Fourier Transform Lecture

Matthew Zahr (mzahr@stanford.edu)

Schwartz class $S(\mathbb{R}^n)$ consists of functions $f \in C^\infty(\mathbb{R}^n)$ such that for any pair of multiindices α, β:

$$p_{\alpha\beta}(f) := \sup_x |x^\alpha D^\beta f(x)| < +\infty$$

where

$$\alpha = (\alpha_1, \ldots, \alpha_n), \quad \beta = (\beta_1, \ldots, \beta_n) \qquad \text{[multi-index]}$$

and

$$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \qquad D^\beta f(x) = \frac{\partial^{\|\beta\|_1}}{\partial x_1^{\beta_1} \cdots \partial x_n^{\beta_n}} f(x).$$

$$\|\beta\|_1 = \sum_{i=1}^n |\beta_i| = \text{order of derivative}$$

Example:
$n = 2, \quad \beta = (1, 2)$

$$D^\beta f(x) = \frac{\partial^3}{\partial x_1 \partial x_2^2} f(x)$$

* If $f \in S(\mathbb{R}^n)$, then f is infinitely differentiable and rapidly decreasing.
- Every derivative of f goes to zero faster than any polynomial.

$$C_c^\infty(\mathbb{R}^n) \subset S(\mathbb{R}^n) \subset C^\infty(\mathbb{R}^n)$$

Fourier Transform: Let $f \in L^1(\mathbb{R}^n) = \{ f : \mathbb{R}^n \to \mathbb{R} \mid \int_{\mathbb{R}^n} |f| < +\infty \}$ function,

then the Fourier Transform of f is:

$$\mathcal{F}(\xi; f) = \hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \underline{x} \cdot \xi} \, d\underline{x}.$$

Proposition: For $f \in L^1(\mathbb{R}^n)$,

$$|\hat{f}(\xi)| \leq \int_{\mathbb{R}^n} |f(x)| \, dx < +\infty$$

Proof:

$$|\hat{f}(\xi)| \leq \int_{\mathbb{R}^n} | f(x) e^{-2\pi i \underline{x} \cdot \xi} | \, dx \underset{\uparrow\, |e^{i\alpha}| = 1}{\leq} \int_{\mathbb{R}^n} |f(x)| \, dx \underset{\uparrow\, f \in L^1(\mathbb{R})}{<} +\infty$$

$\blacksquare$

Properties of Fourier Transform:

- Let $f, g \in S(\mathbb{R}^n)$, then

(0) F.T. is linear: $\widehat{f+g} = \hat{f} + \hat{g}$

(i) $\hat{f}(\xi) \in S(\mathbb{R}^n)$

(ii) $\widehat{\frac{\partial f}{\partial x}}(\xi) = 2\pi i \xi \hat{f}(\xi)$ and $(-2\pi i)(\widehat{x_j f})(\xi) = \frac{\partial \hat{f}}{\partial \xi_j}(\xi)$.

(iii) $\int_{\mathbb{R}^n} f(x)\, \hat{g}(x)\, dx = \int_{\mathbb{R}^n} \hat{f}(x)\, g(x)\, dx$

(iv) Inversion formula : $f(x) = \int_{\mathbb{R}^n} \hat{f}(\xi)\, e^{2\pi i\, \underline{x} \cdot \underline{\xi}}\, d\underline{\xi}$

(v) If $f(x) = e^{-\pi |x|^2}$, then $\hat{f}(\underline{x}) = f(\underline{x})$.

<u>Proof</u> : (i), (iv) skip

(ii) (a) $\widehat{\frac{\partial f}{\partial x_j}}(\xi) = \int_{\mathbb{R}^n} \frac{\partial f}{\partial x_j}(x)\, e^{-2\pi i\, \underline{\xi} \cdot \underline{x}}\, d\underline{\xi} = \int_{\mathbb{R}^n} \left[\frac{\partial}{\partial x_j}\left(f(x) e^{-2\pi i\, \underline{\xi}\cdot\underline{x}}\right) - f(x)(-2\pi i \xi_j)\, e^{-2\pi i\, \underline{\xi}\cdot\underline{x}}\right] d\underline{x}$

$$= \lim_{r\to\infty} \cancel{\int_{\partial B_r(0)} f(x)\, e^{-2\pi i\, \underline{\xi}\cdot\underline{x}}\, n_j}\; d\Gamma + 2\pi i \xi_j \int_{\mathbb{R}^n} f(x)\, e^{-2\pi i\, \underline{\xi}\cdot\underline{x}}\, d\underline{x}$$

$= 0.$ (f schwartz)

$$= 2\pi i\, \xi_j\, \hat{f}(\xi). \qquad \blacksquare$$

(b) $\frac{\partial \hat{f}}{\partial \xi_j}(\xi) = \frac{\partial}{\partial \xi_j} \int_{\mathbb{R}^n} f(x)\, e^{-2\pi i\, \underline{\xi}\cdot\underline{x}}\, d\underline{x} = \int_{\mathbb{R}^n} f(x)(-2\pi i x_j)\, e^{-2\pi i\, \underline{\xi}\cdot\underline{x}}\, dx$

$$= (-2\pi i)(\widehat{x_j f})(\xi)$$

(v) $f(x) = e^{-\pi |x|^2} = e^{-\pi x_1^2} \cdots e^{-\pi x_n^2}$

Sufficient to prove for $n=1$ because if $\widehat{e^{-\pi x_i^2}} = e^{-\pi \xi_i^2}$, then

$$\widehat{e^{-\pi|x|^2}} = \widehat{e^{-\pi x_1^2} \cdots e^{-\pi x_n^2}} = \int e^{-\pi x_1^2} \cdots e^{-\pi x_n^2}\, e^{-2\pi i x_1 \xi_1} \cdots e^{-2\pi i x_n \xi_n}\, dx_1 \cdots dx_n$$

$$= \int e^{-\pi x_1^2} e^{-2\pi i x_1 \xi_1}\, dx_1 \cdots \int e^{-\pi x_n^2} e^{-2\pi i x_n \xi_n}\, dx_n$$

$$= \widehat{e^{-\pi x_1^2}} \cdots \widehat{e^{-\pi x_n^2}} = e^{-\pi |\xi|^2}$$

Let $x, \xi \in \mathbb{R}$:

$$\widehat{e^{-\pi x^2}} = \int_{\mathbb{R}} e^{-\pi x^2} e^{-2\pi i \xi x}\, dx = \int_{\mathbb{R}} e^{-\pi x^2} e^{-2\pi i \xi x} e^{-\pi \xi^2} e^{\pi \xi^2}\, dx$$

$$= e^{-\pi \xi^2} \int_{\mathbb{R}} e^{-\pi(x^2 + 2i\xi x - \xi^2)}\, dx$$

$$= e^{-\pi \xi^2} \int_{\mathbb{R}} e^{-\pi(x+i\xi)^2}\, dx$$

$$= e^{-\pi \xi^2} \underbrace{\int_{\mathbb{R}} e^{-\pi y^2}\, dy}_{=1} = e^{-\pi \xi^2}$$

$\blacksquare$

(iii) $\int_{\mathbb{R}^n} f(x)\hat{g}(x)\,dx = \int_{\mathbb{R}^n} f(x)\int_{\mathbb{R}^n} g(z)\,e^{-2\pi i z\cdot x}\,dz\,dx$

$= \iint_{\mathbb{R}^n} f(x)\,g(z)\,e^{-2\pi i z\cdot x}\,dz\,dx$; Fubini b/c both $f, g \in S(\mathbb{R}^n)$

$= \iint_{\mathbb{R}^n} f(x)\,g(z)\,e^{-2\pi i z\cdot x}\,dx\,dz$

$= \int_{\mathbb{R}^n} g(z)\int_{\mathbb{R}^n} f(x)\,e^{-2\pi i z\cdot x}\,dx\,dz.$

$= \int_{\mathbb{R}^n} g(z)\,\hat{f}(z)\,dz$; change of variables

$= \int_{\mathbb{R}^n} \hat{f}(x)\,g(x)\,dx$

∎

Examples

Problem 6, Hwk 4

Let $u(t,x)$ satisfy the heat equation, $u_t - \Delta u = 0$, $t>0$, $x\in\mathbb{R}^n$ (★)
with IC: $u(0,x) = f(x)$

(a) Use F.T. to show $u(t,x) = \int e^{2\pi i \xi\cdot x - 4\pi^2|\xi|^2 t}\,\hat{f}(\xi)\,d\xi$

(b) Convert answer in (a) to the form $u(t,x) = \int G(t,x-y)f(y)\,dy$.

Solution:

(a) Take F.T. in x of (★)

$$\widehat{\Delta u} = \widehat{\sum_{k=1}^{n} \frac{\partial^2 u}{\partial x_k^2}} \underset{\text{linearity}}{=} \sum_{k=1}^{n} \widehat{\frac{\partial}{\partial x_k}\left(\frac{\partial u}{\partial x_k}\right)} = \sum_{k=1}^{n} 2\pi i\,\xi_k\,\widehat{\frac{\partial u}{\partial x_k}} = \sum_{k=1}^{n}(-4\pi^2)\,\xi_k^2\,\hat{u}(\xi)$$

$$= (-4\pi^2)|\xi|^2\,\hat{u}(\xi)$$

$\Rightarrow \hat{u}_t(t,\xi) = -4\pi^2|\xi|^2\,\hat{u}(t,\xi) \xrightarrow{\text{solve ODE}} \hat{u}(t,\xi) = \hat{u}(0,\xi)\,e^{-4\pi^2|\xi|^2 t}$

$\hat{u}(0,\xi) = \hat{f}$ $\qquad = \hat{f}(\xi)e^{-4\pi^2|\xi|^2 t}.$

Inverse F.T.: $\boxed{u(t,x) = \int_{\mathbb{R}^n} \hat{u}(t,\xi)\,e^{2\pi i\xi\cdot x}\,d\xi = \int_{\mathbb{R}^n} e^{2\pi i \xi\cdot x}\,e^{-4\pi^2|\xi|^2 t}\,\hat{f}(\xi)\,d\xi}$ ∎

(b) $u(t,x) = \int_{\mathbb{R}^n} e^{2\pi i\xi\cdot x}\,e^{-4\pi^2|\xi|^2 t}\int_{\mathbb{R}^n} f(y)\,e^{-2\pi i \xi\cdot y}\,dy\,d\xi$; combine integrals ; use Fubini

$= \iint_{\mathbb{R}^n} f(y)\,e^{2\pi i\xi(x-y)}\,e^{-4\pi^2|\xi|^2 t}\,d\xi\,dy$; change of variables: $\xi \to \frac{\xi}{\sqrt{4\pi t}}$, $d\xi \to \frac{d\xi}{(4\pi t)^{n/2}}$.

$= \int_{\mathbb{R}^n} f(y)\int_{\mathbb{R}^n} e^{-\pi|\xi|^2}\,e^{2\pi i\frac{\xi(x-y)}{\sqrt{4\pi t}}}\,\frac{d\xi}{(4\pi t)^{n/2}}\,dy$;

$= \int_{\mathbb{R}^n} f(y)\,e^{-\pi\frac{|x-y|^2}{4\pi t}}\,\frac{1}{(4\pi t)^{n/2}}\,dy = \int_{\mathbb{R}^n} \frac{1}{(4\pi t)^{n/2}}\,e^{-\frac{|x-y|^2}{4t}}\,f(y)\,dy$ ∎

Introduction to Numerical PDEs

- Background : Taylor's Theorem , O notation
- Finite Differences Approximations
- Aside : ODE solvers (implicit vs. explicit)
- Truncation error, consistency, convergence
- Ex : KdV equation

Background : Taylor's Theorem.

Let $k \geq 1$ integer and $f : \mathbb{R} \to \mathbb{R}$ $k+1$ times differentiable at $a \in \mathbb{R}$. Then, $\exists\, h_k : \mathbb{R} \to \mathbb{R}$ such that

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{f^{(k+1)}(\xi)}{(k+1)!}(x-a)^{k+1}$$

for $\xi \in [a, x]$.
$$R_k(x) = \frac{f^{(k+1)}(\xi)}{(k+1)!}(x-a)^{k+1} = O((x-a)^{k+1})$$

O - notation : $f(x) = O(g(x))$ as $x \to a$ iff $\exists\ M, \delta > 0$ such that

$$|f(x)| \leq M|g(x)| \quad \text{for } |x-a| < \delta.$$

* as $x \to a$, f can be bounded by a constant times g.

Finite Differences Approximations

Consider a function $f \in \mathbb{R}$, sufficiently smooth (if I need a derivative, I have it).

1st order approx. to 1st derivatives

Taylor series about x, to first order :

$$f(x+h) = f(x) + f'(x)h + \frac{h^2}{2}f''(\xi) \Rightarrow f'(x) = \frac{f(x+h) - f(x)}{h} + O(h).$$

Similarly, $f'(x) = \dfrac{f(x) - f(x-h)}{h} + O(h)$ as $h \to 0$

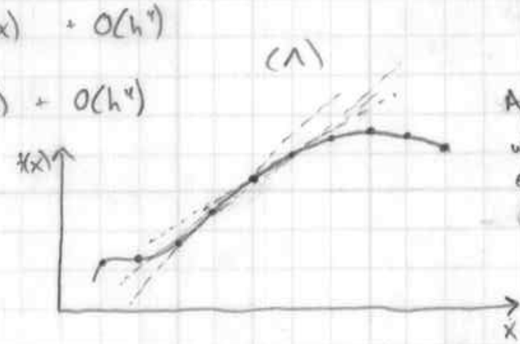
(for a given x, $\frac{1}{2}f''(\xi)$ constant)

2nd order approx. to 2nd derivatives

$$f(x+h) = f(x) + f'(x)h + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + O(h^4)$$

$$f(x-h) = f(x) - f'(x)h + \frac{h^2}{2}f''(x) - \frac{h^3}{3!}f'''(x) + O(h^4)$$

add $\Rightarrow$ $$f''(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}$$



Approx. Derivs with function evaluations

Deriving finite difference approx

Method of Undetermined Coefficients

Ex : One-sided approx. to $f'(x)$ @ equally spaced points : $x, x-h, x-2h$

* fit polynomial to f at a certain point
- evaluate the derivative of the polynomial @ the point of interest.

$$D_2 f(x) = a f(x) + b f(x-h) + c f(x-2h)$$

$$= a f(x) + b\left[f(x) - f'(x)h + \frac{h^2}{2!}f''(x) - f'''(x)\frac{h^3}{3!} + O(h^4)\right]$$
$$+ c\left[f(x) - f'(x)2h + \frac{4h^2}{2!}f''(x) - f'''(x)\frac{8h^3}{3!} + O(h^4)\right]$$

$$= f(x)[a+b+c] + f'(x)[-bh-2ch] + f''(x)\left[\frac{h^2}{2!}b + 2h^2c\right] + f'''(x)\left[\frac{h^3}{3!}b + \frac{4}{3}h^3c\right] + \cdots$$

For $D_2 f$ to agree with f' to high order, we need

$$a+b+c = 0$$
$$-b-2c = 1/h$$
$$\frac{b}{2} + 2c = 0 \rightarrow b = -4c$$

3 eqns, 3 unknowns $\Rightarrow$ $2c = \frac{1}{h} \rightarrow c = \frac{1}{2h}$, $b = -\frac{2}{h}$, $a = -b-c = \frac{2}{h} - \frac{1}{2h} = \frac{3h}{2h^2} = \frac{3}{2h}$

$$\boxed{a = \frac{3}{2h},\ b = -\frac{2}{h},\ c = \frac{1}{2h}}$$

General Approach:

Suppose we want to approximate the k^{th} derivative of f using the value of f at $x_1, \ldots, x_n$.

$$D_k f = c_1 f(x_1) + \cdots + c_n f(x_n) = f^{(k)}(\bar{x}) + O(h^p)$$

(doesn't assume equal spacing!)

Expand about x:

$$f(x_i) = f(x) + f'(x)(x_i - x) + \frac{1}{2!}f''(x)(x_i-x)^2 + \cdots + \frac{1}{k!}f^{(k)}(x_i-x)^k + \cdots + \frac{1}{(n-1)!}f^{(n-1)}(x_i-x)^{n-1}$$

Group by derivative order:

$$D_k f = f(x)\sum_{i=1}^{n} c_i + f'(x)\sum_{i=1}^{n} c_i(x_i - x) + \cdots + f^{(k)}(x)\sum_{i=1}^{n}\frac{1}{k!}c_i(x_i-x)^k + \cdots$$

$$\Rightarrow \frac{1}{(i-1)!}\sum_{j=1}^{n} c_j(x_j - x)^{(i-1)} = \begin{cases} 1 & \text{if } i-1 = k \\ 0 & \text{otherwise} \end{cases}$$

$$i = 1, 2, \ldots, n$$

n unknowns, n equations $\xrightarrow{\text{can be written}}$ $\underset{\sim}{A}\underset{\sim}{c} = \underset{\sim}{b}$

Vandermonde matrix. (non-singular, but ill-conditioned).

$A \in \mathbb{R}^{n\times n}$, $c \in \mathbb{R}^n$, $b \in \mathbb{R}^n$.

$$b = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow (k+1)^{th} \text{ entry}$$

$\Rightarrow$ need $k+1 \le n$!

otherwise, $\underset{\sim}{b} = 0 \Rightarrow \underset{\sim}{c} = 0$!

Why care about 1-sided differences?

- time derivatives
- Neumann B.C.s
- stability (i.e. lin. advection)

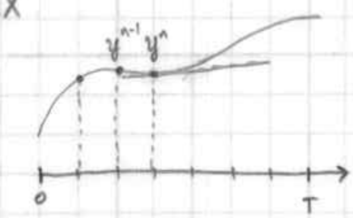
ODE Solvers

General system of ODEs (nonlinear in general)

$$\underline{M}\,\dot{\underline{y}} = \underline{r}(t,\underline{y}) \qquad M \in \mathbb{R}^{n\times n},\ y \in \mathbb{R}^n,\ t \in \mathbb{R}_+,\ \underline{r} : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n$$

velocity $= M^{-1} r(t,y)$

Types of ODE schemes:
- Implicit vs. Explicit vs. IMEX
- Single stage vs. multi-stage
- Single step vs. multi-step
- Serial vs. Parallel

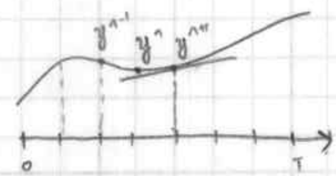


Simplest explicit: forward Euler

$$\underline{M}\,\underline{y}^{n+1} = \underline{M}\,\underline{y}^n + \Delta t\, r(t_n, \underline{y}^n)$$
$$\downarrow$$
$$\boxed{\underline{y}^{n+1} = \underline{y}^n + \Delta t\, M^{-1} r(t_n, \underline{y}^n)}$$

usually $\underline{M}$ diagonal or block diagonal (or approximated as so) $\Rightarrow M^{-1}$ cheap!

Simplest implicit: backward Euler



$$\underline{M}\,\underline{y}^{n+1} = \underline{M}\,\underline{y}^n + \Delta t\, r(t_{n+1}, \underline{y}^{n+1})$$
$$\underline{y}^{n+1} = \underline{y}^n + \Delta t\, \underline{M}^{-1} r(t_{n+1}, \underline{y}^{n+1})$$

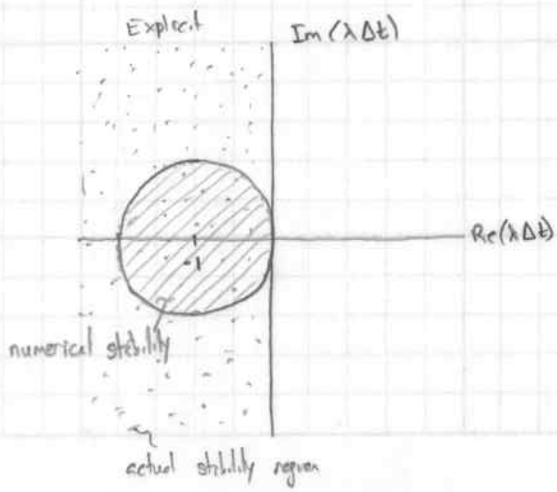
Analysis on simple model problem: $\dot{y} = \lambda y \rightarrow$ exact solution: $y = y(0)\, e^{\lambda t}$
$\hookrightarrow$ stable if $\mathrm{Re}\,\lambda < 0$.

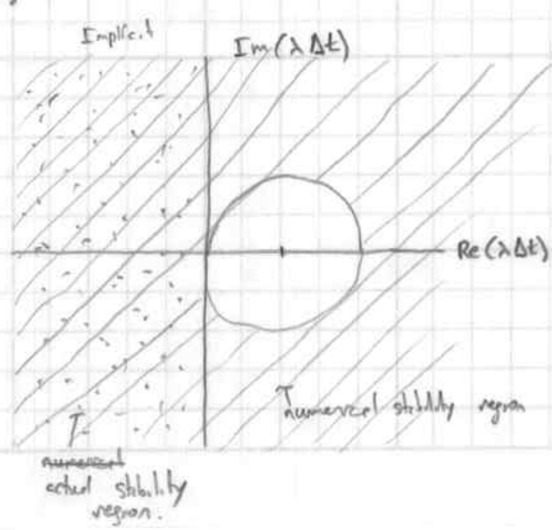
explicit: $y^{n+1} = y^n + \lambda \Delta t\, y^n = (1 + \lambda \Delta t)\, y^n$

$\Rightarrow y^{n+1} = (1+\lambda\Delta t)^{n+1} y^0 \rightarrow$ goes to infinity (unstable) if $|1+\lambda\Delta t| > 1$

implicit: $y^{n+1} = y^n + \lambda \Delta t\, y^{n+1}$

$\Rightarrow y^{n+1} = \dfrac{1}{1-\lambda\Delta t}\, y^n = \dfrac{1}{(1-\lambda\Delta t)^{n+1}}\, y^0 \rightarrow$ unstable if $\dfrac{1}{|1-\lambda\Delta t|} > 1$



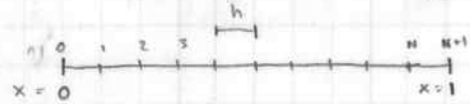


Truncation Error, Consistency, Convergence

Consider the PDE : $u''(x) = f(x) \qquad 0 < x < 1$

$u(0) = \alpha, \quad u(1) = \beta$

Finite Difference approximation :



(☆) $\frac{1}{h^2}\left(\bar{u}_{j+1} - 2\bar{u}_j + \bar{u}_{j-1}\right) = f(x_j) \quad j = 1, \ldots, N$

$\bar{u}_0 = \alpha, \quad \bar{u}_{N+1} = \beta$

$\bar{u}_j$ = fin. diff. approx. to $u(x_j)$

Local truncation error :

Replace u_j w/ exact solution $u(x_j)$:

(☆☆) $\tau_j = \left(u(x_{j+1}) - 2u(x_j) + u(x_{j-1})\right)\frac{1}{h^2} - f(x_j) \qquad j = 1, \ldots, N.$

$= u''(x_j) + \frac{h^2}{12} u''''(x_j) + O(h^4) - f(x_j)$ from (A) on pg. 1 (Taylor series).

$= \frac{h^2}{12} u''''(x_j) + O(h^4)$ b/c $u''(x_j) - f(x_j) = 0$. since u exact sol'n

$\tau_j = O(h^2)$ as $h \to 0$

Matrix notation : $AU = F$ $\quad A = \frac{1}{h^2}\begin{bmatrix} -2 & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & -2 \end{bmatrix} \quad \underline{U} = \begin{bmatrix} \bar{u}_1 \\ \vdots \\ \bar{u}_N \end{bmatrix} \quad \underline{F} = \begin{bmatrix} f(x_1) \\ \vdots \\ f(x_N) \end{bmatrix} - \frac{1}{h^2}\begin{bmatrix} \alpha \\ 0 \\ \vdots \\ 0 \\ \beta \end{bmatrix}.$

Global error : Subtract (☆☆) from (☆) [fin. diff eqn. from local trunc. eqn] :

and define $e_j = \bar{u}_j - u(x_j)$. [clearly $e_0 = e_N = 0$] $\qquad \underline{\tau} = \begin{bmatrix} \tau_1 \\ \vdots \\ \tau_N \end{bmatrix}$

$\frac{1}{h^2}\left(e_{j+1} - 2e_j + e_{j-1}\right) = -\tau_j \qquad j = 1, \ldots, N$ $\quad\Big|\quad AE = -\tau.$ $\quad E = U - U_{exact}$

$e_0 = e_N = 0.$

$\Rightarrow$ error satisfies finite difference equation nearly identical to original equation :

interpret as discretization of :

$$e''(x) = -\tau(x) \qquad x \in (0,1)$$

$$e(0) = e(1) = 0.$$

Since $\tau(x) \approx \frac{h^2}{12} u''''(x)$, integration shows that

$$e(x) \approx -\frac{1}{12}h^2 u''(x) + \frac{1}{12}h^2\left(u''(0) + x\left(u''(1) - u''(0)\right)\right).$$

$$\Rightarrow \boxed{e(x) = O(h^2)} \quad ▨$$

5) Stability :

The error equation can be written $\underline{A}^h \underline{E}^h = -\underline{\tau}^h$ in matrix form.

Since the size of these matrices depends on the grid size, we make it explicit.

$$\Rightarrow \underline{E}^h = -(\underline{A}^h)^{-1} \underline{\tau}^h$$

$$\|\underline{E}^h\| \le \|(A^h)^{-1}\| \cdot \|\tau^h\| \le C \|\tau\|^h$$

$$\text{if } \|(A^h)\| \le C \leftarrow \underline{\text{stability}}$$

<u>Consistency</u> : A numerical scheme is consistent if $\|\tau^h\| \to 0$ as $h \to 0$.

<u>Convergence</u> : A method is convergent if $\|E^h\| \to 0$ as $h \to 0$.

Consistency + stability $\Rightarrow$ convergence.

$$\|E^h\| \le \|(A^h)^{-1}\| \|\tau^h\| \le C \|\tau^h\| \to 0.$$

<u>Time - Dependent Example</u>

1 d KdV equation : $\frac{\partial \phi}{\partial t} + \frac{\partial^3 \phi}{\partial x^3} + 6\phi \frac{\partial \phi}{\partial x} = 0 \qquad x \in [0, L]$

$$\phi(t, 0) = \phi(t, L) \qquad \text{periodic B.C.s}$$

$$\phi(0, x) = f(x)$$

$$\frac{\partial^3 \phi}{\partial x^3} = \frac{1}{8h^3}\left[-\phi_{j+3} + 8\phi_{j+2} - 13\phi_{j+1} + 13\phi_{j-1} - 8\phi_{j-2} + \phi_{j-3}\right]$$

(could derive using methodology from earlier in lec)

$$\frac{\partial \phi}{\partial x} = \frac{\phi_{j+1} - \phi_{j-1}}{2h}$$

$$\phi = \begin{bmatrix} \phi_0 \\ \vdots \\ \phi_{N+1} \end{bmatrix}$$

$$\Rightarrow \frac{\partial^3 \phi}{\partial x^3} \approx D_3 \phi$$

$$\frac{\partial \phi}{\partial x} = D_1 \phi$$

$$D_3 = \frac{1}{8h^3}\begin{bmatrix} 0 & 13 & -8 & 1 & & & \\ -13 & \ddots & \ddots & \ddots & \ddots & & \\ 8 & \ddots & \ddots & \ddots & \ddots & & 1 \\ -1 & \ddots & \ddots & \ddots & \ddots & & -8 \\ & \ddots & \ddots & \ddots & \ddots & & 13 \\ & & & -1 & 8 & -13 & 0 \end{bmatrix} + \frac{1}{8h^3}\begin{bmatrix} & & & & & 1 & -8 & 13 \\ 13 & & & & & & 1 & -8 \\ -8 & 13 & & & & & & 1 \\ & -1 & & & & & & \\ 8 & -1 & 0 & 0 & 0 & \cdots & 0 & -13 \\ -13 & 8 & -1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

$$D_1 = \frac{1}{2h}\begin{bmatrix} 0 & 1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & -1 & 0 \end{bmatrix} + \frac{1}{2h}\begin{bmatrix} 0 & \cdots & -1 \\ & & \\ 1 & \cdots & 0 \end{bmatrix}$$

these terms encode the periodic BCs

$$\frac{\partial \phi}{\partial t} + \frac{\partial^3 \phi}{\partial x^3} + 6\phi \frac{\partial \phi}{\partial x} = 0 \Rightarrow \frac{\partial \phi}{\partial t} + D_3 \underset{\sim}{\phi} + 6(\mathrm{diag}\, \underset{\sim}{\phi})(\underline{D}_1 \underset{\sim}{\phi}) = 0.$$

explicit : $\underset{\sim}{\phi}^{n+1} = \underset{\sim}{\phi}^n - \Delta t \left[\underline{D}_3 \underset{\sim}{\phi}^n + 6(\mathrm{diag}\, \underset{\sim}{\phi})(\underline{D}_1 \underset{\sim}{\phi}) \right]$

implicit : $\underline{\phi}^{n+1} = \underline{\phi}^n - \Delta t \left[\underline{D}_3 \underline{\phi}^{n+1} + 6(\mathrm{diag}\, \underline{\phi}^{n+1})(D_1 \underline{\phi}^{n+1}) \right].$

↑

need derivatives for implicit b/c you must apply Newtons method to solve the equation $R(\phi) = 0$:

$$\boxed{\phi^{n+1}_{K+1} = \phi^{n+1}_{K} - \frac{\partial R}{\partial \phi}(\phi^{n+1}_K)^{-1} \cdot R(\phi^{n+1}_K)}$$

$$\frac{\partial}{\partial \phi_{j-1}} \left[\phi_j (\phi_{j+1} - \phi_{j-1}) \frac{1}{2h} \right] = \phi_j (-1) \frac{1}{2h}$$

$$\frac{\partial}{\partial \phi_{j+1}} \left[\phi_j (\phi_{j+1} - \phi_{j-1}) \frac{1}{2h} \right] = \phi_j (1) \frac{1}{2h}$$

$$\frac{\partial}{\partial \phi_j} [\cdot] = \quad = (\phi_{j+1} - \phi_{j-1}) \frac{1}{2h}$$

$$\left[\phi_j (-\phi_{j-1}) \frac{1}{2h},\ (\phi_{j+1} - \phi_{j-1}) \frac{1}{2h},\ \phi_j (\phi_{j+1}) \frac{1}{2h} \right]$$

$$= D_1 \phi + \mathrm{diag}\,\phi \cdot D_1$$