Direct Products, continued

- **Video**: In $G \oplus H$, $= G \times H$

  $X \times Y$
  "cross product"
  "Cartesian product"

$$\text{ord}((g,h)) = \text{lcm}(\underset{\text{in } G}{\text{ord}(g)}, \underset{\text{in } H}{\text{ord}(h)})$$

- **Ex**: In $\mathbb{Z}_2 \oplus \mathbb{Z}_3$,

$$\begin{aligned} \text{ord}((1,1)) &= \text{lcm}\left(\underset{\text{in } \mathbb{Z}_2}{\text{ord}(1)}, \underset{\text{in } \mathbb{Z}_3}{\text{ord}(1)}\right) \\ &= \text{lcm}(2,3) \\ &= 6 \end{aligned}$$

Check:

$(1,1)$

$(1,1) + (1,1) = (0,2)$

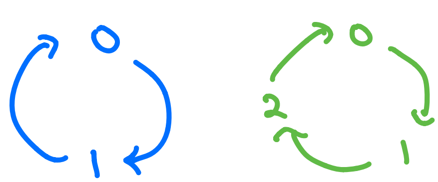
$(1,1) + (1,1) + (1,1) = (1,0)$

$(1,1) + (1,1) + (1,1) + (1,1) = (0,1)$

$5 \cdot (1,1) = (1,2)$

$6 \cdot (1,1) = (0,0)$

Picture:



Worksheet 35: Direct Products of $\mathbb{Z}_n$'s

Math 335

Reporter:

Recorder:

Equity Manager:

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1. Calculate the order of the element $(1,1) \in \mathbb{Z}_2 \oplus \mathbb{Z}_6$ in two ways:

   (i) by using the theorem from today's video, and

   (ii) by writing down all the elements of $\langle (1,1) \rangle \subseteq \mathbb{Z}_2 \oplus \mathbb{Z}_6$.

(i) $\operatorname{ord}((1,1)) = \operatorname{lcm}\left(\begin{matrix}\operatorname{ord}(1) \\ \text{in } \mathbb{Z}_2\end{matrix}, \begin{matrix}\operatorname{ord}(1) \\ \text{in } \mathbb{Z}_6\end{matrix}\right)$
$= \operatorname{lcm}(2, 6) = \boxed{6}$

(ii) $\langle (1,1) \rangle = \{ (0,0), (1,1), (0,2), (1,3), (0,4), (1,5) \}$

2. Pause and check in; make sure everyone in the group agrees on Problem 1. Then, each group member choose one of the following direct products and calculate the order of $(1,1)$ in this direct product in at least one of the above two ways:

   (a) $\mathbb{Z}_2 \oplus \mathbb{Z}_5$ $\operatorname{ord}(1,1) = \operatorname{lcm}(2,5) = 10$

   (b) $\mathbb{Z}_5 \oplus \mathbb{Z}_{10}$ $\operatorname{ord}(1,1) = \operatorname{lcm}(5,10) = 10$

   (c) $\mathbb{Z}_4 \oplus \mathbb{Z}_6$ $\operatorname{ord}(1,1) = \operatorname{lcm}(4,6) = 12$

   (d) $\mathbb{Z}_3 \oplus \mathbb{Z}_4$ $\operatorname{ord}(1,1) = \operatorname{lcm}(3,4) = 12$

3. In each case of the four cases in Problem 2, does $(1,1)$ generate the whole group $\mathbb{Z}_n \oplus \mathbb{Z}_m$?

(a) Yes, because $\langle(1,1)\rangle$ has 10 elements, which is all of $\mathbb{Z}_2 \oplus \mathbb{Z}_5$

(b) No

(c) No

(d) Yes, because $\langle(1,1)\rangle$ has 12 elements, which is all of $\mathbb{Z}_3 \oplus \mathbb{Z}_4$

4. Based on your findings in Problem 3, under what conditions on n and m do you think $(1,1)$ generates the whole group $\mathbb{Z}_n \oplus \mathbb{Z}_m$?

$$\langle(1,1)\rangle = \mathbb{Z}_n \oplus \mathbb{Z}_m \iff \gcd(n,m) = 1$$

(or $\operatorname{lcm}(n,m) = n \cdot m$)

(Proof next video! Key fact is:

$$\gcd(n,m) \cdot \operatorname{lcm}(n,m) = n \cdot m$$)

Challenge: Prove that, if n and m meet the conditions of Problem 4, then $\mathbb{Z}_n \oplus \mathbb{Z}_m \cong \mathbb{Z}_{nm}$.