# The First Isomorphism Theorem

- **Announcements:**

  - Math Dept Graduation Ceremony 5/24 – fill out survey by Monday if graduating!
  - Q&A for Math Dept Scholarships today 3-4pm

- **Video:** Let

$$\varphi : G \longrightarrow H$$

be a homomorphism and let $K = \ker(\varphi)$. Then

$$\boxed{G/K \cong \operatorname{im}(\varphi)}.$$

- This is a super useful way to prove two groups are isomorphic without necessarily finding an isomorphism!

- **Comprehension Question:** If $\varphi : G \longrightarrow H$ is a surjective homomorphism, then

$$G/\ker(\varphi) \cong H.$$

# Worksheet 33: The First Isomorphism Theorem

Math 335

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**Reporter**:

**Recorder**:

**Equity Manager**:

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1. Consider the homomorphism

$$\varphi : \mathbb{R}^* \to \mathbb{R}^*$$
$$\varphi(x) = |x|,$$

where $\mathbb{R}^*$ is a group under multiplication and $|x|$ stands for the absolute value of $x$.

(a) What is $\ker(\varphi)$? What is $\operatorname{im}(\varphi)$?

$\ker(\varphi) = \{1, -1\}$

$\operatorname{im}(\varphi) = \mathbb{R}^+$

(b) What does the First Isomorphism Theorem say in this case? In other words, fill in the blanks with the appropriate groups in this example:

$$\underline{\ \mathbb{R}^*\ } / \underline{\ \{1,-1\}\ } \cong \underline{\ \mathbb{R}^+\ }.$$

(c) Write down a few different elements of the left-hand side of part (b), and what they correspond to on the right-hand side.

$1K \quad \{-1 \quad 1\} \longrightarrow 1$

$2K \quad \{-2 \quad 2\} \longrightarrow 2$

$2.5K \quad \{-2.5 \quad 2.5\} \longrightarrow 2.5$

(d) If you wanted to "list" all the elements of the quotient group $\mathbb{R}^*/\ker(\varphi)$, with each element listed only once, what would that list look like? Discuss with each other why that list would look the same as a list of the elements in $\operatorname{im}(\varphi)$.

Elements of $\mathbb{R}^*/K$ are $aK = \{a, -a\}$, so there's one for each $a \in \mathbb{R}^+$.

2. Consider the homomorphism

$$\varphi : \mathbb{Z} \to \mathbb{Z}_5$$
$$\varphi(x) = x \text{ reduced modulo } 5,$$

where $\mathbb{Z}$ is a group under addition and $\mathbb{Z}_5$ is a group under addition modulo 5.

(a) What is $\ker(\varphi)$? What is $\operatorname{im}(\varphi)$?

$\ker(\varphi) = \{\text{multiples of } 5\} = 5\mathbb{Z}$

$\operatorname{im}(\varphi) = \mathbb{Z}_5$

(b) What does the First Isomorphism Theorem say in this case?

$\mathbb{Z}/5\mathbb{Z} \cong \mathbb{Z}_5$

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3. **Vague discussion questions**: What does the statement of the First Isomorphism Theorem become if $\varphi : G \to G$ is the function

$$\varphi(x) = x?$$

How about if $\varphi : G \to H$ is an isomorphism?

For $\varphi(x) = x$,

$\ker(\varphi) = \{e\}$

$\operatorname{im}(\varphi) = G$,

so F.I.T. says

$G/\{e\} \cong G$.

(Similarly for $\varphi$ an isomorphism.)