Equivalence Relations

- **Reminders:** • Exam 2 on Wed 4/7 (review materials on iLearn by tomorrow)
  - No class/OH Wednesday
  - This week's HW due Friday

- **Video:** An equivalence relation on a set S is a "notion of equivalence", denoted $a \sim b$.

(Prototypical example: $S = \mathbb{Z}$ and $a \sim b$ means $a \equiv b \mod 3$)

-3 -2 -1 0 1 2 3 4 5 6 7

(blue) = things ~ 0
(orange) = things ~ 1
(green) = things ~ 2

Worksheet 25: Equivalence Relations

Math 335

Reporter:

Recorder:

Equity Manager:

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1. Let $G = \mathbb{Z}_{12}$ (which is a group under addition modulo 12), and let

$$H = \{0, 3, 6, 9\}.$$

Define an equivalence relation $\sim$ on G by

$$a \sim b \quad \Leftrightarrow \quad b - a \in H.$$

For example, $1 \sim 7$ because $7 - 1 = 6 \in H$, and $11 \sim 2$ because $2 - 11 = 3 \in H$. (The fact that $2 - 11 = 3$ is because we're in $\mathbb{Z}_{12}$.)

(a) Find all the elements of [1].

$$[1] = \{ b \in \mathbb{Z}_{12} \mid b \sim 1 \}$$

$$= \{1, 4, 7, 10\}$$

$4 \sim 1$ because
$4 - 1 = 3 \in H$
$1 - 4 = 9 \in H$

(b) Find all the elements of [2].

$$[2] = \{ b \in \mathbb{Z}_{12} \mid b \sim 2 \}$$

$$= \{2, 5, 8, 11\}$$

$5 \sim 2$ because
$5 - 2 = 3 \in H$
$2 - 5 = 9 \in H$

(c) Do you recognize the sets showing up in parts (a) and (b)?

$$[1] = 1 + H$$

$$[2] = 2 + H$$

2. Generalizing Problem 1, let G be any group and let $H \subseteq G$ be a subgroup. Define a relation on G by

$$a \sim b \quad \Leftrightarrow \quad a^{-1}b \in H.$$

Let's try to prove that $\sim$ really is an equivalence relation. That is, for any $a, b, c \in G$, prove:

Reflexive (a) $a \sim a$

This means $a^{-1}a \in H$, which is true because H is a subgroup so $e \in H$.

Symmetric (b) $a \sim b \Rightarrow b \sim a$

Suppose $a \sim b$

$\Longrightarrow a^{-1}b \in H$

$\Longrightarrow (a^{-1}b)^{-1} \in H$ ← (because H is a subgroup so it contains inverses)

$\Longrightarrow b^{-1}a \in H$, i.e., $b \sim a$.

Transitive (c) $\left(a \sim b \text{ and } b \sim c\right) \Rightarrow a \sim c.$

Suppose $a \sim b$ and $b \sim c$

$\Longrightarrow a^{-1}b \in H$ and $b^{-1}c \in H$

$\Longrightarrow a^{-1}\cancel{b}\cancel{b^{-1}}c \in H$ ← (because H is a subgroup so it's closed)

$\Longrightarrow a^{-1}c \in H$, i.e., $a \sim c$.

3. **Challenge**: Let $\sim$ be the equivalence relation from Problem 2, and let $a \in G$. I claim that $[a]$ is something we've met before...what is it? Can you prove your answer?

$[a] = aH$

(Proof next video)