Homomorphisms

- **Video**: A homomorphism
$$\varphi : G \longrightarrow H$$
is like an isomorphism, except we don't require it to be a bijection.

$$\varphi(g * h) = \varphi(g) * \varphi(h)$$

Worksheet 31: Homomorphisms

Math 335

Reporter: INJECTIVE: Never have two inputs go to same output

Recorder:

Equity Manager: SURJECTIVE: Everything in codomain is φ(something)

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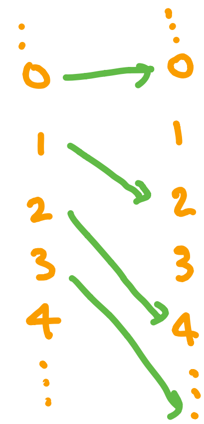
1. Below are the two examples of homomorphisms we saw in the video for today's class. Since they're homomorphisms and not isomorphisms, they don't need to be either injective or surjective.

   (a) Is

   $$\varphi : \mathbb{Z} \to \mathbb{Z}$$
   $$\varphi(x) = 2x.$$

   injective? Is it surjective?

   - Injective (if $2x = 2y$, then $x = y$)
   - Not surjective (odd #s aren't $\varphi(x)$ for any x)

   (b) Is

   $$\varphi : \mathbb{Z} \to \mathbb{Z}_5$$
   $$\varphi(x) = x \text{ reduced modulo } 5.$$

   injective? Is it surjective?

   - Not injective (e.g. $\varphi(0) = \varphi(5)$)
   - Surjective ($0 = \varphi(0)$, $1 = \varphi(1)$, ...)

2. For one more example of a homomorphism, let $\mathbb{R}^*$ be the group of nonzero real numbers (under multiplication), and let

$$\varphi : \mathbb{R}^* \to \mathbb{R}^*$$
$$\varphi(x) = x^2$$

(a) Prove that φ is a homomorphism.

$\varphi(x \cdot y) = (x \cdot y)^2 = x^2 \cdot y^2 = \varphi(x) \cdot \varphi(y)$

(b) Is φ injective? Is it surjective?

- Not injective (e.g. $\varphi(2) = \varphi(-2)$)
- Not surjective (negative #s aren't $\varphi(x)$ for any x)

3. By definition, the **kernel** of a homomorphism $\varphi : G \to H$ is the set of all elements in G that are sent by φ to the identity element $e \in H$. In other words,

$$\ker(\varphi) = \{g \in G \mid \varphi(g) = e\}.$$

Calculate the kernel of each of the homomorphisms in the above examples.

#1a : $\ker(\varphi) = \{x \in \mathbb{Z} \mid 2x = 0\} = \{0\}$

#1b : $\ker(\varphi) = \{x \in \mathbb{Z} \mid x \bmod 5 = 0\} = 5\mathbb{Z}$

multiples of 5

#2 : $\ker(\varphi) = \{x \in \mathbb{R}^* \mid x^2 = 1\} = \{1, -1\}$

Challenge: Do you see a relationship between your answer to Problem 3 and your answers to the previous problems?

φ is injective $\iff \ker(\varphi) = \{e_G\}$

identity in G