Chapter 3

UNIT-III

BIVARIATE DISTRIBUTIONS

§ 3.1. BIVARIATE DISTRIBUTION

Suppose we measure the heights and weight of a certain group of persons then here we have two variables–one variable relating to height and other relating to weight, such a distribution is called a bivariate distribution.

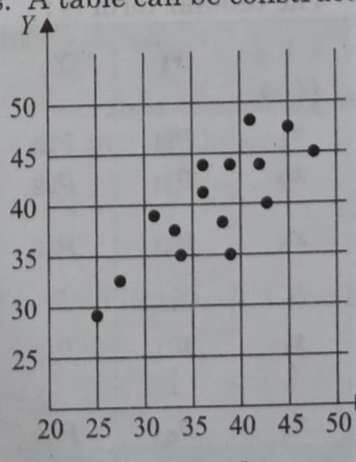
Definition. A universe, every member of which bears one of the values of each of two variates is called **bivariate distribution.**

§ 3.2. BIVARIATE FREQUENCY DISTRIBUTION

It may be possible that a particular value occurs more than once so that corresponding to one value (x_1, y_1) (say) there will be a number of dots. Suppose (x_1, y_1) occurs f_1 times then f_1 is called the frequency of (x_1, y_1). If these values are grouped according to the class-intervals then the frequency distribution so obtained is called a *bivariate frequency distribution.*

§ 3.3. BIVARIATE FREQUENCY TABLE

If the number of measurements is large, then it is convenient to choose some class intervals for the measurement of such variables. A table can be constructed from the dot diagram by sub-dividing the co-ordinate area into equal rectangular compartments and then writing within each compartment the number of dots which fall within it.



Let x and y be two variables and suppose that there exists a correlation between them.

Let the points $(x_1, y_1), (x_2, y_2), (x_3, y_3), \ldots$ correspond to the values of two variables. Plot these points on a graph paper referred to two perpendicular axes. Here the values of one of the variables (independent variable) are taken along x-axis and the other along y-axis. Such a graphical representation is said to be a **Scatter** or **Dot diagram**, in other words the diagram of dots so obtained is called a **scatter** or **dot diagram**.

For example. Following are data of marks of students in Analysis and Statistics at the B.E. examination, maximum marks of each subject being 50, to draw a dot diagram :

Roll No.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Marks in Analysis	28	42	39	41	25	38	45	33	37	34	47	34	39	36	41
Marks in Statistics	32	40	35	47	39	43	47	39	38	35	45	36	43	41	43

Now plot the points (28, 32), (42, 40), (39, 35), ... on a graph paper as shown in the figure on previous page. Here you are free to choose x and y. We have taken marks in Analysis as x's while marks in Statistics as y's.

By inspection of the graph, we find that there is tendency for small values of x to be related with small values of y and for large values of x to be related with large values of y.

Also the general trend of the dots is of a straight line.

For example. In the example above, if we take intervals 25–30, 30–35, 35–40, 40–45, 45–50 for the both x and y then the dots can be arranged in a tabular form as follows :

x / y	25–30	30–35	35–40	40–45	45–50
25–30	1				
30–35	1				
35–40		3	2		
40–45			3	2	
45–50				1	2

§ 3.4. JOINT PROBABILITY

A discrete bivariate distribution represents the joint probability distribution of a pair of random variables.

Two random variable U and V are said to be jointly distributed if they are defined on same probability spaces. The joint probability function is denoted by $P_{UV}(x, y)$ or $f_{UV}(x, y)$ where $x \in U$, $y \in V$.

§ 3.5. JOINT PROBABILITY MASS FUNCTION

The tabular form is :

$V \downarrow$ $U \rightarrow$	y_1	y_2	y_3	...	y_j	...	y_m	Total	
x_1	p_{11}	p_{12}	p_{13}	...	p_{1j}	...	p_{1m}	p_1	$f_U(x_1)$
x_2	p_{21}	p_{22}	p_{23}	...	p_{2j}	...	p_{2m}	p_2	$f_U(x_2)$
x_3	p_{31}	p_{32}	p_{33}	...	p_{3j}	...	p_{3m}	p_3	$f_U(x_3)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$	$\vdots$
x_i	p_{i1}	p_{i2}	p_{i3}	...	p_{ij}	...	p_{im}	p_i	$f_U(x_i)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$				$\vdots$	$\vdots$	$\vdots$
x_n	p_{n1}	p_{n2}	p_{n3}	...	p_{nj}	...	p_{nm}	p_n	$f_U(x_n)$
Total	p_1	p_2	p_3	...	p_j	...	p_m		
	$f_V(y_1)$	$f_V(y_2)$	$f_V(y_3)$	...	$f_V(y_j)$		$f_V(y_m)$		

It must be true that

$$\sum_{i=1}^{n}\sum_{j=1}^{m} p(x_i, y_j) = \sum_{i=1}^{n}\sum_{j=1}^{m} P[U = x, V = y] = 1 \text{ and } p(x_i, y_j) \geq 0.$$

Let U and V be random variables on some, probability sample space S, where

$$U = \{x_1, x_2, \ldots, x_n\} \in S \text{ and } V = \{y_1, y_2, \ldots, y_n\} \in S.$$

Then joint probability mass function of U and V defined as

$$p_{ij} = P(U = x_i \cup V = y_j) = p(x_i, y_j) \quad \ldots(1)$$

Cross product $U(S) \times V(S) = \{x_1, x_2, \ldots, x_n\} \times \{y_1, y_2, \ldots, y_n\}$.

Thus, the function $p(x, y) = P[U = x, V = y]$ for all real number x and y is the joint probability distribution of U and V.

Note. The function $p(x_i, y_j)$ which gives the relative frequencies of the occurrence of the pair (x_i, y_j) is called bivariate frequencies function.

In case of discrete random variables, bivariate distribution represents in a tabular form containing m rows and n columns, the elements in the cell denotes the joint probability of the x and y values.

As, the sum of all probabilities is 1. Since $p(x_i, y_j)$ or simply $p(x, y)$ is probability distribution, it must sum to 1 and lies between 0 and 1.

Note. $p(x_i, y_j)$ may be written as $p(x, y)$ for simplicity.

Definition. Let U and V be two discrete random variable and S denotes sample space. Then, the function $p(x_i, y_j) = p(x, y) = P[U = x, V = y]$ is a joint probability mass function (p.m.f.) if it satisfies.

(i) $0 \leq p(x, y) \leq 1$ (ii) $\sum_{x}\sum_{y} p(x, y) = 1$

(iii) $P[(U, V) \in A] = \sum_{x}\sum_{y} p(x, y)$, where A is subset of space S.

§ 3.6. MARGINAL PROBABILITY DISTRIBUTION

The sum of the probabilities of the row gives the probability distribution of the random variables U (called the marginal probability distribution of U) *i.e.*,

$$P_U(x) = f_U(x) = p_x(x_i) = p(U = x_i) = \sum_{j=1}^{m} p_{ij} = p_{i\cdot} \quad \ldots(3)$$

Adding probabilities down the column you get the probability distribution of random variable V (called the marginal distribution of V)

$$P_V(y) = f_V(y) = p_y(y_j) = p(V = y_j) = \sum_{i=1}^{n} p_{ij} = p_{\cdot j} \quad \ldots(4)$$

$f_U(x)$ and $f_V(y)$ is called Marginal probability distribution of U and V respectively. Some time it is also written as $P_U(x)$ and $P_V(y)$.

In other words,

$$f_U(x) = \sum_y p(x, y) = \sum_y P[U = x, V = y]$$

$$f_V(y) = \sum_x p(x, y) = \sum_x P[U = x, V = y].$$

◈ § 3.7. CONDITIONAL PROBABILITY DISTRIBUTION

The conditional probability function of U, when $V = y_j$ is given by

$$f_{U/V} = (x / y) = P(U = x / V = y)$$

$$= \frac{P(U = x_i \cap V = y_j)}{P(V = y)}$$

$$= \frac{P[U = x_i, V = y_j]}{P[V = y]} = \frac{p(x_i, y_j)}{p(y_j)}$$

$$= \frac{p_{ij}}{p_j}. \quad \text{...(5)}$$

The conditional probability function of V when $U = x_i$ is given by

$$f_{V/U}(y / x) = P(V = y_j / U = x_i)$$

$$= \frac{P(V = x_i \cap Y = y_j)}{P(X = x_i)}$$

$$= \frac{P(U = x_i, Y = y_j)}{P(X = x_i)}$$

$$\frac{p(x_i, y_j)}{p(x_i)} = \frac{p_{ij}}{p_i} \quad \text{...(6)}$$

Equation (5) gives the conditional probability of the event $[U = x]$ given the event $[V = y]$.

Equation (6) gives the conditional probability of the event $[V = y]$ given the event $[U = x]$.

◈ § 3.8. INDEPENDENT

Two discrete random variable U and V are said to be independent if

$$P(U = x_i, V = y_j) = P(U = x_i)\, P(V = y_j) \quad \text{...(7)}$$

Otherwise U and V are called dependent.

◈ § 3.9. CONTINUOUS BIVARIATE RANDOM VARIABLE

Continuous bivariate joint density function defines the probability distribution for a pair of random variables.

Let U and V be two continuous random variables on any sample space S.

Then the function (U, V) is called a two dimensional random variable or bivariate random variable. The joint probability density function or joint distribution function of U and V, denoted by $f_{UV}(x, y)$ end defines as

$$F_{UV}(x, y) = P[U \le x \cap V \le y]$$

or $$F_{UV}(x, y) = P(U \le x, V \le y)\ \forall\ x, y \in R \quad \text{...(1)}$$

Two continuous variables U and V are jointly continuous if there exists a non-negative function $f_{UV}(x, y)$ or simply $f(x, y)$ defined on real number x and y such that

$$F_{UV}(x, y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{UV}(u, v)\, du\, dv \quad -\infty < x, y < \infty$$

or
$$F_{UV}(x, y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(u, v)\, du\, dv, \;\; \forall \;\; (x, y) \in R^2. \quad \text{...(2)}$$

The function $f_{UV}(x, y)$ or $f(x, y)$ is called the joint probability density function of U and V. Note that $f_{UV}(x, y) \geq 0$ and $-\infty < x, y < \infty$.

The function $f_{UV}(x, y)$ or $f(x, y)$ is joint probability density function of U and V, which satisfied the following properties :

(i) $f_{UV}(x, y) \geq 0$ or $f(x, y) \geq 0, \;\; \forall \;\; (x, y) \in R^2$

(ii) $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{U,V}(x, y)\, dx\, dy = 1$

(iii) $P\,[(U, V) \in A] = \iint_A f(x, y)\, dx\, dz$

or $\iint_A f_{U,V}(x, y)\, dx\, dy,$

where $(U, V) \in A$ is an event in the xy-plane.

◈ § 3.10. ELEMENTARY PROPERTIES OF JOINT DISTRIBUTION FUNCTION

The joint distribution function $f_{U,V}(x, y) = P\,[U \leq x, V \leq y]$ of two variables has the following properties :

1. If $a_1 < b_1$ and $a_2 \leq b_2$ then

$$P\,[a_1 \leq U < b_1, a_2 < V < b_2]$$
$$= F(b_1, b_2) - F(b_1, a_2) - F(a_1, b_2) + F(a_1, a_2) \geq 0. \quad \text{...(3)}$$

This properties is known as rectangle rule

2. $F_{U,V}(-\infty, -\infty) = \lim_{\substack{x \to -\infty \\ y \to -\infty}} F_{U,V}(x, y) = 0$ ...(4)

3. $F_{U,V}(\infty, \infty) = \lim_{\substack{x \to \infty \\ y \to \infty}} F_{U,V}(x, y) = 1$ ...(5)

4. $F_{U,V}(-\infty, y) = \lim_{x \to -\infty} F_{UV}(x, y) = 0 \;\forall\; x, y \in R.$ ...(6)

5. $F_{U,V}(x, -\infty) = \lim_{y \to -\infty} F_{UV}(x, y) = 0 \;\forall\; x \in R.$ ...(7)

6. $F_{U,V}(x, y)$ is right continuous in each variable *i.e.*,

$$\lim_{h \to 0^+} F_{U,V}(x + h, y) = \lim_{h \to 0^+} F_{U,V}(x, y + h) = F_{U,V}(x, y) \quad \text{...(8)}$$

7. If the density function $f_{U,V\,(x,\,y)}$ or $f(x, y)$ is continuous at point (x, y), then

$$\frac{\partial^2 F}{\partial x\, \partial y} = f(x, y) \text{ or } \frac{\partial^2 F_{U,V}(x, y)}{\partial x\, \partial y} = f_{U,V}(x, y). \quad \text{...(9)}$$

§ 3.11. MARGINAL PROBABILITY DENSITY FUNCTION

Let U and V be two continuous jointly continuous variable with joint probability density function $f(x, y)$ or $f_{U,V}(x, y)$.

The marginal distribution of U is $F_U(x)$ is written as :

$$F_U(x) = P[U \le x] = P[U \le x, V < \infty]$$
$$= \lim_{y \to \infty} P[U \le x, V \le y]$$
$$= \lim_{y \to \infty} F_{U,V}(x, y). \quad \text{...(10)}$$

The marginal distribution of V is $F_V(y)$ is written as

$$F_V(y) = P[V \le y] = P[U < \infty, V \le y]$$
$$= \lim_{x \to \infty} P[U \le x, V \le y]$$
$$= \lim_{x \to \infty} F_{U,V}(x, y). \quad \text{...(11)}$$

Note. $P[U \le x, V \le y] = P[(U, V) \in (-\infty, x) \times (-\infty, y)] = F_{U,V}(x, y)$

$$= \int_{-\infty}^{x} \int_{-\infty}^{y} f_{U,V}(u, v)\, du\, dv.$$

Now, we have,

$$F_U(x) = F_{U,V}(x, +\infty)$$
$$= \int_{-\infty}^{x} \int_{-\infty}^{\infty} f_{U,V}(u, v)\, du\, dv \quad \text{[by definition]}$$
$$= \int_{-\infty}^{x} \left[\int_{-\infty}^{\infty} f_{U,V}(u, v)\, dv\right] du$$
$$= \int_{-\infty}^{x} f_U(u)\, du$$

where $f_U(u) = \int_{-\infty}^{\infty} f_{U,V}(u, v)\, dv.$

Hence $f_U(x) = \int_{-\infty}^{\infty} f_{U,V}(x, y)\, dy, \; -\infty < x < \infty \quad \text{...(12)}$

$$\left[\because \text{Using } \frac{d}{dx} F_U(x) = f_U(x)\right]$$

The function $f_U(x)$ is called the marginal density function of U.

Again, we have

$$F_V(y) = F_{U,V}(\infty, y)$$
$$= \int_{-\infty}^{\infty} \int_{-\infty}^{y} f_{U,V}(u, v)\, du\, dv$$
$$= \int_{-\infty}^{y} \left[\int_{-\infty}^{\infty} f_{U,V}(u, v)\, du\right] dv$$
$$= \int_{-\infty}^{y} f_V(v)\, dv$$

where $f_V(v) = \int_{-\infty}^{\infty} f_{U,V}(u, v)\, du.$

Hence $f_V(y) = \int_{-\infty}^{\infty} f_{U,V}(x, y)\, dx, \; -\infty < y < \infty \quad \text{...(13)}$

$$\left[\because \text{Using } \frac{d}{dy} F_V(y) = f_U(y)\right]$$

The function $f_V(y)$ is called the marginal density function of V.

Equation (12) gives if $f(U, V)$ has a joint density function $f_{U,V}(x, y)$, then U has a probability density function $f_U(x)$ and $f_U(x)$ is called marginal density function of U.

Equation (13) gives if (U, V) has a joint density function $f_{U,V}(x, y)$, then V has a probability density function $f_V(y)$ and $f_V(y)$ is called marginal density function of V.

◈ § 3.12. CONDITIONAL PROBABILITY DENSITY FUNCTION

Let U and V be bivariate continuous random variables with joint probability density function $f_{U,V}(x, y)$. Let $f_U(x)$ and $f_V(y)$ denotes the marginal density function of U and V respectively. Then the conditional density of U given $V = y$ is denote by $f_{U/V}(x/y)$ and defined as

$$f_{U/V}(x/y) = \frac{f_{U,V}(x, y)}{f_V(y)}, \ -\infty < x < \infty \qquad ...(14)$$

for any y such that $f_V(y) > 0$.

Again the conditional density function of V given $U = x$ is denote by $f_{V/U}(y/x)$ and defined as

$$f_{V/U}(y/x) = \frac{f_{U,V}(x, y)}{f_U(x)}, \ -\infty < y < \infty \qquad ...(15)$$

for any x such that $f_U(x) > 0$.

◈ § 3.13. CONDITIONAL CUMULATIVE DISTRIBUTION FUNCTION

If U and V are bivariate continuous random variables with joint probability density function $f_{U,V}(x, y)$, then conditional cumulative distribution function of U, given $V = y$ is defined by

$$F_{U/V}(x/y) = \int_{-\infty}^{x} f_{U/V}(z/y)\, dz, \ -\infty < x < \infty \qquad ...(16)$$

for all x such that $f_U(x) > 0$.

The conditional cumulative distribution function of V, given $U = x$ is defined by

$$F_{V/U}(y/x) = \int_{-\infty}^{y} f_{V/U}(z/x)\, dz, \ -\infty < y < \infty \qquad ...(17)$$

for all y such that $f_V(y) > 0$.

Example. ***Show that for the conditional densities of U given V = y and of V given U = x satisfies the following conditions :***

(i) $f_{U/V}(x/y) \geq 0$ ***and*** $f_{V/U}(y/x) \geq 0$

(ii) $\int_{-\infty}^{\infty} f_{U/V}(x/y)\, dx = 1$ ***and*** $\int_{-\infty}^{\infty} f_{V/U}(y/x)\, dy = 1$.

Solution. (i) The conditional density of U given $V = y$ is defined by

$$f_{U/V}(x/y) = \frac{f_{U,V}(x, y)}{f_V(y)}.$$

Since $f_{U/V}(x/y)$ is probability density function, and from the definition, we get $f_{U/V}(x/y)$ is greater than or equal to zero. *i.e.*, non-negative for all x. This shows result.

Similarly for $f_{V/U}(y/x) \geq 0$, for all y.

(ii) Show that $\int_{-\infty}^{\infty} f_{U/V}(x/y)\,dx = 1.$

By the definition,

$$f_{U/V}(x/y) = \frac{f_{U,V}(x, y)}{f_V(y)}$$

$$\Rightarrow \quad \int_{-\infty}^{\infty} f_{U/V}(x/y)\,dx = \int_{-\infty}^{\infty} \frac{f_{U,V}(x, y)}{f_V(y)}\,dx$$

$$= \frac{1}{f_V(y)} \int_{-\infty}^{\infty} f_{U,V}(x, y)\,dx$$

$$= \frac{f_V(y)}{f_V(y)} = 1$$

[$\because$ By the definition of marginal density function of V]

for any y such that $f_V(y) > 0$.

Now, to show that $\int_{-\infty}^{\infty} f_{V/U}(y/x)\,dy = 1.$

By the definition,

$$f_{V/U}(y/x) = \frac{f_{U,V}(x, y)}{f_U(x)}$$

$$\Rightarrow \quad \int_{-\infty}^{\infty} f_{V/U}(y/x)\,dy = \int_{-\infty}^{\infty} \frac{f_{U,V}(x, y)}{f_U(x)}\,dy$$

$$= \frac{1}{f_U(x)} \int_{-\infty}^{\infty} f_{U,V}(x, y)\,dy$$

$$= \frac{f_U(x)}{f_U(x)} = 1$$

[$\because$ By the definition of marginal density function of U]

for any x such that $f_U(x) > 0$.

§ 3.14. INDEPENDENCE

Two continuous random variables U and V are independent if and only if the joint p.d.f. of U and V factors into the product of their marginal p.d.f. *i.e.*,

$$f_{U,V}(x, y) = f_U(x) \cdot f_V(y), \; -\infty < x, y < \infty \qquad \text{...(18)}$$

where $f_{U,V}(x, y)$ is joint probability density function, $f_U(x)$ is marginal density of U and $f_V(y)$ is marginal density of V.

In other words, two random variables U and V are said to be independent, if, for all x, y, we have

$$P[U \le x, V \le y] = P[U \le x]\,P[V \le y]. \qquad \text{...(19)}$$

This shows, that the events $[U \le x]$ and $[V \le y]$ are independent events.

Example 1. ***If U and V are two independent random variables, then show that***

$$F_{U,V}(x, y) = F_U(x) \cdot F_V(y), \; -\infty < x, y < \infty$$

where $F_{U,V}(x, y)$ ***is joint distribution of*** (U, V), $F_V(y)$ ***is the marginal distribution of*** V ***and*** $f_U(x)$ ***is the margine distribution of*** U.

Solution. We know that from the definition of $F_{U,V}(x, y)$,

$$F_{U,V}(x, y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{U,V}(u, v)\, du\, dv$$

$$= \int_{-\infty}^{x} \int_{-\infty}^{y} f_{U}(u) \cdot f_{V}(v)\, du\, dv$$

$$= \left\{\int_{-\infty}^{x} f_U(u)\, du\right\} \left\{\int_{-\infty}^{y} f_V(v)\, dv\right\}$$

$$= F_U(x) \,.\, F_V(y)$$

This shows, if U and V are independent random variables.

Then $\quad F_{U,V}(x, y) = F_{U}(x) \cdot F_V(y); \ -\infty < x, y < \infty$

§ 3.15. EXPECTATIONS

Expected values of functions of two continuous random variables.

Let U, V be the continuous random variables with joint probability density function $f_{U,V}(x, y)$ and $g(U, V)$ is a function of two random variables, then expectation of $g(U, V)$ denoted by $E[g(U, V)]$ and defined as

$$E[g(U, V)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) \,.\, f_{U,V}(x, y)\, dx\, dy \qquad \text{...(20)}$$

Note. (i) $E(U) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x \,.\, f(x, y)\, dx\, dy$ or $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x \,.\, f_{U,V}(x, y)\, dx\, dy$

(ii) $E(V) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y \,.\, f(x, y)\, dx\, dy$ or $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y \,.\, f_{U,V}(x, y)\, dx\, dy$

(iii) $E(UV) = \int_{-\infty}^{\infty} xy \,.\, f(x, y)\, dx\, dy$ or $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy \,.\, f_{U,V}(x, y)\, dx\, dy$

(iv) $E(U) = \int_{-\infty}^{\infty} x \,.\, f_U(x)\, dx \qquad \left[\because f_U(x) = \int_{-\infty}^{\infty} f_{U,V}(x, y)\, dy\right]$

(v) $E(V) = \int_{-\infty}^{\infty} y \,.\, f_V(y)\, dy \qquad \left[\because f_V(x) = \int_{-\infty}^{\infty} f_{U,V}(x, y)\, dx\right]$

Example 2. ***If U and V are two continuous random variables then find expectation of***

$$g(U, V) = U + V \quad \textit{i.e. find} \quad E[U + V].$$

Solution. Given that $g(U, V) = U + V$

$$E[g(U, V)] = E(U + V)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x + y)\, f_{U,V}(x, y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x\, f_{U,V}(x, y)\, dx\, dy + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y\, f_{U,V}(x, y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty} x \left[\int_{-\infty}^{\infty} f_{U,V}(x, y)\, dy\right] dx + \int_{-\infty}^{\infty} y \left[\int_{-\infty}^{\infty} f_{U,V}(x, y)\, dx\right] dy$$

$$= \int_{-\infty}^{\infty} x\, f_U(x)\, dx + \int_{-\infty}^{\infty} y\, f_V(y)\, dy$$

$$= E(U) + E(V).$$

Note 1. If U and V are two random variables, then for any constants a, b, we have

$$E[aU + bV] = aE(U) + bE(V).$$

Note 2. If $X_1, X_2, X_3, \ldots, X_n$ are 'n' random variables, then for any 'n' constants $a_1, a_2, a_3, \ldots, a_n$, we have

$$E[a_1X_1 + a_2X_2 + \ldots + a_n X_n] = a_1E(X_1) + a_2E(X_2) + \ldots + a_n E(X_n).$$

Proposition 1. *If U and V are independent, then for any function h and k*

$$E[g(U)\, k(V)] = E[g(U)]\, E[k(V)].$$

Proof. We assume that U and V are jointly continuous random variables,

Then $E[g(U) + k(V)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x)\, k(y)\, f_{U,V}(x, y)\, dx\, dy$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x)\, k(y)\, f_{U(x)} \cdot f_V(y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x)\, f_U(x)\, dx + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} k(y)\, f_V(y)\, dy$$

$$= E[g(U)]\, E[k(V)].$$

§ 3.16. CONDITIONAL EXPECTATION

Let U and V are two continuous random variables with joint continuous density function $f_{U,V}(x, y)$, then the conditional expectation of $g(U, V)$ given $U = x$, denoted by $E[g(U, V) \mid U = x]$ and is defined by

$$E[g(U, V) \mid U = x] = \int_{-\infty}^{\infty} g(x, y) \cdot f_{V/U}(y/x)\, dy. \quad \ldots(21)$$

Now (i) $E[U / V = y] = \int_{-\infty}^{\infty} x \cdot f_{U/V}(x/y)\, dx.$

(ii) $E[V / U = x] = \int_{-\infty}^{\infty} y f_{V/U}(y/x)\, dy.$

§ 3.17. JOINT MOMENT GENERATING FUNCTION (MGF OF U AND V)

Let U and V be two continuous random variables with joint probability density function $f_{UV}(x, y)$, then moment generating function (MGF) of U and V defined by $M_{UV}(t_1, t_2)$ and is defined as

$$M_{UV}(t_1, t_2) = E[e^{t_1U + t_2V}]$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{(t_1U + t_2V)} f_{UV}(x, y)\, dx\, dy \quad \ldots(22)$$

where t_1, t_2 are real numbers.

Note 1. Suppose $M_U(t_1)$ is moment generating function of variable U and t_1 is real number, then we can write

$$m_U(t_1) = m_{UV}(t_1, 0) = \lim_{t_2 \to 0} m_{U,V}(t_1, t_2).$$

2. Suppose $m_V(t_2)$ is moment generating function of variable V and t_2 is real number, then we can write

$$m_V(t_2) = m_{U,V}(0, t_2) = \lim_{t_1 \to 0} m_{U,V}(t_1, t_2).$$

Theorem 1. *Let U and V are two independent continuous variables, then moment generating function (MGF) of a pair of independent random variables U and V is the product of the moment generating function (MGF) of the corresponding marginal distributions.*

i.e., If $m_{UV}(t_1, t_2)$ is the MGF of independent bivariate variable U and V with parameter t_1 and t_2, then prove that

$$m_{UV}(t_1, t_2) = m_U(t_1) \,.\, m_V(t_2)$$

where $m_U(t_1)$ and $m_V(t_2)$ are the marginal distribution of U and V respectively.

Proof. By the definition of MGF, we have

$$m_{UV}(t_1, t_2) = E\,[e^{t_1U + t_2V}]$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{(t_1U + t_2V)} f_{UV}(x, y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (e^{t_1U} \,.\, e^{t_2V}) f_{UV}(x, y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty} e^{t_1U} f_U(x)\, dx \int_{-\infty}^{\infty} e^{t_2V} f_V(y)\, dy$$

$$= m_U(t_1) \,.\, m_V(t_2)$$

which is product of marginal distribution of independent variable U and V.

Alternatively,

By the definition of MGF for two independent random continuous variables, we have

$$m_{U,V}(t_1, t_2) = E\,[e^{t_1U + t_2V}] = E\,[e^{t_1U} \,.\, e^{t_2V}].$$

Using the concept of independent of U and V, we have

$$= E\,[e^{t_1U}] \,.\, E\,[e^{t_1V}]$$

$$= m_U(t_1) \,.\, m_V(t_2).$$

§ 3.18. MARGINAL MOMENT GENERATING FUNCTION (MARGINAL MGF)

Let U and V and two independent random variables and $m_{UV}(t_1, t_2)$ are the moment generating function (MGF) of U and V, then the moment generating function (MGF) of the marginal distribution of U and V are given by $m_{UV}(t_1, 0)$ and $m_{UV}(0, t_2)$ respectively.

Proof. Let $m_U(t)$ denote the marginal distribution of variable U, then, we have

$$m_U(t) = E(e^{tU}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{tU} f_U(x)\, dx$$

$$= \int_{-\infty}^{\infty} (e^{tU}) \left[\int_{-\infty}^{\infty} f_{UV}(x, y)\, dy\right] dx$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{tU} f_{UV}(x, y)\, dy\, dx$$

$$= m_{UV}(t, 0).$$

Similarly, $m_V(t) = E(e^{tV}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{tV} f_V(y)\, dy$

$$= \int_{-\infty}^{\infty} (e^{tV}) \left[\int_{-\infty}^{\infty} f_{UV}(x, y)\, dx\right] dy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{tV} f_{UV}(x, y)\, dx\, dy$$

$$= m_{UV}(0, t).$$

Theorem. *Suppose U and V are two independent random variables with their expectation E(U) and E(V) respectively. If E(U) and E(V) are finite, then E(UV) exists and*

$$E(UV) = E(U) \,.\, E(V).$$

Proof. Given that U and V are independent variables with their expectation $E(U)$ and $E(V)$. Let $f_U(x)$ and $f_V(y)$ be the density functions (pdf) of U and V respectively.

By the definition of independent, joint density of (U, V) is given by

$$f_{U,V}(x, y) = f_U(x) \,.\, f_V(y).$$

Suppose expectation of U and V exists; *i.e.*, $E(U)$ and $E(V)$ exists. Then

$$E(UV) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy\, f_{UV}(x, y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy\, f_U(x)\, f_V(y)\, dx\, dy$$

[$\because f_{UV}(x, y) = f_U(x)\, f_V(y)$ from independence of U and V]

$$= \int_{-\infty}^{\infty} x f_U(x) \left\{\int_{-\infty}^{\infty} y\, f_V(y)\, dy\right\} dx$$

$$= \left[\int_{-\infty}^{\infty} x f_U(x)\, dx\right] E(V)$$

$$= E(U) \,.\, E(V).$$

◈ § 3.19. COVARIANCE OF TWO RANDOM VARIABLES

Let U and V are two random variables, the covariance of U and V, denoted by cov (U, V) and is defined as

$$\text{Cov}(U, V)] = E[\{U - E(U)\}\{V - E(V)\}]$$

$$= E[UV - VE(U) - UE(V) + E(U)\, E(V)]$$

$$= E(UV) - E(V)\, E(U) - E(U)\, E(V) + E(U)\, E(V)$$

$$\text{Cov}(U, V) = E(UV) - E(U)\, E(V). \qquad \text{...(23)}$$

Note. If U and V are independent with finite expectation, then covariance of U and V is zero *i.e.*, Cov $(U, V) = 0$.

This gives,

$$\text{Cov}(U, V) = E(UV) - E(U)\, E(V) = 0$$

or

$$E(UV) = E(U)\, E(V). \qquad \text{...(24)}$$

Covariance cov (U, V) measures the degree to which two variables U and V tend to be large (or small) at the some time or the degree to which one tends to be large which the other is small.

The above relation in equation (22) are expressed using the notation of expectations operator E.

In other words,

If U and V are random variables with finite variances, then their correspondence is the relation

$$\text{Cov }(UV) = E\,[(U - \mu U)\,(V - \mu V)]$$

where $\mu U = E(U)$ and $\mu V = E(V)$. The covariance is a measure of the extent to which U and V are linearly related. Because

$$(U - \mu U)\,(V - \mu V) = UV - \mu UV - \mu VU - \mu U\,\mu V$$

the covariance can also be expressed as

$$\text{Cov }(U, V) - E(UV) - E\,[U]\,.\,E\,[V] = E\,[UV] - \mu U\,\mu V \qquad \text{...(25)}$$

i.e.,

$$\text{Cov }(U, V) = E(UV) - \mu U\,\mu V.$$

Note that if U and V are independent, then $E\,[UV] = \mu U\,\mu V$. Therefore

$$\text{Cov }(U, V) = 0.$$

Note. If U and V are independent, then

$$E(UV) = E(U)\,.\,E(V).$$

Also,

$$\begin{aligned} \text{Cov }(U, V) &= E(UV) - \mu U\,\mu V \\ &= E(U)\,.\,E(V) - \mu U\,\mu V \\ &= \mu U\,.\,\mu V - \mu U\,\mu V \\ &= 0. \end{aligned}$$

Note 1. If U, V are independent, then the covariance is $\text{Cov }(U, V) = 0$ but converse is not true.

i.e., The condition $\text{Cov }(U, V) = 0$ does not imply that U, V are independent.

Note 2. If U and V are discrete random variables with joint support on sample space S, then the covariance of U and V *i.e.*,

$$\text{Cov }(U, V) = \sum_{(x,\, y)\, \in\, S} (U - \mu U)\,(V - \mu V)\,P(x, y).$$

Note 3. If U and V are continuous random variables on sample space S_1, S_2, then covariance of U and V is

$$\text{Cov }(U, V) = \int_{S_2} \int_{S_1} (U - \mu U)\,(V - \mu V)\, f_{UV}\,(xy)\,dx\,dy.$$

§ 3.20. VARIANCE

If U is any random variable, then

$$\text{Cov}\,(U, U) = E\,[(U - E[U])^2] = \text{Var }(U) \qquad \text{...(26)}$$

§ 3.21. VARIANCE OF SUM OF TWO RANDOM VARIABLES

If U and V be the two variables, then variance of sum of U and V is given by

$$\text{Var }(U + V) = \text{Var }(U) + \text{Var }(V) + 2\,\text{Cov }(U, V)$$

If U and V are two independent variable, then

$$\text{Cov }(U, V) = 0.$$

Above equation becomes,

$$\text{Var }(U + V) = \text{Var }(U) + \text{Var }(V) \qquad \text{...(27)}$$

Now, if U and V are two continuous random variable, then covariance of U and V, Cov (U, V), given by

$$\text{Cov}(U, V) = \int_x \int_y \{U - E(U)\}\{V - E(V)\} f_{UV}(x, y)\, dx\, dy \qquad ...(28)$$

If U and V are independent, such that

$$f_{UV}(x, y) = f_U(x)\, f_V(y)$$

Then their joint moment can be expressed as the product of their separate moments.

Thus, if U and V are independent, then we can write

$$E\{[U - E(U)]^2 [V - E(V)]^2\}$$
$$= \int_x \int_y [U - E(U)]^2 [V - E(V)]^2 f_U(x) f_V(y)\, dy\, dx$$
$$= \int_x [U - E(U)]^2 f_U(x)\, dx \int_y [V - E(V)]^2 f_V(y)\, dy$$
$$= \text{Var}(U) . \text{Var}(V).$$

In case of covariance of U, V, when they are independent, we have

$$\text{Cov}(U, V) = E\{[U - E(U)][V - E(V)]\}$$
$$= \int_x \int_y [U - E(U)][V - E(V)] f_{UV}(x, y)\, dx\, dy$$
$$= \int_x \int_y [U - E(U)][V - E(V)] f_U(x) f_V(y)\, dx\, dy$$
$$= \int_x [U - E(U)] f_U(x)\, dx \int_y [V - E(V)] f_V(y)\, dy$$
$$= \{[E(U) - E(U)][E(V) - E(V)]\}$$
$$= 0$$

Theorem. *Prove that the variance of sum of two independent variables is expressed as the sum of their separate variance.*

Or

If U and V are independent random variables with finite variance, then

$$Var(U + V) = Var(U) + Var(V).$$

Or

Find the variance of sum of two independent random variables.

Proof. Let $E(U) = \mu_U$ and $E(V) = \mu_V$, then

$$E(U + V) = E(U) + E(V)$$

$$\Rightarrow \quad E(U + V) = \mu_U + \mu_V$$

Now $\text{Var}(U + V) = E[(U + V - (E(U + V)))^2]$

$$= E[(U + V - (\mu_U + \mu_V))^2]$$
$$= E\{[(U - \mu U) + (V - \mu V)]^2\}$$
$$= E\{(U - \mu U)^2 + (V - \mu V)^2 + 2(U - \mu U)(V - \mu V)\}$$
$$= E(U - \mu U)^2 + E(V - \mu V)^2 + 2E\{(U - \mu U)(V - \mu V)\}$$
$$= \text{Var}(U) + \text{Var}(V) + 2\,\text{Cov}(U, V).$$

Since given that U and V are random independent variables, this gives,

$$\text{Cov}(U, V) = 0.$$

From, above expression, put Cov $(U, V) = 0$, we get

$$\text{Var}(U + V) = \text{Var}(U) + \text{Var}(V).$$

§ 3.22. PROPERTIES OF COVARIANCE

If X and Y are two random variables. Then

1. $\text{Cov}(X, Y) = E[(X - \mu_x (Y - \mu_y)] = E(XY) - \mu_x \mu_y$
2. $\text{Cov}(X, Y) = \text{Cov}(Y, X)$.
3. $\text{Cov}(X, Y) = 0$ if X and Y are independent.
4. $\text{Cov}(X, c) = 0$.
5. $\text{Cov}(X, X) = \text{Var}(X)$.
6. $\text{Cov}(aX, bY) = ab\,\text{Cov}(X, Y)$.
7. $\text{Cov}(X + a, Y + b) = \text{Cov}(X, Y)$.
8. $\text{Cov}(X, Y + Z) = \text{Cov}(X, Y) + \text{Cov}(X, Z)$.

Proof of property 8 : $\text{Cov}(X, Y + Z) = \text{Cov}(X, Y) + \text{Cov}(X, Z)$

$$\begin{aligned}
\text{L.H.S.} = \text{Cov}(X, Y + Z) &= E[X(Y + Z)] - E(X)\,E(Y + Z) \\
&= E[(XY + XZ)] - E(X)\,[E(Y) + E(Z)] \\
&= E(XY) + E(XZ) - E(X)\,E(Y) - E(X)\,E(Z) \\
&= E(XY) - E(X)\,E(Y)\,E(XZ) - E(X)\,E(Z) \\
&= \text{Cov}(X, Y) + \text{Cov}(X, Z) \\
&= \text{R.H.S.}
\end{aligned}$$

Example 1. ***For the following bivariate probability distribution of X and Y find***

(i) $P(X \leq 2, Y = 3)$ **(ii)** $P(X \leq 1)$ **(iii)** $P(Y = 4)$

(iv) $P(Y \leq 5)$ **(v)** $P(X < 2, Y \leq 3)$

$\downarrow X$ $\rightarrow Y$	1	2	3	4	5	6
0	0	0	$\frac{1}{32}$	$\frac{2}{32}$	$\frac{2}{32}$	$\frac{3}{32}$
1	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$
2	$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{64}$	0	$\frac{2}{64}$

Solution. The marginal distributions are given below :

$f_X(x) =$ Sum of the probabilities of row

$f_Y(y) =$ Sum of the probabilities of column

↓ X → Y	1	2	3	4	5	6	$f_X(x)$
0	0	0	$\frac{1}{32}$	$\frac{2}{32}$	$\frac{2}{32}$	$\frac{3}{32}$	$\frac{8}{32}$
1	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{10}{16}$
2	$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{64}$	0	$\frac{2}{64}$	$\frac{8}{64}$
$f_Y(y)$	$\frac{3}{32}$	$\frac{3}{32}$	$\frac{11}{64}$	$\frac{13}{64}$	$\frac{6}{32}$	$\frac{16}{64}$	1

(i) $P(X \le 2, Y = 3) = P(X = 0, Y = 3) + P(X = 1, Y = 3) + P(X = 2, Y = 3)$

$$= \frac{1}{32} + \frac{1}{8} + \frac{1}{64} = \frac{2 + 8 + 1}{64} = \frac{11}{64}.$$

(ii) $P(X \le 1) = P(X = 0) + P(X = 1) = \frac{8}{32} + \frac{10}{16} = \frac{7}{8}$

(iii) $P(Y = 4) = \frac{13}{64}$

(iv) $P(Y \le 5) = P(Y = 1) + P(Y = 2) + P(Y = 3) + P(Y = 4) + P(Y = 5)$

$$= \frac{3}{32} + \frac{3}{32} + \frac{11}{64} + \frac{13}{64} + \frac{6}{32}$$

$$= \frac{6 + 6 + 11 + 13 + 12}{64} = \frac{48}{64}.$$

(v) $P(X < 2)(Y \le 3) = + P(X = 0, Y = 1) + P(X = 0, Y = 2) + P(X = 0, Y = 3)$

$$+ P(X = 1, Y = 1) + P(X = 1, Y = 2) + P(X = 1, Y = 3)$$

$$= 0 + 0 + \frac{1}{32} + \frac{1}{16} + \frac{1}{16} + \frac{1}{8} = \frac{1 + 2 + 2 + 4}{32} = \frac{9}{32}.$$

Example 2. ***Let U and V be two random variables having the following joint probability distribution for three values – 1, 0 and 1***

↓ V → U	−1	0	1
−1	0·0	0·1	0·1
0	0·2	0·2	0·2
1	0·0	0·1	0·1

(i) ***Find E(U), E(V) and show that E(U) ≠ E(V). [i.e. expectation of U and expectation of V. Show that they are different.]***

(ii) ***Prove that U and V are uncorrelated.***

(iii) ***Find the conditional probability distribution of U when given that V = 0.***

(iv) ***Find*** Var(X) ***and*** Var(Y).

(v) ***Find the conditional variance*** Var($V \mid U = -1$).

Solution. Given joint probability distribution is

$\downarrow$ $\to U$ V	-1	0	1	Total $f_V(y)$
-1	0·0	0·1	0·1	0·2
0	0·2	0·2	0·2	0·6
1	0·0	0·1	0·1	0·2
Total $f_U(x)$	0·2	0·4	0·4	1·0

(i) Compute $E(U)$ and $E(V)$. From the table above

$$E(V) = \Sigma p_i y_i = p_1 y_1 + p_2 y_2 + p_3 y_3$$
$$= 0\cdot 2 \times -1 + 0\cdot 6 \times 0 + 0\cdot 2 \times 1$$
$$= 0$$
$$E(U) = \Sigma p_j x_j = p_1 x_1 + p_2 x_2 + p_3 x_3$$
$$= 0\cdot 2 \times -1 + 0\cdot 4 \times 0 + 0\cdot 4 \times 1$$
$$= -0\cdot 2 + 0 + 0\cdot 4$$
$$= 0\cdot 2.$$

This shows $E(U) \neq E(V)$ *i.e.*, U and V have different expectations.

(ii) To show U and V are uncorrelated. For this, we have to show

$$\text{Cov}(U, V) = 0$$

Now,
$$E(UV) = \Sigma p_{ij}\, x_i\, y_j$$
$$= p_{11}x_1y_1 + p_{12}x_1y_2 + p_{13}x_1y_3 + p_{21}x_2y_1 + p_{22}x_2y_2$$
$$+ p_{23}x_2y_3 + p_{31}x_3y_1 + p_{32}x_3y_2 + p_{33}x_3y_3$$
$$= (0 \times -1 \times -1 + 0\cdot 1 \times -1 \times 0 + 0\cdot 1 \times -1 \times 1)$$
$$+ (0\cdot 2 \times 0 \times -1 + 0\cdot 2 \times 0 \times 0 + 0\cdot 2 \times 0 \times 1)$$
$$+ (0 \times 1 \times -1 + 0\cdot 1 \times 1 \times 0 + 0\cdot 1 \times 1 \times 1)$$
$$= -0\cdot 1 + 0 + 0\cdot 1$$
$$= 0$$
$$\text{Cov}(U, V) = E(UV) - E(U)\,E(V) = 0 - (0\cdot 2 \times 0) = 0.$$

This shows U and V are uncorrelated.

(iii) Find the conditional probability distribution of U when $V = 0$ *i.e.*, find $P(U = x_i \mid V = 0)$ for $i = 1, 2, 3$

$$P(U = x_1 \mid V = 0) = P(U = -1 \mid V = 0) = \frac{P(U = -1 \cap V = 0)}{P(V = 0)}$$
$$= \frac{0\cdot 2}{0\cdot 6} = \frac{1}{3}$$

$$P(U = x_2 \mid V = 0) = P(U = 0 \mid V = 0) = \frac{P[U = 0 \cap V = 0]}{P(V = 0)}$$

$$= \frac{0 \cdot 2}{0 \cdot 6} = \frac{1}{3}$$

$$P(U = x_3 \mid V = 0) = P(U = 1 \mid V = 0) = \frac{P[U = 1 \cap V = 0]}{P(V = 0)}$$

$$= \frac{0 \cdot 2}{0 \cdot 6} = \frac{1}{3}.$$

(iv) Find variance of U and variance of V

$$E(V^2) = \Sigma p_i y_i^2 = 0 \cdot 2 \times (-1)^2 + 0 \cdot 6 \times (0)^2 + 0 \cdot 2 \times (1)^2$$

$$= 0 \cdot 2 + 0 + 0 \cdot 2 = 0 \cdot 4$$

$$E(U^2) = \Sigma p_j x_j^2 = 0 \cdot 2 \times (-1)^2 + 0 \cdot 4 \times (0)^2 + 0 \cdot 4 \times (1)^2$$

$$= 0 \cdot 2 + 0 + 0 \cdot 4 = 0 \cdot 6$$

$$\text{Var}(U) = E(U^2) - [E(U)]^2 = 0 \cdot 6 - (0 \cdot 2)^2 = 0 \cdot 6 - 0 \cdot 04 = 0 \cdot 56$$

$$\text{Var}(V) = E(V^2) - [E(V)]^2 = 0 \cdot 4 - (0)^2 = 0 \cdot 4$$

(v) Find the conditional variance $\text{Var}(V \mid U = -1)$.

It is given by

$$\text{Var}(V / U = -1) = E(V / U = -1)^2 - [E(V / U = -1)]^2.$$

Now $$E(V / U = -1) = \Sigma P(V = y \mid X = -1) \,.\, y_j$$

$$= 0 \times -1 + 0 \cdot 2 \times 0 + 0 \times 1$$

$$= 0$$

$$E(V / U = -1)^2 = \Sigma P(V = y \mid X = -1) \,.\, y_j^2$$

$$= 0 \times (-1)^2 + 0 \cdot 2 \times (0)^2 + 0 \times (1)^2$$

$$= 0 + 0 + 0 = 0.$$

This gives,

$$\text{Var}(V \mid V = -1) = 0 - (0)^2 = 0,$$

Example 3. ***Two fair dice are tossed. Find the expected value of sum of the obtained number of points.***

Solution. Let X_1 and X_2 denote the number of first and second die respectively. We have to find $E(X_1 + X_2)$.

In single throw of a dice assume the values 1, 2, 3, 4, 5, 6 and the probability in each case 1/6.

$\because$ Given probability distribution is as follows :

u :	1	2	3	4	5	6
p :	1/6	1/6	1/6	1/6	1/6	1/6

For discrete random variable, $E(X_1) = \sum_{i=1}^{6} u_i p_i$

$$E(X_1) = p_1u_1 + p_2u_2 + p_3u_3 + p_4u_4 + p_5u_5 + p_6u_6$$
$$= 1\times\frac{1}{6} + 2\times\frac{1}{6} + 3\times\frac{1}{6} + 4\times\frac{1}{6} + 5\times\frac{1}{6} + 6\times\frac{1}{6}$$
$$= \frac{1}{6}[1+2+3+4+5+6]$$
$$= \frac{21}{6} = \frac{7}{2}.$$

Similarly, $E(X_2) = \frac{7}{2}$.

The required expected value

$$E(X_1 + X_2) = E(X_1) + E(X_2)$$
$$= \frac{7}{2} + \frac{7}{2} = 2\,(7/2) = 7.$$

Example 4. ***Calculated the expected sum obtained when three fair dice are rolled.***

Solution. Let X_1, X_2, X_3 denote the number of first, second and third dice respectively. Then similar to above example :

$$E(X_1) = 7/2,\; E(X_2) = 7/2,\; E(X_3) = 7/2.$$

$$\therefore \quad E[X_1 + X_2 + X_3] = E(X_1) + E(X_2) + E(X_3)$$
$$= 7/2 + 7/2 + 7/2$$
$$= 3\,(7/2) = \frac{21}{2}.$$

Example 5. ***Let (U, V) has joint probability density function***

$$f_{U,V}(x, y) = \begin{cases} 4xy, & \text{if } 0 < x < y < 1 \\ 0, & \text{otherwise} \end{cases}$$

with $E(U) = \frac{8}{15}, E(V) = \frac{4}{5}$. Compute the covariance of U and V.

Solution. We have, by the definition of covariance

$$\text{Cov}(U, V) = E(UV) - E(U)\,E(V).$$

Compute $E(UV)$

$$E(UV) = \int_{x=0}^{1}\int_0^y xy\, f_{UV}(x, y)\, dx\, dy.$$

Given $f_{UV}(x, y) = 4xy$ for $0 < x < y < 1$ otherwise $f_{UV}(x, y) = 0$

$$= \int_0^1\int_0^y xy\,.\,4xy\, dx\, dy$$
$$= \int_0^1\int_0^y 4x^2y^2\, dx\, dy$$
$$= \int_0^1 4y^2\left[\int_0^y x^2\, dx\right] dy$$
$$= \int_0^1 4y^2\left(\frac{x^3}{3}\right)_0^y dy$$

$$= \int_0^1 \frac{4}{3} y^2 \cdot y^3 \, dy = \int_0^1 \frac{4}{3} y^5 \, dy$$

$$= \frac{4}{3}\left[\frac{y^6}{6}\right]_0^1 = \frac{4}{3}\left(\frac{1}{6} - 0\right) = \frac{4}{18} = \frac{2}{9}.$$

Hence, $\text{Cov}(U, V) = E(UV) - E(U)\,E(V)$

$$\Rightarrow \quad \text{Cov}(U, V) = \frac{2}{9} - \frac{8}{15} \times \frac{4}{5} \qquad \left[\because E(U) = \frac{8}{15}, E(V) = \frac{4}{5} \text{ given}\right]$$

$$\Rightarrow \quad \text{Cov}(U, V) = \frac{2}{9} - \frac{32}{75}.$$

Example 6. ***Three fair coins are tossed. Let X denote the number of heads on the first two coins, and let Y denote the numbers of talls on the last two coins :***

(*a*) ***Find the joint distribution of X and Y.***

(*b*) ***Find the conditional distribution of Y given that X = 1.***

(*c*) ***Find Cov (X, Y).***

(*d*) ***Find ρ(X, Y).***

Solution. When we tossed three fair coins, the eight possible outcomes of the experiment as :

	HHH	*HHT*	*HTH*	*HTT*	*THH*	*TTT*	*TTH*	*THT*
X :	2	2	1	1	1	0	0	1
Y :	0	1	1	2	0	2	1	1

(a) The joint distribution of X and Y with marginal distribution is given below :

$\downarrow Y \quad \rightarrow X$	0	1	2	$f_Y(y)$
0	0	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{2}{8} = \frac{1}{4}$ $f_Y(0)$
1	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{1}{8}$	$\frac{4}{8} = \frac{1}{2}$ $f_Y(1)$
2	$\frac{1}{8}$	$\frac{1}{8}$	0	$\frac{2}{8} = \frac{1}{4}$ $f_Y(2)$
$f_X(x)$	$\frac{2}{8} = \frac{1}{4}$ $f_X(0)$	$\frac{4}{8} = \frac{1}{2}$ $f_X(1)$	$\frac{2}{8} = \frac{1}{4}$ $f_X(2)$	1

(b) To find the conditional distribution of Y given that $X = 1$

i.e., To find $f_{Y/X}(y \mid X) = \dfrac{f(1 \cdot 2)}{f_x(1)} = 2f(1, y)$, since $f_X(1) = \dfrac{1}{2}$

$$\text{Hence} \qquad f_{Y \mid X}(X=1) = \begin{cases} 2 \times \dfrac{1}{8} = \dfrac{1}{4} & \text{at } \; y = 0, \\[2mm] 2 \times \dfrac{2}{8} = \dfrac{1}{2} & \text{at } \; y = 1, \\[2mm] 2 \times \dfrac{1}{8} = \dfrac{1}{4} & \text{at } \; y = 2. \end{cases}$$

(c) $E(X) = 0 \times f_X(0) + 1 \times f_X(1) + 2 \times f_X(2) = 0 \times \dfrac{1}{4} + 1 \times \dfrac{1}{2} + 2 \times \dfrac{1}{4} = 1$

$$E(Y) = 0 \times f_Y(0) + 1 \times f_Y(1) + 2 \times f_Y(2) = 0 \times \frac{1}{4} + 1 \times \frac{1}{2} + 2 \times \frac{1}{4} = 1$$

$$E(XY) = 1 \times \frac{2}{8} + 2 \times \frac{1}{8} + 2 \times \frac{1}{8} + 4 \times 0 = \frac{6}{8} = \frac{4}{3}.$$

Hence $\quad \text{Cov}(X, Y) = E(XY) - E[X]\,E[Y] = \dfrac{3}{4} - 1 = -\dfrac{1}{4}$

$$E(X^2) = \Sigma x^2 \, f_X(x) = 0^2 \times f_X(0) + 1^2 \times f_X(1) + (2)^2 \times f_X(2)$$

$$\Rightarrow \qquad E[X^2] = 0^2 \times \frac{1}{4} + 1^2 \times \frac{1}{2} + 2^2 \times \frac{1}{4} = \frac{3}{2}$$

$\text{Var}[X] = E[X^2] - \{E[X]\}^2 = \dfrac{3}{2} - 1 = \dfrac{1}{2}$. Similarly $\text{Var}[Y] = \dfrac{1}{2}$.

Now $\quad \sqrt{\text{Var}(X)} = \sigma_x \Rightarrow \sigma x = \sqrt{1/2}, \; \sigma_y = \sqrt{\text{Var}(Y)} \Rightarrow \sigma_y = \sqrt{1/2}$

$$\rho(X, Y) = \frac{\text{cov}(X, Y)}{\sigma_X \sigma_Y}$$

$$\rho = \frac{-1/4}{\sqrt{\dfrac{1}{2}} \cdot \sqrt{\dfrac{1}{2}}} = \frac{-\dfrac{1}{4}}{\dfrac{1}{2}} = -\frac{1}{2} = -0 \cdot 5.$$

Example 7. ***At a party N men throw their hats into the center of a room. The hats are mixed up and each map randomly selects one. Find the expected number of men who select their own hats.***

Solution. Let X denote the number of men that select their own hats, and

$$X = X_1 + X_2 + \ldots + X_N. \text{ Now compute } E(X)$$

i.e., $\qquad E(X) = E(X_1) + E(X_2) + \ldots + E(X_n).$

Here, $\qquad X_i = \begin{cases} 1, & \text{if the ith man selects his own hat} \\ 0, & \text{otherwise.} \end{cases}$

Now, because the *i*th man is equally likely to select any of the N hats, it follows that

$$P[X_i = 1] = P[i\text{th man selects his own hat}] = \frac{1}{N}.$$

Then $\qquad E[X_i] = 1P[X_i = 1] + 0P[X_i = 0] = \dfrac{1}{N}.$

Hence, from equation

$$E[a_1X_1 + a_2X_2 + \ldots + a_nX_n] = a_1E(X_1) + a_2E(X_2) + \ldots + a_nE(X_n)$$

$$\Rightarrow \quad E[X] = E[X_1] + \ldots + E[X_N] = \left(\frac{1}{N}\right)N = 1.$$

Hence, no matter how many people are at the party, on the average exactly one of the men will select his own hat.

Example 8. ***Let X and Y be jointly normal random variables with parameter*** $\mu_X = 1, \sigma_X^2 = 1, \mu_Y = 0, \sigma_Y^2 = 4$ ***and*** $\rho = 1/2$

(i) Find $P[2X + Y \le 3]$

(ii) Find $Cov\,(X + Y, 2X - Y)$.

Solution. (i) Given that X and Y are jointly normal random variables, so that the random variable $K = 2X + Y$ is also jointly normal random variable, we have

$$E(K) = E(2X + Y) = 2E(X) + E(Y)$$

$$= 2\cdot 1 + 0 = 2 \qquad [\because\ E(X) = \mu_X, E(Y) = \mu_Y]$$

$$\text{Var}(K) = V(2X + Y) = \text{Var}(2X) + \text{Var}(Y) + 2\,\text{cov}(2X, Y)$$

$$= 4\,\text{Var}(X) + \text{Var}(Y) + 4\,\text{cov}(X, Y).$$

Given that $\sigma_X^2 = \text{Var}(X) = 1,\ \sigma_Y^2 = \text{Var}(Y) = 4,\ \rho = \frac{1}{2}$.

Also, $\rho(X, Y) = \dfrac{\text{cov}(X, Y)}{\sigma_x \cdot \sigma_y}$

$$\text{Cov}(X, Y) = \sigma_X \cdot \sigma_Y\ \rho(X, Y)$$

$$\text{Var}(K) = \text{Var}(2X + Y) = 4\,\text{Var}(X) + \text{Var}(Y) + 4\sigma_X\sigma_Y\ \rho(X, Y)$$

$$\text{Var}(K) = \text{Var}(2X + Y)$$

$$= 4 + 4 + 4 \times 1 \times 2 \times \frac{1}{2}$$

$$= 12.$$

$\therefore$

$$\text{Mean} = E(K) = E(2X + Y) = 2$$

$$\text{Variance} = \text{Var}(K) = \text{Var}(2X + Y) = 12.$$

Thus, $K \sim N(2, 12)$. Therefore

$$P(K \le 3) = \phi\left(\frac{3-2}{\sqrt{12}}\right) = \phi\left(\frac{1}{\sqrt{12}}\right) = 0\cdot 6136.$$

(ii) Find $\text{Cov}(X + Y, 2X - Y)$.

We know $\quad \text{Cov}(X, Y) = \sigma_X\sigma_Y\ \rho(X, Y)$

$$\text{Cov}(X, Y) = 1 \times 2 \times \frac{1}{2} = 1.$$

Now, $\text{Cov}(X+Y, 2X-Y) = 2\,\text{Cov}(X, X) - \text{Cov}(X, Y)$

$+ 2\,\text{Cov}(Y, X) - \text{Cov}(Y, Y)$

$= 2\sigma_X^2 - \text{Cov}(X, Y) + 2\,\text{Cov}(Y, X) - \sigma_Y^2$

$[\because \text{Cov}(X, X) = \sigma_X^2, \text{Cov}(Y, Y) = \sigma_Y^2]$

$= 2 \times 1 - 1 + 2 \times 1 - 4$

$= 2 - 1 + 2 - 4$

$= -1.$

Example 9. ***If U and V be jointly (bivariate) normal variable with $Var(U) = Var(V)$. Show that the two random variables $U+V$ and $U-V$ are independent.***

Solution. Since U and V are jointly (bivariate) normal variable, so that the random variables $U+V$ and $U-V$ are also jointly normal. We have

$\text{Cov}(U+V, U-V) = \text{Cov}(U, U) - \text{Cov}(U, V) + \text{Cov}(V, U) - \text{Cov}(V, V)$

$= \text{Cov}(U, U) - \text{Cov}(V, V) \quad [\because \text{Cov}(U, V) = \text{Cov}(V, U)]$

$= \text{Var}(U) - \text{Var}(V)$

$= 0 \quad [\because \text{given Var}(U) = \text{Var}(V)]$

Since $U+V$ and $U-V$ are jointly normal and uncorrelated, they are independent.

Example 10. ***Let U and V are two random variables with joint probability distribution function (pdf) :***

$\downarrow$ V $\rightarrow U$	−1	0	+1
0	a	$2a$	a
1	$3a$	$2a$	a
2	$2a$	a	$2a$

Find (i) Marginal distribution of U and V.

(ii) Conditional distribution of U when given V = 2.

Solution. From the given table :

V \ U	−1	0	1	Total	
0	a	$2a$	a	$4a$	$f_V(0)$
1	$3a$	$2a$	a	$6a$	$f_V(1)$
2	$2a$	a	$2a$	$5a$	$f_V(2)$
Total	$6a$	$5a$	$4a$	$15a$	
$f_U(x)$	$f_U(-1)$	$f_U(0)$	$f_U(1)$		

(Column heading beside Total: $f_V(y)$)

The probability distribution function of above distribution is

$$f_{UV}(x, y) = 15a$$

Since it is pdf, $15a = 1 \Rightarrow a = \frac{1}{15}$

(i) Marginal distribution of U. From the table, we have

$$f_U(-1) = 6a,\ f_U(0) = 5a,\ f_U(1) = 4a$$

Marginal distribution of V, from the table, we have

$$f_V(0) = 4a,\ f_V(1) = 6a,\ f_V(2) = 5a$$

(ii) Conditional distribution of U given $V = 2$.

Conditional distribution of U for V is given by

$$f_{U/V}(U = x / V = y) = \frac{P[U = x \cap V = 2]}{P(V = 2)}$$

$$f_{U/V}(U = -1 / V = 2) = \frac{P[U = -1 \cap V = 2]}{P(V = 2)}$$

$$= \frac{2a}{5a} = \frac{2}{5}$$

$$f_{U/V}(U = 0 / V = 2) = \frac{P[U = 0 \cap V = 2]}{P(V = 2)}$$

$$= \frac{a}{5a} = \frac{1}{5}$$

$$f_{U/V}(U = 1 / V = 2) = \frac{P[U = 1 \cap V = 2]}{P(V = 2)}$$

$$= \frac{2a}{5a} = \frac{2}{5}.$$

Example 11. ***If U and V are two random variables and they are independent with N(0, 1).***

Find Cov (z, w) if $z = 1 + U + UV^2$, $w = 1 + U$.

Solution. Given that U and V are two independent random variables with $N(0, 1)$,

$$z = 1 + U + UV^2,\ w = 1 + U$$

$$\text{Cov}(z, w) = \text{Cov}(1 + U + UV^2, 1 + U)$$

$$= \text{Cov}(U + UV^2, U) \qquad [\because \text{Cov}(U + c, V) = \text{Cov}(U, V)]$$

$$= \text{Cov}(U, U) + \text{Cov}(UV^2, U)$$

$$[\because \text{Cov}(U + V, z) = \text{Cov}(U, z) + \text{Cov}(V, z)]$$

$$= \text{Var}(U) + E(U^2V^2) - E(UV^2)\,E(U)$$

$[\because \text{Cov}(U, V) = \text{Var}(V)$ and by definition of covariance$]$

$$= \text{Var}(U) + E(U^2)\,E(V^2) - E(U)\,E(V^2)\,E(U)$$

$$= \text{Var}(U) + E(U^2)\,E(V^2) - (E(U))^2\,E(V^2)$$

$$= 1 + 1 - 0$$

$$= 2.$$

Example 12. ***The joint probability of U and V is given in the following table :***

$\downarrow$ U $\rightarrow V$	**1**	**3**	**9**	
2	$\frac{1}{8}$	$\frac{1}{24}$	$\frac{1}{12}$	$f_U(2)$
4	$\frac{1}{4}$	$\frac{1}{4}$	0	$f_U(4)$
6	$\frac{1}{8}$	$\frac{1}{24}$	$\frac{1}{12}$	$f_U(6)$
	$f_V(1)$	$f_V(3)$	$f_V(9)$	

(a) ***Find the marginal probability distribution of V.***

(b) ***Find the conditional distribution of V given U = 4.***

(c) ***Find covariance of U and V***

(d) ***Are U and V independent ?***

Solution. In this distribution table,

$f_U(x)$ denote sum of probabilities of row

$f_V(y)$ denote sum of probabilities of column

U \ V	1	3	9	$f_U(x)$
2	$\frac{1}{8}$	$\frac{1}{24}$	$\frac{1}{12}$	$\frac{6}{24}$
4	$\frac{1}{4}$	$\frac{1}{4}$	0	$\frac{2}{4}$
6	$\frac{1}{8}$	$\frac{1}{24}$	$\frac{1}{12}$	$\frac{6}{24}$
$f_V(y)$	$\frac{4}{8}$	$\frac{8}{24}$	$\frac{2}{12}$	1

(a) Marginal probability distribution of V

$$f(V=1)=\frac{4}{8}=\frac{1}{2},\quad f(V=2)=\frac{8}{24}=\frac{1}{3},\quad f(V=3)=\frac{2}{12}=\frac{1}{6}$$

(b) Conditional probability of V when $U = 4$

$$P(V=y/U=4)=\frac{P(V=y\cap U=4)}{P(U=4)}$$

$$P(V=1/U=4)=\frac{P(V=1\cap U=4)}{P(U=4)}=\frac{\frac{1}{4}}{\frac{2}{4}}=\frac{1}{2}$$

$$P(V = 3/U = 4) = \frac{P(V = 3 \cap U = 4)}{P(U = 4)} = \frac{\frac{1}{4}}{\frac{2}{4}} = \frac{1}{2}$$

$$P(V = 9/U = 4) = \frac{P(V = 9 \cap U = 4)}{P(U = 4)} = 0$$

(c) To find Cov (X, Y) we will first find $E(UV)$, $E(U)$ and $E(V)$

$$E(U) = \Sigma x \,.\, f_U(x) = 2 \times \frac{6}{24} + 4 \times \frac{2}{4} + 6 \times \frac{6}{24} = 4$$

$$E(V) = \Sigma y \,.\, f_V(y) = 1 \times \frac{4}{8} + 3 \times \frac{8}{24} + 9 \times \frac{2}{12} = 3$$

$$E(U, V) = \Sigma xy \; f_{UV}(x, y)$$

$$= \left(2 \times \frac{1}{8} + 4 \times \frac{1}{4} + 6 \times \frac{1}{8}\right) + \left(6 \times \frac{1}{24} + 12 \times \frac{1}{4} + 18 \times \frac{1}{24}\right)$$

$$+ \left(18 \times \frac{1}{12} + 36 \times 0 + 54 \times \frac{1}{12}\right) = 12$$

$$\text{Cov}(U, V) = E(UV) - E(U)\,E(V) = 12 - 12 = 0.$$

(d) $f_{U,V}(4, 3) = \frac{1}{4}$ $\quad f_U(4) = \frac{2}{4}$ $\quad f_V(3) = \frac{8}{24}$

$$f_U(4)\, f_V(3) = \frac{2}{4} \times \frac{8}{24} = \frac{1}{6}$$

$\therefore \quad f_{U,V}(4, 3) \neq f_U(4)\, f_V(3) \Rightarrow \frac{1}{4} \neq \frac{1}{6}$

$\therefore \quad U$ and V are not independent.

Example 13. ***Show that the bivariate function***

$$F(x, y) = \begin{cases} e^{-(x+y)} & , \quad x \geq 0, y \geq 0 \\ 0 & , \quad 0 \text{ otherwise} \end{cases}$$

is not a joint distribution function.

Solution. Given function

$$F(x, y) = \begin{cases} e^{-(x+y)} & , \quad x \geq 0, y \geq 0 \\ 0 & , \quad 0 \text{ otherwise} \end{cases}$$

For joint distribution function, show that

$$\lim_{\substack{x \to \infty \\ y \to \infty}} F(x, y) = 1$$

Now

$$\lim_{\substack{x \to \infty \\ y \to \infty}} (x, y) = \lim_{\substack{x \to \infty \\ y \to \infty}} e^{-(x+y)}$$

$$= \lim_{\substack{x \to \infty \\ y \to \infty}} e^{-x} \,.\, e^{-y}$$

$$= \lim_{x \to \infty} e^{-x} \cdot \lim_{y \to \infty} e^{-y}$$

$$= e^{-\infty} \cdot e^{-\infty} = 0 \neq 1$$

Hence $F(x, y)$ is not a joint distribution function.

Example 14. ***The joint density function of X, Y is***

$$f(x, y) = \frac{1}{y} e^{-(y + x/y)}, \quad 0 < x, y < \infty$$

Show that the f (x, y) is a joint density function.

Solution. To show that $f(x, y)$ is a joint density function we need to show it is non-negative, and that $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\, dy\, dx = 1$.

Now, $$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\, dy\, dx = \int_0^{\infty} \int_0^{\infty} \frac{1}{y} e^{-(y + x/y)}\, dy\, dx$$

$$= \int_0^{\infty} e^{-y} \left[\int_0^{\infty} \frac{1}{y} e^{-x/y}\, dx \right] dy$$

$$= \int_0^{\infty} e^{-y} \cdot 1\, dy$$

$$= \int_0^{\infty} e^{-y}\, dy$$

$$= 1$$

Since $f(x, y)$ is non-negative and $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\, dx\, dy = 1$

The given function is joint pdf.

Example 15. ***Find c so that $f(x, y) = cxy \quad 1 \le x \le y \le 2$ will be a joint probability density function.***

Solution. We know $f(, y) = cxy$ is joint probability density function, then

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\, dx\, dy = 1$$

$$\Rightarrow \quad \int_1^2 \int_x^2 cxy\, dx\, dy = 1$$

$$\Rightarrow \quad \int_1^2 \left[\int_x^2 cxy\, dy \right] dx = 1$$

$$\Rightarrow \quad \int_1^2 \left[cx \frac{y^2}{2} \right]_x^2 dx = 1$$

$$\Rightarrow \quad \int_1^2 cx \left[\frac{4}{2} - \frac{x^2}{2} \right] dx = 1$$

$$\Rightarrow \quad \int_1^2 c \left[\frac{4x}{2} - \frac{x^3}{2} \right] dx = 1$$

$$\Rightarrow \quad c \left[\frac{4x^2}{4} - \frac{x^4}{8} \right]_1^2 = 1$$

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$$\Rightarrow \quad \frac{c}{4}\left[4x^2 - \frac{x^4}{2}\right]_1^2 = 1$$

$$\Rightarrow \quad \frac{c}{4}\left[\left(4 \times 4 - \frac{16}{2}\right) - \left(4 - \frac{1}{2}\right)\right] = 1$$

$$\Rightarrow \quad \frac{c}{4}[(16-8)-(7/2)] = 1$$

$$\Rightarrow \quad \frac{c}{4}\left[8 - \frac{7}{2}\right] = 1 \Rightarrow \frac{c}{4}\left[\frac{9}{2}\right] = 1$$

$$\Rightarrow \quad c = 8 \Rightarrow c = 8/9$$

For $c = 8/9$, the function is joint pdf.

Example 16. ***The joint probability density function of (U, V) is given by***

$$f(x, y) = \begin{cases} 2, & 0 < x < 1, 0 < y < x \\ 0, & \text{otherwise} \end{cases}$$

(a) Show that $f(x, y)$ is joint pdf.

(b) Find the marginal density functions of U and V.

(c) Find the conditional density function of V given U = x and that of U given V = y.

(d) Are U and V independent.

Solution. Given that joint pdf is

$$f(x, y) = \begin{cases} 2, & 0 < x < 1, 0 < y < x \\ 0, & \text{otherwise} \end{cases}$$

(a) To show $f(x, y)$ is joint probability density function, we need to show $f(x, y) \geq 0$ and

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\, dx\, dy = 1$$

Now
$$\int_0^1 \int_0^x 2\, dx\, dy = \int_0^1 2\left[\int_0^x dy\right] dx$$

$$= \int_0^1 2\,[y]_0^x\, dx = \int_0^1 2x\, dx = \left[2\frac{x^2}{2}\right]_0^1$$

$$= [x^2]_0^1 = 1 - 0 = 1.$$

Hence
$$\int_0^1 \int_0^x 2\, dx\, dy = 1.$$

It is clear $f(x, y) = 2 \geq 0$.

This shows $f(x, y)$ is point probability distribution function.

(b) Marginal density function of U and V.

Marginal density function of X is given by

$$f_U(x) = \int_{-\infty}^{\infty} f(x, y)\, dy = \int_0^x 2\, dy = [2x]_0^x = 2x$$

$$\therefore \quad f_U(x) = 2x, \quad 0 < x < 1$$
$$= 0, \quad \text{otherwise}$$

Marginal density function of V is given by

$$f_V(y) = \int_{-\infty}^{\infty} f(x, y)\,dx = \int_y^1 2\,dx = [2x]_y^1 = 2\,[1-y]$$

$$\therefore \quad f_V(y) = 2\,(1-y), \quad 0 < y < 1$$
$$= 0, \quad \text{otherwise}$$

(c) The conditional density function V given $U = x$ and that of U given $V = y$.

The conditional density function of V given $U = x$ is given by

$$f_{V/U}(y/x) = \frac{f(x, y)}{f_U(x)} = \frac{2}{2x} = \frac{1}{x}, \quad 0 < x < 1$$

The conditional density function of U given $V = y$ is given by

$$f_{U/V}(x/y) = \frac{f(x, y)}{f_V(y)} = \frac{2}{2\,(1-y)}, \quad 0 < y < 1$$

(d) Are U and V independent. For this show that $f(x, y) = f_U(x) \cdot f_V(y)$, if not, then U and V are not independent.

$$\because \quad f(x, y) = 2,\ f_U(x) \times f_V(y) = 2x \times 2\,(1-y) = 4x - 4xy$$

This $\quad f(x, y) \neq f_U(x) \cdot f_V(y)$.

Example 17. ***Let X and Y be two jointly continuous random variables with joint PDF***

$$f_{XY}(x, y) = \begin{cases} \mathbf{2}, & y + x \leq 1, x > 0, y > 0 \\ \mathbf{0}, & \textit{otherwise} \end{cases}$$

Find Cov (X, Y) and ρ(X, Y) i.e. Find covariance and correlation.

Solution. Given that

$$f_{XY}(x, y) = \begin{cases} 2, & y + x \leq 1, x > 0, y > 0 \\ 0, & \text{otherwise} \end{cases}$$

For $0 \leq x \leq 1$, we have

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y)\,dy$$
$$= \int_0^{1-x} 2\,dy \qquad [\because f_{XY}(x, y) = 2 \text{ for } x \geq 0, y \geq 0, y + x \leq 1]$$
$$= [2y]_0^{1-y} = 2\,[1-x]$$

Thus, $$f_X(x) = \begin{cases} 2\,(1-x), & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

Similarly, we obtain

$$f_Y(y) = \begin{cases} 2\,(1-y), & 0 \leq y \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

Thus, expectation of $X = E(X)$, is

$$E(X) = \int_0^1 x\, f_X(x)\,dx$$
$$= \int_0^1 x \cdot 2\,(1-x)\,dx = \int_0^1 (2x - 2x^2)\,dx$$

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$$= 2\int_0^1 (x - x^2)\, dx$$

$$= 2\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$$

$$= 2\left[\frac{1}{2} - \frac{1}{3}\right] = 2 \times \frac{1}{6} = \frac{1}{3}$$

Similarly, $E(Y) = \int_0^1 y\, f_Y(y)\, dy$

$$= \int_0^1 y \,.\, 2(1 - y)\, dy$$

$$= \int_0^1 2y - 2y^2\, dy$$

$$= 2\int_0^1 (y - y^2)\, dy$$

$$= 2\left[\frac{y^2}{2} - \frac{y^3}{3}\right]_0^1 = \frac{1}{3}$$

$$E(X^2) = \int_0^1 x^2 \,.\, 2(1 - x)\, dx$$

$$= 2\int_0^1 (x^2 - x^3)\, dx = 2\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1$$

$$= 2\left[\frac{1}{3} - \frac{1}{4}\right]$$

$$= 2\left[\frac{4-3}{12}\right] = 2\left(\frac{1}{12}\right)$$

$$= \frac{1}{6}$$

$$E(Y)^2 = \int_0^1 y^2\, f_Y(y)\, dy$$

$$= \int_0^1 y^2 \,.\, 2(1 - y)\, dy$$

$$= 2\int_0^1 y^2\,(1 - y)\, dy$$

$$= 2\int_0^1 (y^2 - y^3)\, dy$$

$$= 2\left[\frac{y^3}{3} - \frac{y^4}{4}\right]_0^1 = 2\left[\frac{1}{3} - \frac{1}{4}\right]$$

$$= 2\left(\frac{1}{12}\right) = \frac{1}{6}$$

$$E(XY) = \int_0^1 \int_0^{1-x} xy\, f_{XY}(x, y)\, dx\, dy$$

$$= \int_0^1 \int_0^{1-x} xy \,.\, 2\, dx\, dy$$

$$= \int_0^1 \int_0^{1-x} 2\, xy\, dx\, dy$$

$$= \int_0^1 2x \left[\int_0^{1-x} y\, dy\right] dx$$

$$= \int_0^1 2x \left[\frac{y^2}{2}\right]_0^{1-x} dx$$

$$= \int_0^1 2x \left[\frac{(1-x)^2}{2}\right] dx$$

$$= \int_0^1 x(1-x)^2\, dx$$

$$= \frac{1}{12}$$

Now $\sigma_x^2 = \text{Var}(X) = E(X^2) - (E(X))^2$

$$= \frac{1}{6} - \left(\frac{1}{3}\right)^2 = \frac{1}{6} - \frac{1}{9}$$

$$= \frac{3-2}{12} = \frac{1}{18}$$

$$\sigma_x = \sqrt{\text{Var}(X} = \sqrt{\frac{1}{18}} = \frac{1}{3\sqrt{2}}$$

Similarly, $\sigma_y^2 = \text{Var}(Y) = E(Y^2) - (E(Y))^2$

$$= \frac{1}{18}$$

Determine $\text{Cov}(X, Y) = E(XY) - E(X)\, E(Y)$

$$= \frac{1}{12} - \left(\frac{1}{3}\right)\left(\frac{1}{3}\right) = \frac{1}{12} - \frac{1}{9} = \frac{3-4}{36}$$

$$= -\frac{1}{36}$$

and $\text{Correlation} = \rho(X, Y) = \dfrac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X)\,\text{Var}(Y)}}$

$$= \frac{-\frac{1}{36}}{\sqrt{\frac{1}{18} \times \frac{1}{18}}}$$

$$= \frac{-\frac{1}{36}}{\frac{1}{18}} = -\frac{1}{36} \times \frac{18}{1} = -\frac{1}{2}$$

Example 18. ***The joint p.d.f. of U and V be the***

$$f_{UV}(x, y) \text{ or } f(x, y) = \begin{cases} (x+y), & 0 \le x \le 1, 0 \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Find (i) $P\left[0 < U < \frac{1}{2}, \; 0 < V < \frac{1}{2}\right]$

(ii) $E(U), E(V), E(UV)$ ***and*** $E(U+V)$

(iii) $\sigma(U, V)$

Solution. Given joint p.d.f.

$$f_{UV}(x, y) \text{ or } f(x, y) = \begin{cases} (x+y), & 0 \le x \le 1, 0 \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$$

(i) $P\left[0 < U < \frac{1}{2}, \; 0 < V < \frac{1}{4}\right]$

$$= \int_0^{1/2} \int_0^{1/4} f_{UV}(x, y)\, dx\, dy$$

$$= \int_0^{1/2} \int_0^{1/4} (x+y)\, dx\, dy = \int_0^{1/2} \left[\int_0^{1/4} (x+y)\, dy\right] dx$$

$$= \int_0^{1/2} \left[xy + \frac{y^2}{2}\right]_0^{1/4} dx$$

$$= \int_0^{1/2} \left(\frac{1}{4}x + \frac{1}{16 \times 2}\right) dx = \int_0^{1/2} \left(\frac{x}{4} + \frac{1}{32}\right) dx$$

$$= \left[\frac{x^2}{8} + \frac{1}{32}x\right]_0^{1/2} = \frac{1}{32} + \frac{1}{64} = \frac{3}{64}$$

(ii) Find $E(U)$, $E(V)$, $E(UV)$ and $E(U+V)$

$E(U)$ is given by

$$E(U) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x\, f_{UV}(x, y)\, dx\, dy$$

or

$$E(U) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x \,.\, f(x, y)\, dx\, dy$$

$$= \int_0^1 \int_0^1 x(x+y)\, dx\, dy$$

$$= \int_0^1 \left[\int_0^1 (x^2 + xy)\, dy\right] dx$$

$$= \int_0^1 \left[x^2 y + \frac{xy^2}{2}\right]_0^1 dx$$

$$= \int_0^1 \left(x^2 + \frac{x}{2}\right) dx$$

$$= \left(\frac{x^3}{3} + \frac{x^2}{4}\right)_0^1 = \frac{1}{3} + \frac{1}{4} = \frac{4+3}{12} = \frac{7}{12}$$

$$E(V) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y\, f_{UV}(x, y)\, dx\, dy$$

$$= \int_0^1 \int_0^1 y\,(x + y)\, dx\, dy$$

$$= \int_0^1 \left[\int_0^1 (yx + y^2)\, dx\right] dy$$

$$= \int_0^1 \left[\frac{x^2 y}{2} + xy^2\right]_0^1 dy$$

$$= \int_0^1 \left[\frac{y}{2} + y^2\right] dy$$

$$= \left(\frac{y^2}{4} + \frac{y^3}{3}\right)_0^1 = \left(\frac{1}{4} + \frac{1}{3}\right) = \frac{3+4}{12} = \frac{7}{12}$$

$$E(UV) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy\, f_{UV}(x, y)\, dx\, dy$$

$$= \int_0^1 \int_0^1 xy\,(x + y)\, dx\, dy$$

$$= \int_0^1 x\left[\int_0^1 y(x + y)\, dy\right] dx$$

$$= \int_0^1 x\left[\int_0^1 (yx + y^2)\, dy\right] dx$$

$$= \int_0^1 x\left[\frac{xy^2}{2} + \frac{y^3}{3}\right]_0^1 dx$$

$$= \int_0^1 x\left(\frac{x}{2} + \frac{1}{3}\right) dx$$

$$= \int_0^1 \left(\frac{x^2}{2} + \frac{1}{3}x\right) dx = \left[\frac{x^3}{6} + \frac{x^2}{6}\right]_0^1$$

$$= \left(\frac{1}{6} + \frac{1}{6}\right) = \frac{2}{6} = \frac{1}{3}$$

$$E(U + V) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x + y)\, f_{UV}(x, y)\, dx\, dy$$

$$= \int_0^1 \int_0^1 (x + y)(x + y)\, dx\, dy$$

$$= \int_0^1 \left[\int_0^1 (x^2 + y^2 + 2xy)\, dy\right] dx$$

$$= \int_0^1 \left[x^2 y + \frac{y^3}{3} + 2x\frac{y^2}{2}\right]_0^1 dx$$

$$= \int_0^1 \left(x^2 + \frac{1}{3} + 2x\frac{1}{2}\right) dx$$

$$= \int_0^1 \left(x^2 + \frac{1}{3} + x\right) dx$$

$$= \left[\frac{x^3}{3} + \frac{1}{3}x + \frac{x^2}{2}\right]_0^1 = \left[\frac{1}{3} + \frac{1}{3} + \frac{1}{2}\right]$$

$$= \frac{2+2+3}{6} = \frac{7}{6}$$

(iii) Find $\sigma(U, V)$

Calculate

$$\text{Cov}(U, V) = E(UV) - E(U)\,E(V)$$

$$= \frac{1}{3} - \frac{7}{12} \times \frac{7}{12} = \frac{1}{3} - \frac{49}{144}$$

$$= \frac{48-49}{144} = -\frac{1}{144}$$

$$E(U^2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^2 f_{UV}(x, y)\, dx\, dy$$

$$= \int_0^1 \int_0^1 x^2 (x+y)\, dx\, dy$$

$$= \int_0^1 \left[\int_0^1 (x^3 + x^2 y)\, dy\right] dx$$

$$= \int_0^1 \left[x^3 y + \frac{x^2 y^2}{2}\right]_0^1 dx$$

$$= \int_0^1 \left[x^3 + \frac{x^2}{2}\right] dx = \left(\frac{x^4}{4} + \frac{x^3}{6}\right)_0^1$$

$$= \frac{1}{4} + \frac{1}{4} = \frac{3+2}{12} = \frac{5}{12}$$

$$E(V^2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y^2 f_{UV}(x, y)\, dx\, dy$$

$$= \int_0^1 \int_0^1 y^2 (x+y)\, dx\, dy$$

$$= \frac{5}{12}$$

Now, $\quad \text{Var}(U) = E(U^2) - [E(U)]^2$

$$= \frac{5}{12} - \left(\frac{7}{12}\right)^2 = \frac{5}{12} - \left(\frac{49}{144}\right)$$

$$= \frac{60-49}{144} = \frac{11}{144}$$

$$\text{Var}(V) = E(V^2) - [E(V)]^2$$

$$= \frac{5}{12} - \left[\frac{7}{12}\right]^2 = \frac{5}{12} - \frac{49}{144}$$

$$= \frac{60-49}{144} = \frac{11}{144}$$

Hence, correlation $\rho(U, V) = \dfrac{\text{Cov}(U, V)}{\sqrt{\text{Var}(U) \times \text{Var}(V)}}$

$$= -\frac{\frac{1}{144}}{\sqrt{\frac{11}{144} \times \frac{11}{144}}} = -\frac{\frac{1}{144}}{\frac{11}{144}} = -\frac{1}{11}$$

Example 19. *Two random variable U and V have the following joint probability density function :*

$$f_{UV}(x, y) \text{ or } f(x, y) = \begin{cases} 2 - x - y, & 0 \le x \le 1, 0 \le y \le 1 \\ 0, & \textit{otherwise} \end{cases}$$

Calculate :

(i) Marginal probability density function of U and V

(ii) Conditional density function

(iii) Variance of U and Variance of V i.e., Var (U) and Var (V)

(iv) Covariance between U and V i.e, Cov (U, V).

Solution. Given joint p.d.f.

$$f_{UV}(x, y) \text{ or } f(x, y) = \begin{cases} 2 - x - y, & 0 \le x \le 1, 0 \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$$

(i) Marginal pdf of U is given by

$$f_U(x) = \int_{-\infty}^{\infty} f_{UV}(x, y)\, dy \text{ or } \int_{-\infty}^{\infty} f(x, y)\, dy$$

$$= \int_0^1 (2 - x - y)\, dy = \left[2y - xy - \frac{y^2}{2}\right]_0^1$$

$$= \left[2 - x - \frac{1}{2}\right] = \frac{4 - 2x - 1}{2} = \frac{3 - 2x}{2} = \frac{3}{2} - x.$$

$$\therefore \quad f_U(x) = \begin{cases} 3/2 - x, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Marginal pdf of V is given by

$$f_V(y) = \int_{-\infty}^{\infty} f_{UV}(x, y)\, dx \text{ or } \int_{-\infty}^{\infty} f(x, y)\, dx$$

$$f_V(y) = \int_0^1 (2 - x - y)\, dx = \left[2x - \frac{x^2}{2} - xy\right]_0^1$$

$$= 2 - \frac{1}{2} - y = \frac{4 - 1 - 2y}{2} = \frac{3 - 2y}{2} = \frac{3}{2} - y.$$

$$\therefore \quad f_V(y) = \begin{cases} 3/2 - x, & 0 < y < 1 \\ 0, & \text{otherwise.} \end{cases}$$

(ii) Conditional density function of U given $V = y$ is given by

$$f_{U/V}(x/y) = \frac{f_{U/V}(x, y)}{f_V(y)} = \frac{f(x, y)}{f_V(y)}$$

$$= \frac{(2 - x - y)}{(3/2 - y)}, \; 0 < x, y < 1.$$

Conditional density function of V given $U = x$ is given by

$$f_{V/U}(y/x) = \frac{f_{U/V}(x, y)}{f_U(x)} = \frac{f(x, y)}{f_U(x)}$$

$$= \frac{(2 - x - y)}{(3/2 - x)}, \; 0 < x, y < 1.$$

(iii) Variance of U *i.e.*, Var (U).

First calculate $E(U)$ and $E(U^2)$

$$E(U) = \int_{-\infty}^{\infty} x \cdot f_U(x)\, dx = \int_0^1 x\,(3/2 - x)\, dx = \int_0^1 \left(\frac{3}{2}x - x^2\right) dx$$

$$= \left[\frac{3}{2}\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \left(\frac{3}{4} - \frac{1}{3}\right) = \frac{5}{12}$$

$$E(U^2) = \int_{-\infty}^{\infty} x^2 f_U(x)\, dx = \int_0^1 x^2 \left(\frac{3}{2} - x\right) dx$$

$$= \int_0^1 \left(\frac{3x^2}{2} - x^3\right) dx = \left(\frac{3x^3}{6} - \frac{x^4}{4}\right)_0^1$$

$$= \left(\frac{3}{6} - \frac{1}{4}\right) = \frac{6 - 3}{12} = \frac{3}{12} = \frac{1}{4}.$$

$$\therefore \; \text{Var}(U) = E(U^2) - [E(U)]^2$$

$$= \frac{1}{4} - \left(\frac{5}{12}\right)^2 = \frac{1}{4} - \frac{25}{144} = \frac{11}{144}.$$

Variance of V *i.e.*, Var (V). First calculate $E(V)$ and $E(V^2)$

$$E(V) = \int_{-\infty}^{\infty} y f_V(y)\, dy$$

$$= \int_0^1 y \cdot \left(\frac{3}{2} - y\right) dy = \int_0^1 \left(\frac{3}{2}y - y^2\right) dy = \left[\frac{3y^2}{4} - \frac{y^3}{3}\right]_0^1$$

$$= \frac{3}{4} - \frac{1}{3} = \frac{5}{12}$$

$$E(V^2) = \int_{-\infty}^{\infty} y^2 f_V(y)\, dy = \int_0^1 y^2 \cdot \left(\frac{3}{2} - y\right) dy = \int_0^1 \left(\frac{3y^2}{2} - y^3\right) dy$$

$$= \left[\frac{3}{6}y^3 - \frac{y^4}{4}\right]_0^1 = \frac{3}{6} - \frac{1}{4} = \frac{6 - 3}{12} = \frac{3}{12} = \frac{1}{4}.$$

$\therefore \text{ Var } (V) = E(V^2) - [E(V)]^2$

$$= \frac{1}{4} - \left(\frac{5}{12}\right)^2 = \frac{1}{4} - \frac{25}{144} = \frac{11}{144}.$$

(iv) Covariance of U and $V = \text{Cov}(U, V)$

Cov (U, V) is given by

$$\text{Cov}(U, V) = E(UV) - E(U)\,E(V).$$

Calculate $E(UV)$.

Now
$$E(UV) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{UV}(x, y)\,dx\,dy$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x, y)\,dx\,dy$$
$$= \int_0^1\int_0^1 xy\,(2 - x - y)\,dx\,dy$$
$$= \int_0^1 y\left[\int_0^1 x\,(2 - x - y)\,dx\right]dy$$
$$= \int_0^1 y\left[\int_0^1 (2x - x^2 - xy)\,dx\right]dy$$
$$= \int_0^1 y\left[\frac{2x^2}{2} - \frac{x^3}{3} - \frac{x^2 y}{2}\right]_0^1 dy$$
$$= \int_0^1 y\left(1 - \frac{1}{3} - \frac{y}{2}\right)dy = \int_0^1 y\left(\frac{2}{3} - \frac{y}{2}\right)dy$$
$$= \int_0^1 \left(\frac{2}{3}y - \frac{y^2}{2}\right)dy = \left[\frac{2y^2}{6} - \frac{y^3}{6}\right]_0^1$$
$$= \left[\frac{2}{6} - \frac{1}{6}\right] = \frac{1}{6}.$$

Hence $\text{Cov}(U, V) = E(UV) - E(U)\,E(V)$
$$= \frac{1}{6} - \frac{5}{12}\cdot\frac{5}{12} = \frac{1}{6} - \frac{25}{144}$$
$$= \frac{24 - 25}{144} = \frac{-1}{144}.$$

Example 20. ***Let X and Y have joint p.d.f.***

↓ Y \ → X	−1	0	1
0	b	$2b$	b
10	$3b$	$2b$	b
2	$2b$	b	$2b$

Find marginal distribution of X and Y. Also find conditional distribution of X given Y = 1.

Kgt. LOKESH Jdikaria

Solution. The marginal distribution of X and Y are given as :

Y \ X	-1	0	1	$P_Y(y)$
0	b	$2b$	b	$4b$
1	$3b$	$2b$	b	$6b$
2	$2b$	b	$2b$	$5b$
$P_X(x)$	$6b$	$5b$	$4b$	$15b$

Marginal distribution of X

$$P_X(-1) = P(X = -1) = 6b,\ P_X(0) = P(X = 0) = 5b,\ P_X(1) = P(X = 1) = 4b.$$

Marginal distribution of Y

$$P_Y(0) = P(Y = 0) = 4b,\ P_Y(1) = P(Y = 1) = 6b,\ P_Y(2) = P(Y = 2) = 5b.$$

Conditional distribution of X when $Y = 1$

$$P(X = x / Y = 1) = \frac{P(X = x \cap Y = 1)}{P(Y = 1)}$$

$$= \frac{P(X = -1 \cap Y = 1)}{P(Y = 1)} = \frac{3b}{6b} = \frac{1}{2} \quad \text{when } X = -1 \mid Y = 1$$

$$= \frac{P(X = 0 \cap Y = 1)}{P(Y = 1)} = \frac{2b}{6b} = \frac{1}{3} \quad \text{when } X = 0 \mid Y = 1$$

$$= \frac{P(X = 1 \cap Y = 1)}{P(Y = 1)} = \frac{b}{6b} = \frac{1}{6} \quad \text{when } X = 1 \mid Y = 1.$$

◆ § 3.23. CALCULATION OF COEFFICIENT OF CORRELATION r FOR A BIVARIATE FREQUENCY DISTRIBUTION

For a bivariate frequency distribution, the coefficient of correlation r is defined as

$$r = \frac{\Sigma f\,uv - \dfrac{\Sigma fu\ \Sigma fv}{\Sigma f}}{\sqrt{\left[\left\{\Sigma fu^2 - \dfrac{(\Sigma fu)^2}{\Sigma f}\right\}\left\{\Sigma fv^2 - \dfrac{(\Sigma fv)^2}{\Sigma f}\right\}\right]}}$$

where u and v have their usual meaning and f is the frequency of a particular rectangle in the table of correlation.

ILLUSTRATIVE EXAMPLES

Example 1. ***Calculate the Karl Pearson's coefficient of correlation for the following table :***

x / y → ↓	0–5	5–10	10–15	15–20	20–25
0–4	1	2			
4–8		4	5	8	
8–12			3	4	
12–16				2	1

Solution. Let the assumed means for y and x series be $A_y = 12\cdot 5$ and $A_x = 10$ respectively and let their scales be $h' = 5$ and $h = 4$.

$\therefore$ The required coefficient of correlation is obtained by

			Class	0–5	5–10	10–15	15–20	20–25				
			Mid Value y	2.5	7.5	12.5	17.5	22.5				
			$\eta = y - 12.5$	− 10	− 5	0	5	10				
Class	Mid-value x	$\xi = x - 10$	$v = \frac{\eta}{h'}$ / $u = \frac{\xi}{h}$	− 2	− 1	0	1	2	Total (f)	fu	fu^2	fuv
0–4	2	− 8	− 2	4 / 1 / 4	2 / 2 / 4				3	− 6	12	8
4–8	6	− 4	− 1		1 / 4 / 4	0 / 5 / 0	−1 / 8 / −8		17	− 17	17	− 4
8–12	10	0	0			0 / 3 / 0	0 / 4 / 0		7	0	0	0
12–16	14	4	1				1 / 2 / 2	2 / 1 / 2	3	3	3	4
			Total (f)	1	6	8	14	1	30	− 20	32	8
			fv	− 2	− 6	0	14	2	8			
			fv^2	4	6	0	14	4	28			
			fuv	4	8	0	− 6	2	8			

$$r = \frac{\Sigma fuv - \dfrac{\Sigma fu\, \Sigma fv}{\Sigma f}}{\sqrt{\left[\left\{\Sigma fu^2 - \dfrac{(\Sigma fu)^2}{\Sigma f}\right\}\left\{\Sigma fv^2 - \dfrac{(\Sigma fv)^2}{\Sigma f}\right\}\right]}} = \frac{8 - \dfrac{(-20)\times 8}{30}}{\sqrt{\left[\left\{32 - \dfrac{(-20)^2}{30}\right\}\left\{28 - \dfrac{8^2}{30}\right\}\right]}}$$

$$= \frac{8 + 5\cdot 333}{\sqrt{[\{32 - 13\cdot 33\}\{28 - 2\cdot 13\}]}}$$

$$= \frac{13\cdot 333}{\sqrt{(18\cdot 67 \times 25\cdot 87)}} = \frac{13\cdot 33}{21\cdot 977} = 0\cdot 6055.$$

Remark. In the above example, there are some squares (nine in this case) which contain three numbers. In each of these squares :

(i) the number 'on the left hand top is the value of the product uv'

(ii) the number in the middle represents the frequency (f) which is given in question, and

(iii) the number in the circle 'on the right hand bottom is the value of the product fuv'.

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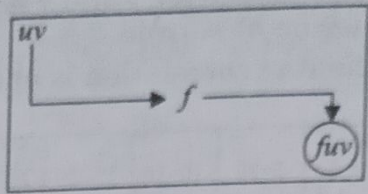
These points may be represented as :



Example 2. *Find the coefficient of correlation from the following table:*

x / y	0–4	4–8	8–12	12–16
0–5	7			
5–10	6	8		
10–15		5	3	
15–20		7	2	
20–25				9

Solution. Let $u = \dfrac{\xi}{h} = \dfrac{x - 10}{4}$ and $v = \dfrac{\eta}{h'} = \dfrac{y - 12 \cdot 5}{5}$, then

			Class	0–4	4–8	8–12	12–16				
			$x \rightarrow$ Mid Value x	2	6	10	14				
	$y \downarrow$		$\xi = x - 10$	– 8	– 4	0	4				
Class	Mid-value y	$\eta = y - 12.5$	$u = \frac{\xi}{h}$ / $v = \frac{\eta}{h'}$	– 2	– 1	0	1	Total (f)	fv	fv^2	fuv
0–5	2.5	– 10	– 2	4 7 28				7	– 14	28	28
5–10	7.5	– 5	– 1	2 6 12	1 8 8			14	– 14	14	20
10–15	12.5	0	0		0 5 0	0 3 0		8	0	0	0
15–20	17.5	5	1		–1 7 –7	0 2 0		9	9	9	– 7
20–25	22.5	10	2				2 9 18	9	18	36	18
			Total (f)	13	20	5	9	47	– 1	87	59
			fu	– 26	-20	0	9	– 37			
			fu^2	52	20	0	9	81			
			fuv	40	1	0	18	59			

$\therefore$ The required coefficient of correlation

$$r = \frac{\Sigma f\,uv - \dfrac{\Sigma fu\,\Sigma fv}{\Sigma f}}{\sqrt{\left[\left\{\Sigma fu^2 - \dfrac{(\Sigma fu)^2}{\Sigma f}\right\}\left\{\Sigma fv^2 - \dfrac{(\Sigma fv)^2}{\Sigma f}\right\}\right]}} = \frac{59 - \dfrac{(-37)\times(-1)}{47}}{\sqrt{\left[\left\{81 - \dfrac{(-37)^2}{47}\right\}\left\{87 - \dfrac{(-1)^2}{47}\right\}\right]}}$$

$$= \frac{59 - 0\cdot7872}{\sqrt{[\{81 - 29\cdot1276\}\{87 - 0\cdot0213\}]}} = \frac{58\cdot2128}{\sqrt{(51\cdot8782\times 86\cdot9787)}} = \frac{58\cdot2128}{67\cdot1697}$$

$= 0\cdot8666.$

Example 3. ***Find the coefficient of correlation from the following table :***

x/ y	67	72	77	82	87	92	97
92				1	2	3	1
87			1	3	8	1	5
82	4	4	6	4	9	1	
77	3	3	7	6	4	0	
72	2	3	5	6	1	1	
67	3	2					
62	1						

Solution. Let $u = \dfrac{x-82}{5}$ and $v = \dfrac{y-77}{5}$. See calculation table.

	x	67	72	77	82	87	92	97				
y	v \ u	−3	−2	−1	0	1	2	3	Total (f)	fv	fv^2	fuv
92	3				0 1 (0)	3 2 (6)	6 3 (18)	9 1 (9)	7	21	63	33
87	2			−2 1 (−2)	0 3 (0)	2 8 (16)	4 1 (4)	6 5 (30)	18	36	72	48
82	1	−3 4 (−12)	−2 4 (−8)	−1 6 (−6)	0 4 (0)	1 9 (9)	2 1 (2)		28	28	28	−15
77	0	0 3 (0)	0 3 (0)	0 7 (0)	0 6 (0)	0 4 (0)			23	0	0	0
72	−1	3 2 (6)	2 3 (6)	1 5 (5)	0 6 (0)	−1 1 (−1)	−2 1 (−2)		18	−18	18	14
67	−2	6 3 (18)	4 2 (8)						5	−10	20	26
62	−3	9 1 (9)							1	−3	9	9
	Total (f)	13	12	19	20	24	6	6	100	54	210	115
	fu	−39	−24	−19	0	24	12	18	−28			
	fu^2	117	48	19	0	24	24	54	286			
	fuv	21	6	−3	0	30	22	39	115			

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$\therefore$ The required coefficient of correlation is given by

$$r = \frac{\Sigma f\, uv - \dfrac{\Sigma fu\, \Sigma fv}{\Sigma f}}{\sqrt{\left[\left\{\Sigma fu^2 - \dfrac{(\Sigma fu)^2}{\Sigma f}\right\}\left\{\Sigma fv^2 - \dfrac{(\Sigma fv)^2}{\Sigma f}\right\}\right]}}$$

$$= \frac{115 - \dfrac{(-28)\times(54)}{100}}{\sqrt{\left[\left(286 - \dfrac{(-28)^2}{100}\right)\left(210 - \dfrac{(54)^2}{100}\right)\right]}} = \frac{115 + 15\cdot 12}{\sqrt{\{286 - 7\cdot 84\}\{210 - 29\cdot 16\}}}$$

$$= \frac{130\cdot 12}{\sqrt{\{(180\cdot 84)(278\cdot 16)\}}} = \frac{130\cdot 12}{\sqrt{(50302\cdot 4544)}} = \frac{130\cdot 12}{224\cdot 48} = \frac{13012}{22448} = 0\cdot 58.$$

EXERCISE 3 (A)

1. Find the coefficient of correlation from the following table :

x / y	0–4	4–8	8–12	12–16
0–5	1			
5–10	2	4	1	2
10–15		5	3	
15–20		8	4	2
20–25	2			1

2. Calculate the coefficient of correlation from the following table giving the ages of 100 husbands and their wives in years :

Age of husbands → Age of wives ↓	20–30	30–40	40–50	50–60	60–70	Totals
15–25	5	9	3			17
25–35		10	25	2		37
35–45		1	12	2		15
45–55			4	16	5	25
55–65				4	2	6
Totals	5	20	44	24	7	100

3. Calculate the coefficient of correlation from the following table giving the ages of husbands and their wives :

Age of wives (y)	Age of husbands (x)					Totals
	10–20	20–30	30–40	40–50	50–60	
15–25	6	3	—	—	—	9
25–35	3	16	10	—	—	29
35–45	—	10	15	7	—	32
45–55	—	—	7	10	4	21
55–65	—	—	—	4	5	9
Totals	9	29	32	21	9	100

4. The following table gives according to age the frequency of marks obtained by 100 students in an intelligence test :

Age in years → Marks ↓	18	19	20	21	Totals
10–20	4	2	2		8
20–30	5	4	6	4	19
30–40	6	8	10	11	35
40–50	4	4	6	8	22
50–60		2	4	4	10
60–70		2	3	1	6
Totals	19	22	31	28	100

Calculate the coefficient of correlation between age and intelligence.

5. Calculate from the data produced below pertaining to 66 selected village in Raipur district, the value of r, between 'total cultivated area' 'the area under wheat.'

Total cultivated area→ (In Bighas) ↓	0–500	500–1000	1000–1500	1500–2000	2000–2500	Totals
0–200	12	6				18
200–400	2	18	4	2	1	27
400–600		4	7	3		14
600–800		1		2	1	4
800–1000				1	2	3
Totals	14	29	11	8	4	66

6. Calculate the coefficient of correlation for the following table :

y/x	94·5	96·5	98·5	100·5	102·5	104·5	106·5	108·5	110·5
94·5			4	3		4	1		1
59·5	1	3	6	18	6	9	2	3	1
89·5	7	3	16	16	4	4	1		1
119·5	5	9	10	9	2		1	2	
149·5	3	5	8	1		1			
179·5	4	2	3	1					
209·5	4	4		1					
239·5	1	1							

ANSWERS

1. 0·12 2. 0·796 3. 0·802 4. 0·25
5. 0·749 6. –0·49.

§ 3.24. BAYES THEOREM OR BAYES RULE

The posterior probability of an event based on some prior probability by utilizing conditional probabilities.

Consider, the conditional probability formula discussed earlier :

$$P_{V/U}(y/x) = \frac{P_{UV}(x, y)}{P_U(x)}$$

$$\Rightarrow \quad P_{UV}(x, y) = P_{V/U}(y/x) \,.\, P_U(X)$$

and

$$P_{UV}(x, y) = P_{U/V}(x/y) \,.\, P_V(y).$$

§ 3.25. BAYE'S THEOREM FOR DISCRETE VARIABLE

3.25.1. Discrete Variable :

If U and V are two discrete random variables, then conditional probability function of U, when $V = y_j$, given by Bayes rule as :

$$P_{U/V}(x/y) = \frac{P_{UV}(x, y)}{P_V(y)}$$

$$\Rightarrow \quad P_{U/V}(x/y) = \frac{P_{V/U}(y/x)\, P_U(x)}{P_V(y)} \qquad \text{...(1)}$$

where $P_{V/U}(y/x)$ is conditional probability function of V, when $U = x_i$ is given, $P_U(x)$ is marginal probability distribution of U and $P_V(y)$ is being marginal distribution of V.

Using, the law of total probability, we have

$$P_V(y) = \sum_K P_{U/V}(y/k)\, P_U(k) \qquad \text{...(2)}$$

with new $P_V(y)$, equation (1) becomes,

$$P_{U/V}(x/y) = \frac{P_{V/U}(y/x) \,.\, P_U(x)}{\sum_k P_{V/U}(y/k)\, P_U(k)} \,. \qquad \text{(using chain)} \quad \text{...(3)}$$

Equation (3) gives Bayes rule for two discrete variable U and V.

§ 3.26. BAYES THEOREM FOR CONTINUOUS VARIABLE

If U and V are continuous random variables with joint probability density function $f_{UV}(x, y)$. We know that

$$f_{U,V}(x, y) = f_{V/U}(y/x)\, f_U(x)$$
$$= f_{U/V}(x/y)\, f_V(y). \qquad \text{...(1)}$$

Then conditional probability function of U, when $V = y$, is given by

$$f_{U/V}(x/y) = \frac{f_{V/U}(y/x)}{f_V(y)}\, f_U(x) \qquad \text{...(2)}$$

where $f_{V/U}(y/x)$ is conditional probability function of V when $U = x$.

$f_U(x)$ is marginal probability distribution of U, $f_V(y)$ is marginal distribution of V.

Using the fact

$$f_V(y) = \int_{-\infty}^{\infty} f_{UV}(u, y)\, du = \int_{-\infty}^{\infty} f_U(u)\, f_{V/U}(y/u)\, du. \quad ...(3)$$

Equation (2), becomes

$$f_{U/V}(x/y) = \frac{f_{V/U}(y/x)}{\int_{-\infty}^{\infty} f_U(u)\, f_{V/U}(y/u)\, du} f_U(x). \quad ...(4)$$

Equation (4) gives Bayes rule for two continuous variables U and V.

§ 3.27. BAYES RULE FOR MIXED CONTINUOUS-DISCRETE VARIABLE

If U is discrete random variable and $(U = x)$ is well defined and V is continuous random variable, then a mixed continuous discrete variable can be represent in Bayes rule is

$$P_{U/V}(x/y) = P(U = x \mid V = y) \quad ...(1)$$

$$= \frac{f_{V/U}(y/x)\, P_U(x)}{f_V(y)}$$

$$= \frac{f_{V/U}(y/x)\, P_U(x)}{\sum_k f_{V/U}(y/k)\, P_U(k)} \quad ...(2)$$

where $P_U(x)$ is marginal probability distribution of discrete variable U, $f_V(y)$ is marginal probability distribution of continuous variable V, $f_{V/U}(y/x)$ is conditional probability distribution function of continuous variable V when $U = x_i$.

EXERCISE 3 (B)

1. Discuss bivariate distribution, joint and marginal probability distribution.
2. Explain in term of discrete random variable :
   (i) Conditional probability distribution function.
   (ii) Independent.
   (iii) Joint probability mass function.
3. Define probability distribution for a pair of continuous random variables.
4. Discuss elementry properties of joint distribution function.
5. Explain in term of continuous random variables :
   (i) Marginal probability density function.
   (ii) Conditional probability density function.
   (iii) Conditional cummulative distribution function.
   (iv) Independence.
6. Explain expectations and conditional expectation with example.
7. Explain joint MGF and prove that
$$m_{UV}(t_1, t_2) = M_U(t_1) \cdot M_V(t_2).$$
8. Explain covariance and variance of two random variables.
9. Prove that the variance of sum of two independent variables is equal to the sum of their separate variable prove that
i.e., $$\text{Var}(U + V) = \text{Var}(U) + \text{Var}(V).$$
10. Discuss Bayes theorem or Bayes rule for bivariate continuous and discrete random variable.

11. Find K so that the function

$$f(x, y) = K(x + y),\ 0 < x < 1,\ 0 < y < 1$$

is a joint probability density function.

12. Two tetrahedral with sides numbered 1 to 4 are tossed. Let U denotes the number on the down turned face of the first tetrahedron and U and the larger of the down turned numbers. Find :

(i) The joint density function U and V.

(ii) The marginal density function of U and V.

(iii) $P[U \le 2, V \le 3]$.

(iv) The conditional distribution of V given $U = 2$ and $V = 3$.

(v) $E[V/U = 2]$ and $E[V/U = 3]$.

(vi) $\rho(U, V)$ *i.e.*, correlation of U and V.

[**Hint.** Sample space is $S = \{1, 2, 3, 4\} \times \{1, 2, 3, 4\}$ and each has the outcomes of probability $= \frac{1}{16}$.

Let U = number on the first dice (down turned face)

V = larger of the numbers on the two dice (two down turned faces)

(i) The joint discrete density function of U and V is given below :

(x, y):	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(2, 2)	(2, 3)	(2, 4)	(3, 3)	(3, 4)	(4, 4)
$f_{UV}(x, y)$:	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{3}{16}$	$\frac{1}{16}$	$\frac{4}{16}$

In tabular form

U \ V	1	2	3	4	$f_U(x)$ (Marginal of U)
1	1/16	1/6	1/16	1/16	4/16
2	0	2/16	1/16	1/16	4/16
3	0	0	3/16	1/16	4/16
4	0	0	0	4/16	4/16
$f_V(y)$ Marginal of V	1/16	3/16	5/16	7/16	1

(ii) Marginal distribution of U and V

$$f_U(1) = 4/16,\ f_U(2) = 4/16,\ f_U(3) = 4/16,\ f_U(4) = 4/16$$
$$f_V(1) = 1/16,\ f_V(2) = 3/16,\ f_V(3) = 5/16,\ f_V(4) = 7/16.$$

(iii) $P[U \le 2, V \le 3]$

$$= P[U = 1, V = 1] + P[U = 1, V = 2] + P[U = 1, V = 3]$$
$$+ P[U = 2, V = 1] + P[U = 2, V = 2] + P[U = 2, V = 3]$$
$$= \left(\frac{1}{16} + \frac{1}{16} + \frac{1}{16}\right) + \left(0 + \frac{2}{16} + \frac{1}{16}\right)$$
$$= \frac{6}{16} = \frac{3}{8}.$$

(iv) The conditional distribution of V given $U = 2$

$$P[V = y \mid U = 2] = \frac{P[V = y \cap U = 2]}{P[U = 2]}$$

$$y = 1, \quad P[V = 1 | U = 2] = \frac{P[V = 1 \cap U = 2]}{P[U = 2]} = \frac{P[1, 2]}{P(U = 2) = f_U(2)} = \frac{0}{\frac{4}{16}} = 0$$

$$y = 2, \quad P[V = 2 | U = 2] = \frac{P[V = 2 \cap U = 2]}{P[U = 2]} = \frac{1}{2}$$

$$y = 3, \quad P[V = 3 | U = 2] = \frac{P[V = 3 \cap U = 2]}{P[U = 2]} = \frac{1}{4}$$

$$y = 4, \quad P[V = 4 | U = 2] = \frac{P[V = 4 \cap U = 2]}{P[U = 2]} = \frac{1/16}{4/16} = \frac{1}{4}.$$

Now the conditional distribution of V given $U = 3$

$$P[V = y | U = 3] = \frac{P[V = y \cap U = 3]}{P[U = 3]}$$

$$y = 1, \; P[V = 1 | U = 3] = \frac{P[V = 1 \cap U = 3]}{P[U = 3] = f_U(3)} = \frac{0}{4/16}$$

$$y = 2, \; P[V = 2 | U = 3] = \frac{P[V = 2 \cap U = 3]}{P[U = 3]} = 0$$

$$y = 3, \; P[V = 3 | U = 3] = \frac{P[V = 3 \cap U = 3]}{P[U = 3]} = \frac{3}{4}$$

$$y = 4, \; P[V = 4 | U = 3] = \frac{P[V = 4 \cap U = 3]}{P[U = 3]} = \frac{1}{4}.$$

(v) The conditional expectation $E[V / U = 2]$

$$= \sum_j y_j \, P[V = y_j | U = 2] = \Sigma y_j \, . \, f(y / x = 2)$$

$$= \Sigma \, | \, y_j \, . \, \frac{f[x = 2 \cap y]}{f(x = 2)}$$

$$= 1 \, . \, P[V = 1 | U = 2] + 2 \, . \, P[V = 2 | U = 2]$$

$$+ 3 \, . \, P[V = 3 | U = 2] + 4 \, . \, P[V = 4 | U = 2]$$

$$= 1 \, . \, 0 + 2 \, . \, \frac{1}{2} + 3 \, . \, \frac{1}{4} + 4 \, . \, \frac{1}{4} = \frac{11}{4}.$$

The conditional expectation $E(V / U = 3)$

$$= \sum_j y_j \, P[V = y_j | U = 3]$$

$$= 1 \, . \, P[V = 1 | U = 3] + 2 \, . \, P[V = 2 | U = 3]$$

$$+ 3 \, . \, P[V = 3 | U = 3] + 4 \, . \, P[V = 4 | U = 3]$$

$$= 1 \times 0 + 2 \times 0 + 3 \times \frac{3}{4} \times 4 \times \frac{1}{4} = \frac{13}{4}.$$

(vi) Correlation coefficient of U and V *i.e.*, $\rho(U, V)$ First find Cov (U, V) *i.e.*, covariance between U and V

$$E(U) = \Sigma x_i \, f_U(x)$$

$$= 1 \, . \, f_U(1) + 2 \, . \, f_U(2) + 3 \, . \, f_U(3) + 4 \, . \, f_U(4)$$

$$= 1 \times \frac{4}{16} + 2 \times \frac{4}{16} + 3 \times \frac{4}{16} + 4 \times \frac{4}{16}$$

$$= \frac{5}{2}$$

$$E(U^2) = \Sigma x^2 \, . \, f_U(x) = (1)^2 \times \frac{4}{16} + (2)^2 \times \frac{4}{16} + (3)^2 \times \frac{4}{16} + (4)^2 \times \frac{4}{16} = \frac{15}{2}$$

$$E(V) = \Sigma y \,.\, f_V(y) = 1 \,.\, f_V(1) + 2 \,.\, f_V(2) + 3 \,.\, f_V(3) + 4 \,.\, f_V(4)$$
$$= 1 \times \frac{1}{16} + 2 \times \frac{3}{16} + 3 \times \frac{5}{16} + 4 \times \frac{7}{16}$$
$$= \frac{25}{8}$$
$$E(V^2) = \Sigma y^2 \,.\, f_V(y) = \frac{85}{8}$$
$$E(UV) = \sum_x \sum_y xy \,.\, f_{UV}(x, y)$$
$$= \left(1 \times \frac{1}{16} + 2 \times \frac{1}{16} + 3 \times \frac{1}{16} + 4 \times \frac{1}{16}\right) + \left(2 \times 0 + 4 \times \frac{2}{16} + 6 \times \frac{1}{16} + 8 \times \frac{1}{16}\right) + \left(3 \times 0 + 6 \times 0 + 9 \times \frac{3}{16} + 12 \times \frac{1}{16}\right) + \left(4 \times 0 + 8 \times 0 + 12 \times 0 + 16 \times \frac{4}{16}\right)$$
$$= \frac{135}{16}.$$
$$\text{Var}(U) = E(U^2) - [E(U)]^2 = \frac{15}{2} - \left(\frac{5}{2}\right)^2 = \frac{5}{4}$$
$$\text{Var}(V) = E(V^2) - (E(V))^2 = \frac{85}{8} - \left(\frac{25}{8}\right)^2 = \frac{55}{64}.$$

Now
$$\text{Cov}(U, V) = E(UV) - E(U)\,E(V)$$
$$= \frac{135}{16} - \left(\frac{5}{8} \times \frac{25}{8}\right) = \frac{5}{8}$$

and
$$\rho(U, V) = \frac{\text{Cov}(U, V)}{\sqrt{\text{Var}(U) \times \text{Var}(V)}}$$
$$= \frac{\frac{5}{8}}{\sqrt{\frac{5}{4} \times \frac{55}{64}}} = \frac{2}{\sqrt{11}}.$$

13. Two tetrahedral with sides numbered 1 to 4 are tossed. Let X denote the smaller of down turned face of two tetrahedral and Y the larger of two :

(a) Find joint density function of X and Y

(b) Find $P(X \geq 2, Y \geq 2)$.

(c) Find mean and variance of X and Y.

(d) Find the conditional distribution of Y given X for each possible value of X.

(e) Find correlation coefficient of X and Y.

14. Consider a single of size 2 drawn without replacement from an urn containing three balls, numbered 1, 2 and 3. Let X be the number on the first ball drawn and Y the larger of the two numbers drawn

(a) Find joint discrete density function of X and Y

(b) Find $P[X = 1 \mid Y = 3]$

(c) Find cov (X, Y)

(d) Find $\rho(X, Y)$.

15. An own contain four balls. Two of the balls are numbered with 1 and other two are numbers with 2. Two balls are drawn from the urn without replacement. Let X denote the smallest of the numbers on drawn balls and Y the large :

(a) Find joint density of X and Y

(b) Find marginal distribution of Y

(c) Find cov (X, Y).

16. Three coins are tossed. Let X denote the number of heads on first two and Y denote the number of heads on last two :

(a) Find the joint distribution of X and Y.

(b) Find $E[Y / X = 1]$ (b) Find $\rho(X, Y)$.

17. Two discrete random variables U and V have

$$P[U=0, V=0]=\frac{2}{9},\ P[U=0, V=1]=\frac{1}{9},$$

$$P[U=1, V=0]=\frac{1}{9},\ P[U=1, V=1]=\frac{5}{9}.$$

Test whether U and V are independent.

18. If $f(x, y) = e^{-(x+y)},\ x \geq 0,\ y \geq 0$

$= 0,$ otherwise

is joint probability density function random variable X and Y, find :

(a) $P[X < 1]$ (b) $P[X > 4]$ (c) $P[X + Y < 1]$.

19. Given that (U, V) has joint density function

$$f_{UV}(x, y) \text{ or } f(x, y) = \begin{cases} 2e^{-x}e^{-2y} & , \ 0 < x, y < \infty \\ 0 & , \ \text{otherwise} \end{cases}$$

Find (i) the marginal density $f_U(x)$ (ii) $P[U < y]$

[**Hint.** We know $f_U(x) = \int_{-\infty}^{\infty} f_{UV}(x, y)\, dy$

or

$$f_U(x) = \int_{-\infty}^{\infty} f(x, y)\, dy$$

$$= 2e^{-x} \int_0^{\infty} e^{-2y}\, dy$$

$$= 2e^{-x}\left[\frac{1}{2}e^{-2y}\right]_0^{\infty}$$

$$f_U(x) = e^{-x} \text{ for } x > 0$$

and

$$f_U(x) = e^{-x} \text{ for } x \leq 0.$$

(ii) To compute $P[U < y]$. Then for $0 < U < y$

$0 < V < \infty$

$$P[U < y] = \iint_C f_{U,V}(u, v)\, du\, dv \qquad \text{[by definition]}$$

$$= \int_0^y \int_0^{\infty} f_{UV}(x, y)\, dx\, dy$$

$$= \int_0^{\infty} \left\{\int_0^y 2e^{-x}e^{-2y}\, dx\right\} dy$$

$$= \int_0^{\infty} 2e^{-2y}(-e^{-x})_0^y\, dy$$

$$= \int_0^\infty 2e^{-2y}(1 - e^{-y})\, dy$$

$$= \int_0^\infty 2e^{-2y} - \int_0^\infty 2e^{-3y}\, dy$$

$$= \left[-e^{-2y}\right]_0^\infty - \left[-\frac{2}{3}e^{-3y}\right]_0^\infty$$

$$= 1 - \frac{2}{3} = \frac{1}{3}.$$

20. Suppose the joint density function of (X, Y) is

$$f(x, y) \text{ or } f_{XY}(x, y) = \begin{cases} C(x + y^2) & , \quad 0 < x, y < 1 \\ 0 & , \quad \text{otherwise} \end{cases}$$

Find the conditional probability density function of X given $Y = y$ for $0 < y < 1$, then compute

$$P\left[X < \frac{1}{2} \middle| Y < \frac{1}{2}\right]$$

[**Hint.** We know $\quad f_{X/Y}(x/y) = \dfrac{f_{X,Y}(x, y)}{f_Y(y)}$

$$\because \quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dx\, dy = \int_0^1\int_0^1 c\,(x + y^2)\, dx\, dy = 1$$

It gives $c = 1$

Now
$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dx$$

$$= \int_0^1 (x + y^2)\, dx$$

$$= \left[\frac{x^2}{2} + \frac{y^2}{x}\right]_0^1$$

$$= \frac{1}{2} + y^2, \quad 0 < x < 1$$

$$\therefore \quad f_{X/Y}(x/y) = \frac{f_{X,Y}(x, y)}{f_Y(y)}$$

$$= \frac{x + y^2}{\frac{1}{2} + y^2}, \quad 0 < x < 1$$

and $\quad P\left[x < \frac{1}{2} \middle/ Y < \frac{1}{2}\right] = \int_0^{1/2} f_{X/Y = 1/2}(x/y = 1/2)\, dx = 1/3$

21. Suppose that the random vector (U, V) has the following joint density function

$$f(x, y) \text{ or } f_{UV}(x, y) = \begin{cases} \frac{1}{\pi} & , \quad x^2 + y^2 \le 1 \\ 0 & , \quad \text{otherwise} \end{cases}$$

Find (i) Marginal density function of U and V

(ii) Covariance of U and V

(iii) $E(U^2)$

[**Hint :** (i) Marginal density function of $U = f_U(x) = \int_{-\infty}^{\infty} f_{UV}(x, y)\, dy$

$$= 2\int_0^{\sqrt{1-x^2}} \frac{1}{\pi}\, dy$$

$$= \frac{2}{\pi}\sqrt{1-x^2}, \quad 0 < y < 1$$

Similarly, Marginal density function of $V = f_V(y) = \frac{2}{\pi}\sqrt{1-y^2}, \quad 0 < y < 1$

(ii) We know that covariance U and V is given by

$$\text{Cov}(U, V) = E(UV) - E(U)\, E(V)$$

$$E(U) = \int_{-\infty}^{\infty} x\, f_U(x)\, dx = \int_0^{\sqrt{1-y^2}} x \frac{2}{\pi}\sqrt{1-x^2}\, dx$$

$$= 0$$

$$E(V) = \int_{-\infty}^{\infty} y\, f_V(y)\, dy = \int_0^{\sqrt{1-x^2}} y \frac{2}{\pi}\sqrt{1-y^2}\, dy$$

$$= 0$$

and $$E(UV) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_{U,V}(x, y)\, dx\, dy$$

Using polar coordinates, we have

$$E(UV) = \frac{1}{\pi}\int_0^1 \int_0^{2\pi} r^3 \sin\theta \cos\theta\, d\theta\, d\theta$$

$$= \frac{1}{\pi}\int_0^1 r^3 \left(-\frac{\cos 2\theta}{4}\right)_0^{2\pi} d\theta$$

$$= 0$$

$\because$ $$\text{Cov}(U, V) = E(UV) - E(U)\, E(V)$$

$$= 0 - 0 = 0$$

22. Suppose (X, Y) has joint density function

$$f_{X,Y}(x, y) = \begin{cases} 1 & \text{if } -y < x < y,\ 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

(i) Compute the marginal densities $f_X(x)$ and $f_Y(y)$ respectively

(ii) Compute Covariance of X and Y.

[**Hint (i)** We know

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dx$$

$$= \int_{-y}^{y} dx$$

$$= 2y \quad \text{for } 0 < y < 1$$

$$= 0 \quad \text{otherwise}$$

and $$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dy$$

$$= \int_{-x}^{1} dy \quad \text{if } -1 < x < 0$$

Then $$f_X(x) = \int_{-x}^{1} dy, \quad \text{if } -1 < x < 0$$

$$= \int_x^1 dy, \quad \text{if } 0 < x < 1$$

$$= 0, \quad \text{otherwise}$$

That is, $$f_X(x) = \begin{cases} 1+x & \text{if } -1 < x < 0 \\ 1-x & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

It is clear that

$$f_{X,Y}(x, y) \neq f_X(x)\, f_Y(y).$$

Hence X and Y are not independent.

(ii) Compute $\quad C(X, Y)$

$$E(X) = 0 \text{ and } E(Y) = 2/3$$

and $$E(XY) = \int_0^1 \left[\int_{-y}^{y} x\,dx\right] dy$$

$$= \int_0^1 y \left[\frac{x^2}{2}\right]_{-y}^{y} dy$$

$$= \int_0^1 y \left[\frac{y^2}{2} - \frac{y^2}{2}\right] dy$$

$$= 0.$$

Hence $\quad \text{Cov}(X, Y) = 0.$

23. X and Y are two random variable having joint density function $= \frac{1}{27}(2x + y)$ where and y can assume only integer values 0, 1 and 2. Find conditional distribution of Y fo $X = 1$.

[Hint. $f(x, y) = \frac{1}{27}(2x + y),\ x = 0, 1, 2;\ y = 0, 1, 2$

The joint probability distribution with marginal distribution is given by

X \ Y	0	1	2	$f_X(x)$
0	0	$\frac{1}{27}$	$\frac{2}{27}$	$\frac{3}{27}$
1	$\frac{2}{27}$	$\frac{3}{27}$	$\frac{4}{27}$	$\frac{9}{27}$
2	$\frac{4}{27}$	$\frac{5}{27}$	$\frac{6}{27}$	$\frac{15}{27}$
$f_Y(y)$	$\frac{6}{27}$	$\frac{9}{27}$	$\frac{12}{27}$	1

The marginal probability distribution of X is

$$f_X(x) = \sum_y f(x, y)$$

$$f_X(0) = \frac{3}{27},\ f_X(1) = \frac{9}{27},\ f_X(2) = \frac{15}{27}.$$

ANSWERS

11. 1

13. (b) $\frac{9}{16}$ (c) $\overline{X} = \frac{15}{8}$, $\overline{Y} = \frac{25}{8}$, variance $X = \frac{55}{64}$ variance $Y = \frac{55}{64}$ (c) $0 \cdot 45$

14. (a)

Y \ X	1	2	3
2	$\frac{1}{6}$	$\frac{1}{6}$	0
3	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{6}$

(b) $\frac{1}{4}$ (c) $\frac{1}{6}$ (d) $\frac{\sqrt{3}}{4}$

15. (a)

Y \ X	1	2
1	$\frac{1}{6}$	0
2	$\frac{4}{6}$	$\frac{1}{6}$

(b) $\frac{5}{6}$ (c) $\frac{1}{5}$

16. (a)

Y \ X	0	1	2
0	$\frac{1}{8}$	$\frac{1}{8}$	0
1	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{1}{8}$
2	0	$\frac{1}{8}$	$\frac{1}{8}$

(b) 1 (c) $0 \cdot 5$

17. No

18. (a) $1 - e^{-1}$, (b) $\frac{1}{2}$ (c) $1 - 2e^{-1}$

20. (i) $c = 1$ (ii) $f_{X/Y}(x/y) = \dfrac{x + y^2}{\frac{1}{2} + y^2}$ (iii) $P\left[X < \frac{1}{2} \middle| Y < \frac{1}{2}\right] = \frac{1}{3}$

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