Quotient Groups, continued

- **Video**: If

$$G = \text{group}$$
$$H \subseteq G \text{ "normal" subgroup,}$$

always true if G is abelian

then

$$G/H = \left\{ \begin{array}{c} \text{left cosets of} \\ H \text{ in } G \end{array} \right\}$$

forms a group.

- **Another perspective on quotient groups:**

The group table for G/H is like the group table for G, arranged into "blocks".

Quotient Groups, continued (Example)

Math 335

Consider the group $G = \mathbb{Z}_9$, under addition modulo 9, and the subgroup $H = \{0, 3, 6\}$. There are three left cosets of H in G:

$$0 + H = \{0, 3, 6\}$$
$$1 + H = \{1, 4, 7\}$$
$$2 + H = \{2, 5, 8\},$$

so

$$G/H = \{0 + H, 1 + H, 2 + H\}.$$

Here is the group table for G/H:

	$0+H$	$1+H$	$2+H$
$0+H$	$0+H$	$1+H$	$2+H$
$1+H$	$1+H$	$2+H$	$0+H$
$2+H$	$2+H$	$0+H$	$1+H$

And here is the group table for G, arranged so that elements of the same coset are adjacent:

	0	3	6	1	4	7	2	5	8
0	0	3	6	1	4	7	2	5	8
3	3	6	0	4	7	1	5	8	2
6	6	0	3	7	1	4	8	2	5
1	1	4	7	2	5	8	3	6	0
4	4	7	1	5	8	2	6	0	3
7	7	1	4	8	2	5	0	3	6
2	2	5	8	3	6	0	4	7	1
5	5	8	2	6	0	3	7	1	4
8	8	2	5	0	3	6	1	4	7

Each block of the big table corresponds to one entry of the small table!

Worksheet 30: Quotient Groups, continued (Version 1)

Math 335

Reporter:

Recorder:

Equity Manager:

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1. Consider the group D_4 of symmetries of a square, whose elements are

$$D_4 = \{e, r_{90}, r_{180}, r_{270}, h, v, d, d'\},$$

and consider the subgroup

$$H = \{e, r_{180}\} \subseteq D_4.$$

Here's a group table for D_4, for your reference:

	e	r_{90}	r_{180}	r_{270}	h	v	d	d'
e	e	r_{90}	r_{180}	r_{270}	h	v	d	d'
r_{90}	r_{90}	r_{180}	r_{270}	e	d'	d	h	v
r_{180}	r_{180}	r_{270}	e	r_{90}	v	h	d'	d
r_{270}	r_{270}	e	r_{90}	r_{180}	d	d'	v	h
h	h	d	v	d'	e	r_{180}	r_{90}	r_{270}
v	v	d'	h	d	r_{180}	e	r_{270}	r_{90}
d	d	v	d'	h	r_{270}	r_{90}	e	r_{180}
d'	d'	h	d	v	r_{90}	r_{270}	r_{180}	e

(a) I claim that H is a normal subgroup of D_4. Check this in an example by seeing whether $aba^{-1} \in H$ for $a \in D_4$ and $b \in H$. For example:

$$h \circ r_{180} \circ h^{-1} = h \circ r_{180} \circ h$$
$$= h \circ v$$
$$= r_{180} \in H \quad \checkmark$$

(b) Here are the left cosets of H in D_4:

$$eH = \{e, r_{180}\}$$
$$r_{90}H = \{r_{90}, r_{270}\}$$
$$hH = \{h, v\}$$
$$dH = \{d, d'\}.$$

In the blank table below, re-write the group table for D_4, but this time arrange it so that elements of the same coset are adjacent.

	e	r_{180}	r_{90}	r_{270}	h	v	d	d'
e	e	r_{180}	r_{90}	r_{270}	h	v	d	d'
r_{180}	r_{180}	e	r_{270}	r_{90}	v	h	d'	d
r_{90}	r_{90}	r_{270}	r_{180}	e	d'	d	h	v
r_{270}	r_{270}	r_{90}	e	r_{180}	d	d'	v	h
h	h	v	d	d'	e	r_{180}	r_{90}	r_{270}
v	v	h	d'	d	r_{180}	e	r_{270}	r_{90}
d	d	d'	v	h	r_{270}	r_{90}	e	r_{180}
d'	d'	d	h	v	r_{90}	r_{270}	r_{180}	e

(c) Look at the Limnu Board of a group of classmates who has Version 2 of the worksheet, and compare their table with yours.

(b) Here are the left cosets of H in D_4:

$$eH = \{e, r_{180}\}$$
$$r_{90}H = \{r_{90}, r_{270}\}$$
$$hH = \{h, v\}$$
$$dH = \{d, d'\}.$$

In the blanks below, make a group table for D_4/H; in other words, your table should show how to compose any two of the above left cosets to get another one.

	eH	$r_{90}H$	hH	dH
eH	eH	$r_{90}H$	hH	dH
$r_{90}H$	$r_{90}H$	eH	dH	hH
hH	hH	dH	eH	$r_{90}H$
dH	dH	hH	$r_{90}H$	eH

$r_{90}H \circ r_{90}H$
$= r_{180}H$
$= eH$

(c) Look at the Limnu Board of a group of classmates who has Version 1 of the worksheet, and compare their table with yours.

2. Now consider the group S_3, and the subgroup

$$H = \{e, (1,2)\},$$

which we saw in the video for today's class is *not* normal. Here's a group table for S_3:

	e	$(1,2)$	$(1,3)$	$(2,3)$	$(1,2,3)$	$(1,3,2)$
e	e	$(1,2)$	$(1,3)$	$(2,3)$	$(1,2,3)$	$(1,3,2)$
$(1,2)$	$(1,2)$	e	$(1,3,2)$	$(1,2,3)$	$(2,3)$	$(1,3)$
$(1,3)$	$(1,3)$	$(1,2,3)$	e	$(1,3,2)$	$(1,2)$	$(2,3)$
$(2,3)$	$(2,3)$	$(1,3,2)$	$(1,2,3)$	e	$(1,3)$	$(1,2)$
$(1,2,3)$	$(1,2,3)$	$(1,3)$	$(2,3)$	$(1,2)$	$(1,3,2)$	e
$(1,3,2)$	$(1,3,2)$	$(2,3)$	$(1,2)$	$(1,3)$	e	$(1,2,3)$

And here are the left cosets of H in S_3:

$$eH = \{e, (1,2)\}$$
$$(1,3)H = \{(1,3), (1,2,3)\}$$
$$(2,3)H = \{(2,3), (1,3,2)\}.$$

In the blank table below, re-write the group table for S_3, but this time arrange it so that elements of the same coset are adjacent.

	e	(1,2)	(1,3)	(1,2,3)	(2,3)	(1,3,2)
e	e	(1,2)	(1,3)	(1,2,3)	(2,3)	(1,3,2)
(1,2)	(1,2)	e	(1,3,2)	(2,3)	(1,2,3)	(1,3)
(1,3)	(1,3)	(1,2,3)	e	(1,2)	(1,3,2)	(2,3)
(1,2,3)	(1,2,3)	(1,3)	(2,3)	(1,3,2)	(1,2)	e
(2,3)	(2,3)	(1,3,2)	(1,2,3)	(1,3)	e	(1,2)
(1,3,2)	(1,3,2)	(2,3)	(1,2)	e	(1,3)	(1,2,3)

Look at the Limnu Board of a group who has Version 2 of the worksheet, and discuss how the table you just made relates to their findings on Problem 2.

2. Now consider the group S_3, and the subgroup

$$H = \{e, (1,2)\},$$

which we saw in the video for today's class is *not* normal. Here's a group table for S_3, for your reference:

	e	$(1,2)$	$(1,3)$	$(2,3)$	$(1,2,3)$	$(1,3,2)$
e	e	$(1,2)$	$(1,3)$	$(2,3)$	$(1,2,3)$	$(1,3,2)$
$(1,2)$	$(1,2)$	e	$(1,3,2)$	$(1,2,3)$	$(2,3)$	$(1,3)$
$(1,3)$	$(1,3)$	$(1,2,3)$	e	$(1,3,2)$	$(1,2)$	$(2,3)$
$(2,3)$	$(2,3)$	$(1,3,2)$	$(1,2,3)$	e	$(1,3)$	$(1,2)$
$(1,2,3)$	$(1,2,3)$	$(1,3)$	$(2,3)$	$(1,2)$	$(1,3,2)$	e
$(1,3,2)$	$(1,3,2)$	$(2,3)$	$(1,2)$	$(1,3)$	e	$(1,2,3)$

(a) Here are the left cosets of H in S_3:

$$\begin{aligned} eH &= (1,2)H = \{e, (1,2)\} \\ (1,3)H &= (1,2,3)H = \{(1,3), (1,2,3)\} \\ (2,3)H &= (1,3,2)H = \{(2,3), (1,3,2)\}. \end{aligned}$$

Look back at Worksheet 29 to discuss how we know that composition of left cosets is not well-defined.

$$\begin{array}{ccccc} eH & \circ & (1,3)H & = & (1,3)H \\ \shortparallel & & & & \neq \\ (1,2)H & \circ & (1,3)H & = & (1,3,2)H \end{array}$$

(b) Turn to a group of your classmates who has Version 1 of the worksheet, and discuss how the table they made relates to your answer to part (a).