Exam 2 Review:

- Finishing from Friday: (NOTE: This material is not on Wednesday's exam.)

Given a "function",

$$f : \text{equivalence classes} \longrightarrow$$

we must check that it's well-defined:

$$[a] = [b] \implies f([a]) = f([b]).$$

If not, it's not a function!

- Analogy:

$$f : \mathbb{Q} \longrightarrow \mathbb{Q}$$
$$f\left(\frac{a}{b}\right) = a+b$$

isn't really a function, because

$$\frac{1}{2} = \frac{3}{6} \quad \text{but} \quad \underbrace{f\left(\tfrac{1}{2}\right)}_{1+2} \neq \underbrace{f\left(\tfrac{3}{6}\right)}_{3+6}.$$

- **Exam Logistics**:
  - Any 90 minutes on Wednesday
  - Use any non-human resource (books, notes, videos, etc.)
  - No class or office hours on Wednesday

- **Topics** (see iLearn for detailed list):
  - Cyclic groups (esp. orders of elements in cyclic groups and the Fundamental Theorem)
  - Isomorphisms
  - Cosets + Lagrange's Theorem
  - Equivalence relations + equivalence classes

- **Final Project**: Guidelines on iLearn, more detail (+ PDFs of readings) to come!

Math 335, Midterm 2

November 6, 2019

Name: ______________________________

Problem	Points Scored	Total Points Possible
1		8
2		8
3		8
4		8
5		8
Total		40

Problem 1:

(a) Carefully define what it means for a group G to be **cyclic**.

G is cyclic if $\exists\ g \in G$ such that $\langle g \rangle = G$.

(b) Let

$$U_4 = \{1, -1, i, -i\} \subseteq \mathbb{C}^*,$$

which is a group under multiplication of complex numbers. Explain how you know that U_4 is cyclic.

$$\langle i \rangle = \{1, i, i^2, i^3\}$$
$$= \{1, i, -1, -i\}$$
$$= U_4$$

(c) Let

$$\mathbb{Z}_5 = \{0, 1, 2, 3, 4\},$$

which is a group under addition modulo 5. In this group, what is $\langle 3 \rangle$? List all of its elements.

$$\langle 3 \rangle = \{0, 3, 1, 4, 2\}$$

Problem 2: You do not need to show your work or give justifications on this problem.

(a) Let

$$G = \langle a \rangle = \{1, a, a^2, a^3, \ldots, a^{19}\},$$

a group under multiplication in which a has order 20. List all the subgroups of G, being sure to list each one only once.

One of size d for each $d \mid 20$:

- $d=1$: $\langle a^{20/1} \rangle = \langle 1 \rangle = \{1\}$
- $d=2$: $\langle a^{20/2} \rangle = \langle a^{10} \rangle = \{1, a^{10}\}$
- $d=4$: $\langle a^{20/4} \rangle = \langle a^5 \rangle = \{1, a^5, a^{10}, a^{15}\}$
- $d=5$: $\langle a^{20/5} \rangle = \langle a^4 \rangle = \{1, a^4, a^8, a^{12}, a^{16}\}$
- $d=10$: $\langle a^{20/10} \rangle = \langle a^2 \rangle = \{1, a^2, a^4, a^6, \ldots, a^{18}\}$
- $d=20$: $\langle a^{20/20} \rangle = \langle a^1 \rangle = \{1, a, a^2, a^3, \ldots, a^{19}\}$

(b) Let G be the same group as in part (a). For which element $b \in G$ does $\langle b \rangle$ have exactly two elements?

$b = a^{10}$

(c) Give an example of a group G with no proper, nontrivial subgroups.

$G = \{1, a, a^2, a^3, a^4$ in which $\mathrm{ord}(a) = 5$

(or, more generally, any group with a prime number of elements)

Problem 3: In each of the following cases, is the group G isomorphic to the group H? If so, write down an isomorphism $\phi : G \to H$. (You do **not** need to prove that it is an isomorphism.) If not, briefly explain how you know.

(a) $G = D_4$ (under composition) and $H = \mathbb{Z}_8$ (under addition modulo 8).

No: $\mathbb{Z}_8$ is abelian (or cyclic) and D_4 is not.

(b) $G = \mathbb{Z}_3$ (under addition modulo 3) and $H = \{0, 2, 4\} \subseteq \mathbb{Z}_6$ (under addition modulo 6) .

Yes: an isomorphism is

$$\phi : \mathbb{Z}_3 \longrightarrow H$$

$$\phi(0) = 0$$

$$\phi(1) = 2$$

$$\phi(2) = 4$$

Problem 4: Let $G = S_3$ (under the operation of composition), and let $H \subseteq G$ be the subgroup generated by the element $(1, 2, 3)$.

(a) What is H? Write down all of its elements.

$$H = \{e,\ (1,2,3),\ (1,2,3)\circ(1,2,3)\}$$
$$= \boxed{\{e,\ (1,2,3),\ (1,3,2)\}}$$

(b) What is the left coset $(1, 2)H$? Write down all of its elements.

$$(1,2)H = \{(1,2),\ (1,2)\circ(1,2,3),\ (1,2)\circ(1,3,2)\}$$
$$= \boxed{\{(1,2),\ (2,3),\ (1,3)\}}$$

(c) Apply Lagrange's Theorem to determine the number of left cosets of H in G.

$$\frac{|G|}{|H|} = \frac{6}{3} = \boxed{2}$$

(d) List all of the left cosets of H in G, being sure to list each one only once.

$$eH = \{e,\ (1,2,3),\ (1,3,2)\}$$
$$(1,2)H = \{(1,2),\ (2,3),\ (1,3)\}$$

Problem 5:

(a) Let S be a set with an equivalence relation $\sim$, and let T be any set. Carefully define what it means to say that a function $f : S \to T$ is well-defined under $\sim$.

f is <u>well-defined</u> if

$$a \sim b \implies f(a) = f(b)$$

(b) Consider the equivalence relation on the nonzero real numbers $\mathbb{R}^*$ defined by

$$a \sim b \quad \Leftrightarrow \quad ab > 0.$$

What is [1]?

$$\begin{aligned}
[1] &= \{ b \in \mathbb{R}^* \mid 1 \sim b \} \\
&= \{ b \in \mathbb{R}^* \mid 1 \cdot b > 0 \} \\
&= \{ \text{positive real numbers} \}
\end{aligned}$$

(c) Let $\sim$ be the same equivalence relation as in part (b). Give an example to show that

$$f : \left\{ \begin{matrix} \text{equivalence classes} \\ \text{under } \sim \end{matrix} \right\} \to \mathbb{R}$$
$$f([a]) = a + 1$$

is **not** well-defined.

$[1] = [2]$ but $f([1]) \neq f([2])$