In this case, we have for all a and b in the group $ab = (ab)^{-1} = b^{-1}a^{-1} = ba$, so that G is Abelian. Then, for any nonidentity elements $a, b \in G$ with $a \neq b$, the set $\{e, a, b, ab\}$ is closed and therefore is a subgroup of G of order 4. Since this contradicts Lagrange's Theorem, we have proved that G must have an element of order p; call it a.

Now let b be any element not in $\langle a \rangle$. Then by Lagrange's Theorem and our assumption that G does not have an element of order $2p$, we have that $|b| = 2$ or p. Because $|\langle a \rangle \cap \langle b \rangle|$ divides $|\langle a \rangle| = p$ and $\langle a \rangle \neq \langle b \rangle$ we have that $|\langle a \rangle \cap \langle b \rangle| = 1$. But then $|b| = 2$, for otherwise, by Theorem 7.2 $|\langle a \rangle\langle b \rangle| = |\langle a \rangle||\langle b \rangle| = p^2 > 2p = |G|$, which is impossible. So, any element of G not in $\langle a \rangle$ has order 2.

Next consider ab. Since $ab \notin \langle a \rangle$, our argument above shows that $|ab| = 2$. Then $ab = (ab)^{-1} = b^{-1}a^{-1} = ba^{-1}$. Moreover, this relation completely determines the multiplication table for G. [For example, $a^3(ba^4) = a^2(ab)a^4 = a^2(ba^{-1})a^4 = a(ab)a^3 = a(ba^{-1})a^3 = (ab)a^2 = (ba^{-1})a^2 = ba$.] Since the multiplication table for all noncyclic groups of order $2p$ is uniquely determined by the relation $ab = ba^{-1}$, all noncyclic groups of order $2p$ must be isomorphic to each other. But of course, D_p, the dihedral group of order $2p$, is one such group. ∎

As an immediate corollary, we have that the non-Abelian groups S_3, the symmetric group of degree 3, and $GL(2, Z_2)$, the group of 2×2 matrices with nonzero determinants with entries from Z_2 (see Example 19 and Exercise 51 in Chapter 2) are isomorphic to D_3.

An Application of Cosets to Permutation Groups

Lagrange's Theorem and its corollaries dramatically demonstrate the fruitfulness of the coset concept. We next consider an application of cosets to permutation groups.

Definition Stabilizer of a Point

Let G be a group of permutations of a set S. For each i in S, let $\mathrm{stab}_G(i) = \{\phi \in G \mid \phi(i) = i\}$. We call $\mathrm{stab}_G(i)$ the *stabilizer of i in G*.

The student should verify that $\mathrm{stab}_G(i)$ is a subgroup of G. (See Exercise 35 in Chapter 5.)

Definition Orbit of a Point

Let G be a group of permutations of a set S. For each s in S, let $\mathrm{orb}_G(s) = \{\phi(s) \mid \phi \in G\}$. The set $\mathrm{orb}_G(s)$ is a subset of S called the *orbit of s under G*. We use $|\mathrm{orb}_G(s)|$ to denote the number of elements in $\mathrm{orb}_G(s)$.

Example 7 should clarify these two definitions.

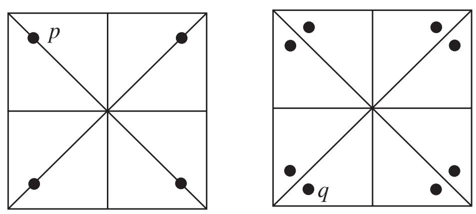
EXAMPLE 7 Let

$$G = \{(1), (132)(465)(78), (132)(465), (123)(456), (123)(456)(78), (78)\}.$$

Then,

$$\begin{aligned}
&\text{orb}_G(1) = \{1, 3, 2\}, && \text{stab}_G(1) = \{(1), (78)\},\\
&\text{orb}_G(2) = \{2, 1, 3\}, && \text{stab}_G(2) = \{(1), (78)\},\\
&\text{orb}_G(4) = \{4, 6, 5\}, && \text{stab}_G(4) = \{(1), (78)\},\\
&\text{orb}_G(7) = \{7, 8\}, && \text{stab}_G(7) = \{(1), (132)(465), (123)(456)\}.
\end{aligned}$$

EXAMPLE 8 We may view D_4 as a group of permutations of a square region. Figure 7.1(a) illustrates the orbit of the point p under D_4, and Figure 7.1(b) illustrates the orbit of the point q under D_4. Observe that $\text{stab}_{D_4}(p) = \{R_0, D\}$, whereas $\text{stab}_{D_4}(q) = \{R_0\}$.



(a) (b)

Figure 7.1

The preceding two examples also illustrate the following theorem.

Theorem 7.4 Orbit-Stabilizer Theorem

Let G be a finite group of permutations of a set S. Then, for any i from S, $|G| = |\text{orb}_G(i)|\,|\text{stab}_G(i)|$.

PROOF By Lagrange's Theorem, $|G|/|\text{stab}_G(i)|$ is the number of distinct left cosets of $\text{stab}_G(i)$ in G. Thus, it suffices to establish a one-to-one correspondence between the left cosets of $\text{stab}_G(i)$ and the elements in the orbit of i. To do this, we define a correspondence T by mapping the coset $\phi\text{stab}_G(i)$ to $\phi(i)$ under T. To show that T is a well-defined function, we must show that $\alpha\text{stab}_G(i) = \beta\text{stab}_G(i)$ implies $\alpha(i) = \beta(i)$. But $\alpha\text{stab}_G(i) = \beta\text{stab}_G(i)$ implies $\alpha^{-1}\beta \in \text{stab}_G(i)$, so that $(\alpha^{-1}\beta)(i) = i$ and, therefore, $\beta(i) = \alpha(i)$. Reversing the argument from the last step to the first step shows that T is also one-to-one. We conclude the proof by showing that T is onto $\text{orb}_G(i)$. Let $j \in \text{orb}_G(i)$. Then $\alpha(i) = j$ for some $\alpha \in G$ and clearly $T(\alpha\text{stab}_G(i)) = \alpha(i) = j$, so that T is onto. ∎

We leave as an exercise the proof of the important fact that the orbits of the elements of a set S under a group partition S (Exercise 43).

The Rotation Group of a Cube and a Soccer Ball

It cannot be overemphasized that Theorem 7.4 and Lagrange's Theorem (Theorem 7.1) are *counting* theorems.† They enable us to determine the numbers of elements in various sets. To see how Theorem 7.4 works, we will determine the order of the rotation group of a cube and a soccer ball. That is, we wish to find the number of essentially different ways in which we can take a cube or a soccer ball in a certain location in space, physically rotate it, and then have it still occupy its original location.

EXAMPLE 9 Let G be the rotation group of a cube. Label the six faces of the cube 1 through 6. Since any rotation of the cube must carry each face of the cube to exactly one other face of the cube and different rotations induce different permutations of the faces, G can be viewed as a group of permutations on the set $\{1, 2, 3, 4, 5, 6\}$. Clearly, there is some rotation about a central horizontal or vertical axis that carries face number 1 to any other face, so that $|\text{orb}_G(1)| = 6$. Next, we consider $\text{stab}_G(1)$. Here, we are asking for all rotations of a cube that leave face number 1 where it is. Surely, there are only four such motions—rotations of 0°, 90°, 180°, and 270°—about the line perpendicular to the face and passing through its center (see Figure 7.2). Thus, by Theorem 7.4, $|G| = |\text{orb}_G(1)|\,|\text{stab}_G(1)| = 6 \cdot 4 = 24$. ∎

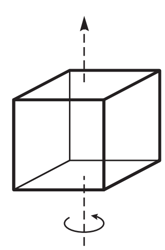


Figure 7.2 Axis of rotation of a cube.

Now that we know how many rotations a cube has, it is simple to determine the actual structure of the rotation group of a cube. Recall that S_4 is the symmetric group of degree 4.

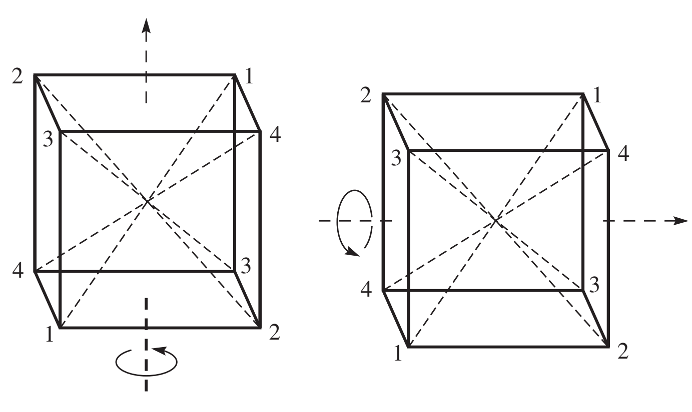
†"People who don't count won't count" (Anatole France).

Theorem 7.5 The Rotation Group of a Cube

The group of rotations of a cube is isomorphic to S_4.

PROOF Since the group of rotations of a cube has the same order as S_4, we need only prove that the group of rotations is isomorphic to a subgroup of S_4. To this end, observe that a cube has four diagonals and that the rotation group induces a group of permutations on the four diagonals. But we must be careful not to assume that different rotations correspond to different permutations. To see that this is so, all we need do is show that all 24 permutations of the diagonals arise from rotations. Labeling the consecutive diagonals 1, 2, 3, and 4, it is obvious that there is a 90° rotation that yields the permutation $\alpha = (1234)$; another 90° rotation about an axis perpendicular to our first axis yields the permutation $\beta = (1423)$. See Figure 7.3. So, the group of permutations induced by the rotations contains the eight-element subgroup $\{\varepsilon, \alpha, \alpha^2, \alpha^3, \beta^2, \beta^2\alpha, \beta^2\alpha^2, \beta^2\alpha^3\}$ (see Exercise 63) and $\alpha\beta$, which has order 3. Clearly, then, the rotations yield all 24 permutations, since the order of the rotation group must be divisible by both 8 and 3. ■

EXAMPLE 10 A traditional soccer ball has 20 faces that are regular hexagons and 12 faces that are regular pentagons. (The technical term for this solid is *truncated icosahedron*.) To determine the number of rotational symmetries of a soccer ball using Theorem 7.4, we may choose our set S to be the 20 hexagons or the 12 pentagons. Let us say that S is the set of 12 pentagons. Since any pentagon can be carried to any other



$\alpha = (1234)$ $\beta = (1423)$

Figure 7.3

pentagon by some rotation, the orbit of any pentagon is S. Also, there are five rotations that fix (stabilize) any particular pentagon. Thus, by the Orbit-Stabilizer Theorem, there are $12 \cdot 5 = 60$ rotational symmetries. (In case you are interested, the rotation group of a soccer ball is isomorphic to A_5.) ∎

In 1985, chemists Robert Curl, Richard Smalley, and Harold Kroto caused tremendous excitement in the scientific community when they created a new form of carbon by using a laser beam to vaporize graphite. The structure of the new molecule was composed of 60 carbon atoms arranged in the shape of a soccer ball! Because the shape of the new molecule reminded them of the dome structures built by the architect R. Buckminster Fuller, Curl, Smalley, and Kroto named their discovery "buckyballs." Buckyballs are the roundest, most symmetric large molecules known. Group theory has been particularly useful in illuminating the properties of buckyballs, since the absorption spectrum of a molecule depends on its symmetries and chemists classify various molecular states according to their symmetry properties. The buckyball discovery spurred a revolution in carbon chemistry. In 1996, Curl, Smalley, and Kroto received the Nobel Prize in chemistry for their discovery.

An Application of Cosets to the Rubik's Cube

Recall from Chapter 5 that in 2010 it was proved via a computer computation, which took 35 CPU-years to complete, that every Rubik's cube could be solved in at most 20 moves. To carry out this effort, the research team of Morley Davidson, John Dethridge, Herbert Kociemba, and Tomas Rokicki applied a program of Rokicki, which built on early work of Kociemba, that checked the elements of the cosets of a subgroup H of order $(8! \cdot 8! \cdot 4!)/2 = 19{,}508{,}428{,}800$ to see if each cube in a position corresponding to the elements in a coset could be solved within 20 moves. In the rare cases where Rokicki's program did not work, an alternate method was employed. Using symmetry considerations, they were able to reduce the approximately 2 billion cosets of H