Homework 10

Monday, April 26, 2021 10:57 AM

1. Topic #2 EP3

It seems that the number of rotational symmetries is divisible by the number of elements of rotation of a particular face:

← # of elements of stab(F)

$6 \cdot 4 = 24$

$4 \cdot 3 = 12$

↖ # of elements of stab(g)

Octahedron: 8 sides, 3 rotations

$8 \cdot 3 = 24$

Dodecahedron: $12 \cdot 5 = 60$

2. By definition G & H are groups.

$\varphi : G \to H$

$\varphi(a * b) = \varphi(a) * \varphi(b)$

Need to show $im(\varphi) \subseteq H$ is a group

Closure: By definition, $im(\varphi)$ is everything in H that φ results in. Given $a, b \in G$, by definition, $\varphi(a*b) = \varphi(a) * \varphi(b) \in im(\varphi) \subseteq H$. Hence $im(\varphi)$ is closed. If $\varphi(a)$ and $\varphi(b)$ are in $im(\varphi)$, then so is $\varphi(a*b)$.

Identity: $\varphi(e_G) = e_H \in H$. Since $im(\varphi)$ contains everything the φ results in, e_H is the identity of $im(\varphi)$. $\varphi(e_G * g) = \varphi(e_G) * \varphi(g) = e_H * \varphi(g) = \varphi(g) \in im(\varphi)$

Inverse: Suppose $g \in G$. By definition $\varphi(g) \in im(\varphi)$ $\forall g \in G$

$$\varphi(e_G) = \varphi(g * g^{-1}) = \varphi(g) * \varphi(g^{-1})$$

$$= \varphi(g) * \varphi(g)^{-1} = e_{im(\varphi)} \;\forall g \in G$$

Hence, inverses exist for all $g \in G$ such that $\varphi(g)^{-1} \in im(\varphi)$

3. $ker(\varphi) = \{g \in G \mid \varphi(g) = e_H\} \leq G$

Closure: Suppose $a, b \in ker(\varphi)$. Then

$$\varphi(a) = e_H \text{ and } \varphi(b) = e_H$$

$$\varphi(a*b) = \varphi(a) * \varphi(b) = e_H * e_H = e_H$$

$\forall a, b \in ker(\varphi)$

Identity: Suppose $g \in ker(\varphi)$; then $\varphi(g) = e_H$. Hence the identity of $ker(\varphi)$ is $\varphi(g)$.

Inverse: Given our identity above and because φ is a homomorphism,

$$\varphi(g^{-1}) = \varphi(g)^{-1} = e_H^{-1} = e_H$$

Hence $g^{-1} \in ker(\varphi)$

To prove it is normal have to show $\forall g \in G$ and $\forall k \in ker(\varphi)$

$$gkg^{-1} \in ker(\varphi)$$

Let $g \in G$ and $k \in ker(\varphi)$. Then

$$\varphi(gkg^{-1}) = \varphi(g) * \varphi(k) * \varphi(g)^{-1} \quad (\varphi \text{ is a homomorphism})$$

$$= \varphi(g) * e_H * \varphi(g)^{-1} \quad (\varphi(k) = e_H)$$

$$= \varphi(g) * \varphi(g)^{-1} = e_H$$

Hence $gkg^{-1} \in ker(\varphi)$

4. a. $(0,0)$ $\qquad G = \mathbb{Z} \oplus \mathbb{Z}$

b. $ker(\varphi) = \{g \in G \mid \varphi(g) = e_H\} \leq G \qquad H = \mathbb{Z}$

$\Rightarrow \varphi((a,b)) = a - b \qquad a - b = 0$ when $a = b$

$ker(\varphi) = \{(a,b) \in \mathbb{Z} \oplus \mathbb{Z} \mid a = b\} = \{(0,0),(1,1),(2,2),\ldots(n,n)\}$

$\qquad 0 \quad 1 \quad 2 \quad \ldots \quad n$

$ker(\varphi) \cong \mathbb{Z}$

c. $im(\varphi) = \{h \in H \mid h = \varphi(g)$ for some $g \in G\} \leq H$

$im(\varphi) = \{z \in \mathbb{Z} \mid z = \varphi((a,b))$ for some $(a,b) \in \mathbb{Z} \oplus \mathbb{Z}\}$

$= \{z \in \mathbb{Z} \mid z = a - b$ for some $(a,b) \in \mathbb{Z} \oplus \mathbb{Z}\}$

$= \{z \in \mathbb{Z} \mid z + b = a$ for some $(a,b) \in \mathbb{Z} \oplus \mathbb{Z}\}$.

$\varphi(a,b) = \varphi(z+b, b) = (z+b) - b = z$ for all $z \in \mathbb{Z}$

Hence the $im(\varphi) = \mathbb{Z}$

d. $G/K \cong im(\varphi) \qquad K = ker(\varphi)$

$\Rightarrow \mathbb{Z} \oplus \mathbb{Z}/K \cong im(\varphi) = \mathbb{Z} \cong ker(\varphi)$

$\mathbb{Z} \oplus \mathbb{Z}/K \cong K \cong im(\varphi)$