Isomorphisms

- **Video**: Definition of isomorphism $\varphi : G \to H$.

- **Idea**: An isomorphism is a perfect "dictionary", e.g.

$$G = \{a, b, c, d\}$$

$$H = \{\alpha, \beta, \gamma, \delta\}$$

$\varphi(a) = \alpha$
$\varphi(b) = \beta$
$\varphi(c) = \gamma$
$\varphi(d) = \delta$

And that dictionary respects the operation! E.g. if

$$a * b = d \quad \text{in } G$$

then

$$\alpha * \beta = \delta \quad \text{in } H.$$

$$\boxed{\underbrace{\varphi(a * b)}_{\substack{\varphi(d) \\ = \delta}} = \underbrace{\varphi(a) * \varphi(b)}_{\alpha * \beta}}$$

- **Theorem**: If $\varphi : G \longrightarrow H$ is an isomorphism, then φ sends the identity in G to the identity in H.

Proof: Let

e_G = identity in G

e_H = identity in H.

Then

$$e_H *_H \varphi(\cancel{e_G}) = \varphi(e_G *_G e_G) = \varphi(e_G) *_H \varphi(\cancel{e_G})$$

$$\Longrightarrow e_H = \varphi(e_G) \quad \text{(by cancellation)}.$$

$\boxtimes$

Worksheet 20: Isomorphisms

Math 335

Reporter:

Recorder:

Equity Manager:

You may remember that the sample midterm involved a group $G = \{a, b, c, d\}$, where the group operation was defined by the following table:

	a	b	c	d
a	d	c	a	b
b	c	d	b	a
c	a	b	c	d
d	b	a	d	c

I claim that there's an isomorphism

$$\varphi : G \to \mathbb{Z}_4,$$

where $\mathbb{Z}_4 = \{0, 1, 2, 3\}$ is a group under addition modulo 4. Let's try to cook up that isomorphism piece by piece.

1. We've just learned that an isomorphism must send the identity element of one group to the identity element of the other. Use this to figure out what $\varphi(c)$ must be.

$$\boxed{\varphi(c) = 0}$$

(c is the identity in G because $c * x = x \ \forall x$)

2. How about $\varphi(d)$? Notice that

$$d * d = c,$$

according to the table. Apply φ to both sides of this equation, and use the definition of isomorphism to figure out what $\varphi(d)$ must be.

$$\varphi(d * d) = \varphi(c)$$

$$\Longrightarrow \varphi(d) + \varphi(d) = \varphi(c)$$

$$\Longrightarrow \varphi(d) + \varphi(d) = 0 \quad \Longrightarrow \boxed{\varphi(d) = 2}$$

3. The last step is to figure out $\varphi(a)$ and $\varphi(b)$. Make a random choice for these, remembering that φ needs to be a bijection.

E.g. try
$\varphi(a) = 1$
$\varphi(b) = 3$
$\varphi(c) = 0$
$\varphi(d) = 2$

(in fact, either choice works)

4. Now test to see whether your random choice worked. That is, if φ is going to be an isomorphism, you need

$$\varphi(x * y) = \varphi(x) + \varphi(y)$$

for all $x, y \in G$. Pick some specific values of x and y and see if this equation holds.

E.g. $\varphi(a * b) \stackrel{?}{=} \varphi(a) + \varphi(b)$
$\varphi(c) \stackrel{\checkmark}{=} 1 + 3$

$\varphi(a * d) \stackrel{?}{=} \varphi(a) + \varphi(d)$
$\varphi(b) \stackrel{\checkmark}{=} 1 + 2$

5. In the space below, copy over the table for the group G, but replacing each a with $\varphi(a)$, each b with $\varphi(b)$, and so on. For example, if you chose that $\varphi(a) = 1$, you would replace all the a's in the table with 1's.

	a	b	c	d
a	d	c	a	b
b	c	d	b	a
c	a	b	c	d
d	b	a	d	c

	1	3	0	2
1	2	0	1	3
3	0	2	3	1
0	1	3	0	2
2	3	1	2	0

What do you notice about this new table?

It <u>is</u> the group table for $\mathbb{Z}_4$, just with rows/columns in a weird order.