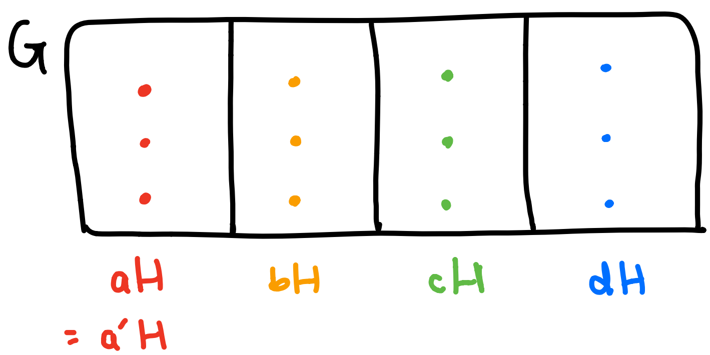
Cosets, continued

- **Note:** Next extra credit posted (on HW7), due 4/7 (the day of our second exam).

- **Video:** Let G be a group and let $H \subseteq G$ be a subgroup. Then:



These cosets are:

① disjoint if they're not equal

② all of size $|H|$ ← # of elements in H

- **Ex:** $G = \mathbb{Z}$

$H = \{0, 3, 6, 9, 12, \ldots, -3, -6, -9, -12, \ldots\}$

$0 + H$

$1 + H$

$2 + H$

Worksheet 23: Cosets, Continued

Math 335

Reporter:

Recorder:

Equity Manager:

Illustration of this example:

$\mathbb{Z}_{12}$

(0+H)	(1+H)	(2+H)
0	1	2
3	4	5
6	7	8
9	10	11

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1. Let $G = \mathbb{Z}_{12}$, which is a group under addition modulo 12, and let H be the subgroup

$$H = \{0, 3, 6, 9\}.$$

(a) What are all the left cosets of H in G? If two are the same (for instance, you should find that $0 + H = 3 + H$) then you only need to write it once.

$$\boxed{\begin{aligned} 0+H &= \{0,3,6,9\} \\ 1+H &= \{1,4,7,10\} \\ 2+H &= \{2,5,8,11\} \end{aligned}}$$

$= 3+H = 6+H = 9+H$

$= 4+H = 7+H = 10+H$

$= 5+H = 8+H = 11+H$

(b) How many distinct left cosets of H in G are there? How many elements does each one have?

3 distinct left cosets

4 elements in each (because $|H| = 4$)

(c) Pause for a moment and think about the relationship (in this example) between the three numbers $|G|$, $|H|$, and # of distinct left cosets of H in G.

$|G| = |H| \cdot$ (# of distinct left cosets of H in G)

$12 = 4 \cdot 3$

What you've just done is an illustration of...

Lagrange's Theorem: Let G be a finite group, and let $H \subseteq G$ be a subgroup. Then the number of distinct left cosets of H in G equals $|G|/|H|$.

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2. In this problem, you'll prove Lagrange's Theorem. There are three ingredients:

   (a) **Ingredient #1**: Every element of G is in some left coset aH. Why is this true?

   E.g. in #1, $7 \in 7+H$

   For any $g \in G$,

   $g = g \cdot e$ in H because H is a subgroup

   $\Rightarrow g \in gH$

   (b) **Ingredient #2**: No element of G is in more than one left coset. Why is this true?

   Distinct left cosets are disjoint, by theorem from today's video.

   (c) **Ingredient #3**: The number of elements in any left coset is equal to $|H|$. Why is this true?

   Theorem from today's video.

   (d) Let r denote the number of distinct left cosets of H in G. Use the above three ingredients to convince yourself that

   $$|G| = \underbrace{|H| + |H| + \cdots + |H|}_{r \text{ times}},$$

   and from here, to conclude that Lagrange's Theorem is true.

   If $a_1H, a_2H, \ldots, a_rH$ are the distinct left cosets of H in G, then

   $G = a_1H \cup \cdots \cup a_rH$ (Ing. #1)

   $\Rightarrow |G| = |a_1H| + \cdots + |a_rH|$ (Ing. #2)

$$\implies |G| = |H| + \cdots + |H| \quad \text{(Ing. \#3)}$$

$$\implies |G| = r \cdot |H|$$

$$\implies r = |G| / |H| .$$

$\boxtimes$