AME40541/60541: Finite Element Methods
Homework 4: Due Monday, April 13, 2020

Problem 1: (50 points) Consider the standard second-order PDE in one-dimension

$$-\frac{d}{dx}\left(a(x)\frac{du}{dx}\right) + c(x)u(x) = f(x), \qquad 0 < x < L$$
$$u(0) = \bar{u}_0, \quad u(L) = \bar{u}_L$$

where $a(x)$, $c(x)$, and $f(x)$ are polynomials over $(0, L)$ of degrees p_a, p_c, and p_f, respectively. The element stiffness matrix and force vector are

$$K_{ij}^e = \int_{\Omega^e} \left(\frac{d\phi_i^e}{dx}a(x)\frac{d\phi_j^e}{dx} + \phi_i^e(x)c(x)\phi_j^e(x)\right) dV, \quad f_i^e = \int_{\Omega^e} \phi_i^e(x)f(x)\, dV,$$

assuming a general element basis $\{\phi_i^e\}_{i=1,\dots,p+1}$.

(a) If we use basis functions of order p in the physical domain, how many Gaussian quadrature points are needed to integrate the stiffness matrix exactly? How many are needed to integrate force vector exactly? Your answer should be in terms of p, p_a, p_c, p_f. What happens if either $a(x)$, $c(x)$, or $f(x)$ are non-polynomial functions?

(b) Instead, suppose we use basis functions of order p in the reference domain and use an isoparametric mapping to define basis functions over the physical element. That is, let $\{\psi_i\}_{i=1,\dots,p+1}$ be the basis functions of order p defined over the master element $\Omega_\square := (-1, 1)$ with equally spaced nodes. The isoparametric mapping is

$$\mathcal{G}_e(\xi) = \sum_{i=1}^{p+1} \hat{x}_i^e \psi_i(\xi),$$

where $\{\hat{x}_i^e\}_{i=1,\dots,p+1}$ are the positions of the nodes of element e in the physical space and $\mathcal{G}_e(\xi)$ is the position in the physical element corresponding to the position ξ in the reference element. The basis functions in the physical space are then defined as

$$\phi_i^e(x) = \psi_i(\mathcal{G}_e^{-1}(x)),$$

where $\mathcal{G}_e^{-1}$ is the inverse of the isoparametric mapping. Re-write the integrals over the physical element defining the element stiffness matrix and force vector to integrals over the reference domain. You will need to use the following relationship:

$$\frac{d\phi_i^e}{dx} = \frac{d\psi_i}{d\xi}\frac{d(\mathcal{G}_e^{-1})}{dx} = \frac{d\psi_i}{d\xi}\left(\frac{d\mathcal{G}_e}{d\xi}\right)^{-1}.$$

How many quadrature nodes are needed to exactly integrate the stiffness matrix and force vector in the isoparametric setting? What is special about $p = 1$?