Math 335, Homework 7

Due Friday, April 2 (note extended deadline!)

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1. Let
$$H = \{e,\ (1,2)(3,4),\ (1,3)(2,4),\ (1,4)(2,3)\} \subseteq S_4.$$
List all left cosets of H in S_4, being sure to list each one only once..

   Answer:
   There are $\dfrac{|S_4|}{|H|} = \dfrac{24}{4} = 6$ unique left cosets of H in S_4.

eH	$(1,2)H$	$(1,3)H$	$(2,3)H$	$(1,2,3)H$	$(1,2,4)H$
e	$(1,2)$	$(1,3)$	$(2,3)$	$(1,2,3)$	$(1,2,4)$
$(1,2)(3,4)$	$(3,4)$	$(1,2,3,4)$	$(1,3,4,2)$	$(1,3,4)$	$(1,4,3)$
$(1,3)(2,4)$	$(1,3,2,4)$	$(2,4)$	$(1,2,4,3)$	$(2,4,3)$	$(1,3,2)$
$(1,4)(2,3)$	$(1,4,2,3)$	$(1,4,3,2)$	$(1,4)$	$(1,4,2)$	$(2,3,4)$

2. Let $G = \langle g \rangle$ be a cyclic group with 30 elements. List all of the left cosets of $\langle g^4 \rangle$ in G, being sure to list each one only once.

   Answer:
   There are $\dfrac{|\langle g \rangle|}{|\langle g^4 \rangle|} = \dfrac{30}{15} = 2$ unique left cosets of $\langle g^4 \rangle$ in G. The two left cosets of $\langle g^4 \rangle$ in G are:
$$\langle g^4 \rangle = \{g^2, g^4, g^6, g^8, g^{10}, g^{12}, g^{14}, g^{16}, g^{18}, g^{20}, g^{22}, g^{24}, g^{26}, g^{28}, g^{30}\}$$
$$g \cdot \langle g^4 \rangle = \{g^1, g^3, g^5, g^7, g^9, g^{11}, g^{13}, g^{15}, g^{17}, g^{19}, g^{21}, g^{23}, g^{25}, g^{27}, g^{29}\}$$

3. Let $\mathbb{C}^*$ be the group of nonzero complex numbers (under multiplication). Recall that the *norm* of a complex numbers is defined as its distance from the origin in the complex plane; in other words, it's
$$|a + bi| := \sqrt{a^2 + b^2}.$$
Two facts about the norm, which you may assume for this problem, are
$$|z \cdot w| = |z| \cdot |w| \quad \text{and} \quad \left|\frac{1}{z}\right| = \frac{1}{|z|}.$$
Given this, let
$$H = \left\{ z \in \mathbb{C}^* \,\middle|\, |z| = 1 \right\} \subseteq \mathbb{C}^*.$$
(The next page shows a picture of H, for your reference.)

   (a) Prove that, for $v, w \in \mathbb{C}^*$, we have $vH = wH$ if and only if $|v| = |w|$.

Proof. Let $v, w \in \mathbb{C}^*$. Suppose $vH = wH$ and $x_1, x_2 \in H$ such that $vx_1 = wx_2$. With some algebraic manipulation, we can state that $vw^{-1} = x_2x_1^{-1}$. Using the properties of the norm, it can be asserted that

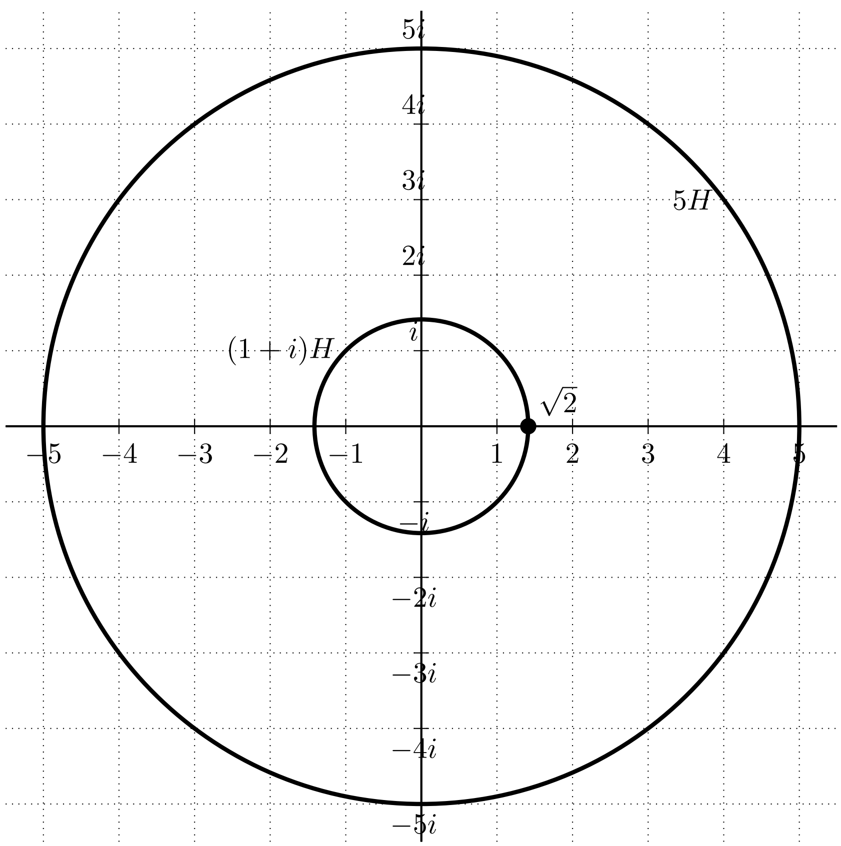
$$\left|x_2x_1^{-1}\right| = |x_2| \cdot \left|x_1^{-1}\right| = \frac{|x_2|}{|x_1|} = 1.$$

Notice that this also means that

$$\left|vw^{-1}\right| = |v| \cdot \left|w^{-1}\right| = \frac{v}{w} = 1,$$

which allows us to conclude that $|v| = |w|$.
Conversely, suppose that $|v| = |w|$ for $v, w \in \mathbb{C}^*$. Then $\frac{|v|}{|w|} = 1$ and by using the properties of the norm, we can conclude that $\left|vw^{-1}\right| = 1$. Hence, $vw^{-1} \in H$ and $vH = wH$. $\square$

(b) Given part (a), draw a picture of the left coset $5H$, and of the left coset $(1+i)H$.
Answer:



4. Let G be a group with 8 elements.

(a) What are the possible orders of elements of G?
Answer:
By corollary, the possible orders of elements of G are divisors of $|G|$, which are 1, 2, 4, and 8.

(b) Prove that G must have an element of order 2.

(**Hint**: Start by choosing a non-identity element of G at random. If it doesn't have order 2, try to cook up an element of order 2 out of it.)

Proof. Suppose $g \in G$ and $g \neq e$. If $\mathrm{ord}(g) = 8$, then $\mathrm{ord}(g^4) = 2$. Similarly, if $\mathrm{ord}(g) = 4$, then $\mathrm{ord}(g^2) = 2$. Hence, G must have an element of order 2. □

5. Prove that a group with a prime number of elements must be cyclic.

Proof. Suppose that $\mathrm{ord}(G)$ is prime. Let $g \in G$ and $g \neq e$. Since G is a finite group, by corollary, $\mathrm{ord}(\langle g \rangle)$ divides $|G|$. Additionally, since $g \neq e$, $\mathrm{ord}(\langle g \rangle) \neq 1$. Since the only two divisors of a prime number are 1 and itself, $\mathrm{ord}(\langle g \rangle) = |G|$, and it follows that $\langle g \rangle = G$. □