Gyan Ganga Institute of Technology & Sciences, Jabalpur

Gyan Ganga College of Technology, Jabalpur

Imp Questions of Engineering Mathematics-I (BT-202)

Unit-1

1. Solve $(1+y^2)dx-(\tan^{-1}y-x)dy=0$
2. Solve $(1+e^y)\cos x\, dx+e^y \sin x\, dy=0$
3. Solve $(xy^5+y)dx-dy=0$
4. Solve $\frac{dy}{dx}=\frac{x^3+y^3}{xy^2}$
5. Solve $2ydx+(2x\log x-xy)dy=0$
6. Solve $\frac{d^3y}{dx^3}+3\frac{d^2y}{dx^2}+2\frac{dy}{dx}=x^2$
7. Solve $(D^2-2D+2)y=x \quad , \frac{d}{dx}=D$
8. Solve $(D^3-D^2-6D)y=1+x^2$
9. Solve $(D^2-2D+1)y=e^x+\sin x \; , \frac{d}{dx}=D$
10. Solve $\frac{d^2y}{dx^2}+4y=\sin^2x$

Maths Mid Sem - I

Q1> Solve $(1+y^2)\,dx - (\tan^{-1}y - x)\,dy = 0$

Sol

$$(1+y^2)\,dx = (\tan^{-1}y - x)\,dy$$

$$\frac{dx}{dy} = \frac{\tan^{-1}y}{1+y^2} - \frac{x}{1+y^2}$$

$$\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1}y}{1+y^2}$$

which is a linear differential equation in x

$\therefore$ On comparing with $\frac{dx}{dy}\ Px + Q$

$$P = \frac{1}{1+y^2}, \quad Q = \frac{\tan^{-1}y}{1+y^2}$$

Integrating factor i.e $IF = e^{\int P\,dy} = e^{\int \frac{1}{1+y^2}dy}$

$$\boxed{IF = e^{\tan^{-1}y}}$$

The complete solution is

$$x\,IF = \int Q\,IF\,dy + C$$

Let $t = \tan^{-1}y$

$\frac{dt}{dy} = \frac{1}{1+y^2}$

$dy = (1+y^2)\,dt$

$$x\,e^{\tan^{-1}y} = \int \frac{\tan^{-1}y}{1+y^2}\,e^{\tan^{-1}y}\,dy + C$$

$$x\,e^{\tan^{-1}y} = \int \frac{t\,e^t}{\cancel{(1+y^2)}}\,\cancel{(1+y^2)}\,dt + C$$

$$x\,e^{\tan^{-1}y} = t\,e^t - \int 1\,e^t\,dt + C$$

$$= t\,e^t - e^t + C$$

$$= e^t(t-1) + C$$

$$x = \frac{e^{\tan^{-1}y}}{e^{\tan^{-1}y}}(\tan^{-1}y - 1) + \frac{C}{e^{\tan^{-1}y}}$$

$$\boxed{x = (\tan^{-1} y - 1) \pm \frac{C}{e^{\tan^{-1} y}}}$$

which is the required solution

Q2> Solve $(1+e^y)\cos x\, dx + e^y \sin x\, dy = 0$

Sol

$$(1+e^y)\cos x\, dx + e^y \sin x\, dy = 0$$

On comparing with $M\,dx + N\,dy = 0$

$$M = (1+e^y)\cos x \quad , \quad N = e^y \sin x$$

Condition for exact diff eq^n is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(1+e^y)\cos x \quad , \quad \frac{\partial N}{\partial x} = \frac{\partial}{\partial x} e^y \sin x$$

$$\boxed{\frac{\partial M}{\partial y} = \cos x\, e^y} \qquad \boxed{\frac{\partial N}{\partial x} = e^y \cos x}$$

$$\because \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\therefore$ The given eq^n is exact diff eq^n

$\therefore$ The complete Solution is

$$\int_{y = \text{const}} M\, dx + \int N\, dy \; (\text{Free from } x) = C$$

$$\int \cos x (1+e^y)\, dx + \int 0\, dy = C$$

$$\boxed{\sin x (1+e^y) = C}$$

Q3> Solve $(xy^5 + y)\, dx - dy = 0$

Sol

$$(xy^5 + y)\, dx = dy$$

$$\frac{dy}{dx} = xy^5 + y$$

$$\frac{dy}{dx} - y = xy^5$$

Dividing both sides by y^5

$$\frac{1}{y^5}\frac{dy}{dx} - \frac{y}{y^5} = \frac{xy^5}{y^5}$$

$$\frac{1}{y^5}\frac{dy}{dx} - y^{-4} = x$$

Let $t = y^{-4}$

$$\frac{dt}{dx} = -4y^{-5}\frac{dy}{dx}$$

$$\frac{1}{y^5}\frac{dy}{dx} = \frac{-1}{4}\frac{dt}{dx}$$

$$\frac{-1}{4}\frac{dt}{dx} - t = x$$

Multiplying -4 both sides

$$\boxed{\frac{dt}{dx} + 4t = -4x}$$ which is linear diff eq^n in t

On comparing with $\frac{dt}{dx} + P(1-n)t = (1-n)Q$

$$P = 4 \quad , \quad Q = -4x$$

Integrating factor, $IF = e^{\int P\, dx} = e^{\int 4\, dx} = \boxed{e^{4x}}$

Complete sol^n is $t \cdot I.F = \int Q\, I.F\, dx + C$

$$y^{-4} e^{4x} = \int -4x\, e^{4x}\, dx + C$$

$$y^{-4} e^{4x} = -4\left[x \frac{e^{4x}}{4} - \int 1 \frac{e^{4x}}{4} dx\right] + C$$

$$y^{-4} e^{4x} = -4\left[\frac{e^{4x}}{4}\left(x - \frac{1}{4}\right)\right] + C$$

$$\boxed{\frac{1}{y^4} = \left(-x + \frac{1}{4}\right) + \frac{C}{e^{4x}}}$$

Q4) Solve $\frac{dy}{dx} = \frac{x^3+y^3}{xy^2}$

Sol $dy(xy^2) = (x^3+y^3)\,dx$

$$(x^3+y^3)\,dx - (xy^2)\,dy = 0 \quad -①$$

On Comparing with $M\,dx + N\,dy = 0$

$M = x^3+y^3$, $N = -xy^2$

$\frac{\partial M}{\partial y} = 3y^2$, $\frac{\partial N}{\partial x} = -y^2$

$\because \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

$\therefore$ The given eqn is not exact diff eqn but it is homogeneous

$\therefore IF = \frac{1}{Mx+Ny} = \frac{1}{(x^3+y^3)x + (-xy^2)y}$

$IF = \frac{1}{x^4+xy^3-xy^3}$

$$\boxed{IF = \frac{1}{x^4}}$$

Multiplying I.F. in eq ①

$$\left(\frac{1}{x} + \frac{y^3}{x^4}\right)dx - \frac{y^2}{x^3}\,dy = 0$$

On comparing with $M\,dx + N\,dy = 0$

$M = \frac{1}{x} + \frac{y^3}{x^4}$, $N = -\frac{y^2}{x^3}$

$\frac{\partial M}{\partial y} = \frac{3y^2}{x^4}$, $\frac{\partial N}{\partial x} = \frac{3y^2}{x^4}$

$$\because \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\therefore$ Now the eq^n is exact diff eq^n

Complete Sol^n is $\int_{y=const} M\,dx + \int N\,dy$ (free from x) $= C$

$$\int \left(\frac{1}{x} + \frac{y^3}{x^4}\right) dx + \int 0\,dy = C$$

$$\log x + y^3\left(\frac{x^{-3}}{-3}\right) = C$$

$\boxed{\log x - \frac{y^3}{3x^3} = C}$ which is the required sol^n

Q5> Solve $2y\,dx + (2x\log x - xy)\,dy = 0$

Sol $2y\,dx + (2x\log x - xy)\,dy = 0$ —(1)

On Comparing with $M\,dx + N\,dy = 0$

$$M = 2y, \quad N = 2x\log x - xy$$

$$\frac{\partial M}{\partial y} = 2, \quad \frac{\partial N}{\partial x} = 2\left[x \times \frac{1}{x} + \log x \times 1\right] - y$$

$$\frac{\partial N}{\partial x} = 2(1 + \log x) - y$$

$$\because \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$\therefore$ The given eqn is not exact diff eqn but it is reducable to exact diff eqn

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{2x\log x - xy}\left[2 - (2(1+\log x) - y)\right]$$

$$= \frac{\cancel{2} - \cancel{2} - 2\log x + y}{2x\log x - xy}$$

$$= \frac{-2\log x + y}{-x(-2\log x + y)}$$

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = -\frac{1}{x}$$

$$\therefore \boxed{f(x) = -\frac{1}{x}}$$

$$I.F. = e^{\int f(x)\,dx} = e^{\int -\frac{1}{x}dx} = e^{-\log x}$$

$$\boxed{I.F. = x^{-1}}$$

Multiplying I.F. in eqn (1)

$$\frac{2y}{x}\,dx + (2\log x - y)\,dy = 0$$

On comparing with $M\,dx + N\,dy = 0$

$$M = \frac{2y}{x}, \quad N = 2\log x - y$$

$$\frac{\partial M}{\partial y} = \frac{2}{x}, \quad \frac{\partial N}{\partial x} = \frac{2}{x}$$

$$\because \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\therefore$ Now the eqn is exact diff eqn

The complete solⁿ is

$$\int_{y=\text{const}} M\,dx + \int N\,dy \text{ (free from } x) = C$$

$$\int \frac{2y}{x}\,dx + \int -y\,dy = C$$

$$\boxed{2y\log x - \frac{y^2}{2} = C}$$ which is the required solⁿ

Q 6> Solve $\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = x^2$

Sol $(D^3 + 3D^2 + 2D)y = x^2$

The complete solⁿ is $\boxed{y = CF + PI}$

For C.F. Put $D = m$, $y = 1$

Auxillary eqⁿ is $m^3 + 3m^2 + 2m = 0$

$m(m^2 + 3m + 2) = 0$

$m(m^2 + 2m + m + 2) = 0$

$m(m(m+2) + 1(m+2)) = 0$

$m(m+1)(m+2) = 0$

$\therefore m = 0, -1, -2$

$\because$ roots are real and unequal

$CF = C_1 e^{m_1 x} + C_2 e^{m_2 x} + C_3 e^{m_3 x}$

$\therefore$

$$\boxed{CF = C_1 + C_2 e^{-x} + C_3 e^{-2x}}$$

For PI

$$PI = \frac{1}{(D^3+3D^2+2D)} x^2$$

$$= \frac{1}{2D\left(1+\frac{D^3}{2D}+\frac{3D^2}{2D}\right)} x^2$$

$$= \frac{1}{2D}\left[1+\left(\frac{D^2}{2}+\frac{3D}{2}\right)\right]^{-1} x^2$$

$$\left[(1+x)^{-1} = 1-x+x^2-x^3+---\right]$$

$$PI = \frac{1}{2D}\left[1-\left(\frac{D^2}{2}+\frac{3D}{2}\right)+\left(\frac{D^2}{2}+\frac{3D}{2}\right)^2+---\right]x^2$$

$$PI = \frac{1}{2D}\left[x^2-\frac{D^2x^2}{2}-\frac{3Dx^2}{2}+\frac{D^4x^2}{4}+\frac{9D^2x^2}{4}+\frac{3}{2}D^3x^2\right]$$

$$PI = \frac{1}{2D}\left[x^2-\frac{2}{2}-\frac{6x}{2}+0+\frac{18}{4}+0\right]$$

$$PI = \frac{1}{2D}\left[x^2-1-3x+\frac{9}{2}\right]$$

$$PI = \frac{1}{2}\int\left(x^2-3x-\frac{7}{2}\right)dx$$

$$\boxed{PI = \frac{1}{2}\left[\frac{x^3}{3}-\frac{3x^2}{2}+\frac{7}{2}x\right]}$$

$$y = CF + PI$$

$$\boxed{y = C_1 + C_2e^{-x} + C_3e^{-2x} + \frac{1}{2}\left[\frac{x^3}{3}-\frac{3}{2}x^2+\frac{7}{2}x\right]}$$ Ans

Q 7> Solve $(D^2 - 2D + 2)\, y = x$, $\frac{d}{dx} = D$

Sol $(D^2 - 2D + 2)\, y = x$

The complete solⁿ is $\boxed{y = C.F. + P.I.}$ —(1)

For CF Put $D = m$, $y = 1$

Auxillary eqⁿ is $m^2 - 2m + 2 = 0$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{2 \pm \sqrt{4-8}}{2}$$

$$m = \frac{2 \pm \sqrt{-4}}{2} = \frac{1 \pm 2i}{2}$$

$$m = 1 \pm 1i$$

$$= \alpha \pm i\beta$$

$$\therefore \alpha = 1, \quad \beta = 1$$

$$CF = e^{\alpha x}\left[C_1 \cos \beta x + C_2 \sin \beta x\right]$$

$$\therefore \boxed{CF = e^{x}\left[C_1 \cos x + C_2 \sin x\right]}$$

For PI

$$PI = \frac{1}{(D^2 - 2D + 2)}\, x$$

$$= \frac{1}{2\left(1 + \left(\frac{D^2}{2} - \frac{2D}{2}\right)\right)}\, x$$

$$= \frac{1}{2}\left[1 + \left(\frac{D^2}{2} - D\right)\right]^{-1} x$$

$$\left[(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots\right]$$

$$PI = \frac{1}{2}\left[1 - \left(\frac{D^2}{2} - D\right) + \left(\frac{D^2}{2} - D\right)^2 - \cdots\right] x$$

$$PI = \frac{1}{2}\left[x - \frac{D^2}{2}x + Dx + \frac{D^4}{4}x + D^2 x - D^3 x\right]$$

$$PI = \frac{1}{2}\left[x - 0 + 1 + 0 + 0 - 0\right]$$

$$\boxed{PI = \frac{x+1}{2}}$$

$$y = CF + PI$$

$$\boxed{y = e^{x}\left[C_1 \cos x + C_2 \sin x\right] + \frac{x+1}{2}}$$

Q8> Solve $(D^3 - D^2 - 6D)y = 1 + x^2$

Sol $(D^3 - D^2 - 6D)\,y = 1 + x^2$

Complete solⁿ is $\boxed{y = CF + PI}$

For CF Put $D = m$, $y = 1$

Auxillary eqⁿ is $m^3 - m^2 - 6m = 0$

$m(m^2 - m - 6) = 0$

$m(m^2 - 3m + 2m - 6) = 0$

$m(m-3)(m+2) = 0$

$m = -2, 0, 3$

$\because$ roots are real & unequal

$\therefore CF = C_1 e^{m_1 x} + C_2 e^{m_2 x} + C_3 e^{m_3 x}$

$$\boxed{CF = C_1 e^{-2x} + C_2 + C_3 e^{3x}}$$

For PI $PI = \frac{1}{(D^3 - D^2 - 6D)}(1 + x^2)$

$$PI = \frac{1}{-6D\left(1 + \frac{D^2}{6D} - \frac{D^3}{6D}\right)}(1 + x^2)$$

$$PI = \frac{-1}{6D}\left[1 + \left(\frac{D}{6} - \frac{D^2}{6}\right)\right]^{-1}(1 + x^2)$$

$\left[(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots\right]$

$$PI = \frac{-1}{6D}\left[1 - \left(\frac{D}{6} - \frac{D^2}{6}\right) + \left(\frac{D}{6} - \frac{D^2}{6}\right)^2 - \cdots\right](1 + x^2)$$

$$PI = \frac{-1}{6D}\left[1 - \frac{D}{6}(1+x^2) + \frac{D^2}{6}(1+x^2) + \frac{D^2}{36}(1+x^2) + \frac{D^4}{36}(1+x^2) - \frac{D^3}{18}(1+x^2)\right]$$

$$PI = \frac{-1}{6D}\left[1 + x^2 - 0 - \frac{2x}{6} + 0 + \frac{2}{6} + 0 + \frac{2}{36} + 0 - 0\right]$$

$$PI = \frac{-1}{6D}\left[1 + x^2 - \frac{x}{3} + \frac{1}{3} + \frac{1}{18}\right]$$

$$PI = \frac{-1}{6D}\left[x^2 - \frac{x}{3} + \frac{18 + 6 + 1}{18}\right]$$

$$PI = \frac{-1}{6}\int\left(x^2 - \frac{x}{3} + \frac{25}{18}\right)dx$$

$$\boxed{PI = \frac{-1}{6}\left(\frac{x^3}{3} - \frac{x^2}{6} + \frac{25}{18}x\right)}$$

$$y = CF + PI$$

$$\boxed{y = C_1 e^{-2x} + C_2 + C_3 e^{3x} - \frac{1}{6}\left[\frac{x^3}{3} - \frac{x^2}{6} + \frac{25}{18}x\right]}$$

Q9> $(D^2 - 2D + 1)\,y = e^x + \sin x$

Sol $(D^2 - 2D + 1)\,y = e^x + \sin x$

Complete solⁿ is $\boxed{y = CF + PI}$

For CF Put $D=m$, $y=1$

Auxillary eq^n is $m^2-2m+1=0$

$$m^2-m-m+1=0$$

$$m(m-1)-1(m-1)=0$$

$$(m-1)(m-1)=0$$

$$m=1,1$$

$\because$ roots are real & equal

$$CF = e^{mx}[C_1 + C_2 x]$$

$$\boxed{CF = e^{x}[C_1 + xC_2]}$$

For PI

$$PI = \frac{1}{(D^2-2D+1)}(e^x+\sin x)$$

$$PI = \frac{1}{(D^2-2D+1)}e^x + \frac{1}{(D^2-2D+1)}\sin x$$

$$\left[\frac{1}{F(D)}e^x = \frac{1}{F(a)}e^x, \quad \frac{1}{F(D)}\sin ax = \frac{1}{F(-a^2)}\sin ax\right]$$

$F(a)\neq 0$ $\qquad F(-a^2)\neq 0$

$$PI = \frac{1}{\cancel{1}-2+\cancel{1}}e^x + \frac{1}{\cancel{-1}-2D+\cancel{1}}\sin x$$

$\because$ it fails

$$\therefore \frac{1}{F(D)}e^x = \frac{x}{F'(D)}e^x = \frac{x}{F'(a)}e^x, \quad F'(a)\neq 0$$

$$PI = \frac{x}{2D-2}e^x - \frac{1}{2D}\sin x$$

$\because$ it fails again

$$\therefore \frac{1}{F(D)}e^x = \frac{x^2}{F''(D)}e^x = \frac{x^2}{F''(a)}e^x, \quad F''(a)\neq 0$$

$$PI = \frac{x^2}{2}e^x - \frac{1}{2}\int \sin x\, dx$$

$$\boxed{PI = \frac{x^2 e^x}{2} + \frac{\cos x}{2}}$$

$y = CF + PI$

$$\boxed{y = e^x(C_1 + x C_2) + \frac{x^2 e^x}{2} + \frac{\cos x}{2}}$$

Q 10) $\frac{d^2y}{dx^2} + 4y = \sin^2 x$

Sol $(D^2 + 4)\, y = \sin^2 x$

Complete solⁿ is $\boxed{y = C.F. + P.I.}$

For CF Put $D = m$, $y = 1$

Auxillary eqⁿ is $(m^2+4) = 0$

$m = \pm 2i$

$= \alpha \pm i\beta$

$\alpha = 0$, $\beta = 2$

$CF = e^{\alpha x}[C_1 \cos \beta x + C_2 \sin \beta x]$

$\therefore \boxed{CF = C_1 \cos 2x + C_2 \sin 2x}$

For PI $PI = \frac{1}{(D^2+4)} \sin^2 x = \frac{1}{(D^2+4)} \frac{(1-\cos 2x)}{2}$ $1-\cos 2x = 2\sin^2 x$

$\sin^2 x = \frac{1}{2} - \frac{\cos 2x}{2}$

$PI = \frac{1}{2(D^2+4)} e^{0x} - \frac{1}{2(D^2+4)} \cos 2x$

$\left[\cancel{PI = \frac{1}{f(D)} Q, \text{ If } Q = 0 \quad PI = 0} \quad , \frac{1}{F(D)} \cos ax = \frac{1}{F(-a^2)} \cos ax, \ F(-a^2) \neq 0 \right]$

$PI = \frac{1}{2(0^2+4)} - \frac{1}{2(-4+4)} \cos 2x$

$\because$ It fails,

$$PI = \frac{1}{F(D)} \cos ax = \frac{x}{F'(D)} \cos ax = \frac{x}{F'(-a^2)} \cos ax, \quad F'(-a^2) \neq 0$$

$$PI = \frac{1}{8} - \frac{x}{2(2D)} \cos 2x$$

$$PI = \frac{1}{8} - \frac{x}{4D} \cos 2x$$

$$PI = \frac{1}{8} - \frac{x}{4} \int \cos 2x \, dx$$

$$\boxed{PI = \frac{1}{8} - \frac{x}{8} \sin 2x}$$

$y = CF + PI$

$$\boxed{y = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{8} - \frac{x}{8} \sin 2x}$$