Equivalence Relations + Cosets

- **Reminder:** Exam next Wednesday (any 90 minutes), review Monday

- **Video:** If

$$G = \text{group}$$
$$H \subseteq G \text{ subgroup},$$

we can define an equivalence relation on G by

$$a \sim b \iff \boxed{a^{-1}b \in H \iff aH = bH}$$

$(-a+b \in H)$

And in this equivalence relation,

$$\boxed{[a] = aH}$$

Worksheet 26: Equivalence Relations and Cosets

Math 335

Reporter:

Recorder:

Equity Manager:

---

1. $G = S_3$ be the symmetric group (under the operation of composition), and let

$$H = \{e, (1,3)\}.$$

Consider the equivalence relation on G defined by

$$a \sim b \quad \Leftrightarrow \quad a^{-1}b \in H.$$

$b \in H$

$e^{-1}b \in H$

(a) Find all elements of $[e]$. (Remember, these are the elements $b \in S_3$ such that $e \sim b$.)

$[e] = \{e, (1,3)\}$

(b) Similarly, find all elements of $[(1,2)]$. $(1,2) \sim b \iff (1,2)^{-1} b \in H$

$[(1,2)] = \{(1,2), (1,2) \circ (1,3)\}$

$= \{(1,2), (1,3,2)\}$

(c) Compare your answers to parts (a) and (b) to the calculations in Problem 2 of Worksheet 22. Do you find that $[a] = aH$?

Yes: $[e] = eH$

$[(1,2)] = (1,2)H$

2. Let $S = \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$, with the equivalence relation

$$(a, b) \sim (c, d) \quad \Leftrightarrow \quad \frac{a}{b} = \frac{c}{d}.$$

For example, $(1, 2) \sim (3, 6)$ because $\frac{1}{2} = \frac{3}{6}$. Remember, this means that

$$[(1, 2)] = [(3, 6)].$$

In each of the following cases, I'm going to try to define a function

$$f : \left\{ \begin{matrix} \text{equivalence classes of} \\ S \text{ under } \sim \end{matrix} \right\} \to \mathbb{Q}.$$

Which ones are well-defined?

(a) $f\big([(a, b)]\big) = a + b$

Not well-defined, e.g.

$[(1,2)] = [(3,6)]$ but $1+2 \neq 3+6$

(b) $f\big([(a, b)]\big) = \frac{a}{b}$

Well-defined, e.g.

$[(1,2)] = [(3,6)]$ and $\frac{1}{2} = \frac{3}{6}$

(c) $f\big([(a, b)]\big) = \frac{b}{a}$

Well-defined, e.g.

$[(1,2)] = [(3,6)]$ and $\frac{2}{1} = \frac{6}{3}$

3. **Challenge**: What's another way to think of the domain of the functions in Problem 2? Using this perspective, can you express any of the functions in Problem 2 more succinctly?

The domain is $\mathbb{Q}$! In this sense, the functions in parts (b) + (c) are $f(x) = x$ and $f(x) = \frac{1}{x}$.