AME50541: Finite Element Methods
Note 3: Variational Methods

1 Variational formulations of PDEs

Consider a PDE of order m in residual form:

$$\text{find } u(x) \in C^m(\Omega) \text{ such that } R(x; u) = 0 \text{ for all } x \in \Omega, \tag{1}$$

where $\Omega \subset \mathbb{R}^d$ (open set) and $R(x; u)$ is a differential operator on u of order m defining the PDE. Using the canoncial example in one-dimension, we have $d = 1$, $\Omega = (0, L)$, $m = 2$, and $R(x; u) = -\frac{d}{dx}\left[a(x)\frac{du}{dx}\right] - f(x)$, where $a(x)$ and $f(x)$ are known (smooth) functions. This is called the *strong formulation* of the PDE because it enforces the governing equations pointwise throughout the domain and has strict regularity requirements on the solution (m continuous derivatives). While this formulation is easy to understand and usually relates to physical principles (conservation of mass, momentum, energy), it is not always convenient to use as a foundation for numerical methods. For example, consider the PDE and boundary conditions

$$\begin{aligned} &-\frac{d}{dx}\left[x^2\frac{du}{dx}\right] = x(1-x) \quad 0 < x < 1, \\ &u(0) = 1, \quad \left[x^2\frac{du}{dx}(x)\right]_{x=1} = 0 \end{aligned} \tag{2}$$

that we approximate with the function $U_2(x) := 1 + c_1(x^2 - 2x) + c_2(x^3 - 3x)$, which clearly satisfies the boundary conditions $U_2(0) = 1$, $\left[x^2 U_2'(x)\right]_{x=1} = 0$. The scalars c_1 and c_2 are unknown; substitution of the approximation $U_2(x)$ into the governing equations yields the system of equations

$$-12c_2 = 0, \; -6c_1 = -1, \; 4c_1 + 6c_2 = 1,$$

which are inconsistent, i.e., there is no c_1, c_2 that satisfies all three equations. This shows that, by using the strong formulation, we cannot find an approximation to (2) of the form $1 + c_1(x^2 - 2x) + c_2(x^3 - 3x)$. It is not surprising that this approach failed: we are requiring the PDE be satisfied pointwise, but using an approximation that cannot represent its solution. To avoid this issue, we turn to variational formulations of the PDE, which will be the foundation of a slew of numerical methods, including the finite element method.

1.1 Weighted-residual formulation

The weighted-integral or weighted-residual formulation corresponding to the strong form(ulation) of the PDE is:

$$\text{find } u(x) \in C^m(\Omega) \text{ such that } \int_\Omega w(x) R(x; u)\, dV = 0 \text{ for all } w \in C^0(\Omega). \tag{3}$$

This weighted-integral formulation of the PDE is *equivalent* to the strong formulation, i.e., if $u(x)$ is a solution of the strong form, it is a solution of the weighted-integral form and vice versa. This equivalence can be proven by invoking the fundamental lemma of variational calculus (Lemma 1), with $G(x)$ taken to be $R(x; u)$.

Lemma 1 (Fundamental Lemma of Variational Calculus)**.** *Suppose $G \in C^0(\Omega)$, where $\Omega \subset \mathbb{R}^d$ (open). Then, $G(x) = 0$ for all $x \in \Omega$ is equivalent to the weighted-integral statement*

$$\int_\Omega G(x)\eta(x)\, dV = 0, \quad \forall \eta \in C^0(\Omega). \tag{4}$$

Proof. $G(x) = 0$ immediately implies the weighted-integral statement. To show the converse is true, suppose the weighted-integral statement holds. Taking $\eta(x) = G(x)$ (valid since η can be any function in $C^0(\Omega)$), leads to

$$\int_\Omega G^2\, dV = 0.$$

Since the integrand is positive, this implies $G(x) = 0$ for all $x \in \Omega$. □

It is important to note that the weighted-integral formulation is equivalent to enforcing the PDE over its domain Ω, but does not incorporate any of the boundary conditions of the problem. This implies that any numerical method based on the weighted-integral formulation must explicitly enforce all boundary conditions. In the next section, we introduce the weak formulation of the problem, which relaxes the regularity requirements on the solution $u(x)$ and incorporates the Neumann or natural boundary conditions into the formulation.

1.2 Weak formulation

The construction of the weak formulation of a partial differential equation begins with the weighted-residual formulation of the PDE (3) and, assuming the PDE is of order $2m$, moves m derivatives from the PDE solution variable u onto the test function w using integration-by-parts. The final step in the derivation of the weak form is to incorporates the natural boundary conditions from the problem statement into boundary terms that arise.

For concreteness, consider the canonical second-order PDE in one-dimension

$$\begin{aligned} -\frac{d}{dx}\left[a(x)\frac{du}{dx}\right] + c(x)u(x) = f(x), \quad 0 < x < L, \\ u(0) = u_0, \quad \left[a(x)\frac{du}{dx}(x)\right]_{x=L} = Q_L, \end{aligned} \tag{5}$$

where $a(x)$, $c(x)$, and $f(x)$ are given, smooth functions and u_0 and Q_L are given constants. The weighted-integral form reads

$$\int_0^L w(x)\left[-\frac{d}{dx}\left[a(x)\frac{du}{dx}\right] + c(x)u(x) - f(x)\right] dx. \tag{6}$$

Applying integration-by-parts to move one derivative from u to w leads to

$$\int_0^L \left[\frac{dw}{dx}a(x)\frac{du}{dx} + w(x)(c(x)u(x) - f(x))\right] dx - \left[w(x)a(x)\frac{du}{dx}\right]_0^L = 0. \tag{7}$$

To incorporate natural boundary conditions into the weak formulation, we need a concrete definition of essential and natural boundary conditions, which stems from the definition of primary and secondary variables. Essential, or Dirichlet, boundary conditions are conditions on *primary* variables along boundaries, while natural, or Neumann, boundary conditions are conditions on *secondary* variables along boundaries. A primary variable can be identified from the weak formulation: replace the test function variables (w in the case) with the PDE solution variable (u in this case) in the *boundary terms* of the weak form to identify primary variables. Secondary variable multiply the test functions (or their derivatives) in the boundary terms of the weak form. From these definitions, it is clear that PDEs of order $2m$ will have m primary and secondary variables since there will be m boundary terms resulting from m applications of integration-by-parts to move derivatives from the solution variable to the test variable. There will also be a secondary variable *corresponding* to each primary variable. The secondary variable corresponding to a primary variable can be identified by replacing the primary variable with its corresponding test function in the boundary terms of the weak form and identifying the variable that multiplies it. With these definitions, it follows that u is the only primary variable for the PDE in (5) and $a(x)\frac{du}{dx}$ is the only secondary variable.

With these definitions, we incorporate the natural boundary conditions by substituting their expression into the boundary term of the weak formulation, i.e., $a(L)\frac{du}{dx}(L) = Q_L$, to yield

$$\int_0^L \left[\frac{dw}{dx}a(x)\frac{du}{dx} + w(x)\left(c(x)u(x) - f(x)\right)\right] dx - w(L)Q_L + w(0)\left[a(x)\frac{du}{dx}\right]_{x=0} = 0. \tag{8}$$

Recall from the weighted-residual formulation that the weighted-integral equation must hold for all continuous test functions on Ω (open set, i.e., without boundary). This implies that, regardless of the value of the test functions on the boundary, the weighted-integral formulation still enforces the PDE. We leverage this flexibility and choose test functions that are zero everywhere a primary variable is specified. For the PDE in (5), this implies $w(0) = 0$ since the primary variable (u) is prescribed as $x = 0$ to give the final version of the weak formulation

$$\text{find } u(x) \text{ such that } \int_0^L \left[\frac{dw}{dx} a(x) \frac{du}{dx} + w(x)(c(x)u(x) - f(x))\right] dx - w(L)Q_L = 0 \text{ for all } w(x). \tag{9}$$

Notice that we intentionally did not choose the test functions to be zero where natural boundary conditions are specified because allowing the test functions to vary at those points enforces the natural boundary conditions in the weak formulation. We close this section with the derivation of the weak formulation for more complicated PDEs.

For convenience, we will often write the weak form as a bilinear form: find u such that $B(w, u) = l(w)$ for all w, where $B(w, u)$ is a bilinear function defined by collecting all terms of the weak form that depend on both u and w and $l(w)$ is a linear function defined by collection all terms in the weak form that only depend on w.

1.2.1 Example: Fourth-order PDE in 1d

Consider the fourth-order partial differential equation in one dimension

$$\frac{d^4 w}{dx^4} = 0, \quad 0 < x < L \tag{10}$$

with boundary conditions $w(0) = w_0$, $w(L) = w_L$, $\frac{dw}{dx}(0) = w_0'$, and $\frac{d^2w}{dx^2}(L) = w_L''$. Multiplying the governing equation by a test function v, integrating over the domain, and using integration-by-parts twice to move two derivatives from w onto v yields

$$0 = \int_0^L v \frac{d^4 w}{dx^4}\, dx = \int_0^L -\frac{dv}{dx}\frac{d^3 w}{dx^3}\, dx + \left[v \frac{d^3 w}{dx^3}\right]_0^L = \int_0^L \frac{d^2 v}{dx^2}\frac{d^2 w}{dx^2}\, dx + \left[v\frac{d^3 w}{dx^3}\right]_0^L - \left[\frac{dv}{dx}\frac{d^2 w}{dx^2}\right]_0^L.$$

From examining the boundary terms, we identify the primary variables as w and $\frac{dw}{dx}$ and the corresponding secondary variables as $\frac{d^3w}{dx^3}$ and $\frac{d^2w}{dx^2}$, respectively. Since both primary variables are specified at $x = 0$, we set the corresponding test functions to zero $v(0) = \frac{dv}{dx}(0) = 0$. In addition, the primary variable w is specified as $x = L$ so we set the corresponding test function to be zero $v(L) = 0$. Using these choices for the test function and incorporating the natural boundary condition $\frac{d^2w}{dx^2}(L) = w_L''$ into the boundary terms above, we arrive at the weak form

$$\int_0^L \frac{d^2 v}{dx^2}\frac{d^2 w}{dx^2}\, dx - \frac{dv}{dx}(L) w_L'' = 0.$$

1.2.2 Example: Timoshenko beam

Consider the equations for the deflection of a beam using the Timoshenko theory

$$\begin{aligned} -\frac{d}{dx}\left[S\left(\frac{dw}{dx} + \phi_x\right)\right] + c_f w = q \\ -\frac{d}{dx}\left(D\frac{d\phi_x}{dx}\right) + S\left(\frac{dw}{dx} + \phi_x\right) = 0 \end{aligned} \tag{11}$$

with boundary conditions $w(0) = \phi_x(0) = 0$, $\left[S\left(\frac{dw}{dx} + \phi_x\right)\right]_{x=L} = F_L$, and $\left[D\frac{d\phi_x}{dx}\right]_{x=L} = M_0$. The unknown functions are $w(x)$ and $\phi_x(x)$; the remaining terms $S(x)$, $D(x)$, $c_f(x)$, and $q(x)$ are known (smooth) functions. Multiply the first equation (for w) by the test function v_1 and the second equation (for ϕ_x) by the test function v_2, integrate each over the domain, and add them to yield the weighted-integral form

$$\int_0^L \left(v_1\left\{-\frac{d}{dx}\left[S\left(\frac{dw}{dx} + \phi_x\right)\right] + c_f w - q\right\} + v_2\left\{-\frac{d}{dx}\left(D\frac{d\phi_x}{dx}\right) + S\left(\frac{dw}{dx} + \phi_x\right)\right\}\right) = 0.$$

Apply integration-by-parts to move a derivative from w onto v_1 and from ϕ_x onto v_2

$$\int_0^L \left\{ \frac{dv_1}{dx} S \left(\frac{dw}{dx} + \phi_x \right) + \frac{dv_2}{dx} D \frac{d\phi_x}{dx} + v_1(c_f w - q) + v_2 S \left(\frac{dw}{dx} + \phi_x \right) \right\} dx - \left[v_1 S \left(\frac{dw}{dx} + \phi_x \right) \right]_0^L - \left[v_2 D \frac{d\phi_x}{dx} \right]_0^L = 0.$$

From this form, we can identify the primary variables as w (replace v_1 with w in the boundary terms to identify) and ϕ_x (replace v_2 with ϕ_x in the boundary terms to identify) and the corresponding secondary variables as $S\left(\frac{dw}{dx} + \phi_x\right)$ and $D\frac{d\phi_x}{dx}$, respectively. Since both primary variables are specified as $x = 0$, we take $v_1(0) = v_2(0) = 0$ and substitute the natural boundary conditions at $x = L$ to arrive at the final version of the weak formulation

$$\int_0^L \left\{ \frac{dv_1}{dx} S \left(\frac{dw}{dx} + \phi_x \right) + \frac{dv_2}{dx} D \frac{d\phi_x}{dx} + v_1(c_f w - q) + v_2 S \left(\frac{dw}{dx} + \phi_x \right) \right\} dx - v_1(L) F_L - v_2(L) M_L = 0.$$

1.2.3 Example: Poisson equation in Nd

Our final example considers the Poisson equation in an N-dimensional space

$$\begin{aligned} -\Delta u &= 0 && \text{in } \Omega \\ u &= g && \text{on } \partial\Omega_1 \\ \nabla u \cdot n &= h && \text{on } \partial\Omega_2, \end{aligned} \tag{12}$$

where $\Omega \subset \mathbb{R}^N$ is the domain with boundary $\partial\Omega$ that is partitioned into $\partial\Omega_1$ and $\partial\Omega_2$, i.e. $\partial\Omega = \overline{\partial\Omega_1 \cup \partial\Omega_2}$. For convenience, we convert this equation to indicial notation: $-u_{,ii} = 0$ in Ω, $u = g$ on $\partial\Omega_1$, $u_{,i} n_i = h$ on $\partial\Omega_2$. Next, we follow the standard weak form derivation procedure and setup the weighted-integral equation

$$\int_\Omega w(-u_{,ii}) \, dV = 0.$$

Applying integration-by-parts (for simplicity, use the identity $(wu_{,i})_{,i} = w_{,i} u_{,i} + w u_{,ii}$ and apply the divergence theorem) yields

$$\int_\Omega \left(w_{,i} u_{,i} - (w u_{,i})_{,i} \right) \, dV = \int_\Omega w_{,i} u_{,i} \, dV - \int_{\partial\Omega} w u_{,i} n_i \, dS = 0.$$

By examining the boundary terms, we see that u is the primary variable (from replacing the test function with u in the boundary term) and $u_{,i} n_i$ is the secondary variable (multiplies the test function in the boundary term). Next, we choose $w(x) = 0$ for $x \in \partial\Omega_1$ because the primary variable is specified on $\partial\Omega_1$. This causes the integral over the entire boundary to become an integral over only $\partial\Omega_2$ because of the additive property of integration

$$\int_{\partial\Omega} w u_{,i} n_i \, dS = \int_{\partial\Omega_1} w u_{,i} n_i \, dS + \int_{\partial\Omega_2} w u_{,i} n_i \, dS = \int_{\partial\Omega_2} w u_{,i} n_i \, dS,$$

where the last equality used $w(x) = 0$ on $\partial\Omega_1$. Finally, we substitute the natural boundary condition $u_{,i} n_i = h$ on $\partial\Omega_2$ into the weak form to yield

$$\int_\Omega w_{,i} u_{,i} \, dV - \int_{\partial\Omega_2} w \, h \, dS = 0.$$

2 Variational methods to approximation solutions of PDEs

With the variational formulations of partial differential equations introduced in the previous section, we turn to constructing numerical methods based on the weighted-integral formulation, called the method of weighted-residuals, and on the weak formulation, called the Ritz method.

2.1 The method of weighted residuals

Recall the weighted-integral formulation of a PDE in (3)

$$\text{find } u(x) \in C^m(\Omega) \text{ such that } \int_\Omega w(x) R(x;u)\, dV = 0 \text{ for all } w \in C^0(\Omega).$$

It is obvious that in a numerical method (intended to be implemented on a computer or computed by hand), we have no hope of enforcing this condition for all functions in $C^0(\Omega)$ (an infinite-dimensional function space). Instead, we will settle for enforcing the weighted-integral equation on an N-dimensional subspace $S \subset C^0(\omega)$ spanned by the functions $\{w_1(x), \dots, w_N(x)\}$. By virtue of the weight function appearing linearly in the weighted-integral equation, the equation holds for all $w \in S$ if it holds for each w_i, $i = 1, \dots, N$. Suppose

$$\int_\Omega w_i(x) R(x;u)\, dV = 0,$$

then it follows that for any $w \in S$, $w = \alpha_i w_i$ and

$$\int_\Omega w(x) R(x;u)\, dV = \alpha_i \int_\Omega w_i(x) R(x;u)\, dV = 0.$$

Therefore, we have reduced the problem from testing the residual against infinitely many weight functions w to only having to test it against N functions and the weighted-integral formulation reduces to

$$\text{find } u(x) \in C^m(\Omega) \text{ such that } \int_\Omega w_i(x) R(x;u)\, dV = 0 \text{ for } i = 1, \dots, N.$$

While we have reduced our problem to only having to test the residual against a finite number of weighting functions, we still must search an infinite-dimensional function space $C^m(\Omega)$ for the solution. To simplify this task to one that can be performed on a computer (or with hand calculations), we expand the solution is a basis and solve for the unknown coefficients

$$u(x) \approx U_N(x) := \varphi(x) + c_i \phi_i(x), \tag{13}$$

where $\varphi(x)$, $\phi_i(x)$ for $i = 1, \dots, N$ are known functions and c_i are unknown scalars. Since the weighted-integral form does not enforce the boundary conditions of the problem, the only hope of obtaining an approximation that satisfies the boundary conditions is to only search for solution that do so. To this end, we require $\varphi(x)$ to satisfy the boundary conditions of the problem and $\phi_i(x)$ to satisfy the homogeneous form of the boundary conditions. For example, for the boundary conditions in (5), we require $\varphi(0) = u_0$, $a(L)\varphi'(L) = Q_L$, $\phi_i(0) = 0$, and $a(L)\phi_i'(L) = 0$ for $i = 1, \dots, N$. Another requirement on the basis is that each $\phi_i(x)$ must have m non-zero derivatives for a PDE of order m. Otherwise, the term would vanish when substituted into the PDE residual. Substituting the solution approximation into the weighted-integral equation, the final form of the method of weighted residuals is

$$\text{find } \boldsymbol{c} \in \mathbb{R}^N \text{ such that } \int_\Omega w_i(x) R(x; \varphi + c_j \phi_j)\, dV = 0 \text{ for } i = 1, \dots, N.$$

While the solution basis function must be chosen to satisfy the problem boundary conditions and therefore varies from problem to problem, there are four common choices for the test function basis.

2.1.1 The Petrov-Galerkin method

The first, and most general choice, is allowing the test functions to be any continuous function and independent of the choice of $\phi_i(x)$ and the resulting method is called the Petrov-Galerkin method. In the case where $R(x;u)$ is linear in u, we have

$$\int_\Omega w_i \left[R(x;\varphi) + c_j R(x;\phi_j) \right] dV = 0, \tag{14}$$

which can be written as $K_{ij}c_j = F_i$, or as $\boldsymbol{Kc} = \boldsymbol{F}$, in matrix form, where

$$K_{ij} = \int_\Omega w_i R(x;\phi_j)\,dV, \qquad F_i = -\int_\Omega w_i R(x;\varphi). \tag{15}$$

For concreteness, we return to the PDE (2). The solution basis chosen previously is valid for the method of weighted residuals because it has two non-zero derivatives, $\varphi(x)$ satisfies the boundary conditions, and $\phi_i(x)$ satisfies the homongeneous form of the bounday conditions. In addition, we choose the test basis $w_1(x) = 1$ and $w_2(x) = x$ to yield the equations

$$\begin{aligned} \int_0^1 1 \cdot R(x;U_2(x))\,dx = 0 \\ \int_0^1 x \cdot R(x;U_2(x))\,dx = 0, \end{aligned} \tag{16}$$

which leads to the system $\frac{2}{3}c_1 + \frac{5}{4}c_2 = 1$, $\frac{3}{4}c_1 + \frac{31}{20}c_2 = \frac{1}{2}$, which can be solved to yield $c_1 = 222/23$ and $c_2 = -100/23$. Therefore, the approximation to the solution of the PDE using the method of weighted residual (Petrov-Galerkin) is $u(x) \approx U_2(x) = 1 + \frac{222}{23}(x^2 - 2x) - \frac{100}{23}(x^3 - 3x)$. Unlike the strong form, the weighted-integral form resulted in a solvable linear system of equations and a valid approximation of the PDE.

2.1.2 The Galerkin method

The second choices takes $w_i(x) = \phi_i(x)$ and is usually called the Galerkin method. In the case where $R(x;u)$ is linear in u, we have

$$\int_\Omega \phi_i \left[R(x;\varphi) + c_j R(x;\phi_j)\right]\,dV = 0, \tag{17}$$

which can be written as $K_{ij}c_j = F_i$, or as $\boldsymbol{Kc} = \boldsymbol{F}$, in matrix form, where

$$K_{ij} = \int_\Omega \phi_i R(x;\phi_j)\,dV, \qquad F_i = -\int_\Omega \phi_i R(x;\varphi). \tag{18}$$

2.1.3 The collocation method

Another common choice, called the collocation method, takes $w_i(x) = \delta(x - x_i)$, where δ is the Delta function and $x_i \in \mathbb{R}^d$ are selected points throughout the domain $i = 1, \ldots, N$. The collocation method amounts to requiring the residual to be zero at selected points throughout the domain (instead of in the weighted integral sense)

$$R(x_i;\varphi + \phi_j c_j) = 0. \tag{19}$$

In the case where $R(x;u)$ is linear in u, we have

$$R(x_i;\varphi) + c_j R(x_i;\phi_j) = 0 \tag{20}$$

which can be written as $K_{ij}c_j = F_i$, or as $\boldsymbol{Kc} = \boldsymbol{F}$, in matrix form, where

$$K_{ij} = R(x_i;\phi_j), \qquad F_i = -R(x_i;\varphi). \tag{21}$$

2.1.4 The least-square methods

Finally, the least-squares method defines the solution coefficients c_i to be the solution of the minimization problem

$$\underset{\boldsymbol{c}\in\mathbb{R}^N}{\text{minimize}} \quad f(\boldsymbol{c}) := \int_\Omega R(x;\varphi + c_j\phi_j)^2. \tag{22}$$

The first-order optimality condition states that $\frac{\partial f}{\partial c_i} = 0$ or

$$\int_\Omega \frac{\partial R}{\partial c_i}(x;\varphi + c_j\phi_j) R(x;\varphi + c_j\phi_j)\,dV = 0 \tag{23}$$

for $i = 1, \ldots, N$. This is precisely a weighted-integral method with $w_i = \frac{\partial}{\partial c_i} R(x;\varphi + c_j\phi_j)$.

2.2 The Ritz method

While the method of weighted residuals did not suffer from the same drawbacks as methods based on the strong formulation, they have their own disadvantages. In particular, the approximation functions used must have $2m$ non-zero derivatives for PDEs of order $2m$, which eliminates a number of useful and efficient families of approximations. In addition, the weighted-residual form does not incorporate any of the boundary conditions so the solution basis must account for them. To avoid these issues, we construct a numerical method based on the weak formulation of the PDE, which only requires the solution basis have m non-zero derivatives (because half of the derivatives of the PDE were move onto the test functions) and only need to satisfy the essential boundary conditions (the natural boundary conditions are embedded in the weak form).

Similar to the method of weighted residuals, we only require the weak form (written as a bilinear form) holds for test functions in a finite-dimensional subspace of $C^0(\Omega)$, which is equivalent to enforcing it on each of the basis functions for that space $\{w_1(x), \dots, w_N(x)\}$, i.e., find u such that $B(w_i, u) = l(w_i)$ for $i = 1, \dots, N$. Next we introduce a basis for the solution $u(x) \approx U_N(x) := \varphi(x) + c_j\phi_j(x)$; however, unlike the method of weighted residuals, $\varphi(x)$ only needs to satisfy the essential boundary conditions and $\phi_i(x)$ only needs to satisfy the homogeneous form of the essential boundary conditions. Substituting the solution approximation into the weak form and making the Galerkin choice for the test basis $w_i(x) = \phi_i(x)$, the weak form becomes:

$$\text{find } \boldsymbol{c} \text{ such that } K_{ij}c_j = F_i \text{ for } i = 1, \dots, N, \tag{24}$$

where $K_{ij} = B(\phi_i, \phi_j)$ and $F_i = l(\phi_i) - B(\phi_i, \varphi)$. Once this linear system has been solved, the coefficients are substituted back into the expression for U_N to obtain our approximation of the PDE.

To conclude, we apply the Ritz method to the simple PDE in one-dimension

$$-\frac{d^2u}{dx^2} - u + x^2 = 0 \tag{25}$$

with boundary conditions $u(0) = 0$, $\left.\frac{du}{dx}\right|_{x=1} = 1$. The weak form of this PDE is

$$\int_0^1 \left(\frac{dw}{dx}\frac{du}{dx} + w(x^2 - u) \right) dx - w(1) = 0,$$

which can be written as a bilinear form: find u such that $B(w, u) = l(w)$ for all w, where

$$B(w, u) = \int_0^1 \left(\frac{dw}{dx}\frac{du}{dx} - uw \right) dx, \qquad l(w) = \int_0^1 -wx^2 \, dx + w(1). \tag{26}$$

For this example, we choose $\varphi(x) = 0$ and $\phi_i(x) = x^i$, for $i = 1, \dots, N$, which satisfy the requirements of the Ritz method (all $\phi_i(x)$ have at least one non-zero derivative, the approximation satisfies the essential boundary conditions, it is a complete polynomial up to order N). Then, the entries of the stiffness matrix are

$$K_{ij} = B(\phi_i, \phi_j) = B(x^i, x^j) = \int_0^1 \left(ijx^{i+1}x^{j+1} - x^{i+j} \right) dx = \frac{ij}{i+j-1} - \frac{1}{i+j+1} \tag{27}$$

and the force vector are

$$F_i = l(\phi_i) = \int_0^1 -x^{i+2} \, dx + (1)^i = -\frac{1}{i+3} + 1. \tag{28}$$

For the special case of $N = 2$, we setup the linear system $\boldsymbol{Kc} = \boldsymbol{F}$ and solve it to get $c_1 = 1.295$ and $c_2 = -0.1511$ for an approximation $u(x) \approx U_2(x) = 1.295x - 0.1511x^2$.