Kernel + Image

- **Note:** See iLearn for feedback on first Exploration Problem for Final Project.

- **Video:** If

$$\varphi : G \longrightarrow H$$

is a homomorphism,

  - $\ker(\varphi)$ = things in G that φ sends to identity

    φ is injective $\iff \ker(\varphi) = \{e_G\}$

  - $\mathrm{im}(\varphi)$ = things in H that are φ(something)

    φ is surjective $\iff \mathrm{im}(\varphi) = H$

Worksheet 32: Kernel and Image

Math 335

Reporter:

Recorder:

Equity Manager:

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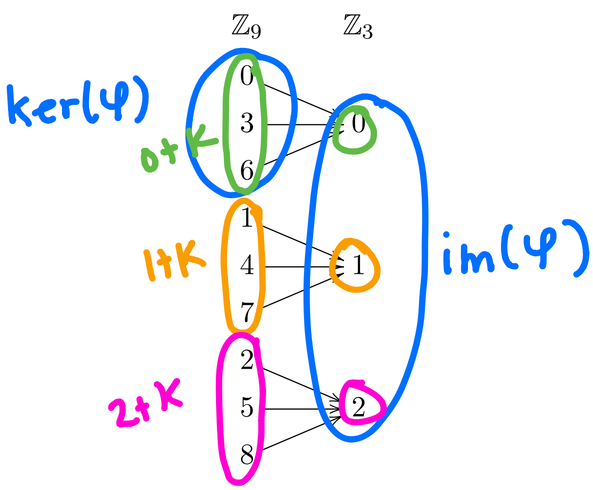
1. Consider the homomorphism

$$\varphi : \mathbb{Z}_9 \to \mathbb{Z}_3$$
$$\varphi(x) = x \text{ reduced modulo } 3.$$

Here's a schematic picture of what φ does:



(a) What is $\ker(\varphi)$? What is $\mathrm{im}(\varphi)$? Label them on the above schematic picture.

$\ker(\varphi) = \{0, 3, 6\}$

$\mathrm{im}(\varphi) = \{0, 1, 2\}$

(b) Let $K = \ker(\varphi) \subseteq \mathbb{Z}_9$. What are the left cosets of K in $\mathbb{Z}_9$?

$0 + K = \{0, 3, 6\}$

$1 + K = \{1, 4, 7\}$

$2 + K = \{2, 5, 8\}$

(c) What relationship do you notice between the left cosets of K in $\mathbb{Z}_9$ and the above schematic picture?

Elements of the same left coset are sent to the same place by φ

2. Thinking of $\mathbb{Z}$ as a group under the operation of addition and $\mathbb{C}^*$ as a group under the operation of multiplication, consider the homomorphism

$$\varphi : \mathbb{Z} \to \mathbb{C}^*$$
$$\varphi(n) = i^n.$$

(For instance, $\varphi(2) = i^2 = -1$ and $\varphi(3) = i^3 = -i$.)

$\varphi(0)=1$ $\varphi(4)=1$
$\varphi(1)=i$ $\varphi(5)=i$
$\varphi(2)=-1$ $\varphi(6)=-1$
$\varphi(3)=-i$ etc...

(a) What is $\ker(\varphi)$? What is $\operatorname{im}(\varphi)$?

$\ker(\varphi) = \{\text{multiples of } 4\}$

$\operatorname{im}(\varphi) = \{1, -1, i, -i\}$

(b) Let $K = \ker(\varphi) \subseteq \mathbb{Z}$. What are the left cosets of K in $\mathbb{Z}$?

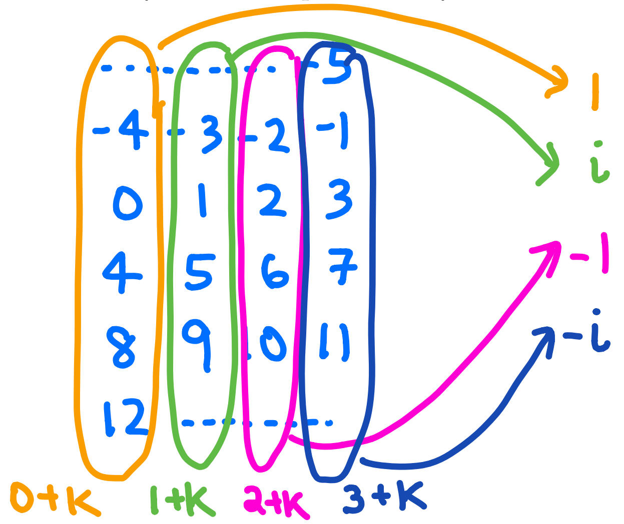
$0 + K = \{\ldots, -8, -4, 0, 4, 8, \ldots\}$

$1 + K = \{\ldots, -7, -3, 1, 5, 9, \ldots\}$

$2 + K = \{\ldots, -6, -2, 2, 6, 10, \ldots\}$

$3 + K = \{\ldots, -5, -1, 3, 7, 11, \ldots\}$

(c) Try your best to make a schematic picture of what φ does, similarly to Problem 1. Do you notice any relationship between your schematic picture and the left cosets of K?



Elements of the same left coset are sent to the same place by φ