Lagrange's Theorem

- **Theorem**: If G is a finite group and $H \subseteq G$ is a subgroup, then

$$\begin{array}{c}\#\text{ of left}\\ \text{cosets of } H\\ \text{in } G\end{array} = \frac{|G|}{|H|}.$$

- This has lots of consequences, perhaps the most important of which is that the size of a subgroup of G always divides the size of G.

Worksheet 24: Lagrange's Theorem and its Consequences

Math 335

Reporter:

Recorder:

Equity Manager:

1. Let g be an element of order 15 in a group, and let

$$G = \langle g \rangle = \{e, g, g^2, g^3, \ldots, g^{14}\}$$
$$H = \langle g^5 \rangle = \{e, g^5, g^{10}\} \subseteq G.$$

   (a) Apply Lagrange's Theorem to determine the number of left cosets of H in G.

$$\# \text{ of left cosets} = \frac{|G|}{|H|} = \frac{15}{3} = \boxed{5}$$

   (b) Sanity-check your answer to part (a) by listing all of the left cosets of H in G, trying to list each one only once.

$$eH = \{e, g^5, g^{10}\}$$
$$gH = \{g, g^6, g^{11}\}$$
$$g^2H = \{g^2, g^7, g^{12}\}$$
$$g^3H = \{g^3, g^8, g^{13}\}$$
$$g^4H = \{g^4, g^9, g^{14}\}$$

2. Now suppose g has order 30 and

$$G = \langle g \rangle,$$
$$H = \langle g^4 \rangle \subseteq G.$$

By Lagrange's Theorem, how many left cosets of H in G are there in this case?

First, calculate $|H| = \text{ord}(g^4) = \dfrac{30}{\gcd(4,30)} = \dfrac{30}{2} = 15.$

Then Lagrange's Theorem says:

$$\# \text{ of left cosets} = \frac{|G|}{|H|} = \frac{30}{15} = \boxed{2}.$$

3. Lagrange's Theorem doesn't apply when the group has infinitely many elements, but we can still study the left cosets. For example, let $G = \mathbb{Z}$ (under addition), and let

$$H = \langle 3 \rangle = \{0, 3, 6, 9, \ldots, -3, -6, -9, \ldots\} \subseteq G.$$

(a) List all of the left cosets of H in G.

$0 + H = \{0, 3, 6, 9, \ldots, -3, -6, -9, \ldots\}$

$1 + H = \{1, 4, 7, 10, \ldots, -2, -5, -8, \ldots\}$

$2 + H = \{2, 5, 8, 11, \ldots, -1, -4, -7, \ldots\}$

(b) Calculate some elements of the left cosets $4 + H$ and $1 + H$, and convince yourself that these two left cosets are actually the same.

$4 + H = \{4, 7, 10, 13, \ldots, 1, -2, -5, \ldots\}$

$1 + H = \{1, 4, 7, 10, \ldots, -2, -5, \ldots\}$

These have the same elements!

(c) Along the same lines, are the left cosets $5 + H$ and $7 + H$ the same?

No:

$5 + H = \{5, 8, 11, \ldots, 2, -1, -4, \ldots\}$

$7 + H = \{7, 10, 13, \ldots, 4, 1, -2, \ldots\}$

which have different elements.

(d) Do you have a conjecture about the conditions under which $a + H = b + H$?

$a + H = b + H \iff a \equiv b \bmod 3$

Another way to say this:

$$\boxed{a + H = b + H \iff b - a \in H}$$