# Direct Products

- **Announcement:** Mathematistas "Happy Hour" with Prof. Sheldon Axler today at 3pm

- **Video:** $G \oplus H = \{(g,h) \mid g \in G, h \in H\}$ \oplus

- **Ex:** $\mathbb{Z}_2 \oplus \mathbb{Z}_4 = \{(0,0), (1,0), (0,1), (1,1), (0,2), (1,2), (0,3), (1,3)\}$

**Note:** $|G \oplus H| = |G| \cdot |H|$

$$(1,2) + (1,3) = (0,1)$$

- **Question:** $\mathbb{Z}_2 \oplus \mathbb{Z}_4 \cong \mathbb{Z}_8$?

  More generally, $\mathbb{Z}_n \oplus \mathbb{Z}_m \cong \mathbb{Z}_{nm}$?

# Worksheet 34: Direct Products

Math 335

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**Reporter**:

**Recorder**:

**Equity Manager**:

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1. We'll start by considering the direct product $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, where $\mathbb{Z}_2$ is a group under addition modulo 2.

   (a) List all the elements of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.

   $(0,0)$ ← identity in $\mathbb{Z}_2 \oplus \mathbb{Z}_2$

   $(0,1)$

   $(1,0)$

   $(1,1)$

   (b) Calculate the order of each of these elements.

   $\text{ord}((0,0)) = 1$ ← because it's the identity

   $\text{ord}((0,1)) = 2$ ← because $(0,1)+(0,1)=(0,0)$

   $\text{ord}((1,0)) = 2$

   $\text{ord}((1,1)) = 2$ ← similar

   (c) Is it true that $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \cong \mathbb{Z}_4$? Why or why not?

   No! In $\mathbb{Z}_4$, there's an element of order 4, but <u>not</u> in $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.

2. Now let's look at a different direct product: $\mathbb{Z}_2 \oplus \mathbb{Z}_3$. We listed the elements of this group in the example we did as a class earlier.

   (a) What is the subgroup $\langle(1,1)\rangle \subseteq \mathbb{Z}_2 \oplus \mathbb{Z}_3$? List all of its elements.

$$\langle(1,1)\rangle = \{(0,0),\ (1,1),\ \underbrace{(1,1)+(1,1)}_{=(0,2)},\ \underbrace{(1,1)+(1,1)+(1,1)}_{=(1,0)},\ \underbrace{(1,1)+(1,1)+(1,1)+(1,1)}_{=(0,1)},\ \underbrace{(1,1)+(1,1)+(1,1)+(1,1)+(1,1)}_{(1,2)}\}$$

$\langle(1,1)\rangle = \mathbb{Z}_2 \oplus \mathbb{Z}_3$

   (b) Use the answer to the previous question to find an isomorphism

$$\varphi : \mathbb{Z}_6 \to \mathbb{Z}_2 \oplus \mathbb{Z}_3.$$

$$\varphi(0) = (0,0)$$
$$\varphi(1) = (1,1)$$
$$\varphi(2) = (1,1)+(1,1) = (0,2)$$
$$\varphi(3) = (1,1)+(1,1)+(1,1) = (1,0)$$
$$\varphi(4) = 4\cdot(1,1) = (0,1)$$
$$\varphi(5) = 5\cdot(1,1) = (1,2)$$

3. **Vague discussion question**: Discuss how Problem 1 and Problem 2 differ, and what conclusions you can draw.

It's sometimes, but not always, true that

$$\mathbb{Z}_n \oplus \mathbb{Z}_m \cong \mathbb{Z}_{nm}.$$

Specifically, it's true $\Longleftrightarrow$ $\mathbb{Z}_n \oplus \mathbb{Z}_m$ has an element of order $nm$... when is this?