Fundamental Theorem of Finite Abelian Groups

- Gives a procedure for "classifying" abelian groups with a given # of elements — i.e., writing a list so that
  - no two groups on the list are $\cong$
  - any abelian group with that # of elements is $\cong$ to something on the list

- <u>Comments</u>:
  1) Surprising! There are a TON of ways to fill in a 36×36 table, and only four of them form abelian groups!
  2) <u>Much</u> harder for non-abelian groups, and there are no general methods. E.g. in Gallian, it's not until Ch. 26 that we can classify all groups of size 8.

Worksheet 37: Fundamental Theorem of Finite Abelian Groups

Math 335

Reporter:

Recorder:

Equity Manager:

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1. (a) Use the Fundamental Theorem to list all abelian groups of size 24.

$24 = 2 \cdot 2 \cdot 2 \cdot 3$

- $2 \cdot 2 \cdot 2 \cdot 3 \rightsquigarrow \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3$
- $2 \cdot 4 \cdot 3 \rightsquigarrow \mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_3$
- $8 \cdot 3 \rightsquigarrow \mathbb{Z}_8 \oplus \mathbb{Z}_3$

(b) The group $\mathbb{Z}_{24}$ must be isomorphic to one of the groups on your list; which one?

$\mathbb{Z}_{24} \cong \mathbb{Z}_8 \oplus \mathbb{Z}_3$

(by Theorem from last time, because $\gcd(8,3) = 1$)

(c) The group $\mathbb{Z}_2 \oplus \mathbb{Z}_{12}$ must be isomorphic to one of the groups on your list; which one?

$\mathbb{Z}_2 \oplus \mathbb{Z}_{12} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_3$

(by Theorem from last time, because $\gcd(4,3) = 1$)

2. On Homework 2, you proved that the set $G = \{5, 15, 25, 35\}$ is a group under the operation of multiplication mod 40. To what direct product of $\mathbb{Z}_n$'s is this group isomorphic?

(**Hint**: What are the possibilities, based on the number of elements of G? Try using orders of elements to figure out which possibility is correct.)

This is an abelian group with 4 elements, so it's isomorphic to either

$\mathbb{Z}_4$ or $\mathbb{Z}_2 \oplus \mathbb{Z}_2$

orders of elements are 1, 2, and 4

orders of elements are 1 and 2

In G, the orders are

$\text{ord}(5) = 2$, $\text{ord}(15) = 2$, $\text{ord}(25) = 1$, $\text{ord}(35) = 2$

so $\boxed{G \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2}$

3. The set

$$G = \{1, 7, 17, 23, 49, 55, 65, 71\}$$

forms a group under multiplication modulo 96. To what direct product of $\mathbb{Z}_n$'s is this group isomorphic?

(Feel free to use Wolfram Alpha, where you can type things like "17^2 mod 96.")

This is an abelian group with 8 elements, so it's isomorphic to either

$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ or $\mathbb{Z}_2 \oplus \mathbb{Z}_4$ or $\mathbb{Z}_8$

orders of elements are 1 and 2

orders of elements are 1, 2, and 4

orders of elements are 1, 2, 4, and 8

In G,

$\text{ord}(1) = 1$, $\text{ord}(7) = 4$, $\text{ord}(17) = 2$, $\text{ord}(23) = 4$,
$\text{ord}(49) = 2$, $\text{ord}(55) = 4$, $\text{ord}(65) = 2$, $\text{ord}(71) = 4$,

so $\boxed{G \cong \mathbb{Z}_2 \oplus \mathbb{Z}_4}$.