The Fundamental Theorem of Cyclic Groups

- **Video**: Let

$$G = \langle g \rangle = \{e, g, g^2, g^3, \dots, g^{n-1}\}$$

be a cyclic subgroup with n elements. Then we can list ALL the subgroups of G: there's one of size d for each $d \mid n$.

Worksheet 19: The Fundamental Theorem of Cyclic Groups

Math 335

Reporter:

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1. Let g be a complex number such that $g^{15} = 1$ but no smaller positive power of a equals 1, and let
$$G = \{1, g, g^2, g^3, \ldots, g^{13}, g^{14}\},$$
which forms a cyclic group under multiplication of complex numbers.

   (a) How many elements does G have? What are all the divisors of this number of elements?

   15 elements, divisons are

   1, 3, 5, 15

   (b) For each divisor d from part (a), use the Fundamental Theorem of Cyclic Groups to write down a subgroup of G with d elements.

   - 1 element: $\langle g^{15/1} \rangle = \langle g^{15} \rangle = \langle 1 \rangle = \{1\}$
   - 3 elements: $\langle g^{15/3} \rangle = \langle g^5 \rangle = \{1, g^5, g^{10}\}$
   - 5 elements: $\langle g^{15/5} \rangle = \langle g^3 \rangle = \{1, g^3, g^6, g^9, g^{12}\}$
   - 15 elements: $\langle g^{15/15} \rangle = \langle g^1 \rangle = \{1, g^1, g^2, \ldots, g^{14}\}$

   Congratulations: according to the Fundamental Theorem, you've just written down a complete list of *all* the subgroups of G!

2. Now consider

$$\mathbb{Z}_{16} = \{0, 1, 2, 3, \ldots, 14, 15\},$$

which forms a cyclic group under addition modulo 16.

(a) How many elements does $\mathbb{Z}_{16}$ have? What are all the divisors of this number?

16 elements, divisors are

1, 2, 4, 8, 16

(b) For each divisor d from part (a), use the Fundamental Theorem of Cyclic Groups to write down a subgroup of G with d elements.

(**Caution**: If the operation is addition, what does an expression like g^2 or g^4 mean?)

$g = 1$

In this case, $\mathbb{Z}_{16} = \langle 1 \rangle$, and an expression like $\langle g^2 \rangle$ means $\langle 1+1 \rangle$. So the subgroups are all $\langle 16/d \rangle$ for divisors d of 16, i.e.:

- 1 element: $\langle \frac{16}{1} \rangle = \langle 0 \rangle = \{0\}$
- 2 elements: $\langle \frac{16}{2} \rangle = \langle 8 \rangle = \{0, 8\}$
- 4 elements: $\langle \frac{16}{4} \rangle = \langle 4 \rangle = \{0, 4, 8, 12\}$
- 8 elements: $\langle \frac{16}{8} \rangle = \langle 2 \rangle$
- 16 elements: $\langle \frac{16}{16} \rangle = \langle 1 \rangle$

Challenge: What do you think the subgroups of G are if $G = \langle g \rangle$ is a cyclic group with infinitely many elements?

There's a subgroup $\langle g^m \rangle \subseteq G$ for each m, and they all have infinitely many elements.