UNIT — 2

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Partial differential equation :-

(I) $z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$ —— (1)

Differentiating z partially w.r.to x and y.

w.r to x ——

$$\frac{\partial z}{\partial x} = 2f'\left(\frac{1}{x} + \log y\right)\left(-\frac{1}{x^2}\right)$$

∴ here, $\frac{\partial z}{\partial x} = p$

$$-px^2 = 2f'\left(\frac{1}{x} + \log y\right) \quad \text{—— (2)}$$

w.r.to y ——

$$\frac{\partial z}{\partial y} = 2y + 2f'\left(\frac{1}{x} + \log y\right)\frac{1}{y}$$

∴ here, $\frac{\partial z}{\partial y} = q$

$$q - 2y = 2f'\left(\frac{1}{x} + \log y\right)\frac{1}{y}$$

$$qy - 2y^2 = 2f'\left(\frac{1}{x} + \log y\right)$$

—— (3)

By Eliminating $2F'\left(\frac{1}{x}+\log y\right)$

From (2) and (3) equation, we get

$$-px^2 = qy - 2y^2$$

$$\boxed{2y^2 = qy + px^2}$$

(2) (i) Solve

$$x(y^2+z)p - y(x^2+z)q = z(x^2-y^2)$$

Here, $P = x(y^2+z)$, $Q = -y(x^2+z)$

$R = z(x^2-y^2)$

Its Auxillary equation is

$$\frac{\partial x}{P} = \frac{\partial y}{Q} = \frac{\partial z}{R}$$

$$\frac{\partial x}{x(x^2+z)} = \frac{\partial y}{-y(x^2+z)} = \frac{\partial z}{z(x^2-y^2)}$$

(1) Now the set of the multipliers is $(l, m, n) = (x, y, -1)$

we get,

each fraction $= \frac{l\,dx + m\,dy + n}{}$

each fraction $= \dfrac{l\,dx + m\,dy + n\,dz}{lP + mQ + nR}$

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each fraction $= \dfrac{x\,dx + y\,dy - 1\,dz}{x\cdot(xy^2+xz) + y(-yx^2 - yz) - (zx^2 - zy^2)}$

each fraction $= \dfrac{x\,dx + y\,dy - 1\,dz}{0}$

$0 = x\,dx + y\,dy - dz$

on intregating, we get

$\dfrac{x^2}{2} + \dfrac{y^2}{2} - z = c_1$

$x^2 + y^2 - 2z = 2c_1$

$\boxed{x^2 + y^2 - 2z = a}$

Now, again the set of multiplier is

$\dfrac{1}{x}, \dfrac{1}{y}, \dfrac{1}{z}$

each fraction $= \dfrac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$

$0 = \dfrac{1}{x}dy + \dfrac{1}{y}dy + \dfrac{1}{z}dz$

On integrating, w.

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$\log x + \log y + \log z = \log c_2$

$\log x + \log y + \log z = \log b$

$\cancel{\log}(x \cdot y \cdot z) = \cancel{\log} b$

$\boxed{x \cdot y \cdot z = b}$

Hence the genral solution is

$F(a, b) = 0$

$\boxed{F(x^2 + y^2 - 2z, \; x \cdot y \cdot z) = 0}$

② (ii) $y^2 p - xyq = x(z - 2y)$ — ①

Here, $P = y^2$, $Q = -xy$, $R = x(z - 2y)$

Its A.E is

$$\frac{\partial x}{P} = \frac{\partial y}{Q} = \frac{\partial z}{R}$$

$$\frac{\partial x}{y^2} = \frac{\partial y}{-xy} = \frac{\partial z}{x(z-2y)} \quad ②$$

(i) Taking first two member we get,

$$\frac{\partial x}{y^2} = \frac{\partial y}{-xy}$$

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$$\frac{\partial x}{y} = \frac{\partial y}{-x}$$

$$-x\,\partial x = y\,\partial y$$

On integrating.

$$\frac{-x^2}{2} + \frac{y^2}{2} = c_1$$

$$x^2 + y^2 = 2c_1$$

$$\boxed{x^2 + y^2 = a}$$

(ii) On taking last two members.

$$\frac{\partial y}{-\cancel{x}y} = \frac{\partial z}{\cancel{x}(z-2y)}$$

$$\frac{\partial y}{-y} = \frac{\partial z}{z-2y}$$

$$\frac{\partial y}{\partial z} = \frac{-y}{z-2y}$$

$$\frac{z-2y}{-y} = \frac{\partial z}{\partial y}$$

$$-\frac{z}{y} + 2 = \frac{\partial z}{\partial y}$$

$$\frac{\partial z}{\partial y} + \frac{z}{y} = 2$$

which is linear differential equation.

on comparing with

$$\frac{\partial z}{\partial y} + P y = Q$$

$$I.F = e^{\int P \, dy}$$

$$I.F = e^{\int \frac{1}{y} dy}$$

$$I.F = e^{\log y}$$

$$\boxed{I.F = y}$$

$$z.I.F = \int Q.I.F \, dy + b$$

$$z.y = \int 2.y \, dy + b$$

$$z.y = \frac{2y^2}{2} + b$$

$$z.y = y^2 + b$$

$$\boxed{zy - y^2 = b}$$

Hence, the genral solution is

$$F(a, b) = 0$$

$$\boxed{F(x^2+y^2, zy-y^2) = 0}$$

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(3) (i) $z = p^2 + q^2$

Given :- $z = p^2 + q^2$ —— (1)

By using standard form II because the given problem is in the form of standard form II.

ie, $F(z, p, q) = 0$

$z = f(X)$

where, $X = x + ay$

$\boxed{\frac{\partial z}{\partial X} = p}$ $\boxed{a\frac{\partial z}{\partial X} = q}$

Putting the value in eq (1)

$$z = \left(\frac{\partial z}{\partial X}\right)^2 + a^2\left(\frac{\partial z}{\partial X}\right)^2$$

$$z = \left(\frac{\partial z}{\partial X}\right)^2(1 + a^2)$$

$$\frac{z}{(1 + a^2)} = \left(\frac{\partial z}{\partial X}\right)^2$$

$$\frac{\sqrt{z}}{\sqrt{1 + a^2}} = \frac{\partial z}{\partial x}$$

$$\frac{\partial z}{\sqrt{z}} = \frac{\partial x}{\sqrt{1+a^2}}$$

on integrating, we get

$$\int z^{-1/2}\, dz = \frac{1}{\sqrt{1+a^2}} \int dx$$

$$\frac{z^{1/2}}{\frac{1}{2}} = \frac{1}{\sqrt{1+a^2}} x + b$$

where,

$b = \; + \sqrt{1+a^2}$

$$2\sqrt{z} = \frac{1}{\sqrt{1+a^2}} (x+b)$$

$$2\sqrt{z} = \frac{x+ay+b}{\sqrt{1+a^2}}$$

$$2\sqrt{z}\sqrt{1+a^2} = x+ay+b$$

$$\boxed{\sqrt{2}\, z \sqrt{1+a^2} = (x+ay+b)^2}$$

$z^2(p^2x^2+q^2y^2)=1$

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given :- $z^2(p^2x^2+q^2y^2)=1$

The given problem in the form of standard form II

$$F(z,p,q)=0$$

$$z=F(X)$$

$$X=x+ay$$

$$p=\frac{\partial z}{\partial X} \qquad q=a\frac{dz}{\partial X}$$

$$z^2\left[\left(\frac{dz}{dX}\right)^2x^2+a^2\left(\frac{dz}{dX}\right)^2y^2\right]=1$$

$$\left(\frac{dz}{dX}\right)^2\left[z^2x^2+a^2y^2z^2\right]=1$$

$$\sqrt{z^2x^2+a^2y^2z^2}=\frac{dX}{dz}$$

$$z\,dz=\frac{dX}{\sqrt{x^2+a^2y^2}}$$

on interigating

$$\frac{z^2}{2}=\frac{X}{\sqrt{x^2+a^2y^2}}+c)$$

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$\therefore \quad b = c_1 \cdot \sqrt{x^2 + a^2 \cdot y^2}$

$$\frac{z^2}{2} = \frac{x + b}{\sqrt{x^2 + a^2 \cdot y^2}}$$

$$\frac{z^2}{2} = \frac{x + ay + b}{\sqrt{x^2 + a^2 \cdot y^2}}$$

$$z = \sqrt{\frac{2x + 2ay + 2b}{x^2 + a^2 \cdot y^2}}$$

$$\boxed{z\,(x^2 + a^2 y^2) = \sqrt{2x + 2ay + 2b}}$$

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4(i) $z^2(p^2z^2+q^2)=1$

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The given problem in the form of Standard form II

$$f(z,p,q)=0$$

$z=f(X)$

where, $X = x+ay$

$p = \frac{dz}{dX}$ $\quad q = a\frac{dz}{dX}$

$$z^2\left[\left(\frac{dz}{dX}\right)^2 z^2 + a^2\left(\frac{dz}{dX}\right)^2\right]=1$$

$$\left(\frac{dz}{dX}\right)^2 z^2(z^2+a^2)=1$$

$$z^2(z^2+a^2) = \left(\frac{dX}{dz}\right)^2$$

$$z\sqrt{z^2+a^2} = \frac{dX}{dz}$$

$$z\sqrt{z^2+a^2}\,dz = dX$$

$$\frac{1}{3}(z^2+a^2)^{3/2} = X+b$$

$$\frac{1}{3}(z^2+a^2)^{3/2} = x+ay+b$$

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$$\sqrt{(z^2+a^2)^{3/2}} = 3x + 3ay + 3b$$

(ii) $x^2 p^2 + y^2 q^2 = z^2$

The given problem in the form of Standard Form II

$$F(z, p, q) = 0$$

$z = F(X)$

where, $X = x + ay$

$$\frac{dz}{dX} = p \qquad a\frac{dz}{dX} = q$$

$$x^2 \left(\frac{dz}{dX}\right)^2 + y^2 a^2 \left(\frac{dz}{dX}\right)^2 = z^2$$

$$\left(\frac{dz}{dX}\right)^2 \left[x^2 + a^2 y^2\right] = z^2$$

$$x^2 + a^2 y^2 = \left(\frac{dX}{dz}\right)^2 z^2$$

$$\sqrt{x^2 + a^2 y^2} = \frac{dX}{dz} \cdot z$$

$$\frac{dz}{z} = \frac{dX}{\sqrt{x^2 + a^2 y^2}}$$

On integrating —

$$\log z = \frac{x+b}{\sqrt{x^2+a^2y^2}}$$

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where $= b = c_1 \cdot \sqrt{x^2+a^2y^2}$

$$\log z = \frac{x+ay+b}{\sqrt{x^2+a^2y^2}}$$

(ii) $x^2p^2 + y^2q^2 = z^2$

Standard form II

$$x^2\left(\frac{dz}{dx}\right)^2 + y^2\left(\frac{dz}{dx}\right)^2 = z^2$$

$$\left(\frac{x}{z}\frac{dz}{dx}\right)^2 + \left(\frac{y}{z}\frac{dz}{d}\right)^2 = 1$$

$$\left(\frac{dz/z}{dx/x}\right)^2 + \left(\frac{dz/z}{d\,y/y}\right)^2 = 1$$

$x = \log x \quad y = \log y \quad z = \log z$

$$dx = \frac{1}{x}dx \quad dy = \frac{1}{y}dy \quad dz = \frac{1}{z}dz$$

$$\left(\frac{dz}{dX}\right)^2 + \left(\frac{dz}{dy}\right)^2 = 1$$

$$\frac{dz}{dX} = a \qquad \frac{dz}{dy} = b$$

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$\boxed{p = a}$ $\boxed{q = b}$

$$p^2 + q^2 = 1$$

$$f(p, q) = 0$$

$$f(a, b) = 0$$

$$a^2 + b^2 = 1 \quad \text{--- } (3)$$

$$z = ax + by + c$$

$$1 - a^2 = b^2$$

$$b = \sqrt{1 - a^2}$$

$$z = ax + \sqrt{1 - a^2}\, y + c$$

$$\boxed{z = a \log x + \sqrt{1 - a^2} \log y + c}$$

(9) Solve $(p^2+q^2)y = qz$ by using charpit's method.

Let $f = f(x, y, z, p, q)$

$$= (p^2+q^2)y - qz = 0.$$

So, that,

$$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = p^2+q^2, \quad \frac{\partial f}{\partial z} = -q$$

$$\frac{\partial f}{\partial p} = 2py \quad , \quad \frac{\partial f}{\partial q} = 2qy - z$$

A.E of charpits :-

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}}$$

$$\frac{dp}{-pq} = \frac{dq}{p^2+q^2+q(-q)} = \frac{dz}{-p(2py) - q(2qy-z)}$$

$$= \frac{dx}{-2py} = \frac{dy}{-2qy+z}$$

$$\frac{dp}{-pq} = \frac{dq}{p^2} = \frac{dz}{-2p^2y - 2q^2y + qz}$$

$$= \frac{dx}{-2py} = \frac{dy}{-2qy+z}$$

Taking first two numbers

$$\frac{dp}{-\cancel{p}q} = \frac{dq}{p^{\cancel{2}}}$$

$$\frac{dp}{-q} = \frac{dq}{p}$$

$$p\,dp = -q\,dq$$

$$p\,dp + q\,dq = 0$$

On integrating we get

$$\frac{p^2}{2} + \frac{q^2}{2} = c_1$$

$$p^2 + q^2 = 2c_1$$

$$\boxed{p^2 + q^2 = a} \quad \text{—} \; (2)$$

putting eq (2) in eq (1)
we get

$$a^2 y = qz$$

$$q = \frac{a^2 y}{z}$$

$$\cancel{\left(p^2 + \left(\frac{a^2 y}{z}\right)^2\right) y = \frac{a^2 y \, z}{z}}$$

$\cancel{p^2 = a^2 y}$

$$p^2 + \left(\frac{a^2 y}{z}\right)^2 = a^2$$

$$p^2 = a^2\left[1 - \frac{a^2 y^2}{z^2}\right]$$

$$p^2 = \frac{a^2}{z^2}\left[z^2 - a^2 y^2\right]$$

$$p = \frac{a}{z}\sqrt{z^2 - a^2 y^2}$$

∴ complete integral

$$dz = p\,dx + q\,dy$$

$$dz = \frac{a}{z}\sqrt{z^2 - a^2 y^2}\,dx + \frac{a^2 y}{z}\,dy$$

$$z \cdot dz = a\sqrt{z^2 - a^2 y^2}\,dx + a^2 y\,dy$$

$$\frac{z\,dz - a^2 y\,dy}{\sqrt{z^2 - a^2 y^2}} = a\,dx$$

$$z^2 - a^2 y^2 = t$$

$$2z\,dz - a^2\,2y\,dy = dt$$

$$2\left[z\,dz - a^2 y\,dy\right] = dt$$

$$\frac{1}{2}\int \frac{dt}{\sqrt{t}} = \int a\,dx$$

$$\frac{1}{2}\frac{t^{-1/2+1}}{-1/2+1} = ax+b$$

$$\cancel{\frac{1}{2}}\frac{t^{1/2}}{\cancel{\frac{1}{2}}} = ax+b$$

$$\sqrt{z^2-a^2b^2} = ax+b$$

$$z^2-a^2b^2 = (ax+b)^2$$

(10) Charpit's

$$px+qy=pq \quad \text{---} (1)$$

Let $F = F(p,q,x,y,z) = px+qy-pq = 0$

So, that

$$\frac{dF}{dx} = 0, \quad \frac{dF}{dy} = q, \quad \frac{dF}{dz} = 0$$

$$\frac{dF}{dp} = x-q \qquad \frac{dF}{dq} = y-p$$

A.E of charpit's

$$\frac{dp}{\frac{dF}{dx}+p\frac{dF}{dz}} = \frac{dq}{\frac{dF}{dy}+q\frac{dF}{dz}} = \frac{dz}{-p\frac{dF}{dp}-q\frac{dF}{dq}} = \frac{dx}{-\frac{dF}{dp}} = \frac{dy}{-\frac{dF}{dq}}$$

$$\frac{dp}{0} = \frac{dq}{q+q(0)} = \frac{dz}{-p(x-q)-q(y-p)} = \frac{dx}{(x-q)}$$

$= \frac{dy}{-y+p}$

$\Rightarrow \frac{dq}{q} = \frac{dz}{-px+pq-qy+qp} = \frac{dx}{-x+q}$

$= \frac{dy}{-y+p}$

Taking first two member

$\frac{dp}{0} = \frac{dq}{q}$

$dp = 0$

on integrating

$\int dp = \int 0$

$\boxed{p = a}$ —— (2)

putting eq (2) in (1)

$ax + qy = aq$

$ax = q(a-y)$

$\frac{ax}{(a-y)} = q$

Complete integrating / integral

$dz = p\,dx + q\,dy$

$dz = a\,dx + \frac{ax}{(a-y)}\,dy$

$$(a-y)\,dz = (a-y)\,a\,dx + q\,dy$$

$$\frac{(a-y)\,dz - q\,dy}{(a-y)} = a\,dx$$

$$dz - \frac{q}{(a-y)}\,dy = a\,dx$$

On Integrating

$$dz - \frac{ax}{(a-y)^2}\,dy = a\,dx$$

$$dz - \frac{ax}{\left(\frac{a}{y}-1\right)^2}\frac{1}{y^2}\,dy = a\,dx$$

$$dz - \frac{ax}{(a-y)^2}\,dy = a\,dx$$

$$z - ax\,\frac{(a-y)^{-1}}{-1}\,\frac{-y^2}{2} = ax+b$$

$$\boxed{z - \frac{1}{2}\frac{ax}{(a-y)}\,y^2 = ax+b}$$

(7) solve

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2} = y\cos x$$

For complete solution $z = C.F + P.I$

For C.F :-

Its A.E is

$$z = 1 \qquad D = m \qquad D' = 1$$

$$m^2 + m - 6 = 0$$
$$m^2 + 3m - 2m - 6 = 0$$
$$m(m+3) - 2(m+3) = 0$$
$$(m+3)(m-2) = 0$$
$$m = -3, 2$$ [roots are real unequal

$$C.F = f_1(y-3x) + f_2(y+2x)$$

$$P.I = \frac{1}{(D+3D')(D-2D')} y\cos x$$

$$= \frac{1}{D+3D'}\left[\frac{y\cos x}{(D-2D')}\right]$$

$D \quad D' = 0 \qquad m \quad = 0$

$m =$

$$y = -mx + c$$
$$y = -2x + c$$

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$$= \frac{1}{D+3D'}\left[\int (c-2x)\cos x \, dx\right]$$

$$= \frac{1}{D+3D'}\left[(c-2x)\sin x - \int\left[-2 \cdot \sin x\right]dx\right]$$

$$= \frac{1}{D+3D'}\left[(c-2x)\sin x \cdot \cos x\right]$$

$$= \frac{1}{D+3D'}\left[y\sin x - 2 \cdot \cos x\right]$$

$$D+3D' = 0$$

$$m+3=0, \quad m=-3$$

$$y = -mx + c$$

$$y = 3x + c$$

$$= \int (3x+c)\sin x \, dx - 2\int \cos x \, dx$$

$$= -(3x+c)\cos x - \int\left[3\,(-\cos x)\right]dx$$

$$-2\sin x$$

$$= -(3x+c)\cos x + 3\sin x - 2\sin x$$

$$= -(3x+c)\cos x + \sin x$$

$$= -y\cos x + \sin x$$

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$z = C.F + P.I$

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$z = f_1(y-3x) + f_2(y+2x) +$

$(-y \cos x + \sin x)$

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