Orders in Cyclic Groups

- **Recall**: • G is cyclic if $G = \langle g \rangle$ for some $g \in G$
  ("every element of G is a power of one single element")
  - **Goal**: If $G = \langle g \rangle$, then
    $\mathrm{ord}(g^k) = \ldots\ldots$ fill this in!
  - **Step 1**:
    $\mathrm{ord}(g) = \#$ of elements in G
  - **Step 2**:
    "Lemma": $\langle g^k \rangle = \langle g^{\gcd(n,k)} \rangle$ $n = \#$ of elements in G

Worksheet 17: Orders of Elements in Cyclic Groups

Math 335

Reporter:

Recorder:

Equity Manager:

Suppose that g is a complex number such that

$$g^6 = 1$$

but no smaller positive power of g equals 1. (Trust me, there are complex numbers like this, though their expressions aren't so simple.) Let

$$G = \{1, g, g^2, g^3, g^4, g^5\},$$

which forms a cyclic group under multiplication of complex numbers.

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1. Write down all of the elements in the subgroup generated by each element of G. That is, fill in the following:

$$\langle 1 \rangle = \{1\}$$

$$\langle g \rangle = \{1, g, g^2, g^3, g^4, g^5\}$$

$$\langle g^2 \rangle = \{1, g^2, g^4\}$$

$$\langle g^3 \rangle = \{1, g^3\}$$

$$\langle g^4 \rangle = \{1, g^4, g^2\}$$

because $g^8 = \cancel{g^6} g^2 = g^2$

$g^{12} = \cancel{g^6} \cdot \cancel{g^6} = 1$

$$\langle g^5 \rangle = \{1, g^5, g^4, g^3, g^2, g\}$$

because $g^{10} = \cancel{g^6} \cdot g^4 = g^4$

2. Are your answers to Problem 1 consistent with the lemma from today's video?

Yes : $\langle g^4 \rangle = \langle g^{\gcd(6,4)} \rangle = \langle g^2 \rangle$

$\langle g^k \rangle = \langle g^{\gcd(6,k)} \rangle$

$\langle g^5 \rangle = \langle g^{\gcd(6,5)} \rangle = \langle g^1 \rangle$

3. Your answer to Problem 1 can be interpreted as a calculation of the order of every element in G. For example, since

$$\langle g^2 \rangle = \{1, g^2, g^4\}$$

and we know that

$$\text{ord}(g^2) = \# \text{ of elements in } \langle g^2 \rangle,$$

we see that $\text{ord}(g^2) = 3$. Repeat this reasoning to calculate the order of each element in G.

$\text{ord}(1) = 1$

$\text{ord}(g) = 6$

$\text{ord}(g^2) = 3$

$\text{ord}(g^3) = 2$

$\text{ord}(g^4) = 3$

$\text{ord}(g^5) = 6$

4. Based on your calculations, can you guess a formula for the order of any element in G? Once you have a guess, check to make sure it agrees with your calculations in Problem 3.

$$\text{ord}(g^k) = \frac{6}{\gcd(6,k)}$$

$$\left(\text{E.g. } \text{ord}(g^2) = \frac{6}{\gcd(6,2)} = \frac{6}{2} = 3\right)$$

5. **Vague Thought Question**: This might feel weirdly similar to what you did on Worksheet 9, where you calculated orders of elements in $\mathbb{Z}_6$. Can you see a relationship between orders of elements of the group G from this worksheet and orders of elements of the group $\mathbb{Z}_6$?

$g^k \in G$ "behaves the same" as $k \in \mathbb{Z}_6$

(We'll make this more precise next week!)