Isomorphisms, Continued

- **Recall**: An isomorphism
$$\varphi : G \longrightarrow H$$
is a "dictionary" between them. If this exists, then G and H are "essentially the same group."

- **Video**: If $\exists$ an isomorphism $\varphi : G \to H$, we say G is <u>isomorphic</u> to H
$$(G \cong H).$$
In this case, they share all their group-theoretic properties.

- **Punchline**: How to know if $G \cong H$:

  - To prove $G \cong H$, find an isomorphism
  $$\varphi : G \longrightarrow H.$$
  - To prove $G \not\cong H$, find a group-theoretic property one has but not the other.

Worksheet 21: Isomorphisms, continued

Math 335

Reporter:

Recorder:

Equity Manager:

Properties
- Abelian
- Cyclic

For each of the following pairs of groups G and H, is $G \cong H$? If so, try to write down an isomorphism $\varphi : G \to H$. If not, try to think of a group-theoretic property that one group has but the other does not.

1. $G = \mathbb{Z}_6$ (under addition modulo 6) and $H = S_3$ (under composition)

No: $\mathbb{Z}_6$ is abelian and S_3 isn't.

2. $G = \mathbb{Z}_6$ (under addition modulo 6) and $H = \{1, g, g^2, g^3, g^4, g^5\}$ (under multiplication), where g is a complex number of order 6

Yes: an isomorphism is

$$\varphi : \mathbb{Z}_6 \longrightarrow H$$
$$\varphi(k) = g^k.$$

$0 \to 1$
$1 \to g^1$
$2 \to g^2$
$3 \to g^3$
$4 \to g^4$
$5 \to g^5$

This is an isomorphism because:

① It's a bijection ✓

② $\underline{\varphi(k+l)} = g^{k+l} = g^k \cdot g^l = \underline{\varphi(k) \cdot \varphi(l)}$

3. $G = \mathbb{Z}$ (under addition) and $H = \mathbb{R}$ (under addition)

No: $\mathbb{Z}$ is cyclic and $\mathbb{R}$ isn't.

(Alternatively, if you know about countability: $\mathbb{Z}$ is countable and $\mathbb{R}$ isn't, so $\nexists$ a bijection between them.)

4. $G = \mathbb{Z}$ (under addition) and $H = \{\text{even integers}\}$ (under addition)

Yes: an isomorphism is

$$\varphi: G \longrightarrow H$$
$$\varphi(x) = 2x.$$

This is an isomorphism because

① It's a bijection ← (e.g. because $\varphi^{-1}: H \to G$ is $\varphi^{-1}(x) = \frac{1}{2}x$)

② $\varphi(x+y) = 2(x+y) = 2x + 2y = \varphi(x) + \varphi(y)$

5. (**Challenge/Discussion Question**) I claim that *any* cyclic group with finitely elements is isomorphic to $\mathbb{Z}_n$ for some n (under addition modulo n), and *any* cyclic group with infinitely many elements is isomorphic to $\mathbb{Z}$ (under addition). Does this match with the cyclic groups you can think of? Do you have any thoughts on how you might prove this statement?

- Finite case: $G = \langle g \rangle = \{e, g, g^2, \ldots, g^{n-1}\}$
  $\quad\quad e \mapsto 0,\ g \mapsto 1,\ g^2 \mapsto 2,\ \ldots,\ g^{n-1} \mapsto n-1$

- Infinite case: $G = \langle g \rangle = \{e, g, g^2, \ldots, g^{-1}, g^{-2}, \ldots\}$
  $\quad\quad e \mapsto 0,\ g \mapsto 1,\ g^2 \mapsto 2, \ldots,\ g^{-1} \mapsto -1,\ g^{-2} \mapsto -2, \ldots$