Math 335, Midterm 1

October 4, 2019

Name: ____

Problem	Points Scored	Total Points Possible
1		8
2		8
3		8
4		8
5		8
Total		40

Problem 1: You do not need to show any work or give justifications on this problem.

(a) Fill in the blank with a number in the range $\{0, 1, 2, 3, 4, 5\}$:

$-7 + 6 = -1 + 6 = 5$

$$-7 \equiv \underline{\quad 5 \quad} \mod 6.$$

(b) We've seen that the set $\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$ is a group under the operation of addition modulo 6. In this group, what is the inverse of 2?

$2 + 4 = 6 \equiv 0 \mod 6$

(c) Give an example of a proper, nontrivial subgroup of $\mathbb{Z}_6$. (Write out your subgroup explicitly, by listing all of its elements.)

$\{0, 3\}$

0. closed under addition
1. associative ✓ (assumed
2. identity $e = 0$
3. inverse

	0	3
0	0	3
3	3	0

(d) Give an example of a subset of $\mathbb{Z}_6$ that is *not* a subgroup.

$\{0, 1\}$ $\quad 1 + 1 = 2 \mod 6 \notin \{0, 1\}$

NOT CLOSED UNDER ADDITION

Problem 2: Which of the following are groups? In each case, circle "group" or "not a group"; you do not need to justify your answer.

(a) $G = \{0, 2\}$, where the operation is addition modulo 4.

closed ✓

	0	2
0	0	2
2	2	0

associative ✓

identity $e = 0$

inverse

group

not a group

(b) $G = \{1, 2, 3\}$, where the operation is multiplication modulo 4.

	1	2	3
1	1	2	3
2	2	0	2
3	3	2	1

group

not a group

inverse of 2 is $\frac{1}{2} \notin G$

(c) $G = \{\text{nonzero integers}\}$, where the operation is multiplication.

~~not closed~~

inverse of $g \in G$ is $\frac{1}{g} \notin G$

group

not a group

(d) $G = \{\text{rotational symmetries of a square}\} = \{I, R_{90}, R_{180}, R_{270}\}$, where the operation is composition.

group

not a group

	I	R_{90}	R_{180}	R_{270}
I	I	R_{90}	R_{180}	R_{270}
R_{90}	R_{90}	R_{180}	R_{270}	I
R_{180}	R_{180}	R_{270}	I	R_{90}
R_{270}	R_{270}	I	R_{90}	R_{180}

Problem 3: Suppose G is a group with four elements,

$$G = \{a, b, c, d\},$$

and the group table for G is the following:

x e

	a	b	c	d	
a	d	c	a	b	
b	c	d	b	a	
c	a	b	c	d	$\leftarrow e$
d	b	a	d	c	

y

In other words, the entry in the top-right corner means that $a * d = b$.

(a) What is the identity element of G? Briefly explain how you know.

$e = c$. By definition $x * e = x$, and $e * x = x$ so notice that $c * x = x$ and $x * c = *$ on the table

(b) What is the inverse of a? Briefly explain how you know.

$a^{-1} = b$. By definition $x * x^{-1} = e$ and $x^{-1} * x = e$
$e = c$, and $a * b = c$ & $b * a = c$, $a^{-1} = b$

(c) Is the operation on G commutative—that is, is G abelian? Briefly explain how you know.

Yes, G is commutative. If you transpose the columns and rows of the table, you get the same identical table.
Specifically, $x * y = y * x$

Problem 4: You do not need to show any work or give justifications on this problem.

(a) Carefully define what it means to say that a function $f : A \to B$ is **injective** (or **one-to-one**).

(b) In the symmetric group S_5, let

$$\alpha = (1,2,3,5), \quad \beta = (1,3,4).$$

What is $\alpha \circ \beta$? Express your answer in cycle notation, as a composition of disjoint cycles.

$\alpha \circ \beta = (1,5)(2,3,4)$

(c) In the symmetric group S_6, what is the order of

$$f = (1,3)\ (2,6,5,4)?$$

$\mathrm{ord}(f) = \mathrm{lcm}(2,4) = 4$

(d) If f is the same permutation as in part (c), what is f^8?

$f^8 = (1)(2)(3)(4)(5)(6)$

Problem 5:

(a) Carefully define what it means for a group G to be **cyclic**.

(b) Let

$$U_4 = \{1, -1, i, -i\} \subseteq \mathbb{C}^*,$$

which is a group under multiplication of complex numbers. Explain how you know that U_4 is cyclic.

(c) Let

$$\mathbb{Z}_5 = \{0, 1, 2, 3, 4\},$$

which is a group under addition modulo 5. In this group, what is $\langle 3 \rangle$? List all of its elements.