Math 335, Homework 2

Due Wednesday, February 10

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1. Define a binary operation $*$ on $\mathbb{Z}$ by
$$a * b = 2a + 2b.$$
So, for example,
$$1 * 3 = 2 \cdot 1 + 2 \cdot 3 = 8.$$
Use a specific example to show that $*$ is *not* associative.

2. Let
$$G = \{5, 15, 25, 35\}.$$
Prove that G is a group under the operation of multiplication modulo 40. You can assume that multiplication is associative, but you should prove closure, the existence of an identity, and the existence of inverses. (**Hint**: Make a multiplication table.)

3. Let n be any positive integer, and let
$$U_n = \{z \in \mathbb{C} \mid z^n = 1\}.$$
(This is called the set of **nth roots of unity**.) For example,
$$U_2 = \{1, -1\}$$
$$U_4 = \{1, -1, i, -i\},$$
or, more weirdly,
$$U_3 = \left\{1,\ -\frac{1}{2} + \frac{\sqrt{3}}{2}i,\ -\frac{1}{2} - \frac{\sqrt{3}}{2}i\right\}.$$
(Don't worry, I'd never expect you to know that last one on your own!) Prove that, for all n, the set U_n is a group under the operation of multiplication. You can assume that multiplication is associative, but you should prove closure, the existence of an identity, and the existence of inverses.

4. Find an example of three elements $a, b, c \in D_4$ such that
$$b \circ a = a \circ c \quad \text{but} \quad b \neq c.$$
What does this tell you about the cancellation property in D_4?

In Problems 5, we use exponents to indicate doing the group operation repeatedly. That is, let G be a group with operation $*$ and let $a \in G$. Then we write a^2 to mean $a * a$, we write a^3 to mean $a * a * a$, and so on.

5. Let G be any group and let $a, b \in G$. Prove that $(a * b)^2 = a^2 * b^2$ if and only if $a * b = b * a$.