Chapter 1

UNIT-I

BASIC PROBABILITY

§ 1.1. INTRODUCTION

If a coin is tossed repeatedly under essentially homogeneous and similar conditions, then one is not sure if a 'head' or a 'tail' will be obtained. Such type of phenomena (*i.e., phenomena which do not lend itself to deterministic approach*) are called '**unpredictable**' or '**probabilistic**' phenomena.

In 1993, A. N. Kolmogrov, a Russian mathematician, tried successfully to relate the theory of probability with the set theory by axiomatic approach. The axiomatic definition of probability includes both the classical and the statistical definitions as particular cases and overcomes the definitions of each of them.

Now we shall have some definitions, before actually discussing 'probability'.

§ 1.2. RANDOM EXPERIMENT, SAMPLE SPACE

Consider a bag containing 4 white and 5 black balls. Suppose 2 balls are drawn at random. Here the natural phenomena is that 'both balls may be white' or 'one white and one black' or 'both black'. Thus there is a *probabilistic situation*.

We feel intuitively in the following statements.

(i) The probability of getting a 'tail' in one toss of an unbiased coin is $\frac{1}{2}$.

(ii) The probability of getting an 'ace' in a single toss of an unbiased die is $\frac{1}{6}$.

Similarly the probability of getting either a '2' or a '3' in a single throw of an unbiased die should be the sum of probabilities of getting a '2' or a '3' *i.e.*, $\frac{1}{6}+\frac{1}{6}=\frac{1}{3}$.

In other words, there should be some kind of additive property in the theory of probabilities. To understand such type of probabilistic situations, we need mathematical models.

Any probabilistic situation is called a random experiment and is denoted by E.

Each performance in random experiment is called a 'trial' and the result of a trial is called an 'outcome' or a 'sample point' or an 'elementary event'.

Sample space. *A sample space of a random experiment is the set of all possible outcomes of that experiment and is denoted by S.*

Example 1. If a coin is tossed, then there are two possibilities : either we shall get a head (H) or a tail (T). Thus the sample space is the set $S = (H, T)$.

Example 2. If a cubical die is rolled, out of six faces, one and only one shall come upwards. Thus sample space is the set $S = \{1, 2, 3, 4, 5, 6\}$.

Sample point. Every element of the sample space is called a *sample point.* In example 1 above, the sample points are H and T. In example 2 above, the sample points are 1, 2, 3, 4, 5 and 6.

Finite sample space. A sample space containing finite number of sample points, is called a *finite sample space*. In examples 1 and 2 above both sample space are finite sample spaces.

Event. Of all the possible outcomes in the sample space of an experiment some outcomes satisfy a specified description, it is called an event. In other words '*every non-empty subset of a sample space is called an event of the sample space*'. It is denoted by E. Several events are denoted by E_1, E_2 etc.

Certain and impossible events. If S is a sample space, then S and ϕ are both subsets of S and so S and ϕ both are events. S is called *certain event* and ϕ is called *impossible event.*

Equally likely events. Two events are considered equally likely if one of them cannot be expected in preference to the other.

For example. If an unbiased coin is tossed then we may get any of head (H) or tail (T), thus the two different events are equally likely.

Exhaustive Events :

All possible outcomes in a trial, are called exhaustive events.

For example. If an unbiased die is rolled, then we may obtain any one of the six numbers 1, 2, 3, 4, 5 and 6. Hence there are six exhaustive events in this trial.

Favourable events. The total number of favourable outcomes (or ways) in a trial, to happen an event, are called favourable events.

For example. If a pair of fair dice is tossed then the favourable events to get the sum 7 are six :

$$(1, 6),\ (2, 5),\ (3, 4),\ (4, 3),\ (5, 2),\ (6, 1).$$

Mutually Exclusive or Incompatible Events :

Two or more than two events are called *mutually exclusive events* if there is no element (or outcome or result) common to these events. In other words, events are called mutually exclusive if the happening of one of them prevents or precludes the happening of the other events.

If E_1 and E_2 are two mutually exclusive events then $E_1 \equiv E_2 = \phi \rightarrow E_1$ and E_2 are mutually exclusive.

EXAMPLES ON SAMPLE SPACE AND EVENT

Example 1. ***In a single toss of a fair die, find (a) sample space; (b) event of getting an even number, (c) event of getting an odd number, (d) event of getting numbers greater than 3, (e) event of getting numbers less than 4.***

Solution. (a) When we toss a die, then we may get any of the six numbers, 1, 2, 3, 4, 5 and 6. Hence the set of these six numbers is the sample space S for this experiment, *i.e.*,

$$S = \{1, 2, 3, 4, 5, 6\}.$$

In the above experiment, to get an even number is an event, say E_1; to get an odd number is an event, say E_2; to get numbers greater than 3 is an event, say E_3; and to get numbers less than 4 is an event, say E_4. Thus

(b) $E_1 = \{2, 4, 6\}$,

(c) $E_2 = \{1, 3, 5\}$,

(d) $E_3 = \{4, 5, 6\}$,

(e) $E_4 = \{1, 2, 3\}$.

Example 2. ***Consider an experiment in which two coins are tossed together. Find the sample space. Find also the following events :***

(a) Heads on the upper faces of both coins, (b) head on one and tail on other, (c) Tails on both, (d) at least one head.

Solution. If H denotes 'head' and T denotes 'tail' then the toss of two coins can lead to four cases $(H, H), (H, T), (T, H), (T, T)$ all equally likely. Hence the sample space S is the set of all these four ordered pairs, thus

$$S = \{(H, H), (H, T), (T, H), (T, T)\}.$$

In this experiment let E_1, E_2, E_3 and E_4 be the events of getting both heads, one head and one tail, both tails and at least on head respectively, then

(a) $E_1 = \{(H, H)\}$,

(b) $E_2 = \{(H, T), (T, H)\}$,

(c) $E_3 = \{(T, T)\}$,

(d) $E_4 = \{(H, H), (H, T), (T, H)\}$.

Example 3. ***Find the sample space, when***

(a) Two dice are tossed.

(b) A coin and a die are tossed together.

Solution. (a) When two dice are tossed together, the sample space S for this experiment is given by

$$\begin{aligned} S &= \{1, 2, 3, 4, 5, 6\} \times \{1, 2, 3, 4, 5, 6\} \\ &= \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6), \\ &\quad (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6), (4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6), \\ &\quad (5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)\}. \end{aligned}$$

Clearly number of elements in the sample S are 36 *i.e.*, $n(S) = 36$.

(b) When a coin and a die are tossed together the sample space S is given by

$$S = \{H, T\} \times \{1, 2, 3, 4, 5, 6\}$$
$$= \{(H, 1), (H, 2), (H, 3), (H, 4), (H, 5), (H, 6), (T, 1), (T, 2), (T, 3), (T, 4), (T, 5), (T, 6)\}.$$

Clearly $n(S) = 12$.

Example 4. ***Find the sample space when three coins are tossed together or a coin is tossed three times in succession.***

Solution. The sample space S for both experiment is same as is given by

$$S = \{H, T\} \times \{H, T\} \times \{H, T\}$$
$$= \{HH, HT, TH, TT\} \times \{H, T\}$$
$$= \{HHH, HTH, THH, TTH, HHT, HTT, THT, TTT\}.$$

Here $n(S) = 2 \times 2 \times 2 = 8$.

§ 1.3. SIMPLE AND COMPOUND EVENTS

Consider a random experiment and let $e_1, e_2, \ldots, e_n$ be the outcomes or sample points so that the sample space S for this experiment is given by $S = \{e_1, e_2, \ldots, e_n\}$.

Let E be an event related to this experiment then $E \subseteq S$. The set E representing the event, may have only one or more elements of S. Based upon this fact, every event can be divided into two following events.

(1) Simple event. If E contains only one element of the sample space S, then E, is called simple event. Thus

$$E = \{e_i\}$$

where $e_i \in S$, is a simple event since it contains only one element of S.

For example. Consider tossing of a coin singly. The possible outcomes for this experiment are H (head) and T (tail). The event E of getting H is a simple event which is denoted by $E = \{H\}$. Similarly the event of getting a tail (T) is a simple event and is represented by $\{T\}$.

Again consider the tossing of a die singly. The possible outcomes for this experiment are the numbers 1 to 6 *i.e.,* the sample space S is given by $S = \{1, 2, 3, 4, 5, 6\}$.

The event of getting the number 4 is a simple event and is represented by $\{3\}$. Similarly $\{1\}, \{2\}, \{3\}, \{4\}, \{5\}$ and $\{6\}$ are all simple events.

(2) Compound event. If E contains more than one element of the sample space S, then E is called *compound event.* Thus

$$E = \{e_i\}, \quad i = 1, 2, \ldots, n \text{ and } e_i \in S$$

is a compound event.

For example. Consider tossing of a die singly. If E denotes the event of getting an odd number then $E = \{1, 3, 5\}$. Since E contains three elements of the sample space.

$S = \{1, 2, 3, 4, 5, 6\}$ and hence E is a compound event.

Relation Between Simple and Compound Events :

Every compound event can be represented as the union of its component simple events.

For example. The sample space S in tossing a die singly is given by

$$S = \{1, 2, 3, 4, 5, 6\}.$$

Let E be the event of getting an odd number

i.e., $E = \{1, 3, 5\}$ is a compound event.

Clearly E is the union of simple events $\{1\}$, $\{3\}$ and $\{5\}$

i.e., $E = \{1, 3, 5\} = \{1\} \cup \{3\} \cup \{5\}$.

◈ § 1.4. MATHEMATICAL OR PRIOR OR CLASSICAL DEFINITION OF PROBABILITY

If an event E can happen in 'a' ways and fail in 'b' ways, and each of these ways is equally likely, the probability (or chance) of the happening of the event E is $\frac{a}{a+b}$ and that of its failing is $\frac{b}{a+b}$.

For example. If in a lottery there are 31 prizes and 40 blanks, the probability that a person holding 1 ticket will win a prize is $\frac{31}{31+40}$ *i.e.*, $\frac{31}{71}$ and his probability of not wining is $\frac{40}{31+40}$ *i.e.*, $\frac{40}{71}$.

The probability of happening an event E is denoted by $P(E)$ and the probability of not happening is denoted by $P(\overline{E})$. Thus

$$P(E) = \frac{a}{a+b} = \frac{\text{Number of ways favourable to event E}}{\text{Number of exhaustive ways}}$$

$$P(\overline{E}) = \frac{b}{a+b} = \frac{\text{Number of ways unfavourable to event E}}{\text{Number of exhaustive ways}}.$$

Note 1. If p denotes the probability of the happening of the event E and q of its not happening, then

$$p + q = \frac{a}{a+b} + \frac{b}{a+b} \Rightarrow p + q = 1.$$

$$\therefore \quad q = 1 - p.$$

Note 2. Instead of saying that the probability of the happening of an event E is $a/(a+b)$, it is sometimes said that *'the odds are a to b in favour of the event E'* or *'the odds are b to a against the event E'*.

Note 3. The odds in favour of and against the event E are defined as follows :

The odds in favour of the event E

$$= \frac{p}{q} = \frac{a/(a+b)}{b/(a+b)} = \frac{a}{b} = \frac{\text{No. of favourable ways}}{\text{No. of unfavourable ways}}.$$

Similarly the odds against the event E

$$= \frac{q}{p} = \frac{b}{a} = \frac{\text{No. of unfavourable ways}}{\text{No. of favourable ways}}.$$

◈ § 1.5. EMPERICAL OR STATISTICAL DEFINITION OF PROBABILITY

If trial be repeated for a large number of times N (say), under the same conditions and a certain event E happens in pN occasions, then the probability of the happening of the event E is defined as

$$\lim_{N \to \infty} \frac{pN}{N} = p.$$

Remark. The two definitions (mathematical and statistical) of probability given above are apparently different. The mathematical definition gives the **relative frequency** of favourable cases to the total number of cases while statistical definition gives the **limit of the relative frequency** of the happening of the event. Symbolically, the probability of the happening of the event E is given by

$$P(E) = \lim_{N \to \infty} \frac{m}{n}$$

where m is the number of time which the event E occurs in a series of n trials.

For example. If 1000 tosses of a coin results in 529 heads (H), then the probability of getting H is $\frac{529}{1000} = 0 \cdot 529$. If another 1000 tosses result in 481 heads (H), then the probability of H is $\frac{529 + 481}{2000} = 0 \cdot 505$. Again if another 1000 tosses result in 497 heads (H), then the probability of heads is $\frac{529 + 481 + 497}{3000} = 0 \cdot 5023$ etc.

Clearly this limit tends to $0 \cdot 5$ *i.e.*, 0·5210, 0·5050, 0·5023, ..., $\to$ 0·5.

Hence by statistical definition, the probability of getting head is 0·5.

◈ § 1.6. PROBABILITY OF AN EVENT

Axiomatic definition. Let S be the sample space consisting of n simple events $e_1, e_2, \ldots, e_n$ *i.e.*, $S = \{e_1, e_2, \ldots, e_n\}$.

The probability of each simple event $e_i \in S$, denoted by $P(e_i)$, is equal to a definite number and it has to satisfy the following two conditions :

(i) The number representating the probability of any event is always positive *i.e.*,

$$P(e_i) \geq 0, \quad i = 1, 2, \ldots, n.$$

(ii) The sum of the probabilities of each event is equal to 1,

i.e., $$P(e_1) + P(e_2) + \ldots + P(e_n) = 1.$$

Hence if p denotes the probability of an event then

$$0 \leq p \leq 1.$$

The above two conditions are the axioms of probability. Although the above axioms give us the restrictions upon the probability measure, they do not clearly state what would be probability of an event. The probability of an event is obtained by intuition, mathematical reasoning or by any other method.

◈ § 1.7. FORMULAE FOR PROBABILITY OF AN EVENT

Consider a random experiment and let S be the sample space (finite) consisting of n elements (or outcomes) for this experiment. Let each outcome be equally likely then the probability of obtaining any one outcome is denoted by $1/n$.

Now let E be an event of S containing m elements of S *i.e.,* $n(E) = m$. If $P(E)$ is the probability of the event E happening then,

$$P(E) = \frac{1}{n} + \frac{1}{n} + \ldots \text{ upto } m \text{ terms} = m/n$$

or

$$P(E) = \frac{n(E)}{n(S)}.$$

◈ § 1.8. COMPLEMENTARY EVENT

Let E be an event then 'not happening of the event E is called the complementary event of E and is denoted by $\overline{E}$. Hence $\overline{E}$ includes all those sample point (or elements) of the sample space not included in E.

The probability of the happening of the event $\overline{E}$ is denoted by $P(\overline{E})$.

Theorem 1. *If E is an event and $\overline{E}$ its complementary event then*

$$P(E) + P(\overline{E}) = 1.$$

Proof. Let S be the sample space for the event E and let S has n elements. Let number of elements in S favourable to the even E be m, then remaining $n - m$ elements in S are favourable to the complementary event $\overline{E}$. Thus

$$n(S) = n,\ n(E) = m,\ n(\overline{E}) = n - m.$$

Now probability of E, $\quad P(E) = \dfrac{n(E)}{n(S)} = \dfrac{m}{n}$

and probability of $\overline{E}$, $\quad P(\overline{E}) = \dfrac{n(\overline{E})}{n(S)} = \dfrac{n-m}{n}$

$$\therefore \quad P(E) + P(\overline{E}) = \frac{m}{n} + \frac{n-m}{n} = \frac{n}{n} = 1.$$

Theorem 2. *The probability of any event always lies between 0 and 1.*

Proof. If E is any event, then we are to prove that

$$0 \le P(E) \le 1.$$

Let $\quad S = \{e_1, e_2, \ldots, e_n\}$ be a sample space

and $\quad E = \{e_1, e_2, \ldots, e_m\},\ m \le n$ be its any event.

Since $E \subseteq S$ and so E cannot have more than n elements and less than 0 elements and hence

$$0 \le m \le n$$

$$\Rightarrow \quad 0/n \le m/n \le n/n \Rightarrow 0 \le m/n \le 1$$

$$\Rightarrow \quad 0 \le P(E) \le 1 \qquad \left[\because P(E) = \frac{n(E)}{n(S)} = \frac{m}{n}\right]$$

ILLUSTRATIVE EXAMPLES

Example 1. ***Find the probability of throwing on even number with a die.***

Solution. Let S be the sample space and the event of getting an even number be E, then

$$S = \{1, 2, 3, 4, 5, 6\} \text{ and } E = \{2, 4, 6\}.$$

$\therefore$ $n(S) = 6,\ n(E) = 3.$

$\therefore$ Probability of event E happening

$$P(E) = \frac{n(E)}{n(S)} = \frac{3}{6} = \frac{1}{2}.$$

Example 2. ***If two coins are tossed, find the chance that there should be heads on both.***

Solution. The possibility of getting head (H) or tail (T) on the two coins are equally likely. Also H on one coin may be associated to H or T on the other coin, and so with T. If S be the sample space for this experiment, then

$$S = \{(H, H)\ (H, T), (T, H), (T, T)\}$$

where (H, T) means head (H) on first coin and tail (T) on second coin.

If E be the event of getting H on both coins, then

$$E = \{(H, H)\} \qquad \therefore n(S) = 4, n(E) = 1.$$

$$\therefore \quad P(E) = \frac{n(E)}{n(S)} = \frac{1}{4}.$$ **Ans.**

Example 3. ***Find the probability of throwing a sum of 7 in a single throw with two dice.***

Solution. The possible number of cases $= 6 \times 6 = 36$. [See example 3, page 103]. Therefore, the sample space S for this experiment has 36 elements.

$\therefore$ $n(S) = 36$. Let E be the event of getting a sum of 7, then

$$E = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$$

where (1, 6) means : 1 on first die and 6 on the second die, etc.

Thus $n(E) = 6.$

Hence probability of E,

$$P(E) = \frac{n(E)}{n(S)} = \frac{6}{36} = \frac{1}{6}.$$ **Ans.**

Example 4. ***From a bag containing 5 white, 7 red and 4 black balls a man draws 3 at random, find the probability of being all white.***

Solution. Total number of balls in the bag $= 5 + 7 + 4 = 16$

$\therefore$ The total number of ways in which 3 'balls' can be drawn

$$= {}^{16}C_3 = \frac{16 \times 15 \times 14}{3 \times 2 \times 1} = 560.$$

Thus sample space S for this experiment has 560 outcomes *i.e.*, $n(S) = 560$.

Let E be the event of all the three balls being white. Total number of white balls is 5. So the number of ways in which 3 white balls can be drawn

$$= {}^{5}C_3 = \frac{5 \times 4 \times 3}{3 \times 2 \times 1} = 10.$$

Thus E has 10 elements of S.

$\therefore$ $$n(E) = 10.$$

$\therefore$ Probability of E, $P(E) = \dfrac{n(E)}{n(S)} = \dfrac{10}{560} = \dfrac{1}{56}$. **Ans.**

Example 5. ***From a pack of 52 cards two cards are drawn at random. Find the probability of the following events :***

(i) both cards are of spade

(ii) one card is of spade and one card is of diamond.

Solution. The total number of ways in which 2 cards can be drawn

$$= {}^{52}C_2 = \frac{52 \times 51}{1 \times 2} = 26 \times 51 = 1326.$$

$\therefore$ Number of elements in the space S are $n(S) = 1326$.

(i) Let the event, that both cards are of space, be denoted by E_1, then

$n(E_1)$ = Number of elements in E_1

= Number of ways in which 2 cards can be selected out of 13 cards of spade

$$= {}^{13}C_2 = \frac{13 \times 12}{1 \times 2} = 78.$$

$\therefore$ Probability of $E_1 = P(E_1)$

$$= \frac{n(E)}{n(S)} = \frac{78}{1326} = \frac{1}{17}.$$

(ii) Let E_2 be the event that one card is of spade and one is of diamond, then

$n(E_2)$ = number of elements in E_2

= number of ways in which one card of spade can be selected out of out of 13 spade cards and the one card of diamond can be selected out be selected out of 13 spade cards

$$= {}^{13}C_1 \times {}^{13}C_1 = 13 \times 13 = 169.$$

$\therefore$ $P(E_2) = \dfrac{n(E_2)}{n(S)} = \dfrac{169}{1326} = \dfrac{13}{102}$. **Ans.**

Example 6. ***From a pack of 52 cards 6 cards are drawn at random. Find the probability of the following events :***

(i) three are red and 3 are black cards,

(ii) three are kings and 3 are queens.

Solution. The total number of ways in which 6 cards can be drawn $= {}^{52}C_6$.

$\therefore$ number of elements in the sample space S are $n(S) = {}^{52}C_6$.

(i) We know that a pack of cards has 26 red and 26 black cards.

Let E_1 be the event that 3 are red and 3 are black cards, then

$n(E_1)$ = Number of ways in which 3 red and 3 black cards can be selected

$$= {}^{26}C_3 \times {}^{26}C_3.$$

$\therefore$ $P(E_1) = \dfrac{n(E_1)}{n(S)} = \dfrac{{}^{26}C_3 \times {}^{26}C_3}{{}^{52}C_6}$. **Ans.**

(ii) We know that a pack of cards has 4 kings and 4 queens. Let E_2 be the event that 3 are kings and 3 are queens, then

$$n(E_2) = {}^4C_3 \times {}^4C_3$$

$$\therefore \quad P(E_2) = \frac{n(E_2)}{n(S)} = \frac{{}^4C_3 \times {}^4C_3}{{}^{52}C_6}.$$ **Ans.**

Example 7. ***A has 3 shares in a lottery containing 3 prizes and 6 blanks ; B has one share in a lottery containing one prize and 2 blanks. Compare their chances of success.***

Solution. Total number of tickets in the first lottery = 3 + 6 = 9.

A may select any three tickets out of these 9 tickets. Therefore, the number of elements in the sample space S is given by

$$n(S) = {}^9C_3 = \frac{9 \times 8 \times 7}{3 \times 2 \times 1} = 84.$$

Let E_1 be the event of winning the prize in the first lottery by A. So $\overline{E}_1$ is the event of not winning the prize of A , then number of elements in $\overline{E}_1$ is given by $n(\overline{E}_1) =$ number of ways of selecting 3 tickets out of six blank tickets

$$= {}^6C_3 = \frac{6 \times 5 \times 4}{1 \times 2 \times 3} = 20.$$

$\therefore$ The probability of not winning the prize of A

$$P(\overline{E}_1) = \frac{n(\overline{E}_1)}{n(S)} = \frac{20}{84} = \frac{5}{21}.$$

Since $\quad P(E_1) + P(\overline{E}_1) = 1.$

$\therefore \quad P(E_1) = 1 - (\overline{E}_1)$

or $\quad P(E_1) = 1 - (5/21) = 16/21.$

For second lottery. $n(S) =$ Number of ways of selecting 1 ticket out of 3 tickets

or $\quad n(S) = {}^3C_1 = 3.$

Let E_2 be the event of winning the prize of B.

$\therefore \quad n(E_2) =$ number of ways of selecting 1 ticket out of 1 prize ticket

$$= {}^1C_1 = 1.$$

$\therefore$ The probability of winning the prize of B.

$$P(E_2) = \frac{n(E_2)}{n(S)} = \frac{1}{3}.$$

$\therefore$ The ratio of the probabilities of winning the prizes by A and B

$$= \frac{P(E_1)}{P(E_2)} = \frac{16/21}{1/3} = \frac{16}{7}.$$ **Ans.**

Example 8. (a) ***What is the chance that a leap year selected at random, will contain 53 Wednesdays ?***

Solution. A leap year consists of 366 days, in which there are 52 complete weeks and 2 days more. 52 weeks contain 52 Wednesdays and so we are to find the possibility of being one Wednesday out of two remaining days.

These two remaining days may make the following seven combinations :

(1) Tuesday and Wednesday
(2) Wednesday and Thursday
(3) Thursday and Friday
(4) Friday and Saturday
(5) Saturday and Sunday
(6) Sunday and Monday
(7) Monday and Tuesday.

Out of these 7 likely cases only first two are favourable. Hence the required probability $= 2/7$.

Example 8. (b) ***What is the probability that a leap year selected at random, will contain 53 sundays.***

Solution. Out or 7 likely cases [see example 8 (a)] only cases (5) and (6) are favourable. Hence the required probability $= 2/7$.

Example 9. ***A card is drawn from an ordinary pack and a gambler bets that it is a spade or an ace. What are the odds against his winning this bet ?***

Solution. Let S be the sample space for this experiment.

Then $n(S) =$ Number of total ways of drawing a card from a pack of 52 cards

$$= {}^{52}C_1 = 52.$$

Let E be the event of the gambler's winning the bet. The number of spade cards is 13 (including one spade ace) and there are 3 more aces. Thus

$\therefore \quad n(E) =$ Number of ways of drawing one spade or an ace from 16 cards

$$= {}^{16}C_1 = 16$$

$$\therefore \quad P(E) = \frac{n(E)}{n(S)} = \frac{16}{52} = \frac{4}{13} = \frac{4}{9+4}.$$

Hence the odds against his winning are 9 to 4.

Example 10. ***From a lottery tickets marked 1, 2, 3, ..., 30 four tickets are drawn, find the chance that the tickets marked 1 and 2 are among them.***

Solution. Total number of ways selecting 4 tickets from 30 tickets

$$= {}^{30}C_4 = \frac{30 \times 29 \times 28 \times 27}{4 \times 3 \times 2 \times 1} = n(S).$$

Since two tickets marked 1 and 2 are always to be included, therefore, number of ways of selecting 2 tickets from remaining 28 tickets

$$= {}^{28}C_2 = \frac{28 \times 27}{2 \times 1} = n(E)$$

$$\therefore \quad \text{The required probability } P(E) = \frac{n(E)}{n(S)}$$

$$= \frac{28 \times 27}{2 \times 1} \times \frac{4 \times 3 \times 2 \times 1}{30 \times 29 \times 28 \times 27} = \frac{2}{145}.$$ **Ans.**

Example 11. ***From 12 tickets marked 1 to 12, one ticket is drawn at random, find the chance that the number on it is a multiple of 3.***

Solution. Let S be the sample space, then

$$S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}.$$

$$\therefore \quad n(S) = 12$$

Let E be the event that a number is a multiple of 3.

$\therefore \quad E = \{3, 6, 9, 12\}, \qquad \therefore \; n(E) = 4.$

$\therefore \quad$ Required probability $P(E) = \dfrac{n(E)}{n(S)} = \dfrac{4}{12} = \dfrac{1}{3}.$

Example 12. ***Let E_1, E_2, E_3 be three possible results (events) in a random experiment. If probability of happening E_1 is 3 times the probability of happening E_2 while the probability of happening E_2 is 5 times the probability of happening E_3, then find the probabilities of happening of E_1, E_2 and E_3.***

Solution. Let $P(E_1)$, $P(E_2)$ and $P(E_3)$ be the probabilities of happening E_1, E_2 and E_3 respectively. Then according to the question

$$P(E_1) = 3\,P(E_2) \qquad \text{...(1)}$$

$$P(E_2) = 5\,P(E_3). \qquad \text{...(2)}$$

Since E_1, E_2 and E_3 are the only possible events in an experiment and so it is definite that one of them will happen. Consequently

$$P(E_1) + P(E_2) + P(E_3) = 1. \qquad \text{...(3)}$$

From (1) and (2),

$$P(E_1) = 15\,P(E_3) \qquad \text{...(4)}$$

$\therefore \quad$ Using (2) and (4) in (3), we get

$$15\,P(E_3) + 5\,P(E_3) + P(E_3) = 1$$

$$\Rightarrow \quad P(E_3) = \frac{1}{21}$$

$$\therefore \quad P(E_1) = \frac{15}{21} = \frac{5}{7} \text{ and } P(E_2) = \frac{5}{21}.$$

Example 13. ***What is the probability that the birthdays of seven great men will fall on seven different days of a week ?***

Solution. The total number of ways that the birthdays of 7 great men will fall on any day of a week $= 7 \times 7 \times 7 \times 7 \times 7 \times 7 \times 7 = 7^7$.

$$\therefore \quad n(S) = 7^7.$$

Let E be the event that the birthdays of 7 great men will fall on 7 different days of week, then

$$n(E) = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 7!$$

$\therefore \quad$ Required probability $P(E) = \dfrac{n(E)}{n(S)} = \dfrac{7!}{7^7}.$

Example 14. ***From a set of seven digits 1, 2, 3, 4, 5, 6, 7 four are drawn at random. Find the chance that the sum of these four digits is less than 12.***

Solution. Let S be the sample space, then

$$n(S) = \text{Total number of ways selecting four digits from 7 digits}$$
$$= {}^7C_4 = \frac{7 \times 6 \times 5 \times 4}{1 \times 2 \times 3 \times 4} = 35.$$

Let E be the even that the sum of four digits is less than 12. The event E can happen in the following two ways :

(1) $1 + 2 + 3 + 4 = 10 < 12$

(2) $1 + 2 + 3 + 5 = 11 < 12$

$\therefore \quad n(E) = 2.$

$\therefore$ The required probability $P(E) = \dfrac{n(E)}{n(S)} = \dfrac{2}{35}$.

Example 15. ***The chance of one event happening is the square of the chance of a second event happening, but the odds against the first are the cube of the odds against the second. Find the chance of happening of each.***

Solution. Let p = probability of happening the second event. Then the probability of happening the first event = p^2.

$\therefore$ The odds against the second event = $\dfrac{1-p}{p}$

and the odds against the first event = $\dfrac{1-p^2}{p^2}$.

Now according to the given problem, we have

$$\frac{1-p^2}{p^2} = \left(\frac{1-p}{p}\right)^3$$

$$\Rightarrow \quad p^3(1-p^2) = p^2(1-p)^3$$
$$\Rightarrow \quad p^2(1-p)[p(1+p) - (1-p)^2] = 0$$
$$\Rightarrow \quad p^2(1-p)[p + p^2 - 1 + 2p - p^2] = 0$$
$$\Rightarrow \quad p^2(1-p)(3p-1) = 0$$
$$\Rightarrow \quad p = 0, \frac{1}{3}, 1.$$

But the values $p = 0$ and $p = 1$ are not possible. Hence $p = \dfrac{1}{3}$.

$\therefore$ The chance (probability) of happening the first event = $p^2 = \dfrac{1}{9}$

and that of second event = $p = \dfrac{1}{3}$.

EXERCISE 1 (A)

1. If one coin is tossed, find the chance that there is head.
2. In a single throw with one die, find the chance that there should be an odd number.
3. In a single throw with one die, find the chance of throwing 5.
4. In a single throw with one die, find the chance of throwing more than 3.

5. Find the chance of throwing a sum of 8 in one throw with 2 dice.
6. In a class of 12 students 5 are boys and rest are girls. Find the chance that a student selected will be a girl.
7. What is the chance that a leap year selected at random, will contain 53 Sundays ?
8. From the letters of the word 'HORSE', two letters are selected at random, find the probabilities of the following events :
   (i) both are vowels
   (ii) at least one is vowel
   (iii) at least one is R.
9. A bag contains 7 white and 9 black balls. Find the chance of drawing a white ball.
10. What is the chance that a non-leap year should have 53 Sundays ?
11. A bag contains 5 red, 3 black and 7 white balls. Two balls are drawn at random. Find the chance that one is red and one is white.
12. One of the two events always happen. If the chance of one event happening is two-third of the chance of the second event, then find the odds in favour of the second event.
13. The chance of one event happening is the square of the chance of a second event happening, but the odds against the first are the cube of the odds against the second. Find the chance of each.
14. A bag contains 7 white, 6 red and 5 black balls. Two balls are drawn at random. Find the chance that both are white balls.
15. (a) From a pack of cards, 3 cards are drawn at random. Find the chance that they are a king, a queen and a knave.
    (b) From a pack of chords, one card is drawn at random. What is the probability that it is not a king.
16. From a set of 17 cards, numbered 1, 2, 3, ..., 17, one is drawn at random. Show that the chance that its number is divisible by 3 or 5 or 7 is 9/17.
17. An integer is chosen at random from the first two hundred digits 1, 2, 3, ..., 199, 200. Find the probability that the integer chosen is divisible by 6 or 8.
18. From a pack of cards, 5 cards are drawn at random. Find the chance of the following events :
    (i) only one ace.
    (ii) at least one ace.
19. State the statistical definition of probability.

ANSWERS

1. $\frac{1}{2}$ 2. $\frac{1}{2}$ 3. $\frac{1}{6}$ 4. $\frac{1}{2}$

5. $\frac{5}{32}$ 6. $\frac{7}{12}$ 7. $\frac{2}{7}$

8. (a) $\frac{1}{10}$ (b) $\frac{7}{10}$ (c) $\frac{2}{5}$ 9. $\frac{7}{16}$

10. $\frac{1}{7}$ 11. $\frac{1}{3}$ 12. 3 : 2 13. $\frac{1}{9}, \frac{1}{3}$

14. $\frac{7}{51}$ 15. (a) $\frac{16}{5525}$ (b) $\frac{12}{13}$ 17. $\frac{1}{4}$

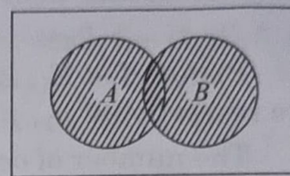
18. (i) $\frac{{}^4C_1 \times {}^{48}C_4}{{}^{52}C_5}$ (ii) $\frac{{}^{48}C_5}{{}^{52}C_5}$.

◈ § 1.9. COMPOSITION OF EVENTS

Following are three fundamental rules to composite two or more events by the help of set notations.

Let S be a sample space and A and B be its any two events :

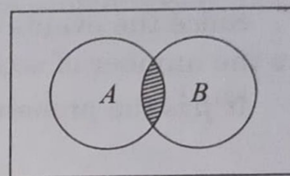
(i) The event represented by $\mathbf{A \cup B}$ or $\mathbf{A + B}$. If the event E happens when A happens or B happens then E is denoted by $A \cup B$ *i.e.*, the event E represented by $A \cup B$ includes all those elements (or outcomes or results) which A or B contain.



$A \cup B =$

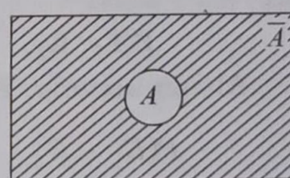
Thus $E = A \cup B$, the union of A and B is represented by shaded area in the figure.

(ii) The event represented by $\mathbf{A \cap B}$ or AB. If the event E happens when the events A and B both happen then the event E is represented by $A \cap B$ *i.e.*, the event E denoted by $A \cap B$ includes elements (or outcomes) common to both A and B. The shaded area in the figure represents the events $E = A \cap B$.



$A \cap B =$

(iii) Complement of event A or the event $\overline{\mathbf{A}}$ or $\mathbf{A^\circ}$. If the event E happens when the event A does not happen then E is denoted by $\overline{A}$ or A°.



$\overline{A} =$

◈ § 1.10. THEOREM OF TOTAL PROBABILITY OR ADDITIVE LAW OF PROBABILITY

Theorem. *If E_1 and E_2 are any two events then*

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2).$$

Proof. Let S be the sample space and n be the number of elements in S. Let l be the number of elements in E_1 and m the number of elements in the event E_2, *i.e.*,

$$n(S) = n, \quad n(E_1) = l, \quad n(E_2) = m.$$

If the events E_1 and E_2 are not mutually exclusive then $E_1 \cap E_2 \neq \phi$. Let $n(E_1 \cap E_2) = r$.

Clearly $\quad n(E_1 \cup E_2) = l + m - r.$

Now the probability of E_1 or E_2 happening denoted by $P(E_1 \cup E_2)$ is given by

$$P(E_1 \cup E_2) = \frac{n(E_1 \cup E_2)}{n(S)} = \frac{l + m - r}{n} = \frac{l}{n} + \frac{m}{n} - \frac{r}{n}$$

$$= \frac{n(E_1)}{n(S)} + \frac{n(E_2)}{n(S)} - \frac{n(E_1 \cap E_2)}{n(S)}$$

or $$\mathbf{P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)}. \quad \text{...(1)}$$

Cor. If E_1 and E_2 **be mutually exclusive events then**

$$E_1 \cap E_2 = \phi \text{ and } n(E_1 \cap E_2) = 0.$$

Now from (1), we have

$$\mathbf{P(E_1 \cup E_2) = P(E_1) + P(E_2)}.$$

§ 1.11. THEOREM OF TOTAL PROBABILITY FOR n MUTUALLY EXCLUSIVE EVENTS

Theorem. *If $p_1, p_2, \ldots, p_n$ be respectively the probabilities of n mutually exclusive events then the probability, that any of these events will happen is* $p_1 + p_2 + \ldots + p_n$.

Proof. Let $E_1, E_2, \ldots, E_n$ be n mutually exclusive events whose probabilities are respectively $p_1, p_2, \ldots, p_n$. Let N be the total number of trials. Then

The number of occasions the event E_1 happens $= p_1 N$ i.e., out of N occasions $p_1 N$ are favourable to the event E_1 happening.

The number of occasions the event E_2 happens $= p_2 N$, and so on.

Since the events are mutually exclusive, therefore $p_1 N + p_2 N + \ldots + p_n N$ are the number of occasions in which any one of the n events can happen.

If p is the probability that any one of the events happen, then

$$p = \frac{p_1 N + p_2 N + \ldots + p_n N}{N}$$

i.e.,

$$\mathbf{p = p_1 + p_2 + \ldots + p_n}$$

$$\therefore \quad \mathbf{P(E_1, E_2, \ldots, E_n) = P(E_1) + P(E_2) + \ldots + P(E_n).}$$

Alternatively. By statistical definition,

$$P(E_1 + E_2 + \ldots + E_n) = \lim_{n \to \infty} \sum_{i=1}^{n} \frac{p_i N}{N}$$

$$= \sum_{i=1}^{n} \lim_{n \to \infty} \frac{p_i N}{N} = \sum_{i=1}^{n} p_i$$

$$= p_1 + p_2 + \ldots + p_n$$

$$= P(E_1) + P(E_2) + \ldots + P(E_n)$$

$$= \sum_{i=1}^{n} P(E_i).$$

ILLUSTRATIVE EXAMPLES

Example 1. ***If $\frac{1}{4}$ is the probability of winning a race by the horse A and $\frac{1}{3}$ be the probability of winning the same race by the horse B. Find the probability that one of these horses will win.***

Solution. Let E_1 and E_2 be the events that the horses A and B wins the race respectively. Then

$$P(E_1) = \frac{1}{4}, \; P(E_2) = \frac{1}{3}.$$

We know that if the horse A wins the race then the horse B cannot win the race and if B wins the race then A can not win. Hence the events E_1 and E_2 are mutually exclusive events. Therefore, the probability that any one of A or B wins the race is given by

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) = \frac{1}{4} + \frac{1}{3} = \frac{7}{12}. \qquad \textbf{Ans.}$$

Example 2. ***Two dice are thrown. Find the probability that the sum of faces is (i) 7 or 8 and (ii) more than 8.***

Solution. (i) The number of elements in the sample S is

$$n(S) = 6^2 = 36.$$

Let E_1 be the event that the sum of faces is 7 and E_2 the event when the sum of faces is 8.

$$\therefore \quad E_1 = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}.$$

$$\therefore \quad n(E_1) = 6. \text{ Hence } P(E_1) = (6/36).$$

$$E_2 = \{(2, 6), (3, 5), (4, 4), (5, 3), (6, 2)\}.$$

$$\therefore \quad n(E_2) = 5. \text{ Hence } P(E_2) = (5/36).$$

Since the events E_1 and E_2 are mutually exclusive hence the probability that the sum of faces is 7 or 8 is given by

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) = \frac{6}{36} + \frac{5}{36} = \frac{11}{36}.$$ **Ans.**

(ii) Proceeding as part (i) above, we have

P (sum is more than 8)

$$= P\,(\text{sum} = 9) + P\,(\text{sum} = 10) + P\,(\text{sum} = 11) + P\,(\text{sum} = 12)$$

$$= \frac{4}{36} + \frac{3}{36} + \frac{2}{36} + \frac{1}{36} = \frac{10}{36} = \frac{5}{18}.$$

Example 3. ***Discuss and criticise the following :***

$$\boldsymbol{P(A) = \frac{1}{6}, P(B) = \frac{1}{4}, P(C) = \frac{2}{3}}$$

for the probabilities of three mutually exclusive events A, B, C.

Solution. Since A, B, C are mutually exclusive events, therefore :

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) = \frac{1}{6} + \frac{1}{4} + \frac{2}{3} = \frac{13}{12}.$$

But $P(A \cup B \cup C)$ can not be greater than 1. Hence the given statement is not correct. Thus either the probabilities are wrong or the events are not mutually exclusive.

Example 4. ***Two cards are drawn at random from a pack of cards. Find the probability that both these cards are of black colour or both are aces.***

Solution. Total number of ways in which 2 cards can be drawn

$$= {}^{52}C_2 = \frac{52 \times 51}{1 \times 2} = 1326.$$

$\therefore$ The number of elements in the sample space S are $n(S) = 1326$. Let E_1 be the event that the 2 cards are black and E_2 the event that both are aces. Since a pack of cards has 26 black cards.

$$\therefore \quad n(E_1) = {}^{26}C_2 = \frac{26 \times 25}{1 \times 2} = 325.$$

Again the pack of cards has 4 aces.

$$\therefore \quad n(E_2) = {}^{4}C_2 = \frac{4 \times 3}{1 \times 2} = 6.$$

But the events E_1 and E_2 are not mutually exclusive as they have 2 black aces in common.

i.e., they have one element in common.

$$\therefore \quad n(E_1 \cap E_2) = 1.$$

Hence the required probability *i.e.,* the probability that both cards are black or aces is given by

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$$

$$= \frac{325}{1326} + \frac{6}{1326} - \frac{1}{1326} = \frac{330}{1326} = \frac{55}{221}.$$

Ans.

EXERCISE 1 (B)

1. A bag contains 6 red, 4 white and 5 black balls. One ball is drawn at random. Find the probability that the ball is red or black.
2. Three horses A, B and C entered for a race. The chance that A wins is two times of B and the chance that B wins is two times of C. What are the probabilities of A, B and C winning the race ? Find also the probability that B or C wins the race.
3. If the probability of a horse A winning the race is $\frac{1}{6}$ and the probability of a horse B winning the same race is $\frac{1}{4}$, what is the probability that one of the horses will win ?
4. In a given race the odds in favour of three horses are $1 : 2$, $1 : 3$, $1 : 4$, find the chance that one of them wins the race.
5. Four teams take part in a game whose chances of winning are respectively $\frac{1}{7}, \frac{1}{8}, \frac{1}{9}$ and $\frac{1}{10}$. Find the chance that one of them wins the game. Also find the chance that none of them wins.
6. Two dice are thrown. Find the chance that the sum of the faces is 7 or 11.
7. Two disc are thrown. Find the chance that the sum of the faces is neither 7 nor 11.
8. State and prove the theorem of additive law of probability.
9. Let A and B be two events in the probability space, then prove that
$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$
10. Discuss and criticise the following :
$$P(A) = \frac{2}{3}\ P(B) = \frac{1}{4}\ P(C) = \frac{1}{6}$$
for the probabilities of three mutually exclusive events A, B, C.

[**Hint.** Here
$$P(A) = \frac{1}{6}$$
$$\frac{2}{3}P(B) = \frac{1}{6} \Rightarrow P(B) = \frac{1}{4}$$
$$\frac{1}{4}P(C) = \frac{1}{6} \Rightarrow P(C) = \frac{2}{3}.$$
Now proceed as Example 3]

ANSWERS

1. $\frac{11}{15}$
2. $\frac{4}{7}, \frac{2}{7}, \frac{1}{7}, \frac{3}{7}$
3. $\frac{3}{20}$
4. $\frac{47}{60}$
5. $\frac{1207}{2520}$; $\frac{1313}{2520}$
6. $\frac{2}{9}$
7. $\frac{7}{9}$.

Alternative Definition. If E_1 and E_2 be any two events of a sample space S then E_1 and E_2 are called independent events if

Probability of E_1 = Conditional probability of E_1 when E_2 has occurred

i.e., $$P(E_1) = P(E_1 / E_2).$$

Theorem. *If A and B are two independent events, then*

$$P(A \cap B) = P(A \text{ and } B) = P(AB) = P(A) \,.\, P(B).$$

Proof. Let $P(A)$ and $P(B)$ be the probabilities of two independent events A and B respectively. Let N be the total number of trials.

Now out of total number of trials N, the event A happens in $P(A)\ N$ occasions. The event A is now to be followed by event B. The second event B happens (along with the first event A) on $P(B)\ \{P(A)\,.\,N\}$ occasions. Thus out of a total number of trials N, the number of occasions in which both events A and B will occur

$$= P(B) \,.\, P(A) \,.\, N.$$

Hence the probability that both events A and B will occur is given by

$$P(A \cap B) = P(A \text{ and } B) = P(AB)$$

$$= \frac{P(A)\ P(B) \,.\, N}{N} = P(A) \,.\, P(B)$$

or $$\mathbf{P(AB) = P(A) \,.\, P(B).}$$

If there are n independent events whose probabilities are $p_1, p_2, \ldots, p_n$ then the probability p that all of them will occur is given by

$$\mathbf{p = p_1\ p_2 \cdots p_n \,.}$$

Remark 1. Let p be the probability that an event will occur in one trial, then the probability that it will occur in succession of r trials is = **p. p... r times** = $\mathbf{p^r}$.

Remark 2. If $p_1, p_2, \ldots, p_n$ be the probabilities of n independent events, then the probability of all of these failing is

$$= \mathbf{(1 - p_1)\,(1 - p_2) \ldots (1 - p_n).}$$

Hence the probability that at least one of these must happen is

$$= \mathbf{1 - (1 - p_1)\,(1 - p_2) \ldots (1 - p_n).}$$

ILLUSTRATIVE EXAMPLES

Example 1. ***If A and B are two events, where $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{3}$, and $P(A \cap B) = \frac{1}{4}$, then evaluate the following :***

(a) $P(A / B)$, ***(b)*** $P(B / A)$, ***(c)*** $P(A \cup B)$.

Solution. (a) We have, from theorem of compound probability,

$$P(A \cap B) = P(B) \,.\, P(A / B).$$

$$\therefore \quad P(A / B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{4}}{\frac{1}{3}} = \frac{3}{4}.$$

(b) $$P(B/A)=\frac{P(B\cap A)}{P(A)}=\frac{P(A\cap B)}{P(A)}=\frac{\frac{1}{4}}{\frac{1}{2}}=\frac{2}{4}=\frac{1}{2}.$$

(c) $$P(A\cup B)=P(A)+P(B)-P(A\cap B)=\frac{1}{2}+\frac{1}{3}-\frac{1}{4}=\frac{7}{12}.$$ **Ans.**

Example 2. ***Explain with reason that***

$$P(A)=\frac{2}{3},\ P(B)=\frac{1}{4},\ P(C)=\frac{1}{6}$$

are respectively the probabilities of three mutually exclusive event, A, B, C.

Solution. Here $P(A)+P(B)+P(C)=\frac{2}{3}+\frac{1}{4}+\frac{1}{6}=\frac{13}{12}>1.$

Since the events A, B, C are mutually exclusive, therefore, the sum of their probabilities cannot be greater than 1.

Hence the given statement is false.

Example 3. ***What is the chance of throwing a total of 11 with two dice, if the digit on first die is 5 ?***

Solution. Let E_1 be the event when the total of 11 occurs on the two dice and E_2 the event when 5 comes on the first die

$$E_2=\{(5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6)\}.$$

Since the E_1 happens when E_2 has already happened, therefore, E_2 is the required sample space for E_1. There is only one pair (5, 6) in E_2 whose sum of digits is 11 :

$$E_1=\{(5, 6)\}.$$

Now $E_1\cap E_2=\{(5, 6)\},\ n(E_1\cap E_2)=1$ and $n(E_2)=6.$

Hence $$P(E_1/E_2)=\frac{n(E_1\cap E_2)}{n(E_2)}=\frac{1}{6}.$$ **Ans.**

Example 4. ***In a college 25% students in Mathematics, 15% students in Physics and 10% students in Mathematics and Physics both are failed, A student is selected at random :***

(i) if he is failed in Physics, then find the chance of his failure in Mathematics,

(ii) if he is failed in Mathematics, then find the chance of his failure in Physics,

(iii) find the chance of his failure in Mathematics or Physics.

Solution. Let E_1 and E_2 be the events of failure in Mathematics and Physics respectively. Let the total number of students appearing in the examination be 100. So $n(S)=100$.

Since 25% students are failed in Mathematics, so

$$n(E_1)=25.$$

$$\therefore\quad P(E_1)=\frac{n(E_1)}{n(S)}=\frac{25}{100}=\frac{1}{4}.$$

Since 15% students are failed in Physics, so $n(E_2)=15.$

$$\therefore\quad P(E_2)=\frac{n(E_2)}{n(S)}=\frac{15}{100}=\frac{3}{20}.$$

Again 10% students are failed in Physics and Mathematics both, so $n(E_1 \cap E_2) = 10.$

$$\therefore \quad P(E_1 \cap E_2) = \frac{n(E_1 \cap E_2)}{n(S)} = \frac{10}{100} = \frac{1}{10}.$$

(i) The chance of failure in Mathematics while he is failed in Physics is given by

$$P(E_1 / E_2) = \frac{P(E_1 \cap E_2)}{P(E_2)} = \frac{1/10}{3/20} = \frac{2}{3}.$$ **Ans.**

(ii) The chance of failure in Physics while he is failed in Mathematics is given by

$$P(E_2 / E_1) = \frac{P(E_1 \cap E_2)}{P(E_1)} = \frac{1/10}{1/4} = \frac{2}{5}.$$ **Ans.**

(iii) The chance of failure in Mathematics or Physics is :

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \equiv E_2)$$

[since both events are not mutually exclusive]

$$= \frac{1}{4} + \frac{3}{20} - \frac{1}{10} = \frac{3}{10}.$$ **Ans.**

Example 5. ***The probability that a man A will be alive for next 30 years more is $\frac{1}{4}$ and the probability that the man B will be alive for next 30 years more is $\frac{1}{3}$. Find the probabilities of the following events :***

(i) both will be alive in the next 30 years,

(ii) at least one will be alive in the next 30 years.

Solution. Let E_1 and E_2 be the events of the men A and B respectively to be alive for next 30 years, then

$$P(E_1) = \frac{1}{4}, P(E_2) = \frac{1}{3}.$$

(i) Since the events E_1 and E_2 are mutually independent, the probability that both will be alive for next 30 years is $P(E_1 \cap E_2)$ and is given by

$$P(E_1 \cap E_2) = P(E_1)\, P(E_2) = \frac{1}{4} \cdot \frac{1}{3} = \frac{1}{12}.$$ **Ans.**

(ii) The probability that at least one will be alive for next 30 years is $P(E_1 \cup E_2)$ and is given by

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2) = \frac{1}{4} + \frac{1}{3} - \frac{1}{12} = \frac{1}{2}.$$ **Ans.**

Example 6. ***One bag contains 4 white, 6 red and 15 black balls and a second bag contains 11 white, 5 red and 9 black balls. One ball from each bag is drawn. Find the probabilities of the following events :***

(i) both balls are white,

(ii) both balls are red,

(iii) both balls are black,

(iv) both balls are of the same colour.

Solution. Total number of balls in the first bag

$$= 4 \text{ white} + 6 \text{ red} + 15 \text{ black} = 25 \text{ balls.}$$

Total number of balls in the second bag

$$= 11 \text{ white} + 5 \text{ red} + 9 \text{ black} = 25 \text{ balls.}$$

(i) Let E_1 and E_2 be events of drawing one white ball from first bag and one white ball from second bag respectively.

$$\therefore \qquad P(E_1) = 4/25, \quad P(E_2) = 11/25.$$

Since the events E_1 and E_2 are mutually independent, therefore the probability that both balls are white is $P(E_1 \cap E_2)$ and is given by

$$P(E_1 \cap E_2) = P(E_1)\, P(E_2) = \frac{4}{25} \cdot \frac{11}{25} = \frac{44}{625}. \qquad \textbf{Ans.}$$

(ii) Proceed as in part (i) above

$$\left\{\frac{6}{25} \cdot \frac{5}{25} = \frac{6}{125}\right\}. \qquad \textbf{Ans.}$$

(iii) Proceed as in part (i) above

$$\left\{\frac{15}{25} \cdot \frac{9}{25} = \frac{27}{125}\right\}. \qquad \textbf{Ans.}$$

(iv) The events that both balls are white or red or black are mutually exclusive. Therefore the probability that both balls are of the same colour

= the probability of both being white + the probability of being red + the probability of both being black

$$= \frac{44}{625} + \frac{6}{125} + \frac{27}{125} = \frac{209}{625}. \qquad \textbf{Ans.}$$

Example 7. ***The odds against a certain event are 5 to 2 and the odds in favour of another event, independent of the former, are 6 to 5, find the odds that one at least of the events will happen.***

Solution. Let the first and second events be respectively represented by E_1 and E_2.

Since the odds against the event E_1 are $= 5 : 2$.

$\therefore$ The probability that E_1 will not happen

$$= P(\overline{E}_1) = \frac{5}{5+2} = \frac{5}{7}.$$

The odds in favour of the event E_2 are $= 6 : 5$.

$$\therefore \qquad = P(\overline{E}_2) = \frac{5}{6+5} = \frac{5}{11}.$$

Thus the probability that events E_1 and E_2 both will not happen

$$= P(\overline{E}_1) \,.\, P(\overline{E}_2) = \frac{5}{7} \cdot \frac{5}{11} = \frac{25}{77}.$$

Here the probability that one at least of the events E_1 and E_2 will happen

$$= 1 - \frac{25}{77} = \frac{52}{77}. \qquad \textbf{Ans.}$$

Example 8. ***A problem in statistics is given to three students A, B and C whose chances of solving are $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ respectively. If they all try to solve the problem, what is the probability that the problem will be solved ?***

Solution. Chance of A's not solving the problem $= 1 - \frac{1}{2} = \frac{1}{2}$.

Chance of Bs not solving the problem $= 1 - \frac{1}{3} = \frac{2}{3}$.

Chance of Cs not solving the problem $= 1 - \frac{1}{4} = \frac{3}{4}$.

Since the above three events are mutually independent, therefore, A, B, C all may not solve the problem, the probability for which is $= \frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} = \frac{1}{4}$.

Hence the probability that the problem will be solved $= 1 - \frac{1}{4} = \frac{3}{4}$. **Ans.**

Example 9. ***A, B and C try to hit a target. A can hit the target 3 times in 5 shots, B 2 times in 5 shots, C 3 times in 4 shots. Find the probabilities of the following event :***

(*a*) ***two of them hit,*** (*b*) ***two at least hit.***

Solution. Chance of A's hitting $= 3/5$

Chance of B's hitting $= 2/5$

Chance of C's hitting $= 3/4$.

(a) Any two A, B and C can hit in the following three cases :

(i) First case. A and B hit and C may lose, for which the probability is

$$= \left(\frac{3}{5}\right) \cdot \left(\frac{2}{5}\right) \cdot \left(1 - \frac{3}{4}\right) = \frac{3}{5} \cdot \frac{2}{5} \cdot \frac{1}{4} = \frac{6}{100}.$$

(ii) Second case. A, C may hit and B may lose, for which the probability is

$$= \frac{3}{5} \cdot \frac{3}{4} \cdot \left(1 - \frac{2}{5}\right) = \frac{3}{5} \cdot \frac{3}{4} \cdot \frac{3}{5} = \frac{27}{100}.$$

(iii) Third case. B, C may hit and A may lose, for which the probability is

$$= \frac{2}{5} \cdot \frac{3}{4} \cdot \left(1 - \frac{3}{5}\right) = \frac{2}{5} \cdot \frac{3}{4} \cdot \frac{2}{5} = \frac{12}{100}.$$

Since above three events are mutually exclusive, hence the probability that any one of the above three events may happen (*i. e.*, any two will hit)

$$= \frac{6}{100} + \frac{27}{100} + \frac{12}{100} = \frac{45}{100} = \frac{9}{20}.$$

Ans.

(b) Two at least may hit in the following four cases :

(i) A, B may hit and C may lose.

(ii) A, C may hit and B may lose.

(iii) B, C may hit and A may lose.

(iv) A, B and C may hit.

For (i), (ii) and (iii) see part (a) above.

(iv) A, B and C may hit, for which the probability is

$$= \frac{3}{5} \cdot \frac{2}{5} \cdot \frac{3}{4} = \frac{18}{100}.$$

Since these events are mutually exclusive, the required probability is

$$= \frac{6}{100} + \frac{27}{100} + \frac{12}{100} + \frac{18}{100} = \frac{63}{100}. \qquad \textbf{Ans.}$$

Example 10. ***In a given race the odds in favour of four horses A, B, C, D are 1 : 3, 1 : 4, 1 : 5, 1 : 6 respectively. Assuming that a dead heat is impossible, find the chance that one of them wins the race.***

Solution. Let $P(A), P(B), P(C), P(D)$ be the probabilities of winning of the horses A, B, C, D respectively, then

$$P(A) = \frac{1}{4}, P(B) = \frac{1}{5}, P(C) = \frac{1}{6}, P(D) = \frac{1}{7}.$$

Since the above events are mutually exclusive, the chance that one of them wins

$$= P(A \cup B \cup C \cup D)$$
$$= P(A) + P(B) + P(C) + P(D)$$
$$= \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} = \frac{319}{420}. \qquad \textbf{Ans.}$$

Example 11. ***A person writes 4 letters and addresses 4 envelopes. If the letters are placed in the envelopes at random, what is the probability that all letters are not placed in the right envelopes ?***

Solution. The total number of ways in which 4 letters can be placed in 4 envelopes $= 4!$

We know that all the four letters can be placed rightly in 4 envelopes in only one way, the probability for which is $= 1/4!$

Hence the probability that all letters are not placed in right envelopes

$$= 1 - (1/4!) = 23/24. \qquad \textbf{Ans.}$$

Example 12. ***From a pack of cards, four are drawn at random. What is the chance that there is one card of each suit ?***

Solution. A pack of cards has 52 cards. Let S be the sample space, then

$$n(S) = {}^{52}C_4 = \frac{52 \times 51 \times 50 \times 49}{4 \times 3 \times 2 \times 1}$$
$$= 13 \times 17 \times 25 \times 49.$$

Let E be the event of drawing one card from each suit.

There are four different suits and each suit has 13 cards.

$n(E) =$ number of total ways of drawing one card from each suit

$$= {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1$$
$$= 13 \times 13 \times 13 \times 13$$

$\therefore$ Required probability $= P(E) = n(E)/n(S)$

$$= \frac{13 \times 13 \times 13 \times 13}{13 \times 17 \times 25 \times 49} = \frac{2197}{20825}. \qquad \textbf{Ans.}$$

Example 13. ***From a pack of 52 cards three are drawn at random. Find the chance that they are a king, a queen and a knave.***

Solution. Let S be the sample space. Then

$$n(S) = \text{Total number of drawing 3 cards from a pack of 52 cards}$$
$$= {}^{52}C_3 = \frac{52 \times 51 \times 50}{3 \times 2 \times 1} = 26 \times 17 \times 50.$$

There are 4 kings, 4 queens and 4 knaves. Let E be event of drawing one king, one queen and one knave.

$$\therefore \quad n(E) = \text{number of ways of drawing (one king from 4 kings} \times \text{one queen from 4 queens} \times \text{one knave from 4 knaves)}$$
$$= {}^4C_1 \times {}^4C_1 \times {}^4C_1 = 4 \times 4 \times 4.$$

Hence required probability

$$= P(E) = n(E)/\,n(S) = \frac{4 \times 4 \times 4}{26 \times 17 \times 50} = \frac{16}{5525}. \qquad \textbf{Ans.}$$

Example 14. ***A number is chosen from each of two sets :***

{1, 2, 3, 4, 5, 6, 7, 8, 9}; {1, 2, 3, 4, 5, 6, 7, 8, 9}.

If p_1 denotes the probability that the sum of the two numbers be 10 and p_2 the probability that their sum is 8; find $p_1 + p_2$.

Solution. Here each set contains 9 numbers. Hence the total number of ways of choosing one from each set are

$$n(S) = {}^9C_1 \times {}^9C_1 = 9 \times 9 = 81.$$

Let E_1 and E_2 be the events of getting a sum of 10 and 8 respectively.

Now, a sum of 10 can be obtained in the following ways :

(1, 9); (2, 8); (3, 7); (4, 6); (5, 5); (6, 4); (7, 3); (8, 2); (9, 1).

$$\therefore \quad n(E_1) = 9.$$

Hence $$p_1 = p(E_1) = \frac{n(E_1)}{n(S)} = \frac{9}{81} = \frac{1}{9}.$$

Again a sum of 8 can be obtained in the following ways :

(1, 7); (2, 6); (3, 5); (4, 4); (5, 3); (6, 2); (7, 1).

$$\therefore \quad n(E_2) = 7.$$

Hence $$p_2 = P(E_2) = \frac{n(E_2)}{n(S)} = \frac{7}{81}.$$

Thus $$p_1 + p_2 = \frac{1}{9} + \frac{7}{81} = \frac{16}{81}. \qquad \textbf{Ans.}$$

Example 15. ***A coin is tossed three times. Find the chance of getting head (H) and tail (T) alternately.***

Solution. We know that the chance of getting a, $H = \dfrac{1}{2}$

and the chance of getting a, $T = \dfrac{1}{2}$.

The event of getting H and T alternately may be any one of the following two events :

(i) Head, Tail, Head (*i.e.*, H, T, H), the probability of which is

$$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8} = P(E_1), \text{(say)}.$$

(ii) Tail, Head Tail (*i.e.*, T, H, T), the probability of which is

$$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8} = P(E_2), \text{(say)}$$

Since above both events are mutually exclusive, hence the probability of happening any one of E_1 and E_2 is

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}. \qquad \textbf{Ans.}$$

Example 16. ***Four persons are chosen at random from a group containing 3 men, 2 women and 4 children. Show that the chance that exactly two of them will be children is 10/21.***

Solution. Total number of persons $= 3 + 2 + 4 = 9$.

Let S be the sample space. Then

$$n(S) = \text{number of choosing 4 persons out of 9}$$

$$= {}^9C_4 = \frac{9 \times 8 \times 7 \times 6}{4 \times 3 \times 2 \times 1} = 9 \times 2 \times 7.$$

Let E be the event that 2 of them are children, then

$n(E) =$ Number of choosing 2 persons out of 5 (men and women) and 2 children out of 4

$$= {}^5C_2 \times {}^4C_2 = \frac{5 \times 4}{2 \times 1} \times \frac{4 \times 3}{2 \times 1} = 60.$$

$\therefore$ The required probability $P(E) = \dfrac{n(E)}{n(S)}$

$$= \frac{60}{9 \times 2 \times 7} = \frac{10}{21}. \qquad \textbf{Ans.}$$

Example 17. ***Three groups of children contain respectively 3 girls and 1 boy; 2 girls and 2 boys, 1 girl and 3 boys. One child is selected at random from each group. Show that the chance that the three selected consist of 1 girl and 2 boys is 13/32.***

Solution. One girl and two boys can be selected in the following ways :

(i) Girl from the first group, boy from the second group, boy from the third group. The probability of which is

$$p_1 = \frac{{}^3C_1}{{}^4C_1} \times \frac{{}^2C_1}{{}^4C_1} \times \frac{{}^3C_1}{{}^4C_1} = \frac{3}{4} \times \frac{2}{4} \times \frac{3}{4} = \frac{9}{32}.$$

(ii) Boy from the first group, girl from the second, boy from the third. The probability of which is

$$p_2 = \frac{1}{4} \times \frac{2}{4} \times \frac{3}{4} = \frac{3}{32}.$$

(iii) Boy from the first group, boy from the second group, girl from the third. The probability of which is

$$p_3 = \frac{1}{4} \times \frac{2}{4} \times \frac{1}{4} = \frac{1}{32}.$$

Since all the above three events are mutually exclusive, hence the chance that any of these events happen

$$= p_1 + p_2 + p_3 = \frac{9}{32} + \frac{3}{32} + \frac{1}{32} = \frac{13}{32}.$$ **Ans.**

Example 18. ***One bag contains 5 white balls and 7 black balls, another contains 3 white balls and 5 black balls. If a bag is chosen at random and a ball is drawn from it, then :***

(i) What is the chance that it is white ?

(ii) What is the chance that it is black ?

Solution. Here there are two bags. The chance of selecting the first bag $= \frac{1}{2}$.

Now the chance that a white ball is drawn from the first bag $= \frac{{}^5C_1}{{}^{12}C_1} = \frac{5}{12}$.

Hence if first bag is chosen and a white ball is drawn from it, the probability of this event E_1 is

$$P(E_1) = \frac{1}{2} \times \frac{5}{12} = \frac{5}{24}.$$

Similarly if second bag is chosen and a white ball is drawn from it, the probability of this event E_2 is

$$P(E_2) = \frac{1}{2} \times \frac{3}{8} = \frac{3}{16}.$$

Since both above events are mutually exclusive, the required probability is

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) = (5/24) + (3/16) = 19/48.$$ **Ans.**

(ii) Proceed as in part (i) above.

Example 19. ***If four squares are chosen at random on a chess-board, find the chance that they should be in a diagonal line.***

Solution. A chess board has $8 \times 8 = 64$ squares, as shown in the figure. Out of 64 squares, 4 can be chosen in ${}^{64}C_4$ ways. Thus if S is the sample space, then

$$n(S) = {}^{64}C_4.$$

Let E be the event that 4 squares are in a diagonal line.

First of all consider the $\Delta\, ABC$.

Along the diagonal AB 4 squares can be chosen in ${}^8C_4$ ways.

Along the diagonal $A_1 B_1$, 4 squares can be chosen in ${}^7C_4$ ways.

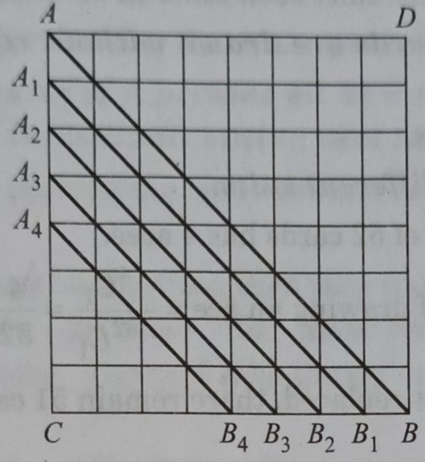
Similarly the number of ways in which 4 squares can be chosen along the diagonals $A_2 B_2$, $A_3 B_3$, $A_4 B_4$ are ${}^6C_4$, ${}^5C_4$, ${}^4C_4$ respectively.

Similarly in $\Delta\, ABD$, the squares can be chosen parallel to AB in an equal number of ways.

Hence the total number of ways in which four squares can be chosen in a diagonal parallel to AB is

$$= {}^8C_4 + 2\,[{}^7C_4 + {}^6C_4 + {}^5C_4 + {}^4C_4].$$

Since the line AB is common to both the triangles.



The same argument applies to squares parallel to CD and hence total number of favourable ways

$$n(E) = 2\,.\,{}^8C_4 + 4\,[\,{}^7C_4 + {}^6C_4 + {}^5C_4 + {}^4C_4]$$

$$= 2 \times 70 + 4\,[35 + 15 + 5 + 1] = 364.$$

Hence the required probability

$$= P(E) = \frac{n(E)}{n(S)}$$

$$= \frac{364}{{}^{64}C_4} = \frac{364 \times 4\,!}{64 \times 63 \times 62 \times 61} = \frac{91}{158844}.$$ **Ans.**

Example 20. ***A speaks the truth in 75% cases and B speaks the truth in 80% of the cases. In what percentage of cases are they likely to contradict each other in stating the same fact ?***

Solution. The probability of A speaking truth

$$P(A) = 75\% = 75/\,100 = 3/\,4.$$

The probability of B speaking truth

$$P(B) = 80\% = 80/\,100 = 4/\,5.$$

A and B will contradict each other if one speaks the truth and the other does not.

(i) Let E_1 be the event when A speaks truth and B tells a lie, the probability of it is

$$P(E_1) = \frac{3}{4} \times \left(1 - \frac{4}{5}\right) = \frac{3}{20}.$$

(ii) Let E_2 be the event when A tells a lie and B speaks truth, the probability of which is

$$P(E_2) = \left(1 - \frac{3}{4}\right) \times \frac{4}{5} = \frac{1}{5}.$$

Since the events E_1 and E_2 both are mutually exclusive, hence the required probability is

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) = \frac{3}{20} + \frac{1}{5} = \frac{7}{20} = \frac{35}{100}.$$

Hence A and B will contradict each other in 35% cases.

Example 21. ***Four cards are drawn without replacement. What is the probability that***

(a) they are all aces,

(b) they are all of different suits.

Solution. (a) A pack of 52 cards has 4 aces.

$\therefore$ The probability of drawing an ace $= \frac{^4C_1}{^{52}C_1} = \frac{4}{52}$.

Since the cards are not replaced, there remain 51 cards and 3 aces, if one ace has already been drawn.

$\therefore$ The probability of drawing an ace second time $= 3/51$.

Similarly the probability of drawing third ace $= 2/50$ and the probability of drawing fourth ace $= 1/49$.

Here the events are dependent, hence the required probability

$$= \frac{4}{52} \cdot \frac{3}{51} \cdot \frac{2}{50} \cdot \frac{1}{49} = \frac{1}{270725}.$$

(b) A pack of 52 cards has 4 different suits and each suit has 13 cards.

The first card may be of any suit *i.e.,* it may be any card out of 52 cards.

$\therefore$ The probability of drawing a card of any suit $= 52/52 = 1$.

In the second draw there should be card of different suit *i.e.,* this card should be any one of $52 - 13 = 39$ cards out of remaining 51 cards.

$\therefore$ The probability of drawing a second card of different suit $= 39/51$.

Similarly the probability of drawing the third card of different suit $= 26/50$ and that of drawing the fourth card $= 13/49$.

The events are dependent, therefore, the required probability is

$$= \frac{52}{52} \cdot \frac{39}{51} \cdot \frac{26}{50} \cdot \frac{13}{49} = \frac{2197}{20825}.$$ **Ans.**

Example 22. ***If four whole numbers taken at random are multiplied together, show that the chance that the last digit in the product is 1, 3, 7 or 9 is 16/625.***

Solution. In any whole number the last digit can be 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. Out of these ten digits there are six *i.e.,* 0, 2, 4, 5, 6, 8 which are divisible by 2 or 5.

Therefore, the probability that out of four whole numbers any one or all may be divisible by 2 or 5 is $= 6/10 = 3/5$.

Now the probability that a number is not divisible by 2 or 5 is $= 1 - \frac{3}{5} = \frac{2}{5}$.

In order that the product is not divisible by 2 or 5, none of the constituent numbers (*i.e.*, all the four numbers should not be divisible by 2 or 5 and its probability) is

$$= \frac{2}{5} \times \frac{2}{5} \times \frac{2}{5} \times \frac{2}{5} = \frac{16}{625}.$$

This is the required probability that in the product the last digit is 1, 3, 7 or 9.

Example 23. ***A party of n persons sit at a round table, find the odds against two specified individuals sitting next to each other.***

Solution. Since n persons can sit at a round table in $(n-1)!$ ways, therefore $n(S) = (n-1)!$.

If two persons always sit together, then they (taking two specified persons as one unit) can sit in $(n-2)!$ ways. But these two specified individuals can interchange their positions. Hence if E be the event that two persons sit next to each other, then

$$n(E) = 2.(n-2)!.$$

$\therefore$ The probability $P(E) = \dfrac{n(E)}{n(S)} = \dfrac{2(n-2)!}{(n-1)!} = \dfrac{2(n-2)!}{(n-1)(n-2)!} = \dfrac{2}{n-1}$

Again $\quad P(\overline{E}) = 1 - \dfrac{2}{n-1} = \dfrac{n-3}{n-1}.$

$\therefore$ The odds against are $= \dfrac{P(\overline{E})}{P(E)} = \dfrac{(n-3)/(n-1)}{2/(n-1)} = \dfrac{n-3}{2}.$ **Ans.**

Example 24. ***If 1/100 be the probability of success, then find the number of trials so that the probability of success in at least one is greater than half (i.e. 0.5).***

Solution. Here $p =$ probability of success $= 1/100$

$\therefore$ $q =$ the probability of not success

$$= 1 - \frac{1}{100} = \frac{99}{100}.$$

Let n be the total number of trials.

$\therefore$ The probability of not success in all the n trials $= (99/100)^n$.

Hence the probability of success in at least one trial $= 1 - (99/100)^n$.

Therefore, according to the given condition

$$1 - \left(\frac{99}{100}\right)^n > \frac{1}{2} \Rightarrow \left(\frac{99}{100}\right)^n < \frac{1}{2}$$

$\Rightarrow \quad n\,[\log 99 - \log 100] < \log 1 - \log 2$

$\Rightarrow \quad n\,[1 \cdot 9956 - 2] < 0 \cdot 3010$

$\Rightarrow \quad n > \dfrac{0 \cdot 3010}{0 \cdot 0044} \Rightarrow n > 68 \cdot 4.$

Hence required number of trials $= 69$. **Ans.**

Example 25. ***A is one of the eight horses entered for a race, and is to be ridden by one of the two jockeys B and C, it is 4 : 3 that B rides A, in which case all horses are equally likely to win; if C rides A his chances of winning are four times. What are the odds against his winning ?***

Solution. Let $E_1 = B$ rides A, $E_2 = C$ rides A, $E_3 = A$ wins.

The probability of B riding $A = P(E_1) = \frac{4}{4+3} = \frac{4}{7}$.

In this case (where B rides A) all eight horses are equally likely to win. Hence the probability that A wins is

$$= P(E_3 / E_1) = 1/8.$$

$\therefore$ The probability that B rides A and wins

$$= P(E_1 \cap E_3) = P(E_1) . P(E_3 / E_1)$$
$$= \frac{4}{7} \times \frac{1}{8} = \frac{4}{56}.$$

Again the probability of C riding A

$$= P(E_2) = 1 - \frac{4}{7} = \frac{3}{7} \qquad \text{[since when } C \text{ rides, } B \text{ cannot ride]}$$

In this case the chance of A's win is four times of the chance in the previous case

i.e., $$P(E_3 / E_2) = 4 \times \frac{1}{8} = \frac{4}{8}.$$

$\therefore$ The probability that C rides A and wins

$$= P(E_2 \cap E_3) = P(E_2) . P(E_3 / E_2)$$
$$= \frac{3}{7} \cdot \frac{4}{8} = \frac{12}{56}.$$

Since the horse A will win when either B rides it or C rides it, therefore

$$E_3 = (E_1 \cap E_3) \cup (E_2 \cap E_3)$$
$$\Rightarrow \quad P(E_3) = P\,\{(E_1 \cap E_3) \cup (E_2 \cap E_3)\}$$
$$= P(E_1 \cap E_3) + P(E_2 \cap E_3) \quad \text{[since events are mutually exclusive]}$$
$$= \frac{4}{56} + \frac{12}{56} = \frac{16}{56} = \frac{2}{7} = \frac{2}{2+5}.$$

Hence the odds against A's winning are 5 : 2.

Ans.

EXERCISE 1 (C)

1. Find the chance of throwing a prime number with an ordinary die.
2. From a pack of cards, three cards are drawn one by one without replacement, find the chance that they are a king, a queen and a knave (in this order).
3. The odds against a certain event are 3 : 2 and the odds in favour of another event independent of the former are 5 : 4, find the odds that one at least of the events will happen.
4. If the odds in favour of solving a problem by A are 7 : 5 and the odds against solving the same problem by B are 4 : 3. If they both try to solve the problem, what is the probability that the problem will be solved ?

5. There are three events A, B, C one of which must and only one can happen, the odds are 8 to 3 against A, 5 to 2 against B; find the odds against C.
6. A, B and C try to hit a target. A can hit the target 4 times in 5 shots; B, 3 times in 4 shots, C, 2 times in 3 shots. If they all hit, find the chance that two atleast hit.
7. From a pack of cards, face cards (3 of each colour) are removed, then 4 cards are drawn at random from the remaining 40 cards; find the probability of the following events.
   (a) All the four cards are of different suits.
   (b) All the four cards are of different suits and of different denominations.
8. Three candidates appear in an examination of Mathematics. Their chances of success are $\frac{1}{3}, \frac{1}{4}, \frac{1}{5}$ respectively. Find the chance that at least two will succeed.
9. A coin is tossed three times. Find the chance of getting head and tail alternately.
10. A bag contains 3 white balls and 3 black balls. These are drawn without replacement successively. Find the chance that the colours of the balls changes alternately.
11. Three groups of children contain respectively 3 girls and 1 boy; 2 girls and 2 boys; 1 girl and 3 boys. One child selected at random from each group. Show that the chance that three selected, consist of 1 girl and 2 boys is 13/32.
12. n letters to each of which corresponds an envelope are placed in the envelopes at random. What is the probability that no letter is placed in the right envelope ?
13. A and B stand in a ring with 10 other persons. If the arrangement of the 12 persons is at random; find the chance that there are exactly three persons between A and B.
14. A party of 23 persons take their seats at a round table show that the odds are 10 to 1 against two specified persons sitting together.
15. The probability that a person who is now 50 years old living till he is 60 is 0·83 and the probability that his wife who is now 45 years old living till she is 55 is 0·87. Find the probability that both will be alive 10 years hence.
16. It is 8 : 5 against a person who is now 40 years old living till he is 70 and 4 : 3 against a person 50 living till he is 80. Find the probability that at least one of these persons will be alive 30 years hence.
17. The probability that a boy will pass an examination is 3/5 and it is 2/5 for a girl. Find the probability that at least one of these will pass.
18. Find the probability of drawing an ace or a spade or both from a pack of cards.
    [**Hint** : $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
    $= P(A) + P(B) - P(A) \,.\, P(B)$].
19. The probability of happening an event A is $0 \cdot 7$ and the probability of not happening an other even B is $0 \cdot 5$ and the probability of not happening at least one of A and B is $0 \cdot 6$. Find the probability that at least one of A and B will happen.
20. If $P(A) = 0 \cdot 3$, $P(B) = 0 \cdot 5$, $P(A \equiv B) = 0 \cdot 2$, then find $P(A \cup B)$.
21. The probabilities that a student will get first, second or third division are respectively 0·25, 0·40 and 0·30. Find the probability of getting at least 2nd division.
22. If three squares are chosen at random on a chess-board, show that the chance that they should be in a diagonal line is 7/144.
23. One bag contains 3 white balls and 2 black balls, another contains 5 white and 3 black balls. If a bag is chosen at random and a ball is drawn from it, what is the chance that it is white ?

24. A is one of the six horses entered for a race and is to be ridden by one of the two Jockeys B and C, it is 2 to 1 that B rides A in which case all horses are equally likely to win; if C rides A his chance is trebled. What are the odds against his winning ?

25. Explain with examples, the following terms :

(i) Random variable, (ii) Conditional Probability,

(iii) Compound Probability, (iv) Equally like Events.

26. Explain the concept of conditional probability.

ANSWERS

1. $\frac{2}{3}$	**2.** $\frac{8}{13 \times 51 \times 25}$	**3.** $\frac{11}{15}$	**4.** $\frac{16}{21}$
5. 43 : 34	**6.** $\frac{5}{6}$	**7.** (a) $\frac{1000}{9139}$	(b) $\frac{504}{9139}$
8. $\frac{1}{6}$	**9.** $\frac{1}{4}$	**10.** $\frac{1}{10}$	**11.** $1 - \frac{1}{n!}$
13. $\frac{2 \times {}^{10}C_3 \times 3! \times 7!}{11!}$	**13.** 0·7221	**16.** $\frac{59}{91}$	**17.** $\frac{19}{25}$
18. $\frac{4}{13}$	**19.** 0·8	**20.** $\frac{3}{5}$	**21.** $\frac{13}{20}$
23. $\frac{49}{80}$	**24.** 13 : 5.		

◈ § 1.14. (A) USE OF BINOMIAL THEOREM

Theorem. *If the probability of success in one trial is p and that of failure is q so that $p + q = 1$, then the probability of r success in n trials is given by ${}^nC_r \cdot p^r q^{n-r}$ or the $(r + 1)$th term in the expansion of $(q + p)^n$.*

Proof. Total number of trials are $= n$.

Out of n trials, r can be chosen in ${}^nC_r$ ways.

Again the chance that the event happens in r trials and fails in $n - r$ trials $= p^r q^{n-r}$.

Therefore, the chance of exactly r successes $= {}^nC_r \; p^r \; q^{n-r}$. ...(1)

Putting $r = 0$ in (1), we get the probability of happening the event never.

Putting $n = 1, 2, 3, \ldots$ in relation (1), we get the probability of happening exactly once, twice, thrice, ... in n trials. Hence the probability that the event will happen exactly r times in n trials is ${}^nC_r p^r q^{n-r}$ which is the $(r + 1)$th term in the expansion of $(q + p)^n$.

Cor. If P is the chance that an event happens at least r times in n trials then

$$P = p^n + {}^nC_1 \; p^{n-1} \; q + {}^nC_2 \; p^{n-2} \; q^2 + \ldots + {}^nC_{n-r} \; p^r \; q^{n-r}.$$

ILLUSTRATIVE EXAMPLES

Example 1. ***In four throws, with a pair of dice, what is the chance of throwing doublets twice at least ?***

Solution. Here $n(S) = 6^2 = 36.$

There are only six doublet namely (1, 1); (2, 2); (3, 3); (4, 4); (5, 5); (6, 6).

$\therefore$ The chance of throwing a doublet in one throw

$$p = \frac{6}{36} = \frac{1}{6}.$$

Thus the chance of not throwing a doublet

$$q = 1 - \frac{1}{6} = \frac{5}{6}.$$

Hence the chance of at least two doublets in 4 throws :

$$= p^4 + {}^4C_1\, p^3 q + {}^4C_2\, p^2 q^2$$

$$= \left(\frac{1}{6}\right)^4 + 4\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right) + \frac{4\cdot 3}{1\cdot 2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^2$$

$$= \frac{1 + 20 + 150}{6\times 6\times 6\times 6} = \frac{171}{36\times 36} = \frac{19}{144}.$$

Example 2. ***If on an average 1 vessel in every 10 is wrecked find the chance that out of p vessels expected to arrive, 4 at least will arrive safely.***

Solution. Let the probability of wrecking a vessel be q and safe arrival be p. Then

$$q = 1/10,\quad p = 1 - (1/10) = 9/10.$$

The various probabilities are given by the terms in the binomial expansion of $(q + p)^5$. We are required to find the chance of survival of 4 and 5 vessels.

The chance of 4 vessels arriving safely $= {}^5C_4\, qp^4.$

The chance of 5 vessels arriving safely $= {}^5C_5\, p^5.$

$\therefore$ The required probability $= {}^5C_4\, qp^4 + {}^5C_5\, p^5$

$$= {}^5C_4\,(1/10)(9/10)^4 + {}^5C_5\,(9/10)^5$$

$$= \frac{9^4}{10^5}(5 + 9) = \frac{9^4\times 4}{10^5} = \frac{45927}{50000}.$$ **Ans.**

Example 3. ***If m things are distributed among a men and b women, show that the chance the number of things received by men is odd, is*** $\frac{1}{2}\frac{(b+a)^m - (b-a)^m}{(b+a)^m}$.

Solution. The probability of one thing going to a man is,

$$p = a/(a + b)$$

and the probability of one thing going to a woman

$$q = b/(a + b).$$

Here
$$p + q = \frac{a}{a + b} + \frac{b}{a + b} = \frac{a + b}{a + b} = 1.$$

Thus the probabilities of 1, 2, 3, ... things going to men are the terms having $p, p^2, p^3, \ldots$ in the binomial expansion of $(q + p)^m$.

Therefore, the probability that r things out of m are received by men is ${}^mC_r\, p^r\, q^{m-r}$. But in the present problem, we are required to find the probability that an odd number of things are received by men.

$\therefore$ The required probability

$$= {}^mC_1\, pq^{m-1} + {}^mC_3\, p^3\, q^{m-3} + {}^mC_5\, p^5\, q^{m-5} + \ldots$$

$$= {}^mC_1 \left(\frac{a}{a+b}\right)\left(\frac{b}{a+b}\right)^{m-1} + {}^mC_3 \left(\frac{a}{a+b}\right)^3 \left(\frac{b}{a+b}\right)^{m-3} + {}^mC_5 \left(\frac{a}{a+b}\right)^5 \left(\frac{b}{a+b}\right)^{m-5} + \ldots$$

$$= \frac{1}{(a+b)^m} [{}^mC_1\, ab^{m-1} + {}^mC_3\, a^3\, b^{m-3} + {}^mC_5\, a^5\, b^{m-5} + \ldots]$$

$$= \frac{1}{2} \cdot \frac{(b+a)^m - (b-a)^m}{(a+b)^m}.$$

Proved.

Example 4. ***10 coins are tossed together, find the probability of having at least 7 heads.***

Solution. If one coin is tossed, let p be the probability of having head and q that of not having head.

$$\therefore \qquad p = \frac{1}{2},\ q = \frac{1}{2}.$$

The probability of having at least 7 heads.

= sum of the probabilities of having exactly 7, 8, 9 and 10 heads

$$= \sum_{r=7}^{10} {}^{10}C_r \left(\frac{1}{2}\right)^r \left(\frac{1}{2}\right)^{10-r}$$

$$= \sum_{r=7}^{10} {}^{10}C_r \left(\frac{1}{2}\right)^{10} = \left(\frac{1}{2}\right)^{10} \sum_{r=7}^{10} {}^{10}C_r$$

$$= \left(\frac{1}{2}\right)^{10} [{}^{10}C_7 + {}^{10}C_8 + {}^{10}C_9 + {}^{10}C_{10}]$$

$$= \left(\frac{1}{2}\right)^{10} [{}^{10}C_3 + {}^{10}C_2 + {}^{10}C_1 + {}^{10}C_0]$$

$$= \frac{1}{32 \times 32}\left[\frac{10 \times 9 \times 8}{3 \times 2 \times 1} + \frac{10 \times 9}{2 \times 1} + \frac{10}{1} + 1\right]$$

$$= \frac{1}{32 \times 32} [120 + 45 + 10 + 1] = \frac{176}{32 \times 32} = \frac{11}{64}.$$

Ans.

EXERCISE 1 (D)

1. An experiment succeeds twice as often as it fails. Find the chance that in the next six trials there will be at least four successes.
2. In tossing 10 coins, what is the probability of having exactly 5 heads ?
3. The probability of winning of game of A against B is 2/3, find the chance that out of 5 games, A will win in at least 3 games.
4. Six dice are tossed 729 times. How many times do you expect, that at least 3 dice will show 5 or 6 ?
5. Show that if a coin is tossed n times, the probability of not more than r heads is
$$\left(\frac{1}{2}\right)^n ({}^nC_0 + {}^nC_1 + \dots + {}^nC_r).$$
6. If p is the chance that an odd number of aces turn up when n ordinary dice are thrown, show that
$$1 - 2p = (2/3)^n.$$
[**Hint** : The probability of throwing an ace by an ordinary die = 1/6 and 5/6 that of not throwing an ace.

$\therefore$ The probabilities of throwing 1, 3, 5, ... aces in throw of n dice is given by the alternative terms of the binomial expansion $\left(\frac{5}{6} + \frac{1}{6}\right)^n$, so that required probability
$$= {}^nC_1 (5/6)^{n-1} (1/6) + {}^nC_3 (5/6)^{n-3} (1/6) + \dots \text{etc.}]$$

ANSWERS

1. $\frac{496}{729}$ 2. $\frac{63}{256}$ 3. $\frac{64}{81}$ 4. 233.

§ 1.15. USE OF MULTINOMIAL THEOREM

Theorem. *If a die has f faces marked with 1, 2, ..., f, the probability of throwing a total p with n dice is given by the coefficient of x^n in the expansion $(x^1 + x^2 + \dots + x^f)^n$ divided by f^n.*

Proof. A dice has n faces and, therefore, any one of the f faces may be exposed on any one of the n dice. Hence the total number of ways in which the n dice may fall is f^n

$$\therefore \qquad n(S) = f^n.$$

Let A be the event of throwing a total p with n dice.

$\therefore$ $n(A)$ = total number of ways favourable to the event A.

= total number of ways selecting n numbers from 1, 2, 3, ..., f so as to make p as their sum

= coefficient of x^p in the expansion of $(x^1 + x^2 + x^3 + \dots + x^f)^n$.

The required probability $P(A)$ is given by
$$P(A) = \frac{n(A)}{n(S)}$$

or
$$P(A) = \frac{\text{Coeff. of } x^p \text{ in } (x^1 + x^2 + \ldots + x^f)^n}{f^n}$$
$$= \frac{\text{coeff. of } x^p \text{ in } x^n (1 + x + x^2 + \ldots + x^{f-1})^n}{f^n}$$
$$= \frac{\text{coeff. of } x^p \text{ in } x^n \dfrac{(1 - x^f)^n}{(1 - x)^n}}{f^n}$$
$$= \frac{\text{coeff. of } x^{p-n} \text{ in } (1 - x^f)^n (1 - x)^{-n}}{f^n}.$$

ILLUSTRATIVE EXAMPLES

Example 1. ***Find the chance of throwing 10 exactly in one throw with three dice.***

Solution. Total ways of throwing 3 dice $= 6^3$.

Favourable ways of throwing exactly 10
$$= \text{coeff. of } x^{10} \text{ in } (x^1 + x^2 + \ldots + x^6)^3$$
$$= \text{coeff. of } x^{10} \text{ in } x^3 \left(\frac{1 - x^6}{1 - x}\right)^3$$
$$= \text{coeff. of } x^{10} \text{ in } x^3 (1 - x^6)^3 (1 - x)^{-3}$$
$$= \text{coeff. of } x^7 \text{ in } (1 - 3x^6 + \ldots)\Big(1 + 3x + 6x^2 + 10x^3 + 15x^4$$
$$+ 21x^5 + 28x^6 + 36x^7 + \ldots + \frac{(r+1)(r+2)}{2} x^r + \ldots\Big)$$
$$= 36 - 9 = 27.$$

$\therefore$ The required probability $= 27/6^3 = 1/8$. **Ans.**

Example 2. ***Four dice are thrown. What is the probability that the sum of the numbers appearing on the dice is 18 ?***

Solution. Total ways of throwing 4 dice $= 6^4$.

Favourable ways of throwing a sum of 18
$$= \text{coeff. of } x^{18} \text{ in } (x^1 + x^2 + \ldots + x^6)^4$$
$$= \text{coeff. of } x^{18} \text{ in } x^4 (1 + x + \ldots + x^5)^4$$
$$= \text{coeff. of } x^{14} \text{ in } (1 - x^6)^4 (1 - x)^{-4}$$
$$= \text{coeff. of } x^{14} \text{ in } (1 - 4x^6 + 6x^{12} \ldots)$$
$$(1 + 4x + 10x^2 + \ldots + 165x^8 + \ldots + 680x^{14} + \ldots)$$
$$= 680 - 660 + 60 = 80.$$

The required probability $= 80/6^4 = 5/81$. **Ans.**

Example 3. ***Determine the probability of throwing more than 8 with 3 perfectly symmetrical dice.***

Solution. Total ways of throwing 3 dice $= 6^3$.

First we shall find the probability of getting a sum upto 8. But from 3 dice a sum less than 3 cannot be obtained.

$\therefore$ The number of ways obtaining a sum from 3 to 8

$$= \text{Sum of the coeffs. of } x^3, x^4, x^5, x^6, x^7, x^8 \text{ in } (x + x^2 + \dots + x^6)^3$$

$$= \text{Sum of the coeffs. of } x^0, x^1, \dots, x^5 \text{ in } (1 + x + \dots + x^5)^3$$

$$= \text{Sum of the coeffs. of } x^0, x^1, \dots, x^5 \text{ in } \left(\frac{1-x^6}{1-x}\right)^3$$

$$= \text{Sum of the coeffs. of } x^0, x^1, \dots, x^5 \text{ in } (1-x^6)^3 (1-x)^{-3}$$

$$= \text{Sum of the coeffs. of } x^0, x^1, \dots, x^5 \text{ in } (1 - 3x^6 + \dots)(1 + 3x + 6x^2 + 10x^3 + 15x^4 + 21x^5 + \dots)$$

$$= 1 + 3 + 6 + 10 + 15 + 21 = 56.$$

$\therefore$ The probability of obtaining a sum of 3 to 8 $= 56/6^3 = 7/27$.

Hence the required probability *i.e.*, the probability of throwing more than 8

$= 1 - (7/27) = 20/27$. **Ans.**

Example 4. ***A person throws two dice, one the common cube, and the other regular tetrahedron, the number in the lowest face being taken in the case of tetrahedron. What is the chance that the sum of the numbers thrown is not less than 5.***

Solution. We know that a common cube and a regular tetrahedron has respectively 6 and 4 faces. Therefore, the total ways of throwing the cube and tetrahedron is 6×4 *i.e.*, 24.

Here in the present case the minimum sum obtained is 2. Therefore, number of ways of getting a sum of 2, 3 and 4

$$= \text{the sum of coeffs. of } x^2, x^3, x^4 \text{ in } (x^1 + x^2 + \dots + x^6)^1 (x + x^2 + x^3 + x^4)^1$$

$$= \text{the sum of coeffs. of } x^2, x^3, x^4 \text{ in } (x^2 + 2x^3 + 3x^4 + 4x^5 + \dots)$$

$$= 1 + 2 + 3 = 6.$$

$\therefore$ The probability of getting a sum of 2 to 4 $= 6/24 = 1/4$.

Hence the required chance $= 1 - (1/4) = 3/4$. **Ans.**

EXERCISE 1 (E)

1. Find the chance of throwing 10 with 4 dice.
2. Four tickets marked 00, 01, 10, 11 respectively are placed in a bag. A ticket is drawn at random five times, being replaced each time. Find the probability that the sum of the numbers on tickets thus drawn is 23.

   [**Hint.** Total ways of drawing 4 tickets in 5 trials $= 4^5$.

   Favourable ways = coeff. of x^{23} in $(x^0 + x^1 + x^{10} + x^{11})^5 = 100$].

3. Counters marked 1, 2, 3 are placed in a bag, one is withdrawn and replaced three times. What is the chance of obtaining a total of 6 ?
4. The four faces of a regular tetrahedron are numbered 1, 2, 3, 4. It is thrown five times and the figure on the lowest face is noted down. What is the probability that the sum will be 12 ?
5. Find the chance of throwing more than 16 in one throw with 3 dice.
6. There are 10 tickets, 5 of which are blanks, and the others are marked with numbers 1, 2, 3, 4, 5. What is chance of drawing 10 in three trials, the tickets are replaced after every trial.
7. Five coins whose faces are marked 2, 3 are thrown ; what is the chance of obtaining a total of 12 ?

   [**Hint.** Total ways of throwing five coins $= 2^5 = 32$

   Favourable ways = coeff. of x^{12} in $(x^2 + x^3)^5 = 10$

   Required chance $= 10/32 = 5/16$] **Ans.**

ANSWERS

1. $\frac{5}{81}$ 2. $\frac{25}{256}$ 3. $\frac{7}{27}$ 4. $\frac{154}{1024}$

5. $\frac{5}{192}$ 6. $\frac{33}{1000}$ 7. $\frac{5}{16}$.

§ 1.16. RANDOM VARIABLE

A real valued function defined on a sample space is called a random variable. In other words, a variable which takes a definite set of values with a definite probability associated with each value of the variable is called the random variable. For example, if we toss a coin and we are interested to get a head on upper side, then we define random variable X with two values 1 and 0. 1 for head and 0 for tail.

Sample space $S = \{H, T\}$

$$X = 1, \quad P(H) = P(X = 1) = \frac{1}{2}$$

$$X = 0, \quad P(T) = P(X = 0) = \frac{1}{2}.$$

$P(H)$ or $P(X = 1)$ be the probability that head appears on upper side. $P(T)$ or $P(X = 0)$ be the probability that tail appears on upper side.

In tossing of two coins, we have sample space as

$$S = \{(HH), (HT), (TH), (TT)\}.$$

Let X be the random variable which show the number of heads appears on the upper side. Then X takes three values 0, 1, 2, *i.e.*,

$$P(TT) = P(X = 0) = 1/4$$

$$P(HT, TH) = P(X = 1) = 2/4 = 1/2$$

$$P(HH) = P(X = 2) = 1/4.$$

Suppose if we throw a single die and we are interested in getting an even number. Then the sample space is

$$S = \{1, 2, 3, 4, 5, 6\}.$$

Let X be the random variable which shows the even number on the upper side. So X takes only two values 1 and 0. 1 for even number and 0 for odd number.

$$P(X = 1) = P(2, 4, 6) = \frac{3}{6} = \frac{1}{2}$$

$$P(X = 0) = P(1, 3, 5) = \frac{3}{6} = \frac{1}{2}.$$

Suppose if we toss three coins simultaneously and we are interested in getting the number of heads appearing on the upper side. Let X be the random variable which denotes the number of heads and takes only 4 values 0, 1, 2, 3. Here we have the sample space

$$S = \{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT\}$$

$$P(X = 0) = P(TTT) = \frac{1}{8}$$

$$P(X = 1) = P(TTH, THT, HTT) = \frac{3}{8}$$

$$P(X = 2) = P(HHT, HTH, THH) = \frac{3}{8}$$

$$P(X = 3) = P(HHH) = \frac{1}{8}.$$

◈ § 1.17. TYPES OF A RANDOM VARIABLE

There are two types of random variables :

(a) Discrete random variables

(b) Continuous random variables

(a) Discrete Random Variable. A variable which takes only countable number of values and is defined on a discrete sample space is called a discrete random variable. For example, suppose a perfect die in thrown then X, the number of points on the die is a random variable, since X has the following two properties :

(i) X takes only a set of discrete values 1, 2, 3, 4, 5, 6.

(ii) The values which X takes depends on the chance.

Actually X takes values 1, 2, 3, 4, 5, 6 each with probability 1/6. The set of values 1, 2, 3, 4, 5, 6 with their probability 1/6 is called the **Probability Distribution** of the variate X.

In general. Suppose that corresponding to X, exhaustive and mutually exclusive cases are obtained from a trial, the variate X takes n values $x_1, x_2, \ldots, x_n$ with their probabilities $P_1, P_2, \ldots, P_n$.

The set of values x_i (for $i = 1, 2, 3, \ldots, n$) with their probabilities p_i (for $i = 1, 2, 3, \ldots, n$) is called the probability distribution of the random variable X.

In rolling of a perfect die, the probability distribution of the number on the die will be

x	1	2	3	4	5	6
$P(X = x)$	1/6	1/6	1/6	1/6	1/6	1/6

In tossing two perfect coins. Let X be r.v. of number of heads on upper sides. The probability distribution of X will be

x	0	1	2
$P(X = x)$	1/ 4	1/ 4	1/ 4

In tossing two perfect dice simultaneously, let X be random variable of sum of numbers on the upper side of two dice. Then X takes only 11 values 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12. The probability distribution of X will be

X	Sample space point	P(X = x)
2.	(1, 1)	1/ 36
3.	(1, 2), (2, 1)	2/ 36
4.	(1, 3), (2, 2), (3, 1)	3/ 36
5.	(1, 4), (2, 3), (3, 2), (4, 1)	4/ 36
6.	(1, 5), (2, 4), (3, 3), (4, 2), (5, 1)	5/ 36
7.	(1, 6) (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)	6/ 36
8.	(2, 6), (3, 5), (4, 4), (5, 3), (6, 2)	5/ 36
9.	(3, 6), (4, 5), (5, 4), (6, 3)	4/ 36
10.	(4, 6), (5, 5), (6, 4)	3/ 36
11.	(5, 6), (6, 5)	2/ 36
12.	(6, 6)	1/ 36

Binomial, poisson negative binomial, hyper geometric are the examples of discrete probability distribution.

The probability distribution of discrete random variable is called discrete probability distribution and it is expressed in terms of probability mass function.

Note. The sum of all the probabilities of a probability distribution of X is always one.

(b) Continuous Random Variable. A random variable X is said to be continuous if it can take all possible values between certain limits. When we deal with variates like weights and temperature then we know that these variates can take on infinite number of values in a given interval. Such type of variate is known as continuous random variable.

Let X be a continuous random variable and let the probability of X falling in very small interval $\left(x - \frac{1}{2}\,dx, x + \frac{1}{2}\,dx\right)$ be expressed by $f(x)\,dx$, *i.e.*,

$$P\left[x - \frac{1}{2}\,dx < X < x + \frac{1}{2}\,dx\right] = f(x)\,dx$$

where $f(x)$ is a continuous function of x and satisfies the following two conditions :

(i) $f(x) \geq 0$

(ii) $\int_a^b f(x)\,dx = 1;\ a \leq x \leq b.$

Then the function $f(x)$ is called the **probability density function** of the continuous random variable X.

The probability distribution of continuous random variable is called continuous probability distribution and it is expressed in terms of **probability density function.**

Remarks :

(i) If the range of X be finite, then also it can be expressed as infinite range. For example

$$\begin{aligned} f(x) &= 0 \quad \text{for } x < a \\ f(x) &= \phi(x) \text{ for } a \le x \le b \\ f(x) &= 0 \quad \text{for } x > b. \end{aligned}$$

(iii) The probability that a value of continuous variable X lies within the interval (c, d) is given by

$$P(c \le x \le d) = \int_c^d f(x)\, dx.$$

(iii) If X is a continuous random variable then probability of any point will be zero, *i.e.*,

$$P(X = K) = 0$$

where K is a constant quantity.

◈ § 1.18. CUMULATIVE DISTRIBUTION FUNCTION

The probability that the value of a random variable X is "x or less than x", is called the cumulative distribution function of X and usually denoted by $F(x)$. In symbolic notation, the cumulative distribution function of discrete random variate X is given by

$$F(x) = P(X \le x) = \sum_{x \le x_i} P(x_i).$$

The cumulative distribution function of a continuous random variate is given by

$$F(x) = P[X \le x] = \int_{-\infty}^{x} f(x)\, dx.$$

Some Properties of Cumulative Distribution Function :

(i) $F(-\infty) = 0$

(ii) $F(x)$ is non decreasing function, *i.e.*, $x_1 < x_2 \Rightarrow F(x_1) < F(x_2)$

(iii) For a discontinuous variate

$$P(a < x < b) = F(b) - F(a) = \Sigma\, p(x_i);\ a < x_i < b.$$

And for a continuous variate

$$\begin{aligned} P(a < x \le b) &= P(a \le X \le b) = P(a \le X \le b) \\ &= P(a < X < b) = F(b) - F(a) \\ &= \int_a^b f(x)\, dx. \end{aligned}$$

(iv) $F(+\infty) = 1$

(v) $F(x)$ is a discontinuous function for a discontinuous variate and $F(x)$ is continuous function for a continuous variate.

Normal distribution, Gamma distribution, Beta distribution, t-distribution, F-distribution, χ^2-distribution are some examples of continuous random variables.

There are several types of distribution, but the following two types of distribution are commonly used in Statistics.

◈ § 1.19. BERNOULLI DISTRIBUTION

Suppose a random experiment is performed. Let E be an event. We shall call the occurrence of the event E as a success and its non occurrence as a failure. If p denotes the probability of a success and q denotes the probability of its failure then $p + q = 1$. A random variable X which takes two values 0 and 1 with probability q and p respectively. Then probability distribution of X will be

$$P(X = 1) = p$$
$$P(X = 0) = q.$$

Then, X is called the Bernoulli variate and the probability distribution of X is called Bernoulli distribution.

The mean and variance of this distribution are p and pq respectively.

◈ § 1.20. THEORETICAL FREQUENCY DISTRIBUTION

Definition. *When frequency distributions of some universes are not based on actual observations or experiments, but can be derived mathematically from certain pre-determined hypothesis, then such distributions are said to be theoretical distributions.*

For example. Four coins are tossed 80 times, then according to the principle of probability, the expected frequency distribution is obtained as given in the following table :

No. of heads	Probability	Expected frequency distribution in 80 tosses
0	1/16	$80 \times (1/16) = 5$
1	4/16	$80 \times (4/16) = 20$
2	6/16	$80 \times (6/16) = 30$
3	4/16	$80 \times (4/16) = 20$
4	1/16	$80 \times (1/16) = 5$
Total	**1**	**80**

Types of Theoretical Distribution. There are several types of theoretical distributions, but following two types of theoretical distribution are usually used in statistics.

1. Discrete Probability Distribution :

(a) Binomial Distribution,

(b) Poisson Distribution.

2. Continuous Probability Distribution :

(a) Normal Distribution,

(b) Rectangular Distribution,

(c) Exponential Distribution.

§ 1.21. BINOMIAL DISTRIBUTION

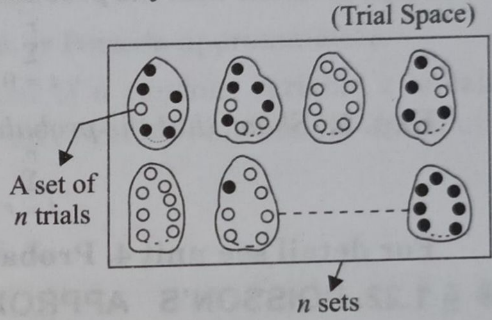
Binomial Distribution was discovered by James Bernoulli (1654–1705) in 1700.

Suppose a random experiment is performed repeatedly. Let E be an event. We shall call the occurrence of the event E a 'success and its non-occurrence a failure'. If p denotes the probability of a success and q denotes the probability of its failure, then $p + q = 1$. Then the event E be tried n times, where n is finite. The hypothesis for binomial distribution are :

(i) All the trials are independent *i.e.*, the result of one trial will not effect the results of succeeding trials.

(ii) The number (n) of trials is finite.

(iii) The probability p of successes is the same in every trial.



The number of successes in trials may be $0, 1, 2, 3, ..., r, ..., n$ and is clearly a random variate. Suppose that the set of n trials is repeated N times, where N is very large. Obviously out of these N sets there will be a few sets with no success, a few sets with one success, a few sets with two successes, a few sets with 3 successes, ..., etc.

Now suppose that first r trials are successes and the remaining $n - r$ trials are failures. Its probability is $p^r q^{n-r}$. Clearly it is also the probability for r successes and $n - r$ failures which may occur in any order. But we are to consider all the possible cases where any r trials are successes and thus r can be chosen out of n in ^nC_r mutually exclusive ways.

Hence the probability, $P(r)$ of r successes in a series of n independent trials is given by the following formula [using theorem of total probability].

$$P(r) = {}^nC_r\ p^r\ q^{n-r}.$$

Definition. A random variable X is said to follow binomial distribution if it takes only non-negative values and its probability mass function is given by the following formula

$$P(X = r) = \begin{cases} {}^nC_r\ p^r\ q^{n-r} & ; \quad r = 0, 1, 2, ..., n,\ q = 1 - p \\ 0 & ; \quad r = 0, 1, 2, ..., n. \end{cases}$$

Here, the two independent constants n and p (or q) in the distribution are called **parameters** of distribution.

Since we have considered N sets, each of n trials, therefore the number of sets with r successes $= N.\ {}^nC_r\ p^r q^{n-r}$ or $N\ .\ P(r)$. Hence we obtain the following frequency distribution :

No. of successes :	0	1	2	...	r	...	n
No. of sets :	Nq^n	$N.\ {}^nC_1\ pq^{n-1}$	$N.\ {}^nC_2\ p^2q^{n-2}$	...	$N\ {}^nC_r\ p^rq^{n-r}$	...	Np^n

Thus we see that, for N sets of n trials the frequencies of 0, 1, 2, ..., r,..., n successes are the successive terms of the following expression :

$$N \cdot q^n + N \cdot {}^nC_1 pq^{n-1} + N \cdot {}^nC_2 q^2 q^{n-2} + \ldots + N \cdot {}^nC_r p^r q^{n-r} + \ldots + N \cdot p^n$$

i.e., $$N\,[q^n + {}^nC_1 pq^{n-1} + {}^nC_2 p^2 q^{n-2} + \ldots + {}^nC_r p^r q^{n-r} + \ldots + p^n]$$

which is the binomial expansion of $N \cdot (q+p)^n$. It is called the **Binomial frequency distribution.**

Remark 1. In the above binomial distribution n and p (or q) are said to be parameters.

Remark 2. If $p = q = \frac{1}{2}$, the binomial distribution is called **symmetrical distribution** otherwise it is called **skew distribution**.

Exp. 1. *Show that the probability of at most r successes in n trials is*

$$\sum_{t=0}^{r} {}^nC_t\, p^t\, q^{n-t}.$$

Exp. 2. *Show that the probability of at least r successes in n trials is*

$$\sum_{t=r}^{n} {}^nC_t\, p^t\, q^{n-t}.$$

For detail see unit 4, Probability Distribution.

◈ § 1.22. POISSON'S APPROXIMATION AS A LIMITING FORM OF BINOMIAL DISTRIBUTION

The Poisson's approximation is a particular limiting form of the binomial distribution when p (or q) is very small and n is large so that the average number of successes np is a finite constant m (say).

We know that, in the binomial distribution, the probability of r successes is given by

$$b(r; n, p) = {}^nC_r\, p^r\, q^{n-r}$$

$$= {}^nC_r\, p^r\, (1-p)^{n-r} \qquad [\because\ p + q = 1]$$

$$= \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}\, p^r \left(1 - \frac{np}{n}\right)^{n-r}$$

$$= \frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{r-1}{n}\right)}{r!} \frac{(np)^r}{\left(1-\frac{np}{n}\right)^r} \left(1 - \frac{np}{n}\right)^n.$$

$\therefore$ $P(r)$ = the probability of r successes in Poisson's distribution

$$= \lim_{p \to 0,\, n \to \infty,\, np = m} b(r; n, p)$$

$$= \lim_{n \to \infty} \frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{r-1}{n}\right)}{r!} \frac{m^r}{\left(1-\frac{m}{n}\right)^r} \left(1 - \frac{m}{n}\right)^n$$

$$= \frac{m^r \cdot e^{-m}}{r!} \qquad \left[\because \lim_{n\to\infty}\left(1-\frac{m}{n}\right)^n = \lim_{n\to\infty}\left\{\left(1-\frac{m}{n}\right)^{-n/m}\right\}^{-m} = e^{-m}\right]$$

and $$\lim_{n\to\infty}\left(1-\frac{m}{n}\right)^r = (1-0)^r = 1.$$

It is called *Poisson's distribution* or *Poisson approximation.*

Therefore, the chances of 0, 1, 2, ... r successes are respectively

$$e^{-m}, \frac{me^{-m}}{1!}, \frac{m^2 e^{-m}}{2!}, \ldots, \frac{m^r e^{-m}}{r!}.$$

$\therefore$ The limiting form of binomial distribution

$$(q+p)^n \text{ where } p \to 0, n \to \infty$$

so that $np = m$ is called Poisson's distribution or Poisson approximation.

Definition. The probability distribution of a random variable x is called Poisson's distribution if x can assume non-negative values only and its distribution is given by

$$P(x=r) = \begin{cases} \dfrac{e^{-m} m^r}{r!} & , r = 0, 1, 2, \ldots \\ 0 & , r \neq 0, 1, 2, \ldots \end{cases}$$

Note 1. m is known as parameter of Poisson's distribution.

Note 2. Following are some examples of Poisson variate :

(a) The number of deaths in a city in one year by a rat disease as by heart attack or cancer.

(b) The number of telephone calls per minute in a switch board.

(c) The number of suicides in a city in one year.

(d) The number of accidents in a district in one year.

(e) The number of misprints on a page of a book.

(f) The number of defective blades in a packet of 100 blades.

(g) The number of cars passing through a certain street in time t.

◆ § 1.23. BERNOULLI TRIAL EXPERIMENTS

When a coin tossed ten times. What is the probability that heads occurs eight out of the ten times ?

If you guess at the answers of ten multiple choice questions, what are your chance for a passing grade ?

Such type of problem of a certain type of probability problem are known as *Bernoulli trials.*

Such problem involved **repeated trials** of an experiment with only two possible outcomes true or false–heads or tails, right or wrong, and so on. The two outcomes are called as **success** or **failure.**

◈ § 1.23.1. SOME DEFINITIONS

A *binomial trial* is a single binomial experiment or observation. The treatment of a single patient with the antibiotic is a binomial trial. The trial must result in only one of two outcomes, where the two outcomes are *mutually exclusive.* In the above example, the only possible outcomes are that a patient is either cured or not cured. In addition, only one of these outcomes is possible after treatment. A patient cannot be both cured and not cured after treatment. Each binomial trial must be *independent.* The result of a patient's treatment does not influence the outcome of the treatment for a different patient.

◈ § 1.23.2. PROPERTIES OF BERNOULLI EXPERIMENT

(1) The experiment is repeated a fixed number of times (n times).

(2) Each trial has only two possible outcomes : success and failure. The outcomes are exactly the same for each trial.

(3) The probability of success remains the same for each trial. (p for probability of success and $q = 1 - p$ for probability of failure).

(4) The trials are independent.

(5) We have to find the total number of successes, not the order in which they occur.

There nay be 0, 1, 2, ... or n successes in n trials.

Example 1. A student guesses at all the answers on a ten multiple choice quiz (four choices of an answer on each question). This fulfills the properties of a Bernoulli trial because

(1) Each choices is a trial ($n = 10$)

(2) There are two possible outcomes; correct and incorrect, known as success or failure.

(3) The probability of a correct answers is $p = \frac{1}{4}$, and the probability of an incorrect answer is $q = 1 - p = \frac{3}{4}$ on each trial, and

(4) The choices are independent because guessing an answer on one question gives no information on other questions.

We Now give the general formula for computing the probability using Bernoulli's experiment.

◈ § 1.23.3. PROBABILITY OF A BERNOULLI EXPERIMENT

Given a Bernoulli experiment and n independent repeater trials, p is the probability of success in a single trial.

$q = 1 - p$ is the probability of failure in a single trial, r is the number of successes ($r \leq n$).

Then the probability of r successes in n trials is–

$P(r \text{ successes in } n \text{ trials}) = {}^nC_r\, p^r q^{n-r}$

$P(x \text{ successes in } n \text{ trials})$ may be written $P(r = x)$.

Example 1. ***A coin is tossed ten times. What is the probability that heads occurs six times ?***

Solution. In this case, $n = 10, r = 6$

When we throw a coin, the probability of success $p = \frac{1}{2}$, $q = 1 - p = 1 - \frac{1}{2} = \frac{1}{2}$

$$P(r = 6) = 10C_6 \left(\frac{1}{2}\right)^6 \left(\frac{1}{2}\right)^{10-6}$$

$$= 10C_6 \left(\frac{1}{2}\right)^6 \left(\frac{1}{2}\right)^6$$

$$= \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5}{1 \times 2 \times 3 \times 4 \times 5 \times 6} \left(\frac{1}{2}\right)^6 \left(\frac{1}{2}\right)^6$$

$$= 210 \left(\frac{1}{64}\right) \left(\frac{1}{16}\right)$$

$$= 0 \cdot 205$$

The probability that head occurs six times will be $0 \cdot 205$.

◈ § 1.24. MATHEMATICAL EXPECTATION OR THE EXPECTED VALUE OF DISCRETE RANDOM VARIABLE

Definition. Let x be a discrete random variable and let its frequency distribution be as follows :

variate	:	x_1	x_2	x_3	...	x_n
probability	:	p_1	p_2	p_3	...	p_n

where $p_1 + p_2 + p_3 + \ldots + p_n = \Sigma p = 1$, then the mathematical expectation of x (or simply expectation of x) is denoted by $E(x)$ and is defined by

$$E(x) = p_1 x_1 + p_2 x_2 + p_3 x_3 + \ldots + p_n x_n = \sum_{i=1}^{n} p_i x_i = \Sigma px.$$

If $\phi(x)$ is a **probability density function*** corresponding to the variate x, then

$$E(x) = \int_{-\infty}^{\infty} x \phi(x) \, dx.$$

If $\phi(x)$ is such function of x that it takes values $\phi(x_1), \phi(x_2), \ldots$ when x takes the values of $x_1, x_2, \ldots$.

Let $p_1, p_2, \ldots$ be their respective probabilities, then the mathematical expectation of $\phi(x)$, denoted by $E[\phi(x)]$, is defined as

$$E[\phi(x)] = p_1 \phi(x_1) + p_2 \phi(x_2) + \ldots + p_n \phi(x_n) \quad \ldots(1)$$

where

$$\Sigma p = 1.$$

*Let the probability of the variate x falling in the infinitesimal interval $\left(x - \frac{1}{2} dx, x + \frac{1}{2} dx\right)$ be expressed in the form $\phi(x) \, dx$, where $\phi(x)$ is a continuous function of x. The function $\phi(x)$ is called the **probability density function.**

If $\phi(x) = x^r$. Then (1) gives,

$$E(x^r) = p_1 x_1^r + p_2 x_2^r + \ldots + p_n x_n^r \quad \ldots(2)$$

where $\Sigma p = 1$.

(2) is defined as the rth moment of the discrete probability distribution about $x = 0$ and it is denoted by μ_r'; and μ_r' is the expected value of the rth power of the variate.

In particular,

$$\mu_1' = E(x) = p_1 x_1 + p_2 x_2 + \ldots + p_n x_n = \sum_{i=1}^{n} p_i x_i.$$

Here if p_i is replaced by $\frac{f_i}{N}$, where $\sum_{i=1}^{n} f_i = N$, then

$$E(x) = \frac{\sum_{i=1}^{n} f_i x_i}{N} = \frac{\Sigma fx}{N} = \text{mean}.$$

Here $E(x)$ denotes the mean in random experiments when x take values $x_1, x_2, \ldots, x_n$ with frequencies $f_1, f_2, \ldots, f_n$.

This moment $E(x)$ is called the **mean value** of the variate or the distribution (just as for frequency).

§ 1.25. VARIANCE

Definition. The relation $\mu_2 = E[\{x - E(x)\}^2] = E(x^2) - [E(x)]^2$ denoted by Var (x), or $V(x)$ is defined as the **variance** of the distribution of x. μ_r and μ_r' have the same relation as of the frequency distribution. But here the expected value of the deviation of the variate x from its mean vanishes *i.e.*,

$$E[x - E(x)] = 0.$$

§ 1.26. MATHEMATICAL EXPECTATION FOR CONTINUOUS RANDOM VARIABLE

Suppose x is a continuous random variable with probability density function $f(x)$, then mathematical expectation $E(x)$ of x with certain restrictions is given by

$$E(x) = \int_{-\infty}^{\infty} x f(x)\, dx.$$

Example. *Show that* $|E(x)| \le E(|x|)$.

Solution. We know that (see § 1.7)

$$E(x) = \sum_{i=1}^{n} p_i x_i = \Sigma px$$

$\Rightarrow \quad |E(x)| \le |\Sigma px| \le \Sigma |px|$

$\Rightarrow \quad |E(x)| \le \Sigma pE(|x|)$

$\Rightarrow \quad |E(x)| \le E|x|.$ $\quad [\because p \ge 0]$

§ 1.27. EXPECTATION OF A SUM

Theorem. *The expectation of the sum of two variates is equal to the sum of their expectations, i.e., if x and y are two variates then*

$$E(x+y) = E(x) + E(y).$$

Proof. Let $x_1, x_2, \ldots, x_m$ be the values of the variate p and $p_1, p_2, \ldots, p_m$ be the corresponding probabilities. Also let $y_1, y_2, \ldots, y_n$ be the n values of y and $p_1', p_2', \ldots, p_n'$ be the corresponding probabilities then the sum $x + y$ is a random variable which can assume mn values $x_i + y_j$ $(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n)$. Now let p_{ij} be the probability of the variate x when x assumes the value x_i and the variate y assumes the value y_j.

Since x takes a definite value x_i, y takes one of the n values $y_1, y_2, \ldots, y_n$ so that the sum $\sum_{j=1}^{n} p_{ij}$ represents the probability p_i of x assuming the values x_i, that is, we shall have

$$\sum_{j=1}^{n} p_{ij} = p_i. \qquad \ldots(1)$$

Similarly discussing as above, $\sum_{i=1}^{m} p_{ij}$ represents the probability of y assuming the value y_j, that is, we shall have

$$\sum_{i=1}^{m} p_{ij} = p'_j. \qquad \ldots(2)$$

Thus, by definition, we have

$$E(x+y) = \sum_{i=1}^{m} \sum_{j=1}^{n} p_{ij}\,(x_i + y_j)$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} p_{ij}\, x_i + \sum_{i=1}^{m} \sum_{j=1}^{n} p_{ij}\, y_j$$

$$= \sum_{i=1}^{m} x_i \left(\sum_{j=1}^{n} p_{ij}\right) + \sum_{j=1}^{n} y_j \left(\sum_{i=1}^{m} p_{ij}\right)$$

$$= \sum_{i=1}^{m} x_i\, p_i + \sum_{j=1}^{n} y_j\, p_j' \qquad \text{[use (1) and (2)]}$$

$$= E(x) + E(y).$$

Hence $\mathbf{E(x+y) = E(x) + E(y).}$ $\ldots(3)$

The **general form** of the above theorem is

$$E(x + y + z + \ldots) = E(x) + E(y) + E(z) + \ldots.1$$

Example 1. ***Show that E(ax + by) = a E(x) + b E(y) where a and b are constants.***

Solution. With the notations of § 1.10, we have

$$E(ax + by) = \sum_{i=1}^{m} \sum_{j=1}^{n} p_{ij}\,(ax_i + by_j)$$

$$= a \sum_{i=1}^{m} x_i \left(\sum_{j=1}^{n} p_{ij}\right) + b \sum_{j=1}^{n} y_j \left(\sum_{i=1}^{m} p_{ij}\right)$$

$$= a \sum_{i=1}^{m} x_i\ p_i + b \sum_{j=1}^{n} y_j\ p_j'$$

$$= aE(x) + bE(y).$$

Example 2. ***E(ax + b) = a E(x) + b, a and b being constants.***

Solution. $E(ax+b) = \sum_{i=1}^{m} p_i\ (ax_i + b) = a \sum_{i=1}^{m} p_i\ x_i + b \sum_{i=1}^{m} p_i$

$$= a\ E(x) + b. \qquad \left[\because \sum_{i=1}^{m} p_i = 1\right]$$

◈ § 1.28. JOINT PROBABILITY DISTRIBUTION

Let x and y be two random variables. If x assume values $x_1, x_2, \ldots, x_m$ with corresponding probabilities $f(x_1), f(x_2), \ldots, f(x_m)$ and y assume values $y_1, y_2, \ldots, y_n$ with corresponding probabilities $g(x_1), g(x_2), \ldots, g(x_n)$, then the system of relations

$$p(x = x_i, y = y_j) = h\ (x_i, y_j)$$

is defined as the joint (or compound) probability distribution.

◈ § 1.29. INDEPENDENT VARIATES

Definition. Two random variates x and y on a sample space are called independent if the probability that either of them will assume a prescribed value does not depend on the value assumed by the other.

◈ § 1.30. PRODUCT OF EXPECTATIONS

Theorem. *The expectation of the product of two independent variates is equal to the product of their expectations*

i.e., $$E(xy) = E(x)\,.\,E(y).$$

Proof. Let x and y be two independent random variables. Let x assume m values $x_1, x_2, \ldots, x_m$, with corresponding probabilities $p_1, p_2, \ldots, p_m$. Let y assume n values $y_1, y_2, \ldots, y_n$ with corresponding probabilities $p_1', p_2', \ldots, p_n'$. Thus p_i is the probability of x assuming the value x_i and p_j' is the probability of y assuming the value y_j. Since x and y are independent, the probability that x assumes the value x_i and at the same time y assumes the value y_j is $= p_i\, p_j'$ (by theorem of compound probability).

$\therefore$ By definition, we have

$$E(xy) = \sum_{i=1}^{m} \sum_{j=1}^{n} p_i\, p_j'\ x_i\ y_j$$

$$= \sum_{i=1}^{m} p_i\ x_i \sum_{j=1}^{n} p_j'\ y_j = \sum_{i=1}^{m} p_i\ x_i\ E(y)$$

$$= E(y) \sum_{i=1}^{m} p_i\ x_i = E(y)\ E(x).$$

$\therefore$ $\mathbf{E(xy) = E(x)\ E(y).}$

The **general form** of the above theorem is

$$E(xyz \ldots) = E(x)\ E(y)\ E(z) \ldots$$

where $x, y, z, \ldots$ are independent variates.

§ 1.31. COVARIANCE

Definition. Let x and y be two random variables and $\bar{x}$ and $\bar{y}$ be their expected values (or means) respectively. The co-variance between x and y, denoted by cov (x, y), is defined as

$$\text{cov}(x, y) = E[(x - \bar{x})(y - \bar{y})]$$

Thus the covariance between x and y is the product of their deviations from the means.

Theorem. *The co-variance of two independent variates is zero.*

Proof. Let x and y be two independent variates, and $\bar{x}$, $\bar{y}$ be their expected values (or means) respectively, then

$$E(x - \bar{x}) = E(x) - E(\bar{x}) = \bar{x} - \bar{x} = 0$$

$$E(y - \bar{y}) = E(y) - E(\bar{y}) = \bar{y} - \bar{y} = 0.$$

$$\therefore \quad \text{cov}(x, y) = E(x - \bar{x})(y - \bar{y}) = E(x - \bar{x})\, E(y - \bar{y}) = 0 \,.\, 0 = 0.$$

Note :

$$\begin{aligned} \text{cov}(x, y) &= E\{(x - \bar{x})(y - \bar{y})\} \\ &= E(xy - \bar{x}y - \bar{y}x + \bar{x}\,\bar{y}) \\ &= E(xy) - E(\bar{x}y) - E(\bar{y}x) + E(\bar{x}\,\bar{y}) \\ &= E(xy) - \bar{x}\,E(y) - \bar{y}\,E(x) + \bar{x}\,\bar{y} \\ &= E(xy) - \bar{x}\,\bar{y} - \bar{y}\,\bar{x} + \bar{x}\,\bar{y} \\ &= E(xy) - \bar{x}\,\bar{y} \\ &= E(xy) - E(x)\,E(y). \end{aligned}$$

§ 1.32. VARIANCE OF n VARIATES

Theorem 1. *If the variance of variates $x_1, x_2, \ldots, x_n$ are $\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2$ respectively then to find the variance of u, where*

$$u = a_1 x_1 + a_2 x_2 + \ldots + a_n x_n$$

and $a_1, a_2, \ldots, a_n$ are constants.

Proof.

$$\begin{aligned} E(u) &= E(a_1 x_1 + a_2 x_2 + \ldots + a_n x_n) \\ &= a_1 E(x_1) + a_2 E(x_2) + \ldots + a_n E(x_n). \end{aligned}$$

$$\therefore \quad u - E(u) = a_1 [x_1 - E(x_1)] + a_2 [x_2 - E(x_2)] + \ldots + a_n [x_n - E(x_n)]$$

$$\Rightarrow \quad [u - E(u)]^2 = a_1^2 [x_1 - E(x_1)]^2 + a_2^2 [x_2 - E(x_2)]^2 + \ldots + a_n^2 (x_n - E(x_n)]^2 + 2a_1 a_2 [x_1 - E(x_1)][x_2 - E(x_2)] + \ldots.$$

Take expected values of both sides, we get

$$\text{var}(u) = a_1^2 \text{ var}(x_1) + a_2^2 \text{ var}(x_2) + \ldots + a_n^2 \text{ var}(x_n) + 2a_1 a_2 \text{ cov}(x_1, x_2) + \ldots + 2a_{n-1} a_n \text{ cov}(x_{n-1}, x_n).$$

Cor. 1. If $a_1 = a_2 = 1$, $a_3 = a_4 = \ldots = a_n = 0$, then

$$\text{var}(x_1 + x_2) = \text{var}(x_1) + \text{var}(x_2) + 2 \text{ cov}(x_1, x_2).$$

Cor. 2. If $a_1 = 1, a_2 = -1, a_3 = a_4 = \ldots = a_n = 0$, then

$$\text{var}(x_1 - x_2) = \text{var}(x_1) + \text{var}(x_2) - 2 \text{ cov}(x_1, x_2).$$

Cor. 3. If $x_1, x_2, \ldots, x_n$ are independent, then

$$\text{var}(u) = a_1^2 \text{ var}(x_1) + a_2^2 \text{ var}(x_2) + \ldots + a_n^2 \text{ var}(x_n).$$

Theorem 2. *There is a series of n independent trials and p_i is the probability of the ith trial. To find the mean number of successes and the variance of the number of successes.*

Proof. Suppose, that the variable associated with the ith trial is x_i and it has the value 1 in case of success and 0 is case of failure. Total number of independent trials are n. Let m be the number of successes

$$m = x_1 + x_2 + \ldots + x_n.$$

$$\therefore \quad E(m) = E(x_1 + x_2 + \ldots + x_n) = E(x_1) + E(x_2) + \ldots + E(x_n). \quad \ldots(1)$$

Since x_i assume value 1 with probability p_i, and x_i assume value 0 with probability $(1 - p_i)$, therefore, we have

$$E(x_i) = 1,\ p_i + 0\,(1 - p_i) = p_i. \quad \ldots(2)$$

Putting values in (1), we get

$$E(m) = p_1 + p_2 + \ldots + p_n = \sum_{i=1}^{n} p_i. \quad \ldots(3)$$

Again $\quad \text{var}\,(m) = \text{var}\,(x_1 + x_2 + \ldots + x_n)$

$$= \text{var}\,(x_1) + \text{var}\,(x_2) + \ldots + \text{var}\,(x_n) \quad \ldots(4)$$

Since $x_i^2 = 1$ for success and $x_i^2 = 0$ for failure,

$$\therefore \quad E(x_i^2) = 1 \,.\, p_i + 0\,(1 - p_i) = p_i \quad \ldots(5)$$

$$\therefore \quad \text{var}\,(x_i) = E\,[x_i - E(x_i)]^2 = E(x_i^2) - \{E(x_i)]^2$$

$$= E(x_i^2) - \{E(x_i)\}^2 = p_i - p_i^2 \quad \text{[use (2) and (5)]}$$

$$= p_i\,(1 - p_i) = p_i\,q_i \quad \text{where } q_i = 1 - p_i$$

Putting values in (4).

$$\text{var}\,(m) = p_1\,q_1 + p_2\,q_2 + \ldots + p_n\,q_n = \sum_{i=1}^{n} p_i\,q_i. \quad \ldots(6)$$

Cor. 1. If $p_1 = p_2 = p_3 = \ldots = p_n = p$ (say), then (6) gives

$$\text{var } m = pq + pq + \ldots + \text{upto } n \text{ terms} = npq.$$

Cor. 2. *To prove var* $m \le \left(\frac{1}{4}\right)$.

We have $\quad pq = p\,(1 - p) = p - p^2 = p - p^2 + \frac{1}{4} - \frac{1}{4}$ $\quad$ [Add and subtract $\frac{1}{4}$]

$$= \frac{1}{4} - \left(p^2 - p + \frac{1}{4}\right)$$

$$= \frac{1}{4} - \left(p - \frac{1}{2}\right)^2. \quad \ldots(7)$$

Since $\left(p - \frac{1}{2}\right)^2$ is always non-negative, hence (7) implies that

$$pq \le \frac{1}{4}.$$

ILLUSTRATIVE EXAMPLES

Example 1. (a) ***What is the expected value of the number of points that will be obtained in a single throw with an ordinary die ? Find variance also.***

Solution. The variate *i.e.,* numbers showing on a die assumes the values 1, 2, 3, 4, 5, 6 and probability in each case is $\frac{1}{6}$.

$\therefore$ given probability distributions is as follows :

x :	1	2	3	4	5	6
p :	1/6	1/6	1/6	1/6	1/6	1/6

$$\therefore \quad E(x) = \sum_{i=1}^{6} p_i\, x_i = p_1\, x_1 + p_2\, x_2 + p_3\, x_3 + \ldots + p_6\, x_6$$

$$= \frac{1}{6}.1 + \frac{1}{6}.2 + \frac{1}{6}.3 + \frac{1}{6}.4 + \frac{1}{6}.5 + \frac{1}{6}.6$$

$$= \frac{1}{6}[1+2+3+4+5+6] = \frac{1}{6} \times 21 = \frac{7}{2}.$$

Also var $(x) = E(x^2) - [E(x)]^2$

$$= \frac{1}{6}(1^2 + 2^2 + \ldots + 6^2) - (7/2)^2 = 35/82.$$

Example 1. (b) ***Two fair dice are tossed. Find the expected value of sum of the obtained number of points.***

Solution. Let x and y denote the number on first and second die respectively. *Then similar to Ex. 1 (a), we have,*

$$E(x) = 7/2 \text{ and } E(y) = 7/2.$$

$\therefore$ The required expected value $= E(x+y)$

$$= E(x) + E(y) = \frac{7}{2} + \frac{7}{2} = 7.$$ **Ans.**

Example 2. ***A bag contains two one rupee coins and 3 twenty paise coins. A person is asked to draw two coins randomly. Find the value of his expectation.***

Solution. The possible ways of drawing two coins randomly are as follows :

(i) Both are one rupee coins, its probability $= \frac{^2C_2}{^5C_2} = \frac{1}{10}$.

(ii) Both are 20 paise coins, its probability $= \frac{^3C_2}{^5C_2} = \frac{3.2}{5.4} = \frac{6}{20} = \frac{3}{10}$.

(iii) One is 1 rupee coin and other is 20 paise coin, its probability

$$= \frac{^2C_1 \cdot {^3C_1}}{^5C_2} = \frac{2 \times 3 \times 2 \times 1}{5 \times 4} = \frac{3}{5}.$$

We know that two, one rupee coins = 10 twenty paise coins.

$\therefore$ 1, one rupee coin and 1, twenty paise coin = 6 twenty paise coins

$\therefore$ The required expectation $= \Sigma p_i \ x_i$

$$= \left[\left(\frac{1}{10} \times 10\right) + \left(\frac{3}{10} \times 2\right) + \left(\frac{3}{5} \times 6\right)\right] \text{ twenty paise coins}$$

$$= 1 + \frac{3}{5} + \frac{18}{5} = \frac{26}{5} \text{ twenty paise coin}$$

$$= (26/5) \times 20 \text{ Paise} = 104 \text{ Paise} = 1 \cdot 04 \text{ Rs.}$$

Ans.

Example 3. ***Thirteen cards are drawn simultaneously from a pack of 52. If aces count 1, face cards 10 and others according to their denomination, find the expectations of the total score on 13 cards.***

Solution. Thirteen cards are drawn out of 52 cards. Let x_i be the number corresponding to the ith card. Therefore x_i assumes the values 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 10, 10, 10 (since knave, queen, king count 10), each with probability 1/13. Let $E(x_i)$ be the expectation of drawing a single card in a single draw.

$$\text{Then } E(x_i) = \sum_{i=1}^{13} p_i \ x_i$$

$$= \frac{1}{13}.1 + \frac{1}{13}.2 + \frac{1}{13}.3 + \ldots + \frac{1}{13}.9 + \frac{1}{13}.10 + \frac{1}{13}.10 + \frac{1}{13}.10 + \frac{1}{13}.10$$

$(1/13)(1 + 2 + 3 + \ldots + 9 + 10 + 10 + 10 + 10) = 85/13.$

Hence the required expectation of drawing 13 cards $= 13 \times (85/13) = 85.$

Example 4. ***In four tosses of a coin. Let x be the number of heads. Calculate the expected values of x and x^2.***

Solution. In four tosses of a coin, there is only one way of getting all heads, *i.e.*,

$$P(x = 4) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{16}$$

$$P(x = 3) = \frac{{}^4C_3}{16} = \frac{4}{16} = \frac{1}{4}$$

$$P(x = 2) = \frac{{}^4C_2}{16} = \frac{6}{16} = \frac{3}{8}$$

$$P(x = 1) = \frac{{}^4C_1}{16} = \frac{4}{16} = \frac{1}{4}$$

$$P(x = 0) = \frac{{}^4C_0}{16} = \frac{1}{16}.$$

Therefore the probability distribution of x is as follows :

x :	0	1	2	3	4
$P(x)$:	1/16	1/4	3/8	1/4	1/16

$\therefore$ The required expectation $= E(x) = \sum_{x=0}^{4} x \ p(x)$

$$= 0.\frac{1}{16} + 1.\frac{1}{4} + 2.\frac{3}{8} + 3.\frac{1}{4} + 4.\frac{1}{16} = 2$$

$$E(x^2) = \sum_{x=0}^{4} x^2 \cdot p(x) = 0 \cdot \frac{1}{16} + (1)^2 \cdot \frac{1}{4} + (2)^2 \cdot \frac{3}{8} + (3)^2 \cdot \frac{1}{4} + (4)^2 \cdot \frac{1}{16}$$

$$= \frac{1}{4} + \frac{3}{2} + \frac{9}{4} + 1 = \frac{20}{4} = 5.$$ **Ans.**

Example 5. ***Find $E(x)$, $E(x^2)$, $E\{(x - \bar{x})^2\}$ for the following probability distribution :***

x :	8	12	16	20	24
$P(x)$:	1/8	1/6	3/8	1/4	1/12

Solution. The mean of probability distribution is

$$E(x) = \Sigma\, x \,.\, p(x) = 8.\frac{1}{8} + 12.\frac{1}{6} + 16.\frac{3}{8} + 20.\frac{1}{4} + 24.\frac{1}{12} = 16.$$

$$E(x^2) = \Sigma x^2 \,.\, p(x)$$

$$= (8)^2 \cdot \frac{1}{8} + (12)^2 \cdot \frac{1}{6} + (16)^2 \cdot \frac{3}{8} + (20)^2 \cdot \frac{1}{4} + (24)^2 \cdot \frac{1}{12}$$

$= 276 =$ The second moment about the origin

$$E\,\{(x - \bar{x})^2\} = E(x - \bar{x})^2 \; p(x)$$

$$= (8-16)^2 \cdot \frac{1}{8} + (12-16)^2 \cdot \frac{1}{6} + (16-16)^2 \cdot \frac{3}{8} + (20-16)^2 \cdot \frac{1}{4} + (24-16)^2 \cdot \frac{1}{12}$$

$= 20 =$ variance of the distribution.

Example 6. ***A bag contains 5 black, 6 white and 7 red balls. Four balls are drawn at random from it. If x denote the numbers of white balls then find $E(x)$.***

Solution. The total number of balls in the bag $= 5 + 6 + 7 = 18$.

Now x assumes the values 0, 1, 2, 3, 4.

Thus $$P(x) = \frac{{}^6C_x \times {}^{12}C_{4-x}}{{}^{18}C_4}, \text{ where } x = 0, 1, 2, 3, 4$$

Then $$P(x=0) = \frac{{}^6C_0 \times {}^{12}C_4}{{}^{18}C_4} = \frac{495}{3060}$$

$$P(x=1) = \frac{{}^6C_1 \times {}^{12}C_3}{{}^{18}C_4} = \frac{1320}{3060}$$

$$P(x=2) = \frac{{}^6C_2 \times {}^{12}C_2}{{}^{18}C_4} = \frac{990}{3060}$$

$$P(x=3) = \frac{{}^6C_3 \times {}^{12}C_1}{{}^{18}C_4} = \frac{240}{3060}$$

$$P(x=4) = \frac{{}^6C_4 \times {}^{12}C_0}{{}^{18}C_4} = \frac{15}{3060}$$

$$\therefore \quad E(x) = \sum_{x=0}^{4} x \,.\, P(x)$$

$$= 0 \,.\, \frac{495}{3060} + 1 \,.\, \frac{1320}{3060} + 2 .\frac{990}{3060} + 3 \,.\, \frac{240}{3060} + 4 \,.\, \frac{15}{3060} = \frac{4080}{3060} = \frac{68}{51}.$$

Example 7. ***A bag contains a coin of value M and a number of other coins whose aggregate value is m. A person draws one at a time till he draws the coin M, find his expectation.***

Solution. Let the bag contain $n + 1$ coins in all, one is of value M and other n coins, each of value m / n. The person can draw the M valued coin in first chance, second chance, ..., $(n + 1)$th chance and their respective probabilities are

$$\frac{1}{n+1}, \left(1 - \frac{1}{n+1}\right) . \frac{1}{n}, \left(1 - \frac{1}{n+1}\right)\left(1 - \frac{1}{n}\right) . \frac{1}{n-1}, \ldots$$

i.e.,

$$\frac{1}{n+1}, \frac{1}{n+1}, \frac{1}{n+1} \ldots .$$

The values corresponding to the random variable x respectively are

$$M, \frac{m}{n} + M, \frac{2m}{n} + M, \ldots, \frac{nm}{n} + M.$$

$\therefore$ Required expectation $= E(x)$

$$= M \times \frac{1}{(n+1)} + \left(\frac{m}{n} + M\right) . \frac{1}{n+1} + \ldots + \left(\frac{mn}{n} + M\right) . \frac{1}{n+1}$$

$$= \frac{1}{(n+1)} \left[M(n+1) + \frac{m}{n} (1 + 2 + \ldots + n) \right]$$

$$= \frac{1}{(n+1)} \left[M(n+1) + \frac{m}{n} . \frac{n\,(n+1)}{2} \right] = M + \frac{m}{2}.$$

Ans.

Example 8. ***Balls are taken out of an urn containing a white and b black balls until the first black ball is drawn. Show that the expectation of the number of white balls preceding the first black ball is $a / (1 + b)$.***

Solution. Let

x_1 = The number of white balls before the first black ball is drawn

x_2 = The number of white balls before the first and second black balls are drawn.

Similarly x_i = the number of white balls following the $(i - 1)$ th black ball and preceding the ith black ball.

Let x_{b+1} be the number of white balls left after the bth black ball has been drawn.

Then $\quad x_1 + x_2 + \ldots . + x_{b+1} = a$

$\Rightarrow \quad E(x_1 + x_2 + \ldots + x_{b+1}) = E(a)$

$\Rightarrow \quad E(x_1) + E(x_2) + \ldots + E(x_{b+1}) = E(a). \qquad \ldots(1)$

Since the number of ways to draw the $(a + b)$ balls one by one $= \dfrac{(a+b)\,!}{a\,!\,b\,!}$.

The probability of every system of $x_1, x_2, \dots, x_{b+1}$ is $= \dfrac{1}{\dfrac{(a+b)!}{a!\,b!}} = \dfrac{a!\,b!}{(a+b)!}$.

Thus we can say that the probability of every system of numbers $x_1, x_2, \dots, x_{b+1}$ is same *i.e.*,

$$E(x_1) = E(x_2) = \dots = E(x_{b+1}) = k \text{ (say).} \quad \dots(2)$$

$\therefore$ From equation (1) and (2), we have,

$$k + k + \dots + (b+1) \text{ terms} = a$$

$\Rightarrow$ $(b+1)\,k = a \Rightarrow k = a/(b+1)$.

$\therefore$ The required expectation $= E(x_1) = a/(b+1)$.

Example 9. ***A box contains a white and b black balls; c balls are drawn. Show that the expectation of the number of white balls drawn is ca / (a + b).***

Solution. Consider a variate x_i defined as follows

$$x_i = x_1, x_2, x_3, \dots, x_c.$$

In this case, we have

$$x_i = 1 \text{ if the } i\text{th ball drawn is white}$$
$$= 0 \text{ if } i\text{th ball drawn is black.}$$

If x is the number of white balls out of c drawn balls, then

$$x = x_1 + x_2 + x_3 + \dots + x_c. \quad \dots(1)$$

Then the probability to draw 1 white ball from the box

$$= P(x_i = 1) = a/(a+b)$$

and the probability to draw the black ball from the box $= P(x_i = 0) = b/(a+b)$.

$$\therefore \quad E(x_i) = \{P(x_i = 1)\}\,1 + \{P(x_i = 0)\}\,.\,0$$
$$= \frac{a}{a+b}\,.\,1 + \frac{b}{a+b}\,.\,0 = \frac{a}{a+b}$$

i.e., $\quad E(x_i) = \dfrac{a}{a+b}$ and it is free from i.

Now from (1), we get,

$$E(x) = E(x_1) + E(x_2) + E(x_3) + \dots + E(x_c)$$
$$= \left(\frac{a}{a+b}\right) + \left(\frac{a}{a+b}\right) + \dots c \text{ terms}$$
$$= \frac{ca}{a+b}.$$

Example 10. ***Show that the expectation of the number of failures preceding first success in an infinite series of independent trials with constant probability p of success in a trial is (1/ p) – 1 or q / p.***

Solution. Let the random variate X denote the number of failures preceding the first success and q be the probability of failures such that $q = 1 - p$. Now X takes the values $0, 1, 2, 3, \dots,$ and their probabilities are $p, pq, q^2 p, q^3 p, \dots,$ respectively.

$$\therefore \quad E(X) = p \,.\, 0 + qp \,.\, 1 + q^2 p \,.\, 2 + q^3 p \,.\, 3 + \ldots$$
$$= pq\,(1 + 2q + 3q^2 + \ldots)$$
$$= pq\,(1 - q)^{-2} = pq\,(1/\,p^2) = q/\,p$$

or
$$E(X) = \frac{q}{p} = \frac{1-p}{p} = \frac{1}{p} - 1.$$

Example 11. ***A random variate X takes the values***
$$x_k = (-1)^k\, 2^k / k \text{ for } k = 1, 2, 3, \ldots,$$
with probabilities $p_k = 1/2^k$, ***find*** $E(X)$.

Solution. $E(x) = p_1\, x_1 + p_2\, x_2 + p_3\, x_3 + \ldots = \sum_{k=1}^{\infty} x_k\, p_k$

$$= \sum_{k=1}^{\infty} (1/2^k) \,.\, (-1)^k\, 2^k / k = \sum_{k=1}^{\infty} (-1)^k / k$$
$$= -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \ldots$$
$$= -\left[1 - \frac{1}{2} + \frac{1}{3} - \ldots\right]$$
$$= -\log(1+1) = -\log 2 = \log\frac{1}{2}.$$

Example 12. ***If n fair dice are tossed and X denotes the sum of points on n dice, then find the expectation of X.***

Solution. Let x_i denote the number of the ith dice, then $i = 1, 2, 3, \ldots, n$.

The sum of points on n dice is
$$X = x_1 + x_2 + x_3 + \ldots + x_n.$$

Therefore $E(X) = E(x_1 + x_2 + \ldots + x_n) = E(x_1) + E(x_2) + \ldots + E(x_n)$.

But for ith dice, x_i can assume the values 1, 2, 3, 4, 5, 6 each with probability $\frac{1}{6}$, thus

$$E(x_i) = \frac{1}{6} \,.\, 1 + \frac{1}{6} \,.\, 2 + \frac{1}{6} \,.\, 3 + \frac{1}{6} \,.\, 4 + \frac{1}{6} \,.\, 5 + \frac{1}{6} \,.\, 6 = \frac{21}{6} = \frac{7}{2}$$
$$\therefore \quad E(x_1) = E(x_2) = \ldots = E(x_n) = 7/2.$$

Hence $\quad E(x) = \frac{7}{2} + \frac{7}{2} + \ldots \; n \text{ times} = \frac{7n}{2}.$

Example 13. ***Find the expected value of the product of points on n dice.***

Solution. Let x_i denote the number of points on ith dice, then the product of points on n dice is $= x_1 \cdot x_2 \cdot x_3 \cdot \ldots \cdot x_n$.

For every single dice, we have

$$E(x_i) = \frac{7}{2},\; i = 1, 2, 3, \ldots, n$$ [See Ex. 1 (a) above]

$$\therefore \quad E(x_1\, x_2 \ldots x_n) = E(x_1) \,.\, E(x_2) \ldots E(x_n)$$
$$= \frac{7}{2} \cdot \frac{7}{2} \cdot \frac{7}{2} \ldots n \text{ factors}$$
$$= (7/2)^n.$$

Example 14. ***A coin is tossed until the head appears. What is expectation of the number of tosses ?***

Solution. Let x denote the number of tosses until the first head appears. The values of variates with their probabilities are tabulated as follows :

$$\text{event}: \ H, TH, T\,T\,H, \ldots$$

$$x: \ 1, \ 2, \ 3, \ldots$$

$$p(x): \ \frac{1}{2}, \left(\frac{1}{2}\right)^2, \left(\frac{1}{2}\right)^3, \ldots$$

$$\therefore \quad E(x) = \frac{1}{2} \cdot 1 + \left(\frac{1}{2}\right)^2 2 + \left(\frac{1}{2}\right)^3 \cdot 3 + \ldots$$

(Arithmetico-geometric series)

Also
$$\frac{1}{2} E(x) = \left(\frac{1}{2}\right)^2 \cdot 1 + \left(\frac{1}{2}\right)^3 \cdot 2 + \ldots$$

Subtracting :
$$\frac{1}{2} E(x) = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots$$

$$= \frac{\frac{1}{2}}{1 - \left(\frac{1}{2}\right)} = 1 \Rightarrow E(x) = 2.$$

Ans.

§ 1.33. CORRELATION COEFFICIENTS

Consider the annual rainfall in a certain region and the agricultural yield so as to find some sort of relation between the two-whether increase in rainfall results in greater agricultural yield.

We may find that a change in one variable results in change of second variable or does not have any effect on second variable.

Definition. *Whenever there exists a relationship between two variables such that a change in one variable results in a positive or negative change in the other and also greater change in one variable results in a corresponding greater change in the other, the relationship is called correlation and the two variables are called correlated.*

According to Croxton and Cowden :

"*When the relationship is of quantitative nature, the appropriate statistical tool for discovering and measuring the relationship and expressing it in brief formula is known as* **correlation.**"

According to Professor King. "*If it is proved true that in a large number of instances two variables tend always to fluctuate in the same or in opposite directions, we consider that the fact is established and that a relationship exists. This relationship is called* **correlation.**"

Positive and Negative Correlation :

Definition. Two variables are called positively correlated if corresponding to an increase (or decrease) in one variable results in an increase (or decrease) in the other.

Two variables are called negatively correlated if corresponding to an increase (or decrease) in one variable results in decrease (or increase) in the other.

Examples. (i) Demand and price are positively correlated.

(ii) Supply and price are negatively correlated.

§ 1.34. KARL PEARSON'S COEFFICIENT OF CORRELATION

It is the best mathematical method to find correlation since it is based on mean and standard deviation. By this method we can find not only the direction and magnitude of correlation but also its positive measure. Karl Pearson (1867–1936) developed a formula (in 1890) called correlation coefficient.

Correlation coefficient between two random variables X and Y, denoted by r, is a numerical measure of linear relationship between X and Y and is defined (by Karl Pearson) by

$$r = \frac{\Sigma xy}{\sqrt{(\Sigma x^2 \ \Sigma y^2)}} = \frac{\Sigma xy}{n\sigma_x \ \sigma_y} = \frac{p}{\sigma_x \ \sigma_y}$$

where $x = X - M_x$ = deviation of variable X measured from its mean M_x

$y = Y - M_y$ = deviation of variable Y measured from its mean M_y

σ_x = standard deviation of X-series

σ_y = standard deviation of Y-series

$p = \{\Sigma (xy) / n\}$

n = number of pairs of two variables

r is called the **product moment correlation coefficient.**

§ 1.35. CHANGE OF ORIGIN AND SCALE

Consider two new variables defined by $u = \frac{X - X_0}{h}, v = \frac{Y - Y_0}{k}$; then

$$X = uh + X_0, \ Y = kv + Y_0.$$

If $\overline{u}$ and $\overline{v}$ denote the means of u series and v series respectively, then

Mean of X series $M_X = \overline{u}h + X_0$;

Mean of Y series $M_Y = \overline{v}k + Y_0$.

Also $\sigma_x{}^2 = E(X - M_x)^2 = h^2 \ E(u - \overline{u})^2 = h^2 \sigma_u{}^2$.

Similarly, $\sigma_y{}^2 = k^2 \sigma_v{}^2$. Therefore, $\sigma_x = |\, h \,| \, \sigma_u, \sigma_y = |\, k \,| \, \sigma_v$ since σ_x and σ_y are always positive irrespective of the signs of h and k.

$$\text{Now, } r_{XY} = \frac{\Sigma xy}{n\sigma_x \sigma_y} = \frac{\Sigma (X - M_X)(Y - M_Y)}{n\sigma_x \sigma_y} \qquad \text{[See § 1.34, take } r_{XY} \text{ for } r]$$

$$= \frac{hk\,\Sigma (u - \bar{u})(v - \bar{v})}{n\,|h||k|\,\sigma_u\,\sigma_v} = \frac{hk}{|h||k|} r_{uv}$$

$$= r_{uv}$$ [if h and k are of same sign, which is possible by changing the units of measurement only]

Hence coefficient of correlation is independent of X_0, Y_0 (change of origin) and h, k (change of scale).

Remark 1. If h and k are of opposite signs, then we have

$$r_{xy} = -r_{uv}.$$

Remark 2. Coefficient of correlation has no dimension and is a real number.

Remark 3. Since coefficient of correlation is independent of change of origin and scale, therefore, this property is useful in computation of r, as we can conveniently select any origin and scale.

Problem. Define coefficient of correlation. Show that the coefficient of correlation is independent of change of scale and origin of the variables.

◈ § 1.36. DEGREE OF CORRELATION

Since coefficient of correlation determines numerical measure of correlation, therefore they may be positive or negative.

Measure of Correlation at a Glance

S. No.	Measure of Correlation	Positive	Negative
1	Perfect	+ 1	− 1
2	High Degree	Between + 0·75 and + 1	Between − 0·75 and − 1
3	Mediate Degree	Between + 0·5 and + 0·75	Between − 0·5 and − 0·75
4	Low Degree	Between 0 and + 0·5	Between − 0 and − 0·5
5	No Correlation	0	0

ILLUSTRATIVE EXAMPLES

Example 1. ***The students got the following percentage of marks in principles of Economics and Statistics :***

Roll No. :	1	2	3	4	5	6	7	8	9	10
Marks in Economics :	78	36	98	25	75	82	90	62	65	39
Marks in Statistics :	84	51	91	60	68	62	86	58	53	47

Calculate the coefficient of Correlation.

Solution. Suppose the marks of two subjects are denoted by variables X and Y. Then the mean of X-series

$$= M_x = \Sigma x / n = 650/10 = 65$$

and the mean for Y-series $= M_y = \Sigma y / n = 660/10 = 66.$

If the deviations of X's and Y's from their respective means be x and y, then the data may be arranged as shown in the table :

X	Y	$x = X - M_x$	$y = Y - M_y$	x^2	y^2	xy
78	84	13	18	169	324	234
36	51	−29	−15	841	225	435
98	91	33	25	1089	625	825
25	60	−40	−6	1600	36	240
75	68	10	2	100	4	20
82	62	17	−4	289	16	−68
90	86	25	20	625	400	500
62	58	−3	−8	9	64	24
65	53	0	−13	0	169	0
39	47	−26	−19	676	361	494
650	660	0	0	5398	2224	2704

Thus we have,

$$\Sigma x^2 = 5398,\ \Sigma y^2 = 2224,\ \Sigma xy = 2704.$$

∴ The coefficient of correlation

$$r = \frac{\Sigma xy}{\sqrt{\{(\Sigma x^2\ \Sigma y^2)\}}} = \frac{2704}{\sqrt{\{(5398 \times 2224)\}}}$$

$$= \frac{2704}{73 \cdot 4 \times 47 \cdot 1} = \frac{2704}{3457 \cdot 14} = \frac{270400}{345714} = 0 \cdot 78 \text{ (nearly).}$$

Example 2. ***Calculate the Karl Pearson's coefficient of correlation between X and Y series :***

X :	**17**	**18**	**19**	**19**	**20**	**20**	**21**	**21**	**22**	**23**
Y :	**12**	**16**	**14**	**11**	**15**	**19**	**22**	**16**	**15**	**20**

Solution. The data may be arranged in the following form :

X	Y	$x = X - M_x$	$y = Y - M_y$	x^2	y^2	xy
17	12	−3	−4	9	16	12
18	16	−2	0	4	0	0
19	14	−1	−2	1	4	2
19	11	−1	−5	1	25	5
20	15	0	−1	0	1	0
20	19	0	3	0	9	0
21	22	1	6	1	36	6
21	16	1	0	1	0	0
22	15	2	−1	4	1	−2
23	20	3	4	9	16	12
200	160	0	0	30	108	35

If the mean of X's and Y's are M_x and M_y respectively, then

$$M_x = \frac{\Sigma X}{n} = \frac{200}{10} = 20 \text{ and } M_y = \frac{\Sigma Y}{n} = \frac{160}{10} = 16.$$

If the standard deviations of X's and Y's are σ_x and σ_y then

$$\sigma_x = \sqrt{(\Sigma x^2 / n)} = \sqrt{(30/10)} = \sqrt{3} = 1\cdot 73$$

$$\sigma_y = \sqrt{(\Sigma y^2 / n)} = \sqrt{(108/10)} = \sqrt{(10\cdot 8)} = 3\cdot 28.$$

$\therefore$ The coefficient of correlation r is given by

$$r = \frac{\Sigma xy}{\sqrt{\{(\Sigma x^2\ \Sigma y^2)\}}} = \frac{\Sigma xy}{n\ \sigma_x\ \sigma_y}$$

$$= \frac{35}{10 \times 1\cdot 73 \times 3\cdot 28} = \frac{3\cdot 5}{1\cdot 75 \times 3\cdot 25} = 0\cdot 616.$$

Example 3. ***A computer while calculating the correlation coefficient between two variates x and y from 25 pairs of observations obtain the following constants :***

$$n = 25, \Sigma x = 125, \Sigma x^2 = 650, \Sigma y = 100, \Sigma y^2 = 960, \Sigma xy = 508.$$

It was however, later discovered at the time of checking that he had copied down two pairs as $\begin{array}{c|c} x & y \\ 6 & 14 \\ 8 & 6 \end{array}$ ***while the correct values were*** $\begin{array}{c|c} x & y \\ 8 & 12 \\ 6 & 8 \end{array}$. ***Obtain the correct value of the correlation coefficient.***

Solution. Since $8 + 6 = 6 + 8$ and $8^2 + 6^2 = 6^2 + 8^2, 14 + 6 = 12 + 8$ therefore, on account of mistake there is no change in Σx, Σy and Σx^2. But there will be change in Σy^2 and Σxy.

In Σy^2, instead of old value $14^2 + 6^2 = 232$, the new value $12^2 + 8^2 = 208$ is to be substituted.

In Σxy, instead of old value $(6 \times 14 + 8 \times 6 = 132)$, the new value $(8 \times 12 + 6 \times 8 = 144)$ is to be substituted.

Σy^2 (correct value) $= 960 - 232 + 208 = 960 - 24 = 936.$

Similarly Σxy (correct value) $= 508 - (132) + 144 = 508 + 12 = 520.$

Hence the coefficient of correlation r is given by

$$r = \frac{\Sigma xy}{\sqrt{(\Sigma x^2\ \Sigma y^2)}} = \frac{520}{\sqrt{\{(650 \times 936)\}}} = \frac{520}{\sqrt{(608400)}} = \frac{520}{780} = \frac{2}{3} = 0\cdot 666.$$

EXERCISE 1 (F)

1. (a) If x and y are variates and a and b are constants, then show that :

   (i) $E(a) = a$, (ii) $E(ax) = a\ E(x)$,

   (iii) $E(ax + by) = aE(x) + bE(y)$, (iv) $\text{var}\ (ax + b) = a^2\ \text{var}\ (x)$,

   (v) $\text{cov}\ (ax, by) = ab\ \text{cov}\ (x, y)$, (vi) $\text{cov}\ (x + a, y + a) = \text{cov}\ (x, y)$,

   (vii) $\text{cov}\ (x, x) = \text{var}\ (x)$.

   (b) Define the mathematical expectation of a random variate and prove that $V(x + y) = V(x) + V(y)$ where x and y are independent variates.

   (c) If X_1 and X_2 are two independent random variables then show that

   $$E(X_1 X_2) = E(X_1)\ E(X_2)$$

   where $E(X)$ denotes the mathematical expectation of random variable X.

2. Prove that $\text{cov}(ax+by, cx+dy) = ac \text{ var}(x) + bd \text{ var}(y) + (ad+bc) \text{ cov}(x, y)$.
3. In a lottery m tickets are drawn out of n tickets numbered from 1 to n. What is the expectation of the sum of numbers drawn ?

   Also find the variance.
4. Balls are taken one by one out of an urn containing a white and b black balls until the first white ball is drawn. Show that the expectation of the number of black balls preceding the first white ball is $\{b/(a+1)\}$.
5. Three urns contain respectively 3 green, 2 white balls; 5 green and 6 white balls, 2 green and 4 white balls. One ball is drawn from each urn. Find the expected number of white balls drawn out.

   [**Hint** : $0\,(3/5 \,.\, 5/11 \,.\, 1/6) + 1\,.\,(2/5 \,.\, 2/5 \,.\, 2/6 + 3/5 \,.\, 5/11 \,.\, 4/6 + 3/5 \,.\, 6/12 \,.\, 2/6)$
   $+ 2\,.\,(2/5 \,.\, 6/11 \,.\, 2/6 + 2/5 \,.\, 5/11 \,.\, 4/6 + 3/5 \,.\, 6/11 \,.\, 4/6)$
   $+ 3\,.\,(2/5 \,.\, 6/11 \,.\, 4/6) = 266/165$]
6. Show by an example that the mathematical expectation need not be finite.

   [**Hint** : Consider the discontinuous random variable x having values $0!, 1!, 2!, 3!, \ldots \infty$ and let the probability for it be $\dfrac{e^{-1}}{x!}$, then

$$E(x!) = 0!\frac{e^{-1}}{0!} + 1!\frac{e^{-1}}{1!} + 2!\frac{e^{-1}}{2!} + \ldots$$
$$= e^{-1} + e^{-1} + e^{-1} + \ldots$$
$$= e^{-1}(1+1+1+1+\ldots\infty)$$
$$= e^{-1}\sum_{0}^{\infty} 1, \text{ which is not finite.}]$$

7. Two unbiased dice are thrown. Find the expected value of the sum of numbers of points drawn.
8. Let x denote the profit that a man can earn in business. He may earn Rs. 2000 with the probability 0·5, he may lose Rs. 1500 with probability 0·3 and he may neither lose not gain with probability 0·2. Find the value of $E(x)$.

   [**Hint** : $E(x) = -1500 \times (0\cdot3) + 0 \times (0\cdot2) + 2000 \times (0\cdot5)$]
9. Let x denote the profit that a man can earn in business. He may earn Rs. 2,800 with probability $\frac{1}{2}$, he may lose Rs. 5,000 with probability 3/10 and he may neither lose nor gain with probability 1/5. Show that the mathematical expectation is -100.
10. If x is a random variable for which $E(x) = 10$ and var $(x) = 25$. Find the positive values of a and b such that $y = ax - b$ has expectation 0 and variance 1.
11. Show that
$$E(x) = \sum_{x=1}^{\infty} P(X \geq x), x = 0, 1, 2, \ldots.$$
12. A player throwing an ordinary die is to receive Rs. $1/2^n$ when n is the number of throws that he takes to throw the first sum. Find his expectation.

ANSWERS

3. $\frac{1}{2}m(n+1); \frac{1}{12}m(n+1)(n-2)$ 7. 7 10. $a = 1/5, b = 2$.

EXERCISE 1 (G)

Objective Type Questions

1. Variance μ_2 is equal to :
   (a) $\mu_2 = E(x)$ (b) $\mu_2 = \{E(x)\}^2$
   (c) $\mu_2 = E\{x - E(x)\}$ (d) $\mu_2 = E\{x - E(x)\}^2$.
2. The value of $E(x - \bar{x})$ is :
   (a) 0 (b) 1
   (c) x (d) None of these.
3. If x is random variate and a and b are constants and if all following expectations exist then which of the following is true :
   (a) $E(ax + b) = a\,E(x)$ (b) $E(ax + b) = E(x) + b$
   (c) $E(ax + b) = a\,E(x) + b$ (d) None of these.
4. If x and y are two random variate such that $y \le x$ and if expectations $E(x)$ and $E(y)$ both exist then which of the following is true :
   (a) $E(y) = E(x)$ (b) $E(y) \le E(x)$
   (c) $E(y) \ge E(x)$ (d) None of these.
5. If expectations $E(x)$ and $E(|x|)$ both exist, then which of the following is true :
   (a) $E(y) \le E(|x|)$ (b) $E|(x)| \le E(x)$
   (c) $|E(x)| \le E(|x|)$ (d) $E(|x|) \le |E(x)|$.
6. If x is a non- zero random variable, such that $E(x)$ and $E\left(\frac{1}{x}\right)$ both exist, then which of the following is true :
   (a) $E(1/x) = 1/E(x)$ (b) $E(1/x) < 1/E(x)$
   (c) $E(1/x) \ge 1/E(x)$ (d) $E(1/x) \le 1/E(x)$.
7. For two random variates x and y the relation $E(xy) = E(x)\,E(y)$ is true :
   (a) If x and y are independent (b) For all values of x and y
   (c) If x and y are identical (d) None of these.
8. n dice are thrown together, then the mathematical expectation of the sum of their numbers, is :
   (a) 7/2 (b) $7n/2$
   (c) 7 (d) $(7/2)^n$.
9. n dice are tossed together, then the mathematical expectation of the product of their number, is :
   (a) 7/2 (b) $7n/2$
   (c) 7 (d) $(7/2)^n$.
10. Thirteen cards are drawn together from a pack of 52 cards. if aces count 1, and others according to their denomination then the expectation of total score on the 13 cards, is :
    (a) 13 (b) 13/85
    (c) 85/13 (d) 85.
11. A bag contains 9 balls numbered from 1 to 9. Three balls are drawn without replacement, then the expectation of the sum of numbers written on the balls, is :
    (a) 5 (b) 15
    (c) 45 (d) None of these.

12. Five coins are tossed together their faces are marked 2, 3. The expectation of obtaining sum on all the five coins, is :
    (a) 5/2
    (b) 5
    (c) 25/2
    (d) None of these.
13. In a lottery m ($< n$) tickets are drawn out of n tickets numbered from 1 to n, then the expectation of the sum of numbers on the drawn tickets is :
    (a) $\frac{n+1}{2}$
    (b) $\frac{m+1}{2}$
    (c) $\frac{m(n+1)}{2}$
    (d) $\frac{n(m+1)}{2}$.
14. A bag contains a white and b black balls c balls are drawn at random, then the mathematical expectation of the number of white balls, is :
    (a) $\frac{ab}{a+b}$
    (b) $\frac{ac}{a+b}$
    (c) $\frac{bc}{a+b}$
    (d) $\frac{a}{a+b}$.
15. For two random variates x and y which of following relation is false :
    (a) $E(x+y) = E(x) + E(y)$
    (b) $E(x+y) < E(x) + E(y)$
    (c) $E(x) \le E(|x|), -E(y) \le E(|y|)$
    (d) $[E(xy)]^2 \le E(x^2) . E(y^2)$.
16. If x and y are two random variates, then which of following relation defines cov (x, y) :
    (a) $E(x) + E(y)$
    (b) $E(x) . E(y)$
    (c) $E(xy) + E(x) E(y)$
    (d) $E(xy) - E(x) E(y)$.
17. If x and y are two independent random variates, then which of the following relations is false :
    (a) $E(xy) = E(x) E(y)$
    (b) $\text{cov}(x, y) = 0$
    (c) $\text{var}(x+y) = \text{var}(x) + \text{var}(y)$
    (d) $\text{cov}(x+a, y+a) = \text{cov}(x, y) + a$.
18. If x and y are two random variates and a and b are two constants, then which of the following relation is false :
    (a) $\text{cov}(x, y) = E[(x-\bar{x})(y-\bar{y})]$
    (b) $\text{cov}(ax, by) = ab \text{ cov}(x, y)$
    (c) $\text{cov}(x+a, y+a) = \text{cov}(x, y)$
    (d) $\text{cov}(x, y) = \text{var}(x+y)$
19. If a random variate x assumes the values 1, 2, 3, ..., n and each with probability $1/n$, then the value of $E(x)$, is :
    (a) $\frac{n+1}{2}$
    (b) $\frac{n(n+1)}{2}$
    (c) $\frac{(n+1)(2n+1)}{6}$
    (d) $\frac{n(n+1)(2n+1)}{6}$.
20. If a random variate x assumes the values 1, 2, 3, ..., n and each with equal probability $1/n$, then the value of $E(x^2)$, is :
    (a) $\frac{n+1}{2}$
    (b) $\frac{n(n+1)}{2}$
    (c) $\frac{(n+1)(2n+1)}{6}$
    (d) $\frac{n(n+1)(2n+1)}{6}$.

21. If random variate x assumes the values 1, 2, 3, ..., n and each with probability $1/n$, then the value of var (x), is :

(a) $\frac{n+1}{2}$ (b) $\frac{(n+1)(2n+1)}{6}$

(c) $\frac{n^2-1}{12}$ (d) None of these.

22. The mathematical expectation of a random variable may be :

(a) Indeterminate (b) Infinite

(c) None of these.

23. If $V(x) = \sigma^2$, then the value of $V(x + c)$, c being a constant, is :

(a) σ^2 (b) $\sigma^2 + c$

(c) $c\sigma^2$ (d) $c^2\sigma^2$.

24. If Var $(X) = 1$, then Var $(2X \pm 3)$ is :

(a) 5 (b) 13

(c) 4 (d) None of these.

25. If X is a continuous random variable with probability density function $f(x)$ and median M, then which of the following options is not correct ?

(a) $\int_{-\infty}^{M} f(x)\,dx = \frac{1}{2}$ (b) $\int_{M}^{+\infty} f(x)\,dx = \frac{1}{2}$

(c) $\int_{-\infty}^{M} f(x)\,dx = \int_{M}^{+\infty} f(x)\,dx$ (d) None of these.

26. If X is a random variable whose mathematical expectation $E(X) = 2$, then $E(2X + 3) =$

(a) 2 (b) 4

(c) 7 (d) None of these.

27. If $E(X) = 3$, then $E(X^2) =$

(a) 2 (b) 9

(c) 3 (d) None of these.

28. If $f(x) = Cx^2$; $0 < x < 1$ is a probability density function then the value of C is :

(a) 1 (b) 1/3

(c) 3 (d) None of these.

ANSWERS

1.	(d)	2.	(a)	3.	(c)	4.	(b)	5.	(c)	6.	(c)	7.	(a)	8.	(b)
9.	(d)	10.	(d)	11.	(b)	12.	(c)	13.	(c)	14.	(b)	15.	(b)	16.	(d)
17.	(d)	18.	(d)	19.	(a)	20.	(c)	21.	(c)	22.	(b)	23.	(a)	24.	(c)
25.	(d)	26.	(c)	27.	(b)	28.	(c)								