Orders of Elements in S_n

- **Reminder**: Exam 1 in one week

  Logistics:
  - On iLearn, take during any 90-minute time period on Wednesday 2/24 (12:01am – 11:59pm)
  - Submit via PDF (e.g. hand-write and scan)
  - Can use any non-human resource (books, notes, videos, etc.)
  - Review materials on iLearn (ignore last problem in sample exam)

- **Video**: Almost finished proof of:

Theorem: If

$$f = \underbrace{(a_1, \dots, a_k)}_{\alpha} \circ \underbrace{(b_1, \dots, b_\ell)}_{\beta}$$

where α and β are disjoint cycles, then

$$\operatorname{ord}(f) = \operatorname{lcm}(k, \ell).$$

- In fact, this theorem works for compositions of more than two cycles, e.g.

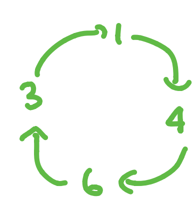
$$\operatorname{ord}\Big((1,4,2)(5,6)(3,8,9,7)\Big)$$

$$= \operatorname{lcm}(3, 2, 4) \quad = 12$$

- This gives a way to calculate the order of <u>any</u> element in S_n.

- **Ex**: $(1,4,6,3) \in S_6$

$$\operatorname{ord}\big((1,4,6,3)\big) = 4$$



Worksheet 11: Orders of Elements in the Symmetric Group

Math 335

Reporter:

Recorder:

Equity Manager:

Our goal for this worksheet is to prove that if

This worksheet: $\mathrm{ord}(f) \mid \mathrm{lcm}(k,\ell)$

Video: $\mathrm{lcm}(k,\ell) \mid \mathrm{ord}(f)$

$$f = \alpha \circ \beta,$$

where α and β are disjoint cycles in S_n of lengths k and ℓ, respectively, then

$$\mathrm{ord}(f) \mid \mathrm{lcm}(k, \ell).$$

(With what we proved in the video for today's class, this will imply that $\boxed{\mathrm{ord}(f) = \mathrm{lcm}(k, \ell).)}$

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1. First, convince yourself that if m is any power, then

$$f^m = \alpha^m \circ \beta^m.$$

(Which of the given assumptions are you using here?)

$$f^m = (\alpha \circ \beta) \circ (\alpha \circ \beta) \circ \dots \circ (\alpha \circ \beta)$$
$$= (\alpha \circ \dots \circ \alpha) \circ (\beta \circ \dots \circ \beta)$$
$$= \alpha^m \circ \beta^m$$

using that α and β are disjoint, so they commute

2. Now, suppose specifically that $m = \mathrm{lcm}(k, \ell)$. What does α^m equal? (**Hint**: Remember that m is a multiple of both k and ℓ.)

$$\alpha^m = e$$

(because m is a multiple of k
$\Rightarrow k \mid m$
$\Rightarrow \mathrm{ord}(\alpha) \mid m$)

3. Similarly, if $m = \mathrm{lcm}(k, \ell)$, what does β^m equal?

$$\beta^m = e$$

$$\left(\begin{array}{l} \text{because } m \text{ is a multiple of } \ell \\ \implies \ell \mid m \\ \implies \mathrm{ord}(\beta) \mid m \end{array} \right)$$

4. Combine the results of Problems 1–3 to compute f^m (where m still stands for $\mathrm{lcm}(k, \ell)$).

$$\begin{aligned} f^m &= \alpha^m \circ \beta^m \\ &= e \circ e \\ &= e \end{aligned}$$

5. Apply a theorem we learned recently to conclude that $\mathrm{ord}(f) \big| m$, which was our goal.

By theorem from Monday,

$$f^m = e$$

$$\implies \mathrm{ord}(f) \mid m. \qquad \boxtimes$$

Sanity Check: Try tracing through what you've just done for a specific f, like

$$f = (1, 4, 3) \circ (2, 5).$$

In this case, $k = 3$ and $\ell = 2$, so $\mathrm{lcm}(k, \ell) = 6$. Try writing out f^6 and then rearranging it so all the $(1, 4, 3)$'s are grouped together and all the $(2, 5)$'s are grouped together. What happens?

(See below)

$$
\begin{aligned}
f^6 &= (1,4,3)(2,5) \circ (1,4,3)(2,5) \circ (1,4,3)(2,5) \\
&\quad \circ (1,4,3)(2,5) \circ (1,4,3)(2,5) \circ (1,4,3)(2,5) \\
&= \overbrace{(1,4,3)(1,4,3)(1,4,3)}^{=e}\overbrace{(1,4,3)(1,4,3)(1,4,3)}^{=e} \\
&\quad \circ \underbrace{(2,5)(2,5)}_{=e}\underbrace{(2,5)(2,5)}_{=e}\underbrace{(2,5)(2,5)}_{=e} \\
&= e
\end{aligned}
$$