Math 335, Exam 3

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Name: Mark Kim

Problem	Points Scored	Total Points Possible
1		8
2		8
3		8
4		8
5		8
Total		40

Problem 1: Let

$$G = \mathbb{Z}_{12} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\},$$

which is a group under addition modulo 12, and let

$$H = \langle 3 \rangle \subseteq G. \qquad \{0, 3, 6, 9\}$$

You do **not** need to explain your answers or provide any justification on this problem.

(a) How many elements are there in the quotient group G/H?

$$|G/H| = |G|/|H| = \frac{12}{4} = 3$$

(b) List all of the elements in G/H, being sure to list each one only once.

$$G/H = \{0+H, 1+H, 2+H\}$$

(c) What is $(2 + H) + (2 + H)$? Express your answer as one of the elements in your list from part (b).

$$(2+H) + (2+H) = (4+H) = (1+H)$$

Problem 2: This problem concerns the homomorphism

$$\begin{aligned} &\overset{G}{} \quad \overset{H}{} \\ &\varphi : \mathbb{Z}_{12} \to \mathbb{Z}_3 \\ &\varphi(a) = a \text{ reduced modulo 3.} \end{aligned}$$

			0	1	2	0	1
0	1	2	3	4	5	6	7
2	0	1	2				
8	9	10	11				

(So, for example, $\varphi(4) = 1$ and $\varphi(8) = 2$.) You do **not** need to explain your answers or provide any justification on this problem.

(a) What is the kernel of φ? List all of its elements.

$$\ker(\varphi) = \{g \in G \mid \varphi(g) = e\} = \{x \in \mathbb{Z}_{12} \mid \varphi(x) = 0\}$$
$$= \{0, 3, 6, 9\} = \langle 3 \rangle$$

(b) What is the image of φ? List all of its elements.

$$\operatorname{im}(\varphi) = \{h \in H \mid h = \varphi(g) \text{ for some } g \in G\} = \{x \in \mathbb{Z}_3 \mid x = \varphi(g) \text{ for some } g \in \mathbb{Z}_{12}\}$$
$$= \{0, 1, 2\}$$

(c) Which theorem now tells us that $\mathbb{Z}_{12}/\langle 3 \rangle \cong \mathbb{Z}_3$?

The First Isomorphism Theorem

(d) Write down an explicit isomorphism

$$f : \mathbb{Z}_{12}/\langle 3 \rangle \to \mathbb{Z}_3$$

by saying where f sends each element of $\mathbb{Z}_{12}/\langle 3 \rangle$. (You may find it helpful to look back at Problem 1(b), where you listed all of the elements of $\mathbb{Z}_{12}/\langle 3 \rangle$.)

$$\mathbb{Z}_{12}/\{0,3,6,9\} = \{0+K, 1+K, 2+K\}$$
$$\mathbb{Z}_3 = \{0, 1, 2\}$$
$$\{0+K, 1+K, 2+K\} \cong \{0, 1, 2\}$$
$$\therefore \mathbb{Z}_{12}/\langle 3 \rangle \cong \mathbb{Z}_3$$

G		H
$0+K$	0, 3, 6, 9	0
$1+K$	1, 4, 7, 10	1
$2+K$	2, 5, 8, 11	2

$\mathbb{Z}_3$

Problem 3: Let

$$G = D_4 = \{e, r_{90}, r_{180}, r_{270}, h, v, d, d'\}$$

be the group of symmetries of a square, and let

$$H = \{e, h\} \subseteq G,$$

where e denotes the identity and h denotes the reflection across a horizontal axis. Here's a table for the group D_4, for your reference:

	e	r_{90}	r_{180}	r_{270}	h	v	d	d'
e	e	r_{90}	r_{180}	r_{270}	h	v	d	d'
r_{90}	r_{90}	r_{180}	r_{270}	e	d'	d	h	v
r_{180}	r_{180}	r_{270}	e	r_{90}	v	h	d'	d
r_{270}	r_{270}	e	r_{90}	r_{180}	d	d'	v	h
h	h	d	v	d'	e	r_{180}	r_{90}	r_{270}
v	v	d'	h	d	r_{180}	e	r_{270}	r_{90}
d	d	v	d'	h	r_{270}	r_{90}	e	r_{180}
d'	d'	h	d	v	r_{90}	r_{270}	r_{180}	e

(a) Show that H is **not** a normal subgroup of G.

$\forall g \in G \ \forall h \in H$
$ghg^{-1} \in H$

$$r_{90} \circ h \circ r_{90}^{-1} = r_{90} \circ h \circ r_{270}$$
$$= r_{90} \circ d'$$
$$= v \notin H$$

$\therefore H$ is not a subgroup

(b) Use an example to show that composition of left cosets

$$aH \circ bH = (a \circ b)H$$

is **not** well-defined.

$H = \{e, h\} \leq G$

$eH = hH = \{eH\} \subseteq G$

$$eH \circ r_{90}H = r_{90}H \neq$$
$$hH \circ r_{90}H = (h \circ r_{90})H = dH$$

$\therefore$ not well defined

Problem 4: You do **not** need to explain your answers or provide any justification on this problem.

(a) List all of the elements of the group $\mathbb{Z}_2 \oplus \mathbb{Z}_4$.

$(0,0), (0,1), (0,2), (0,3)$
$(1,0), (1,1), (1,2), (1,3)$

(b) Is it true that $\mathbb{Z}_2 \oplus \mathbb{Z}_4 \cong \mathbb{Z}_8$? Briefly explain your answer, using any of the theorems we have learned in class.

By theorem, $\mathbb{Z}_2 \oplus \mathbb{Z}_4 \cong \mathbb{Z}_8 \iff \gcd(n,m)=1$

$\gcd(2,4) = 2 \quad \therefore \quad \mathbb{Z}_2 \oplus \mathbb{Z}_4 \not\cong \mathbb{Z}_8$

(c) List all abelian groups with 36 elements, making sure that no two of the groups in your list are isomorphic to one another.

$36 = 2 \cdot 2 \cdot 3 \cdot 3$

$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$

$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_9$

$\mathbb{Z}_4 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$

$\mathbb{Z}_4 \oplus \mathbb{Z}_9$

Problem 5: You do **not** need to explain your answers or provide any justification on this problem.

(a) For each of the following rings, circle "integral domain", "field", or "neither". **Circle more than one if more than one applies.**

$R = \mathbb{R}$	integral domain	field	neither
$R = M_2(\mathbb{R})$	integral domain	field	neither
not all 2x2 matrices are invertible			
$R = \mathbb{Z}_8$	integral domain	field	neither
$R = \mathbb{Z}_7$	integral domain	field	neither

(b) Give an example of a subset $I \subseteq \mathbb{Z}[x]$ that is an ideal.

I = { polynomials with constant term 0 }

(c) Give an example of a subset $I \subseteq \mathbb{Z}[x]$ that is **not** an ideal.

I = { linear polynomials }