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Imp Questions of Engineering Mathematics-I (BT-202)
Unit-1

1. Solve $(1+y^2)dx - (\tan^{-1}y - x)dy = 0$
2. Solve $(1+e^y)\cos x\, dx + e^y \sin x\, dy = 0$
3. Solve $(xy^5 + y)dx - dy = 0$
4. Solve $\frac{dy}{dx} = \frac{x^3+y^3}{xy^2}$
5. Solve $2y dx + (2x\log x - xy)dy = 0$
6. Solve $\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = x^2$
7. Solve $(D^2 - 2D + 2)y = x \quad , \frac{d}{dx} = D$
8. Solve $(D^3 - D^2 - 6D)y = 1 + x^2$
9. Solve $(D^2 - 2D + 1)y = e^x + \sin x \ , \frac{d}{dx} = D$
10. Solve $\frac{d^2y}{dx^2} + 4y = \sin^2 x$

Maths Mid Sem - I

Q1) Solve $(1+y^2)\,dx - (\tan^{-1}y - x)\,dy = 0$

Sol

$$(1+y^2)\,dx = (\tan^{-1}y - x)\,dy$$

$$\frac{dx}{dy} = \frac{\tan^{-1}y}{1+y^2} - \frac{x}{1+y^2}$$

$$\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1}y}{1+y^2}$$

which is a linear differential equation in x

$\therefore$ On comparing with $\frac{dx}{dy}$ $Px + Q$

$$P = \frac{1}{1+y^2}, \quad Q = \frac{\tan^{-1}y}{1+y^2}$$

Integrating factor i.e $IF = e^{\int P\,dy} = e^{\int \frac{1}{1+y^2}dy}$

$$\boxed{IF = e^{\tan^{-1}y}}$$

The complete sollution is

$$x\,IF = \int Q\,IF\,dy + C$$

$$x\,e^{\tan^{-1}y} = \int \frac{\tan^{-1}y}{1+y^2}\,e^{\tan^{-1}y}\,dy + C$$

Let $t = \tan^{-1}y$

$\frac{dt}{dy} = \frac{1}{1+y^2}$

$dy = (1+y^2)\,dt$

$$x\,e^{\tan^{-1}y} = \int \frac{t\,e^t}{\cancel{(1+y^2)}}\,\cancel{(1+y^2)}\,dt + C$$

$$x\,e^{\tan^{-1}y} = t\,e^t - \int 1\,e^t\,dt + C$$

$$= t\,e^t - e^t + C$$

$$= e^t(t-1) + C$$

$$x = \frac{e^{\tan^{-1}y}}{e^{\tan^{-1}y}}(\tan^{-1}y - 1) + \frac{C}{e^{\tan^{-1}y}}$$

$$\boxed{x = (\tan^{-1} y - 1) + \frac{C}{e^{\tan^{-1} y}}}$$

which is the required solution

Q2> Solve $(1+e^y)\cos x \, dx + e^y \sin x \, dy = 0$

Sol

$(1+e^y)\cos x \, dx + e^y \sin x \, dy = 0$

On comparing with $M\,dx + N\,dy = 0$

$M = (1+e^y)\cos x$, $N = e^y \sin x$

Condition for exact diff eq^n is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(1+e^y)\cos x$, $\frac{\partial N}{\partial x} = \frac{\partial}{\partial x} e^y \sin x$

$\boxed{\frac{\partial M}{\partial y} = \cos x \, e^y}$ $\boxed{\frac{\partial N}{\partial x} = e^y \cos x}$

$\because \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\therefore$ The given eq^n is exact diff eq^n

$\therefore$ The complete Solution is

$$\int_{y=\text{const}} M\,dx + \int N\,dy \text{ (Free from } x) = C$$

$$\int \cos x(1+e^y)\,dx + \int 0\,dy = C$$

$$\boxed{\sin x(1+e^y) = C}$$

Q3> Solve $(xy^5 + y)\,dx - dy = 0$

Sol

$(xy^5 + y)\,dx = dy$

$\frac{dy}{dx} = xy^5 + y$

$\frac{dy}{dx} - y = xy^5$

Dividing both sides by y^5

$\frac{1}{y^5}\frac{dy}{dx} - \frac{y}{y^5} = \frac{xy^5}{y^5}$

$\frac{1}{y^5}\frac{dy}{dx} - y^{-4} = x$

Let $t = y^{-4}$

$\frac{dt}{dx} = -4y^{-5}\frac{dy}{dx}$

$\frac{1}{y^5}\frac{dy}{dx} = \frac{-1}{4}\frac{dt}{dx}$

$\frac{-1}{4}\frac{dt}{dx} - t = x$

Multiplying -4 both sides

$\boxed{\frac{dt}{dx} + 4t = -4x}$ which is linear diff eq^n in t

~~On comparing with $\frac{dt}{dx} + P(1-n)t = (1-n)Q$~~

$P = 4$, $Q = -4x$

Integrating factor, $IF = e^{\int P\,dx} = e^{\int 4\,dx} = \boxed{e^{4x}}$

Complete sol^n is $t \cdot I.F = \int Q \, I.F \, dx + C$

$y^{-4} e^{4x} = \int -4x \, e^{4x}\,dx + C$

$y^{-4} e^{4x} = -4\left[x \frac{e^{4x}}{4} - \int 1 \frac{e^{4x}}{4}\,dx\right] + C$

$y^{-4} e^{4x} = -4\left[\frac{e^{4x}}{4}\left(x - \frac{1}{4}\right)\right] + C$

$$\boxed{\frac{1}{y^4} = \left(-x + \frac{1}{4}\right) + \frac{C}{e^{4x}}}$$

Q4) Solve $\frac{dy}{dx} = \frac{x^3+y^3}{xy^2}$

Sol $dy\,(xy^2) = (x^3+y^3)\,dx$

$$(x^3+y^3)\,dx - (xy^2)\,dy = 0 \quad -①$$

On Comparing with $M\,dx + N\,dy = 0$

$M = x^3+y^3$, $N = -xy^2$

$\frac{\partial M}{\partial y} = 3y^2$, $\frac{\partial N}{\partial x} = -y^2$

$\because \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

$\therefore$ The given eqn is not exact diff eqn but it is homogeneous

$\therefore IF = \frac{1}{Mx + Ny} = \frac{1}{(x^3+y^3)x + (-xy^2)y}$

$IF = \frac{1}{x^4 + xy^3 - xy^3}$

$$\boxed{IF = \frac{1}{x^4}}$$

Multiplying I.F. in eq ①

$$\left(\frac{1}{x} + \frac{y^3}{x^4}\right)dx - \frac{y^2}{x^3}\,dy = 0$$

On comparing with $M\,dx + N\,dy = 0$

$M = \frac{1}{x} + \frac{y^3}{x^4}$, $N = -\frac{y^2}{x^3}$

$\frac{\partial M}{\partial y} = \frac{3y^2}{x^4}$, $\frac{\partial N}{\partial x} = \frac{3y^2}{x^4}$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\therefore$ Now the eq^n is exact diff eq^n

Complete Sol^n is $\int_{y=\text{const}} M\,dx + \int N\,dy$ (free from x) $= C$

$$\int \left(\frac{1}{x} + \frac{y^3}{x^4}\right) dx + \int 0\,dy = C$$

$$\log x + y^3 \left(\frac{x^{-3}}{-3}\right) = C$$

$\boxed{\log x - \dfrac{y^3}{3x^3} = C}$ which is the required sol^n

Q5> Solve $2y\,dx + (2x\log x - xy)\,dy = 0$

Sol $2y\,dx + (2x\log x - xy)\,dy = 0$ —(1)

On Comparing with $M\,dx + N\,dy = 0$

$$M = 2y, \quad N = 2x\log x - xy$$

$$\frac{\partial M}{\partial y} = 2, \quad \frac{\partial N}{\partial x} = 2\left[x \times \frac{1}{x} + \log x \times 1\right] - y$$

$$\frac{\partial N}{\partial x} = 2(1 + \log x) - y$$

$$\because \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$\therefore$ The given eq^n is not exact diff eq^n but it is reducable to exact diff eq^n

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{2x\log x - xy}\left[2 - (2(1+\log x) - y)\right]$$

$$= \frac{\cancel{2} - \cancel{2} - 2\log x + y}{2x\log x - xy}$$

$$= \frac{-2\log x + y}{-x(-2\log x + y)}$$

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = -\frac{1}{x}$$

$$\therefore \boxed{f(x) = -\frac{1}{x}}$$

$$I.F. = e^{\int f(x)\,dx} = e^{\int -\frac{1}{x}dx} = e^{-\log x}$$

$$\boxed{I.F. = x^{-1}}$$

Multiplying I.F. in eqn (1)

$$\frac{2y}{x}\,dx + (2\log x - y)\,dy = 0$$

on comparing with $M\,dx + N\,dy = 0$

$$M = \frac{2y}{x}, \quad N = 2\log x - y$$

$$\frac{\partial M}{\partial y} = \frac{2}{x}, \quad \frac{\partial N}{\partial x} = \frac{2}{x}$$

$$\because \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\therefore$ Now the eqn is exact diff eqn

The complete soln is

$$\int_{y=\text{const}} M\,dx + \int N\,dy \text{ (free from } x) = C$$

$$\int \frac{2y}{x}\,dx + \int -y\,dy = C$$

$$\boxed{2y\log x - \frac{y^2}{2} = C}$$ which is the required soln

Q 6> Solve $\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = x^2$

Sol $(D^3 + 3D^2 + 2D)y = x^2$

The complete sol^n is $\boxed{y = CF + PI}$

For C.F. Put $D = m$, $y = 1$

Auxillary eq^n is $m^3 + 3m^2 + 2m = 0$

$m(m^2 + 3m + 2) = 0$

$m(m^2 + 2m + m + 2) = 0$

$m(m(m+2) + 1(m+2)) = 0$

$m(m+1)(m+2) = 0$

$\therefore m = 0, -1, -2$

$\because$ roots are real and unequal

$CF = C_1 e^{m_1 x} + C_2 e^{m_2 x} + C_3 e^{m_3 x}$

$\therefore$

$\boxed{CF = C_1 + C_2 e^{-x} + C_3 e^{-2x}}$

For PI

$$PI = \frac{1}{(D^3+3D^2+2D)} x^2$$

$$= \frac{1}{2D\left(1+\frac{D^3}{2D}+\frac{3D^2}{2D}\right)} x^2$$

$$= \frac{1}{2D}\left[1+\left(\frac{D^2}{2}+\frac{3D}{2}\right)\right]^{-1} x^2$$

$$\left[(1+x)^{-1} = 1-x+x^2-x^3+---\right]$$

$$PI = \frac{1}{2D}\left[1-\left(\frac{D^2}{2}+\frac{3D}{2}\right)+\left(\frac{D^2}{2}+\frac{3D}{2}\right)^2+---\right] x^2$$

$$PI = \frac{1}{2D}\left[x^2 - \frac{D^2x^2}{2} - \frac{3Dx^2}{2} + \frac{D^4x^2}{4} + \frac{9D^2x^2}{4} + \frac{3}{2}D^3x^2\right]$$

$$PI = \frac{1}{2D}\left[x^2 - \frac{2}{2} - \frac{6x}{2} + 0 + \frac{18}{4} + 0\right]$$

$$PI = \frac{1}{2D}\left[x^2 - 1 - 3x + \frac{9}{2}\right]$$

$$PI = \frac{1}{2}\int\left(x^2-3x-\frac{7}{2}\right)dx$$

$$\boxed{PI = \frac{1}{2}\left[\frac{x^3}{3} - \frac{3x^2}{2} + \frac{7}{2}x\right]}$$

$$y = CF + PI$$

$$\boxed{y = C_1 + C_2e^{-x} + C_3e^{-2x} + \frac{1}{2}\left[\frac{x^3}{3} - \frac{3}{2}x^2 + \frac{7}{2}x\right]}$$ Ans

Q 7> Solve $(D^2 - 2D + 2)\, y = x$, $\frac{d}{dx} = D$

Sol $(D^2 - 2D + 2)\, y = x$

The complete solⁿ is $\boxed{y = C.F. + P.I.}$ —①

For CF Put $D = m$, $y = 1$

Auxillary eqⁿ is $m^2 - 2m + 2 = 0$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{2 \pm \sqrt{4-8}}{2}$$

$$m = \frac{2 \pm \sqrt{-4}}{2} = 1 \pm \frac{2i}{2}$$

$$m = 1 \pm 1i$$

$$= \alpha \pm i\beta$$

$$\therefore \alpha = 1, \quad \beta = 1$$

$$CF = e^{\alpha x}\left[C_1 \cos \beta x + C_2 \sin \beta x\right]$$

$$\therefore \boxed{CF = e^{x}\left[C_1 \cos x + C_2 \sin x\right]}$$

For PI $PI = \frac{1}{(D^2 - 2D + 2)}\, x$

$$= \frac{1}{2\left(1 + \left(\frac{D^2}{2} - \frac{2D}{2}\right)\right)}\, x$$

$$= \frac{1}{2}\left[1 + \left(\frac{D^2}{2} - D\right)\right]^{-1} x$$

$$\left[(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots\right]$$

$$PI = \frac{1}{2}\left[1 - \left(\frac{D^2}{2} - D\right) + \left(\frac{D^2}{2} - D\right)^2 - \cdots\right] x$$

$$PI = \frac{1}{2}\left[x - \frac{D^2}{2}x + Dx + \frac{D^4}{4}x + D^2 x - D^3 x\right]$$

$$PI = \frac{1}{2}\left[x - 0 + 1 + 0 + 0 - 0\right]$$

$$\boxed{PI = \frac{x+1}{2}}$$

$y = CF + PI$

$$\boxed{y = e^{x}\left[C_1 \cos x + C_2 \sin x\right] + \frac{x+1}{2}}$$

Q8) Solve $(D^3 - D^2 - 6D)y = 1 + x^2$

Sol $(D^3 - D^2 - 6D)y = 1 + x^2$

Complete solⁿ is $\boxed{y = CF + PI}$

For CF Put $D = m$, $y = 1$

Auxillary eqⁿ is $m^3 - m^2 - 6m = 0$

$m(m^2 - m - 6) = 0$

$m(m^2 - 3m + 2m - 6) = 0$

$m(m-3)(m+2) = 0$

$m = -2, 0, 3$

$\because$ roots are real & unequal

$\therefore CF = C_1 e^{m_1 x} + C_2 e^{m_2 x} + C_3 e^{m_3 \cancel{x}}$

$$\boxed{CF = C_1 e^{-2x} + C_2 + C_3 e^{3x}}$$

For PI $PI = \frac{1}{(D^3 - D^2 - 6D)}(1+x^2)$

$$PI = \frac{1}{-6D\left(1 + \frac{D^2}{6D} - \frac{D^3}{6D}\right)}(1+x^2)$$

$$PI = \frac{-1}{6D}\left[1 + \left(\frac{D}{6} - \frac{D^2}{6}\right)\right]^{-1}(1+x^2)$$

~~PI~~ $\left[(1+x)^{-1} = 1 - x + x^2 - x^3 + - - -\right]$

$$PI = \frac{-1}{6D}\left[1 - \left(\frac{D}{6} - \frac{D^2}{6}\right) + \left(\frac{D}{6} - \frac{D^2}{6}\right)^2 - - - - -\right](1+x^2)$$

$$PI = \frac{-1}{6D}\left[1 - \frac{D}{6}(1+x^2) + \frac{D^2}{6}(1+x^2) + \frac{D^2}{36}(1+x^2) + \frac{D^4}{36}(1+x^2) - \frac{D^3}{18}(1+x^2)\right]$$

$$PI = \frac{-1}{6D}\left[1+x^2 - 0 - \frac{2x}{6} + 0 + \frac{2}{6} + 0 + \frac{2}{36} + 0 - 0\right]$$

$$PI = \frac{-1}{6D}\left[1+x^2 - \frac{x}{3} + \frac{1}{3} + \frac{1}{18}\right]$$

$$PI = \frac{-1}{6D}\left[x^2 - \frac{x}{3} + \frac{18+6+1}{18}\right]$$

$$PI = \frac{-1}{6}\int\left(x^2 - \frac{x}{3} + \frac{25}{18}\right)dx$$

$$\boxed{PI = \frac{-1}{6}\left(\frac{x^3}{3} - \frac{x^2}{6} + \frac{25}{18}x\right)}$$

$y = CF + PI$

$$\boxed{y = C_1 e^{-2x} + C_2 + C_3 e^{3x} - \frac{1}{6}\left[\frac{x^3}{3} - \frac{x^2}{6} + \frac{25}{18}x\right]}$$

Q9) $(D^2 - 2D + 1)y = e^x + \sin x$

Sol $(D^2 - 2D + 1)y = e^x + \sin x$

Complete solⁿ is $\boxed{y = CF + PI}$

For CF: Put $D = m$, $y = 1$

Auxillary eq^n is $m^2 - 2m + 1 = 0$

$$m^2 - m - m + 1 = 0$$

$$m(m-1) - 1(m-1) = 0$$

$$(m-1)(m-1) = 0$$

$$m = 1, 1$$

$\because$ roots are real & equal

$$CF = e^{mx}[C_1 + C_2 x]$$

$$\boxed{CF = e^{x}[C_1 + xC_2]}$$

For PI

$$PI = \frac{1}{(D^2 - 2D + 1)}(e^x + \sin x)$$

$$PI = \frac{1}{(D^2 - 2D + 1)}e^x + \frac{1}{(D^2 - 2D + 1)}\sin x$$

$$\left[\frac{1}{F(D)}e^x = \frac{1}{F(a)}e^x, \quad \frac{1}{F(D)}\sin ax = \frac{1}{F(-a^2)}\sin ax\right]$$

$$F(a) \neq 0 \qquad F(-a^2) \neq 0$$

$$PI = \frac{1}{\cancel{1} - 2 + \cancel{1}}e^x + \frac{1}{\cancel{-1} - 2D + \cancel{1}}\sin x$$

$\because$ it fails

$$\left[\therefore \frac{1}{F(D)}e^x = \frac{x}{F'(D)}e^x = \frac{x}{F'(a)}e^x, \quad F'(a) \neq 0\right]$$

$$PI = \frac{x}{2D - 2}e^x - \frac{1}{2D}\sin x$$

$\because$ it fails again

$$\left[\therefore \frac{1}{F(D)}e^x = \frac{x^2}{F''(D)}e^x = \frac{x^2}{F''(a)}e^x, \quad F''(a) \neq 0\right]$$

$$PI = \frac{x^2}{2}e^x - \frac{1}{2}\int \sin x \, dx$$

$$\boxed{PI = \frac{x^2}{2}e^x + \frac{\cos x}{2}}$$

$$y = CF + PI$$

$$\boxed{y = e^x(C_1 + xC_2) + \frac{x^2}{2}e^x + \frac{\cos x}{2}}$$

Q. Solve $\frac{d^2y}{dx^2}+4y=\sin^2x$

Solve $(D^2+4)y=\sin^2x$

$(D^2+4)y=0$

put $D=m$

$m^2+4=0$

$m=\pm 2i$

$CF=c_1\cos 2x+c_2\sin 2x$

$P.I.=\frac{1}{(D^2+4)}(\sin^2x)$ | $\cos 2x=1-2\sin^2x$, $\sin^2x=\frac{1-\cos 2x}{2}$

$=\frac{1}{D^2+4}\left(\frac{1-\cos 2x}{2}\right)$

$=\frac{1}{2}\left[\frac{1}{D^2+4}e^{0\cdot x}-\frac{1}{D^2+4}\cos 2x\right]$

$=\frac{1}{2}\left[\frac{1}{0+4}e^{0\cdot x}-\frac{1}{-4+4}\cos 2x\right]$

$=\frac{1}{2}\left[\frac{1}{4}e^{0x}-\frac{1}{2D}\cos 2x\right]$

$=\frac{1}{2}\left[\frac{1}{4}\cdot 1-\frac{1}{2}\int\cos 2x\,dx\right]$

| $\frac{1}{D}f(x)=\int f(x)dx$

$=\frac{1}{2}\left[\frac{1}{4}-\frac{1}{2}\cdot\frac{\sin 2x}{2}\right]$

$PI=\frac{1}{8}(1-\sin 2x)$

The complete solution $y=CF+PI$

$y=c_1\cos 2x+c_2\sin 2x+\frac{1}{8}(1-\sin 2x)$

Ans