

## The First Isomorphism Theorem

### - Announcements:

- Math Dept Graduation Ceremony 5/24 - fill out survey by Monday if graduating!
- Q&A for Math Dept Scholarships today 3-4pm

### - Video: Let

$$\psi: G \rightarrow H$$

be a homomorphism and let  $K = \ker(\psi)$ .

Then

$$\boxed{G/K \cong \text{im}(\psi)}.$$

- This is a super useful way to prove two groups are isomorphic without necessarily finding an isomorphism!

- Comprehension Question: If  $\psi: G \rightarrow H$  is a surjective homomorphism, then

$$G/\ker(\psi) \cong H.$$

# Worksheet 33: The First Isomorphism Theorem

## Math 335

Reporter:

Recorder:

Equity Manager:

1. Consider the homomorphism

$$\begin{aligned}\varphi : \mathbb{R}^* &\rightarrow \mathbb{R}^* \\ \varphi(x) &= |x|,\end{aligned}$$

where  $\mathbb{R}^*$  is a group under multiplication and  $|x|$  stands for the absolute value of  $x$ .

(a) What is  $\ker(\varphi)$ ? What is  $\text{im}(\varphi)$ ?

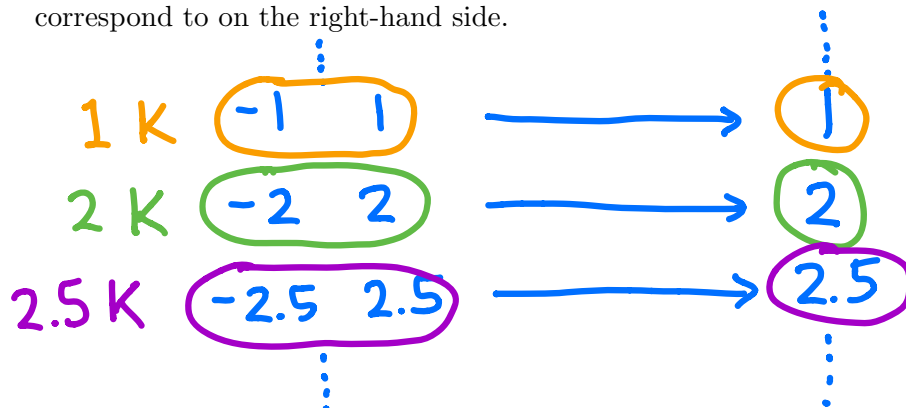
$$\ker(\varphi) = \{1, -1\}$$

$$\text{im}(\varphi) = \mathbb{R}^+$$

(b) What does the First Isomorphism Theorem say in this case? In other words, fill in the blanks with the appropriate groups in this example:

$$\underline{\mathbb{R}^*} / \underline{\{1, -1\}} \cong \underline{\mathbb{R}^+}.$$

(c) Write down a few different elements of the left-hand side of part (b), and what they correspond to on the right-hand side.



(d) If you wanted to “list” all the elements of the quotient group  $\mathbb{R}^*/\ker(\varphi)$ , with each element listed only once, what would that list look like? Discuss with each other why that list would look the same as a list of the elements in  $\text{im}(\varphi)$ .

Elements of  $\mathbb{R}^*/K$  are  $aK = \{a, -a\}$ ,  
so there's one for each  $a \in \mathbb{R}^+$ .

2. Consider the homomorphism

$$\varphi : \mathbb{Z} \rightarrow \mathbb{Z}_5$$

$$\varphi(x) = x \text{ reduced modulo } 5,$$

where  $\mathbb{Z}$  is a group under addition and  $\mathbb{Z}_5$  is a group under addition modulo 5.

(a) What is  $\ker(\varphi)$ ? What is  $\text{im}(\varphi)$ ?

$$\ker(\varphi) = \{\text{multiples of } 5\} = 5\mathbb{Z}$$

$$\text{im}(\varphi) = \mathbb{Z}_5$$

(b) What does the First Isomorphism Theorem say in this case?

$$\mathbb{Z}/5\mathbb{Z} \cong \mathbb{Z}_5$$

3. **Vague discussion questions:** What does the statement of the First Isomorphism Theorem become if  $\varphi : G \rightarrow G$  is the function

$$\varphi(x) = x?$$

How about if  $\varphi : G \rightarrow H$  is an isomorphism?

For  $\varphi(x) = x$ ,

$$\ker(\varphi) = \{e\}$$

$$\text{im}(\varphi) = G,$$

so F.I.T. says

$$G/\{e\} \cong G.$$

(Similarly for  $\varphi$  an isomorphism.)