upsilon_{11}}\xi_2^{\Upsilon_{21}}\xi_3^{\Upsilon_{31}} + \hat{C}_{i2}\xi_1^{\Upsilon_{12}}\xi_2^{\Upsilon_{22}}\xi_3^{\Upsilon_{32}} + \hat{C}_{i3}\xi_1^{\Upsilon_{13}}\xi_2^{\Upsilon_{23}}\xi_3^{\Upsilon_{33}} + \hat{C}_{i4}\xi_1^{\Upsilon_{14}}\xi_2^{\Upsilon_{24}}\xi_3^{\Upsilon_{34}} \\ &= \hat{C}_{i1}\xi_1^0\xi_2^0\xi_3^0 + \hat{C}_{i2}\xi_1^1\xi_2^0\xi_3^0 + \hat{C}_{i3}\xi_1^0\xi_2^1\xi_3^0 + \hat{C}_{i4}\xi_1^0\xi_2^0\xi_3^1 \\ &= \hat{C}_{i1} + \hat{C}_{i2}\xi_1 + \hat{C}_{i3}\xi_2 + \hat{C}_{i4}\xi_3, \end{aligned} \tag{6.100}$$

where the monomial terms are

$$\omega_1(\boldsymbol{\xi}) = 1, \quad \omega_2(\boldsymbol{\xi}) = \xi_1, \quad \omega_3(\boldsymbol{\xi}) = \xi_2, \quad \omega_4(\boldsymbol{\xi}) = \xi_3. \tag{6.101}$$

Denote the $N_{\text{nd}}^{\text{el}}$ nodes of the pth order simplex element as $\{\hat{\boldsymbol{\xi}}_i\}_{i=1}^{N_{\text{nd}}^{\text{el}}}$, where $\hat{\boldsymbol{\xi}}_i = (\hat{\xi}_{1i}, \dots, \hat{\xi}_{di})^T$. The nodal property is

$$\psi_i(\hat{\boldsymbol{\xi}}_j) = \delta_{ij},$$

for $i, j = 1, \dots, N_{\text{nd}}^{\text{el}}$, which leads to

$$\sum_{k=1}^{N_{\text{nd}}^{\text{el}}} \hat{C}_{ik}\omega_k(\hat{\boldsymbol{\xi}}_j) = \delta_{ij}$$

once the expression for $\psi_i(\boldsymbol{\xi})$ is used from (6.92). Let $\hat{V}_{ij} = \omega_j(\hat{\boldsymbol{\xi}}_i) = \prod_{s=1}^{d} \hat{\xi}_{si}^{\Upsilon_{sj}}$ be the Vandermonde matrix corresponding to the d-dimensional, pth order simplex evaluated at $\{\hat{\boldsymbol{\xi}}_i\}_{i=1}^{N_{\text{nd}}^{\text{el}}}$, then the above constraints can be written in matrix form as $\hat{\boldsymbol{V}}\hat{\boldsymbol{C}}^T = \boldsymbol{I}_{N_{\text{nd}}^{\text{el}}}$, where $\hat{\boldsymbol{V}}$, $\hat{\boldsymbol{C}}$ are the matrices with indices $\hat{V}_{ij}$, $\hat{C}_{ij}$, respectively, and $\boldsymbol{I}_{N_{\text{nd}}^{\text{el}}}$ is the $N_{\text{nd}}^{\text{el}} \times N_{\text{nd}}^{\text{el}}$ identity matrix. Once we compute the coefficients, $\hat{\boldsymbol{C}} = \hat{\boldsymbol{V}}^{-T}$, we substitute this expression into (6.92) to give the final expression for

$$\psi_i(\boldsymbol{\xi}) = \sum_{k=1}^{N_{\text{nd}}^{\text{el}}} \hat{C}_{ik}\omega_k(\boldsymbol{\xi}) = \sum_{k=1}^{N_{\text{nd}}^{\text{el}}} \left(\hat{V}^{-1}\right)_{ki} \omega_k(\boldsymbol{\xi}). \tag{6.102}$$

Example 6.19: Linear $\mathcal{P}^1$ tetrahedra nodal basis

To provide a concrete example, we consider the $\mathcal{P}^1$ master tetrahedron ($d = 3$). In Example 6.3.6, we provided a concrete expression for the monomial terms ω_i, $i = 1, \dots, 4$, which we use to construct the Vandermonde matrix as

$$\boldsymbol{V} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \tag{6.103}$$

which leads to the following matrix of coefficients

$$\hat{\boldsymbol{C}} = \begin{bmatrix} 1 & -1 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{6.104}$$

Combining these coefficients with the expansion in (6.92) we have

$$\psi_1(\xi) = 1 - \xi_1 - \xi_2 - \xi_3, \quad \psi_2(\xi) = \xi_1, \quad \psi_3(\xi) = \xi_2, \quad \psi_4(\xi) = \xi_3. \tag{6.105}$$

It is a simple exercise to show these possess the nodal property with respect to the nodes of the $\mathcal{P}^1$ master tetrahedra defined in Example 6.3.6.

Remark 6.1. We have used Vandermonde's procedure to derive the analytical form of the nodal basis functions; however, it is much more useful (and efficient) as a numerical procedure to (numerically) evaluate the nodal basis functions at points throughout the element domain $\Omega_\square$ as we will see.

Remark 6.2. The Vandermonde matrix becomes ill-conditioned for high polynomial degrees p since the monomial basis becomes linearly dependent (numerically). To improve the conditioning of the final system, we could expand the basis functions ψ_i in an orthogonal basis of $\mathcal{P}^p$ (replacing step (6.92)) and repeat the procedure.