

**AME50541: Finite Element Methods**  
**Homework 2: Due Monday, February 18, 2019**

**Problem 1:** (10 points) Re-write the Navier equations using indicial notation and Einstein summation convention. Replace  $x \rightarrow 1, y \rightarrow 2, z \rightarrow 3$ .

$$\begin{aligned}\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} + F_x &= 0 \\ \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z} + F_y &= 0 \\ \frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + F_z &= 0\end{aligned}$$

**Problem 2:** (15 points) The elasticity tensor for a St. Venant-Kirchhoff material is given by  $C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})$ , where  $\lambda, \mu$  are the Lamé parameters. Calculate the stress tensor  $\sigma_{ij}$ , where  $\sigma_{ij} = C_{ijkl} \epsilon_{kl}$  and  $\epsilon_{kl}$  is the strain tensor. Make sure to use the fact that the strain tensor is symmetric ( $\epsilon_{ij} = \epsilon_{ji}$ ). Also, calculate the deviatoric stress  $s_{ij} = \sigma_{ij} - \frac{\sigma_{kk}}{3} \delta_{ij}$ . In both cases, your answer should be in terms of  $\lambda, \mu$ , and the strain tensor  $\epsilon$ .

**Problem 3:** (10 points) From JNR 2.1: Construct the weak form of the nonlinear equation

$$\begin{aligned}-\frac{d}{dx} \left( u \frac{du}{dx} \right) + f &= 0 \quad \text{for } 0 < x < L \\ \left( u \frac{du}{dx} \right) \Big|_{x=0} &= 0, \quad u(L) = \sqrt{2}\end{aligned}$$

**Problem 4:** (20 points) Re-write the incompressible Navier-Stokes equations

$$\begin{aligned}-\frac{\partial}{\partial x} \left( \rho \nu \frac{\partial u}{\partial x} \right) - \frac{\partial}{\partial y} \left( \rho \nu \frac{\partial u}{\partial y} \right) - \frac{\partial}{\partial z} \left( \rho \nu \frac{\partial u}{\partial z} \right) + \rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} + \rho w \frac{\partial u}{\partial z} + \frac{\partial p}{\partial x} &= 0 \\ -\frac{\partial}{\partial x} \left( \rho \nu \frac{\partial v}{\partial x} \right) - \frac{\partial}{\partial y} \left( \rho \nu \frac{\partial v}{\partial y} \right) - \frac{\partial}{\partial z} \left( \rho \nu \frac{\partial v}{\partial z} \right) + \rho u \frac{\partial v}{\partial x} + \rho v \frac{\partial v}{\partial y} + \rho w \frac{\partial v}{\partial z} + \frac{\partial p}{\partial y} &= 0 \\ -\frac{\partial}{\partial x} \left( \rho \nu \frac{\partial w}{\partial x} \right) - \frac{\partial}{\partial y} \left( \rho \nu \frac{\partial w}{\partial y} \right) - \frac{\partial}{\partial z} \left( \rho \nu \frac{\partial w}{\partial z} \right) + \rho u \frac{\partial w}{\partial x} + \rho v \frac{\partial w}{\partial y} + \rho w \frac{\partial w}{\partial z} + \frac{\partial p}{\partial z} &= 0 \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0\end{aligned}$$

with the boundary conditions  $u = \bar{u}, v = \bar{v}, w = \bar{w}$  on  $\partial\Omega_1$  and

$$\begin{aligned}\rho \nu \left( \frac{\partial u}{\partial x} n_x + \frac{\partial u}{\partial y} n_y + \frac{\partial u}{\partial z} n_z \right) - p n_x &= \rho \bar{t}_x \\ \rho \nu \left( \frac{\partial v}{\partial x} n_x + \frac{\partial v}{\partial y} n_y + \frac{\partial v}{\partial z} n_z \right) - p n_y &= \rho \bar{t}_y \\ \rho \nu \left( \frac{\partial w}{\partial x} n_x + \frac{\partial w}{\partial y} n_y + \frac{\partial w}{\partial z} n_z \right) - p n_z &= \rho \bar{t}_z\end{aligned} \quad \text{on } \partial\Omega_2$$

using indicial notation and Einstein summation convention, where  $\mathbf{u} = (u, v, w)^T$  and  $\mathbf{n}$  is the outward normal to  $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2$ . Replace  $x \rightarrow 1, y \rightarrow 2, z \rightarrow 3$  and  $u \rightarrow u_1, v \rightarrow u_2, w \rightarrow u_3$ . Then construct the weak form of the equations.

**Problem 5:** (20 points) Consider a system of  $m$  conservation laws in a  $d$ -dimensional space

$$\nabla \cdot \mathbf{F}(\mathbf{U}, \nabla \mathbf{U}) = \mathbf{S}(\mathbf{U}) \quad \text{in } \Omega,$$

where  $\mathbf{U} \in \mathbb{R}^m$  is the state,  $\mathbf{F}(\mathbf{U}, \nabla \mathbf{U}) \in \mathbb{R}^{m \times d}$  is the flux function,  $\mathbf{S}(\mathbf{U}) \in \mathbb{R}^m$  is a source term, and  $\Omega \subset \mathbb{R}^d$  is the domain. The boundary conditions are  $\mathbf{U} = \bar{\mathbf{U}}$  on  $\partial\Omega_1$  and  $\mathbf{F}(\mathbf{U}, \nabla \mathbf{U})\mathbf{n} = \bar{\mathbf{f}}$  on  $\partial\Omega_2$ , where  $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2$  and  $\mathbf{n}$  is the outward normal.

- Write the conservation law in indicial notation. Drop the arguments to the flux function and source term.
- Construct both the weighted residual and weak formulation of the governing equations.
- What conditions must the approximate solution  $\mathbf{U}_N(\mathbf{x}) \approx \mathbf{U}(\mathbf{x})$  satisfy if applying (a) the method of weighted residuals or (b) the Ritz method? Why is it difficult to construct a solution basis if  $\mathbf{F}$  is nonlinear in  $\mathbf{U}$  or  $\nabla \mathbf{U}$  if using the method of weighted residuals?

**Problem 6:** (40 points) Consider the equations associated with a simply supported beam and subjected to a uniform transverse load  $q = q_0$ :

$$\begin{aligned} \frac{d^2}{dx^2} \left( EI \frac{d^2 w}{dx^2} \right) &= q_0 \quad \text{for } 0 < x < L \\ w = EI \frac{d^2 w}{dx^2} &= 0 \quad \text{at } x = 0, L. \end{aligned}$$

Take  $L = 1$ ,  $EI = 1$ , and  $q_0 = -1$  and approximate the solution using the following methods:

- the method of weighted residuals (Galerkin) using the two-term trigonometric basis  $w(x) \approx w_2(x) = c_1 \sin\left(\frac{\pi x}{L}\right) + c_2 \sin\left(\frac{2\pi x}{L}\right)$ ,
- the method of weighted residuals (collocation) using the same trigonometric basis and collocation nodes  $x_1 = 0.25$  and  $x_2 = 0.75$ ,
- the Ritz method using the same trigonometric basis, and
- the Ritz method using a two-term polynomial basis ( $w(x) \approx w_2(x) = c_1 x(x - L) + c_2 x^2(x - L)^2$ ).

For each method, verify the solution basis satisfies the appropriate conditions. Plot the approximate solution generated by each method as well as the analytical solution. In a separate figure, plot the error of each method  $e(x) = |w(x) - \tilde{w}(x)|$ , where  $\tilde{w}$  is the approximate solution, and the residual over the domain. Finally, quantify the error of each approximation using the  $L^2$ -norm

$$e_{L^2(\Omega)} = \int_{\Omega} |e(x)|^2 dV.$$

I recommend using some symbolic mathematics software (Maple, Mathematica, MATLAB, etc) to assist with the calculations.