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HW 4

1a. $\mathbb{Z}_6 = \{0,1,2,3,4,5\}$

ord $(0) = 1$ | ord $(3) = 2$ | $6 \cdot 1 = 6 = 0$ | $4 \cdot 3 = 12 = 0$
ord $(1) = 6$ | ord $(4) = 3$ | $2 \cdot 3 = 6 = 0$ | $5 \cdot 6 = 30 = 0$
ord $(2) = 3$ | ord $(5) = 6$ | $3 \cdot 2 = 6 = 0$

1b. $\mathbb{Z}_5 = \{0,1,2,3,4\}$

ord $(0) = 1$ | ord $(3) = 5$ | $5 \cdot 1 = 5 = 0$ | $4 \cdot 5 = 20 = 0$
ord $(1) = 5$ | ord $(4) = 5$ | $2 \cdot 5 = 10 = 0$
ord $(2) = 5$ | | $3 \cdot 5 = 15 = 0$

2a. Let G be a group and let $g \in G$ be an element with infinite order. By definition, this means that there is no k such that $g^k = e$. Towards a contradiction, suppose that $i < j$ and $g^i = g^j$. Then

$$g^i * g^{-j} = g^j * g^{-j}$$

$$g^{i-j} = e.$$

But there exists no $k = i - j$ such that $g^k = e$, which contradicts our assumption. Hence

$$g^i = g^j \Rightarrow i = j$$

b. $\mathbb{Z}_3 = \{0,1,2\}$ under addition mod 3

$$\left.\begin{array}{l} 1^3 = 1+1+1 = 3 \equiv 0 \bmod 3 \\ 1^6 = 6 \equiv 0 \bmod 3 \end{array}\right\} \begin{array}{l} 1^3 = 1^6 \text{ but} \\ 3 \neq 6 \end{array}$$

3a. By definition G has to be closed under its operation and every element must have an inverse. But G is finite and since $g^{-k} = g^{n} \in G$ we can find $g^{k} * g^{-k} = g^{k} * g^{n} = g^{k+n} = e$

Since both k and n are finite, for all elements g there exists a finite number such that $g^{k+n} = e$.

3b. Let $G = \mathbb{R}^{*} = \{\text{non-zero } \mathbb{R}\}$ under operation of multiplication, which is an infinite group.

$ord(-1) = 2$ e.g. $(-1)\cdot(-1) = (-1)^{2} = e$

Hence in group G, there exists an element with a finite order.

4. $f = (1,2,3,5) \circ (2,4,5,6,7)$ in S_7

$f = (1,2,4)(3,5,6,7)$

$ord(f) = \operatorname{lcm}(4,3) = 12$

5. S_5 possible combinations:

one element $(\langle e \rangle) \Rightarrow ord(\langle e \rangle) = 1$

two element $((1,2)(3,4)(5)) \Rightarrow ord(f) = \operatorname{lcm}(2,2,1) = 2$

3 element $((1,2,3)(4,5)) \Rightarrow ord(f) = \operatorname{lcm}(3,2) = 6$

4 element $(1,2,3,4)(5) \Rightarrow ord(f) = \operatorname{lcm}(4,1) = 4$

5 element $(1,2,3,4,5) \Rightarrow ord(f) = 5$

5. Possible orders

$1, 2, 4, 5, 6$

Elements in S_5 can come in five different configurations, each which have the same order. The actual contents of each configuration is irrelevant because they all share the same order.