Properties of Groups

- Finish discussing Worksheet 5.

- Video: Some properties of groups:

  - Only one identity element
  - Each element has only one inverse
  - $(a * b)^{-1} = b^{-1} * a^{-1}$

Worksheet 6: Properties of Groups

Math 335

Reporter (person whose first name comes alphabetically first): ____________________

Recorder (person whose first name comes alphabetically second): ____________________

Equity Manager (person whose first name comes alphabetically last): ____________________

Get to know each other: Have you seen any good movies or TV shows recently?

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1. Your comprehension question for today asked whether it's necessarily the case that

$$a * b = b * a$$

in a group. The correct answer is that this is true for some groups but not for all of them. If $a * b = b * a$ for all $a, b \in G$, we say that G is an **abelian group**.

(a) What are some examples of abelian groups?

$\mathbb{Z}$ (operation $+$), or $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{C}$

$\mathbb{R}^*$ (operation $\cdot$), or $\mathbb{Q}^*$, $\mathbb{C}^*$

$\mathbb{Z}_n$ (operation $+$ mod n)

$\{1, 2, 3, 4\}$ (operation $\cdot$ mod 5)

(b) What are some examples of groups that are not abelian?

symmetries of square $\rightarrow$ D_4 (operation $\circ$)

$GL(n, \mathbb{R})$ (operation matrix $\cdot$)

2. In the remainder of the worksheet, we'll work on proving the following theorem:

Theorem: Let G be a group and let $a \in G$. Then a has only one inverse.

   (a) Since we're proving there's only one of something, start by assuming there are two. So, let b and c be two inverses of a. Given that b is an inverse of a, what do we know about b?

$$a \cdot b = e$$
$$b \cdot a = e$$

   (b) Similarly, what do we know about c?

$$a \cdot c = e$$
$$c \cdot a = e$$

   (c) Try to combine your answers to parts (a) and (b), together with the cancellation property, to deduce that $b = c$. This finishes the proof.

$a \cdot b = e$ and $a \cdot c = e$
$\Longrightarrow a \cdot b = a \cdot c$
$\Longrightarrow b = c$ by cancellation.

Semi-serious thought question: The last theorem you learned in today's video stated that

$$(a * b)^{-1} = b^{-1} * a^{-1}.$$

This is sometimes called the "socks and shoes" property of groups. Do you see why?

Formal Written Proof for #2:

Suppose b and c are both inverses of a. The fact that b is an inverse of a means, by definition, that

$$a \cdot b = e \quad \text{and} \quad b \cdot a = e.$$

Similarly, the fact that c is an inverse of a means that

$$a \cdot c = e \quad \text{and} \quad c \cdot a = e.$$

Since $a \cdot b = e$ and $a \cdot c = e$, we have

$$a \cdot b = a \cdot c,$$

and applying the Cancellation Property implies

$$b = c.$$

Thus, our two inverses were the same, which proves that a has only one inverse.