Math 335, Homework 6

Due Wednesday, March 17

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1. Apply the Fundamental Theorem of Cyclic Groups to list all of the subgroups of $\mathbb{Z}_{30}$, which is a group under addition modulo 30.

   Answer:
   Divisors of 30: 1, 2, 3, 5, 6, 10, 15. Hence all the subgroups of $\mathbb{Z}_{30}$ are:

$$\begin{aligned} \langle 1 \rangle &: 30 \text{ elements} \\ \langle 2 \rangle &: 15 \text{ elements} \\ \langle 3 \rangle &: 10 \text{ elements} \\ \langle 5 \rangle &: 6 \text{ elements} \\ \langle 10 \rangle &: 3 \text{ elements} \\ \langle 15 \rangle &: 2 \text{ elements} \end{aligned}$$

2. Let $\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$, which is a group under addition modulo 6, and let S_3 be the symmetric group, which is a group under composition. Consider the following function $\varphi : \mathbb{Z}_6 \to S_3$:

$$\begin{aligned} \varphi(0) &= e \\ \varphi(1) &= (1, 2) \\ \varphi(2) &= (1, 3) \\ \varphi(3) &= (2, 3) \\ \varphi(4) &= (1, 2, 3) \\ \varphi(5) &= (1, 3, 2). \end{aligned}$$

   Is φ an isomorphism? Prove your answer.

   Proof. As outlined above, we know that $\varphi : \mathbb{Z}_6 \to S_3$ is a bijection. Nevertheless, recall that $\mathbb{Z}_6$ under addition modulo 6 is abelian and S_3 (whose operation is composition) is not. Therefore, by the Theorem that explains the abelian properties of isomorphic groups, φ is not an isomorphism. $\square$

3. Let G and H be two groups, and suppose that there exists an isomorphism $\varphi : G \to H$. Prove that G is abelian if and only if H is abelian.

   Proof. Let G and H be groups and suppose $G \cong H$. Let G be abelian, meaning for all $a, b \in G$,

$$a * b = b * a.$$

Applying the function φ to ab results in

$$\varphi(a * b) = \varphi(a) * \varphi(b).$$

Similarly,

$$\varphi(b * a) = \varphi(b) * \varphi(a).$$

Hence

$$\varphi(a) * \varphi(b) = \varphi(b) * \varphi(a).$$

The argument for the converse is similar, but with $\varphi^{-1} : H \to G$. □

4. Prove that the function $\varphi(x) = 10^x$ is an isomorphism from the group $\mathbb{R}$ (under addition) to the group $\mathbb{R}^+ = \{\text{positive real numbers}\}$ (under multiplication).

Proof. Let $\varphi : \mathbb{R}(\text{under addition}) \to \mathbb{R}^+(\text{under multiplication})$ and $\varphi(x) = 10^x$. To prove that φ is an isomorphism, we must first prove that it is a bijection. Recall that a function is a bijection if and only if it has an inverse. Notice that the inverse of $\varphi(x) = 10^x$ is $\varphi^{-1}(x) = \log_{10}(x)$. Since φ^{-1} exists, φ is bijective.

We must now prove that

$$\varphi(a * b) = \varphi(a) * \varphi(b) \quad \text{for all } a, b \in G.$$

Replacing the operations in our function to match the operations in our domain and co-domain results in the following:

$$\begin{aligned}
\varphi(a * b) &= \varphi(a) * \varphi(b) \\
\varphi(a + b) &= \varphi(a) \cdot \varphi(b) \\
\varphi(a + b) &= 10^a \cdot 10^b \\
\varphi(a + b) &= 10^{a+b}.
\end{aligned}$$

Hence, by definition, φ is an isomorphism. □

5. In each of the following cases, decide whether G and H are isomorphic, and prove your answer. (**Hint**: In both cases, calculating orders of elements will be helpful.)

   (a) $G = \mathbb{Z}_4$ (under addition modulo 4) and $H = \{1, a, b, c\}$, under the operation described by the following table:

	1	a	b	c
1	1	a	b	c
a	a	1	c	b
b	b	c	1	a
c	c	b	a	1

   (The group H is called the *Klein four-group*.)

Proof. Let $G = \mathbb{Z}_4$ (under addition modulo 4) and $H = \{1, a, b, c\}$, under the operation described by the above table. Recall that by Theorem, given $G \cong H$, G is cyclic if and only if H is cyclic. Recollect that G is cyclic, and notice from the table that H is not. Explicitly stated,

$$\begin{aligned}
\langle 1 \rangle &= \{1\} \\
\langle a \rangle &= \{1, a\} \\
\langle b \rangle &= \{1, b\} \\
\langle c \rangle &= \{1, c\}.
\end{aligned}$$

Therefore, there is no element $h \in H$ such that $\langle h \rangle = H$. Hence, $G \ncong H$. □

(b) $G = S_4$ and $H = D_{12} = \{\text{symmetries of a regular 12-gon}\}$ (under composition)

Proof. Let $G = S_4$ and $H = D_{12} = \{\text{symmetries of a regular 12-gon}\}$ (under composition). Notice that G does not have an order greater than 4. This can be illustrated as

$$\begin{aligned}
\text{ord}(I) &= 1 \\
\text{ord}\{(1,2)\} &= 2 \\
\text{ord}\{(1,2,3)\} &= 3 \\
\text{ord}\{(1,2,3,4)\} &= 4.
\end{aligned}$$

This remains true for any permutation of G. But observe that for H, $\text{ord}(R_{30}) = 12$ (30 degree rotation). Recall that by Theorem, that given $G \cong H$, the number of elements in G of $\text{ord}(d)$ must equal the number of elements in H of $\text{ord}(d)$. But the number of elements in G of $\text{ord}(d) = 12$ is zero, while the number of elements in H of $\text{ord}(d) = 12$ is *not* zero. Therefore, $G \ncong H$. □