Math 335, Exam 2

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Problem	Points Scored	Total Points Possible
1		8
2		8
3		8
4		8
5		8
Total		40

Problem 1: You do **not** need to explain your answers or provide any justification on this problem.

(a) In the group

$$\mathbb{Q} = \{\text{rational numbers}\},$$

where the operation is **addition**, what is $\langle 2 \rangle$? (Describe your answer explicitly, either in words or by listing enough elements to clearly indicate the answer.)

$\langle 2 \rangle = \{0, 2, 4, 6, 8, \ldots, -2, -4, -6, -8, \ldots\}$

(b) In the group

$$\mathbb{Q}^* = \{\text{nonzero rational numbers}\},$$

where the operation is **multiplication**, what is $\langle 2 \rangle$? (Describe your answer explicitly, either in words or by listing enough elements to clearly indicate the answer.)

$\langle 2 \rangle = \{1, 2, 2^2, 2^3, 2^4, \ldots, 2^{-1}, 2^{-2}, 2^{-3}, 2^{-4}, \ldots\}$

$= \{1, 2, 4, 8, 16, \ldots, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots\}$

(c) In the symmetric group S_3, where the operation is composition, what is

$$\langle (1, 3, 2) \rangle?$$

List all of its elements.

$\langle (1,3,2) \rangle = \{e, (1,3,2), (1,2,3)\}$

Problem 2: This problem concerns the group

$$\mathbb{Z}_{12} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\},$$

under the operation of addition modulo 12. You do **not** need to explain your answers or provide any justification on this problem.

(a) List all of the subgroups of $\mathbb{Z}_{12}$, being sure to list each one only once.

One of size d for each $d \mid 12 \Rightarrow 12, 6, 4, 3, 2, 1$

$\langle \frac{12}{1} \rangle = \langle 12 \rangle = \langle 0 \rangle = \{0\}$

$\langle \frac{12}{2} \rangle = \langle 6 \rangle = \{0, 6\}$

$\langle \frac{12}{3} \rangle = \langle 4 \rangle = \{0, 4, 8\}$

$\langle \frac{12}{4} \rangle = \langle 3 \rangle = \{0, 3, 6, 9\}$

$\langle \frac{12}{6} \rangle = \langle 2 \rangle = \{0, 2, 4, 6, 8, 10\}$

$\langle \frac{12}{12} \rangle = \langle 1 \rangle = \{0, 1, 2, 3, 4, \dots, 11\}$

(b) How many subgroups with 3 elements does $\mathbb{Z}_{12}$ have?

One subgroup with 3 elements. Specifically, it is the subgroup $\langle 4 \rangle$.

(c) Give **two** examples of elements of $\mathbb{Z}_{12}$ with order 3.

$4+4+4 = 12 = 0 \bmod 12 = e \Rightarrow \operatorname{ord}(4) = 3$

$8+8+8 = 24 = 0 \bmod 12 = e \Rightarrow \operatorname{ord}(8) = 3$

$4 \,\&\, 8 \in \mathbb{Z}_{12}$

Problem 3:

(a) Let

$$G = \{1, g, g^2, g^3\} \quad \text{(where ord}(g) = 4 \text{ and the operation is multiplication)}$$

and

$$H = U_4 = \{1, -1, i, -i\} \quad \text{(where the operation is multiplication).}$$

Use the spaces below to write down an isomorphism $\phi : G \to H$. You do **not** need to prove that ϕ is an isomorphism.

$\phi(1) =$ 1

$\phi(g) =$ i

$\phi(g^2) =$ -1

$\phi(g^3) =$ $-i$

(b) Explain how you know that the group $\mathbb{Z}_{24}$ (where the operation is addition modulo 24) is **not** isomorphic to the symmetric group S_4 (where the operation is composition).

The group $\mathbb{Z}_{24}$ is abelian and the group S_4 is not abelian. $\therefore$ by theorem, $\mathbb{Z}_{24}$ is not isomorphic to S_4.

(c) Explain how you know that the group D_4 of symmetries of a square (where the operation is composition) is **not** isomorphic to the symmetric group S_4 (where the operation is composition).

$\left.\begin{array}{l} D_4 \text{ has } 8 \text{ elements} \\ S_4 \text{ has } 4! = 24 \text{ elements} \end{array}\right\}$ They must be a bijection which they clearly are not.

$\therefore$ D_4 is not isomorphic to S_4.

Problem 4: This problem concerns the group

$$G = \{1, g, g^2, g^3, \ldots, g^{11}\},$$

where $\mathrm{ord}(g) = 12$ and the operation is multiplication, and the subgroup

$$H = \langle g^4 \rangle. = \{1, g^4, g^8\}$$

You do **not** need to explain your answers or provide any justification on this problem.

(a) List all elements of the left coset g^3H.

$$g^3H = \{g^3 \cdot 1, g^3 \cdot g^4, g^3 \cdot g^8\}$$
$$= \{g^3, g^7, g^{11}\}$$

(b) Apply Lagrange's Theorem to calculate the number of distinct left cosets of H in G.

$$\frac{|G|}{|H|} = \frac{12}{3} = 4$$

(c) List all of the left cosets of H in G, being sure to list each one only once.

$$eH = \{1, g^4, g^8\}$$
$$gH = \{g, g^5, g^9\}$$
$$g^2H = \{g^2, g^6, g^{10}\}$$
$$g^3H = \{g^3, g^7, g^{11}\}$$

Problem 5: This problem concerns the group

$$\mathbb{Z}_8 = \{0, 1, 2, 3, 4, 5, 6, 7\}, \overset{0}{8}\ \overset{1}{9}$$

where the operation is addition modulo 8, and the subgroup

$$H = \{0, 2, 4, 6\}.$$

You do **not** need to explain your answers or provide any justification on this problem.

(a) Give an example of two distinct elements $a, b \in \mathbb{Z}_8$ such that

$$a + H = b + H.$$

$2 \quad eH = \{0,2,4,6\}$

$3 \quad 1+H = \{1,3,5,7\}$

$$\boxed{\begin{aligned} a &= 1 \\ b &= 3 \end{aligned}}$$

$1+H = \{1,3,5,7\}$

$3+H = \{3,5,7,1\}$

(b) Define an equivalence relation on G by

$$a \sim b \quad \text{means} \quad b - a \in H.$$

What is [3]? List all of its elements.

$$\begin{aligned} [3] &= \{\mathbb{Z}_8 \text{ under } + \bmod 8 \mid 3 \sim b\} \\ &= \{\mathbb{Z}_8 \text{ under } + \bmod 8 \mid b-3 \in H\} \\ &= \{3,5,7,1\} \\ &= \{1,3,5,7\} \end{aligned}$$

(c) Congratulations! Take a breath, look back through your solutions if you have some extra time, and submit your work. Then take a moment to congratulate yourself on how much you've learned in this course so far!