Math 335, Exam 1

February 24, 2021

Name: Mark S. Kim

Problem	Points Scored	Total Points Possible
1		8
2		8
3		8
4		8
5		8
Total		40

Problem 1: You do not need to show any work or give justifications on this problem.

(a) The set $\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$ forms a group under the operation of addition modulo 5. In this group, what is the inverse of 3?

2 because $2+3=0$ & $3+2=0$

(b) The set $\{1, 2, 3, 4\}$ forms a group under the operation of *multiplication* modulo 5. In this group, what is the inverse of 3?

2 because $3 \cdot 2 = 1 \bmod 5$ & $2 \cdot 3 = 1 \bmod 5$

(c) True or false: for any positive integer $n \geq 2$, the set

$$\mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\}$$

forms a group under addition modulo n.

True

(d) True or false: for any positive integer $n \geq 2$, the set

$$\{1, 2, \ldots, n-1\}$$

forms a group under multiplication modulo n.

False $n=4 \Rightarrow \{1,2,3\}$

	1	2	3
1	1	2	3
2	2	0	
3	3		

Problem 2:

(a) Is the set

$$G = \{1, 3, 7, 9\}$$

a group under the operation of multiplication modulo 10? If so, give a brief proof (you don't need to prove associativity). If not, explain why not.

	1	3	7	9
1	1	3	7	9
3	3	9	1	7
7	7	1	9	3
9	9	7	3	1

0. closed: See table ✓
1. associative: assumed ✓
2. identity: $e = 1$, see table ✓
3. inverse: see table ✓

$g \cdot h = e$ & $h \cdot g = e$

$\forall\, g, h \in G$

Yes! G is a group under the operation of multiplication mod 10!

(b) Is the set

$$G = \{\text{nonzero odd integers}\} = \{1, 3, 5, 7, \ldots, -1, -3, -5, -7, \ldots\}$$

a group under the operation of multiplication? If so, give a brief proof (you don't need to prove associativity). If not, explain why not.

0. closed: Yes! Any nonzero odd integer multipled to another nonzero odd integer is an odd integer

$$(2a+1)(2b+1) = 4ab + 2a + 2b + 1$$
$$= \underbrace{2(2ab+a+b)}_{\text{even}} + 1 = \text{odd}$$

1. associative: assumed
2. identity: $e = 1$; $1 \cdot a = a$, $a \cdot 1 = a$ $\forall a \in G$
3. inverses: $a \cdot \frac{1}{a} = 1$, but $\frac{1}{a} \notin G$

$\therefore$ not every element has an inverse

No! G is NOT a group under the operation of multiplication because

Problem 3:

(a) Give a specific example to show that the group

$$D_4 = \{\text{symmetries of a square}\}$$

is not abelian.

For a group to be abelian, it must be commutative. But notice from the table of symmetries of a square that

$$H \circ R_{90} = D \quad \text{and} \quad R_{90} \circ H = D',$$

which means $\boxed{H \circ R_{90} \neq R_{90} \circ H.}$

(b) If $a, b, c \in D_4$ satisfy

$$a \circ b = a \circ c,$$

is it necessarily the case that $b = c$? Briefly explain how you know.

Yes. Since D_4 is a group, the cancellation property applies.

$$a \circ b = a \circ c$$

$$(a^{-1} \circ a) \circ b = (a^{-1} \circ a) \circ c \quad \text{(inverses \& associativity)}$$

$$e \circ b = e \circ c \quad \text{(identity)}$$

$$b = c$$

(c) Give an example of a proper, nontrivial subgroup of D_4.

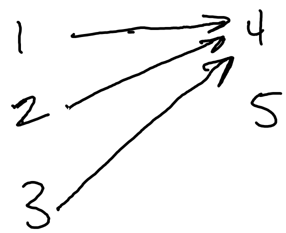
$$\{I, R_{90}, R_{180}, R_{270}\}$$

Problem 4: You do not need to show any work or give justifications on this problem.

(a) Give an example of a function

$$f : \{1, 2, 3\} \to \{4, 5\}$$

that is neither injective nor surjective. (A picture with dots and arrows is fine.)



(b) In the symmetric group S_5, calculate

$$g = (1, 5, 2) \circ (1, 3, 2, 4),$$

expressing your answer in cycle notation.

$g = (1,3)(2,4,5)$

(c) What is the order of the element g in part (b)?

$\operatorname{ord}(g) = \operatorname{lcm}(2,3) = 6$

(d) If g is as above, for which powers k do we have $g^k = e$?

$k = 6, 12, 18, \ldots$

$k = 6n, \quad n = 1, 2, 3, \ldots$

Problem 5: You do not need to show any work or give justifications on this problem.

(a) In the group $\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$ under the operation of addition modulo 6, what is the order of the element 2?

$$2 = 2 = 2$$

$$2 \cdot 2 = 2 + 2 = 4$$

$$3 \cdot 2 = 2 + 2 + 2 = 6 \equiv 0 \bmod 6$$

$$\therefore \text{ord}(2) = 3$$

(b) In the group $\mathbb{R}^* = \{\text{nonzero real numbers}\}$ under the operation of multiplication, give an example of an element whose order is ∞.

$2^1 = 2$ $\qquad$ $\text{ord}(2) = \infty$

$2^2 = 4$

$2^3 = 8$

$2^4 = 16$

$2^5 = 32$

etc..

(c) In the group $\mathbb{R}^* = \{\text{nonzero real numbers}\}$ under the operation of multiplication, give an example of an element whose order is finite.

$1 \qquad \text{ord}(1) = 1$

and

$-1 \qquad \text{ord}(-1) = 2$

(d) Give an example of a proper, nontrivial subgroup of $\mathbb{R}^*$ under the operation of multiplication.

$\{1, -1\}$

	-1	1
-1	1	-1
1	-1	1

closed: see table

assoc: ✓

identity: $e = 1$ ✓

inverse:

$1 \cdot 1 = 1$ ✓

$-1 \cdot -1 = 1$