HAWA MAHAL
PAGE NO:
DATE:

UNIT- 5
Vector Calculus

Some definations:

Position vector at Point $P(x,y,z)$:

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

its magnitude $|\vec{r}| = r = \sqrt{x^2+y^2+z^2}$
$= (x^2+y^2+z^2)^{1/2}$

constant vector: $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$
where a_1, a_2, a_3 are constants

Dot product of two vectors:

$$\vec{a}.\vec{b} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}).(b_1\hat{i} + b_2\hat{j} + b_3\hat{k})$$
$$= a_1b_1 + a_2b_2 + a_3b_3$$

Cross product of two vectors:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \hat{i}(a_2b_3 - a_3b_2) - \hat{j}(a_1b_3 - b_1a_3) + \hat{k}(a_1b_2 - a_2b_1)$$

Scaler Point function ϕ:
A quantity which possesses only magnitude is called scalar quantity. A function which

Scanned with CamScanner

is not involving $\hat{i}, \hat{j}, \hat{k}$ is called scaler point function.

$$\underline{\underline{Ex}} \quad \phi = xy^2 + 3z^2$$

Vector Point function: $\vec{F}$

$$\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$$

$$\underbrace{F_1, F_2, F_3}_{\text{scalar functions}} \text{ are the components of } \vec{F}$$

$$\underline{\underline{Ex}} \quad \vec{F} = x^2\hat{i} + (x-y^2)\hat{j} + z^2\hat{k}$$

Vector Differentiation:

Let, $\vec{a}, \vec{b}, \vec{c}$ be differentiable vector function of a scalar variable t & ϕ be a differentiable scalar funct. of t then

① $$\frac{d}{dt}(\vec{a} \pm \vec{b}) = \frac{d\vec{a}}{dt} \pm \frac{d\vec{b}}{dt}$$

② $$\frac{d}{dt}(\vec{a}.\vec{b}) = \vec{a}.\frac{d\vec{b}}{dt} + \vec{b}.\frac{d\vec{a}}{dt}$$

③ $$\frac{d}{dt}(\vec{a} \times \vec{b}) = \vec{a} \times \frac{d\vec{b}}{dt} + \vec{b} \times \frac{d\vec{a}}{dt}$$

Scanned with CamScanner

$$\frac{d}{dt}[\vec{a}\ \vec{b}\ \vec{c}] = \left[\frac{d\vec{a}}{dt}\ \vec{b}\ \vec{c}\right] + \left[\vec{a}\ \frac{d\vec{b}}{dt}\ \vec{c}\right] + \left[\vec{a}\ \vec{b}\ \frac{d\vec{c}}{dt}\right]$$

$$\frac{d}{dt}\{\vec{a}\times(\vec{b}\times\vec{c})\} = \frac{d\vec{a}}{dt}\times(\vec{b}\times\vec{c}) + \vec{a}\times\left[\frac{d\vec{b}}{dt}\times\vec{c}\right] + \vec{a}\times\left[\vec{b}\times\frac{d\vec{c}}{dt}\right]$$

velocity & acceleration:

$$\vec{v} = \frac{d\vec{r}}{dt}, \quad \vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$$

Scanned with CamScanner

Example 5.01 : If $\vec{r} = 3\hat{i} - 6t^2\hat{j} + 4t\hat{k}$, find $\dfrac{d\vec{r}}{dt}$ and $\dfrac{d^2\vec{r}}{dt^2}$.

Solution. Given : $\vec{r} = 3\hat{i} - 6t^2\hat{j} + 4t\hat{k}$

$$\therefore \qquad \frac{d\vec{r}}{dt} = \frac{d}{dt}(3)\hat{i} + \frac{d}{dt}(-6t^2)\hat{j} + \frac{d}{dt}(4t)\hat{k}$$

$$= 0\hat{i} - 12t\hat{j} + 4\hat{k} = -12t\hat{j} + 4\hat{k}.$$

$$\frac{d^2\vec{r}}{dt^2} = \frac{d}{dt}(-12t)\hat{j} + \frac{d}{dt}(4)\hat{k}$$

$$= -12\hat{j} + 0\hat{k} = -12\hat{j}.$$

Example 5.02 : If $\vec{r} = a\cos t\,\hat{i} + \sin t\,\hat{j} + t\,\hat{k}$, find $\dfrac{d\vec{r}}{dt}$, $\dfrac{d^2\vec{r}}{dt^2}$ and $\left|\dfrac{d^2\vec{r}}{dt^2}\right|$.

Solution. Given : $\vec{r} = a\cos t\,\hat{i} + a\sin t\,\hat{j} + t\,\hat{k}$

Scanned with CamScanner

$\Rightarrow \quad \frac{d\vec{r}}{dt}=\frac{d}{dt}(a\cos t)\hat{i}+\frac{d}{dt}(a\sin t)\hat{j}+\frac{d}{dt}(t)\hat{k}$

$=-a\sin t\,\hat{i}+a\cos t\,\hat{j}+\hat{k}$ **Ans.**

and $\quad \frac{d^2\vec{r}}{dt^2}=\frac{d}{dt}(-a\sin t)\hat{i}+\frac{d}{dt}(a\cos t)\hat{j}+\frac{d}{dt}(1)\hat{k}$

$=-a\cos t\,\hat{i}-a\sin t\,\hat{j}$ **Ans.**

Now $\quad \left|\frac{d^2\vec{r}}{dt^2}\right|=\sqrt{(-a\cos t)^2+(-a\sin t)^2}$

$=a\sqrt{\cos^2 t+\sin^2 t}=a.\ [\because\ \cos^2\theta+\sin^2\theta=1]$ **Ans.**

Example 5.03 : *If If $\vec{a}=t^2\hat{i}-t\hat{j}+(2t+1)\hat{k}$ and $\vec{b}=(2t-3)\hat{i}+\hat{j}-t\hat{k}$, find $\frac{d}{dt}(\vec{a}.\vec{b})$ at $t=1$.*

[RGPV June 2015]

Solution. We have $\frac{d}{dt}(\vec{a}.\vec{b})=\vec{a}.\frac{d\vec{b}}{dt}+\frac{d\vec{a}}{dt}.\vec{b}$

$=\{t^2\hat{i}-t\hat{j}+(2t+1)\hat{k}\}.\{2\hat{i}-\hat{k}\}+\{(2t\hat{i}-\hat{j}+2\hat{k})\}.\{(2t-3)\hat{i}+\hat{j}-t\hat{k}\}$

$=2t^2-2t-1+4t^2-6t-1-2t=6t^2-10t-2$

$\therefore \quad \left[\frac{d}{dt}(\vec{a}.\vec{b})\right]_{at\,t=1}=-6.$ **Ans.**

Example 5.04 : *A particle moves along the curve $x=4\cos t$, $y=4\sin t$, $z=6t$. Find the velocity and acceleration at time $t=0$*

Solution. Let $P(x,y,z)$ be a point on the curve, then

$\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}=4\hat{i}\cos t+4\hat{j}\sin t+6\hat{k}t.$

Then Velocity : $\vec{v}=\frac{d\vec{r}}{dt}=-4\hat{i}\sin t+4\hat{j}\cos t+6\hat{k};$

At $\quad t=0,\ \vec{v}=4\hat{j}+6\hat{k};\ |\vec{v}|=\sqrt{16+36}=2\sqrt{13};$ **Ans.**

Acceleration : $\vec{a}=\frac{d^2\vec{r}}{dt^2}=-4\hat{i}\cos t-4\hat{j}\sin t.$

And at $t=0$; $\vec{a}=-4\hat{i};\ |\vec{a}|=4.$ **Ans.**

Example 5.05 : *A particle moves along the curve $x=t^3+1$, $y=t^2$ and $z=2t+5$, where t is the time. Find the component of its velocity and acceleration at $t=1$ in the direction $\hat{i}+\hat{j}+3\hat{k}$.*

Solution. Velocity $=\frac{d\vec{r}}{dt}=\frac{d}{dt}(x\hat{i}+y\hat{j}+z\hat{k})$

Scanned with CamScanner

$$=\frac{d}{dt}[(t^3+1)\,\hat{i}+t^2\,\hat{j}+(2t+5)\,\hat{k}]=3t^2\hat{i}+2t\hat{j}+2\hat{k}$$

i.e., $\vec{v}=3\hat{i}+2\hat{j}+2\hat{k}$, at $t=1$.

Again unit vector in the direction of $\hat{i}+\hat{j}+3\hat{k}$ is

$$\hat{n}=\frac{\hat{i}+\hat{j}+3\hat{k}}{\sqrt{1^2+1^2+3^2}}=\frac{\hat{i}+\hat{j}+3\hat{k}}{\sqrt{11}}.$$

$\therefore$ Component of velocity at $t=1$ in the direction of $\hat{i}+\hat{j}+3\hat{k}$

$$=\vec{v}.\hat{n}=(3\hat{i}+2\hat{j}+2\hat{k})\,.\,\frac{(\hat{i}+\hat{j}+3\hat{k})}{\sqrt{11}}=\frac{3+2+6}{\sqrt{11}}=\sqrt{11}.\quad \text{[by (1) \& (2)]}$$

Now acceleration $\vec{a}=\frac{d^2\vec{r}}{dt^2}=\frac{d\vec{v}}{dt}=6t\hat{i}+2\hat{j}=6\hat{i}+2\hat{j}$ at $t=1$.

$\therefore$ Component of acceleration at $t=1$ in the direction $\hat{i}+\hat{j}+3\hat{k}$

$$=\vec{a}.\hat{n}=(6\hat{i}+2\hat{j})\,.\,\frac{(\hat{i}+\hat{j}+3\hat{k})}{\sqrt{11}}=\frac{6+2}{\sqrt{11}}=\frac{8}{\sqrt{11}}.\quad \text{[by (2) \& (3)]}$$

Scanned with CamScanner

The operator ∇ (Del)

∇ is also known as Vector differential operator and is ~~also~~ defined as

$$\nabla \equiv \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

operating on scalar point function ϕ is given by $\nabla\phi = \hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}$

Scanned with CamScanner

Gradiant of a Scalar point function ϕ.

Let $\phi = \phi(x, y, z)$ be a scalar point function then

$$\text{grad } \phi = \nabla\phi = \left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right)$$

For Ex: If $\phi = 3x^2y - y^3z^2$ then

$$\text{grad } \phi = \nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$$= \hat{i}\frac{\partial}{\partial x}(3x^2y - y^3z^2)$$

$$+ \hat{j}\frac{\partial}{\partial y}(3x^2y - y^3z^2)$$

$$+ \hat{k}\frac{\partial}{\partial z}(3x^2y - y^3z^2)$$

$$\nabla\phi = \hat{i}(6xy) + \hat{j}(3x^2 - 3y^2z^2) + \hat{k}(2y^3z)$$

☆ If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ then

$$r = |\vec{r}| = \sqrt{x^2+y^2+z^2} = (x^2+y^2+z^2)^{1/2}$$

$$\frac{\partial r}{\partial x} = \frac{1}{\cancel{2}}(x^2+y^2+z^2)^{-1/2} \cdot \cancel{2}x$$

$$= \frac{x}{r}$$

$$\frac{\partial r}{\partial y} = \frac{1}{\cancel{2}}(x^2+y^2+z^2)^{-1/2} \cdot \cancel{2}y$$

$$= \frac{y}{r}$$

Scanned with CamScanner

HAWA MAHAL
PAGE NO.:
DATE:

$$\frac{\partial r}{\partial z} = \frac{1}{2}(x^2+y^2+z^2)^{-1/2}\, 2z$$

$$= \frac{z}{r}$$

$$\therefore \frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\nabla^2 = \nabla.\nabla = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Q. If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ then P.T

(1) $\nabla f(r) = f'(r)\dfrac{\vec{r}}{r}$

Sol. Since, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\Rightarrow r = \sqrt{x^2+y^2+z^2} = (x^2+y^2+z^2)^{1/2}$$

$$\therefore \frac{\partial r}{\partial x} = \frac{1}{2}(x^2+y^2+z^2)^{-1/2}\, 2x$$

$$= \frac{x}{r}$$

Similarly, $\dfrac{\partial r}{\partial y} = \dfrac{y}{r}$ & $\dfrac{\partial r}{\partial z} = \dfrac{z}{r}$

$$\nabla f(r) = \hat{i}\frac{\partial}{\partial x}f(r) + \hat{j}\frac{\partial}{\partial y}f(r) + \hat{k}\frac{\partial}{\partial z}f(r)$$

$$= \hat{i} f'(r)\frac{\partial r}{\partial x} + \hat{j} f'(r)\frac{\partial r}{\partial y} + \hat{k} f'(r)\frac{\partial r}{\partial z}$$

Scanned with CamScanner

HAWA MAHAL
PAGE NO:
DATE:

$$= \hat{i} f'(r) \frac{x}{r} + \hat{j} f'(r) \frac{y}{r} + \hat{k} f'(r) \frac{z}{r}$$

$$= \frac{f'(r)}{r} [x\hat{i} + y\hat{j} + z\hat{k}]$$

$$= \frac{f'(r)}{r} \vec{r} \quad // \qquad [\because \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

2011, 2015

② $\nabla r^n = n r^{n-2} \vec{r}$

Since, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$r = \sqrt{x^2+y^2+z^2} = (x^2+y^2+z^2)^{1/2}$$

$$\therefore \frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\therefore \nabla r^n = \hat{i} \frac{\partial (r^n)}{\partial x} + \hat{j} \frac{\partial (r^n)}{\partial y} + \hat{k} \frac{\partial}{\partial z}(r^n)$$

$$= \hat{i}\, n r^{n-1} \frac{\partial r}{\partial x} + \hat{j}\, n r^{n-1} \frac{\partial r}{\partial y} + \hat{k}\, n r^{n-1} \frac{\partial r}{\partial z}$$

$$= \hat{i}\, n r^{n-1} \frac{x}{r} + \hat{j}\, n r^{n-1} \frac{y}{r} + \hat{k}\, n r^{n-1} \frac{z}{r}$$

$$= \hat{i}\, n r^{n-2} x + \hat{j}\, n r^{n-2} y + \hat{k}\, n r^{n-2} z$$

$$= n r^{n-2} (x\hat{i} + y\hat{j} + z\hat{k})$$

$$= n r^{n-2} \vec{r}$$

//

Scanned with CamScanner

PAGE NO:
DATE:

Q. If $\phi(x,y,z) = 3x^2y - y^3z^2$ find $\nabla\phi$ at the point $(1,-2,-1)$

2014

Given, $\phi(x,y,z) = 3x^2y - y^3z^2$

$$\therefore \nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$$= \hat{i}\frac{\partial}{\partial x}(3x^2y - y^3z^2) + \hat{j}\frac{\partial}{\partial y}(3x^2y - y^3z^2) + \hat{k}\frac{\partial}{\partial z}(3x^2y - y^3z^2)$$

$$\nabla\phi = \hat{i}\,6xy + \hat{j}(3x^2 - 3y^2z^2) + \hat{k}\,2y^3z$$

at $(1,-2,-1)$

$$\nabla\phi = \hat{i}\,6(1)(-2) + \hat{j}(3(1)^2 - 3(-2)^2(-1)^2) + \hat{k}\,2(-2)^3(-1)$$

$$\nabla\phi = -12\hat{i} - 9\hat{j} - 16\hat{k}$$

//

Properties of gradient:

(1) Nomal vector of ϕ is

$$\vec{n} = \text{grad}\,\phi = \nabla\phi$$

Scanned with CamScanner

HAWA MAHAL
PAGE NO.:
DATE :

(2) Directional derivative of a scalar pt funct^n ϕ at $P(x, y, z)$ in the direction of ϕ is

$$\frac{d\phi}{ds} = \text{grad}\,\phi \cdot \hat{a}$$

$$\Rightarrow \frac{d\phi}{\partial s} = \nabla\phi \cdot \hat{a}$$

where, $\hat{a} = \dfrac{\vec{a}}{|\vec{a}|}$

(3) The maximum value of directional derivative of ϕ is grad ϕ

$\therefore$ Unit vector normal to the surface is $\dfrac{\text{grad}\,\phi}{|\text{grad}\,\phi|} = \dfrac{\nabla\phi}{|\nabla\phi|}$

Scanned with CamScanner

Q. Find the Directional Derivative of the function $\phi = x^2 - y^2 + 2z^2$ at the pt. $P(1,2,3)$ in the direction of the line PQ where, Q is the pt. $(5,0,4)$

Sol

Position vector of $P = \hat{i} + 2\hat{j} + 3\hat{k}$
Position vector of $Q = 5\hat{i} + 0\hat{j} + 4\hat{k}$

$\therefore \vec{a} = \vec{PQ}$ = Position vector of Q − Position vector of P

$$= (5\hat{i} + 0\hat{j} + 4\hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k})$$

$$\vec{PQ} = 4\hat{i} - 2\hat{j} + \hat{k}$$

$$\therefore \hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{4\hat{i} - 2\hat{j} + \hat{k}}{\sqrt{16+4+1}} = \frac{1}{\sqrt{21}}(4\hat{i} - 2\hat{j} + \hat{k})$$

$$\text{grad}\,\phi = \nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$$= \hat{i}(2x) + \hat{j}(-2y) + \hat{k}(4z)$$

$$= 2\hat{i} - 4\hat{j} + 12\hat{k} \text{ at pt. } (1,2,3)$$

Directional derivative of ϕ

$$= \text{grad}\,\phi \cdot \hat{a}$$

$$= (2\hat{i} - 4\hat{j} + 12\hat{k}) \frac{1}{\sqrt{21}} (4\hat{i} - 2\hat{j} + \hat{k})$$

Scanned with CamScanner

Q. Find a unit normal vector to the surface $\phi = x^2 + y^2 - z$ at pt. (1, 2, 5)

(2014)

Sol^n Given $\phi = x^2 + y^2 - z$

$$\text{grad}\,\phi = \nabla \phi$$

$$= \hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}$$

$$= \hat{i}(2x) + \hat{j}(2y) + \hat{k}(-1)$$

$$= 2x\hat{i} + 2y\hat{j} - \hat{k}$$

at (1, 2, 5).

$\therefore$ unit normal vector $= 2\hat{i} + 4\hat{j} - \hat{k}$

$$\hat{n} = \frac{\text{grad}\,\phi}{|\text{grad}\,\phi|}$$

$$= \frac{2\hat{i} + 4\hat{j} - \hat{k}}{\sqrt{2^2 + 4^2 + (-1)^2}} = \frac{2\hat{i} + 4\hat{j} - \hat{k}}{\sqrt{4 + 16 + 1}}$$

$$= \frac{2\hat{i} + 4\hat{j} - \hat{k}}{\sqrt{21}}$$

Extra Practice

Q. Find a unit normal vector to the surface $xy^3z^2 = 4$ at the pt. (-1, -1, 2)

2010
2011
2013

Scanned with CamScanner

HAWA MAHAL
PAGE NO.
DATE

Sol ②

Given $\phi = xy^3z^2 - 4$

$$\text{grad}\,\phi = \nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$$= \hat{i}(y^3z^2) + \hat{j}(3xy^2z^2) + \hat{k}(2xy^3z)$$

at $(-1,-1,2)$ $[\because x=-1, y=-1, z=2]$,

$$= \hat{i}(-4) - 12\hat{j} + 4\hat{k}$$

$$\hat{n} = \frac{\text{grad}\,\phi}{|\text{grad}\,\phi|} = \frac{-4\hat{i} - 12\hat{j} + 4\hat{k}}{\sqrt{4^2 + (-12)^2 + 4^2}}$$

$$= \frac{-4\hat{i} - 12\hat{j} + 4\hat{k}}{\sqrt{16+144+16}}$$

$$= \frac{4(-\hat{i} - 3\hat{j} + \hat{k})}{\sqrt{176}}$$

$$= \frac{\cancel{4}(-\hat{i} - 3\hat{j} + \hat{k})}{\cancel{4}\sqrt{11}}$$

$$= \frac{-\hat{i} - 3\hat{j} + \hat{k}}{\sqrt{11}}$$

Scanned with CamScanner

Q. Find D.D of $\phi = xy + yz + zx$ at the pt $(1, 2, 0)$ in the direction of vector $\hat{i} + 2\hat{j} + 2\hat{k}$

2017
2015

Sol

D.D of $\phi = \hat{a}\,\text{grad}\,\phi$, given $\phi = xy + yz + zx$

$$\text{grad}\,\phi = \nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$$= \hat{i}(y+z) + \hat{j}(x+z) + \hat{k}(y+x)$$

$$= 2\hat{i} + \hat{j} + 3\hat{k} \text{ at the pt } (1,2,0)$$

$\vec{a} = \hat{i} + 2\hat{j} + 2\hat{k}$ (directional vector given)

$$\therefore \hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{\hat{i} + 2\hat{j} + 2\hat{k}}{\sqrt{1^2 + 2^2 + 2^2}}$$

$$= \frac{\hat{i} + 2\hat{j} + 2\hat{k}}{\sqrt{9}}$$

$$= \frac{1}{3}(\hat{i} + 2\hat{j} + 2\hat{k})$$

$\therefore$ DD of $\phi = \frac{1}{3}(\hat{i} + 2\hat{j} + 2\hat{k}) \cdot (2\hat{i} + \hat{j} + 3\hat{k})$

$$= \frac{1}{3}(2 + 2 + 6) = \frac{10}{3}$$

Scanned with CamScanner

HAWA MAHAL
PAGE NO.:
DATE:

Divergence of a Vector Point function:

Let, $\vec{F} = \vec{F_1}\hat{i} + \vec{F_2}\hat{j} + \vec{F_3}\hat{k}$ be a vector point function then

$$\operatorname{div}\vec{F} = \nabla.\vec{F}$$
$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(\vec{F_1}\hat{i} + \vec{F_2}\hat{j} + \vec{F_3}\hat{k}\right)$$
$$= \frac{\partial \vec{F_1}}{\partial x} + \frac{\partial \vec{F_2}}{\partial y} + \frac{\partial \vec{F_3}}{\partial z}$$

which is a scalar quantity

* $\nabla.\vec{F} \neq \vec{F}.\nabla$

* **Solenoidal vector:** If $\operatorname{div}\vec{F} = 0$ then $\vec{F}$ is called solenoidal vector

Curl of a vector point function:

If $\vec{F} = \vec{F_1}\hat{i} + \vec{F_2}\hat{j} + \vec{F_3}\hat{k}$ then

$$\operatorname{curl}\vec{F} = \nabla \times \vec{F}$$
$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \vec{F_1} & \vec{F_2} & \vec{F_3} \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial}{\partial y}\vec{F_3} - \frac{\partial}{\partial z}\vec{F_2}\right) - \hat{j}\left(\frac{\partial}{\partial x}\vec{F_3} - \frac{\partial}{\partial z}\vec{F_1}\right) + \hat{k}\left(\frac{\partial}{\partial x}\vec{F_2} - \frac{\partial}{\partial y}\vec{F_1}\right)$$

Scanned with CamScanner

which is a vector quantity

* $\nabla \times \vec{F} \neq \vec{F} \times \nabla$

* Irrotational vector: If $\text{curl}\,\vec{F} = 0$ then $\vec{F}$ is called irrotational vector

Note: $\text{div}\,\vec{F} = \nabla \cdot \vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\vec{F}$

$$= \hat{i}\frac{\partial \vec{F}}{\partial x} + \hat{j}\frac{\partial \vec{F}}{\partial y} + \hat{k}\frac{\partial \vec{F}}{\partial z}$$

$$= \sum \hat{i} \cdot \frac{\partial \vec{F}}{\partial x}$$

$\text{curl}\,\vec{F} = \nabla \times \vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \vec{F}$

$$= \hat{i} \times \frac{\partial \vec{F}}{\partial x} + \hat{j} \times \frac{\partial \vec{F}}{\partial y} + \hat{k} \times \frac{\partial \vec{F}}{\partial z}$$

$$= \sum \hat{i} \times \frac{\partial \vec{F}}{\partial x}$$

Q. If $\vec{F} = xz^3\hat{i} - 2x^2yz\hat{j} + 2yz^4\hat{k}$
find $\text{div}\,\vec{F}$ & $\text{curl}\,\vec{F}$

Sol Given, $\vec{F} = xz^3\hat{i} - 2x^2yz\hat{j} + 2yz^4\hat{k}$

$\text{div}\,\vec{F} = \nabla \cdot \vec{F}$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(xz^3\hat{i} - 2x^2yz\hat{j} + 2yz^4\hat{k}\right)$$

$$= \frac{\partial}{\partial x}(xz^3) - 2\frac{\partial}{\partial y}(x^2yz) + 2\frac{\partial}{\partial z}(yz^4)$$

Scanned with CamScanner

HAWA MAHAL
PAGE NO:
DATE:

$\left[\because \hat{i}.\hat{i} = 1, \ \hat{j}.\hat{j} = 1, \ \hat{k}.\hat{k} = 1\right.$

$= z^3 - 2x^2z + 8yz^3$

& $\text{curl}\,\vec{F} = \nabla \times \vec{F}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz^3 & -2x^2yz & 2yz^4 \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial}{\partial y}(2yz^4) + \frac{\partial}{\partial z}(2x^2yz)\right)$$
$$- \hat{j}\left(\frac{\partial}{\partial x}(2yz^4) - \frac{\partial}{\partial z}(xz^3)\right)$$
$$+ \hat{k}\left(\frac{\partial}{\partial x}(-2x^2yz) - \frac{\partial}{\partial y}(xz^3)\right)$$

$$= \hat{i}(2z^4 + 2x^2y) - \hat{j}(0 - 3xz^2) + \hat{k}(-4xyz - 0)$$

$$= (2z^4 + 2x^2y)\hat{i} + 3xz^2\hat{j} - 4xyz\hat{k}$$

//

Prove that : $\text{curl grad}\,\phi = \vec{0}$

$\text{curl grad}\,\phi = \nabla \times \nabla\phi$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \left(\hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}\right)$$

Scanned with CamScanner

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial^2 \phi}{\partial y \partial z} - \frac{\partial^2 \phi}{\partial z \partial y}\right) - \hat{j}\left(\frac{\partial^2 \phi}{\partial x \partial z} - \frac{\partial^2 \phi}{\partial z \partial x}\right) + \hat{k}\left(\frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial y \partial x}\right)$$

$$= 0\hat{i} + 0\hat{j} + 0\hat{k} = \vec{0}$$

//

Q: P.T $\text{div curl} \vec{F} = 0$

Sol: $\text{div curl} \vec{F} = \nabla.(\nabla \times \vec{F})$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left(\vec{i} \times \frac{\partial \vec{F_1}}{\partial x} + \hat{j} \times \frac{\partial \vec{F_2}}{\partial y} + \hat{k}\frac{\partial \vec{F_3}}{\partial z}\right)$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \vec{F_1} & \vec{F_2} & \vec{F_3} \end{vmatrix}$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left[\hat{i}\left(\frac{\partial \vec{F_3}}{\partial y} - \frac{\partial \vec{F_2}}{\partial z}\right) - \hat{j}\left(\frac{\partial \vec{F_3}}{\partial x} - \frac{\partial \vec{F_1}}{\partial z}\right) + \hat{k}\left(\frac{\partial \vec{F_2}}{\partial x} - \frac{\partial \vec{F_1}}{\partial y}\right)\right]$$

Scanned with CamScanner

$$= \frac{\partial^2 \vec{f_3}}{\partial x \partial y} - \frac{\partial^2 \vec{f_2}}{\partial x \partial z} - \frac{\partial^2 \vec{f_3}}{\partial y \partial x} + \frac{\partial^2 \vec{f_1}}{\partial y \partial z} + \frac{\partial^2 \vec{f_2}}{\partial z \partial x} - \frac{\partial^2 \vec{f_1}}{\partial z \partial y}$$

$$= 0$$

Prove that $div(\vec{A} \times \vec{B}) = \vec{B}.curl A - \vec{A}.curl \vec{B}$

$$div(\vec{A} \times \vec{B}) = \nabla.(\vec{A} \times \vec{B})$$

$$= \sum \hat{i} \frac{\partial}{\partial x}(\vec{A} \times \vec{B})$$

$$= \sum \hat{i} \left(\frac{\partial \vec{A}}{\partial x} \times \vec{B} + \vec{A} \times \frac{\partial \vec{B}}{\partial x} \right)$$

$$= \sum \hat{i}. \left(\frac{\partial \vec{A}}{\partial x} \times \vec{B} \right) + \sum \hat{i} \left(\vec{A} \times \frac{\partial \vec{B}}{\partial x} \right)$$

$$= \sum \hat{i}. \left(\frac{\partial \vec{A}}{\partial x} \times \vec{B} \right) - \sum \hat{i} \left(\frac{\partial \vec{B}}{\partial x} \times \vec{A} \right)$$

$$= \sum \left(\hat{i} \times \frac{\partial \vec{A}}{\partial x} \right) . \vec{B} - \sum \left(\hat{i} \times \frac{\partial \vec{B}}{\partial x} \right) . \vec{A}$$

$$\left[\because a.(b \times c) = (a \times b).c \right]$$

$$= curl \vec{A} . \vec{B} - curl \vec{B} . \vec{A}$$

$$= \vec{B}\ curl \vec{A} - \vec{A}\ curl \vec{B}$$

//

Scanned with CamScanner

Scanned with CamScanner

HAWA MAHAL
PAGE NO:
DATE:

Q. Prove that $\operatorname{div}(r^n\vec{r}) = (n+3)r^n$

Sol. $\operatorname{div}(r^n\vec{r}) = \nabla\cdot r^n\vec{r}$

$$= \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)r^n\left(x\hat{i}+y\hat{j}+z\hat{k}\right)$$

$$= \frac{\partial}{\partial x}(r^n x)+\frac{\partial}{\partial y}(r^n y)+\frac{\partial}{\partial z}(r^n z)$$

Since, $\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$

$$|\vec{r}| = r = \sqrt{x^2+y^2+z^2}$$

$$r = (x^2+y^2+z^2)^{1/2}$$

$$\therefore \frac{\partial r}{\partial x} = \frac{1}{\cancel{2}}\cdot\cancel{2}x\,(x^2+y^2+z^2)^{-1/2}$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}$$

Similarly, $\frac{\partial r}{\partial y} = \frac{y}{r}$, $\frac{\partial r}{\partial z} = \frac{z}{r}$

$$\therefore \operatorname{div}(r^n\vec{r}) = r^n + xnr^{n-1}\frac{\partial r}{\partial x}$$
$$+ r^n + ynr^{n-1}\frac{\partial r}{\partial y}$$
$$+ r^n + znr^{n-1}\frac{\partial r}{\partial z}$$

$$= r^n + xnr^{n-1}\cdot\frac{x}{r} + r^n + ynr^{n-1}\frac{y}{r}$$
$$+ r^n + znr^{n-1}\cdot\frac{z}{r}$$

$$= 3r^n + x^2 n r^{n-2} + y^2 n r^{n-2} + z^2 n r^{n-2}$$

$$= 3r^n + (x^2+y^2+z^2) n r^{n-2}$$

$$= 3r^n + r^2 . n r^{n-2}$$

$$= 3r^n + n r^n$$

$$= (3+n) r^n = (n+3) r^n$$

// Ans

Q. P.T. curl $r^n \vec{r} = 0$

$$\text{curl } r^n . \vec{r} = \nabla \times r^n \vec{r}$$

$$= \nabla \times r^n (x\hat{i} + y\hat{j} + z\hat{k})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ r^n x & r^n y & r^n z \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial}{\partial y}(r^n z) - \frac{\partial}{\partial z}(r^n y)\right) - \hat{j}\left(\frac{\partial}{\partial x}(r^n z) - \frac{\partial}{\partial z}(r^n x)\right) + \hat{k}\left(\frac{\partial}{\partial x}(r^n y) - \frac{\partial}{\partial y}(r^n x)\right)$$

$$= \hat{i}\left(z n r^{n-1} \frac{\partial r}{\partial y} - y n r^{n-1} \frac{\partial r}{\partial z}\right) - \hat{j}\left(z n r^{n-1} \frac{\partial r}{\partial x} - x n r^{n-1} \frac{\partial r}{\partial z}\right) + \hat{k}\left(y n r^{n-1} \frac{\partial r}{\partial x} - x n r^{n-1} \frac{\partial r}{\partial y}\right)$$

Scanned with CamScanner

HAWA MAHAL
PAGE NO
DATE :

$$= nr^{n-1}\left[\hat{i}\left(z\frac{y}{r} - y\frac{z}{r}\right) - \hat{j}\left(z\frac{x}{r} - x\frac{z}{r}\right) + \hat{k}\left(y\frac{x}{r} - x\frac{y}{r}\right)\right]$$

$$= nr^{n-1}\left[\hat{i}\left(\frac{yz - yz}{r}\right) - \hat{j}\left(\frac{xz - xz}{r}\right) + \hat{k}\left(\frac{yx - xy}{r}\right)\right]$$

$$= nr^{n-1}\left\{0\hat{i} + 0\hat{j} + 0\hat{k}\right\} = \vec{0}$$

//

Q. Prove that

2012

$$\vec{F} = (y^2 - z^2 + 3yz - 2x)\hat{i} + (3xz + 2xy)\hat{j} + (3xy - 2xz + 2z)\hat{k}$$

is both solenoidal & irrotational

Sol

Given, $\vec{F} = (y^2 - z^2 + 3yz - 2x)\hat{i} + (3xz + 2xy)\hat{j} + (3xy - 2xz + 2z)\hat{k}$

To prove that given $\vec{F}$ is solenoidal we have to $\text{div}\vec{F} = 0$

$$\therefore \text{div}\vec{F} = \frac{\partial \vec{F_1}}{\partial x} + \frac{\partial \vec{F_2}}{\partial y} + \frac{\partial \vec{F_3}}{\partial z}$$

$$= \frac{\partial}{\partial x}(y^2 - z^2 + 3yz - 2x) + \frac{\partial}{\partial y}(3xz + 2xy) + \frac{\partial}{\partial z}(3xy - 2xz + 2z)$$

Scanned with CamScanner

Scanned with CamScanner

$= -2 + 2x - 2x + 2 = 0$

$\therefore \vec{F}$ is solenoidal

To prove that given $\vec{F}$ is irrotational

$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 - z^2 + 3yz - 2x & 3xz + 2xy & 3xy - 2xz + 2z \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(3xy - 2xz + 2z) - \frac{\partial}{\partial z}(3xz + 2xy)\right]$$

$$- \hat{j}\left[\frac{\partial}{\partial x}(3xy - 2xz + 2z) - \frac{\partial}{\partial z}(y^2 - z^2 + 3yz - 2x)\right]$$

$$+ \hat{k}\left(\frac{\partial}{\partial x}(3xz + 2xy) - \frac{\partial}{\partial y}(y^2 - z^2 + 3yz - 2x)\right)$$

$$= \hat{i}(3x - 3x) - \hat{j}(3y - 2z - 2z + 3y) + \hat{k}(3z + 2y - 2y - 3z)$$

$$= 0\hat{i} - 0\hat{j} + 0\hat{k} = \vec{0}$$

$\therefore \vec{F}$ is irrotational

//

Scanned with CamScanner

2012 Q. If $\vec{F} = (x+y+1)\hat{i} + \hat{j} - (x+y)\hat{k}$
Prove that $\vec{F}\cdot \text{curl}\,\vec{F} = 0$

Sol^n: $\text{curl}\,\vec{F} = \nabla \times \vec{F}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y+1 & 1 & -(x+y) \end{vmatrix}$$

$$= \hat{i}(-1-0) + \hat{j}(0+1) + \hat{k}(0-1)$$
$$= -\hat{i} + \hat{j} - \hat{k}$$

$\therefore \vec{F}\cdot \text{curl}\,\vec{F} = \{(x+y+1)\hat{i} + \hat{j} - (x+y)\hat{k}\}(-\hat{i} + \hat{j} - \hat{k})$
$$= -(x+y+1) + 1 + (x+y) = 0$$

Q. 2014 If vector $\vec{F} = (x+3y)\hat{i} + (y-2z)\hat{j} + (x+az)\hat{k}$ is a solenoidal vector then find the value of a

Sol^n: If $\vec{F}$ is a solenoidal vector then
$\text{div}\,\vec{F} = 0$
$$\Rightarrow \frac{\partial}{\partial x}(x+3y) + \frac{\partial}{\partial y}(y-2z) + \frac{\partial}{\partial z}(x+az) = 0$$
$$\Rightarrow 1+1+a = 0$$
$$\Rightarrow 2+a = 0$$
$$\Rightarrow \boxed{-2 = a}$$

Scanned with CamScanner

Prove that $\operatorname{div}\operatorname{grad} r^m = m(m+1)r^{m-2}$

or $\nabla \cdot \nabla r^m = m(m+1)r^{m-2}$

Sol: LHS $\operatorname{div}\operatorname{grad} r^m = \nabla \cdot \nabla r^m$

$$\nabla r^m = \hat{i}\frac{\partial}{\partial x}r^m + \hat{j}\frac{\partial}{\partial y}r^m + \hat{k}\frac{\partial}{\partial z}r^m$$

$$= \hat{i}\, m r^{m-1}\frac{\partial r}{\partial x} + \hat{j}\, m r^{m-1}\frac{\partial r}{\partial y} + \hat{k}\, m r^{m-1}\frac{\partial r}{\partial z}$$

$$= \hat{i}\, m r^{m-1}\frac{x}{r} + \hat{j}\, m r^{m-1}\frac{y}{r} + \hat{k}\, m r^{m-1}\frac{z}{r}$$

$$= m r^{m-1}\left[\hat{i}\frac{x}{r} + \hat{j}\frac{y}{r} + \hat{k}\frac{z}{r}\right]$$

$$\left[\begin{aligned} & r^2 = x^2+y^2+z^2 \\ & \Rightarrow 2r\frac{\partial r}{\partial x} = 2x \\ & \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r} \\ & \text{similarly, } \frac{\partial r}{\partial y} = \frac{y}{r} \\ & \frac{\partial r}{\partial z} = \frac{z}{r} \end{aligned}\right]$$

$$= m r^{m-2}\left\{\hat{i}x + \hat{j}y + \hat{k}z\right\} \quad \text{--- (1)}$$

$\therefore \operatorname{div}\operatorname{grad} r^m = \nabla \cdot \nabla r^m$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(m r^{m-2}\left(x\hat{i} + y\hat{j} + z\hat{k}\right)\right)$$

$$= \frac{\partial}{\partial x}\left(m r^{m-2}x\right) + \frac{\partial}{\partial y}\left(m r^{m-2}y\right) + \frac{\partial}{\partial z}\left(m r^{m-2}z\right) \quad \text{--- (2)}$$

$$\left[\because \frac{\partial}{\partial x}\left(m r^{m-2}x\right) = m r^{m-2}\cdot 1 + m x\frac{\partial}{\partial x}r^{m-2}\right.$$

$$= m r^{m-2} + m x(m-2)r^{m-3}\frac{\partial r}{\partial x}$$

$$= m r^{m-2} + m x(m-2)r^{m-3}\frac{x}{r}$$

HAWA MAHAL

PAGE NO:

DATE:

$$= mr^{m-2} + m(m-2)r^{m-4}x^2 \Big]$$

Similarly for other expansions.

Eq^n. (2) becomes

$$\nabla.\nabla r^m = 3mr^{m-2} + m(m-2)r^{m-4}(x^2+y^2 ...$$

$$= 3mr^{m-2} + m(m-2)r^{m-2}$$

$$= mr^{m-2}(3+m-2)$$

$$= m(m+1)r^{m-2}$$

//

Q. (2012) Prove that $\nabla^2 f(r) = f''(r) + \frac{2}{r}f'(r)$

Sol.

$$\nabla^2 \equiv \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)$$

$$= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\nabla^2 f(r) = \frac{\partial^2}{\partial x^2}f(r) + \frac{\partial^2}{\partial y^2}f(r) + \frac{\partial^2}{\partial z^2}f(r) \quad \text{--- (1)}$$

$$\frac{\partial^2}{\partial x^2}f(r) = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}f(r)\right]$$

$$= \frac{\partial}{\partial x}\left\{f'(r)\frac{\partial r}{\partial x}\right\} \quad \left[\because \frac{\partial r}{\partial x} = \frac{x}{r}\right]$$

$$= \frac{\partial}{\partial x}\left\{f'(r).\frac{x}{r}\right\}$$

$$= \frac{\left[\frac{\partial}{\partial x}(x.f'(r))\right]r - x f'(r)\frac{\partial r}{\partial x}}{r^2}$$

Scanned with CamScanner

HAWA MAHAL
PAGE NO.
DATE

$$= \frac{r\left[x f''(r)\frac{\partial r}{\partial x} + f'(r)\right] - x f'(r)\frac{x}{r}}{r^2}$$

$$= \frac{r x f''(r)\frac{x}{r} + r f'(r) - \frac{x^2}{r} f'(r)}{r^2}$$

$$\therefore \frac{\partial^2}{\partial x^2} f(r) = \frac{f''(r)}{r^2} x^2 + \frac{1}{r} f'(r) - \frac{x^2}{r^3} f'(r)$$

Similarly, $\frac{\partial^2}{\partial y^2} f(r) = \frac{f''(r)}{r^2} y^2 + \frac{1}{r} f'(r) - \frac{y^2}{r^3} f'(r)$

& $\frac{\partial^2}{\partial z^2} f(r) = \frac{f''(r)}{r^2} z^2 + \frac{1}{r} f'(r) - \frac{z^2}{r^3} f'(r)$

From eq^n. (1) we have,

$$\nabla^2 f(r) = \frac{3}{r} f'(r) + \frac{f''(r)}{r^2}(x^2+y^2+z^2) - \frac{f'(r)}{r^3}(x^2+y^2+z^2)$$

$$= \frac{3}{r} f'(r) + \frac{f''(r)}{r^2} \cdot r^2 - \frac{f'(r)}{r^3} \cdot r^2$$

$$= \frac{3}{r} f'(r) + f''(r) - \frac{1}{r} f'(r)$$

$[\because r^2 = x^2+y^2+z^2]$

$$= f''(r) + \frac{2}{r} f'(r)$$

#

Scanned with CamScanner

Scanned with CamScanner

Vector Integration:

If $\vec{F}(t)$ is defined in the interval $[a, b]$ then the integral $\int_a^b \vec{F}(t)\,dt = [\vec{\phi}(t)]_a^b$

$$= \vec{\phi}(b) - \vec{\phi}(a)$$

is called definite integral

Q. If $\vec{F}(t) = e^t\hat{i} + t^2\hat{j} - e^{-2t}\hat{k}$ then evaluate $\int_1^2 \vec{F}(t)\,dt$

Sol $\int_1^2 \vec{F}(t)\,dt = \int_1^2 (e^t\hat{i} + t^2\hat{j} - e^{-2t}\hat{k})\,dt$

$$= \hat{i}\left[e^t\right]_1^2 + \hat{j}\left[\frac{1}{3}t^3\right]_1^2 - \hat{k}\left[-\frac{1}{2}e^{-2t}\right]_1^2$$

$$= (e^2 - e)\hat{i} + \frac{1}{3}(8-1)\hat{j} + \frac{1}{2}(e^{-4} - e^{-2})\hat{k}$$

$$= (e^2 - e)\hat{i} + \left(\frac{7}{3}\right)\hat{j} + \frac{1}{2}(e^{-4} - e^{-2})\hat{k}$$

//

Q. Given that

$\vec{r}(t) = 2\hat{i} - \hat{j} + 2\hat{k}$ where $t = 2$

$= 4\hat{i} - 2\hat{j} + 3\hat{k}$ where $t = 3$

Show that $\int_2^3 \vec{r} \cdot \frac{d\vec{r}}{dt}\,dt = 10$

Scanned with CamScanner

Scanned with CamScanner

$$\frac{d}{dt}\vec{r}^2 = 2\vec{r}\frac{d\vec{r}}{dt} \Rightarrow \vec{r}\frac{d\vec{r}}{dt} = \frac{1}{2}\frac{d}{dt}(\vec{r}^2)$$

we have, $\int_2^3 \vec{r}\,\frac{d\vec{r}}{dt}\,dt = \int_2^3 \frac{1}{2}\frac{d}{dt}\left(\vec{r}^2(t)\right)dt$

$$= \left[\frac{1}{2}\vec{r}^2(t)\right]_2^3$$

$$= \frac{1}{2}\left[\vec{r}^2(3) - \vec{r}^2(2)\right]$$

$$= \frac{1}{2}\left[(4\hat{i} - 2\hat{j} + 3\hat{k})^2 - (2\hat{i} - \hat{j} + 2\hat{k})^2\right]$$

$$= \frac{1}{2}\left[(16+4+9) - (4+1+4)\right] = 10$$

#

Q If $\vec{r}(t) = 5t^2\hat{i} - t\hat{j} - t^3\hat{k}$ show that

$$\int_1^2 \vec{r}\times\frac{d^2\vec{r}}{dt^2}\,dt = -14\hat{i} + 75\hat{j} - 15\hat{k}$$

Given $\vec{r}(t) = 5t^2\hat{i} - t\hat{j} - t^3\hat{k}$

$$\frac{d\vec{r}}{dt} = 10t\,\hat{i} + \hat{j} - 3t^2\hat{k}$$

$$\frac{d^2\vec{r}}{dt^2} = 10\hat{i} + 0\hat{j} - 6t\hat{k}$$

$$\therefore \vec{r}\times\frac{d^2\vec{r}}{dt^2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5t^2 & t & -t^3 \\ 10 & 0 & -6t \end{vmatrix}$$

$$= \hat{i}(-6t^2+0) - \hat{j}(-30t^3+10t^3) + \hat{k}(0-10t)$$

$$= -6t^2\hat{i} + 20t^3\hat{j} - 10t\hat{k}$$

Scanned with CamScanner

$$\therefore \int_1^2 \vec{r} \times \frac{d^2\vec{r}}{dt^2}\, dt$$

$$= \int_1^2 \left[-6t^2\,\hat{i} + 20t^3\,\hat{j} - 10t\,\hat{k}\right] dt$$

$$= \left[-6\left(\frac{t^3}{3}\right)\hat{i} + 20\left(\frac{t^4}{4}\right)\hat{j} - 10\left(\frac{t^2}{2}\right)\hat{k}\right]_1^2$$

$$= -2t^3\,\hat{i} + 5t^4\,\hat{j} - 5t^2\,\hat{k}$$

$$= -2(8-1)\hat{i} + 5(16-1)\hat{j} - 5(4-1)\hat{k}$$

$$= -2(7)\hat{i} + 5(15)\hat{j} - 5(3)\hat{k}$$

$$= -14\hat{i} + 75\hat{j} - 15\hat{k}$$

//

LINE INTEGRAL :

Line integral of a vector $\vec{F}$ along the curve C is

$$\int_C \vec{F}\cdot d\vec{r} \quad \text{or} \quad \int_C \vec{F}\cdot\frac{dr}{dt}\, dt$$

★ If $\vec{F} = \vec{F_1}\,\hat{i} + \vec{F_2}\,\hat{j} + \vec{F_3}\,\hat{k}$

& $d\vec{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$ then

$$L.I = \int_C \vec{F}\cdot d\vec{r} = \int_C (\vec{F_1}\,\hat{i} + \vec{F_2}\,\hat{j} + \vec{F_3}\,\hat{k})(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$$

Scanned with CamScanner

HAWA MAHAL
PAGE NO
DATE

$$= \int_C F_1\,dx + F_2\,dy + F_3\,dz$$

Q. Find the work done when a force $\vec{F} = (x^2 - y^2 + x)\hat{i} - (2xy + y)\hat{j}$ moves a particle in the xy-plane from $(0,0)$ to $(1,1)$ along the parabola $y^2 = x$

$$W.D = \int_C \vec{F}\cdot\vec{dr} \quad \text{——} \quad (1)$$

$$= \int_C F_1\,dx + F_2\,dy + F_3\,dz$$

$$= \int_C (x^2 - y^2 + x)\,dx - (2xy + y)\,dy$$

Given curve: $C: y^2 = x$

$$\Rightarrow y = \sqrt{x}$$

$$\Rightarrow dy = \frac{1}{2\sqrt{x}}\,dx$$

limit $x \to 0$ to 1 & $y \to 0$ to 1

$$\therefore W.D = \int_{x=0}^{1} (x^2 - x + x)\,dx - \int_{y=0}^{1} (2y^2 \cdot y + y)\,dy$$

$[\because y^2 = x]$

$$= \left[\frac{x^3}{3}\right]_0^1 - \left[\frac{2y^4}{4} + \frac{y^2}{2}\right]_0^1$$

$$= \frac{1}{3} - \left[\left(\frac{1}{2} + \frac{1}{2}\right) - 0\right]$$

$$= \frac{1}{3} - 1 = -\frac{2}{3} \quad \text{(neglecting -ve sign)}$$

$$= 2/3$$

Scanned with CamScanner

Again when path is line $y = x$
i.e, $C: y = x \Rightarrow dy = dx$
& $x =$ to 1

Again

$$W.D = \int_0^1 (x^2 - x^2 + x)dx - (2x.x + x)dx$$

$$= \int_0^1 (x - 2x^2 - x)\, dx$$

$$= \int_0^1 -2x^2\, dx = -2\left[\frac{x^3}{3}\right]_0^1$$

$$= -\frac{2}{3}$$

(neglecting -ve sign)

$$W.D = \frac{2}{3}$$

Thus, work done is not different along the path $y^2 = x$ & $y = x$

Q. Find the total work done in moving a particle in a force field given by $\vec{F} = 3xy\hat{i} - 5z\hat{j} + 10x\hat{k}$ along the curve $x = t^2+1$, $y = 2t^2$, $z = t^3$ from $t=1$ to $t=2$

Sol^n

Given $C: x = t^2+1, \; y = 2t^2, \; z = t^3$

$dx = 2t\, dt$

$dy = 4t\, dt$

$dz = 3t^2 dt$

& limit of $t \to 1$ to 2

Scanned with CamScanner

HAWA MAHAL
PAGE NO:
DATE:

Work done $\int_C \vec{F}\, d\vec{r}$

$$= \int_C (3xy\,dx - 5z\,dy + 10x\,dz)$$

$$= \int_{t=1}^{2} \Big[3(t^2+1)\,2t^2 \cdot 2t\,dt - 5(t^3\,4t)\,dt + 10\,(t^2+1)\,3t^2\,dt \Big]$$

$$= \int_1^2 \left[12t^5 + 10t^4 + 12t^3 + 30t^2 \right] dt$$

$$= \left[12\frac{t^6}{6} + 10\frac{t^5}{5} + 12\frac{t^4}{4} + 30\frac{t^3}{3} \right]_1^2$$

$$= 303$$

Q If $\vec{F} = 2y\hat{i} - z\hat{j} + x\hat{k}$ evaluate $\int_C \vec{F} \times d\vec{r}$ along the curve $x = \cos t$, $y = \sin t$, $z = 2\cos t$ from $t = 0$ to $\pi/2$

A: Given, $\vec{F} = 2y\hat{i} - z\hat{j} + x\hat{k}$

$$= 2\sin t\,\hat{i} - 2\cos t\,\hat{j} + \cos t\,\hat{k}$$

$$d\vec{r} = dx\,\hat{i} - dy\,\hat{j} + dz\,\hat{k}$$

$$= -\sin t\,dt\,\hat{i} + \cos t\,dt\,\hat{j} - 2\sin t\,dt\,\hat{k}$$

Scanned with CamScanner

Scanned with CamScanner

then $\vec{F} \times d\vec{r}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2\sin t & -2\cos t & \cos t \\ -\sin t\,dt & \cos t\,dt & -2\sin t\,dt \end{vmatrix}$$

$$= \hat{i}(4\sin t\cos t\,dt - \cos^2 t\,dt)$$
$$- \hat{j}(-4\sin^2 t\,dt + \sin t\cos t\,dt)$$
$$+ \hat{k}(\cancel{2\sin t\cos t\,dt} - \cancel{2\sin t\cos t\,dt})$$

$$= \hat{i}(4\sin t\cos t\,dt - \cos^2 t\,dt)$$
$$- \hat{j}(-4\sin^2 t\,dt + \sin t\cos t\,dt)$$
$$+ 0\hat{k}$$

$$\therefore \int_C \vec{F}.d\vec{r} = \hat{i}\int_0^{\pi/2}(2\sin 2t - \cos^2 t)\,dt$$

$$+\hat{j}\int_0^{\pi/2}(\cancel{4}\sin^2 t - \frac{\sin 2t}{2})\,dt$$

$$= \hat{i}\,(2 - \pi/4) + (\pi - 1/2)\,\hat{j}$$

//

Scanned with CamScanner

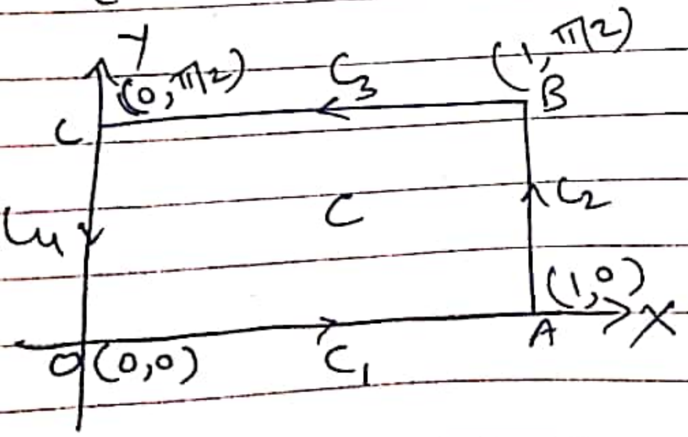
$\frac{}{dt}$ $\frac{}{dt^2}$

Find the circulation of $\vec{F}$ along the curve C, where $\vec{F} = e^x \sin y\,\hat{i} + e^x \cos x\,\hat{j}$ & C is the rectangle whose vertices are $(0,0)$ $(1,0)$ $(1, \pi/2)$ $(0, \pi/2)$

Circulation of $\vec{F}$ = line integral $= \int_c \vec{F}\cdot d\vec{r}$

$$= \int_C \left(e^x \sin y\,\hat{i} + e^x \cos x\,\hat{j}\right)\left(\hat{i}\,dx + \hat{j}\,dy\right)$$

$$= \int_C e^x \sin y\,dx + e^x \cos y\,dy$$



Let, C is the rectangle OABC with curves C_1 C_2 C_3 C_4

then $\int_C = \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4}$

for C_1 : OA, $y = 0$, $dy = 0$

$\Rightarrow x \to 0$ to 1

for C_2 : AB, $x = 1$, $dx = 0$

$\Rightarrow y \to 0$ to $\pi/2$

For C_3 : BC, $y = \pi/2$, $dy = 0$

$\Rightarrow x \to 1$ to 0

For C_4 : CO, $x = 0 \Rightarrow dx = 0$

$\Rightarrow y \to \pi/2$ to 0.

$$\therefore \int_C \vec{F}\,\vec{dr} = \int_{C_1} \vec{F}\,dr + \int_{C_2} \vec{F}\,\vec{dr} + \int_{C_3} \vec{F}\,\vec{dr} + \int_{C_4} \vec{F}\,\vec{dr}$$

$$= \int_{x=0}^{1} 0 + \int_{y=0}^{\pi/2} e\cos y\,dy + \int_{x=1}^{0} e^{x}\sin(\pi/2)\,dx + \int_{y=\pi/2}^{0} \cos y\,dy$$

$$= e[\sin y]_0^{\pi/2} - [e^{x}]_0^{1} + (\sin y)_{\pi/2}^{0}$$

$$= e(1-0) - (e-1) + (0-1) = 0$$

//

Surface Integral

Any integral which is to be evaluated over a surface is called surface integral.

$\therefore$ Surface integral $= \iint_S \vec{F}.\hat{n}\, ds$

or $\int_S \vec{F}.\hat{n}\, ds$

(1) If $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ then cartesian form

$$S.I = \iint_S (F_1\, dy\, dz + F_2\, dz\, dx + F_3\, dx\, dy)$$

(2) If $\vec{F}$ be a vector pt. funct^n along the closed surface S Then

$$S = \iint_S \vec{F}.\hat{n}\, ds \text{ or } \oint_S \vec{F}.\hat{n}\, ds$$

(3) If R_1 be the projection of the surface S on xy plane then

$$\iint_S \vec{F}.\hat{n}\, ds = \iint_{R_1} \frac{\vec{F}.\hat{n}}{|\hat{n}.\hat{k}|}\, dx\, dy$$

R_2 be the projection on the surface S on yz plane then

$$\iint_S \vec{F}.\hat{n}\, ds = \iint_{R_2} \frac{\vec{F}.\hat{n}}{|\hat{n}.\hat{i}|}\, dy\, dz$$

R_3 be the projection on the surface S on zx-plane then

$$\iint_S \vec{F}.\hat{n}\, ds = \iint_{R_3} \frac{\vec{F}.\hat{n}}{|\hat{n}.\hat{j}|}\, dz\, dx$$

Scanned with CamScanner

HAWA MAHAL
PAGE NO.:
DATE:

(4) If given projection of S in xy plane then
$\hat{n} = \hat{k}$ & $ds = dx\,dy$

(5) If the given projection of surface S in xy plane which is a circle $x^2+y^2=a^2$ then we can use polar coordinates (r,θ)
$x = r\cos\theta$, $y = r\sin\theta$, $dx\,dy = r\,dr\,d\theta$
limit $\theta \to 0$ to 2π, $r \to 0$ to radius of circle a.

(6) Let S is the surface of the shape ϕ then normal vector $\vec{n} = \text{grad}\,\phi$
& $\hat{n} = \dfrac{\text{grad}\,\phi}{|\text{grad}\,\phi|}$

Q ~~Evaluate~~ Evaluate $\iint_S \vec{F}\cdot\hat{n}\,ds$ where

2007
2009
2012

$\vec{F} = (x+y^2)\hat{i} - 2x\hat{j} + 2yz\hat{k}$ & S is the surface of the plane $2x+y+2z=6$ in the first octant.

Sol

Let $\phi = 2x+y+2z-6$
surface normal vector $\vec{n} = \text{grad}\,\phi$
$\vec{n} = \nabla\phi$
$= \left(\hat{i}\dfrac{\partial\phi}{\partial x} + \hat{j}\dfrac{\partial\phi}{\partial y} + \hat{k}\dfrac{\partial\phi}{\partial z}\right)$
$= 2\hat{i} + \hat{j} + 2\hat{k}$
Unit normal vector $\hat{n} = \dfrac{\text{grad}\,\phi}{|\text{grad}\,\phi|}$
$= \dfrac{2\hat{i}+\hat{j}+2\hat{k}}{\sqrt{2^2+1^2+2^2}} = 2\hat{i}+\hat{j}+2\hat{k}$

Scanned with CamScanner

Scanned with CamScanner

Let the projection of the surface on the xy-plane be the region R

$R: 2x + y = 6$ & $ds = \dfrac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$

as in the xy plane $z = 0$

for limits :

$\phi = 2x + y + 2z - 6$

$\Rightarrow 2x + y = 6$

$\Rightarrow y = 6 - 2x$

i.e, $y = 0$ to $6 - 2x$

& $x = 0$ & $x = 3$

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_R \frac{\vec{F}\cdot\hat{n}}{|\hat{n}\cdot\hat{k}|}\,dx\,dy$$

$$\iint_R \frac{\left((x+y^2)\hat{i} - 2x\hat{j} + 2yz\hat{k}\right)\cdot\frac{1}{3}(2\hat{i}+2\hat{j}+2\hat{k})}{\left|\frac{2\hat{i}+2\hat{j}+2\hat{k}}{3}\cdot\hat{k}\right|}\,dx\,dy$$

$$= \iint_R \left((\cancel{x}+y^2) - \cancel{x} + 2yz\right)dx\,dy$$

$\because 2z = 6 - 2x - y$

$\Rightarrow z = \dfrac{6 - 2x - y}{2}$

$$= \iint_R (y^2 + 2yz)\,dx\,dy$$

$$\iint_R \left(y^2 + \cancel{2}y\frac{(6 - 2x - y)}{\cancel{2}}\right)dx\,dy$$

Scanned with CamScanner

$$= \int_{x=0}^{3} \int_{y=0}^{6-2x} \text{—} \; dx\,dy$$

$$= 81$$

Scanned with CamScanner

Scanned with CamScanner

HAWA MAHAL
PAGE NO.
DATE:

Q. (1/2) Evaluate $\iint_S \vec{F}.\hat{n}\,ds$ where, $\vec{F} = 18z\hat{i} - 12\hat{j} + 3y\hat{k}$ & S is the surface of the plane $2x+3y+6z=12$ in the first octant.

Sol^n Eq^e. of the plane

$$\phi = 2x+3y+6z = 12$$

i.e, $\phi = 2x+3y+6z-12$

$$\text{grad}\,\phi = \nabla\phi = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\phi$$

$$= 2\hat{i}+3\hat{j}+6\hat{k}$$

$$\therefore \hat{n} = \frac{\text{grad}\,\phi}{|\text{grad}\,\phi|}$$

$$= \frac{2\hat{i}+3\hat{j}+6\hat{k}}{|2\hat{i}+3\hat{j}+6\hat{k}|}$$

$$= \frac{2\hat{i}+3\hat{j}+6\hat{k}}{\sqrt{2^2+3^2+6^2}}$$

$$= \frac{1}{7}(2\hat{i}+3\hat{j}+6\hat{k})$$

$$\vec{F}.\hat{n} = (18z\hat{i}-12\hat{j}+3y\hat{k}).\frac{1}{7}(2\hat{i}+3\hat{j}+6\hat{k})$$

$$= \frac{1}{7}(36z-36+18y)$$

$$= \frac{18}{7}(y+2z-2)$$

Scanned with CamScanner

Scanned with CamScanner

Let R be the projection of S on the xy-plane then

$$dS = \frac{dx\,dy}{|\hat{n}.\hat{k}|}$$

~~R: 2x+3y+1~~

R: $2x + 3y = 12$ $\quad [\because z = 0]$

For limits $y \to 0$ to $(12-2x)/3$

& $x = 0$ to 6

$[\because y = 0]$

$\therefore \hat{n}.\hat{k}$

$$= \frac{1}{7}(2\hat{i} + 3\hat{j} + 6\hat{k})\hat{k}$$

$$= \frac{6}{7}$$

$$\therefore \iint_S \vec{F}.\hat{n}\,dS = \iint_R \vec{F}.\hat{n}\,\frac{dx\,dy}{|\hat{n}.\hat{k}|}$$

$$= \iint_R \frac{18}{7}(y + 2z - 2)\frac{7}{6}\,dx\,dy$$

$$= \iint_R (3y + 6z - 6)\,dx\,dy$$

$$= \iint_R [3y + (12 - 2x - 3y) - 6]\,dx\,dy$$

$[\because 2x + 3y + 6z = 12]$

Scanned with CamScanner

PAGE NO.:
DATE :

$$= \int_{x=0}^{6} \int_{y=0}^{\frac{(12-2x)}{3}} (6-2x)\, dy\, dx$$

$$= \int_{x=0}^{6} (6-2x) \frac{1}{3} (12-2x)\, dx$$

$$= \cancel{\frac{4}{3} \int_0^6 (3-x)(6-x)\, dx}$$

$$= \frac{4}{3} \int_0^6 (18 - 3x - 6x + x^2)\, dx$$

$$= \frac{4}{3} \left[18x - 3.\frac{x^2}{2} - 6.\frac{x^2}{2} + \frac{x^3}{3} \right]_0^6$$

$$= \frac{4}{3} \left[18(6) - \frac{9}{2}(6)^2 + \frac{(6)^3}{3} \right]$$

$$= 24$$

For practice :

1. Evaluate $\iint_S \vec{F}.\hat{n}\, ds$ where, $\vec{F} = z\hat{i} + x\hat{j} - 3y^2z\hat{k}$ & S is the surface of the cylinder $x^2+y^2=16$, included in the first octant b/w $z=0$ & $z=5$

2. Evaluate $\iint_S (yz\hat{i} + zx\hat{j} + xy\hat{k}).ds$, where S is the surface of the sphere $x^2+y^2+z^2=a^2$ in the first octant

Scanned with CamScanner

Scanned with CamScanner

VOLUME INTEGRALS

$$\iiint_V \vec{F}\, dV \quad \text{or} \quad \int_V \vec{F}\, dV$$

* If $\vec{F} = \vec{F_1}\hat{i} + \vec{F_2}\hat{j} + \vec{F_3}\hat{k}$ then the volume integral of Cartesian form

$$\iiint_V \vec{F}\, dV = \hat{i}\iiint F_1\, dx\, dy\, dz + \hat{j}\iiint F_2\, dx\, dy\, dz + \hat{k}\iiint F_3\, dx\, dy\, dz$$

Q. Evaluate $\int_V (2x+y)\, dV$ where, V is the closed region bounded by the cylinder $z = 4-x^2$ & the planes $x=0$, $y=0$, $y=2$ & $z=0$

Sol^n.

The cylinder $z = 4-x^2$ meets the the z-axis at $(0,0,4)$ & the x-axis $(2,2,0)$.

$\therefore$ limits of integration are

$z=0$ to $z = 4-x^2$

for $x = 0$ to 2

$y = 0$ to 2

Scanned with CamScanner

HALWA MAHAL
PAGE NO.:
DATE:

$$\int_V (2x+y)\,dv = \int_{x=0}^{2}\int_{y=0}^{2}\int_{z=0}^{4-x^2} (2x+y)\,dz\,dy\,dx$$

$$= \int_{x=0}^{2}\int_{y=0}^{2} (2x+y)\,[z]_0^{4-x^2}\,dy\,dx$$

$$= \int_{x=0}^{2}\int_{y=0}^{2} (2x+y)\,(4-x^2)\,dy\,dx$$

$$= \int_{x=0}^{2}\int_{y=0}^{2} 2x\,(4-x^2)\,dy\,dx + \int_{x=0}^{2}\int_{y=0}^{2} y\,(4-x^2)\,dy\,dx$$

$$= \left[-\frac{1}{2}(4-x^2)^2\right]_0^2 [y]_0^2 + \left[4x-\frac{1}{3}x^3\right]_0^2 \left[\frac{y^2}{2}\right]_0^2$$

$$= \frac{80}{3}$$

//

For <u>Practice</u>:

Q. Evaluate $\iiint_V \vec{\nabla}\cdot\vec{F}\,dV$ where,

$\vec{F} = (2x^2-3z)\hat{i} - 2xy\hat{j} - 4x\hat{k}$ &

V is bounded by the planes

$x=0,\ y=0,\ z=0$ & $2x+2y+z=4$

Scanned with CamScanner

Stoke's Theorem: Relation b/w Line & Surface integrals

$$\int_C \vec{F}.\vec{dr} = \iint_S \text{curl}\,\vec{F}.\hat{n}\,ds$$

Apply stoke's Th. to evaluate

$\int_C (x+y)dx + (2x-z)dy + (z+y)dz$ where C is the boundary of the triangle with vertices (2,0,0) (0,3,0) & (0,0,6)

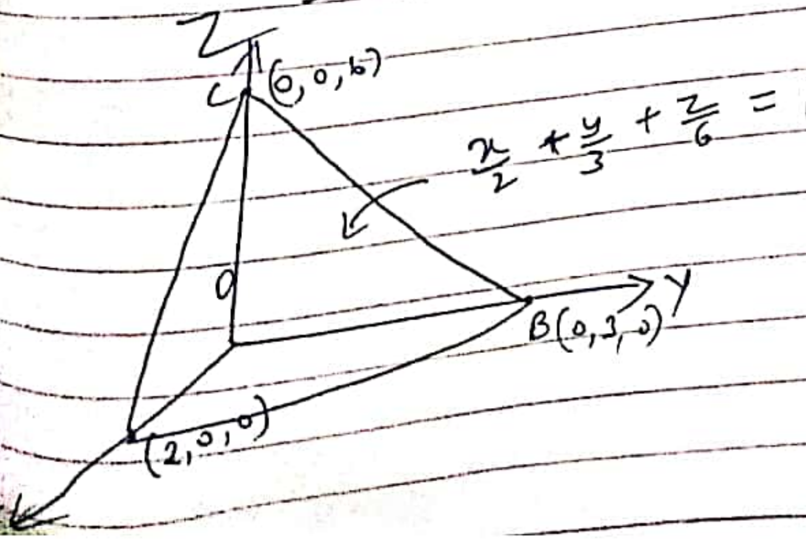
By stoke's Th. $\int_C \vec{F}.\vec{dr} = \iint_S \vec{F}.\hat{n}\,ds$

$$\vec{F} = (x+y)\hat{i} + (2x-z)\hat{j} + (z+y)\hat{k}$$

$$\text{curl}\,\vec{F} = \nabla\times\vec{F}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x+y) & (2x-z) & (z+y) \end{vmatrix}$$

$$= 2\hat{i} + \hat{k}$$



Scanned with CamScanner

Let, ΔABC & S be a surface. Then eq^c. of triangular plane be

$$\frac{x}{2}+\frac{y}{3}+\frac{z}{6}=1$$

$$\Rightarrow 3x+2y+z=6$$

$$\Rightarrow \phi = 3x+2y+z-6 \qquad \vec{n} = \text{grad}\,\phi$$

$$= \frac{3\hat{i}+2\hat{j}+\hat{k}}{}$$

$$\hat{n} = \frac{\text{grad}\,\phi}{|\text{grad}\,\phi|} = \frac{3\hat{i}+2\hat{j}+\hat{k}}{\sqrt{9+4+1}}$$

$$= \frac{1}{\sqrt{14}}(3\hat{i}+2\hat{j}+\hat{k})$$

on the xy plane

$$ds = \frac{dx\,dy}{|\hat{n}.\hat{k}|} = \frac{dx\,dy}{\frac{3\hat{i}+2\hat{j}+\hat{k}}{\sqrt{14}}} = \frac{dx\,dy}{1/\sqrt{14}}$$

$$\int_C \vec{F}.\vec{dr} = \iint_R (2\hat{i}+\hat{k})(3\hat{i}+2\hat{j}+\hat{k})\,dx\,dy$$

$$= \iint_R (6+1)\,dx\,dy = 7\iint_R dx\,dy$$

$$= 7\int_{x=0}^{2}\int_{y=0}^{\frac{6-3x}{2}} dx\,dy = 21$$

#

Verify Stoke's th. for $\vec{F} = (x^2 - y^2)\hat{i} + 2xy\hat{j}$ taken around the rectangle in the plane $z = 0$ bounded by the lines $x = 0$, $y = 0$, $x = a$, $y = b$.

By Stoke's Th.

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \operatorname{curl} \vec{F} \cdot \hat{n}\, ds$$

Let S be the plane region in xy plane bounded by rectangle OABC having 4 sides C_1, C_2, C_3 & C_4

$$\int_C \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r} + \int_{C_3} \vec{F} \cdot d\vec{r} + \int_{C_4} \vec{F} \cdot d\vec{r}$$

$$\boxed{\vec{F} \cdot d\vec{r} = (x^2 - y^2)\,dx + 2xy\,dy}$$

C_1: $y = 0 \Rightarrow dy = 0$, $x \to 0$ to a

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{x=0}^{a} x^2\,dx = \left(\frac{x^3}{3}\right)_0^a = \frac{a^3}{3}$$

C_2: $x = a \Rightarrow dx = 0$, $y \to 0$ to b

$$\int_{C_2} \vec{F} \cdot d\vec{r} = \int_{y=0}^{b} 2ay\,dy = 2a\left(\frac{y^2}{2}\right)_0^b = ab^2$$

$C_3 \quad \int_{C_3} \vec{F}\, d\vec{r} \quad y = b,\ dy = 0$

$$= -\frac{2a^3}{3} - 2ab^2$$

$C_4: \quad x = a,\ dx = 0,\ y \to b \text{ to } a$

$$\int_{C_4} \vec{F}\, d\vec{r} = -ab^2$$

$$\int_C \vec{F}\, d\vec{r} = -4ab^2$$

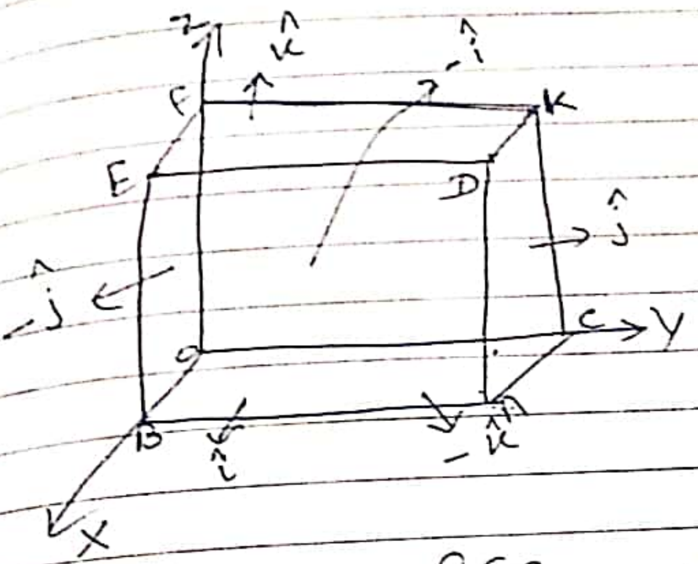
RHS $\quad$ Curl $\vec{F}$ $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+y^2 & -2xy & 0 \end{vmatrix} = 4y\hat{k}$

$$\iint_S \text{Curl } \vec{F} \cdot \hat{n}\, ds = -4ab^2$$

Scanned with CamScanner

Q. 5.1.31) $\vec{F} = x^2\hat{i} + z\hat{j} + yz\hat{k}$

$x=0,\ x=1,\ y=0,\ y=1,\ z=0,\ z=1$



$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \text{div}\,\vec{F}\,.d\vec{v}$$

$\vec{F} = x^2\hat{i} + z\hat{j} + yz\hat{k}$

$x=0,\ x=1,\ y=0,\ y=1,\ z=0,\ z=1$

$\text{div}\,\vec{F} = 2x + y$

$$\int_0^1\int_0^1\int_0^1 (2x+y)\,dx\,dy\,dz = \frac{3}{2}$$

$\iint_S \vec{F}\cdot\hat{n}\,ds$, S consists of 6 planes

S_1: ABCO $\quad \hat{n} = \hat{i} \quad x=1 \quad ds = dy\,dz$

$$\iint_{S_1} \vec{F}\cdot\hat{n}\,ds = \int_{y=0}^{1}\int_{z=0}^{1} (x^2\hat{i} + z\hat{j} + yz\hat{k})\,dy\,dz =$$

S_2 OCKF $\quad \hat{n} = -\hat{i} \quad x=0 \quad ds = dy\,dz$

$\iint \quad = 0$

(3) S_3: ACKD $\hat{n} = \hat{j}$

(4) S_4: OBEF $\hat{n} = -\hat{j}$

(5) S_5: DEFK $\hat{n} = \hat{k}$

(6) S_6: OBAC $\hat{n} = -\hat{k}$

$$= \frac{3}{2}$$

Green's Thm in xy plane

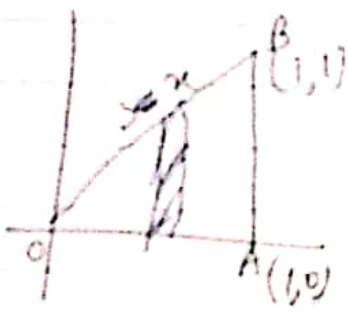
$$\int_C F_1\,dx + F_2\,dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy$$

Ex. 5.94 Green's Theorem

$F_1 = x^2 y$, $F_2 = x^2$

$y \to 0$ to x

$x \to 0$ to 1



$$\int_C (x^2y\,dx + x^2\,dy) = \int_0^1 \int_0^x \frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(x^2y)\,dx\,dy$$

$$= 5/12$$

Scanned with CamScanner

HAWA MAHAL
PAGE NO.
DATE:

Gauss Divergence Theorem:

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \operatorname{div}\vec{F}\,dV$$

Q. Evaluate $\iint_S \vec{F}\cdot d\vec{s}$ where, $\vec{F} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$

2011
2013
& S is the surface bounding the region $x^2+y^2 = 4$, $z=0$ & $z=3$

Sol. By Gauss Divergence Th.

$$\int_S \vec{F}\cdot d\vec{s} = \iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \operatorname{div}\vec{F}\cdot dV \quad \text{--- (1)}$$

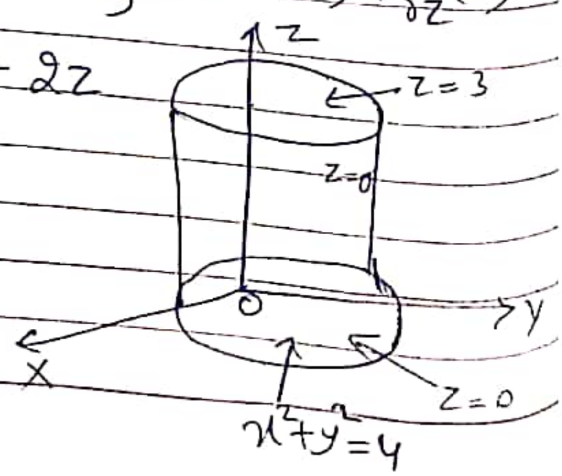
$$\operatorname{div}\vec{F} = \nabla\cdot\vec{F}$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(F_1\hat{i} + F_2\hat{j} + F_3\hat{k}\right)$$

$$= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$= \frac{\partial}{\partial x}(4x) + \frac{\partial}{\partial y}(-2y^2) + \frac{\partial}{\partial z}(z^2)$$

$$= 4 - 4y + 2z$$



Scanned with CamScanner

m π

From eq^{n}. 1

$$\iint_S \vec{F}\cdot\hat{n}\, dS = \iint_S \int_{z=0}^{3} (4-4y+2z)\,dx\,dy\,dz$$

$S: x^2+y^2 = 4$

$$\iint_S \left[4z - 4yz + z^2\right]_{z=0}^{3} dx\,dy$$

$$= \iint_S (12-12y+9)\,dx\,dy$$

$$= \iint_S (21-12y)\,dx\,dy$$

$S: x^2+y^2 = 4$ which is circle using

polar coordinates

substituting $x = r\cos\theta$

$y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$

limit $\theta \to 0$ to 2π

$r \to 0$ to 2 (radius)

$$\iint_S \vec{F}\cdot\hat{n}\, dS = \int_{\theta=0}^{2\pi} \int_{r=0}^{2} [21-12r\sin\theta]\,r\,dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi} \left[21\cdot\frac{r^2}{2} - 12\sin\theta\,\frac{r^3}{3}\right]_{r=0}^{2} d\theta$$

$$= \int_0^{2\pi} [42 - 32\sin\theta]\,d\theta$$

$$= 42\left[\theta + 32\cos\theta\right]_0^{2\pi}$$

$$= 42\,[2\pi + 32(1-1)] = 84\pi$$

Scanned with CamScanner

HAWA MAHAL
PAGE NO:
DATE:

For practice:

Q. Evaluate $\iint_S \vec{F}\cdot d\vec{S}$ where $\vec{F} = yz\hat{i} + 2y^2\hat{j} + xz^2\hat{k}$ & S is

2012 the surface of the cylinder $x^2+y^2=9$ contained in the first octant between the planes $z=0$ & $z=2$

Q Evaluate $\iint_S \vec{F}\cdot\hat{n}\, ds$ over the surface

2011 of the region above the xy-plane bounded by the cone $z^2=x^2+y^2$ & the plane $z=4$ if $\vec{F} = 4xz\hat{i} + xyz^2\hat{j} + 3z\hat{k}$

Sol By Gauss Divergence Th.

$$\iint_S \vec{F}\cdot\hat{n}\, ds = \iiint_V \operatorname{div}\vec{F}\, dV$$

given $\vec{F} = 4xz\hat{i} + xyz^2\hat{j} + 3z\hat{k}$

$$\operatorname{div}\vec{F} = \nabla\cdot\vec{F}$$
$$= \frac{\partial}{\partial x}(4xz) + \frac{\partial}{\partial y}(xyz^2) + \frac{\partial}{\partial z}(3z)$$
$$= 4z + xz^2 + 3$$

Scanned with CamScanner

Given curve $z^2 = x^2 + y^2$ & $z = 4$

$\therefore$ limit of z: $z = \sqrt{x^2+y^2}$ to $z = 4$

& Projection $S: x^2 + y^2 = 4^2$ in xy plane

$\left[x^2 + y^2 = z^2 \ \& \ z = 4 \right.$
$\Rightarrow x^2 + y^2 = 4^2$ in xy plane$\left.\right]$

$$\therefore \iint_S \vec{F}\cdot\hat{n}\, ds = \iint_S \int_{z=\sqrt{x^2+y^2}}^{4} (4z + x z^2 + 3)\, dx\, dy\, dz$$

where, $S: x^2 + y^2 = 4^2$

$$= \iint_S \left[2z^2 + \frac{xz^3}{3} + 3z \right]_{z=\sqrt{x^2+y^2}}^{4} dx\, dy$$

$$= \iint_S \left[\left(32 + \frac{64}{3}x + 12\right) - 2(x^2+y^2) - \frac{x}{3}(x^2+y^2)^{3/2} - 3(x^2+y^2)^{1/2} \right] dx\, dy$$

$S: x^2 + y^2 = 4^2$ which is a circle.

Substituting $x = r\cos\theta$, $y = r\sin\theta$.

$dx\,dy = r\,dr\,d\theta$ [polar coordinates]

limits θ: 0 to 2π

r: 0 to 4

Scanned with CamScanner

HAWA MAHAL
PAGE NO.:
DATE:

$$\iint_S \vec{F}\cdot\hat{n}\,dS = \int_{\theta=0}^{2\pi}\int_{r=0}^{4}\left[44 + \frac{64}{3} r\cos\theta - 2r^2 - \frac{r^4}{3}\cos\theta - 3r\right] r\,dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi}\left[44\frac{r^2}{2} + \frac{64}{3}\cos\theta\,\frac{r^3}{3} - 2\frac{r^4}{4} - \frac{\cos\theta}{3}\frac{r^6}{6} - 3\frac{r^3}{3}\right]_0^4 d\theta$$

$$= \int_{\theta=0}^{2\pi}\left[44.8 + \left(\frac{64}{3}\right)^2\cos\theta - 2(64)\frac{4^6}{18}\cos\theta - 64\right] d\theta$$

$$= \int_0^{2\pi} 160\,d\theta + \left(\frac{64}{3}\right)\int_0^{2\pi}\cos\theta\,d\theta - \frac{4^6}{18}\int_0^{2\pi}\cos\theta\,d\theta$$

$$= 160\left[\theta\right]_0^{2\pi} + 0 \qquad \left[\because \int_0^{2\pi}\cos\theta\,d\theta = 0\right]$$

$$= 160\left[2\pi - 0\right] = 320\pi$$

//

Scanned with CamScanner

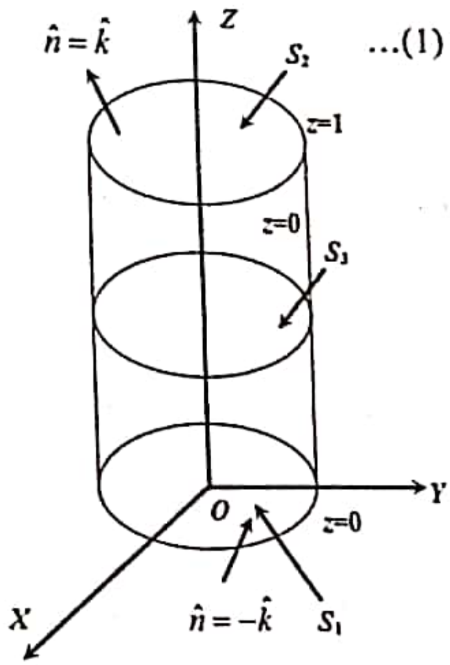
Scanned with CamScanner

$= \frac{\ }{4}a^2 b\pi.$ **Ans.**

Example 5.129 : Evaluate $\iint_S \vec{F}.\hat{n}\,dS$ ***where*** $\vec{F} = x\hat{i} - y\hat{j} + (z^2 - 1)\hat{k}$ ***and S is a closed surface bounded by the planes*** $z = 0,\ z = 1$ ***and the cylinder*** $x^2 + y^2 = 4$.
Also verify Gauss's divergence theorem. ***[RGPV 2001, June 2013]***

Solution. By Gauss's divergence theorem :

$$\iint_S \vec{F}.\hat{n}\,dS = \iiint_V div\,\vec{F}\,dV. \quad \ldots(1)$$



$\therefore \quad div\,\vec{F} = \nabla.\vec{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(-y) + \frac{\partial}{\partial z}(z^2 - 1)$

$= 1 - 1 + 2z = 2z.$

Given curve : $x^2 + y^2 = 4$ and $z = 0, z = 1.$

$\therefore \quad$ limit for $z:\ z = 0$ to $z = 1$

and $\quad$ surface $S : x^2 + y^2 = 4$ is a circle.

Hence (1) becomes :

$$\iint_S \vec{F}.\hat{n}\,dS = \iint_S \int_{z=0}^{1} 2z\,dx\,dy\,dz$$

$$= \iint_S \left[z^2\right]_0^1 dxdy$$

$$= \iint_S dx\,dy = \text{Area of surface } \{S : x^2 + y^2 = 2^2\}$$

$$= \pi(2)^2 = 4\pi. \quad \ldots(2)$$

Scanned with CamScanner

Further for verification of Gauss's divergence theorem :
Here surface S consists of three parts :

(i) *Base* $S_1 : x^2 + y^2 = 4, \quad z = 0, \quad \hat{n} = -\hat{k}$

(ii) *Top face* $S_2 : x^2 + y^2 = 4, \quad z = 1, \quad \hat{n} = \hat{k}$, and

(iii) *Curved surface* $S_3 : x^2 + y^2 = 4$.

$$\therefore \iint_S \vec{F}.\hat{n}\,dS = \iint_{S_1} \vec{F}.\hat{n}\,dS_1 + \iint_{S_2} \vec{F}.\hat{n}\,dS_2 + \iint_{S_3} \vec{F}.\hat{n}\,dS_3 \quad \text{...(3)}$$

For surface S_1 : Circle $x^2 + y^2 = 4, \; z = 0, \; \hat{n} = -\hat{k}, \; dS_1 = dxdy$

Then $$\iint_{S_1} \vec{F}.\hat{n}\,dS_1 = \iint_{S_1} [x\hat{i} - y\hat{j} + (0-1)\hat{k}].(-\hat{k})dxdy$$

$$= \iint_{S_1} 1\,dxdy = \iint_{S_1} dxdy$$

$= \text{Area of circle } S_1 : x^2 + y^2 = 2^2$
$= \pi\,(2)^2 = 4\pi.$

For surface S_2 : Circle $x^2 + y^2 = 4, \; z = 1, \; \hat{n} = \hat{k}, \; dS_2 = dxdy$

Then $$\iint_{S_2} \vec{F}.\hat{n}\,dS_2 = \iint_{S_2} [x\hat{i} - y\hat{j} + 0\hat{k}].(\hat{k})dxdy = 0.$$

For surface S_3: *Curve* $x^2 + y^2 = 4$

Let $\phi = x^2 + y^2 - 4$

$$\therefore \quad \hat{n} = \frac{grad\,\phi}{|grad\,\phi|} = \frac{2x\hat{i} + 2y\hat{j}}{\sqrt{4x^2 + 4y^2}} = \frac{x\hat{i} + y\hat{j}}{2}. \quad [\because x^2 + y^2 = 4]$$

$$\therefore \quad \vec{F}.\hat{n} = [x\hat{i} - y\hat{j} + (z^2 - 1)\hat{k}].\frac{(x\hat{i} + y\hat{j})}{2} = \frac{1}{2}(x^2 - y^2).$$

Then $$\iint_{S_3} \vec{F}.\hat{n}\,dS_3 = \iint_{S_3} \frac{1}{2}(x^2 - y^2)\,dxdy.$$

Since $S_3 : x^2 + y^2 = 4$ is a circle, so using polar co-ordinates.

Put $\quad x = r\cos\theta, \; y = r\sin\theta, \; dxdy = r\,drd\theta$

and $\quad$ limits : $\theta \to 0$ to 2π and $r \to 0$ to 2.

Hence $$\iint_{S_3} \vec{F}.\hat{n}\,dS_3 = \int_{\theta=0}^{2\pi} \int_{r=0}^{2} \frac{1}{2}(r^2\cos^2\theta - r^2\sin^2\theta)\,r\,drd\theta$$

$$= \frac{1}{2}\int_{\theta=0}^{2\pi} \left[\frac{r^4}{4}\right]_0^2 .\cos 2\theta\,d\theta \quad [\because \cos^2\theta - \sin^2\theta = \cos 2\theta]$$

Scanned with CamScanner

$$= 2\int_0^{2\pi} \cos 2\theta\, d\theta = [\sin 2\theta]_0^{2\pi} = 0.$$

Thus, (3) becomes :

$$\iint_S \vec{F}.\hat{n}\, dS = 4\pi + 0 + 0 = 4\pi. \qquad \ldots(4)$$

From (2) and (4), we get

$$\iint_S \vec{F}.\hat{n}\, dS = \iiint_V div\, \vec{F}\, dV.$$

Hence Gauss's divergence theorem is verified.

Scanned with CamScanner

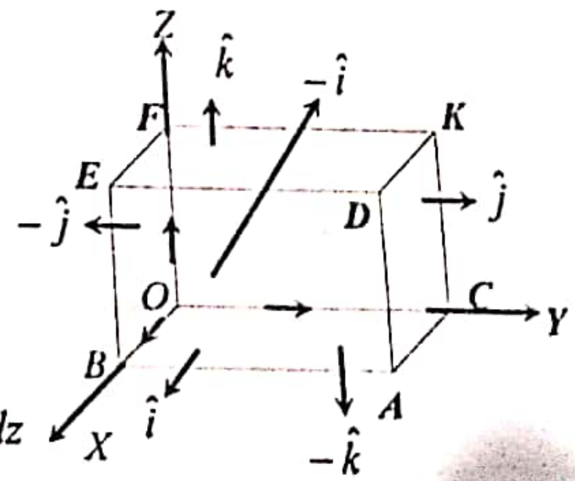
... divergence theorem.

Example 5.131 : Verify Divergence theorem for $\vec{F} = x^2\hat{i} + z\hat{j} + yz\hat{k}$ ***taken over the cube bounded by*** $x = 0, x = 1, y = 0, y = 1, z = 0$ ***and*** $z = 1$. ***[RGPV Dec. 2003, June 2008 (O) & Feb. 2010]***

Solution. By Gauss's divergence theorem :

$$\iint_S \vec{F}.\hat{n}\,dS = \iiint_V div\,\vec{F}\,dV \qquad ...(1)$$

Given : $\vec{F} = x^2\hat{i} + z\hat{j} + yz\hat{k}$



$$div\,\vec{F} = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(z) + \frac{\partial}{\partial z}(yz)$$

$$= 2x + 0 + y = 2x + y.$$

$$\therefore \quad \text{R.H.S.} = \iiint_V div\,\vec{F}\,dV = \int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1} (2x + y)\,dxdydz$$

Scanned with CamScanner

$$= \int_{x=0}^{1} \int_{y=0}^{1} (2x+y)(z)_0^1 \, dxdy$$

$$= \int_{x=0}^{1} \int_{y=0}^{1} (2x+y)\,dxdy = \int_{x=0}^{1} \left(2xy + \frac{y^2}{2}\right)_0^1 dx = \int_0^1 \left(2x + \frac{1}{2}\right) dx$$

$$= \left[x^2 + \frac{1}{2}x\right]_0^1 = 1 + \frac{1}{2} = \frac{3}{2}. \qquad \ldots(2)$$

To evaluate $\iint_S \vec{F}.\hat{n}\,dS$, where S consists of six planes surfaces.

For the face ABCO i.e. $S_1 : \hat{n} = \hat{i},\ \ x = 1,\ \ dS = dydz$

$$\therefore \qquad \iint_{S_1} \vec{F}.\hat{n}\,dS = \int_{y=0}^{1} \int_{z=0}^{1} (1^2\hat{i} + z\hat{j} + yz\hat{k}).\hat{i}\,dydz = \int_0^1 \int_0^1 dydz = 1.$$

For the face OCKF i.e., $S_2 : \hat{n} = -\hat{i},\ \ x = 0,\ \ dS = dydz$

$$\therefore \qquad \iint_{S_2} \vec{F}.\hat{n}\,dS = \int_{y=0}^{1} \int_{z=0}^{1} (0\hat{i} + z\hat{j} + yz\hat{k}).(-\hat{i})\,dydz = 0.$$

For the face OBAC, i.e., $S_3 : \hat{n} = -\hat{k},\ z = 0,\ dS = dxdy$

$$\therefore \qquad \iint_{S_3} \vec{F}.\hat{n}\,dS = \int_{x=0}^{1} \int_{y=0}^{1} \left(x^2\hat{i} + z\hat{j} + yz\hat{k}\right).\left(-\hat{k}\right) dxdy$$

$$= -\int_{x=0}^{1} \int_{y=0}^{1} yz\,dxdy = 0. \qquad [\because z = 0]$$

For the face DEFK, i.e., $S_4 : \hat{n} = \hat{k},\ z = 1,\ dS = dxdy$

$$\therefore \qquad \iint_{S_4} \vec{F}.\hat{n}\,dS = \int_{x=0}^{1} \int_{y=0}^{1} \left(x^2\hat{i} + z\hat{j} + yz\hat{k}\right).\left(\hat{k}\right) dxdy = \int_{x=0}^{1} \int_{y=0}^{1} yz\,dxdy \qquad [\because z = 1]$$

$$= \int_{x=0}^{1} \int_{y=0}^{1} y\,dxdy = \frac{1}{2}.$$

For the face ACKD, i.e., $S_5 : \hat{n} = \hat{j},\ y = 1,\ dS = dxdz$

$$\therefore \qquad \iint_{S_5} \vec{F}.\hat{n}\,dS = \int_{x=0}^{1} \int_{z=0}^{1} \left(x^2\hat{i} + z\hat{j} + yz\hat{k}\right).\left(\hat{j}\right) dxdz$$

Scanned with CamScanner

$$= \int_{x=0} \int_{z=0} z\,dxdz = \frac{1}{2}.$$

For the face OBEF, i.e., $S_6 : \hat{n} = -\hat{j},\ y = 0,\ dS = dxdz$

$$\therefore \qquad \iint_{S_6} \vec{F}.\hat{n}\,dS = \int_{x=0}^{1} \int_{z=0}^{1} \left(x^2\hat{i} + z\hat{j} + yz\hat{k}\right).\left(-\hat{j}\right) dxdz$$

$$= -\int_{x=0}^{1} \int_{z=0}^{1} z\,dxdz = -\int_{x=0}^{1} \left[\frac{z^2}{2}\right]_0^1 dx = -\frac{1}{2}\int_0^1 dx = -\frac{1}{2}.$$

Hence $\quad$ L.H.S. $= \iint_S \vec{F}.\hat{n}\,dS = \iint_{S_1} + \iint_{S_2} + \ldots\ldots\ldots\ldots + \iint_{S_6}$

$$= 1 + 0 + 0 + \frac{1}{2} + \frac{1}{2} - \frac{1}{2} = \frac{3}{2}. \qquad \ldots(3)$$

From (2) and (3), we get

$$\iint_S \vec{F}.\hat{n}\,dS = \iiint_V div\,\vec{F}.dV$$

Proved.

Hence verified Gauss's divergence theorem.

... then evaluate $\iiint \nabla.\vec{F}\,dV$, where V is bounded by

Scanned with CamScanner