Unit – 2

Ordinary Differential Equation (part – II)

* One Integral Know
* Removal of 1^{st} Derivative
* Change of Order of Integration.
* Variation of Parameters

(A) One Integral Known :-

Working Rule :-

(1) Consider 2nd order ODE :- (ordinary Differential Eq^n)

Form:

$$\boxed{\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R} \quad \text{--- (1)}$$

where,

P, Q & R are functions of 'x' alone or constants

(2) On comparing, the given problem with eqⁿ (1)
we get P, Q, R.

(3) Let $y = uv$ be the complete solⁿ.

(4) One of the solⁿ 'u' is known integral (known solⁿ) which we have to find by following conditions :-

(a) $1 + P + Q = 0$, $u = e^{x}$

(b) $1 - P + Q = 0$, $u = e^{-x}$

(c) $P + Qx = 0$, $u = x$

(5) Now, for complete solⁿ, we have to find 'v', by using following eqⁿ

$$\boxed{\frac{d^2v}{dx^2} + \left[\frac{2}{u}\cdot\frac{du}{dx} + P\right]\frac{dv}{dx} = \frac{R}{u}}$$

(6) Now, we take,

$$\boxed{\frac{dv}{dx} = t}$$

Now, Differentiate it,

$$\boxed{\frac{d^2v}{dx^2} = \frac{dt}{dx}}$$

$\frac{(1)}{u}$

sides

(7) On solving this, we will get the value 'v'.

(8) Now, for complete sol^n, we write

$$y = uv$$

Q. Solve:

v.v. imp

$$x\frac{d^2y}{dx^2} - (2x-1)\frac{dy}{dx} + (x-1)y = 0$$

Divide ~~the~~ both sides with x,

we get,

$$\frac{d^2y}{dx^2} - \left(\frac{2x}{x} - \frac{1}{x}\right)\frac{dy}{dx} + \left(\frac{x}{x} - \frac{1}{x}\right)y = 0$$

$$\frac{d^2y}{dx^2} - \left(2 - \frac{1}{x}\right)\frac{dy}{dx} + \left(1 - \frac{1}{x}\right)y = 0 \quad \text{--(1)}$$

So,

on comparing (1) with:

$$\frac{d^2y}{dx^2} + P\frac{dy}{dy} + Qy = R$$

So,

$$P = -\left(2 - \frac{1}{x}\right)$$

$$Q = 1 - \frac{1}{x}$$

$\boxed{R = 0}$

Now,

$$1 + P + Q$$

$$= 1 + \left(-\left(2 - \frac{1}{x}\right) + \left(1 - \frac{1}{x}\right)\right)$$

$$= 1 - 2 + \cancel{\frac{1}{x}} + 1 - \cancel{\frac{1}{x}}$$

$\frac{0}{u}$

$$= 2 - 2$$

$$\boxed{= 0}$$

$\because$ $1 + P + Q = 0$

so, $\boxed{u = e^x}$ — (2)

now, ~~v is give~~

$$\frac{d^2v}{dx^2} + \left[\frac{2}{u}\frac{du}{dx} + P\right]\frac{dv}{dx} = \frac{R}{u}$$

$$\frac{d^2v}{dx^2} + \left[\frac{2}{\cancel{e^x}}\cancel{e^x} + \left(\frac{1}{x} - 2\right)\right]\frac{dv}{dx} = \frac{0}{u}$$

$$\frac{d^2v}{dx^2} + \left[\cancel{2} + \frac{1}{x} - \cancel{2}\right]\frac{dv}{dx} = 0$$

$$\frac{d^2v}{} + \left(\frac{1}{x}\right)\frac{dv}{dx} = 0$$

$$\frac{dv}{dx} = t \quad -(3)$$

differentiating it,

$$\frac{d^2v}{dx^2} = \frac{dt}{dx} \quad -(4)$$

$$\frac{dt}{dx} + \frac{1}{x}(t) = 0$$

$$\frac{dt}{dx} = -\frac{t}{x}$$

$$\frac{dt}{t} = \frac{-dx}{x}$$

Now, integrating both sides, we get-

$$\int \frac{1}{t}\, dt = -\int \frac{1}{x}\, dx$$

$$\log t = -\log x + \log c$$

$$\log t + \log x = \log c$$

$$\log(tx) = \log c$$

So, $tx = c$

$$t = \frac{c}{x}$$

from (3),

$$\frac{dv}{dx} = \frac{c}{x}$$

$$dv = \frac{c}{x}\, dx.$$

Again,

integrating both sides,

we get,

$$\int dv = c \int \frac{1}{x}\, dx.$$

$$\boxed{v = c \log x + \log c_1} \quad — (5)$$

Now,

for complete solⁿ —

$$y = \cancel{ex}\, uv$$

$$\text{So, } \boxed{y = e^{x}(c \log x + \log c_1)} \quad — (6)$$

Solve:

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0 \quad , \quad -① $$

$x + \frac{1}{x}$ is one integral.

now,

$$u = x + \frac{1}{x}$$

Divide eqⁿ ① by 'x^2'

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - \frac{y}{x^2} = 0 \quad -②$$

on comparing eqⁿ ②, with

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$$

So,

$$P = \frac{1}{x}$$

$$Q = -\frac{1}{x^2}$$

$$R = 0$$

Now,

To find 'v'

v is given by:-

$$\frac{d^2v}{dx^2} + \left[\frac{2}{u}\frac{du}{dx} + P\right]\frac{dv}{dx} = \frac{R}{u}$$

Now,

$$u = x + \frac{1}{x}$$

$$\frac{du}{dx} = 1 - \frac{1}{x^2}$$

So,

$$\frac{d^2v}{dx^2} + \left(\frac{2}{\left(x+\frac{1}{x}\right)}\left(1-\frac{1}{x^2}\right) + \frac{1}{x}\right)\frac{dv}{dx} = \frac{0}{u}$$

$$\frac{d^2v}{dx^2} + \left[\frac{2}{\left(\frac{x^2+1}{x}\right)}\left(\frac{x^2-1}{x^2}\right) + \frac{1}{x}\right]\frac{dv}{dx} = 0$$

$$\frac{d^2v}{dx^2} + \left[\frac{2\cancel{x}}{(x^2+1)}\left(\frac{x^2-1}{x^{\cancel{2}}}\right) + \frac{1}{x}\right]\frac{dv}{dx} = 0$$

$$\frac{d^2v}{dx^2} + \left[\frac{2}{(x^2+1)x}(x^2-1) + \frac{1}{x}\right]\frac{dv}{dx} = 0$$

$$\frac{d^2v}{dx^2} + \left[\frac{2x^2-2+x^2+1}{(x^2+1)x}\right]\frac{dv}{dx} = 0$$

$$\frac{d^2v}{dx^2} + \left[\frac{3x^2-1}{(x^2+1)x}\right]\frac{dv}{dx} = 0 \quad -(5)$$

Let $\frac{dv}{dx} = t$ — (4)

& now, differentiate it.

$$\frac{d^2v}{dx^2} = \frac{dt}{dx} \quad \text{— (5)}$$

So,

from (3)

$$\frac{dt}{dx} + \left[\frac{3x^2-1}{(x^2+1)x}\right] t = 0$$

$$\frac{dt}{dx} = -\left[\frac{3x^2-1}{(x^2+1)x}\right] t$$

$$\frac{dt}{t} = -\left[\frac{3x^2-1}{(x^2+1)x}\right] dx$$

$$\frac{dt}{t} + \left[\frac{3x^2-1}{(x^2+1)x}\right] dx = 0$$

Now integrating both sides.

$$\int \frac{dt}{t} + \int \frac{3x^2-1}{(x^2+1)x} dx = \int 0$$

$$\log t + I_1 = \log c_1 \quad \text{— (6)}$$

Now,

For I_1

$$I_1 = \int \frac{3x^2-1}{(x^2+1)\,x}\,dx$$

Now,

By using partial fraction.

$$\frac{3x^2-1}{(x^2+1)\,x} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

For A, put $x=0$ in (x^2+1)

$$\frac{-1}{1} = A$$

$$\boxed{-1 = A}$$

Now,

$$3x^2-1 = A(x^2+1) + (Bx+C)\,x$$
$$3x^2-1 = -1(x^2+1) + Bx^2 + Cx$$
$$3x^2-1 = -x^2-1 + Bx^2 + Cx$$
$$3x^2-1 = x^2(B-1) + Cx - 1$$

$\boxed{C = 0}$

Now, $B-1 = 3$
$B = 3+1$
$\boxed{B = 4}$

Now,

$$\int \frac{3x^2-1}{(x^2+1)x} = \int \frac{A}{x} dx + \int \frac{Bx+C}{x^2+1} dx$$

So, $$I_1 = \int \frac{-1}{x} dx + \int \frac{4x+0}{x^2+1} dx$$

$$I_1 = -\int \frac{1}{x} dx + \int \frac{4x}{x^2+1} dx$$

$$\boxed{I_1 = -\log x + I_2} \quad - (7)$$

Now,

For I_2

$$I_2 = \int \frac{4x}{x^2+1} dx$$

Let,

$$x^2+1 = m$$

So, $$2x = \frac{dm}{dx}$$

$$dx = \frac{dm}{2x}$$

so,

$$I_2 = \int \frac{\cancel{2x}}{m} \cdot \frac{dm}{\cancel{2x}}$$

$$I_2 = \int \frac{2}{m}\, dm$$

$$I_2 = 2\int \frac{1}{m}\, dm$$

$$I_2 = 2 \log m$$

$$\boxed{I_2 = 2\log(x^2+1)} \quad -(8)$$

Now, from (7) & (8).

we get,

$$I_1 = -\log x + I_2$$

$$\boxed{I_1 = -\log x + 2\log(x^2+1)} \quad -(9)$$

Now, from (6) & (9)
we get,

$$\log t + (-\log x + 2\log(x^2+1)) = \log c_1$$

$$\log t - \log x + \log (x^2+1)^2 = \log C_1$$

$$\cancel{\log \left(\frac{t}{(x^2+1)} \right.}$$

$$\log \left(\frac{t(x^2+1)^2}{x} \right) = \log C_1$$

$$\frac{t}{x}(x^2+1)^2 = C_1$$

$$t = \frac{C_1 x}{(x^2+1)^2}$$

$$\frac{dv}{dx} = \frac{C_1 x}{(x^2+1)^2}$$

$$dv = \frac{C_1 x}{(x^2+1)^2} dx$$

Now,

integrating both sides

we get,

$$\int dv = \int \frac{C_1 x}{(x^2+1)^2} dx$$

$$v = C_1 \int \frac{x}{(x^2+1)^2} dx$$

now,

Let, $x^2+1 = n$

$$2x = \frac{dn}{dx}$$

$$dx = \frac{dn}{2x}$$

$$v = c_1 \int \frac{x}{n^2} \frac{dn}{2\cancel{x}}$$

$$v = \frac{c_1}{2} \int \frac{1}{n^2} dn$$

$$v = \frac{c_1}{2} \int n^{-2} dn$$

$$v = \frac{c_1}{2} \left(\frac{n^{-1}}{-1} \right) + \cancel{\log} C_2$$

$$v = \frac{-c_1}{2n} + C_2$$

$$\boxed{v = \frac{-c_1}{2(x^2+1)} + C_2} \quad \text{— (10)}$$

now,

for complete solⁿ.

$$y = uv$$

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$$y = \left(x + \frac{1}{x}\right)\left(\frac{-C_1}{2(x^2+1)} + C_2\right)$$

$$y = \left(\frac{x^2+1}{x}\right)\left(\frac{-C_1}{2(x^2+1)} + \frac{2C_2(x^2+1)}{2(x^2+1)}\right)$$

$$y = \frac{\cancel{x^2+1}}{x}\left(\frac{2C_2(x^2+1) - C_1}{2\cancel{(x^2+1)}}\right)$$

$$y = \frac{2(x^2+1)C_2 - C_1}{2x}$$

$$\boxed{y = \frac{2x^2C_2 + 2C_2 - C_1}{2x}}$$

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Q. Solve!

$$x^2\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 4y = 0$$

when $y(1) = 4$

$$\frac{dy}{dx} = 13$$

$$x = 1$$

Now, divide above eqⁿ by x^2 we get,

$$\frac{d^2y}{dx^2} - \frac{4}{x}\frac{dy}{dx} + \frac{4y}{x^2} = 0$$

$$\frac{d^2y}{dx^2} + \left(\frac{-4}{x}\right)\frac{dy}{dx} + \frac{4}{x^2}y = 0 \quad \text{--- (1)}$$

Now, compare eqⁿ with

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0.$$

we get,

$$P = -\frac{4}{x} \quad \text{--- (2)}$$

$$Q = \frac{4}{x^2} \quad \text{--- (3)}$$

$$R = 0 \quad \text{--- (4)}$$

Now,

<u>For 'u'</u>

$$1+P+Q = 1 + \left(-\frac{4}{x}\right) + \frac{4}{x^2}.$$

$\therefore$ $1+P+Q \neq 0.$

so, $1-P+Q = 1 - \left(-\frac{4}{x}\right) + \frac{4}{x^2}.$

$\therefore$ $1-P+Q \neq 0.$

So,

$$P + Qx = -\frac{4}{x} + \left(\frac{4}{x^{\cancel{2}}}\right)\cancel{x}$$

$$P + Qx = -\cancel{\frac{4}{x}} + \cancel{\frac{4}{x}}$$

$$\boxed{P + Qx = 0}$$

Now,

$$\because \boxed{P + Qx = 0} \longrightarrow (5)$$

$$\text{So, } \boxed{u = x} \longrightarrow (6)$$

Now,

For 'v'

$$\frac{d^2v}{dx^2} + \left[\frac{2}{u}\frac{du}{dx} + P\right]\frac{dv}{dx} = \frac{R}{u}$$

Now,

Let $t = \frac{dv}{dx}$ —— (7)

So, on differentiating it we get

$$\frac{dt}{dx} = \frac{d^2v}{dx^2} \quad \text{— (8)}$$

now;

$\because$ $u = x$ (from 6)

so, $\boxed{\frac{du}{dx} = 1}$ —— 9

now,

$$\frac{dt}{dx} + \left[\frac{2}{x}(1) + \left(-\frac{4}{x}\right)\right]t = \frac{0}{u}$$

$$\frac{dt}{dx} + \left[\frac{2}{x} - \frac{4}{x}\right]t = 0$$

$$\frac{dt}{dx} + \left[\frac{2-4}{x}\right]t = 0$$

$$\frac{dt}{dx} + \left[\frac{-2}{x}\right]t = 0$$

$$\frac{dt}{dx} = \frac{2}{x}t$$

$$\frac{1}{t}dt = \frac{2}{x}dx$$

Now, ~~differentiating it~~, integrating both sides we get,

$$\int \frac{1}{t}\,dt = 2\int \frac{1}{x}\,dx$$

$$\log t = 2\log x + \log c_1$$

$$\log t - \log x^2 = \log c_1$$

$$\log\left(\frac{t}{x^2}\right) = \log c_1$$

so, $\dfrac{t}{x^2} = c_1$

$$t = c_1 x^2 \quad \text{—— } (10)$$

Now,

$$\frac{dv}{dx} = c_1 x^2 \quad (\text{from } (7))$$

$$dv = c_1 x^2\, dx$$

Again, integrating it we get,

$$\int dv = c_1 \int x^2\, dx$$

$$\boxed{v = c_1 \frac{x^3}{3} + c_2} \quad \text{—— } (11)$$

Now,

for complete solⁿ

$y = uv$ $\therefore$ from (6) & (11)

$$y = x\left(\frac{c_1 x^3}{3} + c_2\right)$$

$$\boxed{y = \frac{c_1 x^4}{3} + c_2 x} \quad \text{— (12)}$$

Now, $\because$ $x = 1$

$y = 4$

$\frac{dy}{dx} = 13$

so, $4 = \frac{c_1}{3} + c_2$

$$\boxed{12 = c_1 + 3c_2} \quad \text{— (13)}$$

~~from (7)~~

Now,

$$\frac{dy}{dx} = \frac{4c_1}{3} x^3 + c_2$$

$$13 = \frac{4c_1}{3} + c_2$$

$$\boxed{39 = 4c_1 + 3c_2} \quad \text{— (14)}$$

$$12 = C_1 + \cancel{3C_2}$$

$$39 = 4C_1 + \cancel{3C_2}$$

$$- \quad - \quad -$$

$$-27 = -3C_1$$

$$9 \cancel{\frac{27}{3}} = C_1$$

$$\boxed{C_1 = 9} \quad - (15)$$

Now,

$$12 = 9 + 3C_2$$

$$\cancel{3} = \cancel{3}C_2$$

$$\boxed{C_2 = 1} \quad - (16)$$

So,

$$y = \frac{\cancel{9}^{3} x^4}{\cancel{3}} + (1)x$$

$$\boxed{y = 3x^4 + x} \quad - (17)$$

Q. Solve:

$$x\frac{d^2y}{dx^2} - 2(x+1)\frac{dy}{dx} + (x+2)y = (x-2)e^x$$

METHOD – 2

(Removal of 1st derivative)

Working Rule :-

(1) Consider the 2nd order differential eqn :-

$$\boxed{\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R} \quad \text{— (1)}$$

(2) On comparing the given problem with eqn (1) we, get, P, Q, R.

where, P, Q, R are functions of 'x alone' or constants.

(3) Let $\boxed{y = uv}$ be the complete solⁿ of the given problem.

(4) 'u' is given by

$$\boxed{u = e^{-\frac{1}{2}\int P\,dx}}$$

(5) 'v' is given by :-

$$\boxed{\frac{d^2v}{dx^2} + Iv = \frac{R}{u}}$$

('v' is solved using C.F. + P.I. method)

where,

$$\boxed{I = Q - \frac{1}{2}\frac{dP}{dx} - \frac{P^2}{4}}$$

('I' usually '0' or 1)

(6) After solving 'u' & 'v', we get the complete soln i.e.

$$\boxed{y = uv}$$

Q. Solve:

$$x\frac{d^2y}{dx^2} - 2(x+1)\frac{dy}{dx} + (x+2)y = (x-2)e^x$$

Now, divide the above eq^n by 'x' on both sides.

we get,

$$\frac{d^2y}{dx^2} - 2\left(1+\frac{1}{x}\right)\frac{dy}{dx} + \left(1+\frac{2}{x}\right)y = \left(1-\frac{2}{x}\right)e^x$$

$$\frac{d^2y}{dx^2} + \left(-\frac{2}{x}-2\right)\frac{dy}{dx} + \left(\frac{2}{x}+1\right)y = \left(1-\frac{2}{x}\right)e^x \quad \text{(1)}$$

Now,

on comparing eq^n (1) with

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$$

we get,

$$P = -\frac{2}{x} - 2 \quad \text{— (2)}$$

$$Q = \frac{2}{x} + 1 \quad \text{— (3)}$$

$$R = \left(1-\frac{2}{x}\right)e^x \quad \text{— (4)}$$

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Now,

For 'u'

$$1+P+Q = 1+\left(-\frac{2}{x}-2\right)+\left(1+\frac{2}{x}\right)$$

$$1+P+Q = 1-\cancel{\frac{2}{x}}-2+1+\cancel{\frac{2}{x}}$$

$$1+P+Q = 2-2$$

$$\boxed{1+P+Q=0}$$

$\therefore$ $\boxed{1+P+Q=0}$ — (5)

So, $\boxed{u=e^{x}}$ — (6)

Now,

For 'v'

$$\frac{d^2v}{dx^2}+\left(\frac{2}{u}\cdot\frac{du}{dx}+P\right)\frac{dv}{dx}=\frac{R}{u}$$

So,

Let,

$$\frac{dv}{dx}=t \quad \text{— (7)}$$

Differentiating it, we get

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$$\frac{d^2v}{dx^2} = \frac{dt}{dx} \quad \text{--- (8)}$$

Now,

$u = e^x$ (from (6))

so, $\boxed{\frac{du}{dx} = e^x}$ --- (9)

So,
now

$$\frac{dt}{dx} + \left[\frac{2}{\cancel{e^x}} \cdot \cancel{e^x} + \left(-\frac{2}{x} - 2\right)\right] t = \frac{\left(1-\frac{2}{x}\right)\cancel{e^x}}{\cancel{e^x}}$$

$$\frac{dt}{dx} + \left(\cancel{2} - \frac{2}{x} - \cancel{2}\right)t = 1 - \frac{2}{x}$$

$$\boxed{\frac{dt}{dx} - \frac{2}{x}t = 1 - \frac{2}{x}}$$

$$\cancel{\frac{dt}{dx} = \cancel{1} - \frac{2}{x} + \frac{2t}{x}}$$

Now, $\boxed{\frac{dt}{dx} + \left(\frac{-2}{x}\right)t = 1 - \frac{2}{x}}$ --- (10)

Linear differential eq^n

Now,
on comparing eq^n (10) with

$$\frac{dt}{dx} + mt = n.$$

where, m & n are functions of 'x alone
or constants.

So,
we get.

$$m = -\frac{2}{x} \quad — (11)$$

$$n = 1 - \frac{2}{x} \quad — (12)$$

So,

$$I.F. = e^{\int m\,dx}$$

$$I.F. = e^{\int -\frac{2}{x}\,dx}$$

$$I.F. = e^{-2\int \frac{1}{x}\,dx}$$

$$I.F. = e^{-2\log x}$$

$$I.F. = e^{2\log\left(\frac{1}{x}\right)}$$

$$I.F. = e^{\log\left(\frac{1}{x}\right)^2}$$

So,

$$I.F. = \left(\frac{1}{x}\right)^2$$

$$\boxed{I.F. = \frac{1}{x^2}} \quad - (13)$$

Now,

<u>for solⁿ</u>

$$t \cdot \frac{1}{x^2} = \int \left(1 - \frac{2}{x}\right) \frac{1}{x^2}\, dx$$

$$t \cdot \frac{1}{x^2} = \int \left(\frac{1}{x^2} - \frac{2}{x^3}\right) dx$$

$$t \cdot \frac{1}{x^2} = \int \frac{1}{x^2}\, dx - 2\int \frac{1}{x^3}\, dx$$

$$t \cdot \frac{1}{x^2} = \int x^{-2}\, dx - 2\int x^{-3}\, dx$$

$$t \cdot \frac{1}{x^2} = \frac{x^{-1}}{-1} - \cancel{2}\, \frac{x^{-2}}{\cancel{-2}} + C_1$$

$$I.F. = e^{\log\left(\frac{1}{x}\right)^2}$$

So,

$$I.F. = \left(\frac{1}{x}\right)^2$$

$$\boxed{I.F. = \frac{1}{x^2}} \quad — (13)$$

Now,

for sol^n

$$t \cdot \frac{1}{x^2} = \int\left(1-\frac{2}{x}\right)\frac{1}{x^2}\,dx$$

$$t \cdot \frac{1}{x^2} = \int\left(\frac{1}{x^2} - \frac{2}{x^3}\right)dx$$

$$t \cdot \frac{1}{x^2} = \int\frac{1}{x^2}\,dx - 2\int\frac{1}{x^3}\,dx$$

$$t \cdot \frac{1}{x^2} = \int x^{-2}\,dx - 2\int x^{-3}\,dx$$

$$t \cdot \frac{1}{x^2} = \frac{x^{-1}}{-1} - \cancel{2}\,\frac{x^{-2}}{-\cancel{2}} + C_1$$

$$\frac{t}{x^2} = -\frac{1}{x} + \frac{1}{x^2} + C_1$$

$$t = -x + 1 + C_1 x^2$$

$$\boxed{t = 1 - x + C_1 x^2} \quad - (14)$$

Now,

$$\frac{dv}{dx} = 1 - x + C_1 x^2 \quad (\text{from } (7))$$

$$dv = (1 - x + C_1 x^2)\, dx$$

Now, integrating both sides we get,

$$\int dv = \int (1 - x + C_1 x^2)\, dx$$

$$\boxed{v = x - \frac{x^2}{2} + C_1 \frac{x^3}{3} + C_2} \quad - (15)$$

Now,

for complete solⁿ:

$y = uv$

(from 6 & 15)

$$y = e^x\left(x - \frac{x^2}{2} + C_1\frac{x^3}{3} + C_2\right)$$

$$\boxed{y = xe^x - \frac{x^2e^x}{2} + \frac{C_1x^3e^x}{3} + C_2e^x}$$

GOOD

Solve: by ~~Metho~~ Removal of 1^{st} derivative.

$$x^2\frac{d^2y}{dx^2} - 2(x^2+x)\frac{dy}{dx} + (x^2+2x+2)y = 0$$

Divide the above eqn by 'x^2'

$$\frac{d^2y}{dx^2} - 2\left(1+\frac{1}{x}\right)\frac{dy}{dx} + \left(1+\frac{2}{x}+\frac{2}{x^2}\right)y = 0$$

$$\frac{d^2y}{dx^2} + \left(\frac{-2}{x} - 2\right)\frac{dy}{dx} + \left(1+\frac{2}{x}+\frac{2}{x^2}\right)y = 0 \quad \text{(1)}$$

Now,

On comparing eqn (1) with

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$$

We get,

$$P = -\frac{2}{x} - 2 \quad \text{— (2)}$$

$$Q = 1 + \frac{2}{x} + \frac{2}{x^2} \quad \text{— (3)}$$

$$R = 0 \quad \text{— (4)}$$

Now,

<u>for 'u'</u>

$$u = e^{-\frac{1}{2}\int P\,dx}$$

$$u = e^{-\frac{1}{2}\int\left(-\frac{2}{x}-2\right)dx}$$

$$u = e^{-\frac{1}{2}\cdot 2\int\left(-\frac{1}{x}+1\right)dx}$$

$$u = e^{\int\frac{1}{x}dx+\int 1\,dx}$$

$$u = e^{\log x + x}$$

$$u = e^{\log x}\cdot e^{x}$$

$$\boxed{u = xe^{x}} \quad — \quad (5)$$

Now,

<u>for 'v'</u>

$$\boxed{\frac{d^2v}{dx^2} + Iv = \frac{R}{u}} \quad — \quad (6)$$

where,

$$\boxed{I = Q - \frac{1}{2}\frac{dP}{dx} - \frac{P^2}{4}} \quad - \ (7)$$

$$I \neq 1 + \cancel{\frac{2}{x}} + \cancel{\frac{2}{x^2}} \neq \frac{1}{2}$$

Now,

<u>for 'I'</u>

$$\because P = -\frac{2}{x} - 2$$

$$\frac{dP}{dx} = -2\left(\frac{1}{x} + 1\right)$$

$$\frac{dP}{dx} = -2\left(\frac{-1}{x^2} + 0\right)$$

$$\boxed{\frac{dP}{dx} = \frac{2}{x^2}} \quad - \ (8)$$

Now,

from (7)

$$I = \left(1 + \frac{2}{x} + \frac{2}{x^2}\right) - \frac{1}{\cancel{2}}\left(\frac{\cancel{2}}{x^2}\right) - \frac{\left(-2\left(\frac{1}{x} + 1\right)\right)^2}{4}$$

$$I = 1 + \frac{2}{x} + \frac{2}{x^2} - \frac{1}{x^2} - \frac{\left(\cancel{4}\left(\frac{1}{x} + 1\right)^2\right)}{\cancel{4}}$$

$$I = 1 + \frac{2}{x} + \frac{1}{x^2} - \left(\frac{1}{x^2} + 1 + \frac{2}{x}\right)$$

$$I = \cancel{1} + \cancel{\frac{2}{x}} + \cancel{\frac{1}{x^2}} - \cancel{\frac{1}{x^2}} - \cancel{1} - \cancel{\frac{2}{x}}$$

$$\boxed{I = 0} \quad \text{--- } (9)$$

Now,

from (6)

$$\frac{d^2v}{dx^2} + Iv = \frac{R}{u}$$

$$\frac{d^2v}{dx^2} + 0v = \frac{0}{u}.$$

$$\frac{d^2v}{dx^2} + 0 = 0.$$

$$\boxed{D^2 v = 0} \quad \text{--- } (10)$$

Linear differential eq^n with constant coefficient.

So, The auxilary eq^n is :-

put $D = m$, $v = 1$, R.H.S $= 0$

— (11)

$$\boxed{m = 0, 0}$$

real & equal roots.

So,

$$C.F. = (C_1 + xC_2)e^{mx}$$

$$C.F. = (C_1 + xC_2)e^{0x}$$

$$\boxed{C.F. = (C_1 + xC_2)} \quad \text{—} \quad (12)$$

$\because$ R.H.S $= 0$

So, $\boxed{P.I. = 0}$

So,

Soln is :-

$$v = C.F. + P.I.$$

$$v = (C_1 + xC_2) + 0$$

$$\boxed{v = C_1 + xC_2} \quad \text{—} \quad (13)$$

now,

Complete soln is :-

from (5) & (13)

$$y = uv$$

$$y = xe^x(C_1 + xC_2)$$

$$\boxed{y = xe^x C_1 + x^2 e^x C_2} \quad \text{—} \quad (14)$$

Solve:

$$y'' - \frac{2}{x} y' + \left(1 + \frac{2}{x^2}\right) y = x e^x$$

$$\frac{d^2y}{dx^2} + \left(-\frac{2}{x}\right)\frac{dy}{dx} + \left(1 + \frac{2}{x^2}\right) y = x e^x \quad \text{--- (1)}$$

So,

now, on comparing the given eq^n (1) with

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$$

So, we get

$$P = -\frac{2}{x} \quad \text{--- (2)}$$

$$Q = 1 + \frac{2}{x^2} \quad \text{--- (3)}$$

$$R = x e^x \quad \text{--- (4)}$$

Now,

For 'u'

$$u = e^{-\frac{1}{2}\int P\,dx}$$

$$u = e^{-\frac{1}{2}\int -\frac{2}{x}\,dx}$$

$$u = e^{\int \frac{1}{x}\,dx}$$

Solve:

$$y'' - \frac{2}{x} y' + \left(1 + \frac{2}{x^2}\right) y = xe^x$$

$$\frac{d^2y}{dx^2} + \left(-\frac{2}{x}\right)\frac{dy}{dx} + \left(1 + \frac{2}{x^2}\right) y = xe^x \quad \text{--- (1)}$$

So,

now, on comparing the given eq^n (1) with

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$$

So, we get

$$P = -\frac{2}{x} \quad \text{--- (2)}$$

$$Q = 1 + \frac{2}{x^2} \quad \text{--- (3)}$$

$$R = xe^x \quad \text{--- (4)}$$

Now,

For 'u'

$$u = e^{-\frac{1}{2}\int P\,dx}$$

$$u = e^{-\frac{1}{2}\int -\frac{2}{x}\,dx}$$

$$u = e^{\int \frac{1}{x}\,dx}$$

$$u = e^{\log x}$$

$$\boxed{u = x} \quad \text{— (5)}$$

Now,

<u>For 'v'</u>

$$\boxed{\frac{d^2v}{dx^2} + Iv = \frac{R}{u}} \quad \text{— (6)}$$

where

$$\boxed{I = Q - \frac{1}{2}\frac{dP}{dx} - \frac{P^2}{4}} \quad \text{— (7)}$$

So,

now <u>for 'I'</u>

$$\because P = -\frac{2}{x}$$

$$P = -2\left(\frac{1}{x}\right)$$

$$\frac{dP}{dx} = -2\left(\frac{-1}{x^2}\right)$$

$$\boxed{\frac{dp}{dx} = 2\frac{1}{x^2}} \quad \text{— (8)}$$

so, from (7)

we get

$$I = 1 + \frac{2}{x^2} - \frac{1}{\cancel{2}}\left(\frac{\cancel{2}1}{x^2}\right) - \frac{\left(\frac{-2}{x}\right)^2}{4}$$

$$I = 1 + \frac{2}{x^2} - \frac{1}{\cancel{2}x^2} - \frac{\cancel{4}}{x^2\cancel{4}}$$

$$I = 1 + \frac{2}{x^2} - \frac{1}{\cancel{2}x^2} - \frac{1}{x^2}$$

$$\cancel{I = 1 + \frac{4 - 1 - 2}{2x^2}}$$

$$\cancel{I =}$$

$$I = 1 + \frac{\cancel{2}}{\cancel{x^2}} - \frac{\cancel{2}}{\cancel{x^2}}$$

$$\boxed{I = 1} \quad \text{— (9)}$$

so,

now from (6) & (9)

$$\frac{d^2 v}{dx^2} + 1(v) = \frac{x e^x}{x}$$

$$\boxed{\frac{d^2 v}{dx^2} + v = e^x} \quad \text{--- (10)}$$

This is linear differential eq^n with constant coefficient.

$$(D^2 v + v) = e^x$$

$$\boxed{(D^2+1)\, v = e^x} \quad \text{--- (11)}$$

so, Now,

For C.F.

The auxilary eq^n is :-
(put $D = m$, $v = 1$, R.H.S $= 0$.)

$$m^2+1 = 0$$

$$m^2 = -1$$

$$m = \pm \sqrt{-1}$$

$$\boxed{m = \pm i} \quad \text{--- (12)}$$

$\therefore$, $m = \alpha \pm i\beta$.

$\alpha = 0$, $\beta = 1$

So,

$$C.F. = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x)$$

$$C.F. = e^{0x}(C_1 \cos x + C_2 \sin x)$$

$$\boxed{C.F. = C_1 \cos x + C_2 \sin x} \quad - (13)$$

Now,

For P.I. $\quad \left(P.I. = \frac{1}{f(D)} \cdot f(x) \right)$

$$P.I. = \frac{1}{D^2+1} e^x$$

$$P.I. = \frac{1}{1+1} e^x \qquad e^x = e^{ax}, \text{ so, } a=1$$

$$\boxed{P.I. = \frac{1}{2} e^x} \quad - (14)$$

now,

for sol^n

$$y = C.F. + P.I.$$

$$\boxed{y = C_1 \cos x + C_2 \sin x + \frac{1}{2} e^x} \rightarrow (15)$$

now, ~~for~~

for complete solⁿ

$y = \cancel{e^{x}}\ uv$.

$\cancel{y=}$ so, from (5) & (15)

$$y = x\left(c_1 \cos x + c_2 \sin x\right) + \frac{1}{2}e^{x}$$

$$\boxed{y = x c_1 \cos x + x c_2 \sin x + \frac{1}{2} x e^{x}} \quad - (16)$$

$$\frac{d^2y}{dx^2} - 2\tan x \frac{dy}{dx} + 5y = \sec x \cdot e^x$$

$$\frac{d^2y}{dx^2} + (-2\tan x)\frac{dy}{dx} + 5y = \sec x \cdot e^x \quad \text{— (1)}$$

Now,

on comparing above eq^n (1) with

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$$

we get,

$P = -2\tan x$ — (2)

$Q = 5$ — (3)

$R = \sec x \cdot e^x$ — (4)

Now,

<u>for 'u'</u>

$$u = e^{-\frac{1}{2}\int P dx}$$

$$u = e^{-\frac{1}{2}\int 2\tan x \, dx}$$

$$u = e^{-\frac{1}{2}\int \tan x \, dx}$$

$$u = e^{\log \sec x}$$

$$\boxed{u = \sec x} \quad - (5)$$

Now,

<u>for 'v'</u>

$$\boxed{\frac{d^2v}{dx^2} + Iv = \frac{R}{u}} \quad - (6)$$

where,

$$\boxed{I = S - \frac{1}{2}\frac{dP}{dx} - \frac{P^2}{4}} \quad - (7)$$

Now,

for <u>'I'</u>

$\cancel{I = S - \frac{1}{2}}$

$\because P = -2\tan x$

so $$\boxed{\frac{dP}{dx} = -2\sec^2 x} \quad - (8)$$

$$I = 5 - \frac{1}{\cancel{2}}(-\cancel{2}\sec^2 x) - \frac{\cancel{4}\tan^2 x}{\cancel{4}}$$

$$I = 5 + \sec^2 x - \tan^2 x.$$

$I = 5+1$

$\boxed{I = 6}$ — (9)

Now,

from (9) & (6)

$$\frac{d^2v}{dx^2} + 6v = \frac{\cancel{\sec x} \cdot e^x}{\cancel{\sec x}}$$

$$\frac{d^2v}{dx^2} + 6v = e^x$$

$D^2v + 6v = e^x$

$\boxed{(D^2+6)v = e^x}$ — (10)

This is linear differential eq^n with constant coefficient

So,

For C.F.

The auxilary eq^n is :-

(put $D = m$, $v = 1$, R.H.S $= 0$)

$m^2+6 = 0$

$m^2 = -6$

$m = \pm\sqrt{-6}$

$\boxed{m = \pm\sqrt{6}i}$ — (11)

$\therefore$ ~~m = D ± 6i~~

$\because m = \alpha + i\beta$

$$\boxed{\alpha = 0, \quad \beta = \sqrt{6}}$$

So,

$$C.F. = e^{\alpha x}\left(C_1 \cos \beta x + C_2 \sin \beta x\right)$$

$$\boxed{C.F. = e^{x}\left(C_1 \cos \sqrt{6}x + C_2 \sin \sqrt{6}x\right)} \quad \text{— (11)}$$

Now,

for $\underline{P.I}$

$$P.I. = \frac{1}{f(D)} f(x)$$

$$P.I. = \frac{1}{D^2+6} \cdot e^{x}$$

now, $\because$ (e^{ax}, $P.I. = \frac{1}{f(a)} \cdot f(x)$, $f(a) \neq 0$.

here, $a = 1$ ($\because e^{x}$))

So,

$$P.I. = \frac{1}{1+6} e^{x}$$

$$\boxed{P.I. = \frac{1}{7} e^{x}} \quad \text{— (12)}$$

So,

for $\underline{\text{sol}^n}$: (from (11) & (12))

$$v = C.F. + P.I.$$

$$\boxed{v = e^{x}\left(C_1 \cos\sqrt{6}x + C_2 \sin\sqrt{6}x\right) + \frac{1}{7}e^{x}} \rightarrow (13)$$

So,

now,

for complete sol^n

(from (5) & (13))

$\underline{y = uv}$

$$y = \sec x \cdot \left(e^{x}\left(C_1 \cos\sqrt{6}x + C_2 \sin\sqrt{6}x\right) + \frac{1}{7}e^{x}\right)$$

$$y = \cancel{\sec x}\left(C_1 e^{x} \cos\sqrt{6}x + C_2\right.$$

$$\boxed{y = e^{x}\sec x\left(C_1 \cos\sqrt{6}x + C_2 \sin\sqrt{6}x + \frac{1}{7}\right)}$$

Thursday
21/04/2022

Method 3

Change of Independent Variable

Working Rule :-

(1) Consider 2nd order ordinary differential eqn (ODE) of the form :-

$$\boxed{\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R} \quad \text{--- (1)}$$

(2) Compare the given problem with eqn (1), we get values of P, Q, R

(3) Now choosing new independent variable → 'z'

such that :-

$$\left(\frac{dz}{dx}\right)^2 = |Q|$$

so, $\frac{dz}{dx} = \pm\sqrt{|Q|}$ (neglect (-ve) sign)

we get, $\boxed{\frac{dz}{dx} = \sqrt{Q}}$

so,

now

$$dz = \sqrt{Q}\, dx$$

now,

integrating both sides

$$\int dz = \int \sqrt{Q}\, dx$$

$$\boxed{z = f(x)} \quad \text{or } z = \phi(x)$$

(4) with this 'z', we transform the given problem in the following form

$$\boxed{\frac{d^2y}{dz^2} + P_1 \frac{dy}{dz} + Q_1 y = R_1} \quad — (2)$$

where,

$$\boxed{P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}}$$

$$\boxed{Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2}}$$

$$\boxed{R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2}}$$

NOW,

(5) After calculating P_1, Q_1 & R_1, we have to put these in eqn (2), we get Linear differential eqn with Constant coefficient which we will solve by the previous Method (i.e. $y = C.F + P.I$)

(6) At last, in the complete soln we will put the value 'z', which is the required soln of the given problem.

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Q. Solve:

Ans. $\cos x \frac{d^2y}{dx^2} + \sin x \frac{dy}{dx} - 2\cos^3 x\, y = 2\cos^5 x$

Divide the above eqn by '$\cos x$'

$$\frac{d^2y}{dx^2} + \frac{\sin x}{\cos x}\frac{dy}{dx} - \frac{2\cos^3 x}{\cos x}\cdot y = \frac{2\cos^5 x}{\cos x}$$

$$\frac{d^2y}{dx^2} + \tan x \frac{dy}{dx} + \left(-2\cos^2 x\right) y = 2\cos^4 x - ①$$

now,

Compare eqn ① with

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R.$$

so, we get,

$P = \tan x \quad - ②$

$Q = -2\cos^2 x. \quad - ③$

$R = 2\cos^4 x. \quad - ④.$

Now,

Let 'z' be the new independent variable such that

$$\left(\frac{dz}{dx}\right)^2 = |8|$$

$$\left(\frac{dz}{dx}\right)^2 = |-2\cos^2 x|$$

$$\boxed{\left(\frac{dz}{dx}\right)^2 = 2\cos^2 x} \quad — \quad (5)$$

$$\frac{dz}{dx} = \pm\sqrt{2\cos^2 x}.$$

now, neglect (−ve) ~~to~~ sign to get real values.

$$\boxed{\frac{dz}{dx} = \sqrt{2}\cos x} \quad — \quad (6)$$

$$dz = \sqrt{2}\cos x \, dx$$

now integrating both sides.

$$\int dz = \int \sqrt{2}\cos x \, dx$$

$$\cancel{d}\ \boxed{z = \sqrt{2}\sin x} \quad — \quad (7)$$

with this 'z' we ~~g~~ transform the given eqn ~~by~~ in the following form.

$$\boxed{\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = R_1} \quad (8)$$

where

$$P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}$$

now, for P_1

$\because \frac{dz}{dx} = \sqrt{2}\cos x$ (from (6))

Diff it,

$$\frac{d^2z}{dx^2} = \sqrt{2}(-\sin x)$$

$$\boxed{\frac{d^2z}{dx^2} = -\sqrt{2}\sin x} \quad (9)$$

now,

$$P_1 = \frac{-\sqrt{2}\sin x + \tan x(\sqrt{2}\cos x)}{2\cos^2 x}$$

$$P_1 = \frac{-\sqrt{2}\,\cancel{\sin x} + \sqrt{2}\,\cancel{\sin x}}{2\cos^2 x}$$

$$P_1 = \frac{0}{2\cos^2 x}$$

$$\boxed{P_1 = 0} \quad - (10)$$

Now,

$$Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2}$$

$$Q_1 = \frac{-2\cancel{\cos^2 x}}{2\cancel{\cos^2 x}}$$

$$\boxed{Q_1 = -1} \quad - (11)$$

Now,

$$R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2}$$

$$R_1 = \frac{\cancel{2}\cos^{\cancel{4}} x}{\cancel{2}\cancel{\cos^2 x}}$$

$$\boxed{R_1 = \cos^2 x}$$

$$\boxed{R_1 = 1 - \frac{z^2}{2}} \quad - (12)$$

now,

from (8)

$$\frac{d^2y}{dz^2} + 0\frac{dy}{dz} + (-1)y = 1 - \frac{z^2}{2}$$

$$\boxed{\frac{d^2y}{dz^2} - y = 1 - \frac{z^2}{2}}$$

$$D^2y - 1 = 1 - \frac{z^2}{2}$$

$$\boxed{(D^2 - 1)y = 1 - \frac{z^2}{2}} \quad - (13)$$

This is linear differential eqⁿ with Constant coefficient.

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so,

for C.F.

The auxilary eqⁿ is:-

put ($D = m$, $y = 1$, R.H.S = 0)

$m^2 - 1 = 0$

$m^2 = 1$

$\boxed{m = \pm 1}$ — (14)

real & unequal roots

so,

$C.F. = C_1 e^{m_1 x} + C_2 e^{m_2 x}$

$C.F. = C_1 e^{1x} + C_2 e^{-1x}$

$\boxed{C.F. = C_1 e^{x} + C_2 e^{-x}}$ — ~~(15)~~

so, $\boxed{C.F. = C_1 e^{\sqrt{2}\sin x} + C_2 e^{-\sqrt{2}\sin x}}$ — (15)

Now,

for P.I.

$P.I. = \frac{1}{f(D)} f(z)$

$P.I. = \frac{1}{D^2 - 1} \left(1 - \frac{z^2}{2}\right)$

$$P.I. = \frac{1}{D^2-1}(1) - \frac{1}{2}\frac{1}{D^2-1}(z^2)$$

$$P.I. = \frac{1}{D^2-1}e^{0x} - \frac{1}{2}\frac{1}{D^2-1}(z^2)$$

$$P.I. = \frac{1}{0-1} - \left(\frac{-1}{2}\right)\left(\frac{1}{1-D^2}\right)z^2$$

$$P.I. = -1 + \frac{1}{2}(1-D^2)^{-1}z^2$$

NOW,

$$\left(\because (1-x)^{-1} = 1+x+x^2+x^3+ \cdots\right)$$

$$P.I. = -1 + \frac{1}{2}\left(1+D^2+D^4+D^6+\cdots\right)z^2$$

$$P.I. = -1 + \frac{1}{2}\left(z^2+D^2z^2+D^4z^2+D^6z^2+\cdots\right)$$

$\because D = \frac{d}{dz}$

$Dz^2 = 2z$

$D^2z^2 = 2$

$D^3z^2 = 0$

$D^4z^2 = 0$

$$P.I. = \cancel{1} + \frac{1}{2}[z^2 + 2 + 0 + 0 - \cdots]$$

$$P.I. = -1 + \frac{1}{2}[z^2+2].$$

$$P.I = -1 + \frac{1}{2}(2\sin^2 x + 2) \quad (\text{from } 7)$$

$$P.I. = -1 + \frac{1}{\cancel{2}}(\cancel{2}\sin^2 x) + \frac{1}{\cancel{2}}(\cancel{2})$$

$$P.I. = -\cancel{1} + \sin^2 x + \cancel{1}$$

$$\boxed{P.I. = \sin^2 x} \quad — \quad 16$$

Now,

for complete solⁿ

$$y = C.F + P.I. \quad (\text{from } 15 \ \& \ 16)$$

$$y = C_1 e^{\sqrt{2}\sin x} + C_2 e^{-\sqrt{2}\sin x} + \sin^2 x$$

$$\boxed{y = C_1 e^{\sqrt{2}\sin x} + C_2 e^{-\sqrt{2}\sin x} + \sin^2 x}$$

Q. Solve:

$$\frac{d^2y}{dx^2} + \frac{1}{(1+x)}\frac{dy}{dx} + \frac{1}{(1+x^2)}\cdot y = \frac{4\cos\{\log(1+x)\}}{(1+x)^2} \quad \text{--- (1)}$$

Now,

on comparing above eq^n (1) with

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$$

we get,

$$P = \frac{1}{1+x} \quad \text{--- (2)}$$

$$Q = \frac{1}{1+x^2}. \quad \text{--- (3)}$$

$$R = \frac{4\cos\{\log(1+x)\}}{(1+x)^2} \quad \text{--- (4)}$$

Now,

Let 'z' be the new independent variable such that:

$$\left(\frac{dz}{dx}\right)^2 = |Q|$$

$$\left(\frac{dz}{dx}\right)^2 = \left|\frac{1}{1+x^2}\right|.$$

$$\boxed{\left(\frac{dz}{dx}\right)^2 = \frac{1}{1+x^2}} \quad - (5)$$

$$\frac{dz}{dx} = \pm\sqrt{\frac{1}{1+x^2}}.$$

now, so as to get real values, neglect (-ve) sign,

we get,

$$\frac{dz}{dx} = \sqrt{\frac{1}{1+x^2}}.$$

$$\boxed{\frac{dz}{dx} = \frac{1}{\sqrt{1+x^2}}} \quad - (6).$$

$$\frac{dz}{\cancel{dx}} = \frac{1}{\sqrt{1+x^2}}\,dx$$

$$dz = \frac{1}{\sqrt{1+x^2}}\,dx.$$

Now,

Integrating both sides,

$$\int dz = \int \frac{1}{\sqrt{1+x^2}}\, dx$$

$$z = \frac{1}{2(1)} \log\left[x + \sqrt{1+x^2}\right]$$

$$\boxed{z = \frac{1}{2} \log\left[x + \sqrt{1+x^2}\right]} \quad - (7)$$

Now,

We ~~to~~ transform the eq^n given in the problem to the following form by replacing ~~x~~ independent variable 'x' by ~~d~~ a new independent variable 'z',

We get,

$$\frac{d^2y}{dz^2} + P_1 \frac{dy}{dz} + Q_1 y = R_1$$

where,

$$P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}$$

~~PF~~

Now, for 'P$_1$'

$$\therefore \frac{dz}{dx} = \frac{1}{\sqrt{1+x^2}} \quad (\text{from } (6))$$

Now, on differentiating it,

we get,

$$\frac{d^2z}{dx^2} = \frac{-1}{\cancel{2}} \cdot (1+x^2)^{-3/2} \cdot \cancel{2}x$$

$$\frac{d^2z}{dx^2} = \frac{-x}{(1+x^2)^{3/2}}.$$

$$\frac{\quad}{dz^2} + y$$

$$\Rightarrow \quad (D^2+1)\,y = 0 \qquad \left[\because D \equiv \frac{d}{dz}\right]$$

It's A.E. is

$$m^2+1 = 0 \Rightarrow m = \pm i$$

$$\therefore \quad C.F. = c_1 \cos z + c_2 \sin z$$

and P.I.

$$P.I. = 0$$

Hence the required solutions is : $y = C.F. + P.I.$

$$y = c_1 \cos z + c_2 \sin z$$

or

$$y = c_1 \cos\left[2\log(\tan x/2)\right] + c_2 \sin\left[2\log(\tan x/2)\right]. \text{ Ans.}$$

Example 3.52 : Solve by changing the independent variable:

$$(1+x^2)\frac{d^2y}{dx^2} + 2x(1+x^2)\frac{dy}{dx} + 4y = 0$$

[RGPV June 2012, Dec. 2019]

Solution. Given :

$$\frac{d^2y}{dx^2} + \frac{2x}{(1+x^2)}\cdot\frac{dy}{dx} + \frac{4}{(1+x^2)^2}\,y = 0 \qquad \text{...(1)}$$

Here, $\quad P = \dfrac{2x}{1+x^2},\ Q = \dfrac{4}{(1+x^2)^2},\ R = 0$

Let choosing independent variable z such that:

$$\left(\frac{dz}{dx}\right)^2 = |Q|$$

$$\Rightarrow \quad \frac{dz}{dx} = \frac{2}{1+x^2}$$

$$\Rightarrow \quad \int dz = \int \frac{2}{1+x^2}\,dx$$

$$\Rightarrow \quad z = 2\tan^{-1} x$$

With this variable z, then given equation transform to:

$$\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = R_1, \qquad \text{...(2)}$$

where
$$P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}$$

$$= \frac{\frac{-4x}{(1+x^2)^2} + \frac{2x}{(1+x^2)}\frac{2}{(1+x^2)}}{4/(1+x^2)^2} = 0,$$

$$Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} = \frac{4/(1+x^2)^2}{4/(1+x^2)^2} = 1,$$

and
$$R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2} = 0$$

Hence (2) becomes:

$$\frac{d^2y}{dz^2} + y = 0$$

$\Rightarrow$
$$(D^2+1)y = 0 \qquad \left[\because D \equiv \frac{d}{dz}\right]$$

It's A.E. is $m^2+1=0 \Rightarrow m = \pm i$

$\therefore$ C.F. $= c_1 \cos z + c_2 \sin z$

and P.I. $= 0$

Hence the required solution is : $y = \text{C.F.} + \text{P.I.}$

Or $y = c_1 \cos z + c_2 \sin z + 0$

Or $y = c_1 \cos(2\tan^{-1} x) + c_2 \sin(2\tan^{-1} x)$. **Ans.**

Example 3.53 : Solve by changing the independent variable if

$$x\frac{d^2y}{dx^2} - \frac{dy}{dx} - 4x^3y = x^5$$ *[RGPV Dec. 2012]*

Solution. Given :

$$\frac{d^2y}{dx^2} - \frac{1}{x}\frac{dy}{dx} - 4x^2y = x^4 \qquad ...(1)$$

Example 3.50 : Solve $x\frac{d^2y}{dx^2}-\frac{dy}{dx}-4x^3y=8x^3\sin(x^2)$. *[RGPV Dec. 2013, June 2014]*

Solution. Rewrite given equation :

$$\frac{d^2y}{dx^2}-\frac{1}{x}\frac{dy}{dx}-4x^2y=8x^2\sin(x^2). \quad ...(1)$$

Hene $P=-(1/x)$, $Q=-4x^2$ *and* $R=8x^2\sin(x^2)$.

Choose independent variable z such that :

$$\left(\frac{dz}{dx}\right)^2=|Q|=|-4x^2|=4x^2$$

$$\Rightarrow \quad \frac{dz}{dx}=2x \Rightarrow dz=2x\,dx \Rightarrow z=x^2. \quad ...(2)$$

With this z the given equation transforms to :

$$\frac{d^2y}{dz^2}+P_1\frac{dy}{dz}+Q_1y=R_1 \quad ...(3)$$

where $P_1=\dfrac{\frac{d^2z}{dx^2}+P\frac{dz}{dx}}{(dz/dx)^2}=\dfrac{2+(-\frac{1}{x}).2x}{4x^2}=0,$ $\quad [\because \text{using } 2]$

$$Q_1=\frac{Q}{(dz/dx)^2}=\frac{-4x^2}{4x^2}=-1$$

and $R_1=\dfrac{R}{(dz/dx)^2}=\dfrac{8x^2.\sin(x^2)}{4x^2}=2\sin(x^2)=2\sin z$ $\quad [\because \text{using } 2]$

Hence equation (3) becomes :

$$\frac{d^2y}{dz^2}-y=2\sin z$$

or $\quad (D^2-1)y=2\sin z. \quad ...(4) \quad \left[\because D\equiv\frac{d}{dz}\right]$

It's A.E. is $m^2-1=0 \Rightarrow m=\pm 1$

$\therefore C.F.=c_1e^z+c_2\,e^{-z}$

Now $\quad PI=\dfrac{1}{D^2-1}2\sin z=\dfrac{1}{-1-1}2\sin z=-\sin z$ $\quad [\because D^2=-(1)^2=-1]$

Hence required solution is : $y=C.F.+P.I.$

$$y = c_1 e^{z} + c_2 e^{-z} - \sin z$$

or $$y = c_1 e^{x^2} + c_2 e^{-x^2} - \sin(x^2).$$

Ans.

Example 3.51 : *Solve,* $\dfrac{d^2 y}{dx^2} + \cot x \dfrac{dy}{dx} + 4y \operatorname{cosec}^2 x = 0.$ *[RGPV, Dec. 2011]*

Solution. Comparing the given equation with

$$\frac{d^2 y}{dx^2} + P\frac{dy}{dx} + Qy = R.$$

Here, $P = \cot x,\ Q = 4\operatorname{cosec}^2 x,\ R = 0$

Let choosing independent variable z such that :

$$\left(\frac{dz}{dx}\right)^2 = |Q|$$

$$\Rightarrow \quad \frac{dz}{dx} = 2\operatorname{cosec} x$$

$$\Rightarrow \quad \int dz = \int 2\operatorname{cosec} x\, dx$$

$$\Rightarrow \quad z = 2\log(\tan x/2) \qquad \text{...(1)}$$

With this variable z, then given equation transforms to:

$$\frac{d^2 y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = R_1 \qquad \text{...(2)}$$

Where
$$P_1 = \frac{\dfrac{d^2 z}{dx^2} + P\dfrac{dz}{dx}}{\left(\dfrac{dz}{dx}\right)^2}$$

$$= \frac{-2\operatorname{cosec} x\ \cot x + \cot x \cdot 2\operatorname{cosec} x}{4\operatorname{cosec}^2 x} = 0 \qquad [\because \text{using (1)}]$$

$$Q_1 = \frac{Q}{\left(\dfrac{dz}{dx}\right)^2} = \frac{4\operatorname{cosec}^2 x}{4\operatorname{cosec}^2 x} = 1.$$

and
$$R_1 = \frac{R}{\left(\dfrac{dz}{dx}\right)^2} = 0.$$

(Imp) Method - 4

Variation of Parameters

NOTE:

* This Method is applicable when we don't have the formula for P.I.

$\frac{1}{f(D)} \cot x, \frac{1}{f(D)} \operatorname{cosec} x, \frac{1}{f(D)} \sec x, \frac{1}{f(D)} \tan x$

etc.

Working Rule :-

① Consider second order differential eq^n.

$$\boxed{\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R.} \quad -①$$

where,

P, Q, R are functions of 'x' alone

or Constants.

② Compare the given problem with eq^n ① (standard form) & write the values of P, Q, & R.

(3) We find C.F.

$$\boxed{C.F = Ay_1 + By_2}$$

$(C_1 y_1 + C_2 y_2)$

(4) ~~P.I~~ Now, we find P.I.

$$\boxed{P.I. = uy_1 + vy_2}$$

(5) Now, 'u' is given by :-

$$\boxed{u = \int \frac{-y_2 R}{y_1 y_2' - y_1' y_2} dx}$$

$m^2 + 1 = 0$
$m^2 = -1$
$m = \pm\sqrt{-1}$
$m = \pm i$
C.F. $= C_1 \cos x + C_2 \sin x$
$y_1 = \cos x$
$y_2 = \sin x$

(6) Now, 'v' is given by:

$$\boxed{v = \int \frac{y_1 R}{y_1 y_2' - y_1' y_2} dx}$$

(7) The complete solⁿ of the given problem is.

$$\boxed{y = C.F. + P.I.}$$

x

$$\frac{dx}{} \qquad \frac{}{dx} \qquad \ldots(5)$$

Step 6. : Solving (4) and (5), we get A and B. Putting values of A and B in (3) to obtain the desired complete solution.

Example 3.54 : Solve by the method of variation of parameters

$$\frac{d^2y}{dx^2}+n^2y=\sec nx.$$

[RGPV June 2012]

Solution. Given $\frac{d^2y}{dx^2}+n^2y=\sec(nx)$.

Then find C.F. of $\frac{d^2y}{dx^2}+n^2y=0$.

Its A.E. is $m^2+n^2=0 \Rightarrow m=\pm ni$

$\therefore$ $y=A\cos nx+B\sin nx$ is the C.F., where A and B are constants.

Let $y=A\cos nx+B\sin nx$, ...(1)

be the complete solution of the given equation where A and B are functions of x. Then differentiating (1), we get

$$\frac{dy}{dx}=-An\sin nx+Bn\cos nx+\frac{dA}{dx}\cos nx+\frac{dB}{dx}\sin nx. \qquad \ldots(2)$$

Now choosing A and B, such that

$$\frac{dA}{dx}\cos nx + \frac{dB}{dx}\sin nx = 0 \qquad \ldots(3)$$

Hence (2) becomes :

$$\therefore \quad \frac{dy}{dx} = -An\sin nx + Bn\cos nx.$$

Differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2} = -An^2\cos nx - Bn^2\sin nx - n\frac{dA}{dx}\sin nx - n\frac{dB}{dx}\cos nx.$$

Putting the values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in given equation, we get

$$-An^2\cos nx - Bn^2\sin nx - n\frac{dA}{dx}\sin nx + n\frac{dB}{dx}\cos nx + n^2(A\cos nx + B\sin nx) = \sec nx$$

or
$$-n\frac{dA}{dx}\sin nx + n\frac{dB}{dx}\cos nx = \sec nx. \qquad \ldots(4)$$

Now multiplying (3) by $n\cos nx$ and (4) by $\sin nx$ and subtracting, we get

$$n\frac{dA}{dx} = -\tan nx \Rightarrow dA = -\frac{1}{n}\tan nx\, dx$$

$$\therefore \quad A = \frac{1}{n^2}\log\cos nx + c_1.$$

Again multiplying (3) by $n\sin nx$ and (4) by $\cos nx$ and adding, we get

$$n\frac{dB}{dx} = 1, \quad \Rightarrow dB = \frac{1}{n}dx.$$

$$\therefore \quad B = \frac{x}{n} + c_2.$$

Hence complete solution (1) becomes :

$$y = c_1\cos nx + c_2\sin nx + \frac{1}{n^2}\cos nx.\log\cos nx + \frac{x}{n}\sin nx.$$ **Ans.**

Example 3.56 : Solve by the method of variation of parameters

$$\frac{d^2y}{dx^2}+4y=4\tan 2x.$$

[RGPV Dec. 2004, June 2008 (N), June 2009 & Feb. 2010, June 2016]

Solution. Given $\frac{d^2y}{dx^2}+4y=4\tan 2x$ ...(1)

Then find C.F. of $\frac{d^2y}{dx^2}+4y=0$

$\therefore$ Its A.E. is $m^2+4=0 \Rightarrow m=\pm 2i$

Hence $y=A\cos 2x+B\sin 2x$ is the C.F., where A and B are constants.

Let $y=A\cos 2x+B\sin 2x$ ...(2)

be the complete solution of the given equation (1), where A and B are functions of x.
Differentiating (2) w.r.t. x, we get

$$\therefore \quad \frac{dy}{dx}=-2A\sin 2x+2B\cos 2x+\frac{dA}{dx}\cos 2x+\frac{dB}{dx}\sin 2x.$$

Now choosing A and B, such that :

$$\cos 2x.\frac{dA}{dx}+\sin 2x.\frac{dB}{dx}=0. \qquad ...(3)$$

$$\therefore \quad \frac{dy}{dx}=-2A\sin 2x+2B\cos 2x$$

and $$\frac{d^2y}{dx^2}=-2\frac{dA}{dx}\sin 2x+2\frac{dB}{dx}\cos 2x-4A\cos 2x-4B\sin 2x.$$

Putting the values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in (1), we get

$$-2\frac{dA}{dx}.\sin 2x+2\frac{dB}{dx}\cos 2x-4A\cos 2x-4B\sin 2x+4(A\cos 2x+B\sin 2x)=4\tan 2x.$$

or $$-2\sin 2x\frac{dA}{dx}+2\cos 2x\frac{dB}{dx}=4\tan 2x$$

or $$-\sin 2x\frac{dA}{dx}+\cos 2x\frac{dB}{dx}=2\tan 2x \qquad ...(4)$$

Solving (3) and (4), we get

$$\frac{dA}{dx}=-2\tan 2x.\sin 2x$$

$$= -2\frac{\sin^2 2x}{\cos 2x} = -2\frac{(1-\cos^2 2x)}{\cos 2x}$$

or $$\frac{dA}{dx} = -2\sec 2x + 2\cos 2x.$$

On integrating, we get

$\therefore$ $$A = -\log(\sec 2x + \tan 2x) + \sin 2x + c_1$$

and $$\frac{dB}{dx} = 2\sin 2x.$$

On integrating, we get

$\therefore$ $$B = -\cos 2x + c_2.$$

Hence complete solution (2) becomes :

$$y = c_1 \cos 2x + c_2 \sin 2x - [\log(\sec 2x + \tan 2x)].\cos 2x.$$ **Ans.**

or $\quad y = c_1 + c_2 e \ldots \frac{}{2} \ldots \sin x.$

Example 3.60 : Solve by method of variation of parameters

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = \frac{e^{3x}}{x^2}.$$

[RGPV June 2004, Feb. 2006, June 201

Solution. Given $\dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} + 9y = \dfrac{e^{3x}}{x^2}.$ ...(1

To find C.F. of equation (1)

Its A.E. is $m^2 - 6m + 9 = 0$

$\Rightarrow \quad (m-3)^2 = 0 \Rightarrow m = 3, 3$ (repeated roots)

$\therefore \quad y = (A + Bx)e^{3x}$ is the C.F. of equation (1)

Let $\quad y = (A + Bx)e^{3x}$ ...(2)

be the complete solution of (1), where A and B are functions of x.

Differentiate (2) w.r.t. x, we get

$$\frac{dy}{dx} = 3e^{3x}(A + Bx) + e^{3x}\left(\frac{dA}{dx} + x\frac{dB}{dx} + B\right) \quad \text{...(3)}$$

Let we choose A and B such that :

$$\frac{dA}{dx}+x.\frac{dB}{dx}=0 \qquad \ldots(4)$$

Hence equation (3) becomes :

$$\frac{dy}{dx}=3e^{3x}(A+Bx)+Be^{3x}$$

or
$$\frac{dy}{dx}=e^{3x}[3A+B+3Bx].$$

Differentiate w.r.t. x, we get

$$\frac{d^2y}{dx^2}=3e^{3x}[3A+B+3Bx]+e^{3x}\left[3\frac{dA}{dx}+\frac{dB}{dx}+3B+3x\frac{dB}{dx}\right]$$

$$=e^{3x}\left[9A+6B+9Bx+\frac{dB}{dx}\right]. \qquad [\because \text{ by } (4)]$$

Putting the values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in (1), we get

$$e^{3x}\left(9A+6B+9Bx+\frac{dB}{dx}\right)-6e^{3x}(3A+B+3Bx)+9e^{3x}(A+Bx)=\frac{e^{3x}}{x^2},$$

or
$$9A+6B+9Bx+\frac{dB}{dx}-18A-6B-18Bx+9A+9Bx=\frac{1}{x^2}$$

or
$$\frac{dB}{dx}=\frac{1}{x^2}\Rightarrow dB=\frac{1}{x^2}dx\Rightarrow B=-\frac{1}{x}+c_1.$$

From (4) : $\frac{dA}{dx}+x.\frac{1}{x^2}=0\Rightarrow\frac{dA}{dx}=-\frac{1}{x}\Rightarrow A=-\log x+c_2.$

Hence complete solution (2) becomes :

$$y=\left[-\log x+c_2+x\left(-\frac{1}{x}+c_1\right)\right]e^{3x} \text{ or } y=e^{3x}(-\log x+c_1x+c_2-1). \qquad \textbf{Ans.}$$

...of parameters