Listing Finite Abelian Groups

- **Video:** $\mathbb{Z}_n \oplus \mathbb{Z}_m \cong \mathbb{Z}_{nm} \iff \gcd(n,m)=1$

- **Goal:** What are ALL the abelian groups with 4 elements (or some other # of elements)?

  Write a list so that ANY abelian group with 4 elements is isomorphic to something on your list.

Worksheet 36: Listing Finite Abelian Groups

Math 335

Reporter:

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1. How many abelian groups with four elements can you think of? List as many as you can. (There are hints at the bottom of this worksheet if you want them.)

$\mathbb{Z}_4$, $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, $\cancel{U_4}$,

G

	1	a	b	c
1	1	a	b	c
a	a	1	c	b
b	b	c	1	a
c	c	b	a	1

HW6 #5a ("Klein 4-group")

2. Are any of the abelian groups on your list from Problem 1 isomorphic to each other? Try to make a new list, where you only list something if it's not isomorphic to something you've already written down.

$U_4 \cong \mathbb{Z}_4$

$G \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$

So the new list has only $\mathbb{Z}_4$ and $\mathbb{Z}_2 \oplus \mathbb{Z}_2$

$$\left(\begin{array}{l} \varphi(1) = (0,0) \\ \varphi(a) = (1,0) \\ \varphi(b) = (0,1) \\ \varphi(c) = (1,1) \end{array} \right)$$

3. How many abelian groups with twelve elements can you think of? (Again, see the hints if you want.)

$\mathbb{Z}_{12}$

$\mathbb{Z}_2 \oplus \mathbb{Z}_6$

$\mathbb{Z}_3 \oplus \mathbb{Z}_4$

$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3$

4. Are any of the abelian groups on your list from Problem 3 isomorphic to each other? Try to make a new list, where you only list something if it's not isomorphic to something you've already written down.

$\mathbb{Z}_3 \oplus \mathbb{Z}_4 \cong \mathbb{Z}_{12}$

$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_6$

So the new list has only

$\mathbb{Z}_3 \oplus \mathbb{Z}_4$ and $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3$

Hints:

- One group with four elements is $\mathbb{Z}_4$, and another one appeared on Homework 6.
- Can you build a group with four elements by taking a direct product of $\mathbb{Z}_n$'s?
- In general, can you build a group with a certain number of elements by taking either $\mathbb{Z}_n$ or a direct product of two or more $\mathbb{Z}_n$'s?