Cosets

- **Video:** $H \subseteq G$, $a \in G$

$$\rightsquigarrow aH = \{ah \mid h \in H\} \subseteq G$$

- This is called a <u>left coset</u> of H in G. There's also such a thing as a <u>right coset</u> of H in G:

$$Ha = \{ha \mid h \in H\},$$

but we won't study them because

① if G is abelian, it's the same
($aH = Ha$)

② even if G isn't abelian, all the theory works the same.
($aH \neq Ha$)

Worksheet 22: Cosets

Math 335

Reporter:

Recorder:

Equity Manager:

---

1. Let $G = \mathbb{Z}_6$ (which is a group under addition modulo 6), and let

$$H = \{0, 2, 4\}.$$

(a) Calculate all of the left cosets of H in G. (I've gotten you started.)

$$0 + H = \{0,2,4\} = \{0+0, 0+2, 0+4\}$$

$$1 + H = \{1, 3, 5\}$$

$$2 + H = \{2,4,0\} = \{2+0, 2+2, 2+4\}$$

$$3 + H = \{3,5,1\} = \{3+0, 3+2, 3+4\}$$

$$4 + H = \{4,0,2\}$$

$$5 + H = \{5,1,3\}$$

(b) Is $a + H$ necessarily a subgroup of G? Is it ever a subgroup?

Not necessarily (e.g. $1+H$ isn't) but sometimes (e.g. $0+H$ is).

(c) Is it ever the case that $a + H = b + H$ for two different elements a and b of G?

Yes: $0+H = 2+H = 4+H$ and $1+H = 3+H = 5+H$.

(d) Is it ever the case that $a + H$ and $b + H$ have some but not all of their elements in common?

No! (In this example)

2. Now, we'll do the same thing for the group $G = S_3$ (the symmetric group, under the operation of composition), and the subgroup

$$H = \{e, (1,3)\}.$$

(a) Calculate all of the left cosets of H in G. (Again, I've gotten you started.)

$eH = \{e, (1,3)\}$

$(1,2)H = \{(1,2),\ (1,3,2)\}$

$(1,3)H = \{(1,3), e\}$

$(2,3)H = \{(2,3), (1,2,3)\}$ $\quad (2,3)\cdot(1,3) = (1,2,3)$

$(1,2,3)H = \{(1,2,3), (2,3)\}$ $\quad (1,2,3)\cdot(1,3) = (2,3)$

$(1,3,2)H = \{(1,3,2), (1,2)\}$ $\quad (1,3,2)\cdot(1,3) = (1,2)$

(b) Is a coset necessarily a subgroup of G? Is it ever a subgroup?

As in #1, not necessarily (e.g. $(1,2)H$ isn't) but sometimes (e.g. eH is).

(c) Is it ever the case that the two cosets on the above list are the same?

Yes: $eH = (1,3)H$, $(1,2)H = (1,3,2)H$, $(2,3)H = (1,2,3)H$.

(d) Is it ever the case that two cosets on the above list share some but not all of their elements?

No!

Challenge/Vague Discussion Question: I claim that, if G is a group and $H \subseteq G$ is a subgroup, then "the left cosets partition G up into a bunch of disjoint, equal-sized pieces." Discuss this statement with the people in your Breakout Room. Does it make sense? Is it consistent with the above examples?

#1:

0	1
2	3
4	5

#2:

e	$(1,2)$	$(2,3)$
$(1,3)$	$(1,3,2)$	$(1,2,3)$