Quotient Groups

- Video: If

  G = abelian group
  $H \subseteq G$ subgroup,

  then

  $$G/H = \left\{ \begin{matrix} \text{left cosets of} \\ H \text{ in } G \end{matrix} \right\}$$

  forms a group.

- Note: $\left| G/H \right| = |G|/|H|$ (Lagrange's Theorem)

- Question: What if G isn't abelian?

Worksheet 29: Quotient Groups

Math 335

Reporter:

Recorder:

Equity Manager:

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1. As in the worksheet from last class, let

$$G = \langle g \rangle = \{1, g, g^2, g^3, \ldots, g^{14}\}$$
$$H = \langle g^5 \rangle = \{1, g^5, g^{10}\} \subseteq G,$$

where $\operatorname{ord}(g) = 15$.

(a) List all the elements of the group G/H. (You can look back at the notes from last class to do this.)

$$G/H = \{\underbrace{1H}_{=\{1, g^5, g^{10}\}}, gH, g^2H, g^3H, g^4H\}$$

(b) Make a table that shows how to multiply any element of G/H by any other to get a new element of G/H.

$1H = g^5H$

	$1H$	gH	g^2H	g^3H	g^4H
$1H$	$1H$	gH	g^2H	g^3H	g^4H
gH	gH	g^2H	g^3H	g^4H	$1H$
g^2H	g^2H	g^3H	g^4H	$1H$	gH
g^3H	g^3H	g^4H	$1H$	gH	g^2H
g^4H	g^4H	$1H$	gH	g^2H	g^3H

Looks like the table for the group $\{1, g, g^2, g^3, g^4\}$ where $g^5 = 1$.

2. Now let's look at a case where G is not abelian, to see how the definition of G/H can go wrong. Let $G = S_3$, and let

$$H = \{e,\ (1,2)\} \subseteq G,$$

where e stands for the identity permutation.

(a) Calculate all the elements of each of the following left cosets

$$eH = \{e,\ (1,2)\}$$

$$(1,3)H = \{(1,3)\circ e, (1,3)\circ(1,2)\} = \{(1,3), (1,2,3)\}$$

(b) What is another way to write eH? Write it as aH for some $a \neq e$.

$$eH = (1,2)H$$

(Check: $(1,2)H = \{(1,2)\circ e, (1,2)\circ(1,2)\} = \{(1,2), e\}$)

(c) For the a you found above, what is the coset $\Big(a \circ (1,3)\Big)H$? Write down all of its elements.

$$\Big((1,2)\circ(1,3)\Big)H = (1,3,2)H$$
$$= \{(1,3,2),\ (1,3,2)\circ(1,2)\}$$
$$= \{(1,3,2),\ (2,3)\}$$

(d) Confirm that

$$\underbrace{\Big(e \circ (1,3)\Big)H}_{\{(1,3),\,(1,2,3)\}} \neq \underbrace{\Big(a \circ (1,3)\Big)H}_{\{(1,3,2),\,(2,3)\}}.$$

What does this show?

This shows the operation on cosets isn't well-defined:

$$eH \circ (1,3)H = (e\circ(1,3))H$$
$$\text{"}$$
$$(1,2)H \circ (1,3)H = ((1,2)\circ(1,3))H$$

different!