Math 335, Topic #2 Final Project Rough Draft

Due Wednesday, May 12

Topic 2 is an exploration of the *Orbit-Stabilizer Theorem* and *Symmetries*. Lagrange's Theorem and the Orbit-Stabilizer Theorem are both counting theorems which enable us to find the quantity of elements in different sets, the latter of which allows us to find the number of elements in any finite group of permutations of a set. In fact, Lagrange's Theorem provides us with a method for proving the Orbit-Stabilizer Theorem. Armed with an understanding of the Orbit-Stabilizer Theorem, we find that it can accurately count the number of symmetries of different polyhedra. Specifically, we explored the cube, tetrahedron, octahedron, dodecahedron, and truncated icosehedron. In this paper, I will discuss the Orbit-Stabilizer Theorem, and how it relates to our understanding of cosets and symmetries of the aforementioned polyhedra.

To understand this subject, we must establish a pair of definitions upon which the Orbit-Stabilizer Theorem is built. The stabilizer of a point is defined in Gallian's *Contemporary Abstract Algebra* as follows:

> Let G be a group of permutations of a set S. For each i in S, let $\text{stab}_G(i) = \{\phi \in G \mid \phi(i) = i\}$. We call $\text{stab}_G(i)$ the *stabilizer* of i in G. (152)

Before getting to the definition of "stabilizer", you should say a few words about what we mean by "group of permutations of a set S." By definition, that means G is a set whose elements are bijections $S \to S$, and which forms a group under the operation of composition.

What this first definition describes is the set of elements in G which leave i fixed. To illustrate this, consider a cube with 6 faces–let us refer to this group as C and a single fixed face as f. If we fix one of the faces in place, the stabilizer is the set of facings possible while maintaining the fixed position. In this case, there are 4 different possible facings of the cube as shown in Figure 1 below.

which group?

Beautiful picture!

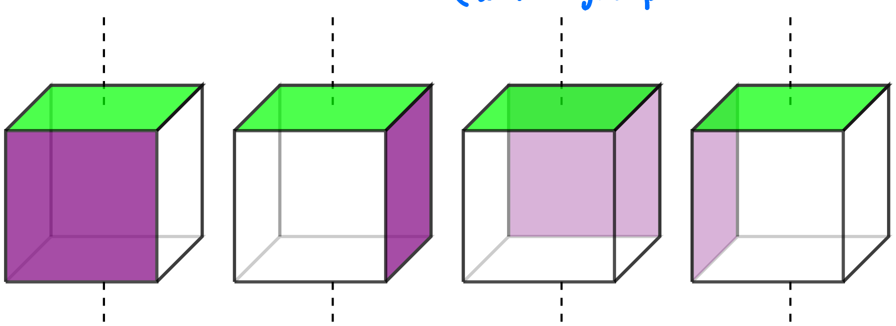

Figure 1: 4 possible facings of a cube with side 1 fixed in place ($|\text{stab}_C(f)| = 4$)

Similarly, we can find the number of elements for a particular face of a tetrahedron–we will call this group T. Fixing the bottom face, a symmetry can be found by composing 120 degree rotations, giving us $\text{stab}_T(f) = \{I, R_{120}, R_{240}\}$ and $|\text{stab}_T(f)| = 3$. The orbit of a point is also defined by Gallian:

> Let G be a group of permutations of a set S. For each s in S, let $\text{orb}_G(s) = \{\phi(s) \mid \phi \in G\}$. The set $\text{orb}_G(s)$ is a subset of S called the *orbit of s under G*. (Gallian 152)

This definition categorizes ~~the~~ all the positions that a particular element in a set S can occupy within the group G. Using our previous example of a cube, the orbit of a face is the set of locations the face can occupy. This is illustrated in Figure 2.

This phrasing is a little misleading—it makes me think S is "within" (a subset of) G, which it's not. I'd phrase the orbit intuitively as something like "all the elements of S to which a particular element can be sent by G."

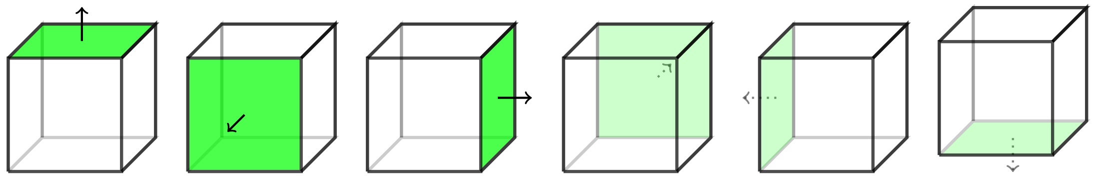

Figure 2: 6 possible locations that each face can occupy ($|\operatorname{orb}_C(f)| = 6$)

Again, the same can be found for a tetrahedron, resulting in $|\operatorname{orb}_T(f)| = 4$.

Great work so far! Your writing is very clear and pleasant to read, and your figures are great.

Be careful to define "C" and "T" precisely. E.g.:

$C = \{$rotational symmetries of a cube$\}$.

Each of these can be viewed as a permutation of the faces of the cube, so we can think of C as a group of permutations of the set

$S = \{$faces of the cube$\}$.

I hope that helps!