25/04/2022

Unit – 3

Partial Differentiation Eqation

* Partial Differentiation Eqn :-

The differential eqn which involves one or more partial derivatives are called partial differential Eqns

* Standard Notation :-

$$p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y} \quad \text{(First order)}$$

$$r = \frac{\partial^2 z}{\partial x^2}, \quad s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}, \quad t = \frac{\partial^2 z}{\partial y^2} \quad \text{(Second order)}$$

* Formation of Partial Differential Eq^n :-

① By the elimination of Arbitary Constants

In this Method, suppose, we have:-
$f(x, y, z, a, b)$, where,
'a' & 'b' are 2 arbitary Constants.

So, now,

$\begin{cases} x, y \rightarrow \text{independent variable} \\ z \rightarrow \text{dependent variable} \end{cases}$

1^{st} we differentiate partially w.r.t 'x' in the given function & differentiate partially w.r.t to 'y' also,

where,

'x' & 'y' are $\longrightarrow$ independent variable

'z' $\longrightarrow$ dependent Variable

* NOTE:

If the no. of arbitary Constants > no. of indenpendent variables

then,

the order of partial differential eqⁿ obtained will be more than '1'.

Form the partial differential eqⁿ of the following eqⁿ by eliminating arbitary constants:-

$$z = ax + by + ab \quad \text{--- (1)}$$

Now,

differentiate eqⁿ (1) w.r.t 'x' partially.

$$\frac{\partial z}{\partial x} = a + 0 + 0$$

$$\boxed{\frac{\partial z}{\partial x} = a} \quad \text{--- (2)}$$

now,

differentiate eqⁿ (1) w.r.t 'y' partially

$$\frac{\partial z}{\partial y} = 0 + b + 0$$

$$\boxed{\frac{\partial z}{\partial y} = b} \quad \text{--- (3)}$$

now,

putting values of eq^n (2) & eq^n (3) in eq^n (1).

we get,

$$z = \left(\frac{\partial z}{\partial x}\right) x + \left(\frac{\partial z}{\partial y}\right) y + \left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right)$$

So, $\boxed{z = px + qy + pq}$

↓

partial differential eq^n of 1^{st} order

---

(2) $z = ax + a^2y^2 + b$.

So,

$$z = ax + a^2y^2 + b \quad \text{——— (1)}$$

Now, differentiate partially eq^n (1) w.r.t 'x'

$$\frac{\partial z}{\partial x} = a + 0 + 0$$

$$\boxed{\frac{\partial z}{\partial x} = a} \quad \text{——— (2)}$$

so,

now,

differentiate partially eqn (1) w.r.t 'y'

we get,

$$\frac{\partial z}{\partial y} = 0 + 2a^2 y + 0$$

$$\boxed{\frac{\partial z}{\partial y} = 2a^2 y} \quad \text{—} ~~(3)~~ \text{— (3)}$$

~~again partially differentiate,~~

~~Now, $\boxed{\frac{\partial^2 z}{\partial y^2} = 2a^2}$ — (3)~~

put values of eqn (2) ~~in~~ eqn (3) ~~in eqn (1)~~

we get,

$$\frac{\partial z}{\partial y} = 2\left(\frac{\partial z}{\partial x}\right)^2 y$$

so,

$$\frac{\partial z}{\partial x} = p$$

$$\frac{\partial z}{\partial y} = q$$

we get,

$$\boxed{q = 2p^2 y}$$

~~$z = \left(\frac{\partial z}{\partial x}\right)x + \frac{\left(\frac{\partial^2 z}{\partial x^2}\right)}{2} y^2$~~

~~$z = px + \frac{q}{2} y$~~

~~$z = px + \frac{q}{2} y$~~

~~$z = \left(\frac{\partial z}{\partial x}\right)x + \frac{\left(\frac{\partial^2 z}{\partial x^2}\right)}{2} y$~~

~~$z = px + qy^2$~~

~~so, $\boxed{z = px + qy^2}$~~

Method - 2

By eliminating Arbitary Functions:-

In this Method, let us consider, 'u' & 'v' are 2 functions of 'x', 'y' & 'z'

Q - Form a P.D.E by eliminating arbitary functions:-

Here, the given eqn involves only (1) functions, so we differentiate partially w.r.t 'x' & w.r.t 'y', we get,

(1) $z = f(x^2 - y^2)$ — (1)

Now, differentiate partially 'z' w.r.t 'x' we get,

$$\frac{\partial z}{\partial x} = f'(x^2 - y^2) \cdot (2x - 0)$$

$$\boxed{\frac{\partial z}{\partial x} = f'(x^2 - y^2) \cdot (2x)} \quad \text{— (2)}$$

Now,

(2)

differentiate partially eqn (1) w.r.t 'y'

$$\frac{\partial z}{\partial y} = f'(x^2-y^2)\cdot(0-2y)$$

$$\boxed{\frac{\partial z}{\partial y} = -f'(x^2-y^2)(2y)} \quad \text{--- (3)}$$

Now,

(2) ÷ (3)

$$\frac{\partial z}{\partial x} \div \frac{\partial z}{\partial y} = \frac{\cancel{f'(x^2-y^2)}(\cancel{2}x)}{-\cancel{f'(x^2-y^2)}(\cancel{2}y)}$$

$$\frac{\partial \cancel{z}}{\partial x} \times \frac{\partial y}{\partial \cancel{z}} = -\frac{x}{y}$$

$$\frac{\partial y}{\partial x} = -\frac{x}{y}$$

$$\frac{p}{q} = \frac{-x}{y}$$

$$py = -xq$$

$$\boxed{py + qx = 0} \quad \text{--- (4)}$$

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(2) $z = f(x+iy) + g(x-iy)$ —— (1)

Here, the eq^n involves, 2 functions of 'x' & 'y'
So, we have to differentiate partially
eq^n (1) w.r.t 'x' & w.r.t 'y'
2 times :

~~we~~ Now,
differentiate eq^n (1) w.r.t 'x' partially.
we get,

$$\frac{\partial z}{\partial x} = f'(x+iy)(1+0) + g'(x-iy)(1+0)$$

$$\boxed{\frac{\partial z}{\partial x} = f'(x+iy) + g'(x-iy)} \quad \text{—— (2)}$$

Now,
again diff.

$$\boxed{\frac{\partial^2 z}{\partial x^2} = f''(x+iy) + g''(x-iy)} \quad \text{—— (3)}$$

Now,

diff eq^n (1) w.r.t 'y' partially

$$\frac{\partial z}{\partial y} = f'(x+iy)(0+i) + g'(x-iy)(0-i)$$

$$\frac{\partial z}{\partial y} = f'(x+iy)(i) + g'(x-iy)(-i)$$

again diff

$$\frac{\partial^2 z}{\partial y^2} = (i) f''(x+iy)\cdot(0+i) + (-i) g''(x-iy)\cdot(0$$

$$\frac{\partial^2 z}{\partial y^2} = f''(x+iy)(i^2) + g''(x-iy)(i^2)$$

$$\boxed{\frac{\partial^2 z}{\partial y^2} = -f''(x+iy) - g''(x-iy) \quad - (4)}$$

Now,

adding (3) & (4).

we get,

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 y}{\partial y^2}$$

$$f''(x+iy) + g''(x-iy) - f''(x+iy) - g''(x-iy)$$

$$= 0.$$

So,

$$\boxed{\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 y}{\partial y^2} = 0}$$

$p = \frac{\partial z}{\partial x}$

$q = \frac{\partial z}{\partial y}$

So, $\boxed{r + t = 0}$

Linear Partial Differential Equations

Lagranges Partial Differential $eq^n \longrightarrow 1^{st}$ order

Linear P.D.E is :-

$$\boxed{Pp + Qq = R}$$

Consider linear P.D.E is :-

where,

P, Q, R function of x, y, z alone or may be constants.

~~Working Rule :-~~

The general ~~form~~ sol^n of the problem,

$f(u, v) = 0$

or,

$f(a, b) = 0$

$u = a$

$v = b$

$f \rightarrow$ arbitary function

or, $f(u) = v$

$u = u(x, y, z)$

$v = v(x, y, z)$

First we write Auxilary eq^n :-

The A.E is:

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

Working Rule :-

① Consider a linear partial differential eq^n :-

$$\boxed{Pp + Qq = R} \quad — ①$$

② Compare eq^n ① with the given problem & find; the values of P, Q & R,
where,
P, Q, R are functions of 'x', 'y' & 'z' or may be constants.

③ Now, we write the Auxilary eq^n. :-

The A.E is :-

$$\boxed{\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}} \quad — ②$$

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(4) For solving this we have 2 ways:-

1st Method is :-

I Method of Grouping

(a) In this method, we take 2 members of Auxilary eqn such ~~that~~ as :-

$$\frac{dx}{P} = \frac{dy}{Q} \quad -(4)$$

or,

$$\frac{dy}{Q} = \frac{dz}{R} \quad -(5)$$

or

$$\frac{dx}{P} = \frac{dz}{R} \quad -(6)$$

Possibilities

$$\boxed{\frac{dx}{P} = \frac{dy}{Q}} \quad -(4)$$

we find a ~~of~~ diff eqn ~~soln~~ in 'x' & 'y' only.

(b) Eqn (4) can easily be solved by integration. (variable separable)

(c) We get ~~1~~ soln.

(d) Illy we can take other 2 members (eqn (5) & eqn (6)) we find 2nd soln.

II 2nd Method is :-

Method of Multipliers

(a) In this Method, we ~~let~~ use a set of multipliers l, m, n (not always constants i.e. can be in the form of x, y, z also), provided that :

$$\boxed{\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l\,dx + m\,dy + n\,dz}{lP + mQ + nR}} \quad \text{— (7)}$$

(from (6) (eg:) l, m, n = 1, 1, 1 resp. $P, Q, R \rightarrow$ from eqn (on comparing))

where,

$$\boxed{lP + mQ + nR = 0} \quad \text{— (8)}$$

so, on substituting values in eqn (8) we should get '0'

So,

& l = function of 'x' alone or constants

m = function of 'y' alone or constant

n = function 'z' alone or constants

(b) the soln is :

$$\boxed{l\,dx + m\,dy + n\,dz = 0} \quad \text{— (9)}$$

$$\cancel{\boxed{lx + my + nz = a}}$$

(c) After integrating eqn (9) we get,

$$\boxed{lx + my + nz = 0} \quad \text{— (10)}$$

(d) Now, we select another set of multipliers l_1, m_1 & n_1 to get the 2nd solⁿ.

So,

$$\boxed{\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l_1 dx + m_1 dy + n_1 dz}{l_1 P + m_1 Q + n_1 R}} \quad (11)$$

where,

$$\boxed{l_1 P + m_1 Q + n_1 R = 0} \quad -(12)$$

(e) again Now,

~~we select~~ the solⁿ is:-

$$\boxed{l_1 dx + m_1 dy + n_1 dz = 0} \quad -(13)$$

(f) After integrating eqⁿ (13)

we get,

$$\int (l_1 dx + m_1 dy + n_1 dz) = \int 0$$

$$\boxed{l_1 x + m_1 y + n_1 z = a} \quad -(14)$$

III

3rd Method is :—

Combination of 'Method 1' & 'Method 2'

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Q. Solve:

$$y^2 z p + x^2 z q = x y^2 \quad \text{--- (1)}$$

On Comparing eq^n (1) with

$$Pp + Qq = R$$

we get

$$P = y^2 z \quad \text{--- (2)}$$

$$Q = x^2 z \quad \text{--- (3)}$$

$$R = x y^2 \quad \text{--- (4)}$$

so,
The auxilary eq^n is :—

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

So,

$$\frac{dx}{y^2 z} = \frac{dy}{x^2 z} = \frac{dz}{xy^2} \quad \text{— (5)}$$

Now,

Taking $\frac{dx}{y^2 z} = \frac{dy}{x^2 z}$

$$\frac{dx}{y^2 \cancel{z}} = \frac{dy}{x^2 \cancel{z}}$$

$$\frac{dx}{y^2} = \frac{dy}{x^2}$$

$$x^2 dx = y^2 dy$$

now,

integrating both sides, we get,

$$\int x^2 dx = \int y^2 dy$$

$$\frac{x^3}{3} = \frac{y^3}{3} + C_1$$

$$\frac{x^3 - y^3}{3} = C_1$$

$$x^3 - y^3 = 3C_1$$

$$\boxed{x^3 - y^3 = a} \quad \text{--- (6)} \quad (a = 3C_1)$$

now,

Taking 1st & last, we get,

$$\frac{dx}{\cancel{y^2} z} = \frac{dz}{x \cancel{y^2}}$$

$$\frac{dx}{z} = \frac{dz}{x}$$

$$x\,dx = z\,dz$$

now, integrating both sides

$$\int x\,dx = \int z\,dz$$

$$\frac{x^2}{2} = \frac{z^2}{2} + C_2$$

$$\frac{x^2 - z^2}{2} = C_2$$

$$x^2 - z^2 = 2C_2$$

$$\boxed{x^2 - z^2 = b} \quad (b = 2C_2)$$

$\rightarrow$ (7)

So,

now,

The general solⁿ is:

$$f(a, b) = 0$$

$$\boxed{f(x^3 - y^3, x^2 - z^2) = 0}$$

Solve:

$$yzp + zxq = xy$$

$$(yz)p + (zx)q = xy \quad \text{---} \quad (1)$$

now,

on comparing eqⁿ (1) with

$$Pp + Qq = R$$

we get

$$P = yz \quad \text{---} \quad (2)$$

$$Q = zx \quad \text{---} \quad (3)$$

$$R = xy \quad \text{---} \quad (4)$$

The auxilary eq^n is:

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

so,

$$\frac{dx}{yz} = \frac{dy}{zx} = \frac{dz}{xy} \quad — (5)$$

now,

Taking 1st & 2nd

we get:

$$\frac{dx}{y\cancel{z}} = \frac{dy}{\cancel{z}x}$$

$$\frac{dx}{y} = \frac{dy}{x}$$

$$x\,dx = y\,dy$$

now integrating both sides.

$$\int x\,dx = \int y\,dy$$

$$\frac{x^2}{2} = \frac{y^2}{2} + C_1$$

$$\frac{x^2-y^2}{2} = c_1$$

$$x^2 - y^2 = 2c_1$$

$$\boxed{x^2 - y^2 = a} \quad (a = 2c_1) \rightarrow (6)$$

now,

Taking 2nd & 3rd.

we get,

$$\frac{dy}{z\cancel{x}} = \frac{dz}{\cancel{x}y}$$

$$\frac{dy}{z} = \frac{dz}{y}$$

$$y\,dy = z\,dz$$

now, integrating both sides,
we get,

$$\int y\,dy = \int z\,dz$$

$$\frac{y^2}{2} = \frac{z^2}{2} + c_2$$

$$\frac{y^2 - z^2}{2} = c_2$$

$$y^2 - z^2 = 2C_2$$

$$\boxed{y^2 - z^2 = b} \quad (b = 2C_2) \rightarrow (7)$$

So, now

The general sol^n is:-

$$f(a, b) = 0$$

$$\boxed{f(x^2 - y^2,\ y^2 - z^2) = 0}$$

Q. Solve:

$$x(y^2 + z)p - y(x^2 + z)q = z(x^2 - y^2)$$

$$x(y^2 + z)p + (-y(x^2 + z)q) = z(x^2 - y^2) \rightarrow (1)$$

now,

on comparing eq^n (1) with

$$Pp + Qq = R$$

we get,

$$P = x(y^2+z) \quad \text{--- (2)}$$

$$Q = -y(x^2+z) \quad \text{--- (3)}$$

$$R = z(x^2-y^2) \quad \text{--- (4)}$$

Now,

The Auxilary eq^n is :-

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{z(x^2-y^2)} \quad \text{--- (5)}$$

Now, we select set of multipliers ~~l_1,~~ l_1, m_1, & n_1,

$$l_1 = \frac{1}{x} \quad \text{--- (6)}$$

$$m_1 = \frac{1}{y} \quad \text{--- (7)}$$

$$n_1 = \frac{1}{z} \quad \text{--- (8)}$$

so,

$$\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{x^2-y^2} = \frac{l_1\,dx + m_1\,dy + n_1\,dz}{l_1 P + m_1 Q + n_1 R}$$

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now,
$$l_1 P + m_1 Q + n_1 R = \frac{1}{\cancel{x}} \cdot \cancel{x}(y^2+z) + \frac{1}{\cancel{y}}(-\cancel{y})(x^2+z) + \frac{1}{\cancel{z}} \cdot \cancel{z}(x^2-y^2)$$

$$l_1 P + m_1 Q + n_1 R = \cancel{y^2} + \cancel{z} - \cancel{x^2} - \cancel{z} + \cancel{x^2} - \cancel{y^2}$$

$$\therefore \boxed{l_1 P + m_1 Q + n_1 R = 0} \quad \text{--- (9)}$$

So,

$$\text{Each fraction} = \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$$

$$\Rightarrow \quad \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0 \quad \text{--- (10)}$$

now,
On integrating both sides,
we get,

$$\int\left(\frac{1}{x}dx\right) + \int\frac{1}{y}dy + \int\frac{1}{z}dz = \int 0$$

$$\log x + \log y + \log z = \log c_1$$

$$\log(xyz) = \log c_1$$

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$xyz = c_1$

$\boxed{xyz = a}$ $(a = c_1)$ → (11)

now,

another set of multipliers be l_2, m_2, n_2,

where,

$l_2 = -x$ — (12)

$m_2 = -y$ — (13)

$n_2 = 1$ — (14)

so,

we get,

$$\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{x^2-y^2} = \frac{l_2\,dx + m_2\,dy + n_2\,dz}{l_2 P + m_2 Q + n_2 R}$$

→ (15)

now,

$l_2 P + m_2 Q + n_2 R =$

$-x(x(y^2+z)) + (-y(-y(x^2+z))) + z(x^2-y^2)$

$$-\cancel{x^2y^2} - \cancel{x^2z} + \cancel{y^2x^2} + \cancel{y^2z} + \cancel{x^2z} - \cancel{y^2z}$$

$$= 0$$

So, $\therefore l_2 P + m_2 Q + n_2 R = 0$. — (16)

$\Rightarrow$

Each fraction $= \dfrac{-x\,dx - y\,dy + 1\,dz}{0}$

$\Rightarrow \quad -x\,dx - y\,dy + dz = 0$ — (17)

on integrating both sides

we get,

$$-\int x\,dx - \int y\,dy + \int dz = \int 0$$

$$-\frac{x^2}{2} - \frac{y^2}{2} + z = C_2$$

$$-x^2 - y^2 + 2z = 2C_2$$

$$x^2 + y^2 - 2z = -2C_2$$

$$\boxed{x^2 + y^2 - 2z = b}$$ — (18)

where, $b = -2c_2$

so,

The general sol^n is:

$$f(a, b) = 0$$

$$\boxed{f(xyz,\ x^2+y^2-2z) = 0}$$

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Q. Solve:

v.v.v. Imp

$$(z^2 - 2yz - y^2)p + (xy + zx)q = xy - zx \quad \text{---} (1)$$

now,

on comparing above eq^n (1) with

$$Pp + Qq = R$$

we get,

$$P = z^2 - 2yz - y^2 \quad \text{---} (2)$$

$$Q = xy + zx \quad \text{---} (3)$$

$$R = xy - zx \quad \text{---} (4)$$

Now,

The auxilary eq^n is :-

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

so,

$$\frac{dx}{z^2-2yz-y^2} = \frac{dy}{xy+zx} = \frac{dz}{xy-zx} \quad \text{— (5)}$$

Now, we select a set of multipliers l, m, n

where,

$l = x$ — (6)

$m = y$ — (7)

$n = z$ — (8)

so,

$$\frac{dx}{z^2-2yz-y^2} = \frac{dy}{xy+zx} = \frac{dz}{xy-zx} = \frac{l\,dx + m\,dy + n\,dz}{lP + mQ + nR}$$

$$\text{Each fraction} = \frac{x\,dx + y\,dy + z\,dz}{x(z^2-2yz-y^2) + y(xy+zx) + z(xy-zx)}$$

so, $lP + mQ + nR = x\cancel{z^2} - 2\cancel{xyz} - \cancel{xy^2} + \cancel{xy^2} + \cancel{xyz}$
$+ \cancel{xyz} - x\cancel{z^2}$

$\therefore lP + mQ + nR = 0$ — (9)

So, each fraction $= \frac{x\,dx + y\,dy + z\,dz}{0}$.

$\Rightarrow x\,dx + y\,dy + z\,dx = 0$ — (10)

now integrating both sides, we get,

$$\int x\,dx + \int y\,dy + \int z\,dz = \int 0$$

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = c_1$$

$$x^2+y^2+z^2 = 2c_1$$

$$\boxed{x^2+y^2+z^2 = a} \quad (\because, a = 2c_1) \rightarrow (11)$$

Now, Taking last 2 members, we get,

$+z(xy-zx)$

$$\frac{dy}{xy+zx} = \frac{dz}{xy-zx}$$

$$\frac{dy}{\cancel{x}(y+z)} = \frac{dz}{\cancel{x}(y-z)}$$

$$\frac{dy}{y+z} = \frac{dz}{y-z}$$

$$\frac{dy}{dz} = \frac{y+z}{y-z} \quad \text{—} \quad (12)$$

Now,

$$f(y,z) = \frac{y+z}{y-z}$$

So,

$$f(\lambda y, \lambda z) = \frac{\lambda y + \lambda z}{\lambda y - \lambda z}$$

$$f(\lambda y, \lambda z) = \frac{\cancel{\lambda}(y+z)}{\cancel{\lambda}(y-z)}$$

$$f(\lambda y, \lambda z) = \frac{y+z}{y-z}$$

$\because f(\lambda y, \lambda z) = f(y,z)$ — (13)

So, above eq^n is homogeneous differential eq^n,

So, $y = vz$ — (14)

So, $$\frac{dy}{dz} = v(1) + z.\frac{dv}{dz}$$

$$\frac{dy}{dz} = v + z\frac{dv}{dz}$$

$$\frac{y+z}{y-z} = v + z\frac{dv}{dz} \quad (\text{from } (12))$$

$$\frac{vz+z}{vz-z} = v + z\frac{dv}{dz} \quad (\text{from } (14))$$

$$\frac{\cancel{z}(v+1)}{\cancel{z}(v-1)} - v = z\frac{dv}{dz}$$

$$\frac{v+1-v^2+v}{v-1} = z\frac{dv}{dz}$$

$$\frac{-v^2+2v+1}{v-1} = z\frac{dv}{dz}$$

$$\int \frac{1}{z}dz = -\int \frac{v-1}{v^2-2v-1}dv \quad -(15)$$

$$\log z = -I_1 \quad -(15)$$

Now, for 'I_1'

$$I_1 = \int \frac{v-1}{v^2-2v-1}dv$$

Let,

$$v^2-2v-1 = t \quad -(16)$$

$$2v-2-0 = \frac{dt}{dv}$$

$$2(v-1) = \frac{dt}{dv}$$

$$dv = \frac{dt}{2(v-1)} \quad -(17)$$

so, $$I_1 = \int \frac{\cancel{v-1}}{t}\frac{dt}{2\cancel{(v-1)}}$$

$$I_1 = \frac{1}{2}\int \frac{1}{t}dt$$

$$I_1 = \frac{1}{2}\log t$$

Now, $I_1 = \frac{1}{2}\log(v^2-2v-1)$

so, from (15)

$$\log z = -\frac{1}{2}\log(v^2-2v-1) + \log c_2$$

$$\log z + \log\sqrt{v^2-2v-1} = \log c_2$$

$$\log z + \log\sqrt{\left(\frac{y}{z}\right)^2 - 2\left(\frac{y}{z}\right) - 1} = \log c_2$$

$$\log z + \log\sqrt{\frac{y^2}{z^2} - \frac{2y}{z} - 1} = \log c_2$$

$$\log z + \log\sqrt{\frac{y^2-2yz-z^2}{z^2}} = \log c_2$$

$$\log z + \log\frac{\sqrt{y^2-2yz-z^2}}{z} = \log c_2$$

$$\log\left(\frac{\cancel{z}}{\cancel{z}}\sqrt{y^2-2yz-z^2}\right) = \log c_2$$

$$\frac{1}{2}\log(y^2-2yz-z^2) = \frac{1}{2}\log b \quad (c_2 = \sqrt{b})$$

$$\boxed{y^2-2yz-z^2 = b} \quad -(18)$$

now, The general soln is:-

$$f(a,b) = 0$$

$$\boxed{f(x^2+y^2-z^2,\ y^2-2yz-z^2) = 0}$$

$(D^2+9)(D^2-9)$

$(D^2)^2 - 81$

$(-9)^2 - 81$

81

$x = e^z$

$\log x = z$

diff

$\frac{1}{x} = \frac{dz}{dx}$

$\frac{9}{2} - 3$

$\frac{9-6}{2}$ $\frac{dy}{dx} = \frac{dy}{\cancel{dz}} \times \frac{dz}{dx} = \frac{dy}{dz} \cdot \frac{1}{x}$

$\frac{3}{2}$

$$\boxed{\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{1}{x}}$$

$x\frac{dy}{dx} = \frac{dy}{dz}$

$x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} \times \frac{dz^2}{dx^2}$ $\frac{d^2y}{dz^2} - \frac{dy}{dz}$

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right)$

$= \frac{-1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d}{dz}\left(\frac{dy}{dz} \cdot \frac{dz}{dx}\right)$

$= -\frac{1}{}$

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Q. Solve:

$$(x^2-yz)p+(y^2-zx)q = z^2-xy \quad \text{— (1)}$$

now,

on comparing eq^n (1) with

$$Pp + Qq = R$$

we get,

$$P = x^2-yz \quad \text{— (2)}$$

$$Q = y^2-zx \quad \text{— (3)}$$

$$R = z^2-xy \quad \text{— (4)}$$

Now,

The Auxilary eq^n is:-

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{x^2-yz} = \frac{dy}{y^2-zx} = \frac{dz}{z^2-xy} \quad \text{— (5)}$$

Now,

$\because$ Method of grouping & method of multipliers fails,

so,

$$\frac{dx-dy}{x^2-y^2-yz+zx} = \frac{dy-dz}{y^2-z^2-zx+xy} = \frac{dz-dx}{z^2-x^2-xy+yz}$$

$$\frac{dx-dy}{(x+y)(x-y)+z(x-y)} = \frac{dy-dz}{(y+z)(y-z)+x(y-z)} = \frac{dz-dx}{(z+x)(z-x)+y(z-x)}$$

$$\frac{dx-dy}{x-y\,\cancel{(x+y+z)}} = \frac{dy-dz}{(y-z)\cancel{(x+y+z)}} = \frac{dz-dx}{(z-x)\cancel{(x+y+z)}}$$

so, $$\frac{dx-dy}{x-y} = \frac{dy-dz}{y-z} = \frac{dz-dx}{z-x} \quad -(7)$$

so,

now,

Taking 1st & 2nd member

$$\frac{dx-dy}{x-y} = \frac{dy-dz}{y-z}.$$

now, integrating both sides we get,

$$\int\frac{dx-dy}{x-y} = \int\frac{dy-dz}{y-z}$$

$$I_1 = I_2 + \log c \quad \text{--- (8)}$$

(6)

so,

<u>for 'I_1'</u>

$$I_1 = \int \frac{dx - dy}{x - y}$$

$$I_1 = \int \frac{dx - dy}{x - y}$$

So,

let $x - y = t$ —— (9)

$$1 - \frac{dy}{dx} = \frac{dt}{dx}$$

$$\frac{dx - dy}{dx} = \frac{dt}{dx}$$

$dy - dx = dt$ —— (10)

$$I_1 = \int \frac{dt}{t}$$

$I_1 = \log t$

$\boxed{I_1 = \log(x-y)}$ — (11)

for 'I_2'

$$I_2 = \int \frac{dy - dz}{y - z}$$

so,

Let

$y - z = f$ — (12)

$dy - dz = df$ — (13)

$$I_2 = \int \frac{df}{f}$$

$I_2 = \log f$

$$\boxed{I_2 = \log(y-z)} \quad - (14)$$

So,

from (8)

$I_1 = I_2 + \log c_1$

$\cancel{I_2} \log(x-y) = \log(y-z) + \log c_1$

$\log(x-y) - \log(y-z) = \log c_1$

$\log\left(\frac{x-y}{y-z}\right) = \log c_1$

$$\frac{x-y}{y-z} = c_1$$

$$\boxed{\frac{x-y}{y-z} = a} \quad (c_1 = a) \rightarrow (15)$$

Now,

Taking taking 2nd & 3rd member

So, we get,

$$\frac{dy-dz}{y-z} = \frac{dz-dx}{z-x}$$

now,

integrating both sides:-

we get

$$\int \frac{dy - dz}{y - z} = \int \frac{dz - dx}{z - x}$$

$$I_3 = I_4 + \log C_2 \quad — (16)$$

Now for 'I_3'

$$I_3 = \int \frac{dy - dz}{y - z}$$

Let

$$y - z = q \quad — (17)$$

$$dy - dz = dq \quad — (18)$$

$$I_3 = \int \frac{dq}{q}$$

$$I_3 = \cancel{\tfrac{q}{q}} \log(q)$$

$$\boxed{I_3 = \log(y - z) \quad — (19)}$$

Now, for I_4

$$I_4 = \int \frac{dz - dx}{z - x}$$

$$z - x = h \quad \text{--- (20)}$$
$$dz - dx = dh \quad \text{--- (21)}$$

$$I_4 = \int \frac{dh}{h}$$

$$I_4 = \log h$$

$$\boxed{I_4 = \log(z - x)} \quad \text{--- (22)}$$

now, from (16)

$$I_3 = I_4 + \log C_2$$

$$\log(y - z) = \log(z - x) + \log C_2$$

$$\log(y - z) - \log(z - x) = \log C_2$$

$$\log\left(\frac{y - z}{z - x}\right) = \log C_2$$

$$\frac{y - z}{z - x} = C_2$$

$$\boxed{\frac{y - z}{z - x} = b} \text{ --- (23)} \quad (b = C_2)$$

so, the general sol^n is: -

$$f(a, b) = 0$$

$$\boxed{f\left(\frac{x - y}{y - z}, \frac{y - z}{z - x}\right) = 0} \quad (\text{from (15) \& (23)})$$

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degree → 1

Linear Partial Differential eq^n with Constant Coefficient.

* Homogeneous Linear Partial Differential Eq^n with Constant Coefficient

degree → 1
order $\geq$ 1

For order > 1 we use C.F + P.I method.

$$\boxed{z = C.F. + P.I.} \quad \text{— (1)}$$

(1) Consider n^{th} order Partial Differential Eq^n

$$\boxed{A_0 \frac{\partial^n z}{\partial x^n} + A_1 \frac{\partial^n z}{\partial x^{n-1} \partial y} + A_2 \frac{\partial^n z}{\partial x^{n-2} \partial y^2} + \cdots + A_n \frac{\partial^n z}{\partial y^n} = f(x, y)}$$

→ (2)

where,
z = dependent
x, y = independent

$$\frac{\partial z}{\partial x} = Dz \;\; \to (3), \qquad \frac{\partial z}{\partial y} = D'z \quad \text{— (4)}$$

$$D = \frac{\partial}{\partial x}, \qquad D' = \frac{\partial}{\partial y}$$

$$\boxed{\left[A_0 D^n + A_1 D^{n-1} D' + A_2 D^{n-2} D'^2 + \cdots + A_n (D')^n\right] z = f(x, y)}$$

→ (5)

(2) The complete sol^n of (5) is :-

1^{st} we find auxilary eq^n by substituting

$D = m$, — (6)

$D' = 1$ — (7)

$z = 1$ — (8)

R.H.S $= 0$

(3) By substituting these values (from (6), (7) & (8))

we will get an eq^n in terms of 'm'

i.e.

$$\boxed{A_0 m^n + A_1 m^{n-1} + A m^{n-2} \cdots + A_n = 0} \quad — (9)$$

(4) On solving eq^n (9), we get values of m.

These values are called roots,

$m = m_1, m_2 \ldots$

(5) Now, Rules (Cases) for finding 'C.F.'

(a) Case I : Roots are distinct (unequal)

$$\boxed{C.F. = f_1(y + m_1 x) + f_2(y + m_2 x)}$$

(b) Case II : When the roots are equal.

$$m_1 = m_2 = m$$

$$\boxed{C.F = f_1(y+mx) + x f_2(y+mx) + x^2 f_3(y+mx)}$$

* (For imaginary roots, if the roots are equal → Case II, if the roots are unequal → Case I)

Now,

(6) Rules for finding 'P.I.'

(a) $$\frac{1}{F(D, D')} e^{ax+by} = \frac{1}{F(a,b)} e^{ax+by} \quad \text{such that } F(a,b) \neq 0$$

(putting $D = a$, & $D' = b$)

(b) * $$\frac{+1}{F(D^2, DD', D'^2)} \sin(ax+by) \pm \frac{1}{F(-a^2, (-ab), -b^2)} \sin(ax+by)$$

$$D^2 = -(a^2)$$
$$DD' = -(ab)$$
$$D'^2 = -(b^2)$$

fails, if $F(-(a^2), (ab), -(b^2)) = 0$

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(c) * $\frac{+1}{F(D^2, DD', D'^2)} \cos(ax+by) = \frac{1}{F(-(a^2), (-ab), (-b)^2)} \cos(ax+by)$

where,
$D^2 = -(a^2)$
$DD' = -(ab)$
$D'^2 = -(b^2)$

& $\boxed{F((-(a^2)), (-ab), (-(b^2)))} \neq 0$

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(d) $\frac{1}{F(D, D')} x^r y^s = [F(D, D')]^{-1} x^r y^s$

which is calculated by expanding $[F(D, D')]^{-1}$ in power of D^* and by using binomial expansion.

(e) * $\frac{1}{F(D^2, D'^2)} \sin ax \sin by = \frac{1}{F(-a^2, -b^2)} \sin ax \sin by$

such that $(F(-a^2, -b^2) \neq 0)$

where, $D^2 = -a^2$
$D'^2 = -b^2$

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(f) * $\frac{1}{f(D^2, D'^2)} \cos ax \cos by = \frac{1}{f(-a^2, -b^2)} \cos ax \cos by$

Such that, $f(-a^2, -b^2) \neq 0$

where, $D^2 = -a^2$
$D'^2 = -b^2$

(g) $\frac{1}{f(D, D')} e^{ax+by} \cdot f(x,y) = e^{ax+by} \frac{1}{f(D+a, D'+b)} f(x,y)$

where, $D = D+a$
$D' = D'+b$

& $f(D+a, D'+b) \neq 0$

(7) Complete Solⁿ is: $\boxed{z = C.F. + P.I.}$

NOTE:

* when these rules fails, then we apply general formula for Partial Differential Eqⁿ. It is not applicable in ordinary

General formula :-

$$\frac{1}{D - mD'} f(x,y) = \int f(x, c-mx)\, dx$$

where, $y = c - mx$.
At the end ~~re~~ substitute $c = y + mx$

Q. Solve:

$$\cancel{(D^2 - 4DD' + 4D'^2)\, z = e^{2x+3y}}$$

$$\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = e^{3x+2y}$$

$\because$

$$D = \frac{\partial}{\partial x}, \quad D' = \frac{\partial}{\partial y}$$

So,

$$D^2 z + 2DD'z + D'^2 z = e^{3x+2y}$$

So, $$(D^2 + 2DD' + D'^2) z = e^{3x+2y} \quad \text{--- (1)}$$

The auxilary eq^n is:-

put $(D = m,\ D' = 1,\ z = 1,\ R.H.S = 0)$

we get,

$$m^2 + 2m(1) + 1^2 = 0 \quad \text{--- (2)}$$

$$m^2 + 2m + 1 = 0$$

$$m^2 + m + m + 1 = 0$$

$$m(m+1) + 1(m+1) = 0$$

$$(m+1)(m+1) = 0$$

$$\boxed{m = -1, -1} \quad \text{--- (3)}$$

$\because$ roots are real & equal

So,

$$C.F. = f_1(y+m_1 x) + x f_2(y+m x)$$

$$C.F. = f_1(y+(-1)x) + x f_2(y+(-1)x)$$

$$\boxed{C.F. = f_1(y-x) + x f_2(y-x)} \quad — (4)$$

Now,

for 'P.I.'

$$P.I = \frac{1}{f(D,D')} \cdot f(x,y)$$

$$P.I. = \frac{1}{D^2+2DD'+D'^2} \cdot e^{3x+2y}$$

$$P.I. \left(\because \frac{1}{f(D,D')} e^{ax+by} = \frac{1}{f(a,b)} e^{ax+by}\right.$$

where, $D=a$, $D'=b$ & $f(a,b) \neq 0$

so here, $\because e^{3x+2y}$

so, $a=3$
$b=2$

so, $D=3$, $D'=2$)

(5) (6)

So,

$$P.I. = \frac{1}{3^2 + 2(3)(2) + 2^2} \cdot e^{3x+2y}$$

$$P.I. = \frac{1}{9 + 12 + 4} e^{3x+2y}$$

$$P.I. = \frac{1}{25} e^{3x+2y}$$

$$\boxed{P.I. = \frac{1}{25} e^{3x+2y}} \quad \text{—} (7)$$

Now, the complete solⁿ is :-

$$z = C.F. + P.I.$$

$$z = f_1(y-x) + x f_2(y-x) + \frac{1}{25} e^{3x+2y}$$

(from (4) & (7))

$$\boxed{z = f_1(y-x) + x f_2(y-x) + \frac{1}{25} e^{3x+2y}}$$

Q. Solve:

$$\frac{\partial^2 z}{\partial x^2} - 4\frac{\partial^2 z}{\partial x \partial y} + 4\frac{\partial^2 z}{\partial y^2} = e^{2x+2y}$$

so,

$\because D = \frac{\partial}{\partial x}$, $D' = \frac{\partial}{\partial y}$

So,

$$D^2 z - 4DD'z + 4D'^2 z = e^{2x+2y}$$

$$(D^2 - 4DD' + 4D'^2)z = e^{2x+2y} \quad \text{—— (1)}$$

For C.F.:

The auxilary eq^n is:-

put ($D = m$, $D' = 1$, $z = 1$, R.H.S $= 0$, in eq^n (1))

we get,

$$m^2 - 4m(1) + 4(1)^2 = 0 \quad \text{—— (2)}$$

$$m^2 - 4m + 4 = 0$$

$$m^2 - 2m - 2m + 4 = 0$$

$$m(m-2) - 2(m-2) = 0$$

$$(m-2)(m-2) = 0$$

$$\boxed{m = 2, 2} \quad \text{—— (3)}$$

$\because$ roots are real & equal

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so,

$$C.F. = f_1(y+mx) + x f_2(y+mx)$$

$$\boxed{C.F. = f_1(y+2x) + x f_2(y+2x)} \quad \text{— (4)}$$

Now,

<u>for P.I.</u>

$$P.I. = \frac{1}{f(D,D')} \cdot f(x,y)$$

$$P.I. = \frac{1}{D^2 - 4DD' + 4D'^2} \cdot e^{2x+2y}$$

$$\left(\because \frac{1}{f(D,D')} e^{ax+by} = \frac{1}{f(a,b)} e^{ax+by} \right.$$

where, $D = a$ & $f(a,b) \neq 0$.
$D' = b$

So, here, $\because e^{2x+2y}$

so, $a = 2$
$b = 2$

i.e, $D = 2$ — (5)
$D' = 2$ — (6))

So,

$$\underline{P.I.} = \frac{1}{2^2 - 4(2)(2) + 4(2)^2} \cdot e^{2x+y}$$

$$P.I. = \frac{1}{4 - \cancel{16} + \cancel{16}} e^{2x+2y}$$

$$\boxed{P.I. = \frac{1}{4} e^{2x+2y}} \quad - (7)$$

So, the complete solⁿ is:-

$$z = C.F. + P.I.$$

$$\boxed{z = f_1(y+2x) + x f_2(y+2x) + \frac{1}{4} e^{2x+2y}}$$

(from (4) & (7))

Q. Find P.I. of the PDE.

$$\frac{\partial^3 z}{\partial x^3} - \frac{7\partial^3 z}{\partial x\,\partial y^2} - 6\frac{\partial^3 z}{\partial y^3} = \sin(x+2y)$$

$$D^3 z - 7DD'^2 z - 6D'^3 z = \sin(x+2y)$$

$$(D^3 - 7DD'^2 - 6D'^3) z = \sin(x+2y) \quad \text{--- (1)}$$

Now, P.I.

$$P.I. = \frac{1}{f(D, D')} f(x, y)$$

$$P.I. = \frac{1}{D^3 - 7DD'^2 - 6D'^3} \sin(x+2y)$$

$$P.I. = \frac{1}{D(D^2) - 7DD'^2 - 6D'(D'^2)} \sin(x+2y)$$

$\because$,

$$\frac{1}{f(D^2, DD', D'^2)} \sin(ax+by) = \frac{1}{f(-a^2, -ab, -b^2)} \sin(ax+by)$$

where, $D^2 = -a^2$

$DD' = -ab$

$D'^2 = -b^2$

& $f(-a^2, -ab, -b^2) \neq 0$

so here,

$\because \sin(x+2y)$

$a = 1$
$b = 2$

so, $D^2 = -1^2$, $D^2 = -1$
$DD' = -(1)(2)$, $DD' = -2$
$D'^2 = -2^2$, $D'^2 = -4$ }

so,
$$P.I. = \frac{1}{D(-1) - 7(-4)D - 6D'(-4)} \sin(ax + by)$$

$$P.I. = \frac{1}{-D + 28D + 24D'} \sin(ax + by)$$

$$P.I. = \frac{1}{27D + 24D'} \sin(ax + by)$$

$$P.I. = \frac{27D - 24D'}{(27D)^2 - (24D')^2} \sin(ax + by)$$

$$P.I. = \frac{27D - 24D'}{729D^2 - 576D'^2} \sin(ax + by)$$

$$P.I. = \frac{27D - 24D'}{729(-1) - 576(-4)} \sin(x+2y)$$

$$P.I. = \frac{27D - 24D'}{-729 + 2304} \sin(x+2y)$$

$$P.I = \frac{27D - 24D'}{1575} \sin(x+2y)$$

$$P.I = \frac{27D(\sin(x+2y)) - 24D'(\sin(x+2y))}{1575}$$

$$P.I. = \frac{27\frac{\partial}{\partial x}(\sin(x+2y)) - 24\frac{\partial}{\partial y}(\sin(x+2y))}{1575}$$

$$P.I. = \frac{27\left(\cos(x+2y)(1+0)\right) - 24\left(\cos(x+2y)2\right)}{1575}$$

$$P.I. = \frac{27(\cos(x+2y)) - 24(2)\cos(x+2y))}{1575}$$

$$P.I. = \frac{27\cos(x+2y) - 48\cos(x+2y)}{1575}$$

$$\boxed{P.I. = \frac{-21}{1575}\cos(x+2y)}$$

Q. Solve!

$$(D^2 - 2DD' + D'^2)z = 12xy \quad \text{--- (1)}$$

or

$$r - 2s + t = 12xy.$$

or

$$\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 12xy.$$

So, Now,

for C.F!

The auxilary eq^n is! —

(put $D = m$, $D' = 1$, $z = 1$, R.H.S $= 0$)

$$\boxed{m^2 - 2m + 1 = 0} \quad \text{--- (2)}$$

$$m^2 - m - m + 1 = 0$$

$$m(m-1) - 1(m-1) = 0$$

$$\boxed{m = 1, 1} \quad \text{--- (3)}$$

$\because$ roots are equal

So,

$$C.F = f_1(y+mx) + x f_2(y+mx)$$

$$\boxed{C.F. = f_1(y+x) + x f_2(y+x)} \quad -(4)$$

Now,

for P.I.

$$P.I = \frac{1}{f(D,D')} f(x,y)$$

$$P.I. = \frac{1}{D^2 - 2DD' + D'^2} \; 12xy$$

$$\left(\because \cancel{\frac{1}{f(D,D')} x^r y^s} = \cancel{\frac{1}{f(D+a, D'+b)}} \frac{1}{f(D+a, D'+b)} \cancel{x^r y^s}\right.$$

$$\left(\because \frac{1}{f(D,D')} x^r y^s = F(D,D')^{-1} x^r y^s\right.$$

$$[\because (m-1)^2, (D-D')^2$$

$$P.I = \frac{1}{(D-D')^2} 12xy$$

$$P.I. = \frac{1}{\left(D\left(1-\frac{D'}{D}\right)\right)^2} 12xy$$

$$P.I. = \frac{1}{D^2\left(1-\frac{D'}{D}\right)^2} 12xy.$$

$$P.I. = \frac{1}{D^2}\left(1-\frac{D'}{D}\right)^{-2} 12xy.$$

~~P.I. =~~

$$\left(\because \ (1-x)^{-2} = 1+2x+3x^2+ \cdots\right)$$

$$P.I. = \frac{1}{D^2}\left(1+2\left(\frac{D'}{D}\right)+3\left(\frac{D'}{D}\right)^2+ \cdots\right)$$

$$P.I. = \frac{1}{D^2}\left(12xy+24\frac{D'}{D}(xy)+36\frac{D'^2}{D^2}(xy)+ \cdots\right)$$

$$\because \quad D = \frac{\partial}{\partial x}, \quad D' = \frac{\partial}{\partial y}$$

$$D(xy) = y, \quad D'(xy) = x$$

$$D^2(xy) = 0, \quad D'^2(xy) = 0$$

So,

$$P.I. = \frac{1}{D^2}\left(12xy + 24\int x\,dx + 36\int 0\,dx + \cdots\right)$$

$$P.I. = \frac{1}{D^2}\left[12xy + \cancel{24}^{12}\cdot\frac{x^2}{\cancel{2}} + 36\,c_1 + \cdots\right)$$

$$P.I. = \frac{1}{D^2}\left[12xy + 12x^2 + 36\,c_1 + \cdots\right)$$

$$P.I. = \frac{1}{D}\int\left(12xy + 12x^2 + 36\,c_1\right)dx$$

$$P.I. = \frac{1}{D}\left(6x^2y + \cancel{12}^{4}\frac{x^3}{\cancel{3}} + c_2 x\right] \qquad c_2 = 36c_1$$

$12xy$

$$P.I. = \frac{1}{D}\left(6x^2y + 4x^3 + c_2 x\right]$$

$$P.I. = \int\left(6x^2y + 4x^3 + c_2 x\right)dx$$

$$P.I. = 2\frac{x^3}{\cancel{3}}y + \cancel{4}\frac{x^4}{\cancel{4}} + \frac{c_2}{2}x^2 \qquad \left(c = \frac{c_2}{2}\right.$$

$$\boxed{P.I. = 2x^3y + x^4 + cx^2}$$

$$P.I. = x^4 + x^3(2y+6)$$

$$\boxed{P.I. = 2x^3y + x^4 + cx^2}$$

complete Solⁿ is :-

$z = C.F. + P.I.$

$$\boxed{z = f_1(y+x) + x f_2(y+x) + 2x^3 y + x^4 + cx^2}$$

Solve:

Q. $(D^2 - 4DD' + 4D'^2)z = e^{2x+y}$ —— (1)

Now,

for 'C.F':

The auxilary eq^n is: –

(put $D = m$, $D' = 1$, $z = 1$, R.H.S $= 0$)

So,

we get,

$$m^2 - 4m + 4 = 0 \quad \text{—— (2)}$$

$$m^2 - 2m - 2m + 4 = 0$$

$$m(m-2) - 2(m-2) = 0$$

$$(m-2)(m-2) = 0$$

$$\boxed{m = 2, 2} \quad \text{—— (3)}$$

So,

$$C.F = f_1(y+mx) + x f_2(y+mx)$$

$$\boxed{C.F. = f_1(y+2x) + x f_2(y+2x)} \quad \text{—— (4)}$$

Now,

for 'P.I'

$$P.I = \frac{1}{f(D, D')} f(x, y)$$

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$$P.I. = \frac{1}{D^2 - 4DD' + 4D'^2} e^{2x+y}$$

~~P.I.~~ so, $\left(\frac{1}{f(D, D')} e^{ax+by} = \frac{1}{f(a,b)} e^{ax+by} \right.$

where, $D = a$
$D' = b$
& $f(a,b) \neq 0$

So, here, e^{2x+y}

$a = 2$
$b = 1$

So, $D = 2$
$D' = 1$. $\left. \right)$

$$P.I. = \frac{1}{(D-2D')^2} e^{2x+y}$$

$$P.I. = \frac{1}{(D-2D')(D-2D')} e^{2x+y}$$

now, P.I. $= \frac{1}{(2-2(1))(2-(2(1))} e^{2x+y}$

$$P.I. = \frac{1}{(2-2)^2} e^{2x+y}$$

$$P.I. = \frac{1}{0} e^{2x+y}$$

so, here the function fails.

so,

$$\left[\because \frac{1}{(D-mx)} f(x,y) = \int f(x, c-mx)\, dx \right.$$

where, $y = c - mx$.]

so, $$P.I. = \frac{1}{(D-2D')(D-2D')} e^{2x+y}$$

$$P.I. = \frac{1}{(D-2D')} \left[\int e^{\cancel{2x}+c-\cancel{2x}} dx\right]$$

$$P.I. = \frac{1}{D-2D'} \left[\int e^{c} dx\right]$$

$$P.I. = \frac{1}{D-2D'} e^{c}\cdot x$$

$$P.I. = \frac{1}{D-2D'} e^{y+2x} \cdot x$$

$$P.I = \frac{1}{D-2D'} e^{2x+y} \cdot x$$

$$P.I = \int e^{\cancel{2x}+c-\cancel{2x}} \cdot x \, dx$$

$$P.I. = \int e^{c} \cdot x \, dx$$

$$P.I. = e^{c} \int x \, dx$$

$$P.I. = e^{c} \frac{x^2}{2}$$

$$P.I. = e^{y+2x} \frac{x^2}{2}$$

$$\boxed{P.I = \frac{x^2}{2} e^{2x+y}}$$

So,

now for complete sol^n

$$\boxed{Z = C.F + P.I}$$

$$\boxed{z = f_1(y+2x) + x f_2(y+2x) + \frac{x^2}{2} e^{2x+y}}$$

Q Solve!

$$\left(D^3 - 2D^2D' - DD'^3 + 2D'^3\right) z = e^{x+y} \quad \text{--- (1)}$$

now,

for C.F.

The auxilary eq^n is! -

put ($D = m$, $D' = 1$, $z = 1$, R.H.S $= 0$)

$$m^3 - 2m^2 - m + 2 = 0 \quad \text{--- (2)}$$

$$m^2(m-2) - 1(m-2) = 0$$

$$(m^2-1)(m-2) = 0$$

$$(m-1)(m+1)(m-2) = 0$$

$$m = 1, -1, 2 \quad \text{--- (3)}$$

$\because$ roots are distinct,

So, $C.F. = f_1(y+m_1x) + f_2(y+m_2x) + f_3(y+m_3x)$

$$\boxed{C.F. = f_1(y-x) + f_2(y+x) + f_3(y+2x)} \quad \text{--- (4)}$$

now,

for P.I.

$$P.I. = \frac{1}{f(D, D')} f(x, y)$$

$$P.I. = \frac{1}{D^3 - 2D^2D' - DD'^3 + 2D'^3} e^{x+y}$$

Now,

$$\left(\frac{1}{f(D, D')} e^{ax+by} = \frac{1}{f(a, b)} e^{ax+by} \right.$$

such that, $D = a$

$D' = b$

& $f(a, b) \neq 0$.

so,

here, $\because$ e^{x+y}

so, $a = 1$

$b = 1$

so, $D = 1$ — (5)

$D' = 1$ — (6)

$$P.I. = \frac{1}{\cancel{1}-\cancel{2}-\cancel{1}+\cancel{2}} e^{x+y}$$

$$P.I = \frac{1}{0} e^{x+y} \quad \text{—} (7)$$

$\because$; $D^2 = 0$
So, This Rule fails

So,
now

$$\frac{1}{D - mD'} f(x,y) = \int f(x, c-mx)\, dx$$

$$y = c - mx \quad \text{———} (8)$$

So,

$$P.I. = \frac{1}{(D-D')(D-(-1)D')(D-2D')} e^{x+y}$$

$$P.I = \frac{1}{(D-D')(D+D')(D-2D')} e^{x+y}$$

$$P.I = \frac{1}{(D-D')(1+1)(1-2)} e^{x+y}$$

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$$P.I. = \frac{1}{(D-D')\,2(-1)} e^{x+y}$$

$$P.I. = \frac{-1}{2\,(D-D')} e^{x+y} \quad \text{—— } (9)$$

$$P.I. = \frac{-1}{2} \int e^{x+c-1x} \, dx$$

$$P.I. = \frac{-1}{2} \int e^{\cancel{x}+c-\cancel{x}} \, dx$$

$$P.I. = \frac{-1}{2} \int e^{c} \, dx$$

$$P.I. = \frac{-1}{2} e^{y+(1)x} \cdot x$$

$$\boxed{P.I. = \frac{-1}{2} x e^{x+y}} \quad \text{—— } (10)$$

Now, The complete solⁿ is: -

$$z = C.F + P.I$$

So, from (9) & (10)

$$\boxed{z = f_1(y-x) + f_2(y+x) + f_3(y+2x) - \frac{1}{2}xe^{x+y}}$$

Q. Solve:

$$(D^2 + 3DD' + 2D'^2)z = 24xy \quad \text{—(1)}$$

Now,

for C.F.

The auxiliary eq^n is:—

(put $D = m$, $D' = 1$, $z = 1$, R.H.S $= 0$)

So,

we get,

$$m^2 + 3m + 2 = 0 \quad \text{—(2)}$$

$$m^2 + m + 2m + 2 = 0$$

$$m(m+1) + 2(m+1) = 0$$

$$(m+1)(m+2) = 0$$

$$m = -1, -2 \quad \text{—(3)}$$

$\because$ roots are real & unequal

So, C.F. $= f_1(y + m_1 x) + f_2(y + m_2 x)$

$$\boxed{C.F. = f_1(y-x) + f_2(y-2x)} \quad \text{--- (4)}$$

now,

for P.I.

$$P.I. = \frac{1}{f(D, D')} \cdot f(x, y)$$

$$P.I = \frac{1}{(D^2 + 3DD' + 2D'^2)} \cdot 24xy$$

$$P.I. = \frac{1}{(D^2 + 3DD' + 2D'^2)} 24xy$$

$$P.I. = \frac{1}{D^2\left(1 + \frac{3D'}{D} + \frac{2D'^2}{D^2}\right)} 24xy$$

$$P.I. = \frac{1}{D^2\left(1 + \left(\frac{3D'}{D} + \frac{2D'^2}{D^2}\right)\right)} 24xy$$

$$P.I. = \frac{1}{D^2}\left(1 + \underbrace{\left(\frac{3D'}{D} + \frac{2D'^2}{D^2}\right)}\right)^{-1} 24xy$$

Now $\left(\because (1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots\right)$

SO,

$$P.I. = \frac{1}{D^2}\left(1 - \left(\frac{3D'}{D} + \frac{2D'^2}{D^2}\right) + \left(\frac{3D'}{D} + \frac{2D'^2}{D^2}\right)^2 \cdots\right) 24xy$$

$$P.I. = \frac{1}{D^2}\left(1 - \left(\frac{3D'}{D} + \frac{2D'^2}{D^2}\right) + \left(\frac{9D'^2}{D^2} + \frac{4D'^4}{D^4} + \frac{12D'^3}{D^3}\right)\right) 24xy$$

$$P.I. = \frac{1}{D^2}\left(1 - \frac{3D'}{D} - \frac{2D'^2}{D^2} + \frac{9D'^2}{D^2} + \frac{4D'^4}{D^4} + \frac{12D'^3}{D^3} + \right) 24xy$$

$$P.I. = \frac{1}{D^2}\left[24xy - \frac{72D'}{D}(xy) - \frac{48D'^2}{D^2}(xy) + \frac{216D'^2}{D^2}(xy) + \frac{96D'^4}{D^4}(xy) + \frac{288D'^3}{D^3}(xy)\right]$$

$$P.I. = \frac{1}{D^2}\left(24xy - 72\frac{D'}{D}(xy) - 48\frac{D'^2}{D^2}(xy) + 216\frac{D'^2}{D^2}(xy) + 96\frac{D'^4}{D^4}(xy) + 288\frac{D'^3}{D^3}(xy)\right)$$

now, $\because$ $D = \frac{\partial}{\partial x}$

$D(xy) = y$

$\therefore$ $D^2(xy) = 0$

$D' = \frac{\partial}{\partial y}$

$\therefore$

$D'(xy) = x$ — (5)

$D'^2(xy) = 0$ — (6)

$D'^3(xy) = 0$ — (7)

$D'^4(xy) = 0$ — (8)

So, now, $\because$ $\frac{1}{D} = \int dx$

So, $\frac{D'(xy)}{D} = \int x\,dx$ (from (5))

$$\boxed{\frac{D'(xy)}{D} = \frac{x^2}{2}} \quad — (9)$$

now,

$$\frac{D'^2(xy)}{D^2} = \frac{1}{D}\int 0\cdot dx$$

$$\frac{D'^2(xy)}{D^2} = \frac{1}{D}C$$

$$\frac{D'^2(xy)}{D^2} = \int c_1\, dx$$

$$\boxed{\frac{D'^2(xy)}{D^2} = c_2 x} \quad -(10)$$

$$\frac{D'^3(xy)}{D^3} = \frac{1}{D^2}\int 0\, dx.$$

$$\frac{D'^3(xy)}{D^3} = \frac{1}{D^2} c_1$$

$$\frac{D'^3(xy)}{D^3} = \frac{1}{D}\int c_1\, dx.$$

$$\frac{D'^3(xy)}{D^3} = \frac{1}{D} c_2 x$$

$$\boxed{\frac{D'^3(xy)}{D^3} = c_2 \frac{x^2}{2}} \quad -(11)$$

So, $P.I. = \frac{1}{D^2}\left(24xy - \cancel{72}^{36}\left(\frac{x^2}{\cancel{2}}\right) - 48c_2x + 216c_2x\right)$

$$P.I. = \frac{1}{D^2}\left(24xy - 36x^2 + 168c_2x\right)$$

$$P.I. = \frac{1}{D}\int\left(24xy - 36x^2 + 168c_2x\right)dx$$

$$P.I. = \frac{1}{D}\left[\cancel{24}^{12} y \frac{x^2}{\cancel{2}} - \cancel{36}^{12}\frac{x^3}{\cancel{3}} + \cancel{168}^{84} C_1 \frac{x^2}{\cancel{2}}\right]$$

$$P.I. = \frac{1}{D}\left[12x^2y - 12x^3 + 84C_1x^2\right]$$

$$P.I. = \int 12x^2y - 12x^3 + 84C_1x^2$$

$$P.I. = \cancel{12}^{4} y \frac{x^3}{\cancel{3}} - \cancel{12}^{3}\frac{x^4}{\cancel{4}} + \cancel{84}^{28} C_1 \frac{x^3}{\cancel{3}},$$

$$\boxed{P.I. = 4x^3y - 3x^4 + 28C_1x^3} \quad \text{— (12)}$$

So,

The complete solⁿ is:—

$$\boxed{z = C.F + P.I.}$$

So,

from (4) & (12)

we get,

$$\boxed{z = f_1(y-x) + f_2(y-2x) + 4x^3y - 3x^4 + 28}$$

Wednesday

04/05/2022

V.V.V.V.V. imp — Q. Solve:-

$$r + s - 6t = y \cos x$$

So, $\frac{d^2 z}{dx^2} + \frac{\partial^2 z}{\partial x \partial y} - 6 \frac{\partial^2 z}{\partial y^2} = y \cos x.$

So,

$$D^2 z + DD' z - 6D'^2 z = y \cos x.$$

$$(D^2 + DD' - 6D'^2) z = y \cos x \quad \text{—— (1)}$$

Now,

for C.F.

(put $D = m$, $D' = 1$, $z = 1$, R.H.S $= 0$)

We get,

$$(m^2 + m(1) - 6(1)) 1 = 0$$

$$m^2 + m - 6 = 0 \quad \text{— (2)}$$

$$\cancel{m^2 + 2m} \; m^2 + 3m - 2m - 6 = 0$$

$$m(m+3) - 2(m+3) = 0$$

$$(m+3)(m-2) = 0$$

$$m = -3, 2 \quad \text{—— (3)}$$

$\because$ roots are distinct.

So,

$$C.F = f_1(y + m_1 x) + f_2(y + m_2 x)$$

$$\boxed{C.F = f_1(y+2x) + f_2(y-3x)} \quad \text{--- (4)}$$

now,
for P.I.

$$P.I. = \frac{1}{f(D,D')} f(x,y)$$

$$P.I. = \frac{1}{D^2 - DD' - 6D'^2}\, y\cos x$$

~~P.I. ≠~~

(∵ $y\cos x$ → don't know,
So applying General formula.

$$\frac{1}{(D-mD')} f(x,y) = \int f(x, c-mx)\, dx$$

where, $\boxed{y = c - mx}$

So,

$$P.I. = \frac{1}{(D-2D')(D+3D')}\, y\cos x$$

(where, $m = 2, -3$)

$$P.I. = \frac{1}{(D-2D')}\left[\int (c+3x)\cos x\, dx\right]$$

$m = -3$

L
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$$P.I. = \frac{1}{D-2D'}\left[\int \underset{2}{(c+3x)} \underset{2}{\cos x}\, dx\right]$$

$$\cancel{P.I. = \frac{1}{D-2D'}\left[\cos x \int (c+3x)\,dx - \int\left(-\sin x \cdot \int (c+3x)\right) dx\right]}$$

$$P.I. = \frac{1}{D-2D'}\left[(c+3x)\sin x - \int 3 \sin x \, dx\right]$$

$$P.I. = \frac{1}{D-2D'}\left[(c+3x)\sin x + 3\cos x\right]$$

$$P.I. = \frac{1}{D-2D'}\left[(y-\cancel{3x}+\cancel{3x})\sin x + 3\cos x\right]$$

$$P.I. = \frac{1}{D-2D'}\left[y \sin x + 3\cos x\right] \quad [\because y+mx=c]$$

now,

$$P.I. = \int\left((c-2x)\sin x + 3\cos x\right)dx.$$

$$P.I. = \int \underset{1}{(c-2x)} \underset{2}{\sin x}\, dx + 3\int \cos x \, dx$$

$$P.I. = (c-2x)(-\cos x) - \int \cancel{\cos x}(-2)(-\cos x)\, dx + 3\sin x$$

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~~P.I. = (c-2x)(sin x)~~

P.I. $= (c-2x)(-\cos x) - 2\int \cos x \, dx \,. + 3\sin x$

P.I. $= (c-2x)(-\cos x) - 2\sin x + 3\sin x$

P.I $= (y+\cancel{2x}-\cancel{2x})(-\cos x) + \sin x$ $\quad (\because c = y+mx)$

$$\boxed{P.I = -y\cos x + \sin x} \quad — (5)$$

So,

The complete sol^n is :-

$z = C.F + P.I.$

from (4) & (5).

$$\boxed{z = f_1(y+2x) + f_2(y-3x) - y\cos x + \sin x}$$

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Imp

Charpit's Method

This is a general method for finding the complete soln of non-linear partial differential eqn of 1st order

~~Let~~ form: $\boxed{f(x, y, z, p, q) = 0}$

Working Rule :-

(1) Transfer all the terms of the given following P.D.E

$$f(x, y, z, p, q) = 0 \quad \text{—— (1)}$$

to the left hand side & denote the entire expression by f.

(2) Write down the Charpit's auxilary eqn :-

The auxilary eqn is :-

$$\frac{df}{0} = \boxed{\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}}}$$

$\hookrightarrow$ (2)

(3) ~~Using the values~~ Now, we differentiate partially eq^n (1) w.r.t 'x', 'y', 'z', 'p', & 'q'.

(4) On putting, above values in eq^n (2), we get auxilary eq^n.

(5) Now simplify auxilary eq^n, we select 2 proper fraction such that:

The resultant integral may come to be simplest relation involving atleast one of 'p' & 'q'

(6) With the previous relation of step 5 is solved with the given problem to find the value of 'p' & 'q' by putting the following formula :-

Complete Integral

$$\boxed{dz = p\,dx + q\,dy} \quad -(3)$$

Thursday

5/5/2022

Q. Solve:

$z = px + qy + p^2 + q^2$ by Charpits Method.

So, $px + qy + p^2 + q^2 - z = 0$ —— ①

$f = 0$.

Now,

differentiating eqⁿ ① partially w.r.t 'x'

$$\frac{\partial f}{\partial x} = p + 0 + 0 + 0 - 0$$

$$\boxed{\frac{\partial f}{\partial x} = p} \quad \text{—— ②}$$

now,

differentiating eqⁿ ① partially w.r.t 'y'

$$\frac{\partial f}{\partial y} = 0 + q + 0 + 0 - 0$$

$$\boxed{\frac{\partial f}{\partial y} = q} \quad \text{—— ③}$$

now,

differentiating eq^n ① w.r.t 'z' partially

$$\frac{\partial f}{\partial z} = 0+0+0+0-1$$

$$\boxed{\frac{\partial f}{\partial z} = -1} \quad — ④$$

now,

differentiating eq^n ① w.r.t 'p' partially.

$$\frac{\partial f}{\partial p} = x+0+2p+0-0$$

$$\boxed{\frac{\partial f}{\partial p} = x+2p} \quad — ⑤$$

now,

differentiating eq^n ① w.r.t 'q' partially.

$$\frac{\partial f}{\partial q} = 0+y+0\ 2q-0$$

$$\boxed{\frac{\partial f}{\partial q} = y+2q} \quad — ⑥$$

now,

Charpit's Auxilary eqⁿ is:-

$$\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}}$$

now,

from (2), (3), (4), (5) & (6)

we get,

$$\frac{dp}{p+p(-1)} = \frac{dq}{q+q(-1)} = \frac{dz}{-p(x+2p)-q(y+2q)} = \frac{dx}{-(x+2p)} = \frac{dy}{-(y+2q)}$$

$$\frac{dp}{p-p} = \frac{dq}{q-q} = \frac{dz}{-px-2p^2-qy-2q^2} = \frac{dx}{-x-2p} = \frac{dy}{-y-2q}$$

$$\frac{dp}{0} = \frac{dq}{0} = \frac{dz}{\cancel{p+q}-px-qy-2(p^2+q^2)} = \frac{dx}{-x-2p} = \frac{dy}{-y-2q} \quad -(7)$$

now

Taking 1th & 5th from (7)

we get,

$$\frac{dp}{0} = \frac{dy}{-y-2q}$$

So, $dp = 0$

now integrating both sides

we get,

$$\int dp = \int 0$$

$$\boxed{p = a} \quad \text{— (8)}$$

now, Taking 2nd & 4th from (7)

we get,

$$\frac{dq}{0} = \frac{dx}{-x-2p}$$

$$dq = 0$$

now, integrating both sides

we get,

$$\int dq = \int 0$$

$$\boxed{q = b} \quad \text{— (9)}$$

now, the complete integral is:-

$$dz = p\,dx + q\,dy$$

$$dz = a\,dx + b\,dy \quad \text{(from ⑧ \& ⑨)}$$

now, integrating both sides

we get,

$$\int dz = \int a\,dx + \int b\,dy$$

$$\boxed{z = ax + by + c} \quad \text{— ⑩}$$

Q. Solve!

$$(p^2+q^2)y = qz$$

now, $(p^2+q^2)y - qz = 0$

$$p^2y + q^2y - qz = 0 \quad \text{— ①}$$

$$f = 0$$

now,

differentiating eqⁿ ① partially w.r.t 'x'

$$\boxed{\frac{\partial f}{\partial x} = 0} \quad \text{— ②}$$

so,

now,

differentiating eq^n 1 partially w.r.t 'y'

$$\frac{\partial f}{\partial y} = p^2+q^2 - 0.$$

$$\boxed{\frac{\partial f}{\partial y} = p^2+q^2} \quad - (3).$$

now,

differentiating eq^n 1 partially w.r.t 'z'

$$\frac{\partial f}{\partial z} = 0 - q$$

$$\boxed{\frac{\partial f}{\partial z} = -q} \quad - (4)$$

now,

differentiating eq^n 1 partially w.r.t 'p'

$$\frac{\partial f}{\partial p} = y\,2p + 0 - 0$$

$$\boxed{\frac{\partial f}{\partial p} = 2py} \quad - (5)$$

now,

differentiating eq^n ① partially w.r.t 'q'

we get,

$$\frac{\partial f}{\partial q} = 0 + y\,2q - z$$

$$\boxed{\frac{\partial f}{\partial q} = 2qy - z} \quad — ⑥$$

now,

∴ Charpit's auxilary eq^n is :-

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}}$$

so,

now, from ②, ③, ④, ⑤ & ⑥.

we get,

$$\frac{dp}{0 + p(-q)} = \frac{dq}{p^2+q^2+q(-q)} = \frac{dz}{-p(2py) - q(2qy - z)} = \frac{dx}{-2py} = \frac{dy}{-2qy+z}$$

$$\frac{dp}{-pq} = \frac{dq}{p^2+\cancel{q^2}-\cancel{q^2}} = \frac{dz}{-2p^2y - 2q^2y + qz} = \frac{dx}{-2py} = \frac{dy}{-2qy+z}$$

$$\frac{dp}{-pq} = \frac{dq}{p^2} = \frac{dz}{-2y(p^2+q^2)+qz} = \frac{dx}{-2py} = \frac{dy}{-2qy+z} \quad \text{— (7)}$$

Now,

Taking 1^{st} & 2^{nd} members from (7)

we get,

$$\frac{dp}{-\cancel{p}q} = \frac{dq}{p^{\cancel{2}}}$$

$$\frac{dp}{-q} = \frac{dq}{p}$$

$$p\,dp = -q\,dq$$

now, integrating both sides,

we get,

$$\int p\,dp = -\int q\,dq$$

$$\frac{p^2}{2} + \frac{q^2}{2} = c^2$$

$$p^2+q^2 = 2c^2$$

$$p^2+q^2 = a^2 \quad (\cancel{a^2 = 2c^2}) \quad \text{— (8)}$$

$$(\because a^2 = 2c^2)$$

now,

from (1) & (8).

we get,

$$a^2y - qz = 0$$

$$a^2y = qz$$

$$\boxed{\frac{a^2y}{z} = q} \quad \text{—— (9)}$$

Squaring both sides,
we get,

$$\frac{a^4y^2}{z^2} = q^2 \quad \text{—— (10)}$$

now,

from (8) & (10).

$$p^2 + \frac{a^4y^2}{z^2} = a^2$$

$$p^2 = a^2 - \frac{a^4y^2}{z^2}$$

$$p^2 = \frac{a^2z^2 - a^4y^2}{z^2}$$

$$p^2 = \frac{a^2(z^2 - a^2y^2)}{z^2}$$

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$$p = \sqrt{\frac{a^2}{z^2}(z^2 - a^2y^2)}$$

$$\boxed{p = \frac{a}{z}\sqrt{z^2 - a^2y^2}} \quad — (11)$$

now, The complete integral is :-

$$dz = p\,dx + q\,dy$$

so, from (9) & (11)

$$dz = \frac{a}{z}\sqrt{z^2 - a^2y^2}\,dx + \frac{a^2y}{z}\,dy$$

$$dz = \frac{a(\sqrt{z^2 - a^2y^2})\,dx + a^2y\,dy}{z}$$

$$z\,dz = a(\sqrt{z^2 - a^2y^2})\,dx + a^2y\,dy$$

$$z\,dz - a^2y\,dy = a(\sqrt{z^2 - a^2y^2})\,dx$$

$$\frac{z\,dz - a^2y\,dy}{\sqrt{z^2 - a^2y^2}} = a\,dx$$

$$\frac{z}{\sqrt{z^2-a^2y^2}}\,dz - \frac{a^2y}{\sqrt{z^2-a^2y^2}}\,dy = a\,dx.$$

now,
integrating both sides,
we get,

$$\int \frac{z}{\sqrt{z^2-a^2y^2}}\,dz - \int \frac{a^2y}{\sqrt{z^2-a^2y^2}}\,dy = \int a\,dx.$$

so,

$$I_1 - I_2 = ax + b \quad \text{--- (12)}$$

so,
now, for 'I_1'

$$I_1 = \int \frac{z}{\sqrt{z^2-a^2y^2}}\,dz.$$

Let,

$$z^2 - a^2y^2 = m^2 \quad \text{--- (13)}$$

$$2z - 0 = 2m\frac{dm}{dz}$$

$$\cancel{2}z = \cancel{2}m\frac{dm}{dz}$$

$$dz = \frac{m}{z}\,dm \quad \text{--- (14)}$$

so,

$$I_1 = \int \frac{\cancel{z}}{\sqrt{m^2}} \frac{m}{\cancel{z}} \, dm$$

$$I_1 = \int \frac{\cancel{m}}{\cancel{m}} \, dm$$

$$I_1 = \int dm$$

$$I_1 = m$$

$$I_1 = \sqrt{z^2 - a^2y^2} \quad (\text{from } (13))$$

$$\boxed{I_1 = \sqrt{z^2 - a^2y^2}} \quad \text{--- } (15)$$

now,

for I_2

$$I_2 = \int \frac{a^2 y}{\sqrt{z^2 - a^2y^2}} \, dy$$

now,

let, $z^2 - a^2y^2 = n^2$ --- (16)

so, $0 - a^2(2y) = 2n \frac{dn}{dy}$

$$0 - 2a^2y = 2n\frac{dn}{dy}$$

$$-\cancel{2}a^2y = \cancel{2}n\frac{dn}{dy}$$

$$dy = \frac{-n}{a^2y}\,dn \quad — (17)$$

So,

$$I_2 = \int \frac{\cancel{a^2y}}{\sqrt{n^2}}\left(\frac{-n}{\cancel{a^2y}}\right)dn$$

$$I_2 = \int \frac{-\cancel{n}}{\cancel{n}}\,dn$$

$$I_2 = -\int dn$$

$$I_2 = -n$$

$$I_2 = -\sqrt{z^2 - a^2y^2} \quad (\text{from } (16))$$

$$\boxed{I_2 = -\sqrt{z^2 - a^2y^2}} \quad — (18)$$

so,

from (12)

$$I_1 - I_2 = ax + b$$

so, from (15) & (18)

we get,

$$\sqrt{z^2 - a^2y^2} - \left(-\sqrt{z^2 - a^2y^2}\right) = ax + b$$

$$\cancel{z^2} \sqrt{z^2 - a^2y^2} + \sqrt{z^2 - a^2y^2} = ax + b$$

$$2\sqrt{z^2 - a^2y^2} = ax + b$$

$$\sqrt{z^2 - a^2y^2} = \frac{ax + b}{2}$$

now, squaring both sides,

we get,

$$\left(\sqrt{z^2 - a^2y^2}\right)^2 = \frac{(ax+b)^2}{2^2}$$

$$z^2 - a^2y^2 = \frac{a^2x^2 + b^2 + 2abx}{4}$$

$$z^2 - a^2y^2 = \frac{a^2}{4}x^2 + \frac{b^2}{4} + \frac{2ab}{4}x$$

$$z^2 - a^2y^2 = c^2x^2 + d^2 + 2cdx.$$

$$\left(c = \frac{a}{2},\ d = \frac{b}{2}\right)$$

so,

$$z^2 - a^2y^2 = c^2x^2 + d^2 + 2cdx.$$

$$\boxed{z^2 - a^2y^2 = (cx + d)^2} \quad \text{—— (19)}$$

Q. Solve:

$$pxy + pq + qy = yz$$

so, $$pxy + pq + qy - yz = 0 \quad \text{—— (1)}$$

$$f = 0.$$

now,

differentiating eqⁿ (1) partially w.r.t 'x'

we get

$$\frac{\partial f}{\partial x} = py + 0 - 0 - 0$$

$$\boxed{\frac{\partial f}{\partial x} = py} \quad \text{—— (2)}$$

now,

differentiating eqⁿ 1 w.r.t 'y' partially.

we get,

$$\frac{\partial f}{\partial y} = px + 0 + q - z$$

$$\boxed{\frac{\partial f}{\partial y} = px + q - z} \quad \text{—— (3)}$$

now,

differentiating eqⁿ 1 w.r.t 'z' partially

we get,

$$\frac{\partial f}{\partial z} = 0 + 0 + 0 - y$$

$$\boxed{\frac{\partial f}{\partial z} = -y} \quad \text{—— (4)}$$

now,

differentiating eqⁿ 1 w.r.t 'p' partially.

we get,

$$\frac{\partial f}{\partial p} = xy + q + 0 - 0$$

$$\boxed{\frac{\partial f}{\partial p} = xy + q} \quad \text{— (5)}$$

now, differentiating eq^n (1) w.r.t (q)

we get,

$$\frac{\partial f}{\partial q} = 0 + p + y - 0$$

$$\boxed{\frac{\partial f}{\partial q} = p + y} \quad \text{— (6)}$$

now,

Charpit's auxilary eq^n is :-

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}}$$

so, from (2), (3), (4), (5) & (6)

we get,

$$\frac{dp}{py + p(-y)} = \frac{dq}{(px + q - z) + q(-y)} = \frac{dz}{-p(xy+q) - q(p+y)} = \frac{dx}{-xy - q} = \frac{dy}{-p - y}$$

$$\frac{dp}{py - py} = \frac{dq}{px + q - z - qy} = \frac{dz}{-pxy - pq - pq - py} = \frac{dx}{-xy - q} = \frac{dy}{-p - q}$$

$$\frac{dp}{0} = \frac{dq}{px+q-z-qy} = \frac{dz}{-pxy-2pq-py} = \frac{dx}{-xy-q} = \frac{dy}{-p-q}$$

Taking 1st & last member.
we get,

$$\frac{dp}{0} = \frac{dy}{-p-q}$$

$$dp = 0.$$

integrating both sides,
we get,

$$\int dp = \int 0\, dp.$$

$$\boxed{p = a} \quad \text{------} \quad (7)$$

So, from (1) & (7),
we get,

$$axy + aq + qy - yz = 0$$

$$axy + aq + qy = yz$$

$$\cancel{axy}\; q(a+y) = yz - axy$$

$$q = \frac{yz - axy}{a+y}$$

$$\boxed{q = \frac{yz - axy}{a+y}} \quad \text{--- (8)}$$

So,

now,

for the complete integral :-

$$dz = p\,dx + q\,dy$$

we get,

$$dz = a\,dx + \frac{yz - axy}{a+y}\,dy$$

$$dz - \cancel{\frac{yz - axy}{a+y}\,dy} \cancel{=} a\,dx$$

~~integrating both sides,~~

~~we get,~~

~~$$\int dz = \int a\,dx + \int \frac{yz - axy}{a+y}\,dy$$~~

~~$$z = \cdot ax +$$~~

$$dz - a\,dx = \frac{yz - axy}{a+y}\,dy$$

$$dz - a\,dx = \frac{y(z - ax)}{a+y}\,dy$$

$$\boxed{q = \frac{yz - axy}{a+y}} \quad \text{---} \quad (8)$$

So,

now,

for the complete integral :-

$$dz = p\,dx + q\,dy$$

we get,

$$dz = a\,dx + \frac{yz - axy}{a+y}\,dy$$

$$\cancel{dz - \frac{yz - axy}{a+y}\,dy = a\,dx}$$

$\cancel{\text{integrating both sides, we get,}}$

$$\cancel{\int dz = \int a\,dx + \int \frac{yz - axy}{a+y}\,dy}$$

$$\cancel{z = ax +}$$

$$dz - a\,dx = \frac{yz - axy}{a+y}\,dy$$

$$dz - a\,dx = \frac{y(z - ax)}{a+y}\,dy$$

$$\frac{dz - a\,dx}{z - ax} = \frac{y}{a+y}\,dy.$$

$$\frac{dz - a\,dx}{z - ax} = \frac{y}{a+y}\,dy.$$

now, integrating both sides,

we get,

$$\int \frac{dz - a\,dx}{z - ax} = \int \frac{y}{a+y}\,dy.$$

$$I_1 = I_2 + \log b \quad \text{—— (9)}$$

so,

now, for I_1,

$$I_1 = \int \frac{dz - a\,dx}{z - ax}$$

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so,

let $\boxed{z - ax = m}$ — (10)

so, differentiating w.r.t 'x'

$$\frac{dz}{dx} - a = \frac{dm}{dx}$$

$$\frac{dz - a\,dx}{\cancel{dx}} = \frac{dm}{\cancel{dx}}$$

$$\boxed{dz - a\,dx = dm} \quad \text{— (11)}$$

so,

$$I_1 = \int \frac{dz - a\,dx}{z - ax}$$

$$I_1 = \int \frac{dm}{m}$$

$$I_1 = \log m$$

$$\boxed{I_1 = \log(z - ax)} \quad \text{— (12)}$$

now,

for 'I_2'

$$I_2 = \int \frac{y}{y+a}\, dy$$

$$I_2 = \int \frac{y+a-a}{y+a}\, dy$$

$$I_2 = \int \frac{\cancel{y+a}}{\cancel{y+a}}\, dy - \int \frac{a}{y+a}\, dy$$

$$I_2 = \int dy - a\int \frac{1}{y+a}\, dy$$

$$\boxed{I_2 = y - a\log(y+a)} \quad \text{— (13)}$$

So,

from (9), (12) & (13)

we get,

$$I_1 = I_2$$

$$\log(z-ax) = y - a\log(y+a) + \log b$$

$$\log(z-ax) = y - \log(y+a)^a + \log b$$

$$\log\left(\frac{(z-ax)}{b}(y+a)^a\right) = y$$

$$e^y = \frac{(z-ax)(y+a)^a}{b}$$

$$\frac{be^y}{(y+a)^a} = z-ax$$

$$\boxed{be^y(y+a)^{-a} = z-ax} \quad \text{— (14)}$$

Q. Solve:

~~pxy~~ $2(z + xp + yq) = yp^2$

$2z + 2xp + 2yq - yp^2 = 0$ — ①

$f = 0$

now,

diff eqⁿ ① w.r.t 'x' partially

we get,

$$\frac{\partial f}{\partial x} = 0 + 2p + 0 - 0$$

$$\boxed{\frac{\partial f}{\partial x} = 2p} \quad \text{— ②}$$

now,

diff eqⁿ ① w.r.t 'y' partially

$$\frac{\partial f}{\partial y} = 0 + 0 + 2q - p^2$$

$$\boxed{\frac{\partial f}{\partial y} = 2q - p^2} \quad \text{— ③}$$

now,

diff eqⁿ ① w.r.t 'z' partially

$$\frac{\partial f}{\partial z} = 2 + 0 + 0 - 0$$

$$\boxed{\frac{\partial f}{\partial z} = 2} \quad \text{— (4)}$$

now, diff eq^n (1) partially w.r.t 'p'

$$\frac{\partial f}{\partial p} = 0 + 2x + 0 - y2p$$

$$\boxed{\frac{\partial f}{\partial p} = 2x - 2py} \quad \text{— (5)}$$

now,

differentiating eq^n (1) partially w.r.t 'q'

$$\frac{\partial f}{\partial q} = 0 + 0 + 2y - 0$$

$$\boxed{\frac{\partial f}{\partial q} = 2y} \quad \text{— (6)}$$

now,

The charpit's auxilary eq^n is :-

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}}$$

so, from (2), (3), (4), (5) & (6)

we get,

$$\frac{dp}{2p+p(2)} = \frac{dq}{(2q-p^2)+q(2)} = \frac{dz}{-p(2x-2py)-q(2y)} = \frac{dx}{2py-2x} = \frac{dy}{-2y}$$

$$\frac{dp}{4p} = \frac{dq}{4q-p^2} = \frac{dz}{-2px+2py-2qy} = \frac{dx}{2py-2x} = \frac{dy}{-2y}$$

now,
Taking 1^{st} & last member

we get,

$$\frac{dp}{4p} = \frac{dy}{-2y}$$

integrating both sides,
we get,

$$\frac{1}{4}\int \frac{dp}{p} = -\frac{1}{2}\int \frac{dy}{y}$$

$$\frac{1}{4}\log p = -\frac{1}{2}\log y + \log a$$

$$\log p^{1/4} + \log y^{1/2} = \log a$$

$$\log (p^{1/4} y^{1/2}) = \log a$$

$$\frac{p^{1/4}}{y^{1/2}} = a$$

$$\int \frac{dp}{p} = -\frac{\cancel{x}2}{\cancel{x}} \int \frac{dy}{y}$$

$$\log p = -2\log y + \log a$$

$$\log p + \log y^2 = \log a$$

$$\log(py^2) = \log a$$

$$py^2 = a$$

$$\boxed{p = \frac{a}{y^2}} \text{ --- } (7)$$

so, from (1) & (7)
we get,

$$2z + 2x\left(\frac{a}{y^2}\right) + 2yq - \cancel{y}\left(\frac{a^2}{y^{\cancel{2}3}}\right) = 0$$

$$2z + \frac{2xa}{y^2} + 2yq - \frac{a^2}{y^3} = 0$$

$$2yq = \frac{a^2}{y^3} - \frac{2xa}{y^2} - 2z.$$

$$2yq = \frac{a^2 - 2xay - 2zy^3}{y^3}$$

$$\boxed{q = \frac{a^2 - 2xay - 2zy^3}{2y^4}} \quad \text{— (8)}$$

now,

the complete solⁿ is :-

$$dz = p\,dx + q\,dy.$$

$$dz = \frac{a}{y^2}dx + \left(\frac{a^2 - 2xay - 2zy^3}{2y^4}\right)dy \quad (\text{from (7) \& (8)})$$

$$dz = \frac{a}{y^2}dx + \frac{a^2}{2y^4}dy - \frac{xa}{y^3}dy - \frac{z}{y}dy$$

$$dz = \frac{1}{y}\left(\frac{a}{y}dx + \frac{a^2}{2y^3}dy - \frac{xa}{y^2}dy - z\,dy\right).$$

$$y\,dz = \frac{a}{y}dx - \frac{xa}{y^2}dy - z\,dy + \frac{a^2}{2}y^{-3}dy$$

$$y\,dz + z\,dy = \frac{a}{y}dx - \frac{xa}{y^2}dy + \frac{a}{2}y^{-3}dy$$

$$d(yz) = \frac{ay\,dx - xa\,dy}{y^2} + \frac{a}{2}y^{-3}dy$$

$$d(yz) = a\frac{(y\,dx - x\,dy)}{y^2} + \frac{a}{2}y^{-3}dy$$

$$d(yz) = a\left(d\left(\frac{x}{y}\right)\right) + \frac{a}{2}y^{-3}dy$$

now,
integrating both sides,
we get,

$$\int d(yz) = a\int d\left(\frac{x}{y}\right) + \frac{a}{2}\int y^{-3}dy$$

$$yz = \frac{ax}{y} + \frac{a}{2}\frac{y^{-2}}{-2} + b$$

$$yz = \frac{ax}{y} - \frac{a}{4y^2} + b$$

$$\boxed{z = \frac{ax}{y^2} - \frac{a}{4y^3} + \frac{b}{y}}$$

Non-Linear Partial Differential Eqn of 1st Order

(I) Standard Form – I

$$\boxed{f(p,q) = 0}$$

Working Rule:

(1) Given that

$$f(p,q) = 0 \quad \text{--- } (1)$$

(2) The Complete soln of (1) is :-

$$\boxed{z = ax + by + c} \quad \text{--- } (2)$$

where,

$$f(a,b) = c$$

(relation of a & b)

(3) Now,

differentiate eqn (2) w.r.t 'x' partially

$$\frac{\partial z}{\partial x} = a + 0 + 0.$$

$$\frac{\partial z}{\partial a} = a.$$

So, $\boxed{p = a}$ — (3)

(4) Now,

differentiating eq^n (2) w.r.t 'y' partially we get,

$$\frac{\partial z}{\partial y} = 0 + b + c$$

$$\boxed{q = b.} \quad — (4)$$

(5) Now, a & b are connected with the relation

$$\boxed{f(a, b) = 0}$$

[suppose $p^2 + q^2 = 0$
$f(p, q)$
$\rightarrow f(a, b) = a^2 + b^2 = 0$]

(6) On solving $f(a, b) = 0$, we find the value of 'b' in terms of 'a'

So, $b = \phi(a)$

(7) Then the required sol^n is :-

$$\boxed{z = ax + \phi(a)y + c}$$

classmate

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Q. Solve:

$$p^2 - q^2 = 25 \quad \text{—} \quad (1)$$

So, $f(p, q) = 0$ — (2)

($\because$, eq^n (1) is of the standard form I)

Now,

$\because$, $z = ax + by + c$ — (3)

now, differentiating eq^n (3) w.r.t 'x' partially

$$\frac{\partial z}{\partial x} = a$$

so, $\boxed{p = a}$ — (4)

now,

differentiating eq^n (3) w.r.t 'y' partially.

$$\frac{\partial z}{\partial y} = b$$

$\boxed{q = b}$ — (5)

So,

now, from (2), (4) & (5)

we get

$$f(a, b) = 0$$

So,

$$\cancel{a^2+b}\quad a^2-b^2=25$$

So,

$$b^2=a^2-25$$

$$\boxed{b=\sqrt{a^2-25}} \quad —— (6)$$

So,

the complete solⁿ is :-

$$z=ax+\phi(a)y+C$$

$$\boxed{z=ax+\sqrt{a^2-25}\;y+C}$$

v.v.v.imp

Q. $x^2p^2+y^2q^2=z^2$

$$x^2\left(\frac{\partial z}{\partial x}\right)^2+y^2\left(\frac{\partial z}{\partial y}\right)^2=z^2$$

$$\frac{x^2}{z^2}\left(\frac{\partial z}{\partial x}\right)^2+\frac{y^2}{z^2}\left(\frac{\partial z}{\partial y}\right)^2=1$$

$$\left(\frac{x}{\partial x}\cdot\frac{\partial z}{z}\right)^2+\left(\frac{y}{\partial y}\,\frac{\partial z}{z}\right)^2=1.$$

$$\left(\frac{\frac{\partial z}{z}}{\frac{\partial x}{x}}\right)^2+\left(\frac{\frac{\partial z}{z}}{\frac{\partial y}{y}}\right)^2=1 \quad —— (1)$$

now,

Let $X = \log x$

So, differentiating it,

$$\frac{dX}{dx} = \frac{1}{x}$$

$$\boxed{dX = \frac{dx}{x}} \quad \text{——— (2)}$$

now,

Let

$$Y = \log y$$

$$\frac{dY}{dy} = \frac{1}{y}$$

$$\boxed{dY = \frac{dy}{y}} \quad \text{— (3)}$$

Let $Z = \log z$

$$\frac{dZ}{dz} = \frac{1}{z}$$

$$\boxed{dZ = \frac{dz}{z}} \quad \text{— (4)}$$

Now,

from (1), (2), (3) & (4).

We get,

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = 1$$

$$P^2 + Q^2 = 1 \quad \text{--- (5)}$$

So, $f(P, Q) = 0$. --- (6)

now,

$\because$ $z = ax + by + c$ --- (7)

so, differentiating eqⁿ (7) partially w.r.t 'x'

$$\frac{\partial z}{\partial x} = a$$

$$\boxed{P = a} \quad \text{--- (8)}$$

now,

differentiating eqⁿ (7) partially w.r.t 'y'

$$\frac{\partial z}{\partial y} = b$$

$$\boxed{Q = b} \quad \text{--- (9)}$$

So,

from (5), (8) & (9).

we get

$$a^2+b^2=1$$

$$b=\sqrt{1-a^2}$$

$$\boxed{b=\sqrt{1-a^2}} \quad -(10)$$

So, the complete solⁿ is :-

$$z=ax+\phi(a)y+c.$$

$$z=ax+\sqrt{1-a^2}\,y+c. \quad -(11)$$

$$z=a\log x+\sqrt{1-a^2}\,(\log y)+c \quad (\because X=\log x \;\&\; Y=\log y)$$

$$\boxed{z=a\log x+\sqrt{1-a^2}\,\log y+c}$$

11/05/2022

II ~~In~~ Standard form — II

~~form~~ Working Rule :-

(1) In this form, the given eqn involves, p, q & z only i.e.

$$f(p, q, z) = 0 \quad \text{—} (1)$$

(2) $$p = \frac{dz}{dx} \quad \text{—} (2)$$

$$q = \frac{a\,dz}{dx} \quad \text{—} (3)$$

where, $z = f(X)$ — (4)

$$X = x + ay \quad \text{—} (5)$$

(3) Solve the resulting differential eqn b/w 'X' & 'z'.
The result so obtained is the complete soln of the given problem.

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Q. Solve:

$$z = p^2 + q^2 \quad \text{--- (1)}$$

The given eq^n involves p, q, z is of the standard form II

i.e. $f(p, q, z) = 0$

On putting $p = \frac{dz}{dX}$ --- (2)

&

$$q = \frac{a\,dz}{dX} \quad \text{--- (3)}$$

where, $X = \cancel{ax}\ x + ay$ --- (4)

$$z = \left(\frac{dz}{dX}\right)^2 + \left(\frac{a\,dz}{dX}\right)^2$$

$$z = \left(\frac{dz}{dX}\right)^2 + a^2\left(\frac{dz}{dX}\right)^2$$

$$z = \left(\frac{dz}{dx}\right)^2 (1 + a^2)$$

$$\frac{z}{1+a^2} = \left(\frac{dz}{dx}\right)^2$$

$$\sqrt{\frac{z}{1+a^2}} = \frac{dz}{dx}$$

$$\frac{1}{\sqrt{1+a^2}}\,dx = \frac{1}{\sqrt{z}}\,dz$$

integrating both sides,

we get,

$$\frac{1}{\sqrt{1+a^2}}\int dx = \int z^{-1/2}\,dz.$$

$$\frac{1}{\sqrt{1+a^2}}\,x + c = \frac{z^{1/2}}{1/2}$$

$$\frac{x}{\sqrt{1+a^2}} + c = 2\sqrt{z}$$

from ④.

$$\frac{x+ay}{\sqrt{1+a^2}} + c = 2\sqrt{z}.$$

$$\frac{x+ay+c\sqrt{1+a^2}}{\sqrt{1+a^2}} = 2\sqrt{z}.$$

$$\frac{x+ay+b}{2\sqrt{1+a^2}} = \sqrt{z} \quad [\text{where, } b = c\sqrt{1+a^2}]$$

so,

$$\boxed{\frac{x+ay+b}{2\sqrt{1+a^2}} = \sqrt{z}}$$

Q. Solve:

v.v.imp

$$z^2(p^2z^2+q^2)=1 \quad \text{--- (1)}$$

$\because$ Eq^n (1) involves p, q, z only

so, it is of the standard form (II).

i.e.

$$f(p,q,z)=0$$

so,

$$p=\frac{dz}{dx} \quad \text{--- (2)}$$

$$q=\frac{a\,dz}{dx} \quad \text{--- (3)}$$

so, from (1), (2) & (3)

we get,

$$z^2\left(\left(\frac{dz}{dx}\right)^2 z^2 + a^2\left(\frac{dz}{dx}\right)^2\right)=1$$

$$z^2\left(\left(\frac{dz}{dx}\right)^2\left[z^2+a^2\right]\right)=1$$

$$\left(\frac{dz}{dx}\right)^2=\frac{1}{z^2(z^2+a^2)}$$

$$\frac{dz}{dx} = \sqrt{\frac{1}{z^2(z^2+a^2)}}$$

$$\frac{dz}{dx} = \frac{1}{z\sqrt{z^2+a^2}}$$

$$z\sqrt{z^2+a^2}\,dz = dx$$

now, integrating both sides,
we get,

$$\int z\sqrt{z^2+a^2}\,dz = \int dx$$

Let, $z^2+a^2 = t^2$ — (4)

$$\cancel{2}z = \cancel{2}t\,\frac{dt}{dz}$$

$$dz = \frac{t\,dt}{z} \quad — (5)$$

$$\int \cancel{z}\,t\,\frac{t\,dt}{\cancel{z}} = \int dx$$

$$\int t^2\,dt = \int dx$$

$$\frac{t^3}{3} = x + C$$

$$\cancel{\frac{(z^2+a^2)^2}{4} = x + c}$$

$$\frac{z^2+a^2\sqrt{z^2+a^2}}{3} = x + c$$

$$(z^2+a^2)^{3/2} = 3x + 3c$$

$$(z^2+a^2)^3 = (3x+3c)^2$$

$$(z^2+a^2)^3 = 3^2(x^2+2xc+c^2)$$

$$\cancel{(z^2+a^2)^3 = 9}$$

$$\boxed{(z^2+a^2)^3 = (3x+b)^2 \qquad (b = 3c)}$$

(III) Standard Form :- III

$$\boxed{f_1(x, p) = f_2(y, q)}$$

Working Rule :-

(1) The problem is of the form :-

$$f_1(x, p) = f_2(y, q) \quad \text{--- (1)}$$

(2) now, put

$$f_1(x, p) = a \quad \text{--- (2)}$$
$$f_2(y, q) = a \quad \text{--- (3)}$$

(3) On solving (2) & (3) we get, 'p', 'q'

where,

we get,

$$p = \phi_1(x, a) \quad \text{--- (4)}$$
$$q = \phi_2(y, a) \quad \text{--- (5)}$$

(4) The complete solⁿ is :-

$$dz = p\,dx + q\,dy$$

$$dz = \phi_1(x, a)\,dx + \phi_2(y, a)\,dy + b.$$

(from (4) & (5))

& $f_2(y,q) = a.$

$$q^2 - y = a.$$

$$q^2 = a+y$$

$$\boxed{q = \sqrt{a+y}} \quad -(4).$$

now,
The complete solⁿ is:-

$$dz = p\,dx + q\,dy$$

$$dz = \sqrt{a+x}\,dx + \sqrt{a+y}\,dy.$$

now,
integrating both sides,
we get,

$$\int dz = \int \sqrt{a+x}\,dx + \int \sqrt{a+y}\,dy.$$

$$z = \int (a+x)^{1/2}dx + \int (a+y)^{1/2}dy$$

$$z = \frac{(a+x)^{3/2}}{3/2} + \frac{(a+y)^{3/2}}{3/2}$$

$$\boxed{z = \frac{2}{3}\left((a+x)^{3/2} + (a+y)^{3/2}\right)}$$

4.4 CHARPIT'S METHOD

This is a general method for finding the complete solution of a non-linear partial differential equation of first order.

Let $f(x, y, z, p, q) = 0$ ...(1)

Working Rule :

Step 1. Transfer all the terms of the given PDE (1) to L.H.S. and denote the entire expression in L.H.S. by f.

Step 2. Write down the Charpit's auxiliary equations :

$$\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}}=\frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}}=\frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}}=\frac{dx}{-\frac{\partial f}{\partial p}}=\frac{dy}{-\frac{\partial f}{\partial q}}=\frac{dF}{0} \quad ...(2)$$

Step 3. Using the value of obtained in *step 1*, write down the values of $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$, etc. occurring in *step 2* and put these in Charpit's auxiliary equations (2).

Step 4. After simplifying the *step 3*, choose two proper fractions such that the resulting integral may come out to be the simplest relation involving at least one of p and q.

Step 5. The simples relation of *step 4*, is solved along with the given equation to find p and q. Put these values of p and q in the complete integral

Example 4.38 : Solve $(p^2 + q^2)y = qz$ by using Charpit's method.

[RGPV June 2007, June 2012, Dec. 2012, Dec. 2013, June 2015]

Solution. Let $f = f(x, y, z, p, q) = (p^2 + q^2)y - qz = 0,$ ...(1)

so that $\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = p^2 + q^2, \frac{\partial f}{\partial z} = -q, \frac{\partial f}{\partial p} = 2py$ and $\frac{\partial f}{\partial q} = 2qy - z.$

$\therefore$ Charpit's auxiliary equations are :

$$\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}}=\frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}}=\frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}}=\frac{dx}{-\frac{\partial f}{\partial p}}=\frac{dy}{-\frac{\partial f}{\partial q}}$$

or $$\frac{dp}{-pq}=\frac{dq}{p^2}=\frac{dz}{-2p^2y+qz-2q^2y}=\frac{dx}{-2py}=\frac{dy}{-2qy+z}. \quad ...(2)$$

Taking the first two members of (2), we get

$$\frac{dp}{-pq}=\frac{dq}{p^2} \Rightarrow p\,dp + q\,dq = 0.$$

Integrating, we get $\frac{1}{2}p^2 + \frac{1}{2}q^2 = \frac{1}{2}a^2$, where a is constant.

or $p^2 + q^2 = a^2$ ...(3)

Putting $p^2 + q^2 = a^2$ in (1), we get $a^2 y = qz.$ ...(4)

From (3) and (4), solve for p and q, we get

$$p = \frac{a}{z}\sqrt{(z^2 - a^2y^2)} \text{ and } q = \frac{a^2 y}{z}. \quad ...(5)$$

$\therefore$ Complete integral, $dz = p\,dx + q\,dy$

$$dz = \frac{a}{z}\sqrt{(z^2 - a^2y^2)}dx + \frac{a^2y}{z}dy \qquad \text{[by (5)]}$$

$$\Rightarrow \quad \frac{z\,dz - a^2y\,dy}{\sqrt{(z^2 - a^2y^2)}} = a\,dx. \qquad \text{[as separation of variable]}$$

Integrating, we get

$$\sqrt{(z^2 - a^2y^2)} = ax + b, \text{ where } b \text{ is constant.}$$

$$z^2 - a^2y^2 = (ax+b)^2.$$

Ans.