AME50541: Finite Element Methods
The Finite Element Method in One Dimension

1 Function spaces

We begin this chapter by introducing two critical *function spaces*, collections of functions between two spaces, as they will faciliate our formulation and analysis of the finite element method. We will focus on real-valued functions on an open set $\Omega \subset \mathbb{R}$. The collection of square-integrable functions over Ω is called the $L^2(\Omega)$ function space. The collection of functions whose derivatives up to order m belong to $L^2(\Omega)$, i.e., are square-integrable, is called the $H^m(\Omega)$ function space. These function spaces are formally defined as

$$\begin{aligned} L^p(\Omega) &:= \left\{ f : \Omega \to \mathbb{R} \;\middle|\; \int_\Omega |f|^p \, dV < \infty \right\} \\ H^m(\Omega) &:= \left\{ f : \Omega \to \mathbb{R} \;\middle|\; \frac{d^s f}{dx^s} \in L^2(\Omega), 0 \leqslant s \leqslant m \right\}. \end{aligned} \tag{1}$$

2 Galerkin approximation of weak formulation

With the concept of a function space at our disposal, we state the weak formulation of a partial differential equation introduce in the previous chapter more rigorously. Consider a linear partial differential equation of order $2m$ defined over the open set $\Omega \subset \mathbb{R}$ with essential boundary conditions prescribed on $\partial\Omega_1 \subset \partial\Omega$

$$\begin{aligned} \mathcal{L}(u) &= 0 \qquad \text{in } \Omega \\ u &= g \qquad \text{on } \partial\Omega_1, \end{aligned} \tag{2}$$

where $\mathcal{L}$ is a differential operator defining the PDE and g is a function defining the essential boundary condition on $\partial\Omega_1$. For the problem to be well-posed, natural boundary conditions must be prescribed on $\partial\Omega_2$ such that $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2$. Let $B(w, u)$ and $\ell(w)$ be the bilinear and linear forms associated with the PDE that results from applying integration-by-part m times to move m derivatives from the primary variable u to the test function w, incorporating the natural boundary conditions into the boundary terms, and requiring the test functions are zero on $\partial\Omega_1$ ($w = 0$ on $\partial\Omega_1$). Then the formal statement of the weak formulation of the PDE is: find $u \in \mathcal{U} := \{u \in H^m(\Omega) \mid u(x) = g(x), x \in \partial\Omega_1\}$ such that

$$B(w, u) = \ell(w) \tag{3}$$

for all $w \in \mathcal{W} := \{w \in H^m(\Omega) \mid w(x) = 0, x \in \partial\Omega_1\}$.

Example: Poisson equation

Recall the Poisson equation over the open set $\Omega := (0, L)$ with essential and natural boundary conditions at $x = 0$ and $x = L$, respectively,

$$\begin{gathered} -\frac{d^2u}{dx^2} = f \quad x \in (0, L) \\ u(0) = 0, \; \left.\frac{du}{dx}\right|_{x=L} = 0. \end{gathered} \tag{4}$$

The weak formulation of this PDE (derivation provided in the previous chapter) is: find $u \in \mathcal{U}$ such that $B(w, u) = \ell(w)$ for all $w \in \mathcal{W}$, where the function spaces are

$$\mathcal{U} = \mathcal{W} = \{f \in H^1(\Omega) \mid f(0) = 0, x \in \partial\Omega_1\} \tag{5}$$

and the bilinear form is

$$B(w, u) = \int_\Omega \frac{dw}{dx}\frac{du}{dx} \, dV, \qquad \ell(w) = \int_\Omega w f \, dV \tag{6}$$

Approximation methods based on the weak form construct *finite-dimensional* subspaces of the *trial space* $\mathcal{U}^h \subset \mathcal{U}$ and *test space* $\mathcal{W}^h \subset \mathcal{W}$ and consider the following finite-dimensional problem: find $u^h \in \mathcal{U}^h$ such that

$$B(w^h, u^h) = \ell(w^h) \tag{7}$$

for all $w^h \in \mathcal{W}^h$. The goal of this approach is to construct subspaces $\mathcal{U}^h$ and $\mathcal{W}^h$ such that solution of the finite-dimensional problem u^h (7) is a good approximation of the solution of the infinite-dimensional problem u (3). In general, the approximation subspaces $\mathcal{U}^h$ and $\mathcal{W}^h$ can be chosen independently; however, we will focus solely on Galerkin methods where the test and trial subspaces are composed of identical functions up to the boundary conditions. To this end, introduce $\mathcal{V}^h \subset H^m(\Omega)$ and define the finite-dimensional subspaces as

$$\mathcal{U}^h := \{u \in \mathcal{V}^h \mid u(x) = g(x), x \in \partial\Omega_1\}, \qquad \mathcal{W}^h := \{w \in \mathcal{V}^h \mid w(x) = 0, x \in \partial\Omega_1\}. \tag{8}$$

In the next section, we define a systematic procedure to define an appropriate approximation subspace $\mathcal{V}^h$ that possess key approximation properties. The spaces $\mathcal{U}^h$ and $\mathcal{W}^h$ will follow readily from $\mathcal{V}^h$ by imposing appropriate conditions along boundaries.

3 Construction of the finite element subspace

The finite element method is a numerical method for approximating solutions of PDEs based on their weak formulation that defines a general, systematic procedure to construct a finite-dimensional subspace $\mathcal{V}^h$ of the test and trial spaces, which we will call the *finite element subspace*. The primary requirements on the finite element subspace are:

(a) $\mathcal{V}^h(\Omega)$ is finite-dimensional, i.e., it is spanned by a finite dimensional basis,

(b) the test $\mathcal{U}^h$ and trial $\mathcal{W}^h$ spaces must be non-empty subsets of $\mathcal{V}^h$, i.e., $\mathcal{V}^h$ must possess functions that satisfy the homogeneous and non-homogeneous essential boundary conditions of the PDE,

(c) $\mathcal{V}^h(\Omega) \subset H^m(\Omega)$ to ensure the finite-dimensional weak formulation is well-defined, and

(d) $\mathcal{V}^h$ possess sufficient "approximation power" with respect to the functions in $\mathcal{U}$.

As we have seen in previous chapters, it is difficult to construct approximation spaces that satisfy essential boundary conditions when considering functions with global support over Ω, particularly when the boundary of the domain $\partial\Omega$ is complicated, which makes condition (b) difficult to satisfy. The finite element method circumvents this difficulty by using functions with *local support*, which greatly simplifies the task of imposing essential boundary conditions as we will see later.

To define functions with local support over $\Omega \subset \mathbb{R}$, we introduce a *mesh* of Ω consisting of nodes $\mathcal{N} := \{\hat{x}_I\}_{I=1}^{N_v}$ and elements $\mathcal{E} := \{\Omega_e\}_{e=1}^{N_e}$ (Figure 1), where each $\Omega_e \subset \Omega$ for each $\Omega_e \in \mathcal{E}$ and N_v and N_e are the number of nodes and elements in the mesh, respectively.

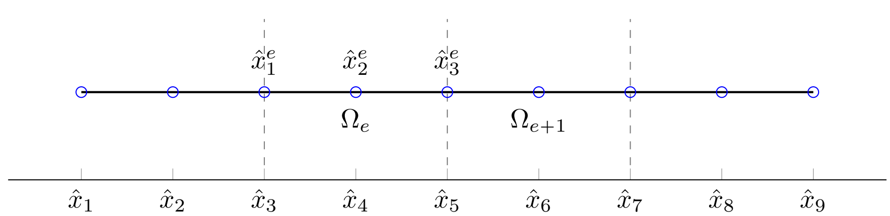

Figure 1: Schematic of mesh of one-dimensional domain

To each element $e = 1, \ldots, N_e$, we associate a set of nodes $\mathcal{N}_e$ consisting of all nodes in $\mathcal{N}$ that lie within $\Omega_e \in \mathcal{E}$, i.e., $\mathcal{N}_e := \{x \in \mathcal{N} \mid x \in \Omega_e\}$. We assume all elements have the same number of nodes, denoted N_v^e. Let us also assume the sets $\mathcal{N}$, $\mathcal{E}$, and $\mathcal{N}_e$ are *ordered*; the ordering of $\mathcal{E}$ is called the *global element numbering*, the ordering of $\mathcal{N}$ is called the *global node numbering*, and the ordering of $\mathcal{N}_e$ is called the *local node numbering*. Since the node sets $\mathcal{N}$ and $\mathcal{N}_e$ contain the same (re-ordered) information, it is convenient to introduce a relationship between the global and local node numbering: define the Ith element of $\mathcal{N}$ as $\hat{x}_I$ and the ith element of $\mathcal{N}_e$ as $\hat{x}_i^e$, then we have $\hat{x}_i^e = \hat{x}_I$ where $I = \Xi_{ie}$ for $i = 1, \ldots, N_v^e$ and $e = 1, \ldots, N_e$. The matrix $\boldsymbol{\Xi} \in \mathbb{R}^{N_v^e \times N_e}$, called the connectivity matrix, relates the local and nodal degrees of freedom and must be constructed from the connectivity of the elements in a given mesh.

Example:
For the mesh in Figure 1, we have the (ordered) node and element sets

$$\mathcal{N} := \{\hat{x}_1, \ldots, \hat{x}_9\}, \qquad \mathcal{E} := \{(\hat{x}_1, \hat{x}_3), (\hat{x}_3, \hat{x}_5), (\hat{x}_5, \hat{x}_7), (\hat{x}_7, \hat{x}_9)\}. \tag{9}$$

The local (ordered) node sets are

$$\mathcal{N}_1 := \{\hat{x}_1, \hat{x}_2, \hat{x}_3\}, \quad \mathcal{N}_2 := \{\hat{x}_3, \hat{x}_4, \hat{x}_5\}, \quad \mathcal{N}_3 := \{\hat{x}_5, \hat{x}_6, \hat{x}_7\}, \quad \mathcal{N}_4 := \{\hat{x}_7, \hat{x}_8, \hat{x}_9\}, \tag{10}$$

which leads to the connectivity matrix

$$\boldsymbol{\Xi} = \begin{bmatrix} 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \\ 3 & 5 & 7 & 9 \end{bmatrix}. \tag{11}$$

Given the finite element mesh $(\mathcal{N}, \mathcal{E})$, we define the element subspace as the space of piecewise polynomial functions of degree $p \geqslant m$ within each element $\Omega_e \in \mathcal{E}$ and globally $C^{m-1}(\Omega)$, i.e.,

$$\mathcal{V}^h = \left\{ f : \Omega \to \mathbb{R} \,\middle|\, f|_{\Omega_e} \in \mathbb{P}^p(\Omega_e),\, \forall \Omega_e \in \mathcal{E} \right\} \cap C^{m-1}(\Omega). \tag{12}$$

See the notes on "Polynomial Approximations" for the definition of the polynomial space $\mathbb{P}^p$. It can be shown that this subspace satsifies all conditions outlined at the beginning of this section. The requirement that the space belongs to $C^{m-1}(\Omega)$ ensure the integrability requirements are satisfied, i.e., $\mathcal{V}^h$ is a subset of $H^m(\Omega)$.

4 The finite element method

Given the construction of the finite element subspace in the previous section, we turn to formulating the finite element method. For concreteness, we restrict attention to the following one-dimensional PDE

$$\begin{aligned} -\frac{d}{dx}\left[a(x)\frac{du}{dx}\right] + c(x)u(x) = f(x), \quad 0 < x < L, \\ u(0) = u_0, \quad \left[a(x)\frac{du}{dx}(x)\right]_{x=L} = Q_L, \end{aligned} \tag{13}$$

where $a(x)$, $c(x)$, and $f(x)$ are given, smooth functions and u_0 and Q_L are given constants. For this problem the weak form is: find $u \in \mathcal{U} := \{f \in H^1(\Omega) \mid f(0) = u_0\}$ such that $B(w, u) = \ell(w)$ for all $w \in \mathcal{W} := \{f \in H^1(\Omega) \mid f(0) = 0\}$, where the bilinear and linear forms are

$$B(w, u) = \int_\Omega \left[\frac{dw}{dx}a(x)\frac{du}{dx} + w(x)c(x)u(x)\right] dV, \qquad \ell(w) = \int_\Omega w(x)f(x)\, dV - w(L)Q_L. \tag{14}$$

First, notice that given a mesh $(\mathcal{N}, \mathcal{E})$ such as the one in Figure 1, the weak form can be re-written, without error, as a sum of element contributions

$$\sum_{e=1}^{N_e} \left[B_e(w, u) - \ell_e(w)\right] = 0, \tag{15}$$

where

$$\begin{aligned} B_e(w, u) &= \int_{\Omega_e} \left[\frac{dw}{dx}a(x)\frac{du}{dx} + w(x)c(x)u(x)\right] dV \\ \ell_e(w) &= \int_\Omega w(x)f(x)\, dV + \begin{cases} -w(L)Q_L & e = N_e \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{16}$$

for each $\Omega_e \in \mathcal{E}$ due to the additive property of integration. Notice that we only included the boundary term in the element touching the boundary to avoid counting it multiple times.

Next, we introduce the approximation of our spaces $\mathcal{U}$ and $\mathcal{W}$ using the finite element subspace $\mathcal{V}^h$ constructed in the previous section. To accomplish this, we introduce a *nodal polynomial basis* (see definition and construction in "Polynomial Approximation" notes) for each element $\Omega_e \in \mathcal{E}$, denoted $\{\phi_i^e\}_{i=1}^{N_v^e}$, where the generating nodes are the nodes of the element $\mathcal{N}_e$. This implies the basis functions possess the Lagrangian property:

$$\phi_i^e(\hat{x}_j^e) = \delta_{ij} \quad \text{(no sum over } e\text{)}, \tag{17}$$

for $i, j = 1, \ldots, N_v^e$, where $\hat{x}_j^e \in \mathcal{N}_e$ are the nodes associated with the element Ω_e. This choice puts a requirements on relationship between the number of nodes in the element and the polynomial degree: $N_v^e = p+1$ since the dimension of $\mathbb{P}^p(\Omega_e) = p+1$. The solution and test function are then approximated elementwise in this basis as

$$u(x)|_{\Omega_e} \approx u^h(x)\big| := \sum_{i=1}^{N_v^e} u_i^e \phi_i^e(x), \qquad w(x)|_{\Omega_e} \approx w^h(x)\big| := \sum_{i=1}^{N_v^e} w_i^e \phi_i^e(x), \tag{18}$$

where $\hat{u}_i^e$ and $\hat{w}_i^e$ are the coefficients of the expansion, which also happen to be the value of $u^h(x)$ and $w^h(x)$ at the nodes $\mathcal{N}_e$ since the basis possesses the Lagrangian property, i.e.,

$$u^h(\hat{x}_j^e) = \sum_{i=1}^{N_v^e} u_i^e \phi_i^e(\hat{x}_j^e) = \sum_{i=1}^{N_v^e} u_i^e \delta_{ij} = u_j^e, \qquad w^h(\hat{x}_j^e) = \sum_{i=1}^{N_v^e} w_i^e \phi_i^e(\hat{x}_j^e) = \sum_{i=1}^{N_v^e} w_i^e \delta_{ij} = w_j^e. \tag{19}$$

Substitution of the solution and test function approximations in (18) into the individual terms in the weak form yields

$$B(w, u) = \sum_{e=1}^{N_e} B_e(w, u) = \sum_{i=1}^{N_v^e} \hat{w}_i^e \sum_{j=1}^{N_v^e} B_e(\phi_i^e, \phi_j^e)\hat{u}_j^e \tag{20}$$

and

$$\ell(w) = \sum_{e=1}^{N_e} \ell_e(w) = \sum_{i=1}^{N_v^e} \hat{w}_i^e \ell_e(\phi_i^e), \tag{21}$$

where we utilized bilinearity of B and linearity of ℓ. Therefore the Galerkin weak form reduces to: find $u^h \in \mathcal{U}^h$ such that

$$\sum_{e=1}^{N_e} \sum_{i=1}^{N_v^e} \hat{w}_i^e \left[\sum_{j=1}^{N_v^e} K_{ij}^e \hat{u}_j^e - F_i^e \right] = 0, \tag{22}$$

for all $w^h \in \mathcal{W}^h$, where the element stiffness matrix $\boldsymbol{K}^e \in \mathbb{R}^{N_v^e \times N_v^e}$ and force vector $\boldsymbol{F}^e \in \mathbb{R}^{N_v^e}$ are defined as

$$\boldsymbol{K}_{ij}^e = B_e(\phi_i^e, \phi_j^e), \qquad \boldsymbol{F}_i^e = \ell_e(\phi_i^e). \tag{23}$$

The approxmiations in (18) only guarantees the functions $u(x)$ and $w(x)$ are piecewise polynomial of degree p, but does not guarantee they are $C^0(\Omega)$ ($C^{m-1}(\Omega)$ where $m = 1$ in this case), nor does it guarantee they satisfy the required boundary conditions. For example, see Figure 2 for a $p = 2$ approximation of the $u(x)$ using 4 elements where all coefficients are chosen independently; the resulting approximation is discontinuous solution at element interfaces and therefore not $C^0(\Omega)$. Both of these issues (globaly continuity and enforcement of boundary conditions) are resolved by placing explicit requirements on the coefficients of u and w.

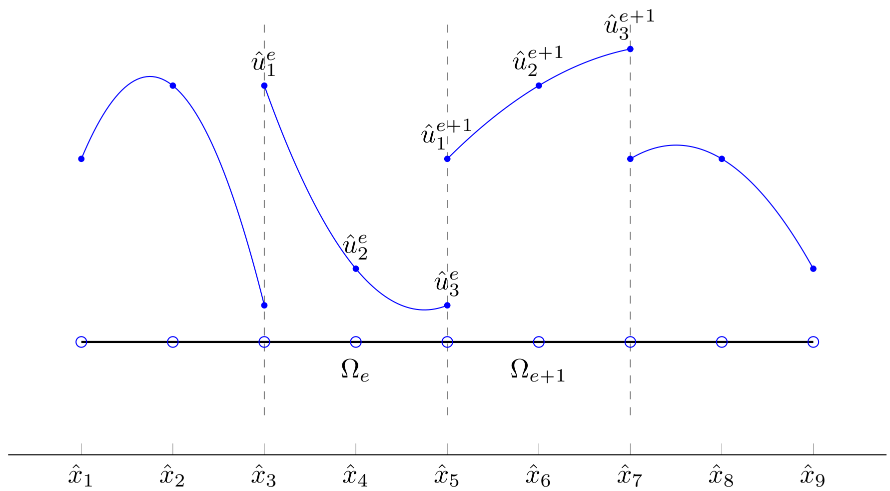

Figure 2: A piecewise polynomial approximation ($\mathbb{P}^2$) of some function $u(x)$ over a mesh of a one-dimensional domain with four elements. When all solution coefficients vary independently, the approximation will not, in general, be globally continuous.

First we consider the issue of global continuity of u and w. Given our choice of piecewise polynomial approximations, it is clear both u and w are guaranteed to be continuous (and smooth) *within* each element; the only possibilities for loss of continuity is at element intersections (Figure 2). Consider two adjacent element e and $e+1$ in a one-dimensional mesh (Figure 2). Given our choice of a nodal basis, continuity of u and w is guaranteed if the nodal coefficients at node I match (Figure 3), i.e.,

$$\hat{u}^e_{N^e_v} = \hat{u}^{e+1}_1, \qquad \hat{w}^e_{N^e_v} = \hat{w}^{e+1}_1. \tag{24}$$

By enforcing a similar condition at all element intersections, global continuity is guaranteed. The above statement is generalized by introducing coefficients associated with each global nodes $\hat{x}_I \in \mathcal{N}$ for $I = 1, \dots, N_v$, i.e., $\{\hat{u}_1, \dots, \hat{u}_{N_v}\}$ for the solution and $\{\hat{w}_1, \dots, \hat{w}_{N_v}\}$ for the test function, and requiring

$$\hat{u}^e_i = u_{\Xi_{ie}} = \sum_{I=1}^{N_v} u_I \delta_{I\Xi_{ie}}, \qquad \hat{w}^e_i = w_{\Xi_{ie}} = \sum_{I=1}^{N_v} w_I \delta_{I\Xi_{ie}}, \tag{25}$$

where $\boldsymbol{\Xi}$ is the connectivity matrix introduced previously. The second relationship is a trivial identity that will be used next. Substituting the compatibility conditions in (25) into the finite element weak form in (22), we have

$$\sum_{e=1}^{N_e} \sum_{i=1}^{N^e_v} \sum_{I=1}^{N_v} w_I \delta_{I\Xi_{ie}} \left[\sum_{j=1}^{N^e_v} K^e_{ij} \sum_{J=1}^{N_v} u_J \delta_{J\Xi_{je}} - F^e_i \right] = 0. \tag{26}$$

After rearranging the order of summation, we arrive at the global weak form that incorporates *global continuity* of both the solution and test function

$$\sum_{I=1}^{N_v} w_I \left[\sum_{J=1}^{N_v} K_{IJ} u_J - F_I \right] = 0, \tag{27}$$

where the *global stiffness matrix* $\boldsymbol{K} \in \mathbb{R}^{N_v \times N_v}$ and *global force vector* $\boldsymbol{F} \in \mathbb{R}^{N_v}$ are

$$K_{IJ} = \sum_{e=1}^{N_e} \sum_{i=1}^{N^e_v} \sum_{j=1}^{N^e_v} K^e_{ij} \delta_{I\Xi_{ie}} \delta_{J\Xi_{je}}, \qquad F_I = \sum_{e=1}^{N_e} \sum_{i=1}^{N^e_v} F^e_i \delta_{I\Xi_{ie}}. \tag{28}$$

The weak form in (26) can be re-written in matrix notation as

$$\boldsymbol{w}^T(\boldsymbol{K}\boldsymbol{u} - \boldsymbol{F}) = 0, \tag{29}$$

where $\boldsymbol{u} = (u_1, \dots, u_{N_v})^T$ and $\boldsymbol{w} = (w_1, \dots, w_{N_v})^T$ contain the nodal values of the solution and test function, resepctively.

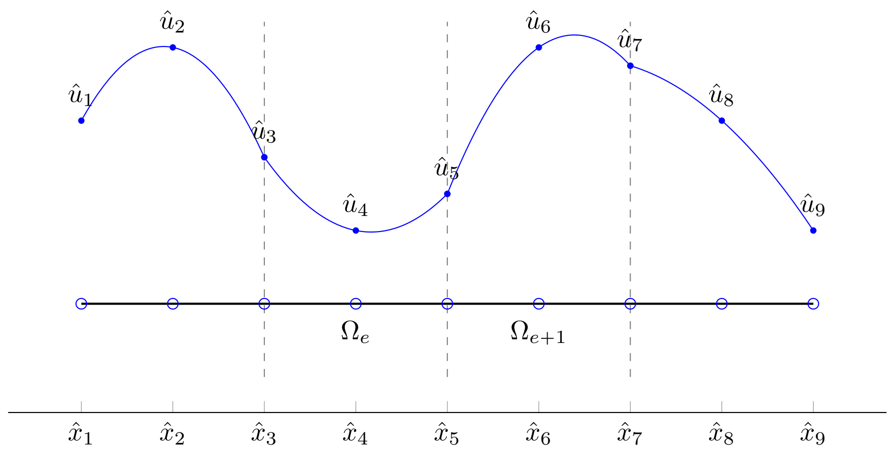

Figure 3: A piecewise polynomial approximation ($\mathbb{P}^2$) of some function $u(x)$ over a mesh of a one-dimensional domain with four elements where the element coefficients that coincide at a given node are required to match. In this case, global continuity is guaranteed since the approximation is interpolatory.

Finally, we turn to the issue of enforcing boundary conditions. Given our use of a nodal basis, it is obvious that the boundary condition $w^h(0) = 0$ and $u^h(0) = \bar{u}_0$ are enforced by requiring

$$w_1 = 0, \qquad u_1 = \bar{u}_0. \tag{30}$$

To generalize this requirement, we will partition all quantities in (26) based on whether an essential boundary condition is prescribed at the corresponding node. Define $\boldsymbol{u}_d$ to be the subvector of $\boldsymbol{u}$ corresponding containing the solution coefficients where an essential boundary condition is prescribed and $\boldsymbol{u}_f$ to be all entries of $\boldsymbol{u}$ not constrained by an essential boundary condition. Then $\boldsymbol{u}$ can be written as

$$\boldsymbol{u} = \begin{bmatrix} \boldsymbol{u}_d \\ \boldsymbol{u}_f \end{bmatrix} \tag{31}$$

after possibly rearranging its entries. In the example under consideration, we have $\boldsymbol{u}_d = \bar{u}_0$ and $\boldsymbol{u}_f = (u_2, \dots, u_{N_v})^T$. Similarly, we partition

$$\boldsymbol{w} = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{w}_f \end{bmatrix}, \qquad \boldsymbol{F} = \begin{bmatrix} \boldsymbol{F}_d \\ \boldsymbol{F}_f \end{bmatrix}, \qquad \boldsymbol{K} = \begin{bmatrix} \boldsymbol{K}_{dd} & \boldsymbol{K}_{df} \\ \boldsymbol{K}_{fd} & \boldsymbol{K}_{ff} \end{bmatrix}, \tag{32}$$

where both the rows and columns of the matrix $\boldsymbol{K}$ are partitioned according to the rule and we directly incorporate $\boldsymbol{0}$ into the partition of the test function since we always choose the test function to be zero wherever an essential boundary condition is prescribed. Substituting these partitions into (26), we have

$$\boldsymbol{w}_f^T \left(\boldsymbol{K}_{ff}\boldsymbol{u}_f + \boldsymbol{K}_{fd}\boldsymbol{u}_d - \boldsymbol{F}_f \right) = 0. \tag{33}$$

At this point, we have imposed all of the required constraints (continuity and enforcement of essential boundary conditions) on the general piecewise polynoimal space such that it satisfies the requirements of our finite element space $\mathcal{V}^h$. The arbitrariness of w^h in the space $\mathcal{W}^h$ is manifested as each coefficient in $\boldsymbol{w}_f$ being arbitrary, i.e., equation (33) must hold for any vector $\boldsymbol{w}_f$ to enforce the weak form in (22) for all $w^h \in \mathcal{W}^h$. This leads to the following system of equations

$$\boldsymbol{K}_{ff}\boldsymbol{u}_f = \boldsymbol{F}_f - \boldsymbol{K}_{fd}\boldsymbol{u}_d, \tag{34}$$

where $\boldsymbol{K}_{ff}$, $\boldsymbol{K}_{fd}$, $\boldsymbol{F}_f$, and $\boldsymbol{u}_d$ are known; we solve for $\boldsymbol{u}_f$ to obtain the coefficients of the finite element approximation, which can be substituted back into (18) to construct $u^h(x)$ pointwise.