Logistics

- **Exam 3:**
  - Wednesday 5/12 any 90 mins
  - Same format, length, + weight as Exams 1 + 2
  - Material since Exam 2
  - Review materials on iLearn today

- **Final Project:**
  - See "Final Project Guidelines" at top of iLearn
  - Report due Thursday 5/20 (~3-page paper or ~10-min video explaining topic as though to our class, based on reading but drawing on Exploration Problems for examples)
    ShowMe
  - Optional rough draft due Weds. 5/12

Integral Domains + Fields

- Video: Two special types of $a \in R$:

  - zero-divisor: $\overset{\text{nonzero}}{a} \cdot \left(\substack{\text{something} \\ \text{nonzero}} \right) = 0$
  - unit: $a \cdot (\text{something}) = 1$

  Two special types of R:

  - integral domain: no zero-divisors
  - field: everything except 0 a unit

- Def: A ring is a set R with two operations, $+$ and $\cdot$, satisfying....

Worksheet 39: Integral Domains and Fields

Math 335

Reporter:

Recorder:

Equity Manager:

1. For each of the following rings R, determine which elements of R are zero-divisors, which are units, and whether R is an integral domain and/or field.

R	Zero-divisors	Units	Integral Domain?	Field?
$\mathbb{Z}_5$	none	1, 2, 3, 4	Yes	Yes
$\mathbb{Z}_9$	3, 6	1, 2, 4, 5, 7, 8	No	No
$\mathbb{Z}$	none	1, −1	Yes	No
$\mathbb{Q}$	none	$\mathbb{Q} \setminus \{0\}$	Yes	Yes
$\mathbb{Z} \oplus \mathbb{Z}$ where the operations are $(a,b)+(c,d)=(a+c,b+d)$ $(a,b)\cdot(c,d)=(a\cdot c,b\cdot d)$	$(a,0)$, $(0,b)$	$(\pm 1, \pm 1)$	No	No

Note: Nothing that's a zero divisor is a unit

2. Based on the examples on the previous page, do you think that

$$R \text{ is an integral domain } \Rightarrow R \text{ is a field?}$$ No (e.g. $\mathbb{Z}$)

Do you think that

$$R \text{ is a field } \Rightarrow R \text{ is an integral domain?}$$ Yes

3. Show that it's impossible for $a \in R$ to be both a zero-divisor and a unit.

Suppose $a \in R$ is both a zero-divisor and a unit. Then $\exists$ nonzero $b \in R$ such that

$$a \cdot b = 0$$

and $\exists$ $a^{-1} \in R$. Thus,

$$a^{-1} \cdot (a \cdot b) = a^{-1} \cdot 0$$

$$\Rightarrow \quad b = 0,$$

contradicting our assumption b is nonzero.

$\boxtimes$

4. Can you use what you proved in Problem 3 to prove your answer to Problem 2?

Suppose R is a field.

Let $a \in R$. Then either

1) $a = 0$, which is not a 0-divisor by definition

2) $a \neq 0$, in which case a is a unit and thus not a 0-divisor.

So <u>no</u> $a \in R$ is a 0-divisor, meaning R is an integral domain.