Midterm Review

- **Logistics**:
  - Exam Wednesday, any 90 mins
  - Use any non-human resource
  - No class on Wednesday
  - Office hours:
    - Today 10-11am
    - Tomorrow 1-2pm
    - (· Not on Wednesday)

- **Topics**:
  - Modular arithmetic
  - Symmetries of a square
  - Definition/properties/examples of groups
  - S_n (injective/surjective, cycle notation, composition, order)
  - Orders of elements (definition/properties/examples)
  - Subgroups (definition/examples)

- **Sample Exam**: Ignore Problem 5

Math 335, Midterm 1

October 4, 2019

Name: ______________________________

Problem	Points Scored	Total Points Possible
1		8
2		8
3		8
4		8
5		8
Total		40

Problem 1: You do not need to show any work or give justifications on this problem.

(a) Fill in the blank with a number in the range $\{0, 1, 2, 3, 4, 5\}$:

$$-7 \equiv \underline{\quad 5 \quad} \mod 6.$$

$(-7 \equiv -1 \equiv 5 \bmod 6)$

(b) We've seen that the set $\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$ is a group under the operation of addition modulo 6. In this group, what is the inverse of 2?

4

(because $2+4=0$, $4+2=0$ ← identity)

(c) Give an example of a proper, nontrivial subgroup of $\mathbb{Z}_6$. (Write out your subgroup explicitly, by listing all of its elements.)

not $\{0\}$

not $\{0,1,2,3,4,5\}$

$\{0, 2, 4\}$ or $\{0, 3\}$

(d) Give an example of a subset of $\mathbb{Z}_6$ that is *not* a subgroup.

$\{0, 1\}$

(Not closed because contains 1 but not $1+1$)

Problem 2: Which of the following are groups? In each case, circle "group" or "not a group"; you do not need to justify your answer.

(a) $G = \{0, 2\}$, where the operation is addition modulo 4.

group　　　　not a group

	0	2
0	0	2
2	2	0

(b) $G = \{1, 2, 3\}$, where the operation is multiplication modulo 4.

group　　　　not a group

Not closed because $2 \in G$ but $2 \cdot 2 = 0$ isn't in G

(c) $G = \{\text{nonzero integers}\}$, where the operation is multiplication.

group　　　　not a group

Doesn't contain inverses, e.g. inv. of $2 = \frac{1}{2} \notin G$

(d) $G = \{\text{rotational symmetries of a square}\} = \{I, R_{90}, R_{180}, R_{270}\}$, where the operation is composition.

group　　　　not a group

	I	R_{90}	R_{180}	R_{270}
I	I	R_{90}	R_{180}	R_{270}
R_{90}	R_{90}	R_{180}	R_{270}	I
R_{180}	R_{180}	R_{270}	I	R_{90}
R_{270}	R_{270}	I	R_{90}	R_{180}

Problem 3: Suppose G is a group with four elements,

$$G = \{a, b, c, d\},$$

and the group table for G is the following:

	a	b	c	d
a	d	c	a	b
b	c	d	b	a
c	a	b	c	d
d	b	a	d	c

$a*d=b$

$b*d=a$

$c*b=b$

$c*a=a$

In other words, the entry in the top-right corner means that $a * d = b$.

(a) What is the identity element of G? Briefly explain how you know.

c, because

$$c*a = a*c = a$$
$$c*b = b*c = b$$
$$c*c = c*c = c$$
$$c*d = d*c = d$$

(b) What is the inverse of a? Briefly explain how you know.

b, because

$$a*b = b*a = c$$

and c is the identity

(c) Is the operation on G commutative—that is, is G abelian? Briefly explain how you know.

Yes, because the table is symmetric across the diagonal (i.e., $g*h = h*g \ \forall g, h \in G$).

Problem 4: You do not need to show any work or give justifications on this problem.

(a) Carefully define what it means to say that a function $f : A \to B$ is **injective** (or **one-to-one**).

$f: A \to B$ is <u>injective</u> if $\forall a_1, a_2 \in A$,

$$f(a_1) = f(a_2) \Rightarrow a_1 = a_2.$$

(b) In the symmetric group S_5, let

$$\alpha = (1, 2, 3, 5), \quad \beta = (1, 3, 4).$$

What is $\alpha \circ \beta$? Express your answer in cycle notation, as a composition of disjoint cycles.

$$(1,2,3,5) \circ (1,3,4) = \boxed{(1,5)(2,3,4)}$$

$1 \to 3 \to 5$
$2 \to 2 \to 3$
3
4
5

(c) In the symmetric group S_6, what is the order of

$$f = (1, 3)\ (2, 6, 5, 4)?$$

$$\text{ord}(f) = \text{lcm}(2,4) = \boxed{4}$$

(d) If f is the same permutation as in part (c), what is f^8?

$$f^8 = (f^4)^2 = e^2 = \boxed{e}$$

SKIP THIS PROBLEM — HAVEN'T COVERED THIS MATERIAL YET

Problem 5:

(a) Carefully define what it means for a group G to be **cyclic**.

(b) Let

$$U_4 = \{1, -1, i, -i\} \subseteq \mathbb{C}^*,$$

which is a group under multiplication of complex numbers. Explain how you know that U_4 is cyclic.

(c) Let

$$\mathbb{Z}_5 = \{0, 1, 2, 3, 4\},$$

which is a group under addition modulo 5. In this group, what is $\langle 3 \rangle$? List all of its elements.