~~Find~~ Reduce ~~matrix~~ matrix A to normal form.

$$A = \begin{bmatrix} 8 & 1 & 3 & 6 \\ 0 & 3 & 2 & 2 \\ -8 & -1 & -3 & 4 \end{bmatrix}$$

$C_2 \leftrightarrow C_1$

$$A = \begin{bmatrix} 1 & 8 & 3 & 6 \\ 3 & 0 & 2 & 2 \\ -1 & -8 & -3 & 4 \end{bmatrix}$$

$R_2 \to R_2 - 3R_1$ $\qquad$ $R_3 \to R_3 + R_1$

$$A = \begin{bmatrix} 1 & 8 & 3 & 6 \\ 0 & -24 & -7 & -16 \\ 0 & 0 & 0 & 10 \end{bmatrix}$$

$C_2 \to C_2 - 8C_1$ $\qquad$ $C_3 \to C_3 - 3C_1$ $\qquad$ $C_4 \to C_4 - 6C_1$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -24 & -7 & -16 \\ 0 & 0 & 0 & 10 \end{bmatrix}$$

$C_2 \to \frac{1}{-24} C_2$, $C_3 \to \frac{1}{-7} C_3$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -16 \\ 0 & 0 & 0 & 10 \end{bmatrix}$$

$C_3 \to C_3 - C_2$ $\quad C_4 \to C_4 + 16 C_2$

$R_3 \to \frac{1}{10} R_3$

$$\cancel{A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 10 \end{bmatrix}}$$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -16 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_3 \to C_3 - C_2$

~~$R_3 \to 16 R_3$~~

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -16 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_4 \to C_4 + 16 C_2$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_4 \leftrightarrow C_3$

$$A = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] = [I_3 \mid 0]$$

$$\boxed{\rho(A) = 3}$$

Q2> Solve system of equation using matrix method

$2x - y + 3z = 0$, $3x + 2y + z = 0$, $x - 4y + 5z = 0$

Sol Matrix form is given by, $AX = 0$ $\because$ equations are in homogeneous form

where, $A = \begin{bmatrix} 2 & -1 & 3 \\ 3 & 2 & 1 \\ 1 & -4 & 5 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Augumented matrix i.e $[A:B] = \left[\begin{array}{ccc:c} 2 & -1 & 3 & 0 \\ 3 & 2 & 1 & 0 \\ 1 & -4 & 5 & 0 \end{array}\right]$

$R_3 \leftrightarrow R_1$

$[A:B] = \left[\begin{array}{ccc:c} 1 & -4 & 5 & 0 \\ 3 & 2 & 1 & 0 \\ 2 & -1 & 3 & 0 \end{array}\right]$

$R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - 2R_1$

$[A:B] = \left[\begin{array}{ccc:c} 1 & -4 & 5 & 0 \\ 0 & 14 & -14 & 0 \\ 0 & 7 & -7 & 0 \end{array}\right]$

$R_2 \to R_2 - 2R_3$, $R_2 \leftrightarrow R_3$

$[A:B] = \left[\begin{array}{ccc:c} 1 & -4 & 5 & 0 \\ 0 & 7 & -7 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\rho[A:B] = 2$, $\rho[A] = 2$

$\because \rho[A:B] = \rho[A]$ which is consistent, $r < n$ i.e having infinite solutions.

$1x - 4y + 5z = 0$ $\Rightarrow$ $1x - 4k + 5k = 0$

$0x + 7y - 7z = 0$ $\quad x + k = 0$

Let $z = k$ (say)

$7y = 7k$

$\boxed{y = k}$ $\boxed{z = k}$ $\boxed{x = -k}$

$\begin{vmatrix} 2 & -1 & 3 \\ 3 & 2 & 1 \\ 1 & -4 & 5 \end{vmatrix}$

$2(14) + 1(14) + 3(-14)$

$14[2 + 1 - 3] = 14(0)$

$= 0$

$\because |A| = 0$

$\therefore \boxed{r < n}$

Q3) Find the Eigen values and Eigen vectors of matrix : $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

Sol The characteristic equation i.e $|A - \lambda I| = 0$

$$\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

$$(8-\lambda)[(7-\lambda)(3-\lambda)-16] + 6[-18+6\lambda+8] + 2[24-14+2\lambda] = 0$$

$$(8-\lambda)[21-7\lambda-3\lambda+\lambda^2-16] - 60+36\lambda + 20 + 4\lambda = 0$$

$$(8-\lambda)[\lambda^2-10\lambda+5] + 40\lambda - 40 = 0$$

$$8\lambda^2 - 80\lambda + 40 - \lambda^3 + 10\lambda^2 - 5\lambda + 40\lambda - 40 = 0$$

$$-\lambda^3 + 18\lambda^2 - 45\lambda = 0$$

$$-\lambda[\lambda^2 - 18\lambda + 45] = 0$$

$-\lambda = 0$ $\qquad$ $\lambda^2 - 18\lambda + 45 = 0$

$\lambda = 0$ $\qquad$ $\lambda^2 - 15\lambda - 3\lambda + 45 = 0$

$$\lambda(\lambda-15) - 3(\lambda-15) = 0$$

$$(\lambda-15)(\lambda-3) = 0$$

$$\lambda = 15, \quad \lambda = 3$$

$\boxed{\lambda = 0, 15, 3}$ eigen values

Eigen vectors

1) When $\lambda = 0$,
$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$8x - 6y + 2z = 0$

$-6x + 7y - 4z = 0$

$2x - 4y + 3z = 0$

Solving last two eqn

$$\frac{x}{21-16} = \frac{-y}{-18+8} = \frac{z}{24-14} = k$$

$$\frac{x}{5} = \frac{y}{10} = \frac{z}{10} = k$$

$$x = 5k, \quad y = 10k, \quad z = 10k$$

$$X_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5k \\ 10k \\ 10k \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

② When $\lambda = 15$,
$$\begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-7x - 6y + 2z = 0$
$-6x - 8y - 4z = 0$
$2x - 4y - 12z = 0$

Solving last two eq^n

$$\frac{x}{96-16} = \frac{-y}{72+8} = \frac{z}{24+16} = k$$

$$\frac{x}{80} = \frac{-y}{80} = \frac{z}{40} = k$$

$x = 80k$, $y = -80k$, $z = 40k$

$$X_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 80k \\ -80k \\ 40k \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

③ When $\lambda = 3$,
$$\begin{bmatrix} 5 & -6 & 2 \\ -6 & 2 & -4 \\ 2 & -4 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$5x - 6y + 2z = 0$
$-6x + 2y - 4z = 0$
$2x - 4y + 0z = 0$

Solving last two eq^n

$$\frac{x}{0-16} = \frac{-y}{0+8} = \frac{z}{24-4} = k$$

$$\frac{x}{-16} = \frac{-y}{8} = \frac{z}{20} = k$$

$x = -16k$, $y = -8k$, $z = 20k$

$$X_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -16k \\ -8k \\ 20k \end{bmatrix} = \begin{bmatrix} -4 \\ -2 \\ 5 \end{bmatrix}$$

Q4> Find eigen values & eigen vectors of matrix $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

Sol The characteristic equation is $|A - \lambda I| = 0$

$$\begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)[(3-\lambda)(2-\lambda) - 2] - 2[2-\lambda-1] + 1[2-3+\lambda] = 0$$

$(2-\lambda)[6-3\lambda-2\lambda+\lambda^2-2]-1+2\lambda-1+\lambda=0$

$(2-\lambda)[\lambda^2-5\lambda+4]+3\lambda-3=0$

$2\lambda^2-10\lambda+8-\lambda^3+5\lambda^2-4\lambda+3\lambda-3=0$

$-\lambda^3+7\lambda^2-11\lambda+5=0$

$\lambda^3-7\lambda^2+11\lambda-5=0$

$(\lambda-1)(\lambda^2-6\lambda+5)=0$

$\lambda=1$ $\quad \cancel{\lambda=\frac{6\pm\sqrt{36-24}}{2(1)}=\frac{6\pm\sqrt{12}}{2}}$

$$\begin{array}{r} \lambda^2-6\lambda+5 \\ \lambda-1 \overline{)\lambda^3-7\lambda^2+11\lambda-5} \\ \underline{\lambda^3 \mp \lambda^2} \\ -6\lambda^2+11\lambda-5 \\ \underline{\mp 6\lambda^2 \pm 6\lambda} \\ 5\lambda-5 \\ \underline{5\lambda \mp 5} \\ 0 \end{array}$$

$(\lambda-5)(\lambda-1)=0$

$\boxed{\lambda=1,1,5}$ Eigen values

Eigen Vectors

① when $\lambda=1$, (repeated roots) $\begin{bmatrix}1&2&1\\1&2&1\\1&2&1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

$R_2\to R_2-R_1,\ R_3\to R_3-R_1$

$\begin{bmatrix}1&2&1\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

$1x+2y+1z=0$

Let $z=k_1$; $y=k_2$

$\therefore x=-2k_2-k_1$

$X_1=\begin{bmatrix}-2k_2-k_1\\k_2\\k_1\end{bmatrix}=\begin{bmatrix}-3\\1\\1\end{bmatrix}$

② When $\lambda=5$, $\begin{bmatrix}-3&2&1\\1&-2&1\\1&2&-3\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

$-3x+2y+z=0$

$x-2y+z=0$

$x+2y-3z=0$

Solving last two eqn

$\frac{x}{6-2}=\frac{-y}{-3-1}=\frac{z}{2+2}=k$

$\frac{x}{4}=\frac{y}{4}=\frac{z}{4}=k$

$x=4k,\ y=4k,\ z=4k$

$X_2=\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}4k\\4k\\4k\end{bmatrix}=\begin{bmatrix}1\\1\\1\end{bmatrix}$

Qs> Find the rank and nullity of matrix, where $A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$

Sol

$$A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix} \quad R_2 \Leftrightarrow R_1$$

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix} \quad R_3 \to R_3 - 3R_1, \; R_4 \to R_4 - R_1$$

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \end{bmatrix} \quad \begin{matrix} R_3 \to R_3 - R_2 \\ R_4 \to R_4 - R_2 \end{matrix}$$

$$\begin{vmatrix} 0 & 1 & -3 \\ 1 & 0 & 1 \\ 3 & 1 & 0 \end{vmatrix}$$

$0(-1) - 1[0-3] - 3[1-0]$

$0 + 3 - 3 = 0$

$\because |A| = 0 \quad \therefore \boxed{r < n}$

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\boxed{\rho[A] = 2}$

Nullity = Order − rank $= 4 - 2 = \boxed{2}$

Q. Reduce the matrix $\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}$ to diagonal form

Sol^n: $A = \begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}$

The characteristic equation is $|A-\lambda I| = 0$

$$\begin{vmatrix}1-\lambda & 1 & 1\\ 1 & 1-\lambda & 1\\ 1 & 1 & 1-\lambda\end{vmatrix} = 0 \quad \text{—— (1)}$$

$\Rightarrow (1-\lambda)\{(1-\lambda)(1-\lambda)-1\} - 1\{1(1-\lambda)-1\} + 1\{1-1+\lambda\} = 0$

$\Rightarrow (1-\lambda)\{\cancel{1}-2\lambda+\lambda^2\cancel{-1}\} - \{\cancel{1}-\lambda-\cancel{1}\} + \lambda = 0$

$\Rightarrow (1-\lambda)\{-2\lambda+\lambda^2\} + 2\lambda = 0$

$\Rightarrow \cancel{-2\lambda} + \lambda^2 + 2\lambda^2 - \lambda^3 + \cancel{2\lambda} = 0$

$\Rightarrow 3\lambda^2 - \lambda^3 = 0$

$\Rightarrow \lambda^3 - 3\lambda^2 = 0$

$\Rightarrow \lambda^2(\lambda-3) = 0 \Rightarrow \boxed{\lambda = 0, 0, 3}$ are eigen values

To find Eigen vectors:

when $\lambda = 0$ (repeated roots)

From (1) we have,

$$\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}$$

Applying Row operations $R_2 \to R_2 - R_1$
$R_3 \to R_3 - R_1$

$$\begin{bmatrix}1&1&1\\0&0&0\\0&0&0\end{bmatrix}$$

Now, $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$x + y + z = 0$

Let, $z = K_1$, $y = K_2 \Rightarrow x = -K_2 - K_1$

Let, $K_1 = K_2 = 1$

$$X_1 = \begin{bmatrix} K_2 - K_1 \\ K_2 \\ K_1 \end{bmatrix} = X_1 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

Let, $K_1 = 0$, $K_2 = 1$

$$X_2 = \begin{bmatrix} -K_2 - K_1 \\ K_2 \\ K_1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

when $\lambda = 3$, from ① we have,

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-2x + y + z = 0$

$x - 2y + z = 0$

$x + y - 2z = 0$

Considering first two eq^n.

$$\frac{x}{1+2} = \frac{-y}{-2-1} = \frac{z}{4-1} = k \text{ (say)}$$

$$\Rightarrow \frac{x}{3} = \frac{y}{3} = \frac{z}{3} = k \text{ (say)}$$

$$\Rightarrow \frac{x}{1} = \frac{y}{1} = \frac{z}{1} = K \text{ (say)}$$

$$\therefore X_3 = \begin{bmatrix} K \\ k \\ k \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$\therefore\ P = \begin{bmatrix} -2 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ Known as Modal matrix

$$P^{-1} = \frac{\text{adj } P}{|P|}$$

$$|P| = -2(1-0) + 1(1-1) + 1(0-1)$$
$$= -2-1 = -3$$

$C_{11} = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1$ | $C_{21} = -\begin{vmatrix} -1 & 1 \\ 0 & 1 \end{vmatrix} = 1$

$C_{12} = -\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$ | $C_{22} = \begin{vmatrix} -2 & 1 \\ 1 & 1 \end{vmatrix} = -3$

$C_{13} = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -1$ | $C_{23} = -\begin{vmatrix} -2 & -1 \\ 1 & 0 \end{vmatrix} = \cancel{1} = -1$

$$C_{31} = \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = -1-1 = -2$$

$$C_{32} = -\begin{vmatrix} -2 & 1 \\ 1 & 1 \end{vmatrix} = 3$$

$$C_{33} = \begin{vmatrix} -2 & -1 \\ 1 & 1 \end{vmatrix} = -1$$

$\text{adj } P = \begin{bmatrix} 1 & 1 & -2 \\ 0 & -3 & 3 \\ -1 & -1 & -1 \end{bmatrix}$ $\quad\therefore P^{-1} = -\frac{1}{3}\begin{bmatrix} 1 & 1 & -2 \\ 0 & -3 & 3 \\ -1 & 1 & -1 \end{bmatrix}$

$$\therefore D = P^{-1}AP = -\frac{1}{3}\begin{bmatrix} 1 & 1 & -2 \\ 0 & -3 & 3 \\ -1 & -1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} -2 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= -\frac{1}{3}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -3 & -3 & -3 \end{bmatrix}\begin{bmatrix} -2 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = -\frac{1}{3}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -9 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

#

Q7) If $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$. find A^{-1} using Cayley-Hamiton theorem and hence evaluate

$$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I.$$

Sol Characteristic equation is $|A - \lambda I| = 0$

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)\left[(1-\lambda)(2-\lambda) - 0\right] - 1\left[0 - 0\right] + 1\left[0 - 1 + \lambda\right] = 0$$

$$(2-\lambda)\left[2-\lambda-2\lambda+\lambda^2\right]-0-1+\lambda=0$$

$$(2-\lambda)(\lambda^2-3\lambda+2)-1+\lambda=0$$

$$2\lambda^2-6\lambda+4-\lambda^3+3\lambda^2-2\lambda-1+\lambda=0$$

$$-\lambda^3+5\lambda^2-7\lambda+3=0$$

$$\boxed{\lambda^3-5\lambda^2+7\lambda-3=0} \quad (1)$$

Using Cayley hamilton theorem to satisfy~~ing~~ eq (1)

$$\boxed{A^3-5A^2+7A-3I=0} \quad (2)$$

$$A^2 = A \cdot A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix}$$

$$A^3 = A^2 \cdot A = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 14 & 13 & 13 \\ 0 & 1 & 0 \\ 13 & 13 & 14 \end{bmatrix}$$

eq (2) becomes,

$$\begin{bmatrix} 14 & 13 & 13 \\ 0 & 1 & 0 \\ 13 & 13 & 14 \end{bmatrix} - 5\begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + 7\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, Cayley hamiton theorem is verified

$$A^8-5A^7+7A^6-3A^5+A^4-5A^3+\overset{7A^2+A^2}{8A^2}\overset{-3A+I}{-2A}+I$$

$$A^5\left(A^3-5A^2+7A-3I\right)+A\left(A^3-5A^2+7A-3I\right)+A^2+A+I$$

$$A^5(0)+A(0)+\begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix}+\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}+\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boxed{= \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}}$$

8> Solve linear equation $5x+3y+7z=4$, $3x+26y+2z=9$,
$$7x+2y+10z=5$$

Sol Matrix form is $AX=B$

where, $A=\begin{bmatrix}5 & 3 & 7\\ 3 & 26 & 2\\ 7 & 2 & 10\end{bmatrix}$, $X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}$, $B=\begin{bmatrix}4\\ 9\\ 5\end{bmatrix}$

Augumented matrix i.e $[A:B]=\left[\begin{array}{ccc:c}5 & 3 & 7 & 4\\ 3 & 26 & 2 & 9\\ 7 & 2 & 10 & 5\end{array}\right]$

$R_1 \leftrightarrow R_2$

$[A:B]=\left[\begin{array}{ccc:c}3 & 26 & 2 & 9\\ 5 & 3 & 7 & 4\\ 7 & 2 & 10 & 5\end{array}\right]$

$R_3 \to R_3 - R_2$ $\quad R_2 \to R_2 - R_1$

$\left[\begin{array}{ccc:c}3 & 26 & 2 & 9\end{array}\right.$

$\begin{vmatrix}5 & 3 & 7\\ 3 & 26 & 2\\ 7 & 2 & 10\end{vmatrix}$

$5(254)-3(16)+7(-176)$

$=0 \quad \therefore |A|=0$

$\therefore r<n$

$R_3 \to R_3 - R_2$

$$[A:B] = \begin{bmatrix} 5 & 3 & 7 & : & 4 \\ 3 & 26 & 2 & : & 9 \\ 4 & -24 & 8 & : & -4 \end{bmatrix} \quad R_1 \to R_1 - R_4$$

$$[A:B] = \begin{bmatrix} 1 & 27 & -1 & : & 8 \\ 3 & 26 & 2 & : & 9 \\ 4 & -24 & 8 & : & -4 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 4R_1 \end{matrix}$$

$$[A:B] = \begin{bmatrix} 1 & 27 & -1 & : & 8 \\ 0 & -55 & 5 & : & -15 \\ 0 & -132 & 12 & : & -36 \end{bmatrix} \quad \begin{matrix} R_2 \to \frac{1}{5} R_2 \\ R_3 \to \frac{1}{12} R_3 \end{matrix}$$

$$[A:B] = \begin{bmatrix} 1 & 27 & -1 & : & 8 \\ 0 & -11 & 1 & : & -3 \\ 0 & -11 & 1 & : & -3 \end{bmatrix} \quad R_3 \to R_3 - R_2$$

$$[A:B] = \begin{bmatrix} 1 & 27 & -1 & : & 8 \\ 0 & -11 & 1 & : & -3 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$

$\rho[A:B] = 2 \qquad \rho[A] = 2$

$\therefore \rho[A:B] = \rho[A]$ which is consistent, $r<2$ i.e having infinitely many solution

$1x + 27y - 1z = 8$

$0x - 11y + 1z = -3$

Let $z = k$

$-11y + k = -3$

$-11y = -3 - k$

$\boxed{y = \frac{3+k}{11}} \qquad \boxed{z = k}$

$x + 27\left(\frac{3+k}{11}\right) - k = 8$

$\frac{11x + 81 + 27k - 11k}{11} = 8$

$11x + 16k + 81 = 88$

$11x = 7 - 16k$

$\boxed{x = \frac{7-16k}{11}}$

Q9) Find that what value of λ & μ the equations $x+y+z=6$, $x+2y+3z=10$, $x+2y+\lambda z = \mu$ have (i) no solution

(ii) a unique solution

(iii) an infinite many solution.

Matrix form is given by $AX = B$ where,

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$$

Augumented matrix i.e $[A:B] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{array}\right]$

$R_2 \to R_2 - R_1 \qquad R_3 \to R_3 - R_1$

$$[A:B] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda-1 & \mu-6 \end{array}\right] \quad R_3 \to R_3 - R_2$$

$$[A:B] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{array}\right]$$

① No solution, $\lambda = 3$ but $\mu \neq 10$

$\rho[A:B] = 3, \quad \rho[A] = 2$

$\because \rho[A:B] \neq \rho[A]$ which is inconsistent

$\therefore$ has no solution.

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & -4 \end{array}\right]$$

② Unique Solution, $\lambda \neq 3$ and $\mu = 10$ or $\mu \neq 10$

$\rho[A:B] = 3, \quad \rho[A] = 3$

$\because \rho[A:B] = \rho[A]$ which is consistent, $r = n$ having Unique solution.

③ Infinite many solution, $\lambda = 3$ and $\lambda = 10$

$\rho[A:B] = 2, \quad \rho[A] = 2$

$\because \rho[A:B] = \rho[A]$ which is consistent, $r < n$ having infinite many solution.

Q10) Show that Cayley-Hamilton Theorem is satisfied by matrix A where

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{bmatrix}$$ and hence find A^{-1}.

Sol) Characteristic equation is $|A - \lambda I| = 0$

$$\begin{vmatrix} -\lambda & 0 & 1 \\ 3 & 1-\lambda & 0 \\ -2 & 1 & 4-\lambda \end{vmatrix} = 0$$

$$-\lambda[(1-\lambda)(4-\lambda) - 0] - 0[\ \ ] + 1[3 + 2 - 2\lambda] = 0$$

$$-\lambda[4 - \lambda - 4\lambda + \lambda^2] + 5 - 2\lambda = 0$$

$$-\lambda[\lambda^2 - 5\lambda + 4] + 5 - 2\lambda = 0$$

$$-\lambda^3 + 5\lambda^2 - 4\lambda + 5 - 2\lambda = 0$$

$$\boxed{\lambda^3 - 5\lambda^2 + 6\lambda - 5 = 0} \quad (1)$$

Using Cayley-Hamilton theorem, satisfying eq (1)

$$\boxed{A^3 - 5A^2 + 6A - 5I = 0} \quad (2)$$

$$A^2 = A.A = \begin{bmatrix} 0 & 0 & 1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 4 \\ 3 & 1 & 3 \\ -5 & 5 & 14 \end{bmatrix}$$

$$A^3 = A^2 . A = \begin{bmatrix} -2 & 1 & 4 \\ 3 & 1 & 3 \\ -5 & 5 & 14 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{bmatrix} = \begin{bmatrix} -5 & 5 & 14 \\ -3 & 4 & 15 \\ -13 & 19 & 51 \end{bmatrix}$$

eq^n (2) becomes,

$$\begin{bmatrix} -5 & 5 & 14 \\ -3 & 4 & 15 \\ -13 & 19 & 51 \end{bmatrix} - 5 \begin{bmatrix} -2 & 1 & 4 \\ 3 & 1 & 3 \\ -5 & 5 & 14 \end{bmatrix} + 6 \begin{bmatrix} 0 & 0 & 1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{bmatrix} - 5 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, Cayley Hamilton theorem is verified.

To determine A^{-1}, Multiplying eq (2) by A^{-1}

$$A^2 - 5A + 6I - 5A^{-1} = 0$$

$$5A^{-1} = A^2 - 5A + 6I$$

$$A^{-1} = \frac{1}{5}\left[\begin{bmatrix} -2 & 1 & 4 \\ 3 & 1 & 3 \\ -5 & 5 & 14 \end{bmatrix} - 5\begin{bmatrix} 0 & 0 & 1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{bmatrix} + 6\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right]$$

$$\boxed{A^{-1} = \frac{1}{5}\begin{bmatrix} 4 & 1 & -1 \\ -12 & 2 & 3 \\ 5 & 0 & 0 \end{bmatrix}}$$

Q2> Show that $\tan^{-1}(x+h) = \tan^{-1}x + h\sin\theta \cdot \frac{\sin\theta}{1} - (h\sin\theta)^2 \cdot \frac{\sin 2\theta}{2} + (h\sin\theta)^3 \cdot \frac{\sin 3\theta}{3} - \cdots + (-1)^n (h\sin\theta)^n \cdot \frac{\sin n\theta}{n} + \cdots$ where $\theta = \cot^{-1}x$.

Sol By Taylor's theorem,

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \cdots \quad — (A)$$

Given, $f(x+h) = \tan^{-1}(x+h)$

$\because \theta = \cot^{-1}x$

$\boxed{x = \cot\theta}$.

$\therefore \boxed{f(x) = \tan^{-1}x}$

$$f'(x) = \frac{1}{1+x^2} = \frac{1}{1+\cot^2\theta} = \frac{1}{\operatorname{cosec}^2\theta}$$

$\theta = \cot^{-1}x$

$$\frac{d\theta}{dx} = \frac{-1}{1+x^2}$$

$$\frac{d\theta}{dx} = \frac{-1}{1+\cot^2\theta}$$

$$\frac{d\theta}{dx} = \frac{-1}{\operatorname{cosec}^2\theta}$$

$$\boxed{\frac{d\theta}{dx} = -\sin^2\theta}$$

$$\boxed{f'(x) = \sin^2\theta}$$

$$f''(x) = 2\sin\theta\cos\theta\frac{d\theta}{dx} = 2\sin\theta\cos\theta(-\sin^2\theta)$$

$$\boxed{f''(x) = -\sin 2\theta \sin^2\theta}$$

$$f'''(x) = -\left[\sin 2\theta \frac{d}{dx}\sin^2\theta + \sin^2\theta\frac{d}{dx}\sin 2\theta\right]$$

$$f'''(x) = -\sin 2\theta \times (-\sin 2\theta \sin^2\theta) - \sin^2\theta\, 2\cos 2\theta\,(-\sin^2\theta)$$

$$f'''(x) = \sin^2 2\theta \sin^2\theta + 2\sin^4\theta\cos 2\theta$$

$$f'''(x) = 2\sin\theta\cos\theta\sin 2\theta\sin^2\theta + 2\sin^3\theta\sin\theta\cos 2\theta$$

$$f'''(x) = 2\sin^3\theta\left[\cos\theta\sin 2\theta + \sin\theta\cos 2\theta\right]$$

$$f'''(x) = 2\sin^3\theta\sin(2\theta+\theta)$$

$$\boxed{f'''(x) = 2\sin^3\theta\sin 3\theta}$$

$\therefore$ eq (A) becomes,

$$\tan^{-1}(x+h) = \tan^{-1}x + h\sin^2\theta + \frac{h^2}{2!}(-\sin 2\theta\sin^2\theta) + \frac{h^3}{3!}(2\sin^3\theta\sin 3\theta)$$

$$\boxed{\tan^{-1}(x+h) = \tan^{-1}x + h\sin\theta \cdot \frac{\sin\theta}{1} - (h\sin\theta)^2\frac{\sin 2\theta}{2} + (h\sin\theta)^3 \cdot \frac{\sin 3\theta}{3} + \cdots}$$

Q 3) Does the function $f(x) = x + \frac{1}{x}$ satisfy the condition of mean value theorem in range $\left[\frac{1}{2}, 3\right]$?

Sol Given, $f(x) = x + \frac{1}{x}$

① $f\left(\frac{1}{2}\right) = \frac{1}{2} + 2 = \frac{5}{2}$

$f(3) = 3 + \frac{1}{3} = \frac{10}{3}$

$\because f(1/2) \neq f(3)$

$\therefore f(a) \neq f(b)$

② $f(x)$ is a polynomial function which is continuous at ~~open~~ closed interval $[1/2, 3]$

③ $f(x)$ is differentiable at ~~closed~~ open interval $(1/2, 3)$ then there exists atleast one real value of point c such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$1 - \frac{1}{c^2} = \frac{\frac{10}{3} - \frac{5}{2}}{3 - \frac{1}{2}} = \frac{\frac{20 - 15}{3 \times 2}}{\frac{6-1}{2}} = \frac{5}{5 \times 3} = \frac{1}{3}$$

$$\frac{c^2 - 1}{c^2} = \frac{1}{3}$$

$$3c^2 - 3 = c^2$$

$$2c^2 - 3 = 0$$

$$c^2 = \frac{3}{2}$$

$$c = \pm\sqrt{\frac{3}{2}}$$

$c = +1.22, -1.22$

$\therefore c = 1.22 \in (1/2, 3)$

$f(x)$ satisfies Lagrange mean value theorem for $[1/2, 3]$

Q 4) Expand $e^{a\sin^{-1}x}$ by Maclaurin's theorem and show that $(1-x^2)y_{n+2}-(2n+1)x\,y_{n+1}-(n^2+a^2)y_n=0$.

Sol Given, $y=e^{a\sin^{-1}x}$

By maclaurin's theorem,

$$y=(y)_0+x(y_1)_0+\frac{x^2}{2!}(y_2)_0+\frac{x^3}{3!}(y_3)_0+\cdots \quad -(A)$$

$y=e^{a\sin^{-1}x}$ $\quad \boxed{(y)_0=1}$

$y_1=e^{a\sin^{-1}x}\left(a\frac{1}{\sqrt{1-x^2}}\right)$ $\quad \boxed{(y_1)_0=a}$ -(2)

$y_1\sqrt{1-x^2}=a\,y$

Squaring both sides

$(y_1)^2(1-x^2)=a^2y^2$

differentiating w.r.t x

$2y_1y_2(1-x^2)+(y_1)^2(-2x)=a^2\,2yy_1$

$\cancel{y_1}\left[2y_2(1-x^2)-2xy_1\right]=a^2\,2y\,\cancel{y_1}$

$2y_2=\frac{a^2 2y+2xy_1}{(1-x^2)}$

$y_2=\frac{a^2y+xy_1}{(1-x^2)}$ $\quad \boxed{(y_2)_0=a^2}$

$(1-x^2)y_2-xy_1-a^2y=0$

Finding general term by using Leibnitz theorem

$\left[(1-x^2)y_{n+2}\,{}^nC_0-2x\,y_{n+1}\,{}^nC_1-2y_{n}\,{}^nC_2\right]-\left[xy_{n+1}\,{}^nC_0+1\,y_n\,{}^nC_1\right]-a^2y_n=0$

$(1-x^2)y_{n+2}-2x\,y_{n+1}\times n-2y_n\frac{n(n-1)}{2}-xy_{n+1}-ny_n-a^2y_n=0$

$(1-x^2)y_{n+2}-xy_{n+1}[2n+1]-y_n[n^2-\cancel{n}+\cancel{n}+a^2]=0$

$$\boxed{(1-x^2)y_{n+2}-(2n+1)xy_{n+1}-(n^2+a^2)y_n=0}$$

Substituting $n=1$

$$y_3(1-x^2) - (2x+1)xy_2 - (a^2+1)y_1 = 0$$

$$(y_3)_0(1-0^2) - (2(0)+1)(0)y_2 - (a^2+1)(y_1)_0 = 0$$

$$(y_3)_0 - (a^2+1)a = 0$$

$$\boxed{(y_3) = (a^2+1)a}$$

$\therefore$ eq (A) becomes,

$$\boxed{e^{a\sin^{-1}x} = 1 + xa + \frac{x^2}{2!}a^2 + \frac{x^3}{3!}(a^2+1)a + \cdots}$$

Q5> Expand function $\log_e x$ in power of $(x-1)$ and hence evaluate $\log(1.1)$ correct to 4 decimal places.

Sol Given, $f(x) = \log_e x$

$$h = x-a = x-1$$

$$a = 1$$

By taylor's theorem,

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \frac{h^3}{3!}f'''(a) + \cdots$$

$$h = x-a$$

$$f(a+x-a) = f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots$$

$\because a=1$

$$f(x) = f(1) + (x-1)f'(1) + \frac{(x-1)^2}{2!}f''(1) + \frac{(x-1)^3}{3!}f'''(1) + \cdots \quad \text{(A)}$$

$f(x) = \log_e x \qquad f(1) = \log_e 1 = \boxed{0}$

$f'(x) = x^{-1} \qquad \boxed{f'(1) = 1}$

$f''(x) = -x^{-2}$ $\quad \boxed{f''(1) = -1}$

$f'''(x) = 2x^{-3}$ $\quad \boxed{f'''(1) = 2}$

$f''''(x) = -6x^{-4}$ $\quad \boxed{f''''(1) = -6}$

$\therefore$ eq (A) becomes,

$$\log x = 0 + (x-1)(1) + \frac{(x-1)^2}{2}(-1) + \frac{(x-1)^3}{6}(2) + \frac{(x-1)^4}{24}(-6)$$

$$\boxed{\log x = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4}$$

$$\log_e(1.1) = (1.1-1) - \frac{1}{2}(1.1-1)^2 + \frac{1}{3}(1.1-1)^3 - \frac{1}{4}(1.1-1)^4$$

$$\log_e(1.1) = 0.1 - \frac{1}{2}\times\frac{1}{100} + \frac{1}{3}\times\frac{1}{1000} - \frac{1}{4}\times\frac{1}{10000}$$

$$\log_e(1.1) = 0.1 - 0.005 + 0.00034 - 0.000025$$

$$\log_e(1.1) = 0.095315$$

$$\boxed{\log_e(1.1) = 0.0953}$$

Q8> If the curve $x^x y^y z^z = c$ then show that $\frac{\partial^2 z}{\partial x \partial y} = -x(\log_e x)^{-1}$ at $x = y = z$

Sol

$c = x^x y^y z^z$

Applying log both sides

$\log c = x\log x + y\log y + z\log z$ — (A)

Partially Differentiating w.r.t x & y

$$0 = \left[\cancel{x}\times\frac{1}{\cancel{x}} + \log x\right] + 0 + \left[z\frac{1}{z} + \log z\right]\frac{\partial z}{\partial x}$$

$$(1+\log x) + (1+\log z)\frac{\partial z}{\partial x} = 0$$

$$\boxed{\frac{\partial z}{\partial x} = \frac{-(1+\log x)}{(1+\log z)}}$$

$$0 + \left[y \times \frac{1}{y} + \log y\right] + \left[z \times \frac{1}{z} + \log z\right] \frac{\partial z}{\partial y} = 0$$

$$(1+\log y) + (1+\log z) \frac{\partial z}{\partial y} = 0$$

$$\boxed{\frac{\partial z}{\partial y} = \frac{-(1+\log y)}{1+\log z}}$$

LHS $\rightarrow \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x} \frac{-(1+\log y)}{(1+\log z)} = \frac{-\partial}{\partial x}\left[(1+\log y) \times (1+\log z)^{-1}\right]$

$$= -\left[(1+\log y) \frac{\partial}{\partial x} (1+\log z)^{-1} + \underbrace{(1+\log z)^{-1} \frac{\partial}{\partial x} (1+\log y)}\right]$$

$$= -(1+\log y)\left(-(1+\log z)^{-2}\right) \times \frac{1}{z} \times \frac{\partial z}{\partial x} + 0$$

$$= (1+\log y)(1+\log z)^{-2} \times \frac{1}{z} \times \frac{-(1+\log x)}{(1+\log z)}$$

$$= \frac{-(1+\log x)(1+\log y)}{z(1+\log z)^3}$$

$[\because x = y = z]$

$$= \frac{-\cancel{(1+\log x)}\cancel{(1+\log x)}}{x(1+\log x)^{\cancel{3}}}$$

$$\boxed{\frac{\partial^2 z}{\partial x \partial y} = -(x\,1+\log x)^{-1}}$$

Q9) Discuss maximum or minimum of function $u = x^3 + y^3 - 3axy$.

Sol) $u = x^3 + y^3 - 3axy$

partially differentiating w.r.t x & y

$\frac{\partial u}{\partial x} = 3x^2 - 3ay$ $\qquad$ $\frac{\partial u}{\partial y} = 3y^2 - 3ax$

Equate to 0, $\frac{\partial u}{\partial x} = 0$ $\qquad$ $\frac{\partial u}{\partial y} = 0$

$3x^2 - 3ay = 0$ $\qquad$ $3y^2 - 3ax = 0$

$\cancel{3}x^2 = \cancel{3}ay$ $\qquad$ $\cancel{3}y^2 = \cancel{3}ax$

$a = \frac{x^2}{y}$ $\quad y = \frac{x^2}{a}$ $\qquad$ $\cancel{a = \frac{x}{y^2} = \frac{x}{\frac{x^4}{a^2}} = \frac{a^2}{x^3}}$.

$$y^2 = ax \qquad\qquad y^2 = a \cdot a$$
$$\frac{x^4}{a^2} = ax \qquad\qquad y^2 = a^2$$
$$\boxed{y = a}$$
$$x^{\cancel{4}\,3} = a^3 \cancel{x}$$
$$\boxed{x = a}$$

$\therefore$ (a, a) is critical point

$$r = \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}(3x^2 - 3ay) = \boxed{6x} \qquad \text{at } (a,a)\ \boxed{r = 6a}$$

$$s = \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}(3y^2 - 3ax) = \boxed{-3a}$$

$$t = \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}(3y^2 - 3ax) = \boxed{6y} \qquad \text{at } (a,a)\ \boxed{t = 6a}$$

$$rt - s^2 = (6a \times 6a) - (-3a)^2 = 36a^2 - 9a^2 = 27a^2$$

$\because$ $rt - s^2 > 0$, $r = 6a > 0$

$\therefore$ $f(x, y)$ is minimum at (a, a)

$$f(a,a) = a^3 + a^3 - 3a \times a \times a = 2a^3 - 3a^3 = \boxed{-a^3}$$

Q1) Discuss Rollè's theorem for function $f(x) = \begin{cases} x^2+1, & 0 \le x \le 1 \\ 3-x, & 1 < x \le 2. \end{cases}$

Sol

$$f(x) = \begin{cases} x^2+1, & 0 \le x \le 1 \\ 3-x, & 1 \le x \le 2 \end{cases}$$

lies in $[0, 2]$

$f(0) = 0^2 + 1 = 1$

$f(2) = 3 - 2 = 1$

$\therefore$ $f(0) = f(2) = 1$

$\therefore$ $f(a) = f(b)$

$\because$ $f(x)$ is continuous in closed interval $[0, 2]$ at $x = 1$

To test differentiability of $f(x)$ in $(0,2)$ at $x=1$

$$Rf'(1) = \lim_{x\to 1} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1} \frac{3-x-2}{}$$

$$= \lim_{x\to 1} \frac{1-x}{x-1}$$

$$= \lim_{x\to 1} \frac{\frac{1}{x}-1}{1-\frac{1}{x}} = \frac{-1}{1}$$

$$\boxed{Rf'(1) = -1}$$

$$Lf'(1) = \lim_{x\to 1} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1} \frac{x^2+1-2}{x-1}$$

left

0 se chalu hone wali limit

$$= \lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} \frac{\cancel{(x-1)}(x+1)}{\cancel{(x-1)}} = 1+1$$

$$\boxed{Lf'(1) = 2}$$

$\because R'f(1) \neq Lf'(1)$

$\therefore$ Rolle's theorem is not verified.

Q6> Show that the rectangular solid of maximum volume that can be inscribed in a sphere is a cube.

Sol Let x, y, z be the length, breadth and height of the rectangular solid.

If V is the volume of solid the $V = xyz$ —(1)

$\because$ each diagonal of rectangular solid passes through the centre of sphere

$\therefore$ each diagonal = diameter of sphere = d

$$\Rightarrow x^2+y^2+z^2 = d^2$$

$$z = \sqrt{d^2-x^2-y^2} \quad \text{—(A)}$$

$\therefore$ eq(1) becomes, $V = xy\sqrt{d^2-x^2-y^2}$

Squaring both sides

$$V^2 = x^2y^2(d^2-x^2-y^2)$$

$$V^2 = x^2y^2d^2 - x^4y^2 - y^4x^2 = f(x,y) \text{ (say)}$$

Differentiating w.r.t x & y

$$\boxed{\frac{\partial f}{\partial x} = 2xy^2d^2 - 4x^3y^2 - 2xy^4} = 2xy^2(d^2 - 2x^2 - y^2)$$

$$\boxed{\frac{\partial f}{\partial y} = 2yx^2d^2 - 2yx^4 - 4y^3x^2} = 2yx^2(d^2 - x^2 - 2y^2)$$

equating to zero, $\frac{\partial f}{\partial x} = 0$ $\qquad$ $\frac{\partial f}{\partial y} = 0$

$$2xy^2(d^2 - 2x^2 - y^2) = 0 \qquad 2yx^2(d^2 - x^2 - 2y^2) = 0$$

$$\boxed{d^2 - 2x^2 - y^2 = 0} \text{ — (2)} \qquad \boxed{d^2 - x^2 - 2y^2 = 0} \text{ — (3)}$$

Subtracting (3) from (2)

$$\cancel{d^2} - 2x^2 - y^2 = 0$$
$$\cancel{d^2} - x^2 - 2y^2 = 0$$
$$\quad + \quad +$$
$$-x^2 + y^2 = 0$$
$$x^2 = y^2$$
$$\boxed{x = y}$$

eq (2) becomes,

$$d^2 - 2x^2 - x^2 = 0$$
$$d^2 - 3x^2 = 0$$
$$d^2 = 3x^2.$$

$$\boxed{x = \frac{d}{\sqrt{3}}} \quad \therefore \quad \boxed{y = \frac{d}{\sqrt{3}}}$$

eq (A), $z = \sqrt{d^2 - x^2 - y^2}$

$$z = \sqrt{d^2 - \frac{d^2}{3} - \frac{d^2}{3}}$$

$$\boxed{z = \frac{d}{\sqrt{3}}}$$

$\therefore \left(\frac{d}{\sqrt{3}}, \frac{d}{\sqrt{3}}\right)$ are critical points

$$r = \frac{\partial^2 f}{\partial x^2} = 2y^2d^2 - 12x^2y^2 - 2y^4$$

at $\left(\frac{d}{\sqrt{3}}, \frac{d}{\sqrt{3}}\right)$, $r = 2\frac{d^2}{3} \times d^2 - 12\frac{d^2}{3}\frac{d^2}{3} - 2\frac{d^4}{9}$

$$= \frac{6d^4 - 12d^4 - 2d^4}{9} = \boxed{\frac{-8d^4}{9}}$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = 4yxd^2 - 8x^3y - 8y^3x.$$

$$\text{at}\left(\frac{d}{\sqrt3}, \frac{d}{\sqrt3}\right) = 4\frac{d}{\sqrt3}\frac{d}{\sqrt3}d^2 - 8\frac{d^3}{3\sqrt3}\frac{d}{\sqrt3} - 8\frac{d^3}{3\sqrt3}\frac{d}{\sqrt3}$$

$$= \frac{4d^4}{\sqrt3\sqrt3}\left[1 - \frac{2}{3} - \frac{2}{3}\right] = \frac{4d^4}{3}\left(\frac{3-2-2}{3}\right)$$

$$= \frac{4}{3}d^4 \times \frac{-1}{3} = \boxed{-\frac{4d^4}{9}}$$

$$t = \frac{\partial^2 f}{\partial y^2} = 2x^2d^2 - 2x^4 - 12y^2x^2$$

$$\text{at}\left(\frac{d}{\sqrt3}, \frac{d}{\sqrt3}\right) = 2\frac{d^2}{3}d^2 - 2\frac{d^4}{9} - 12\frac{d^2}{3}\frac{d^2}{3}$$

$$= \frac{6d^4 - 2d^4 - 12d^4}{9} = \boxed{-\frac{8d^4}{9}}$$

$$rt - s^2 = \frac{-8d^4}{9}\left(-\frac{8d^4}{9}\right) - \left(-\frac{4}{9}d^4\right)^2 = \frac{64d^8}{81} - \frac{16d^8}{81}$$

$$= \frac{\cancel{48}^{16}d^8}{\cancel{81}_{27}} = \boxed{\frac{16d^8}{27}}$$

$\because rt - s^2 > 0$ and $r < 0$

$\therefore$ V is maximum when ~~the~~ $x = y = z$ i.e when the rectangular solid is a cube.

Q7> Expand $\cos x \cos y$ in power of x & y as far as term of fourth degree.

Sol According to taylor's theorem, for two variables functions with:

$$f(a+h, b+k) = f(a,b) + \frac{1}{1!}\left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right)f(a,b) + \frac{1}{2!}\left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right)^2 f(a,b)$$

$$+ \frac{1}{3!}\left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right)^3 f(a,b) + \frac{1}{4!}\left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right)^4 f(a,b)$$

here, $a=0, b=0$, $h=x$, $k=y$

$$f(0+x,0+y)=f(0,0)+\left(x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}\right)f(0,0)+\frac{1}{2!}\left(x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}\right)^2 f(0,0)$$
$$+\frac{1}{3!}\left(x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}\right)^3 f(0,0)+\frac{1}{4!}\left(x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}\right)^4 f(0,0)$$

$$f(x,y)=f(0,0)+\left(x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}\right)f(0,0)+\frac{1}{2!}\left(x^2\frac{\partial^2 f}{\partial x^2}+y^2\frac{\partial^2 f}{\partial y^2}+2xy\frac{\partial^2 f}{\partial x\partial y}\right)$$
$$f(0,0)+\frac{1}{3!}\left(x^3\frac{\partial^3 f}{\partial x^3}+y^3\frac{\partial^3}{\partial y^3}+3x^2y\frac{\partial^3 f}{\partial x^2\partial y}+3xy^2\frac{\partial^3 f}{\partial x\partial y^2}\right)f(0,0)$$
$$+\frac{1}{4!}\left(x^4\frac{\partial^4 f}{\partial x^4}+y^4\frac{\partial^4}{\partial y^4}+4x^3y\frac{\partial^4 f}{\partial x^3\partial y}+4xy^3x\frac{\partial^4 f}{\partial x\partial y^3}+6x^2y^2\frac{\partial^4 f}{\partial x^2\partial y^2}\right)$$
$$f(0,0)$$

Given, $f(x,y)=\cos x\cos y$, $f(0,0)=1$

$f_x=\partial f/\partial x=-\sin x\cos y$, $\dfrac{\partial f(0,0)}{\partial x}=0$

$f_y(x,y)=-\sin y\cos x$, $f_y(0,0)=0$

$f_{xx}(x,y)=-\cos x\cos y$, $f_{xx}(0,0)=-1$

$f_{xy}(x,y)=\sin y\sin x$, $f_{xy}(0,0)=0$

$f_{yy}(x,y)=-\cos y\cos x$, $f_{yy}(0,0)=-1$

$f_{xxx}(x,y)=\sin x\cos y$, $f_{xxx}(0,0)=0$

$f_{xxy}(x,y)=\cos x\sin y$, $f_{xxy}(0,0)=0$

$f_{xyy}(x,y)=\sin x\cos y$, $f_{xyy}(0,0)=0$

$f_{yyy}(x,y)=\sin y\cos x$, $f_{yyy}(0,0)=0$

$f_{xxxx}(x,y)=\cos x\cos y$, $f_{xxxx}(0,0)=1$

$f_{xxxy}(x,y)=-\sin x\sin y$, $f_{xxxy}(0,0)=0$

$f_{xxyy}(x,y)=\cos x\cos y$, $f_{xxyy}(0,0)=1$

$f_{xyyy}(x,y)=-\sin y\sin x$, $f_{xyyy}(0,0)=0$

$f_{yyyy}(x,y)=\cos y\cos x$, $f_{yyyy}(0,0)=1$

∴ eq (A) becomes,

$$f(x,y) = f(0,0) + \left[x f_x(0,0) + y f_y(0,0)\right] f(0,0) + \frac{1}{2!}\left[x^2 f_{xx}(0,0) + y^2 f_{yy}(0,0) + 2xy f_{xy}(0,0)\right] f(0,0)$$

$$+ \frac{1}{3!}\left[x^3 f_{xxx}(0,0) + y^3 f_{yyy}(0,0) + 3x^2y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0)\right] f(0,0)$$

$$+ \frac{1}{4!}\left[x^4 f_{xxxx}(0,0) + y^4 f_{yyyy}(0,0) + 4x^3y f_{xxxy}(0,0) + 4xy^3 f_{xyyy}(0,0) + 6x^2y^2 f_{xxyy}(0,0)\right] f(0,0)$$

$$\cos x \cos y = 1 + \left[x(0) + y(0)\right] 1 + \frac{1}{2!}\left[x^2(-1) + y^2(-1) + 2xy(0)\right] 1$$

$$+ \frac{1}{3!}\left[x^3(0) + y^3(0) + 3x^2y(0) + 3xy^2(0)\right] 1$$

$$+ \frac{1}{4!}\left[x^4(1) + y^4(1) + 4x^3y(0) + 4xy^3(0) + 6x^2y^2(1)\right] 1$$

$$\cos x \cos y = 1 + 0 + \frac{1}{2!}\left[-x^2 - y^2 + 0\right] + \frac{1}{3!}\left[0 + 0 + 0 + 0\right] + \frac{1}{4!}\left[x^4 + y^4 + 6x^2y^2 + 0 + 0\right]$$

$$\boxed{\cos x \cos y = 1 - \frac{x^2}{2} - \frac{y^2}{2} + \frac{x^4}{24} + \frac{y^4}{24} + \frac{x^2y^2}{4}}$$