Math 335, Homework 5

Due Wednesday, March 10

---

1. Is the group D_4 (the group of symmetries of a square, under the operation of composition) cyclic? Carefully explain how you know.

2. (a) Let $G = \langle a \rangle$ be a cyclic group in which $\text{ord}(a) = \infty$. Prove that

$$\langle a^k \rangle \subseteq \langle a^m \rangle$$

   if and only if $m|k$. (**Hint**: A problem from Homework 4 will be helpful here.)

   (b) Give a counterexample to show that part (a) is false if $\text{ord}(a)$ is finite. (**Hint**: Try making m larger than the order of a.)

3. Let G be any group, and let $a \in G$ be an element of order 15. What is the order of a^6? Of a^{10}? Prove your answers.

4. Consider the group $\mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\}$ under addition modulo n. We say that an element k of this group *generates* $\mathbb{Z}_n$ if $\langle k \rangle = \mathbb{Z}_n$.

   (a) List all of the elements of $\mathbb{Z}_9$ that generate $\mathbb{Z}_9$.

   (b) Prove that k generates $\mathbb{Z}_n$ if and only if $\gcd(n, k) = 1$.

5. Consider the group $\mathbb{Z}_p = \{0, 1, 2, \ldots, p-1\}$ under addition modulo p, where p is a prime number. What are all of the subgroups of $\mathbb{Z}_p$? Carefully explain how you know that you've found them all.