Chapter 9 UNIT-VI

TESTS OF SIGNIFICANCE

§ 9.1. TESTING OF SIGNIFICANCE

Testing of significance is an applied form of the theory of testing of hypothesis that we have discussed in previous chapter. In this chapter we will discuss a few mathematical procedures to test the concerned hypothesis, on the grounds of probability. Here, we will make use of test statistics to test whether the result is significant or not. A significant result means that the observed value of the test statistic falls in the rejection region, whereas if this value lies in acceptance region then the result will be insignificant.

As this chapter is an advancement to previous chapter, all the terms and definitions defined there are equally meaningful here also, like level of significance, CR, H_0, H_1 etc. But some terms are again defined with more explanation, which are given below :

Test Statistic. A test statistic is a statistic computed from a random sample taken from the population of interest in a hypothesis test and then used for establishing the probable truth and falsity of the null hypothesis.

Critical Region and acceptance Region. Let us partition the sample space Ω into two disjoint and exhaustive subsets one is called critical region (W) and other is acceptance region (W'). The region in the sample space in which if the computed value of the test statistic lies, we reject the null hypothesis, is called the **critical region or rejection region** (W). The complementary region to the critical region is called the acceptance region (W'). That region in the sample space in which if the computed values of the test statistic lies, we accept the null hypothesis, is called the acceptance region.

The decision as to which values lie in the rejection region and which fall in an acceptance region is made on the basis of the level of significance α. When the rejection region consists of two regions each associated with probability $\frac{\alpha}{2}$ we call it a two tailed test. When the rejection region consists of only one region (either on the right or left) associated with probability α, we call it a one-sided or one-tailed test.

Two-tailed test. A statistical test is said to be two tailed test if it is testing the null hypothesis (H_0) against a two tailed (*i.e.*, $\neq$ type) alternative hypothesis (H_1) *i.e.*, if critical region is lying in both the tails, at once, then such a test is called two tailed test. For example

$$H_0 : \theta = \theta_0$$

against

$$H_1 : \theta_1 \neq \theta_0.$$

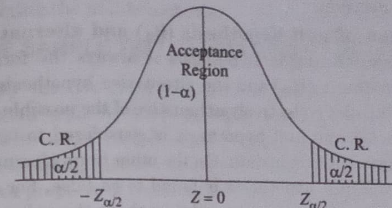


Fig. 9.1

One Tailed test. A statistical test is said to be one tailed test if it is testing the null hypothesis (H_0) against a one tailed (< or > type) alternative hypothesis *i.e.*, if the critical region is lying entirely in right tail then the test will be termed as right tailed test and if critical region lying in left tail then we call it left tailed test.

For example, $H_0 : \theta = \theta_0$ against $H_1 : \theta_1 > \theta_0$

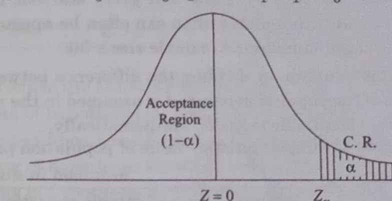


Fig. 9.2

$H_0 : \theta = \theta_0$ against $H_1 : \theta_1 < \theta_0$.

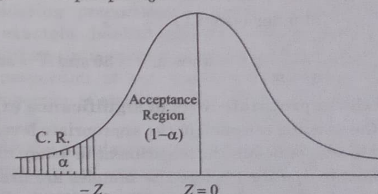


Fig. 9.3

§ 9.2. PROCEDURE FOR TESTING OF SIGNIFICANCE

The following steps are involved in the procedure for testing of significance :

(a) Formulation of null hypothesis and alternative hypothesis.

(b) Selection of a test statistic.

(c) Selection of the appropriate level of significance.

(d) Formulation of critical region.

(e) Making of decision.

(a) Formulation of null hypothesis (H_0) and alternative hypothesis (H_1). The first step in testing of hypothesis is always the formulation of two hypotheses, null hypothesis (H_0) and the alternative hypothesis (H_1) which are mutually exclusive and also collectively exhaustive of the possible states of reality. In testing of hypothesis, the null hypothesis is considered to be true until it is proved false on the basis of sample data. On the other hand alternative hypothesis must be true when the null hypothesis is found to be false. For example, in the testing of the population mean, two hypothesis about the value of a population mean μ, are typically stated in one of three forms by reference to a specified value of the mean μ_0.

Form I $H_0 : \mu = \mu_0$ against $H_1 : \mu \neq \mu_0$ (two tailed test)

Form II $H_0 : \mu = \mu_0$ against $H_1 : \mu > \mu_0$ (right tailed test)

Form III $H_0 : \mu = \mu_0$ against $H_1 : \mu < \mu_0$ (left tailed test)

(b) Selection of a test statistic. The second step in the testing of hypothesis is the selection of a test statistic. We know that every statistic has a sampling distribution of its own, and such a distribution can often be approximated by the normal distribution for large samples (*i.e.*, sample size ≥ 30).

We can find the test statistic by dividing the difference between the sample statistic and the value of the population parameter assumed in the null hypothesis by the standard error of the sample statistic. Mathematically,

$$\text{Test statistic} = \frac{\text{value of sample statistic - value of population parameter assumed in null hypothesis}}{\text{standard error of sample statistic}}.$$

For example, In testing of population mean μ, the test statistic is given by

$$Z = \frac{\overline{X} - \mu_0}{\text{Standar error } (\overline{X})} = \frac{\overline{X} - \mu_0}{\sigma_{\overline{x}}}$$

$$= \frac{\overline{X} - \mu_0}{\sigma / \sqrt{n}} \;;\; \sigma \text{ is known, } n > 30 \text{ and } \overline{X} = \text{sample mean.}$$

(c) Selection of the appropriate level of significance (α). The third step in the testing of hypothesis is the selection of the appropriate level of significance (α). The value of α is always set before the experiment or study under taken. The level of significance must be specified before the samples are drawn because the results obtained should not influence the choice of the decision-maker. The common values of α are $0{\cdot}05$, $0{\cdot}01$ and $0{\cdot}10$.

(d) Formulation of Critical Region. The fourth step in the testing of hypothesis is the formulation of the critical region. The critical region is that part of the sample space in which the numerical values of the test statistic lie when H_0 is rejected. The rejection or critical region consists of all values of the test statistic that are likely to occur if null hypothesis is false.

(e) Making of Decision. On the basis of the calculated value of the test statistic, according to the critical region formed of given size, we take the decision of accepting or rejecting the null hypothesis.

§ 9.3. LARGE SAMPLE TEST (Z-TEST) OR NORMAL TEST

Let us now apply the five-stage procedure introduced above to typical testing situations. We shall begin by assuming that the samples to be taken are large (*i.e.*, $n \geq 30$). Due to large sample size, if population does not have a normal distribution, the sampling distribution of the test statistic is assumed to be normal. Based on the area property of the standard normal distribution the critical region W for the Z-test statistic can be obtained from the following table.

Table of Critical Regions for Z-test Statistic

Level of Significance	Right tailed test	Left tailed test	Two tailed test
$\alpha = 0{\cdot}10$	$Z > 1{\cdot}28$	$Z < -1{\cdot}28$	$\lvert Z \rvert > 1{\cdot}645$
$\alpha = 0{\cdot}05$	$Z > 1{\cdot}645$	$Z < -1{\cdot}645$	$\lvert Z \rvert > 1{\cdot}96$
$\alpha = 0{\cdot}01$	$Z > 2{\cdot}33$	$Z < -2{\cdot}33$	$\lvert Z \rvert > 2{\cdot}58$

We shall discuss the following large sample tests :

(a) Testing of hypothesis for population proportion.

(b) Testing of hypothesis for difference between two proportions.

(c) Testing of hypothesis for single population mean.

(d) Testing of hypothesis for difference between two population means.

(e) Test of hypothesis for difference of standard deviation.

(a) Testing of Hypothesis for Population Proportion. Testing of hypothesis considering proportions of some attributes are useful in many situations. For example, health ministry may be interested in knowing the proportion of drinkers in a certain country, a businessmen may be interest in knowing what proportion of the workers work efficiently etc. In the above examples, the proportion of attributes like efficiency and drinking etc. are denoted by p.

For obtaining the population proportion, a random sample of size n (> 30) is taken and the proportion of successes in the particular sample is given by

$$\hat{p} = \frac{\text{number of successes in the sample}}{\text{sample size}} = \frac{x}{n}.$$

In this case we set up the null hypothesis as :

(i) $H : p = p_0$ against $H_1 : p \neq p_0$ (two tailed test) (where p_0 is some specified value of p)

(ii) $H_0 : p = p_0$ against $H_1 : p > p_0$ (Right tailed test)

(iii) $H_0 : p = p_0$ against $H_1 : p < p_0$ (Left tailed test)

For testing the null hypothesis H_0, it is assumed that (due to large sample size) the sampling distribution of a proportion follows a standard normal distribution with mean zero and variance once. For this testing the Z-test statistic is given by

$$Z = \frac{\hat{p} - p_0}{\sigma_{\hat{p}}} = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0 q_0}{n}}} \quad \text{where } q_0 = 1 - p_0$$

or

$$Z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0 (1 - p_0)}{n}}} \sim N(0, 1).$$

The critical values of Z-test statistic for α values can either be seen from the Table A or from the table for normal distribution in the appendix.

The same rules can be applied for rejection or accepting the null hypothesis in the testing of hypothesis for population proportion as in the testing of hypothesis for population mean.

Example 1. ***In a colony containing 20,000 families, a sample of 1,000 families was selected at random out of these 1,000 families 400 families were found to be consumer of wheat. Does the sample support the assumption that half of the colony is wheat consumer ? Use 5 percent level of significance.***

Solution. Here the null hypothesis is setup as

$$H_0 : p = \frac{1}{2} \text{ Against } H_1 : p \neq \frac{1}{2} \quad \text{(two tailed test)}$$

We are given

$$n = 1000, \; x = 400, \; p_0 = \frac{1}{2} = 0 \cdot 50$$

$$\therefore \quad \hat{p} = \frac{x}{n} = \frac{400}{1000} = 0 \cdot 40, \; \alpha = 5\%.$$

Thus the Z-test statistic is given by

$$Z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0 (1 - p_0)}{n}}} = \frac{0 \cdot 40 - 0 \cdot 50}{\sqrt{\frac{0 \cdot 50 \, (1 - 0 \cdot 50)}{1000}}} = -\frac{0 \cdot 10}{\sqrt{\frac{0 \cdot 25}{1000}}}$$

$$= -\frac{0 \cdot 10}{\frac{0 \cdot 50}{31 \cdot 62}} = -\frac{3 \cdot 162}{0 \cdot 50} = -6 \cdot 324.$$

Therefore $\quad |Z| = 6 \cdot 324.$

Since $|Z_{cal}| = 6 \cdot 324$ is greater than the tabulated value $Z_{\alpha} = 1 \cdot 96$, thus the null hypothesis is rejected.

Hence we conclude that half of colony is not wheat consumer.

Example 2. *An automobile company claims that at least 98 percent of the machines manufactured in the company meet the specifications. A random sample of 50 machines were taken and found that 2 were faulty. Examine the claim of the company at 5 percent level of significance.*

Solution. We are given that the atleast 98 percent of the machines meets to the specifications. So in this case our null hypothesis will be

$$H_0 : p \geq 0 \cdot 98 \text{ Against } H_1 : p < 0 \cdot 98 \text{ (one tailed test)}$$

Here $\quad n = 50, x = 2, p_0 = 0 \cdot 98.$

$\therefore \quad \hat{p} =$ percent of machines meet to specifications

$$= 1 - \frac{x}{n} = 1 - \frac{2}{50} = \frac{48}{50} = 0 \cdot 96.$$

Thus the Z-test statistic is given by

$$Z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1 - p_0)}{n}}} = \frac{(0 \cdot 96 - 0 \cdot 98)}{\sqrt{\frac{0 \cdot 98 \times (1 - 0 \cdot 98)}{50}}}$$

$$= -\frac{0 \cdot 02}{\sqrt{0 \cdot 000392}} = -\frac{0 \cdot 02}{0 \cdot 0198} = -1 \cdot 01.$$

Since $Z_{cal} = -1 \cdot 01$ is greater than the tabulated value $Z_{0 \cdot 05} = -1 \cdot 645$, thus the null hypothesis is rejected. Hence we conclude that the proportion of machines do not meet to specifications.

Example 3. ***A die is thrown 10,000 times and a throw of 5 or 6 was observed 3,800 times. Examine, can the die be regarded an unbiased one ? Use 5% level of significance.***

Solution. Here the null hypothesis is set up as

$$H_0 : p = \frac{1}{3} \text{ vs } H_1 : p \neq \frac{1}{3}$$

We are given

$$n = 10{,}000, \; x = 3800, \; p_0 = \frac{1}{3} = 0 \cdot 333.$$

$$\therefore \quad \hat{p} = \frac{x}{n} = \frac{3800}{10{,}000} = 0 \cdot 38.$$

Thus the Z-test statistic is given by

$$Z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0 (1 - p_0)}{n}}} = \frac{0 \cdot 38 - 0 \;\; 333}{\sqrt{\frac{0 \cdot 333 \, (1 - 0 \cdot 333)}{10{,}000}}}$$

$$= \frac{0 \cdot 047}{0 \cdot 004712} = 9 \cdot 974.$$

Since $Z_{Cal} = 9.974$ is greater than $Z_{tab} = 1.96$. So we reject null hypothesis. The die is not an unbiased one.

Example 4. ***If the expectation is that 3% of men of exact age 70 years will die within a year and out of a group of 1000 such men 36 die within the year. Can this group be regarded as a random sample of such men? Use 5% level of significance.***

Solution. Here null hypothesis is setup as

$$H_0 : p = p_0 = 3\% = 0.03$$

$$H_1 : p \neq p_0.$$

We are given

$\hat{p}$ = Proportion of men of exact age 70 years who will die within a year in in the sample

$$\hat{p} = \frac{36}{1000} = 0.036.$$

The Z-test statistic is given by

$$Z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}} = \frac{0.036 - 0.03}{\sqrt{\frac{0.03(1-0.03)}{1000}}}$$

$$= \frac{0.006}{0.005394} = 1.112.$$

Since $Z_{Cal} = 1.112$ is less than the tabulated value $Z_{0.05} = 1.96$. So we accepted null hypothesis. Hence the observed group may be regarded as a random sample of such men who are of exactly age 70 years and will die within a year.

(b) Testing of Hypothesis for Difference between two Proportions. Testing of hypothesis considering two equal proportions of some attribute is useful in many situations. For example, health ministry may be interested in knowing that proportions of drinkers in two states is same etc.

In this case we setup the null hypothesis as

(i) $H_0 : p_1 = p_2$ or $H_0 : p_1 - p_2 = 0$ against $H_1 : p_1 \neq p_2$

(ii) $H_0 : p_1 = p_2$ Against $H_1 : p_1 > p_2$

(iii) $H_0 : p_1 = p_2$ Against $H_1 : p_1 < p_2$

Now $\hat{p}_1 = \dfrac{x_1}{n_1}$ and $\hat{p}_2 = \dfrac{x_2}{n_2}$

where x_1 = Number of successes in the first sample.

n_1 = First sample size.

x_2 = Number of successes in the second sample.

n_2 = Second sample size.

For the above testing of hypothesis the Z-test statistic is calculated as :

$$Z = \frac{\hat{p}_1 - \hat{p}_2}{\sigma_{(\hat{p}_1 - \hat{p}_2)}} = \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{\hat{p}(1-\hat{p}) \cdot \left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

where $$\hat{p} = \frac{n_1 \hat{p}_1 + n_2 \hat{p}_2}{n_1 + n_2} = \frac{x_1 + x_2}{n_1 + n_2}.$$

The critical values of Z-test statistic for α-level of significance can either be seen from the Table A or from the table for normal distribution in the appendix.

The same rules can be applied for rejection or accepting the null hypothesis in the testing of hypothesis for population proportion as in the testing of hypothesis for population mean.

Example 5. ***In a random sample of 2000 peoples from colony A, 900 were found to be consumers of rice. In a sample of 1500 people from colony B, 800 were found to be consumers of rice. Do the sample support the assumption that there is no significant difference between two colonies A and B, so far as the proportion of rice consumers is concerned.***

Solution. We are given that $n_1 = 2000$, $x_1 = 900$, $n_2 = 1500$, $x_2 = 800$. Therefore the sample proportions of rice consumers in colony A

$$\hat{p}_1 = \frac{x_1}{n_1} = \frac{900}{2000} = 0.45.$$

The sample proportion of rice consumers in colony B.

$$\hat{p}_2 = \frac{x_2}{n_2} = \frac{800}{1500} = 0.53.$$

Also $$\hat{p} = \frac{n_1 \hat{p}_1 + n_2 \hat{p}_2}{n_1 + n_2} = \frac{x_1 + x_2}{n_1 + n_2} = \frac{900 + 800}{2000 + 1500} = \frac{1700}{3500}$$

$$\hat{p} = 0.49 \Rightarrow (1 - \hat{p}) = 1 - 0.49 = 0.51.$$

Here we shall setup the null hypothesis that there is no difference between the proportions of rice consumers of two colonies *i.e.*,

$$H_0 : p_1 = p_2 \text{ against } H_1 : p_1 \neq p_2$$

Now for testing the above null hypothesis H_0 we shall calculate the Z-test statistic as

$$Z = \frac{\hat{p}_1 - \hat{p}_2}{\sigma_{(\hat{p}_1 - \hat{p}_2)}} = \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{\hat{p}(1-\hat{p})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

$$= \frac{0.45 - 0.53}{\sqrt{0.49(1-0.49)\left(\frac{1}{2000} + \frac{1}{1500}\right)}} = -\frac{0.08}{\sqrt{0.2499(0.00117)}}$$

$$Z = -\frac{0.08}{0.0171} = -4.68.$$

$$\therefore \quad |Z| = 4.68.$$

Since $|Z| = 4 \cdot 68 > 3$, the difference between proportions is highly significant. So H_0 is rejected and will say that colony A and colony B differ significantly in respect of rice consumption.

Example 6. ***In a simple sample of 600 men from a certain large city, 400 are found to be smokers. In another sample of 900 from the other city, 450 are smokers. Do the data indicate that the cities are significantly different with respect to prevalence of smoking among men ?***

Solution. We are given that $n_1 = 600, x_1 = 400, n_2 = 900, x_2 = 450.$

Therefore the sample proportions

$$\hat{p}_1 = \frac{x_1}{n_1} = \frac{400}{600} = \frac{2}{3} = 0 \cdot 667$$

$$\hat{p}_2 = \frac{x_2}{n_2} = \frac{450}{900} = \frac{1}{2} = 0 \cdot 5.$$

Also $$\hat{p} = \frac{n_1 \hat{p}_1 + n_2 \hat{p}_2}{n_1 + n_2} = \frac{x_1 + x_2}{n_1 + n_2} = \frac{400 + 450}{600 + 900} = \frac{850}{1500} = 0 \cdot 567$$

$$\Rightarrow \quad (1 - \hat{p}) = (1 - 0 \cdot 567) = 0 \cdot 433.$$

Here we shall set up the null hypothesis that there is no difference between the cities with respect to prevalence of smoking among men *i.e.*,

$$H_0 : p_1 = p_2 \text{ against } H_1 : p_1 \neq p_2.$$

Now for testing the above null hypothesis H_0 we shall calculate the Z-test statistic as

$$Z = \frac{\hat{p}_1 - \hat{p}_2}{\sigma_{(\hat{p}_1 - \hat{p}_2)}} = \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{\hat{p}(1 - \hat{p})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

$$= \frac{0 \cdot 667 - 0 \cdot 5}{\sqrt{0 \cdot 567 \times 0 \cdot 433\left(\frac{1}{600} + \frac{1}{900}\right)}} = \frac{1 \cdot 167}{\sqrt{0 \cdot 2455\,(0 \cdot 0027)}}$$

$$= \frac{0 \cdot 167}{0 \cdot 0257} = 6 \cdot 59.$$

Since $Z = 6 \cdot 49 > 3$. So we reject H_0.

Example 7. ***In a year there are 956 births in a town A of which 52·7% were males while in town A and B combined, this proportion in a total of 1406 births was 0·496. Is there any significant difference in the proportion of male births in the two towns ?***

Solution. We are given that $n_1 = 956, \quad p_1 = 52 \cdot 5\% = 0 \cdot 525$

$$n_1 + n_2 = 1406, \quad \hat{p} = \frac{n_1 p_1 + n_2 p_2}{n_1 + n_2} = 0 \cdot 496$$

$$\Rightarrow \quad n_2 = 1406 - n_1 = 1406 - 956 = 450$$

$$0 \cdot 496 = \frac{956 \times 0 \cdot 525 + 450 p_2}{1406}$$

$$\Rightarrow \quad 0 \cdot 496 \times 1406 = 956 \times 0 \cdot 525 + 450 p_2$$

$$\Rightarrow \quad 697 \cdot 376 - 501 \cdot 9 = 450 p_2$$

$$\Rightarrow \quad 195 \cdot 476 / 450 = p_2$$

$$\Rightarrow \quad p_2 = 0 \cdot 4343.$$

Here we shall setup the null hypothesis that there is no difference between the proportion of male births in the two towns

$$Z = \frac{\hat{p}_1 - \hat{p}_2}{\sigma_{(\hat{p}_1 - \hat{p}_2)}} = \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{\hat{p}(1 - \hat{p})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

$$= \frac{0 \cdot 525 - 0 \cdot 434}{\sqrt{0 \cdot 496 \times 0 \cdot 504\left(\frac{1}{956} + \frac{1}{450}\right)}} = \frac{0 \cdot 091}{\sqrt{0 \cdot 2499 \times 0 \cdot 0032}}$$

$$= \frac{0 \cdot 091}{0 \cdot 0283} = 3 \cdot 32.$$

Since $Z_{Cal} = 3 \cdot 32 > 3$, H_0 is rejected. Thus the data as such indicates that there is significant difference in the proportion of male births in the two towns.

(c) Testing of Hypothesis for single population mean. Let us suppose that we have a population with unknown mean μ and known variance σ^2 (*i.e.*, known S.D. σ). Again let a random sample of size n (≥ 30) be taken from the population. Then our problem is to test the hypothesis that population mean is μ_0 (a hypothesized value of μ), *i.e.*, our problem is to test

$$H_0 : \mu = \mu_0 \text{ against}$$

$$H_1 : \mu \neq \mu_0 \text{ (Two-tailed test)}$$

or $$H_1 : \mu > \mu_0 \text{ (Right tailed test)}$$

or $$H_1 : \mu < \mu_0 \text{ (Left tailed test)}$$

To test the null hypothesis we make use of the test statistic which is defined as

$$Z = \frac{\bar{X} - \mu}{\text{S.E.}(\bar{X})} = \frac{\bar{X} - \mu}{\sigma / \sqrt{n}}$$

where $\bar{X}$ is the sample mean and the test statistic Z follows the standard normal distribution. That is

$$Z = \frac{\bar{X} - \mu}{\sigma / \sqrt{n}} \sim N(0, 1).$$

The above defined Z-test statistic shows how many standard errors $\bar{X}$ is away from μ.

After computing the value of the Z-test statistic, the decision about the null hypothesis H_0 is taken as under :

(i) For two sided alternative $H_1 : \mu \neq \mu_0$

Reject H_0 if calculated $Z < -Z_{\alpha/2}$

or calculated $Z > Z_{\alpha/2}$.

Otherwise accept H_0, where $Z_{\alpha/2}$ is the tabulated value of Z at $\alpha/2$ level of significance.

(ii) For one sided alternative $H_1 : \mu > \mu_0$

Reject H_0 if calculated $Z > Z_\alpha$

Otherwise accept H_0, where Z_α is the tabulated value of Z at α level of significance.

(iii) For one sided alternative $H_1 : \mu < \mu_0$

Reject H_0 if calculated $Z < -Z_\alpha$

Otherwise accept H_0.

Example 8. ***If a sample of size 300, taken from a normal population with mean = 3 and variance = 2, has mean = 3·5 and variance = 2. Test if there is significant difference between the means. ($\alpha = 0 \cdot 05$).***

Solution. (i) Here, $H_0 : \bar{x} = \mu$ v/s $H_1 : \bar{x} \neq \mu$.

(ii) For $\alpha = 0 \cdot 05$, a two tailed CR will be $(Z < -1 \cdot 96 \cup Z > 1 \cdot 96)$

(iii) Here, $\mu = 3$, $\sigma^2 = 2$, $\bar{x} = 3 \cdot 5$ and $n = 300$. Therefore,

$$\sigma = \sqrt{2} = 1 \cdot 414 \text{ and } Z = \frac{3 \cdot 5 - 3}{1 \cdot 414 / \sqrt{300}} = 6 \cdot 125 > 1 \cdot 96.$$

(iv) Since, $Z \in$ CR therefore H_1 will be accepted that sample mean differs significantly from population mean.

Example 9. ***A sample of 225 tube lights is taken from a lot. The mean life time of tube lights is found to be 1940 hours with a standard deviation of 200 hours. Test the hypothesis that the mean life time of the tube lights is 2000 hours against the alternative hypothesis that it is not equal to 2000 hours at 5% level of significance.***

Solution. We shall make the null hypothesis that there is no significant difference between the sample mean and the hypothesized population mean. That is

$$H_0 : \mu = 2000 \text{ Against } H_1 : \mu \neq 2000.$$

We are given that

$$n = 225, \bar{x} = 1940 \text{ hours}, s = 200 \text{ hours and } \alpha = 5\% = 0 \cdot 05.$$

To test the null hypothesis H_0, we calculate the test statistic

$$Z = \frac{\bar{x} - \mu}{\sigma_{\bar{x}}} = \frac{\bar{x} - \mu}{s / \sqrt{n}} = \frac{1940 - 2000}{200 / \sqrt{225}} = -\frac{60}{200/15}$$

$$Z = -4 \cdot 5.$$

$$\therefore \quad |Z| = |-4 \cdot 5| = 4 \cdot 5.$$

Since calculated $|Z| = 4 \cdot 5$ is greater than 3, so the difference between $\bar{x}$ and μ is significant and H_0 is rejected. That is the mean life time of tube lights may not be 2000 hours.

Example 10. ***A random sample of 900 members is found to have a mean of 3·4cm. Could it come from a large population with mean $\mu = 3 \cdot 25$ cm and $\sigma = 2 \cdot 16$ cm?***

Solution. $H_0 : \mu = \mu_0 = 3 \cdot 25$, $H_1 : \mu \neq 3 \cdot 25$

Here, $\bar{x} = 3 \cdot 4$ cm, $\mu_0 = 3 \cdot 25$ cm, $n = 900$
$\sigma = 2 \cdot 16$ cm.

To test the null hypothesis H_0, we calculated the test statistic

$$z = \frac{\bar{x} - \mu}{\sigma / \sqrt{n}} \sim N(0, 1).$$

Under H_0, the value of test statistic z is

$$z = \frac{\bar{x} - \mu}{\sigma / \sqrt{n}}$$

$$= \frac{3 \cdot 4 - 3 \cdot 25}{2 \cdot 16 / \sqrt{900}}$$

$$= \frac{0 \cdot 15 \times 30}{2 \cdot 16}$$

$$= 1 \cdot 72.$$

Sine $Z_{Cal} = 1 \cdot 72$ is less than 3. So the difference between $\bar{x}$ and μ is not significant and H_0 is accepted. That is given sample might have been drawn from a population with mean 3·25 and standard deviation 2·16.

Example 11. ***A sample of 400 students is found to have a mean height of 67·5 inches. Can it be reasonably regarded as a sample from a large population with mean height 67·2 inches and standard deviation 1·2?***

Solution. Here the null hypothesis is setup as :

$$H_0 : \mu = \mu_0 = 67 \cdot 5$$

$$H_1 : \mu = \mu_0 \neq 67 \cdot 5.$$

Here $\quad \bar{x} = 67 \cdot 2$, $\sigma = 1 \cdot 2$, $\mu_0 = 67 \cdot 5$, $n = 400$.

The est statistic is

$$z = \frac{\bar{x} - \mu}{\sigma / \sqrt{n}}$$

$$= \frac{67 \cdot 2 - 67 \cdot 5}{1 \cdot 2 / \sqrt{400}}$$

$$= \frac{-0 \cdot 3 \times 20}{1 \cdot 2}$$

$$= -5.$$

$$\therefore \quad |z| = |-5| = 5.$$

Since $Z_{Cal} = 5$ is greater than 3, so H_0 is rejected. So the given sample can not be regarded as a random sample from a large population with mean height 67·2 and standard deviation 1·2.

(d) Testing of Hypothesis for difference between two population means. Let us suppose that we have two populations with unknown means μ_1 and μ_2 and known variances σ_1^2, and σ_2^2 respectively. Again let two independent random samples of sizes n_1 (> 30) and n_2 (> 30) be taken from the first and second population, respectively. Then our problem is to test the hypothesis that there is no difference between two population means, *i.e.*, our problem is to test

$$H_0 : \mu_1 = \mu_2 \text{ Against } H_1 : \mu_1 \neq \mu_2.$$

To test the null hypothesis we make use of the Z-test statistic which is defined as :

$$Z = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\text{S.E.}(\bar{x}_1 - \bar{x}_2)} = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{\left[\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}\right]}}$$

where $\bar{x}_1$ = mean of the first sample

$\bar{x}_2$ = mean of the second sample

$\bar{x}_1 - \bar{x}_2$ = Difference between two sample means.

$\mu_1 - \mu_2$ = Difference between population means.

S.E. $(\bar{x}_1 - \bar{x}_2)$ = standard error of the statistic $(\bar{x}_1 - \bar{x}_2) = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$.

The Z-test statistic follows the standard normal distribution. That is

$$Z = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{\left[\frac{\sigma_1{}^2}{n_1} + \frac{\sigma_2{}^2}{n_2}\right]}} \sim N(0, 1).$$

The above defined Z-test statistic shows how many standard error $(\bar{x}_1 - \bar{x}_2)$ is away from $(\mu_1 - \mu_2)$. After computing the value of Z-test statistic, the decision about the null hypothesis H_0 is taken as under.

Reject H_0 if calculated $Z < -Z_{\alpha/2}$ or

Calculated $Z > Z_{\alpha/2}$

i.e., $|Z| > Z_{\alpha/2}$.

Otherwise accept H_0, where $Z_{\alpha/2}$ is the tabulated value of Z at $\alpha/2$ level of significance.

Example 12. ***A college conducts both day and night classes intended to be identical. A sample of 100 day students yields examination results as***

$$\bar{x}_1 = 72 \cdot 4, \sigma_1 = 14 \cdot 8.$$

A sample of 200 night students yields examination results as under

$$\bar{x}_2 = 73 \cdot 9, \sigma_2 = 17 \cdot 9.$$

Are the two means statistically equal at 10% level of significance?

Solution. We are given that $n_1 = 100$, $n_2 = 200$, $\bar{x}_1 = 72 \cdot 4$, $\bar{x}_2 = 73 \cdot 9$ and $\sigma_1 = 14 \cdot 8$, $\sigma_2 = 17 \cdot 9$.

Now our problem is to test

$$H_0 : \mu_1 = \mu_2 \text{ Against } H_1 : \mu_1 \neq \mu_2$$

Thus the Z-test statistic is given by

$$Z = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\text{S.E.}(\bar{x}_1 - \bar{x}_2)}$$

$$= \frac{(\bar{x}_1 - \bar{x}_2)}{\sqrt{\left[\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}\right]}} \qquad [\because \text{ under } H_0 : \mu_1 = \mu_2]$$

$$= \frac{(72 \cdot 4 - 73 \cdot 9)}{\sqrt{\left[\frac{(14 \cdot 8)^2}{100} + \frac{(17 \cdot 9)^2}{2}\right]}} = -\frac{1 \cdot 5}{\sqrt{3 \cdot 792}} = -\frac{1 \cdot 5}{1 \cdot 1947} = -0 \cdot 772$$

Thus $|Z| = 0 \cdot 772$ *i.e.*, $|Z| = 0 \cdot 772 < 1 \cdot 645 = Z_{0 10}$.

Since calculated value of $|Z|$ is less than the critical value of Z at 10% level of significance so our null hypothesis H_0 is accepted at 10% level of significance. Thus we conclude that the two means are statistically equal at 10 percent level of significance.

Example 13. ***A washing machine manufacturer wants to test the claim that the mean life times of two types of machines are identical. He puts on test 200 machines and later selects two random samples of machines of each type. The following data were obtained.***

For machine (A)	For machine (B)
$n_1 = 40$	$n_2 = 40$
$\bar{X}_A = 4000$ hours	$\bar{X}_B = 4000$ hours
$s_A = 80$ hours	$s_B = 120$ hours

Are the mean lives of machine A and machine B identical? Test at 5 percent level of significance.

Solution. We are given that $n_1 = 40$, $n_2 = 40$, $\bar{X}_A = 4000$ hours

$\bar{X}_B = 4500$ hours, $s_A = 80$ hours and $s_B = 120$ hours.

Now our problem is to test :

$$H_0 : \mu_A = \mu_B \text{ against } H_1 : \mu_A \neq \mu_B$$

Thus the Z-test statistic is given by

$$Z = \frac{\bar{X}_A - \bar{X}_B}{\text{S.E.}(\bar{x}_A - \bar{x}_B)}$$

$$Z = \frac{\bar{X}_A - \bar{X}_B}{\sqrt{\left[\frac{s_A^2}{n_1} + \frac{s_B^2}{n_2}\right]}}$$

$$= \frac{4000 - 4500}{\sqrt{\left[\frac{80^2}{40} + \frac{120^2}{40}\right]}} = -\frac{500}{\sqrt{520}} = -\frac{500}{22 \cdot 804}$$

$$= -21 \cdot 926$$

$\Rightarrow \quad |Z| = 21 \cdot 926$

$\therefore \quad |Z| = 21 \cdot 926 > 1 \cdot 96$ = critical value at 5% level of significance.

So we shall reject our null hypothesis H_0.

This can also be seen from the figure.

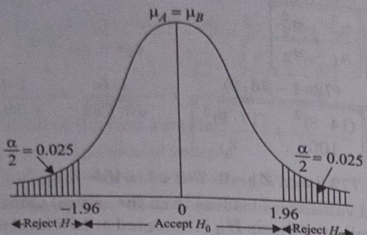


Fig. 9.4

Thus at 5 percent level of significance, the difference is statistically significant. It is very likely that machine B has a large life time than machine A.

Example 14. ***A potential buyer of light bulbs bought 50 bulbs of each of two brands. Upon testing txcvhese bulbs he found that brand A had a mean life of 1282 hours with a S.d. of 80 hours. Where brand B had a mean life of 1208 hours with a S.d. of 94 hours. Can the buyer be quite certain that two brands differ in quality ?***

Solution. We are given that $n_1 = n_2 = 50$

$$\bar{x}_A = 1282, \ \bar{x}_B = 1208$$

$$S_A = 80 \text{ hours}, \ S_B = 94 \text{ hours}.$$

Now our problem is to test

$$H_0 : \mu_A = \mu_B \text{ against } H_1 : \mu_A \neq \mu_B$$

$$z = \frac{\bar{x}_A - \bar{x}_B}{\sqrt{\frac{S_A^2}{n_1} + \frac{S_B^2}{n_2}}}$$

$$= \frac{1282 - 1208}{\sqrt{\frac{80^2}{50} + \frac{94^2}{50}}}$$

$$= \frac{74}{\sqrt{304 \cdot 72}} = \frac{74}{17 \cdot 46}$$

$$= 4 \cdot 24.$$

Since $Z_{Cal} = 4 \cdot 24 > 3$. So we reject H_0. It mean the two brands differ in quality.

Example 15. ***Random samples of 500 and 400 have means 11·5 and 10·9 respectively. Can the samples be regarded as drawn from the population of the standard deviation 5. Use 5% level of significance.***

Solution. We are given that $n_1 = 500, \ n_2 = 400$

$$\bar{x}1 = 11 \cdot 5, \ \bar{x}2 = 10 \cdot 9$$

$$\sigma = \sigma_1 = \sigma_2 = 5.$$

Now our problem is to test

$$H_0 : \mu_1 = \mu_2 \text{ v/s } H_1 : \mu_2 \neq \mu_2$$

$$Z = \frac{\bar{x}1 - \bar{x}2}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

$$= \frac{\bar{x}1 - \bar{x}2}{\sqrt{\sigma^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

$$= \frac{11 \cdot 5 - 10 \cdot 9}{\sqrt{25\left(\frac{1}{500} + \frac{1}{400}\right)}}$$

$$= \frac{0 \cdot 6}{\sqrt{25 \times 0 \cdot 0045}}$$

$$= \frac{0 \cdot 6}{\sqrt{0 \cdot 1125}}$$

$$= 1 \cdot 79.$$

Since $Z_{Cal} = 1 \cdot 79$ is less than tabulated value $Z_{tab} = 1 \cdot 96$. So we accept null hypothesis. The samples can be regarded as drawn from the population of the standard deviation 5 at 5% level of significance.

Example 16. ***Test (at $\alpha = 0 \cdot 05$), $H_0 : \mu_1 = \mu_2$ for the following data,***

Ist sample : $n_1 = 400, \bar{x}_1 = 250$, sample variance $(s_1{}^2) = 1800$

IInd sample : $n_2 = 300, \bar{x} = 200$, sample variance $(s_2{}^2) = 1900$.

Solution. Here, $Z = \dfrac{250 - 200}{\sqrt{\left(\frac{1800}{400} + \frac{1900}{300}\right)}} = 15 \cdot 191.$

Since, $Z > 1 \cdot 96$, H_0 will be rejected in favour of H_1 *i.e.*, both the population means are quite different.

(e) Test of significance for the difference of standard deviation. If s_1 and s_2 are the standard deviations of two independent samples, then under the null hypothesis $H_0 : \sigma_1 = \sigma_2$, *i.e.*, the sample standard deviations do not differ significantly, the statistic is

$$z = \frac{s_1 - s_2}{\sqrt{\frac{\sigma_1^2}{2n_1} + \frac{\sigma_2^2}{2n_2}}}, \text{ where } \sigma_1 \text{ and } \sigma_2 \text{ are population standard deviations}$$

When population standard deviations are not known, then $z = \dfrac{s_1 - s_2}{\sqrt{\frac{s_1^2}{2n_1} + \frac{s_2^2}{2n_2}}}$.

Example 17. ***Random samples drawn from two countries gave the following data relating to the heights of adults males.***

	Country A	Country B
Mean height (in inches)	67·42	67·25
Standard deviation	2·58	2·50
Number in samples	1000	1200

(a) Is the difference between the means significance ?

(b) Is the difference between the standard deviations significance ?

Solution. Given :

$$n_1 = 1000, n_2 = 1200, \bar{x}_1 = 67\cdot42; \bar{x}_2 = 67\cdot25, s_1 = 2\cdot58, s_2 = 2\cdot50$$

Since the samples size are large we can $\sigma_1 = s_1 = 2\cdot58$; $\sigma_2 = s_2 = 2\cdot50$.

(a) Null Hypothesis, $H_0 : \mu_1 = \mu_2$, *i.e.*, sample means do not differ significantly.

Alternative hypothesis : $H_1 : \mu_1 \neq \mu_2$ (Two tailed test)

Under H_0 : $z = \dfrac{\bar{x}_1 - \bar{x}_2}{\sqrt{\dfrac{s_1^2}{n_1} + \dfrac{s_2^2}{n_2}}} = \dfrac{67\cdot42 - 67\cdot25}{\sqrt{\dfrac{(2\cdot58)^2}{1000} + \dfrac{(2\cdot50)^2}{1200}}} = 1\cdot56$

Conclusion. Since $|z| < 1\cdot96$, we accept the null hypothesis at 5% level of significance *i.e.*, there is no significant difference between the sample means.

(b) Null hypothesis, $H_0 : \sigma_1 = \sigma_2$, *i.e.*, the sample S.D.'s do not differ significantly.

Alternative Hypothesis : $H_1 = \sigma_1 \neq \sigma_2$ (Two tailed test)

Under H_0, the test statistic is

$$z = \frac{s_1 - s_2}{\sqrt{\dfrac{\sigma_1^2}{2n_1} + \dfrac{\sigma_2^2}{2n_2}}} = \frac{s_1 - s_2}{\sqrt{\dfrac{s_1^2}{2n_1} + \dfrac{s_2^2}{2n_2}}} \quad [\because \sigma_1 = s_1, \sigma_1 = s_2 \text{ for large samples})]$$

$$= \frac{2\cdot58 - 2\cdot50}{\sqrt{\dfrac{(2\cdot58)^2}{2\times1000} \times \dfrac{(2\cdot50)^2}{2\times1200}}} = \frac{0\cdot08}{\sqrt{\dfrac{6\cdot6564}{2000} + \dfrac{6\cdot25}{2400}}} = 1\cdot0387$$

Conclusion. Since $|z| < 1\cdot96$, we accept the null hypothesis at 5% level of significance, *i.e.*, there is no significant difference between the standard deviation.

Example 18. ***Intelligence test of two groups of boys and girls gives the following results :***

Girls	mean = 84	S.D. = 10	N = 121
Boys	mean = 81	S.D = 12	N = 81

(a) Is the difference in means significant ?

(b) Is the difference between the standard deviations significant ?

Solution. Given :

$$n_1 = 121, n_2 = 81, \bar{x}_1 = 84, \bar{x}_2 = 81, s_1 = 10, {}_2 = 12$$

(a) Null Hypothesis, $H_0 : \mu_1 = \mu_2$, *i.e.*, sample means do not differ significantly.

Alternative hypothesis : $H_1 : \mu_1 \neq \mu_2$ (Two tailed test)

Under H_0 : the test statistic is $z = \dfrac{\bar{x}_1 - \bar{x}_2}{\sqrt{\dfrac{s_1^2}{n_1} + \dfrac{s_2^2}{n_2}}} = \dfrac{84 - 81}{\sqrt{\dfrac{(10)^2}{121} + \dfrac{(12)^2}{81}}} = 0\cdot1859$

Conclusion. Since $|z| < 1\cdot96$, we accept the null hypothesis at 5% level of significance *i.e.*, there is no significant difference between the sample means.

(b) We set up the **Null hypothesis,** $H_0 : \sigma_1 = \sigma_2$, *i.e.*, the sample S.D.'s do not differ significantly.

Alternative Hypothesis : $H_1 = \sigma_1 \neq \sigma_2$ (Two tailed test)

Under H_0, the test statistic is

$$z = \frac{s_1 - s_2}{\sqrt{\dfrac{\sigma_1^2}{2n_1} + \dfrac{\sigma_2^2}{2n_2}}} = \frac{s_1 - s_2}{\sqrt{\dfrac{s_1^2}{2n_1} + \dfrac{s_2^2}{2n_2}}} \quad [\because \sigma_1 = s_1, \sigma_1 = s_2 \text{ for large samples})]$$

$$= \frac{10 - 12}{\sqrt{\dfrac{100}{2\times121} + \dfrac{144}{2\times81}}} = -1\cdot7526$$

Conclusion. Since $|z| = 1\cdot75 < 1\cdot96$, we accept the null hypothesis at 5% level of significance, *i.e.*, there is no significant difference between the standard deviations.

EXERCISE ON TEST OF SIGNIFICANCE ON DIFFERENCE OF STANDARD DEVIATION

1. The mean yiels of two sets of plots and their variability are as given. Examine.
   (i) Whether the difference in the mean yiels of the two sets of plots is significant ?
   (ii) Whether the difference in the variability in yields is significant ?

	Set of 40 plots	Set of 60 plots
Mean yiels per plot	1258 lb	1243 lb
S.D. per plot	34	28

2. The yield of wheat in a random sample of 1000 farms in a certain area has a S.D. of 192 kg. Another random sample of 1000 farms gives a S.D.'s of 224 kg. Are the S.D. significantly different ?

ANSWERS

1. (i) $z = 2{\cdot}321$. Difference significant at 5% level;
   (ii) $z = 1{\cdot}31$. Difference not significant at 5% level.
2. $z = 4{\cdot}851$. The S.D.'s are significantly different.

EXERCISE ON TEST OF SIGNIFICANCE FOR SINGLE PROPORTION

1. A coin is tossed 1000 times and the head come out 550 times. Can the deviation from expected value be due to fluctuations of simple sampling ?
2. A coin is tossed 10,000 times and it turns up head 5195 times. Is it reasonable to think that the coin is unbiased ?
3. An ordinary die is tossed 324 times and odd numbers are obtained 181 times. Is it reasonable to think that the die is unbiased ?
4. A random sample of 500 pineapples was taken from a large consignment and 65 were found to be bad. Show that the S. E. of the proportion of bad ones in a sample of this size is 0·015 and deduce that the percentage of bad pineapples in the consignment almost certainly lies between 8·5 and 17·5.

   [Hint. Here $p = \frac{65}{500}$, $q = 1 - \frac{65}{500} = \frac{435}{500}$

   S. E. of the proportion of bad pineapples

   $$= \sqrt{(pq/n)}$$

   $$= \sqrt{\left(\frac{65}{500} \times \frac{435}{500} \times \frac{1}{500}\right)}$$

   $$= 0{\cdot}015 = 1{\cdot}5\%.$$

   Reqd. limits $\left(\frac{65}{500} \times 100 \pm 3 \times 1{\cdot}5\right)\% = 8{\cdot}5\%$ and $17{\cdot}5\%$]
5. The records of a certain hospital showed the birth of 723 males and 617 females in a certain week. Do these confirm to the hypothesis that the sexes are born in equal proportions.
6. Of 10,000 babies born in M. P. 5,200 are male children. Taking this to be a random samples of the births in M. P., show that it throws considerable doubt on the hypothesis that the sexes are born in equal proportions.
7. Certain crosses of the pea gave 5321 yellow and 1804 green seeds. The expectation is 25% green seeds on a Mendelian hypothesis. Is this divergence significant or might have occurred as due to fluctuations of simple sampling ?

ANSWERS

1. It cannot be due to fluctuations of sampling.
2. No.
3. Biased at 5% level of significance.
5. Figures do not confirm to the given hypothesis.
7. Deviations may be due to fluctuations of sampling.

EXERCISE ON TEST OF SIGNIFICANCE FOR DIFFERENCE OF PROPORTION

1. In a random sample of 500 persons from town A, 200 are found to be consumers of cheese. In a sample of 400 from town B, 200 are also found to be consumers of cheese. Discuss the question whether the data reveal a significant difference between A and B so far as the proportion of cheese consumers is concerned.
2. In a random sample of 800 adults from the population of a certain large city 600 are found to have dark hair. In a random sample of 1000 adults from the inhabitants of another large city, 700 are dark haired. Show that the difference of the proportion of dark haired people is nearly 2·4 times the standard error of the difference for samples of above sizes.
3. In a city 60% men are smokers. On the arrival of new brand cigarette, in a random sample of 100 men, 70 men are found to be smokers. Is the increase in the proportion of smokers significant ?
4. A machine puts out 20 imperfect articles in a sample of 400. After the maximum overhauled puts out 10 imperfect articles in a batch of 300. Has the machine been improved ?
5. One thousand articles from a factory are examined and 30 were found defective. 1500 similar articles from second factory were examined where 300 were found defective. Can it be reasonably concluded that the products of the first factory are inferior to the second ?
6. The following table gives the proportion of dark coloured people in two cities :

Cities	Total population observed	Percentage of Dark coloured
A	450	35
B	600	45

Can the difference observed in the percentage of dark coloured be due to the fluctuation of sampling ?

ANSWERS

1. Yes, the deviation is significant.
3. $z = 2{\cdot}04$, $H_0 : P = 0{\cdot}0$ rejected.
4. $z = 1{\cdot}29$, H_0 Accepted , No improvements.
5. Product of first factory is inferior to second.
6. May be due to fluctuations of sampling.

EXERCISE ON TEST OF SIGNIFICANCE FOR SINGLE MEAN

1. A sample of 1000 students from a university was taken and their average weight was found to be 112 pounds with a S.D. of 20 pounds. Could the mean weight of students in the population be 120 pounds ?
2. A sample of 400 male students is found to have a mean height of 160 cms. Can it be reasonably regarded as a sample from a large population with mean height 162·5 cms and standard deviation 4·5 cms.
3. A random sample of 200 measurements from a large population gave a mean value of 50 and a S.D. of 9. Determine 95% confidence interval for the mean of population.

4. The guaranteed average life of certain type of bulbs si 1000 hours with a S.D. of 125 hours. It is decided to sample the output so as to ensure that 90% of the bulbs do not fall short of the guaranteed average by more than 2·5%. What must be the minimum size of the sample.

5. The heights of college students in a city are normally distributed with S.D. 6 cms. A sample of 1000 students has mean height 158 cms. Test the hypothesis that the mean height of college students in the city is 160 cms.

ANSWERS

1. H_0 is rejected 2. H_0 accepted 3. 48·8 and 51·2 4. $n = 41$
5. H_0 rejected both at 1% to 5% level of significance.

EXERCISE ON TEST OF SIGNIFICANCE FOR DIFFERENCE OF MEAN

1. A random sample of 200 villages was taken from Raipur district and the average population per village was found to be 485 with S. D. of 50. Another random sample of 200 villages from the same district gave an average population of 510 per village with S. D. of 40. Is the difference between the averages of two samples statistically significant? Give reasons.

2. 60 new entrants in Bhopal University are found to have mean height of 171 cms. and 50 seniors a mean height of 173·775 cms. Is the evidence conclusive that the mean height of the seniors is greater than that of the new entrants. Assume the S. D. of the heights to be 6·2 cms.

3. A man buys 100 electric bulbs of each of two well known makes taken at random from stock for testing purposes. He finds that make A has a mean life of 1300 hours with S. D. of 82 hours and make B a mean life of 1248 hours with S. D. of 93 hours. Can the mean be quite certain that the make A is better?

4. Intelligence test on two groups of boys and girls, give the following results. Examine if the difference is significant.

   Girls : Mean = 84, S. D. = 10, number = 121.

   Boys : Mean = 81, S. D. = 12, number = 81.

5. Bricks of the same type produced at two different works, were tested for transverse strength, with the following results :

	Work I	Work II
No. of random sample	300	270
Mean strength in gms (per square cm.)	990	1,000
S. D. in gms. (per square cm.)	240	202

Is the difference between the means significant ?

ANSWERS

1. Significance 2. Fluctuation of sampling 3. Yes
4. Not significant 5. Not significant.

ILLUSTRATIVE EXAMPLES BASED ON SINGLE PROPORTION (LARGE SAMPLES)

Example 1. ***A coin was tossed 400 times and the head turned up 216 times. Test the hypothesis that the coin is unbaised.***

Solution. Null Hypothesis, H_0 : The coin is unbaised *i.e.*, $p_0 = 0\cdot 5 = 1/2$

Alternative Hypothesis, H_1 : The coin is not unbaised *i.e.*, $p_0 \neq 0\cdot 5$.

Here, $n = 400, X =$ Number of success $= 216$

$$\hat{p} = \text{proportion of success in the sample} = \frac{X}{n} = \frac{216}{400} = 0\cdot 54$$

Population proportion $p = 0\cdot 5; q = 1 - p_0 = 1 - 0\cdot 5 = 0\cdot 5$

Under H_0,
$$z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}$$

$$z = \frac{0\cdot 54 - 0\cdot 5}{\sqrt{\frac{0\cdot 5 \times 0\cdot 5}{400}}} = 1\cdot 6 \Rightarrow |z| = 1\cdot 6$$

Conclusion. Since $|z| = 1\cdot 6 < 1\cdot 96$ *i.e.*, $|z| < z_\alpha, z_\alpha$ is the significant value of z at 5% level of significance *i.e.*, H_0 is accepted and hence the coin is unbaised.

Example 2. ***A certain cubical die was thrown 9000 times and 5 or 6 was obtained 3240 times. On the assumption of certain throwing, do the data indicate an unbaised die ?***

Solution. Here, $n = 9000$

$$p_0 = \text{probability of success } (i.e., \text{getting 5 or 6 in die})$$
$$= 2/6 = 1/3, q = 1 - p_0 = 1 - 1/3 = 2/3$$

$$\hat{p} = \frac{X}{n} = \frac{3240}{9000} = 0\cdot 36$$

Null Hypothesis H_0 : The die is unbaised, *i.e.*, $p = 1/3$

Alternative Hypothesis H_1 : $p \neq 1/3$ (two tailed test)

Under H_0,
$$z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}$$

$$= \frac{0\cdot 36 - 0\cdot 33}{\sqrt{\frac{1}{3} \times \frac{2}{3} \times \frac{1}{9000}}} = \frac{(0\cdot 03) \times 284\cdot 60}{1\cdot 4142} = \frac{8\cdot 538}{1\cdot 4143} = 6\cdot 0373$$

$\Rightarrow$ $|z| = 6\cdot 0373$

Conclusion. Since $|z| = 6\cdot 0373 > 1\cdot 96$. Therefore H_0 is rejected *i.e*, the die is not unbaised.

Example 3. ***A manufacturer claims that only 4% of his products supplied by him are defective. A random sample of 600 products contained 36 defectives. Test the claim of the manufacturer.***

Solution. $\hat{p}$ = Observed proportion of success

i.e., $\hat{p}$ = Proportion of defective products in the sample $= \frac{36}{600} = 0 \cdot 06$

p_0 = Proportion of defectives in the population

$= 0 \cdot 04,\ q = 1 - p_0 = 0 \cdot 96$

Null Hypothesis, H_0 : $p_0 = 0 \cdot 04$ is true

Alternative Hypothesis, H_1 : $p_0 \neq 0 \cdot 04$.

Under H_0,
$$z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}} = \frac{0 \cdot 06 - 0 \cdot 04}{\sqrt{\frac{0 \cdot 04 \times 0 \cdot 96}{600}}} = 2 \cdot 5.$$

Conclusion. Since $|z|$ $2 \cdot 5 > 1 \cdot 96$, we reject the hypothesis H_0 at 5% level of significance *i.e.*, the manufacturer's claim is not accepted.

Example 4. ***A manufacturer claims that only 10% of the articles produced are below the standard quality. Out of a random inspection of 300 articles, 37 are found to be of poor quality. Test the manufacturer's claim at 5% level of significance.***

Solution. H_0 = The claim of manufacturer is true

H_1 = The claim of manufacturer is not true

$n = 300$

Given, p_0 = Proportion of defective articles in the population

$= 10\% = \frac{10}{100} = 0 \cdot 1$

$q = 1 - p_0 = 1 - 0 \cdot 1 = 0 \cdot 9$

$\hat{p}$ = Proportion of defective articles in the samples

$= \frac{37}{300} = 0 \cdot 1233$

Under H_0,
$$z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}$$
$$= \frac{0 \cdot 1233 - 0 \cdot 1}{\sqrt{\frac{(0 \cdot 1)(0 \cdot 9)}{300}}} = \frac{0 \cdot 0233}{0 \cdot 0173} = 1 \cdot 3468$$

$\Rightarrow$ $|z| = 1 \cdot 3468 < 1 \cdot 96$

Conclusion. H_0 is accepted *i.e.*, the manufacturer's claim is true.

ILLUSTRATIVE EXAMPLES BASED ON DIFFERENCE OF PROPORTION (LARGE SAMPLES)

Example 1. ***Before an increase in excise duty on tea, 800 people out of a sample of 1000 persons were found to be tea drinkers. After an increase in the duty, 800 persons were known to be tea drinkers in a sample of 1200 people. Do you think that there has been a significant decrease in the consumption of tea after the increase in the excise duty ?***

Solution. $n_1 = 1000,\ n_2 = 1200$

$$\hat{p}_1 = \frac{X_1}{n_1} = \frac{800}{1000} = \frac{4}{5};\quad \hat{p}_2 = \frac{X_2}{n_2} = \frac{800}{1200} = \frac{2}{3}$$

$$\hat{P} = \frac{p_1n_1 + p_2n_2}{n_1 + n_2} = \frac{X_1 + X_2}{n_1 + n_2} = \frac{800 + 800}{1000 + 1200} = \frac{8}{11};\quad 1 - \hat{P} = \frac{3}{11}$$

Null hypothesis H_0 : $p_1 = p_2$ *i.e.*, there is no significant difference in the consumption of tea before and after increase of excise duty.

Alternative hypothesis H_1 : $p_1 > p_2$ (Right tailed test)

Under H_0, the test static
$$z = \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{\hat{p}(1-\hat{p})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$
$$= \frac{0 \cdot 8 - 0 \cdot 6666}{\sqrt{\frac{8}{11} \times \frac{3}{11}\left(\frac{1}{1000} + \frac{1}{1200}\right)}} = 9 \cdot 5$$

$\Rightarrow$ $|z| = 9 \cdot 5$

Conclusion. Since the calculated value of $|z| > 1 \cdot 645$. Also $|z| > 2 \cdot 33$. Hence H_1 is rejected at 5% and 1% level of significance, *i.e.*, there is a significant decrease in the consumption of tea due to increase in excise duty.

Example 2. ***In two large populations there are 30% and 25% respectively of fair haired people. Is this difference likely to be hidden in samples of 1200 and 900 respectively from the two population ?***

Solution. $\hat{p}_1$ = proportion of fair haired people in the first population

$= 30\% = 0 \cdot 3;\ \hat{p}_2 = 25\% = 0 \cdot 25;\ n_1 = 1200,\ n_2 = 900$

Null hypothesis H_0 : Sample proportions are equal *i.e.*, the difference in population proportions is likely to be hidden in sampling.

Alternative hypothesis H_1 : $p_1 \neq p_2$

$\therefore$
$$\hat{p} = \frac{n_1\hat{p}_1 + n_2\hat{p}_2}{n_1 + n_2} = \frac{360 + 225}{2100} = 0 \cdot 2785$$

$$1 - \hat{p} = 1 - 0 \cdot 2785 = 0 \cdot 7215$$

Under H_0, $z = \dfrac{\hat{p}_1 - \hat{p}_2}{\sqrt{\hat{p}(1-\hat{p})\left(\dfrac{1}{n_1}+\dfrac{1}{n_2}\right)}} = \dfrac{0.3-0.25}{\sqrt{0.2785\times 0.7215\left(\dfrac{1}{1200}+\dfrac{1}{900}\right)}}$

$$= \frac{0.05\times 10}{\sqrt{0.20(0.083+0.1111)}} \cdot \frac{0.5}{\sqrt{0.1970}} = 2.538$$

Conclusion. Since $|z| > 1.96$ $\therefore$ H_0 is rejected. However $|z| < 2.58$ $\therefore$ H_0 is accepted. At 5% level, these samples will reveal the difference in the population proportions.

Example 3. ***500 articles from a factory are examined and found to be 2% defective. 800 similar articles from a second factory are found to have only 1.5% defective. Can it reasonably be concluded that the product of the first factory are interior to those of second ?***

Solution. $n_1 = 500$, n_2 800

$\hat{p}_1$ = proportion of defective from first factory = 2% = 0.02

$\hat{p}_2$ = proportion of defective from second factory = 1.5% = 0.015

Null hypothesis H_0 : There is no significant difference between the two products, *i.e.*, the products do not differ in quality.

Alternative hypothesis H_1 : $p_1 < p_2$ (One tailed test)

Under H_0, $z = \dfrac{\hat{p}_1 - \hat{p}_2}{\sqrt{\hat{p}(1-\hat{p})\left(\dfrac{1}{n_1}+\dfrac{1}{n_2}\right)}}$

Here $\hat{p} = \dfrac{n_1 \hat{p}_1 + n_2 \hat{p}_2}{n_1+n_2} = \dfrac{0.02(500)+(0.015)(800)}{500+800} = 0.01692$

$q = 1 - \hat{p} = 0.9830$

$\therefore$ $z = \dfrac{0.02-0.015}{\sqrt{0.01692\times 0.983\left(\dfrac{1}{500}+\dfrac{1}{800}\right)}} = 0.68$

Conclusion. As $|z| < 1.645$ $\therefore$ H_0 is accepted, *i.e.*, the products do not differe in quality.

ILLUSTRATIVE EXAMPLES BASED ON SINGLE MEAN (LARGE SAMPLE)

Example 1. ***A normal population has a mean of 6.8 and standard deviation of 1.5. A sample of 400 members gave a mean of 6.75. Is the difference significant ?***

Solution. Null Hypothesis H_0 = There is no significant difference between $\bar{x}$ and μ *i.e.*, $\bar{x} = \mu$

Alternative Hypothesis H_1 : There is significant difference between $\bar{x}$ and μ *i.e.*, $\bar{x} \neq \mu$

Given, $\mu = 6.8$, $\sigma = 1.5$, $\bar{x} = 6.75$ and $n = 400$

Under H_0; $z = \dfrac{\bar{x}-\mu}{\sigma/\sqrt{n}} = \dfrac{6.75-6.8}{1.5/\sqrt{400}} = -0.67 \Rightarrow |z| = 0.67$

Conclusion. As the calculated value of $|z| < z_\alpha = 1.96$ at 5% level of significance.

$\therefore$ H_0 is accepted, *i.e.*, there is no significant difference between $\bar{x}$ and μ.

Example 2. ***The mean weight obtained from a random sample of size 100 is 64 gms. The S.D. of the weight distribution of the population is 3 gms. Test the statement that the mean weight of the population is 67 gms at 5% level of significance. Also set up 99% confidence limits of the mean weight of the population.***

Solution. Here $n = 100$, $\mu = 67$, $\bar{x} = 64$, $\sigma = 3$

Null Hypothesis H_0 : There is no significant difference between sample and population mean, *i.e.*, $\mu = 67$.

Alternative Hypothesis H_1 : $\mu \neq 67$ (Two tailed test)

Under H_0, $z = \dfrac{\bar{x}-\mu}{\sigma/\sqrt{n}} = \dfrac{64-67}{3/\sqrt{100}} = -10$ $\therefore$ $|z| = 10$

Conclusion. Since the calculated value of $|z| > 1.96$

$\therefore$ H_0 is rejected at 5% level of significance *i.e.*, the sample is not drawn from the population with mean 67.

To find 99% confidence limits

It is given by $\bar{x} \pm 2.58\,\sigma/\sqrt{n} = 64 \pm 2.58 \times 3/\sqrt{100} = 64.774, 63.226$.

Example 3. ***The average marks in Mathematics of a sample of 100 students was 51 with a S.D. of 6 marks. Could this have been a random sample from a population with average marks 50 ?***

Solution. Here $n = 100$, $\bar{x} = 51$, $s = 6$, $\mu = 50$; σ is unknown.

Null Hypothesis H_0 : The sample is drawn from a population with mean 50 *i.e.*, $\mu = 50$

Alternative Hypothesis H_1 : $\mu \neq 50$ (Two tailed test)

Under H_0, $z = \dfrac{\bar{x}-\mu}{s/\sqrt{n}} = \dfrac{51-50}{6/\sqrt{100}} = \dfrac{10}{6} = 1.6666$

Conclusion. Since $|z| = 1.666 < 1.96$

$\therefore$ H_0, is accepted at 5% level of significance *i.e.*, the sample is drawn from the population with mean 50.

ILLUSTRATIVE EXAMPLES BASED ON DIFFERENCE OF MEAN (LARGE SAMPLE)

Example 1. ***The average income of persons was Rs. 210 with a S.D. of Rs. 10 in sample of 100 people of a city. For another sample of 150 persons, the average income was Rs. 220 with S.D. of Rs. 12. The S.D. of incomes of the people of the city was Rs. 11. Test whether there is any significant difference between the average incomes of the localities.***

Solution. Here $n_1 = 100, n_2 = 150, \bar{x}_1 = 210, \bar{x}_2 = 220, s_1 = 10, s_2 = 12$

Null Hypothesis H_0 : The difference is not significant, *i.e.*, there is no difference between the incomes of the localities, *i.e.*, $H_0 : \bar{x}_1 = \bar{x}_2$

Alternative Hypothesis $H_1 : \bar{x}_1 \neq \bar{x}_2$ (Two tailed test)

Under H_0, $$z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} = \frac{210 - 220}{\sqrt{\frac{10^2}{100} + \frac{12^2}{150}}} = -7 \cdot 1428 \quad \therefore |z| = 7 \cdot 1428$$

Conclusion. As the calculated value of $|z| > 1 \cdot 96$.

$\therefore$ H_0 is rejected at 5% level of significance *i.e.*, there is significant difference between the average incomes of the localities.

Example 2. ***Intelligent test were given to two groups of boys and girls.***

	mean	*S.D.*	*Size*
Girls	*75*	*8*	*60*
Boys	*73*	*10*	*100*

Examine if the difference between mean scores is significant.

Solution. Null Hypothesis H_0 : The is not significant difference between mean scores, *i.e.*, $H_0 : \bar{x}_1 = \bar{x}_2$

Alternative Hypothesis $H_1 : \bar{x}_1 \neq \bar{x}_2$ (Two tailed test)

Under H_0, $$z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} = \frac{75 - 73}{\sqrt{\frac{8^2}{60} + \frac{10^2}{100}}} = 1 \cdot 3912$$

Conclusion. As the calculated value of $|z| > 1 \cdot 96$.

$\therefore$ H_0 is accepted at 5% level of significance *i.e.*, there is no significant difference between mean scores.

Example 3. ***A random sample of 200 villages from Coimbatore district gives the mean population per village at 485 with a S.D. of 50. Another random sample of the same size from the same district gives the mean population per village at 510 with a S.D. of 40. Is the difference between the mean values given by the two samples statistically significant ? Justify your answer.***

Solution. Here $n_1 = 200, n_2 = 200, \bar{x}_1 = 485, \bar{x}_2 = 510, s_1 = 50, s_2 = 40.$

Null Hypothesis H_0 : There is no significant difference between the mean values *i.e.*, $\bar{x}_1 = \bar{x}_2$

Alternative Hypothesis $H_1 : \bar{x}_1 \neq \bar{x}_2$ (Two tailed test)

Under H_0, the test statistic is given by

$$z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} = \frac{485 - 510}{\sqrt{\frac{50^2}{200} + \frac{40^2}{200}}} = -5 \cdot 52$$

$$\therefore \quad |z| = 5 \cdot 52.$$

Conclusion. As the calculated value of $|z| > 1 \cdot 96$.

$\therefore$ H_0 is rejected at 5% level of significance *i.e.*, there is significant difference between the mean values of two samples.

Example 4. ***The means of two large samples of 1,000 and 2,000 members are 67·5 inches and 68·0 inches respectively. Can the samples be regarded as drawn from the same population of standard deviation 2·5 inches ?***

Solution. Given $n_1 = 1000, n_2 = 2000, \bar{x}_1 = 67 \cdot 5, \bar{x}_2 = 68, \sigma = 2 \cdot 5$

Assume H_0 : The samples are drawn from the same population *i.e.*, $\bar{x}_1 = \bar{x}_2$

Under H_0, the test statistic is given by

$$z = \frac{\bar{x}_1 - \bar{x}_2}{\sigma\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = \frac{67 \cdot 5 - 68}{(2 \cdot 5)\sqrt{\left(\frac{1}{1000} + \frac{1}{2000}\right)}}$$

$$= \frac{-0 \cdot 5}{(2 \cdot 5)\sqrt{(0 \cdot 001 + 0 \cdot 0005)}} = \frac{-0 \cdot 5}{(2 \cdot 5)(0 \cdot 03872)} = \frac{-0 \cdot}{0 \cdot 09682}$$

$$= -5 \cdot 1642$$

$$\Rightarrow \quad |z| = 5 \cdot 1642.$$

Conclusion. Since $|z| > 1 \cdot 96$.

$\therefore$ H_0 is rejected at 5% level of significance *i.e.*, the samples are not drawn from the same population.

Chapter UNIT-VI

10

ELEMENTARY THEORY OF TESTING OF HYPOTHESIS "SMALL SAMPLE"

§ 10.1. ORIGIN OF THE THEORY OF SAMPLING

First of all we shall define the term population. In statistics, the word **'population'** or **'universe'** means any collection of individuals or any collection of results of operation which can be numerically specified or any collection of attributes of individuals.

In practical problems, usually we have to discuss a population of which we cannot examine every member. Thus there arise a question, "what can be said about a population of which only a limited number of members can be examined?" This question leads to "the origin of the theory of sampling." *A part or small section selected from the population is called a sample and the process of such selection is called sampling.* The sampling should be unbiased.

§ 10.2. OBJECT OF SAMPLING

The fundamental object of sampling is to get as much information as possible about the whole population. By sampling it is tried to get maximum information about the parent population with the minimum effort.

Another aim of the sampling is to establish the reliability of the collected estimates. This can be achieved by comparing the results obtained from the successive samples of the parent population.

§ 10.3. SOME DEFINITIONS. TYPES OF POPULATION (OR UNIVERSE)

A population containing a finite number of individuals (or members) is called a **finite population**, for example : population of heights of students in a class of forty students.

A population containing an infinite number of individuals (or members) is called an **infinite population**, for example; the population of stars.

A population of concrete individuals (or objects) is called an existent **population.**

A hypothetical population is the collection of all possible ways in which an event can happen. For example : The population of heads and tails obtained by tossing a coin an infinite number of times (with the condition that it does not wear out) is a hypothetical population.

Parameter. The statistical constants such as mean, variance, skewness etc. are called *parameters* of population (or universe).

Statistics. It is the statistical measurements obtained from the calculation of samples only.

§ 10.4. STATISTICAL HYPOTHESIS

Definition. A statistical hypothesis is tentative statement about the distribution of one or more random variables. Its object is to specify some parameter to the population. By some test we shall decide whether to accept or to reject a parameter in a pre-assumed sample. This process is called testing of hypothesis. Hypothesis are of two types :

(i) Null Hypothesis,

(ii) Composite Hypothesis.

§ 10.5. NULL HYPOTHESIS

Suppose we are given a sample. Suppose we are to calculate a certain statistic such as mean from this sample. We assume that this sample is taken from a population of known form for which the corresponding parameter is tentatively specified. This tentative specification is called a *null hypothesis.*

For example consider an experiment of tossing a coin. In this experiment, we want to test whether the coin is biased. Consequently the null hypothesis in this case is that the coin is unbiased *i.e.*, $p = \frac{1}{2}$, where p is probability of the occurrence of a head (or tail). If our experiment gives a value which deviates significantly from the value of the parameter $\left(i.e., \ p = \frac{1}{2}\right)$, then we say that the null hypothesis is contradicted and consequently the coin is biased. If this deviation is not significant then the hypothesis is accepted and it is assumed that the deviation is attributed to sampling fluctuations.

Definition. *A null hypothesis is a hypothesis which is tested for possible rejection under the assumption that it is true.* This definition is due to professor R. A. Fisher. This null hypothesis is denoted by H_0.

Composite or Alternative Hypothesis :

Definition. *Composite hypothesis is complementary or alternative to null hypothesis*, we shall denote it by H_1. By definition of composite hypothesis it follows that when H_0 is rejected in an experiment then H_1 is accepted.

For example. In an experiment of null hypothesis $H_0 : \mu = \mu_0$ *i.e.*, if the population has a particular mean μ_0, then we shall have the alternatives.

$H_1 : \mu \neq \mu_0$ *i.e.*, $\mu > \mu_0$ or $\mu < \mu_0 \Rightarrow$ two tailed alternative.

$H_1 : \mu > \mu_0 \Rightarrow$ right tailed alternative.

$H_1 : \mu < \mu_0 \Rightarrow$ left tailed alternative.

§ 10.6. ERRORS OF FIRST AND SECOND KINDS

Two Types of Error :

In a testing a null hypothesis H_0 against an alternative hypothesis, H_1, we can make two possible types of error :

(i) When we reject the null hypothesis H_0, though it is true, then it is known as **error of first kind** or **type I error**.

(ii) We may accept the null hypothesis H_0, though it is false (or it ought to be rejected), then it is known as **error of second kind** or **type II error**. Here we have assumed that the result of rejection of H_0 is the acceptances of H_1.

We shall denote the probabilities of the occurrence of type I and type II, errors by α and β respectively. Thus

$$\alpha = P\,(\text{reject } H_0 \text{ when it is true})$$
$$= P\,(\text{reject } H_0 / H_0)$$
$$\beta = P\,(\text{accept } H_0 \text{ when it is false})$$
$$= P\,(\text{accept } H_0 / H_1)$$

Decision	H_0 is true	H_1 is true (H_0 is false)
H_0 accepted	Right judgement	Type II Error (β)
H_1 accepted (H_0 is rejected)	Type I error (α)	Right judgement

§ 10.7. CRITICAL REGION (OR REJECTION REGION) AND ACCEPTANCE REGION

The fundamental aim of testing of hypothesis is to divide the sample space into two exclusive regions :

(i) Acceptance region and (ii) Rejection region. If the sample point falls within the rejection region, then H_0 is rejected. The region of rejection is called **Critical region**.

Definition. That region of a sample space in which null hypothesis is rejected is called *Critical region* or rejection region and the rest of the sample space is called the *region of acceptance*.

If $X : x_1, x_2, \dots, x_n$ is random observations; if W is the whole region of the sample space; W_c is the critical region, then the region of acceptance W_a is given by

$$W_a = W - W_c.$$

How to choose critical region. Keep type I error fixed at a specified value and, then choose that region which minimizes the type II error. The type I error (which is the same for all these regions) is called the size of these regions.

§ 10.8. LEVEL OF SIGNIFICANCE

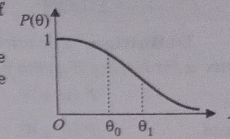
In testing a given hypothesis (here H_0) the maximum probability with which we would be willing to risk an error is called the **level of significance** of the test. The accepted levels of significance are 0·05 and 0·01. Here 0·05 level of significance indicates that there are about 5 chance in 100 that we would reject correct H_0. Similarly 0·01 level of significance indicate that there is about 1 chance in 100 that we would reject a correct H_0.

Definition. α, *the probability of type I error, is called the level of significance of the test.* It is also known as the size of the critical region.

§ 10.9. SIZE AND POWER FUNCTION OF A TEST

Definition. A power function $P(\theta)$ of a test [or power expressed as some function of parameter θ] is defined as the probability of the sample point falling in the critical region of the test where θ is a real parameter. Thus if β is the measure (or size) of type II error, then $\beta(\theta)$ is the probability of the sample point falling in the non-critical region.

Thus $1 - \beta(\theta)$ is also defined as the power function of H_0 against the alternative hypothesis H_1.



Now plot the points taking parameter θ and the corresponding power function $P(\theta)$, then the curve drawn joining these points is called the power curve.

Definition :

$$P(\theta) = P\,(\text{rejection of } H_0/\text{when } H_1 \text{ is true})$$
$$= P\,(w \in W_a / H_1)$$
$$= 1 - P\,(w \in W_c / H_1)$$
$$= 1 - P\,(\text{acceptance of } H_0 / H_1)$$
$$= 1 - \beta(\theta).$$

Power of the Test :

The power of the test is the value of the power function at a specified value of the parameter.

Therefore, if we are to test $H_0 : \theta = \theta_0$ against $H_1 : \theta = \theta_1$, then the power of the test

$$1 - \beta = P\,(w \in W_c / H_0)$$
$$= P\,(\text{rejection of } H_0 /\text{when } H_1 \text{ is true}).$$

§ 10.10. STEPS (RULES) IN SOLVING TESTING OF HYPOTHESIS PROBLEM

Following are the major steps in the solution of a 'testing of hypothesis' problem :

(a) We should have the explicit knowledge of the nature of the population distribution and the parameter(s) about which the hypothesis are set up.

(b) To set up the null hypothesis H_0 and the alternative hypothesis H_1 in terms of the range of the parameter values each one embodies.

(c) The selection of a suitable statistic $t = t\,(x_1, x_2, \dots, x_n)$ known as test statistic, which will best reflect upon the probability of H_0 and H_1.

(d) To partition the set of possible values of the test statistic t into two disjoint sets (or regions) W_c (known as critical or rejection region) and W_0 (known as acceptance region) and setting the following test :

(i) Rejection of H_0 (*i.e.*, acceptance of H_1) if the value of t falls in W_0.

(ii) Acceptance of H_0 if the value of t falls in W_a.

(e) After setting the above test, obtain experimental sample observations, compute the suitable test statistic.

The first two steps and the framing of H_0 and H_1, will become clear from the description of problems. The most important step is to choice the 'best test (*i.e.*, best *t*)' and the critical region W_c. Best test means 'controlling type I error α at any desired low level and minimization of type II error β'. From this we have the following definition.

◈ § 10.11. MOST POWERFUL TEST AND MOST POWERFUL CRITICAL REGION

Definition. *The critical region W_c is called most powerful critical region of size α for testing $H_0 : \theta = \theta_0$ against $H_1 : \theta = \theta_1$ if*

$$P(w \in W_c / H_0) = \int_{W_c} L_0 \, dx = \alpha \qquad \text{...(1)}$$

and

$$P(w \in W_c / H_1) \geq P(w \in W_c'/H_1) \qquad \text{...(2)}$$

[$\because$ other critical region W_c' satisfying (1)]

The corresponding test is called most powerful test of level α.

Best Critical Region :

In testing the hypothesis $H_0 : \theta = \theta_0$ against the alternative hypothesis $H_1 : \theta = \theta_1$ the critical region is best if type II error is minimum when compared to every other possible critical region of size α.

Clearly a test defined by best critical region is said to be *most powerful test*.

ILLUSTRATIVE EXAMPLES

Example. 1. *It is desired to test a hypothesis $H_0 : p = p_0 = \frac{1}{2}$ against the alternative hypothesis $H_1 : p = p_1 = \frac{3}{4}$ on the basis of tossing a coin once, where p is the probability of getting a head in a single trial and agreeing to accept H_0 if a tail appears and to accept H_1 otherwise. Find the values of α and β.*

Solution. Clearly, we have

The probability of committing type I error :

$$\begin{aligned} \alpha &= \text{probability of rejecting } H_0 \text{ when } H_0 \text{ is true} \\ &= P(\text{reject } H_0 / H_0) \\ &= P(\text{head appears}/ p = p_0) \\ &= [p] \text{ when } p = p_0 = \frac{1}{2} \\ &= \frac{1}{2}. \end{aligned}$$

The probability of committing type II error :

$$\begin{aligned} \beta &= \text{probability of accepting } H_0 \text{ when } H_1 \text{ is true} \\ &= P(\text{accept } H_0 / H_1) \\ &= P(\text{tail appears}/ p = p_1) \\ &= [1 - p] \text{ when } p = p_1 = 3/4 \\ &= 1 - (3/4) = \frac{1}{4}. \end{aligned}$$

Example 2. *Given a binomial distribution*

$$f(x, p) = \begin{cases} {}^nC_x \, p^x q^{n-x} & , x = 0, 1, 2, 3, 4 \\ 0 & , \text{elsewhere} \end{cases}.$$

It is desired to test $H_0 : p = p_0 = \frac{1}{3}$ against $H_1 : p = p_1 = \frac{1}{2}$ by agreeing to accept H_0 if $x \leq 2$ in four trials and to reject otherwise. What are the probabilities of committing :

(i) type I error,

(ii) type II error.

Solution. (i) The probability of committing type I error :

$$\begin{aligned} \alpha &= P(\text{type I error}) \\ &= P(\text{reject } H_0 \text{ when } H_0 \text{ is true}) \\ &= P(\text{reject } H_0 / H_0 \text{ is true}) \\ &= P\left(x > 2 / p = p_0 = \frac{1}{3}\right) \\ &= \sum_{x=3}^{4} {}^4C_x \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{4-x} \qquad \left[\because q = 1 - p = 1 - \frac{1}{3} = \frac{2}{3}\right] \\ &= {}^4C_3 \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^{4-3} + {}^4C_4 \left(\frac{1}{3}\right)^4 \left(\frac{2}{3}\right)^0 \\ &= 4 \times \frac{2}{81} + \frac{1}{81} = \frac{9}{81} = \frac{1}{9}. \end{aligned}$$

Hence the hypothesis $H_0 : p = p_0 = \frac{1}{3}$ is being tested at the level of significance $\alpha = \frac{1}{9}$.

(ii) The probability of committing type II error :

$$\begin{aligned} \beta &= P(\text{type II error}) \\ &= P(\text{accept } H_0 \text{ when } H_1 \text{ is true}) \\ &= P\left(x \leq 2 / p = p_1 = \frac{1}{2}\right) \\ &= \sum_{x=0}^{2} {}^4C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{4-x} \qquad \left[\because q = 1 - p = 1 - \frac{1}{2} = \frac{1}{2}\right] \\ &= \sum_{x=0}^{2} {}^4C_x \left(\frac{1}{2}\right)^4 \\ &= {}^4C_0 \left(\frac{1}{2}\right)^4 + {}^4C_1 \left(\frac{1}{2}\right)^4 + {}^4C_2 \left(\frac{1}{2}\right)^4 \\ &= \left(\frac{1}{2}\right)^4 + 4\left(\frac{1}{2}\right)^4 + 6\left(\frac{1}{2}\right)^4 = \frac{11}{2^4} = \frac{11}{16}. \end{aligned}$$

EXERCISE 10 (A)

1. Explain the following terms :
   - (i) Null hypothesis
   - (ii) Alternative hypothesis
   - (iii) Testing of hypothesis
   - (iv) Level of significance
   - (v) Critical region
   - (vi) Most powerful critical region

(vii) Power function (viii) Hypothesis
(ix) Acceptance region (x) Best critical region
(xi) Power of a test (xii) Two types of error.

2. Define the following terms :
(i) Null hypothesis (ii) Two kinds of error
(iii) Level of significance.
3. Write a short note on 'rules of testing a hypothesis'.
4. Write a short note on best critical region.
5. Write a short note on null (zero) hypothesis.
6. Write a short note on testing of hypothesis.
7. Write a short note on type I and type II error.
8. Define the errors of I and II kinds.
9. Explain the following terms :
(i) Statistic (ii) Standard error
(iii) Parameter (iv) Sample.
10. Given a binomial distribution :

$$f(x, p) = \begin{cases} {}^nC_x \, p^x \, q^{n-x} & , \; x = 0, 1, 2, 3 \\ 0 & , \; \text{elsewhere} \end{cases}$$

it is desired to test $H_0 : p = p_0 = \frac{1}{4}$ against $H_1 : p = p_1 = \frac{1}{2}$ by agreeing to accept H_0 if $x \le 1$ in three trials and to accept H_1 otherwise. Find the probability of the following :
(i) Type I error (ii) Type II error.
11. A random variable X follows the exponential distribution

$$f(x, \theta) = \begin{cases} \frac{1}{\theta} e^{-x/\theta} & , \; 0 \le x \le \infty, \theta > 0 \\ 0 & , \; \text{otherwise.} \end{cases}$$

The null hypothesis $H_0 : \theta = 2$ is rejected and alternative hypothesis $H_1 : \theta = 4$ is accepted. If an observation selected at random takes the value 6 or more, then find :
(a) Critical region
(b) Acceptance region
(c) Size of two types of error.
12. A random variable follows the exponential distribution

$$f(x, \theta) = \begin{cases} \frac{1}{\theta} e^{-x/\theta} & , \; 0 \le x \le \infty, \theta > 0 \\ 0 & , \; \text{otherwise} \end{cases}$$

it is desired to test null hypothesis $H_0 : \theta = 2$ against alternative hypothesis $H_1 : \theta = 4$ on the basis of a random sample of size 2. If the critical region W_c is given by

$$W_c : (9{\cdot}5 \le x_1 + x_c \le \infty)$$

then find the power of the test and the sizes of the critical region.
13. A random variable has a following probability density function :

$$f(x, \theta) = \begin{cases} \frac{1}{2} & , \; \theta - 1 \le x \le \theta + 1 \\ 0 & , \; \text{elsewhere} \end{cases}$$

obtain the critical region (range) of size $\alpha = 0{\cdot}25$ for testing $H_0 : \theta = 4$ against $H_1 : \theta = 5$ on the basis of single observed value of x.

14. Given the frequency function

$$f(x, \theta) = \begin{cases} \frac{1}{\theta} & , \; 0 \le x \le \infty, \theta > 0 \\ 0 & , \; \text{elsewhere} \end{cases}$$

and that you are testing the null hypothesis $H_0 : \theta = 1$ against $H_1 : \theta = 2$ by means of a single observed value of x. Then
(a) Find critical regions and errors of type I and type II, if you choose the intervals
(i) $0{\cdot}5 \le x$ and (ii) $1 \le x \le 1{\cdot}5$.
Also obtain the power function of the test.
(b) Find critical region and errors of type I and type II, if you choose the intervals
(i) $x \ge 0{\cdot}7$ and (ii) $0{\cdot}8 \le x \le 1{\cdot}3$.
15. Given

$$f(x, \theta) = \begin{cases} \frac{1}{k} & , \; \theta - 1 < x < \theta + 1 \\ 0 & , \; \text{otherwise.} \end{cases}$$

If $H_0 : \theta = 2$ and $H_1 : \theta = 3$ and the critical range is to be of the size $\alpha = 0{\cdot}25$ and to consist of a single interval, what critical region should be chosen and obtain the value of β, assuming that the test is based on single observed value of X.
16. Find the values of α and β for testing a hypothesis $H_0 : \theta = \theta_0 = \frac{1}{2}$ against alternative hypothesis $H_1 : \theta = \theta_1 = \frac{2}{3}$ on the basis of tossing a coin thrice and it is decided to reject H_0 if 3 head appear and to accept H_1. Here θ is the probability of a head in a single trial.

ANSWERS

10. (i) $\frac{5}{32}$ (ii) $\frac{1}{2}$
11. (a) $W_c : x \ge 6$ (b) $W_a : x > 6$ (c) $\alpha = e^{-3}, \beta = 1 - e^{-3/2}$
12. Power of the test $= 1 - \beta = 0{\cdot}31$, Size of critical region $\alpha = 0{\cdot}05$ (nearly)
13. $4{\cdot}25 \le x \le 4{\cdot}5$
14. (a) (i) $\alpha = 0{\cdot}5, \beta = 0{\cdot}25$ (ii) $\alpha = 0, \beta = 0{\cdot}75$, Power $= 1 - \beta = 0{\cdot}25$
(b) (i) $\alpha = 0{\cdot}3, \beta = 0{\cdot}35$ (ii) $\alpha = 0{\cdot}2, \beta = 0{\cdot}2$
15. $x > 2{\cdot}5, \beta = 0{\cdot}25$
16. $\alpha = \frac{1}{8}, \beta = \frac{19}{27}$.

EXERCISE 10 (B)

Objective Type Questions

1. Rejection of null hypothesis H_0 when it is true, is called :
(a) Type I error (b) Type II error
(c) Both type I and type II errors (d) None of these.
2. Acceptance of null hypothesis when it is false, is called :
(a) Type I error (b) Type II error
(c) Both type I and type II errors (d) None of these.

ANSWERS

1. (a) 2. (b).

◆ § 10.13. TEST OF SIGNIFICANCE BASED ON t-DISTRIBUTION (SMALL SAMPLE)

Now we shall discuss the theory of small samples (<< 30). It is not always possible to have large samples on account of economical factors or some other factors and so we depend on small samples. In case of small samples, it is no longer possible to assume that the random sampling distribution of a statistic is approximately normal. A new technique is therefore necessary to deal with the theory of small samples. In the recent years a theory has been developed which provides more exact methods to discuss with small samples. We shall confine our work to find the estimates of the mean and the standard deviation as these are the two main parameters of interest.

It was William Gasset (W. S. Gasset) who in 1908 under the pen name of 'Student' first found the distribution of t. R. A. Fisher later on defined t correctly and found its distribution in 1926.

Student-t :

Definition. Let $x_1, x_2, \ldots, x_n$ be the members of random sample drawn from a normal population with mean M and variance σ^2. Then t-statistic is defined by

$$t = \frac{(\bar{x} - M)}{s'} \sqrt{n} \qquad \ldots(1)$$

where $\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$ (mean of the sample)

and $s'^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2$

or $(n-1)\, s'^2 = \sum_{i=1}^{n} (x_i - \bar{x})^2 = ns^2.$

Hence
$$\frac{t^2}{\nu} = \frac{t^2}{n-1} = \frac{(\bar{x} - M)^2\, n}{(n-1)\, s'^2} \qquad [\text{where } \nu = n-1]$$

i.e.,
$$\frac{t^2}{\nu} = \frac{(\bar{x} - M)^2 \div \frac{\sigma^2}{n}}{ns^2 / \sigma^2}.$$

Since $x_1, x_2, \ldots, x_n$ are the members of a random sample drawn from a normal population i.e., x_i $(i = 1, 2, \ldots, n)$ is a random sample drawn from the normal population with mean M, $\bar{x} \sim N(M, \sigma / n)$

$$\Rightarrow \qquad \frac{\bar{x} - M}{\sigma / \sqrt{n}} \sim N(0, 1).$$

But we know that $\frac{(\bar{x} - M)^2\, n}{\sigma^2}$ which is the standard normal variation, is distributed as a χ^2 variate with 1 degree of freedom. Therefore t^2 / ν is the ratio of two independent variates, distributed with 1 and ν degree of freedom respectively and thus t^2 / ν is $\beta_2 \left[\frac{1}{2}, \nu / 2\right]$ variate.

If we take different samples, then the distribution of t (established by Fisher) is given by

$$y = \frac{y_0}{(1 + t^2 / \nu)^{(\nu + 1)/2}}, \quad -\infty < x < \infty. \qquad \ldots(2)$$

where y_0 is a constant depending on n and $\nu = n - 1$ is the number of degrees of freedom.

If y_0 is chosen such that the area of the curve given by equation (2) above is unity, then

$$y = \frac{1}{\sqrt{\nu}\, \beta\left(\frac{1}{2}, \frac{1}{2} \nu\right)} \cdot \frac{1}{(1 + t^2 / \nu)^{(\nu + 1)/2}}. \qquad \ldots(3)$$

This equation (3) is called the t-**Probability curve.**

The probability density function of t-distribution with $\nu = n - 1$ degrees of freedom is given by

$$f(t) = \frac{1}{\sqrt{\nu}\, \beta\left(\frac{1}{2}, \frac{\nu}{2}\right)\left(1 + \frac{t^2}{\nu}\right)^{(1+\nu)/2}}, -\infty < t < \infty$$

$$P(|\,t\,| > t_0) = \int_{-\infty}^{t_0} f(t)\, dt + \int_{t_0}^{\infty} f(t)\, dt = 2 \int_{t_0}^{\infty} f(t)\, dt.$$

The values of the above has been tabulated for different degrees of freedom. The critical value of t for null hypothesis can be seen from this table. For $n > 30$, the value z of t becomes equal to $[z \sim N(0, 1)]$.

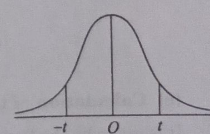
◆ § 10.14. PROBABILITY TABLES

If the value of y_0 is chosen such that the area under the curve be unity then the probability that the value of t for a random sample will lie between two fixed values t_1 and t_2 is given by

$$= \int_{t_1}^{t_2} y_0\, (1 + t^2 / \nu)^{-(\nu + 1)/2}\, dt.$$

The tables for the probability integral $P_S = \int_{-\infty}^{t} y\, dt$ was published by Student in 1925.

But the tables of the probability P_F of the occurrence of deviations lying outside $\pm\, t$ for given values of t and ν were prepared by Fisher and Yate. Now



$$P_F = 2 \int_{t}^{\infty} y\, dt.$$

If
$$P = \int_{-t}^{t} y\, dt = 2 \int_{0}^{t} y\, dt$$

then, we have
$$P_F = 1 - P.$$

But $P_S = \int_{-\infty}^{t} y\,dt = \int_{-\infty}^{0} y\,dt + \int_{0}^{t} y\,dt = \frac{1}{2} + \frac{1}{2}P$

$= \frac{1}{2} + \frac{1}{2}(1 - P_F)$ $[\because P_F = 1 - P]$

$\Rightarrow$ $2P_S = 2 - P_F \Rightarrow P_F = 2(1 - P_S).$

Tabular value of t. The tabular value of $t_{0.05}$ and $t_{0.01}$ corresponding to $\nu (= n-1)$ d.f. are defined as follows :

$$P[|t| \geq t_{0.05}] = 0.05, P[|t| \geq t_{0.01}] = 0.01.$$

For example. The tabular value of t at 5% ($P = 0.05$) significance level for 8 d.f. is 2.3060.

Interpretation. The calculated value of t is compared with the tabular value of t. If the calculated value of t is greater than the tabular of t then the difference is significant and our null hypothesis will be false. If the calculated value of t is less than its tabular value then the difference will be insignificant and so the null hypothesis will be true. Usually the test of significance is done at 5% level though it can be performed at 1% level.

§ 10.15. USE OF t-DISTRIBUTION

There are various uses of t-distribution. Some of them are as follows :

(i) To test the significance of the mean of a small random sample.

(ii) To test the significance of the difference between two means or to compare two samples.

(iii) To test the significance of an observed coefficient of correlation.

(iv) To test the significance of an observed coefficient of regression.

We shall now discuss few of these tests in some detail :

§ 10.15. (A) TO TEST THE SIGNIFICANCE OF THE MEAN OF RANDOM SAMPLE FROM NORMAL POPULATION

To test the significance of the difference between the mean of the small sample and the mean of the population (universe), the following methods are adopted :

(a) Null hypothesis. It is pre-assumed that there is no difference between the means of the small sample and the population *i.e,* the difference between their means is zero.

(b) Calculation of S. D. (s') of the population.

It is calculated from the following formula :

$$s' = \sqrt{\left[\frac{\Sigma(x-\bar{x})^2}{(n-1)}\right]}.$$

(c) Calculation of t-statistic.

It is given by the formula :

$$t = \frac{\bar{x} - M}{s'}\sqrt{n}.$$

But if standard deviation is known, then

$$t = \frac{\bar{x} - M}{\sigma}\sqrt{(n-1)}.$$

(d) Determination of degrees of freedom.

It is obtained by d. f. $= \nu = n - k$

where n = number of independent observations *i.e,* number of total frequencies

k = number of independent constraints.

Now apply the method as explained in § 10.8 above.

ILLUSTRATIVE EXAMPLES

Example 1. ***A machine which produces mica insulating washers of use in electric device is set to turn out washers having a thickness of 10 mils (1 mil = 0.001 inch.). A sample of 10 washers has an average thickness of 9.52 mils with a standard deviation of 0.60 mil. Find out t.***

Solution. We are given : $\bar{x} = 9.52, M = 10, s' = 0.60$, the test t-statistic is

$$t = \frac{\bar{x} - M}{s'}\sqrt{n} = \frac{9.52 - 10}{0.60}\sqrt{10}$$

$$= -\frac{0.48}{0.60} \times 3.1622777 = 0.8 \times 3.1622777$$

$$= 2.5298.$$

Ans.

Example 2. ***Ten individuals are chosen at random from a population and their heights are found to be in inches 63, 63, 64, 65, 66, 69, 69, 70, 70, 71. Discuss the proposal that the mean height in the universe is 65 inches given that for 9 degrees of freedom the value of Student's t at 5 percent level of significance is 2.262.***

Solution. The test-statistic is $t = \frac{(\bar{x} - M)}{s'}\sqrt{n}$

where $\bar{x}$ and s' are to be calculated from the sample values.

The calculation table is :

No of Individuals	x	$x - \bar{x} = x - 67$	$(x - \bar{x})^2$
1	63	−4	16
2	63	−4	16
3	64	−3	9
4	65	−2	4
5	66	−1	1
6	69	2	4
7	69	2	4
8	70	3	9
9	70	3	9
10	71	4	16
$n = 10$	$\Sigma x = 670$		$\Sigma(x - \bar{x})^2 = 88$

$\therefore$ Sample mean $\bar{x} = \frac{\Sigma x}{n} = \frac{670}{10} = 67.$

Standard deviation

$$s' = \sqrt{\left(\frac{\Sigma (x-\bar{x})^2}{n-1}\right)} = \sqrt{\left(\frac{88}{9}\right)}$$

$$= 3 \cdot 13 \text{ inches.}$$

The data are consistent with the assumption of a mean for the universe of 65 inches *i.e.*, $M = 65$.

$$\therefore \quad t = \frac{x - M}{s'}\sqrt{n} = \frac{67-65}{3 \cdot 13}\sqrt{10}$$

$$= 0 \cdot 638 \times 3 \cdot 1622777 = 2 \cdot 02.$$

The number of degree of freedom $= \nu = 10 - 1 = 9.$

Tabulated value for 9 d. f. at 5% level of significance is $2 \cdot 262.$

Since calculated value of t is less than tabulated value for 9 d.f. $(2 \cdot 02 < 2 \cdot 262)$. This error could have arisen due to fluctuations and we may conclude that the data are consistent with the assumption of mean height in the universe of 65 inches.

Example 3. ***Find the Student's t for following variable values in a sample of eight :***

$$-4, -2, -2, 0, 2, 2, 3, 3$$

taking the mean of the universe to be zero.

Solution. Calculations table :

S. No.	x	x − x̄	(x − x̄)²
1	−4	−4·25	18·0625
2	−2	−2·25	5·0625
3	−2	−2·25	5·0625
4	0	−0·25	0·0625
5	2	1·75	3·0625
6	2	1·75	3·0625
7	3	2·75	7·5625
8	3	2·75	7·5625
	Σx = 2		Σ(x − x̄)² = 49·5000

Here $\quad n = 8, \ \bar{x} = \frac{\Sigma x}{n} = \frac{2}{8} = 0 \cdot 25$

and $\quad s' = \sqrt{\left\{\left[\frac{\Sigma (x-\bar{x})^2}{n-1}\right]\right\}} = \sqrt{[49 \cdot 5/7)]}$

$$= \sqrt{(7 \cdot 071428)} = 2 \cdot 659$$

The data are consistent with the assumption that the mean of the universe is 0. We have

$$t = \frac{\bar{x} - M}{s'}\sqrt{n} = \frac{0 \cdot 25 - 0}{2 \cdot 659}\sqrt{8} = 0 \cdot 27.$$ **Ans.**

Example 4. ***Ten individuals are chosen at random from a population and their heights are found to be in cms. 157·5, 157·5, 160, 162·5, 165, 172·5, 172·5, 175, 175, 177·5. Discuss the suggestion that the mean height in the universe is 162·5 cms. given that for 9 degrees of freedom the value of Student's t at 5 percent level of significance is 2·262.***

Solution. Calculation table.

S. No.	x	x − x̄	(x − x̄)²
1	157·5	−10	100
2	157·5	−10	100
3	160	−7·5	56·25
4	162·5	−5	25
5	165	−2·5	6·25
6	172·5	5	25
7	172·5	5	25
8	175	7·5	56·25
9	175	7·5	56·25
10	177·5	10	100
n = 10	Σx = 1675		Σ(x − x̄)² = 550

Here $M = 162 \cdot 5$, Mean $= \frac{\Sigma x}{n} = \frac{1675}{10} = 167 \cdot 5$ cms.

Standard deviation

$$s' = \sqrt{\left\{\left[\frac{\Sigma (x-\bar{x})^2}{n-1}\right]\right\}} = \sqrt{\left(\frac{550}{9}\right)} = 7 \cdot 81 \text{ cms.}$$

Null hypothesis H_0 : The data are consistent with the assumption of a mean of 162·5 cms., *i.e.*, $\mu = \mu_0 = 162 \cdot 5$ cm.

Alternative hypothesis $H_1 : \mu = \mu_1 \neq 162 \cdot 5$ cm.

Test statistic :

$$t = \frac{\bar{x} - \mu}{s'}\sqrt{[(n)]} \sim \mu\ (0, 1)$$

$$= \frac{167 \cdot 5 - 162 \cdot 5}{7 \cdot 81}\sqrt{[(10)]} = \frac{5 \times \sqrt{10}}{7 \cdot 81}$$

$$= \frac{5 \times 3 \cdot 21}{7 \cdot 81} = \frac{15 \cdot 6}{7 \cdot 81} = 2 \text{ (approx.)}$$

$\nu =$ Degree of freedom $= 10 - 1 = 9.$

The tabulated value of 't' for 9 degree of freedom at 5% level of significance is 2·262. The calculated value of t is less than tabulated value at 5% level of significance. (This error could have arisen due to fluctuations of sampling). Thus we may conclude that the mean height in the universe is 162·5 cms.

Example 5. ***A certain stimulus administered to each of 12 patients resulted in the following increases of blood pressure :*** 5, 2, 8, – 1, 3, 0, 6, – 2, 1, 5, 0, 4.

Can it be calculated that the stimulus will be, in general, accompanied by an increase in blood pressure given that for 11 degrees of freedom the value $t_{0.05}$ is 2·201.

Solution. Calculations For Mean and S. D.

No	x	$x-\bar{x}$	$(x-\bar{x})^2$
1	5	+2·4	5·76
2	2	–0·6	0·36
3	8	5·6	29·16
4	–1	–3·6	12·96
5	3	0·4	0·16
6	0	–2·6	6·76
7	6	3·4	11·56
8	–2	–4·6	21·16
9	1	–1·6	2·56
10	5	2·4	5·76
11	0	–2·6	6·76
12	4	1·4	1·96
$n=12$	$\Sigma x = 31$		$\Sigma(x-\bar{x})^2 = 104·92$

Here,
$$\bar{x} = \frac{\Sigma x}{n} = \frac{31}{12} = 2\cdot 6 \text{ (approximately)}$$

$$s' = \sqrt{\left(\frac{\Sigma(x-x')^2}{n-1}\right)} = \sqrt{\left(\frac{10\cdot 92}{12-1}\right)}$$

$$= \sqrt{(9\cdot 54)} = 3\cdot 08 \text{ approximately.}$$

Null hypothesis H_0 : The stimulus will in general not be accompanied by an increase in blood pressure *i.e.*, the mean of universe $M = 0$.

$$\therefore \quad \text{Test statistic } t = \frac{\bar{x}-M}{s'}\sqrt{n} = \frac{2\cdot 6 - 0}{3\cdot 08}\sqrt{(12)} = 2\cdot 94.$$

The calculated value of t (= 2·94) > the tabulated value of t (= 2·201). Thus the value of t is significant and H_0 is rejected.

EXERCISE 10 (C)

1. Six boys are selected at random from a school and their marks in Maths. out of 100 are found to be 63, 63, 64, 66, 60, 68.
   In this light of these marks, discuss the suggestion that the mean marks in the Maths. in school was 66.

2. Nine patients to whom a certain drink was administered registered the following increments in blood pressure : 7, 3, – 1, 4, – 3, 5, 6, – 4, 1. Show that the data do not indicate that the drink was responsible for these increments.
   [The value of Student's P for 8 d. f. for $t = 5\cdot 15$ is ·915].
3. 9 individuals are chosen at random from a population and their weights are found to be in kg. 55, 58, 59, 60, 64, 62, 65, 61, 67. In the light of these data discuss the suggestion that the mean weight of the universe is 60 kg.
4. A sample of size ten has mean 57 and standard deviation 16. Is it half of a population whose mean is 50 ?
5. Find Student's t for the following variate values in a sample of 10 :
   $$-6, -4, -3, -2, -2, 0, 1, 3, 5$$
   taking mean μ to be zero, and find from the tables the probability of getting a value of t as great or greater on random sampling from a normal population.
6. Write a note on t-test.

ANSWERS

1. Mean marks 66 may by taken in Maths. 2. $t = 1\cdot 1$, yes.
4. The hypothesis is not significant and it may be assumed that the sample is half of the population whose mean is 50.
5. $t = -0\cdot 662, \nu = 9, P_S = 0\cdot 738$, so that $P_F = 2(1 - 0\cdot 0788) = 0\cdot 524$.
   $\therefore$ The probability that we should get a value of t greater in absolute value is 0·524.

§ 10.15. (B) TESTING THE SIGNIFICANCE OF THE DIFFERENCE BETWEEN THE SAMPLE MEANS (COMPARISON BETWEEN MEANS OF TWO SAMPLES)

Consider two independent samples from normal populations with same variance, as follows :

First sample $\quad x : x_1, x_2, \ldots, x_{n_1}$

with mean $\bar{x} = \dfrac{\Sigma x}{n_1}$, and variance $\sigma_x^2 = \dfrac{\Sigma(x-\bar{x})^2}{n_1 - 1}$.

Second sample $\quad y : y_1, y_2, \ldots, y_{n_2}$

with mean $\bar{y} = \dfrac{\Sigma y}{n_2}$, and variance $\sigma_y^2 = \dfrac{\Sigma(y-\bar{y})^2}{n_2 - 1}$.

Now we have to test the hypothesis that the populations means are the same ($\mu_1 = \mu_2$) *i.e.*, we are to test the hypothesis that the two independent samples are drawn from the same normal population.

The variate t is defined by the relation

$$t = \frac{\bar{x}-\bar{y}}{s(1/n_1 + 1/n_2)^{1/2}} = \frac{\bar{x} \sim \bar{y}}{s}\sqrt{\left(\frac{n_1 n_2}{n_1 + n_2}\right)} \quad \ldots(1)$$

where
$$s^2 = \frac{\Sigma(x-\bar{x})^2 + \Sigma(y-\bar{y})^2}{n_1 + n_2 - 2}.$$

It can be easily shown that the variate t defined by (1) has Fisher's t-distribution with $n_1 + n_2 - 2$ d. f. since both $\bar{x}$ and $\bar{y}$ are calculated from the data. The significance of t is tested in the same way as given above in § 10.5 for $n_1 + n_2 - 2$ d. f. to test the hypothesis.

ILLUSTRATIVE EXAMPLES

Example 1. ***Two horses A and B were tested according to the time (in seconds) to run a particular track with the following results :***

Horse A :	28	30	32	33	33	29	34
Horse B :	29	30	30	24	27	29	

Test whether you can discriminate between two horses. You can use the fact that 5 percent value of t for 11 degrees of freedom is 2·20.

Solution. Mean of $A = \bar{x} = \dfrac{\Sigma x}{n_1} = \dfrac{219}{7} = 31\dfrac{2}{7}$, $n_1 = 7$,

Mean of $B = \bar{y} = \dfrac{\Sigma y}{n_2} = \dfrac{169}{6} = 28\dfrac{1}{6}$, $n_2 = 6$

Horse A			Horse B		
x	$\mathbf{x - \bar{x}}$	$\mathbf{(x - \bar{x})^2}$	**y**	$\mathbf{y - \bar{y}}$	$\mathbf{(y - \bar{y})^2}$
28	$-3\frac{2}{7}$	$\frac{529}{49}$	29	$\frac{5}{7}$	$\frac{25}{26}$
30	$-1\frac{2}{7}$	$\frac{81}{49}$	30	$1\frac{5}{6}$	$\frac{121}{36}$
32	$\frac{5}{7}$	$\frac{25}{49}$	30	$1\frac{5}{6}$	$\frac{121}{36}$
33	$1\frac{5}{7}$	$\frac{144}{49}$	24	$-4\frac{1}{6}$	$\frac{625}{36}$
33	$1\frac{5}{7}$	$\frac{144}{49}$	27	$-1\frac{1}{6}$	$\frac{49}{36}$
29	$-2\frac{2}{7}$	$\frac{256}{49}$	29	$\frac{5}{6}$	$\frac{25}{36}$
34	$-2\frac{5}{7}$	$\frac{361}{49}$			
219	0	$\frac{1540}{49} = \frac{220}{7}$	169	0	$\frac{966}{36} = \frac{161}{6}$

Standard deviation,

$$s = \sqrt{\left(\frac{\Sigma (x - \bar{x})^2 + \Sigma (y - \bar{y})^2}{n_1 + n_2 - 2}\right)} = \sqrt{\left(\frac{\frac{220}{7} + \frac{161}{6}}{7 + 6 - 2}\right)} = \sqrt{(5 \cdot 29)} = 2 \cdot 3.$$

Under Null hypothesis H_0 : We can not discriminate between two horses.

The test statistic t is given by

$$t = \frac{\bar{x} - \bar{y}}{s} \sqrt{\left(\frac{n_1 n_2}{n_1 + n_2}\right)} = \frac{31\frac{2}{7} - 28\frac{1}{6}}{2 \cdot 3} \sqrt{\left(\frac{7 \times 6}{7 + 6}\right)}$$

$$= \frac{131}{24 \times 2 \cdot 3} \sqrt{\left(\frac{42}{13}\right)} = \frac{131}{2 \cdot 3 \times \sqrt{(42 \times 13)}} = 2 \cdot 43.$$

Also
$$\text{d. f.} = \nu = n_1 + n_2 - 2$$
$$= 7 + 6 - 2 = 11.$$

The tabulated value of t (= 2·20) for 11 d. f. at 5% level of significance < the calculated value of t (= 2·43). The H_0 is rejected since difference is significant. Hence we can discriminate between two horses.

Example 2. ***The heights of six randomly chosen sailors are in inches 63, 65, 68, 69, 71 and 72. Those at 10 randomly chosen soldiers are 61, 62, 65, 66, 69, 69, 70, 71, 72, 73. Discuss the light that these data throw on the suggestion that soldiers are on the average taller than sailors; given that***

$$\nu = 14 \begin{cases} P = 0 \cdot 539 & \textit{for} \quad t = 0 \cdot 10 \\ P = 0 \cdot 527 & \textit{for} \quad t = 0 \cdot 08. \end{cases}$$

Solution. Under null hypothesis H_0 : There is no difference between the mean height of soldiers and sailors.

The test statistic t is given by

$$t = \frac{\bar{x} - \bar{y}}{s} \sqrt{\left(\frac{n_1 n_2}{n_1 + n_2}\right)}$$

Sailors			Soldiers		
x	$\mathbf{x - \bar{x} = x - 68}$	$\mathbf{(x - \bar{x})^2}$	**x**	$\mathbf{y - \bar{y} = y - 67 \cdot 8}$	$\mathbf{(y - \bar{y})^2}$
63	−5	25	61	−6·8	46·24
65	−3	9	62	−5·8	33·64
68	0	0	65	−2·8	7·84
69	1	1	66	−1·8	3·24
71	3	9	69	1·8	1·44
72	4	16	69	1·2	1·44
			70	2·2	4·84
			71	3·2	10·24
			72	4·2	17·64
			73	5·2	27·04
408	0	60	678	0	153·60

Mean height of sailors $= \bar{x} = \dfrac{\Sigma x}{n_1} = \dfrac{408}{6} = 68$, $n_1 = 6$.

Mean height of soldiers $= \bar{y} = \dfrac{678}{10} = 67 \cdot 8$, $n_2 = 10$.

$\therefore$ Standard deviation

$$s = \sqrt{\left(\frac{\Sigma(x-\bar{x})^2 + \Sigma(y-\bar{y})^2}{n_1+n_2-2}\right)} = \sqrt{\left(\frac{60+153\cdot 6}{6+10-2}\right)} = \sqrt{\left(\frac{213\cdot 6}{14}\right)}$$

$$= \sqrt{(15\cdot 257)} = 3\cdot 906.$$

$$\therefore \quad t = \frac{\bar{x}-\bar{y}}{s}\sqrt{\left(\frac{n_1 n_2}{n_1+n_2}\right)} = \frac{68-67\cdot 8}{3\cdot 906}\sqrt{\left(\frac{60}{16}\right)} = \frac{0\cdot 387}{3\cdot 906} = 0\cdot 099$$

d. f. $= \nu = n_1 + n_2 - 2 = 6 + 10 - 2 = 14.$

Now $t = 0\cdot 10$	$P = 0\cdot 539$		$t = 0\cdot 099$
$t = 0\cdot 08$	$P = 0\cdot 527$		$t = 0\cdot 08$
diff. $= 0\cdot 02$	diff. $= 0\cdot 012$		diff. $= 0\cdot 019$

$\because$ For $0\cdot 02$ diff. in t, we have diff. in $P = 0\cdot 012$

$\therefore$ For $0\cdot 019$ diff. in t, we have diff. in $P = \dfrac{0\cdot 012 \times 0\cdot 019}{0\cdot 02} = 0\cdot 011.$

$\therefore$ For $t = 0\cdot 0994$, we have

$$P_S = 0\cdot 527 + 0\cdot 011 = 0\cdot 538.$$

Therefore Fisher's P *i.e.*, $P_F = 2(1 - P_S)$

$$= 2(1 - 0\cdot 538) = 0\cdot 924 > 0\cdot 05$$

the value of t is not significant *i.e.*, we may conclude that the suggestions that the soldiers are on the average taller than sailors is wrong.

Example 3. ***In a Test Examination given to two groups of students, the marks obtained were as follows :***

First group : 18, 20, 36, 50, 49, 36, 34, 49, 41

Second group : 29, 28, 26, 35, 30, 44, 46.

Examine the significance of difference between the arithmetic averages of the marks secured by the students of the above two groups.

[The value of t for 14 d. f. at 5% level of significance = 2·14].

Solution. Calculation For Mean And Standard Deviation of Two Groups.

First Group			Second Group		
x	$x-\bar{x}$	$(x-\bar{x})^2$	y	$y-\bar{y}$	$(y-\bar{y})^2$
18	−19	361	29	−5	25
20	−17	289	28	−6	36
36	−1	1	26	−8	64
50	13	169	35	1	1
49	12	144	30	−4	16
36	−1	1	44	10	100
34	−3	9	46	12	144
49	12	144			
41	4	16			
333	0	1134	238	0	386

Mean of first group $= \bar{x} = \dfrac{\Sigma x}{n_1} = \dfrac{333}{9} = 37,\ n_1 = 9.$

Mean of second group $= \bar{y} = \dfrac{\Sigma y}{n_2} = \dfrac{238}{7} = 34,\ n_2 = 7.$

$\therefore$ $s =$ standard error of difference

$$= \sqrt{\left(\frac{\Sigma(x-\bar{x})^2 + \Sigma(y-\bar{y})^2}{n_1+n_2-2}\right)}$$

$$= \sqrt{\left(\frac{1134+386}{9+7-2}\right)} = \sqrt{(108\cdot 58)} = 10\cdot 42.$$

$$\therefore \quad t = \frac{\bar{x}-\bar{y}}{s}\sqrt{\left[\left(\frac{n_1 \times n_2}{n_1+n_2}\right)\right]}$$

$$= \frac{37-34}{10\cdot 42}\sqrt{\left[\left(\frac{9\times 7}{9+7}\right)\right]} = \frac{3}{10\cdot 42}\sqrt{(3\cdot 9375)} = 0\cdot 577.$$

Also d. f. $= \nu = n_1 + n_2 - 2 = 9 + 7 - 2 = 14.$

Since the tabulated value of t for 14 degree of freedom at 5% level of significance $(= 2\cdot 14) >$ calculated value of $t\ (= 0\cdot 577)$.

Hence the difference between arithmetic averages of the samples is not significant.

Example 4. ***Two independent samples of 8 and 7 items respectively had the following values :***

Sample I :	9	11	13	11	15	9	12	14
Sample II :	10	12	10	14	9	8	10	

Is the difference between the means of the samples significant ? Given that

$$\nu = 13 \begin{cases} P = 0\cdot 874 & \textit{for} \quad t = 1\cdot 2 \\ P = 0\cdot 892 & \textit{for} \quad t = 1\cdot 3. \end{cases}$$

Solution. Calculations For Mean And S. D. of Two Samples.

Sample I		Sample II	
x	x^2	y	y^2
9	81	10	100
11	121	12	144
13	169	10	100
11	121	14	196
15	225	9	81
9	81	8	64
12	144	10	100
14	196	—	—
94	1138	73	785

Mean of sample I $= \bar{x} = \dfrac{\Sigma x}{n_1} = \dfrac{94}{8} = 11 \cdot 75,\ n_1 = 8.$

Mean of sample II $= \bar{y} = \dfrac{\Sigma y}{n_2} = \dfrac{73}{7} = 10 \cdot 43,\ n_2 = 7.$

$$\Sigma(x-\bar{x})^2 = \Sigma x^2 - 2\bar{x}\,\Sigma x + n_1\,\bar{x}^2$$

$$= 1138 - 2 \times 94\,(94/8) + 8 \times (94/8)^2 = 33 \cdot 5$$

$$\Sigma(y-\bar{y})^2 = \Sigma y^2 - 2\bar{y}\,\Sigma y + n_2\,\Sigma\bar{y}^2$$

$$= 785 - 2 \times 73 \times (73/7) + 7 \times (73/7)^2$$

$$= 785 - \frac{(73)^2}{7} = 23 \cdot 7.$$

$\therefore$ Standard deviation $= s = \sqrt{\left[\left(\dfrac{\Sigma(x-\bar{x})^2 + \Sigma(y-\bar{y})^2}{n_1+n_2-2}\right)\right]}$

$$= \sqrt{\left[\left(\frac{33 \cdot 5 + 23 \cdot 7}{8+7-2}\right)\right]}$$

$$= \sqrt{\left[\left(\frac{57 \cdot 2}{13}\right)\right]} = \sqrt{(4 \cdot 4)} = 2 \cdot 1.$$

The test statistic t is, $t = \dfrac{\bar{x}-\bar{y}}{s}\sqrt{\left[\left(\dfrac{n_1\,n_2}{n_1+n_2}\right)\right]}$

$$= \frac{11 \cdot 75 - 10 \cdot 43}{2 \cdot 1}\sqrt{\left[\left(\frac{56}{15}\right)\right]} = 1 \cdot 2144.$$

degree of freedom $= \nu = n_1 + n_2 - 2$

$$= 8 + 7 - 2 = 13$$

$$\left.\begin{array}{ll} t = 1 \cdot 3 & P = 0 \cdot 892 \\ t = 1 \cdot 2 & P = 0 \cdot 874 \end{array}\right\} \qquad \begin{array}{l} t = 1 \cdot 2144 \\ t = 1 \cdot 2 \end{array}$$

$$\text{diff.} = 0 \cdot 1 \qquad \text{diff.} = 0 \cdot 018 \qquad\qquad \text{diff.} = 0 \cdot 0144$$

$\because$ difference in t = difference in P for $0 \cdot 1$

$$= 0 \cdot 018.$$

$\therefore$ difference in t = difference in P for $0 \cdot 0144$

$$= \frac{0 \cdot 018 \times 0 \cdot 0144}{0 \cdot 1} = 0 \cdot 0025.$$

Thus for $t = 1 \cdot 21$,

$$P = 0 \cdot 874 + 0 \cdot 0025 = 0 \cdot 8765 = P_S$$

$$\therefore \quad P_F = 2\,(1 - P_S) = 2\,(1 - 0 \cdot 8765)$$

$$= 0 \cdot 2470 > 0 \cdot 05.$$

Since the value of P_F is considerably greater than $0 \cdot 05$, therefore the difference between the means of two samples is not significant.

Example 5. A group of 7, seven-weeks old chickens, reared on a high protein diet, weight 12, 15, 11, 16, 14, 14, 16 ounces; a second group of 5 chickens, similarly treated except that they receive a low protein diet, weight 8, 10, 14, 10, 13 ounces. Calculate the value of t and test whether there is significant evidence that additional protein has increased the weight of chickens.

Values of t on Levels of Significance "P"

Degree of freedom	$t = 0 \cdot 23$	$P = 0 \cdot 81$
10	2·23	1·01
11	2·20	1·80
12	2·18	1·78

Solution. Calculations for simple means and S. D. of two groups of chickens :

Chickens	Group A			Group B		
	x	$x - \bar{x} = x - 14$	$(x-\bar{x})^2$	y	$y - \bar{y} = y - 11$	$(y-\bar{y})^2$
1	12	−2	4	8	−3	9
2	15	1	1	10	−1	1
3	11	−3	9	14	3	9
4	16	2	4	10	−1	1
5	14	0	0	13	2	4
6	14	0	0			
7	16	2	4			
	98	0	22	55	0	24

Mean of group $A = \bar{x} = 98/7 = 14,\ n_1 = 7.$

Mean of group $B = \bar{y} = 55/5 = 11,\ n_2 = 5.$

$$s = \sqrt{\left[\left\{\frac{\Sigma(x-\bar{x})^2 + \Sigma(y-\bar{y})^2)}{n_1+n_2-2}\right\}\right]}$$

$$= \sqrt{\left[\left(\frac{22+24}{7+5-2}\right)\right]} = \sqrt{(4 \cdot 6)} = 2 \cdot 14.$$

The test statistic t is :

$$t = \frac{\bar{x}-\bar{y}}{s}\sqrt{\left[\left(\frac{n_1\,n_2}{n_1+n_2}\right)\right]}$$

$$= \frac{14-11}{2 \cdot 14}\sqrt{\left[\left(\frac{7 \times 5}{7+5}\right)\right]} = \frac{3}{2 \cdot 14}\sqrt{\left(\frac{35}{12}\right)}$$

$$= 2 \cdot 394$$

degree of freedom $= 7 + 5 - 2 = 10.$

The tabulated value of t for 10 d. f. at 5% level of significance ($= 2 \cdot 23$) is less than the calculated value. Thus the difference of two means is significant. Therefore there is a significant evidence that additional protein has increased the weight of chickens.

Example 6. ***The following data represent the yields in bushes of Indian corn on sub-divisions of equal areas of two agricultural plots in which plot I was a control plot treated the same as plot II, except for the amount of phosphorus applied as a fertilizer :***

Plot I :	6·2	5·7	6·5	6·0	6·3
Plot II :	5·6	5·9	5·6	5·7	5·8
Plot I :	5·8	5·7	6·0	6·0	5·8
Plot II :	5·7	6·0	5·5	5·7	5·5

Is there a significant difference between the yields on the two plots, using the difference between their means by a criterian of judgement.

[Given from Student's table for $\nu = 18$, $P = 0 \cdot 0072$ for the probability that t will fall outside the range $- 3 \cdot 034$ and $+ 3 \cdot 034$].

Solution. Calculation table :

Plot I			Plot II		
x	$\mathbf{x - \bar{x}}$	$\mathbf{(x - \bar{x})^2}$	**y**	$\mathbf{y - \bar{y}}$	$\mathbf{(y - \bar{y})^2}$
6·2	0·2	0·04	5·6	−0·1	0·01
5·7	−0·3	0·09	5·9	0·2	0·04
6·5	0·5	0·25	5·6	−0·1	0·01
6·0	0·0	0	5·7	−0	0
6·3	0·3	0·09	5·8	0·1	0·01
5·8	−0·2	0·04	5·7	0·0	0
5·7	−0·3	0·09	6·0	0·3	0·09
6·0	0·0	0	5·5	−0·2	0·04
6·0	0·0	0	5·7	0	0
5·8	−0·2	0·04	5·5	−0·2	0·04
60·0	0	0·64	57·0	0	0·24

Mean for plot I $= \bar{x} = \dfrac{\Sigma x}{n_1} = \dfrac{60}{10} = 6,\ n_1 = 10.$

Mean for plot II $= \bar{y} = \dfrac{\Sigma y}{n_2} = \dfrac{57}{10} = 5 \cdot 7,\ n_2 = 10.$

$$s = \sqrt{\left[\left(\frac{\Sigma (x - \bar{x})^2 + \Sigma (y - \bar{y})^2}{n_1 + n_2 - 2}\right)\right]}$$

$$= \sqrt{\left[\left(\frac{0 \cdot 64 + 0 \cdot 24}{18}\right)\right]} = \sqrt{\left(\frac{0 \cdot 88}{18}\right)} = \sqrt{\left(\frac{0 \cdot 44}{9}\right)} = 0 \cdot 2211.$$

Test statistic t is

$$t = \frac{\bar{x} - \bar{y}}{s} \sqrt{\left[\left(\frac{n_1 n_2}{n_1 + n_2}\right)\right]} = \frac{6 - 5 \cdot 7}{0 \cdot 2211} \sqrt{\left(\frac{10 \times 10}{10 + 10}\right)}$$

$$= 3 \cdot 03 \text{ (approx.)}$$

$$\text{d. f.} = \upsilon = n_1 + n_2 - 2 = 10 + 10 - 2 = 18.$$

For 18 degree of freedom and for this value of t, $P = 0 \cdot 0072 < 0 \cdot 05$. This shows that there is a significant difference between the yields on two plots.

Example 7. ***A set of 15 observations gives mean $= 68 \cdot 57$, standard deviation $= 2 \cdot 40$, another of 7 observations gives mean $= 64 \cdot 14$, standard deviation $= 2 \cdot 70$.***

Use the t-test to find whether two sets of data were drawn from populations with the same mean, it being assumed that the standard deviations in the two populations were equal.

You may base your answer on the following extract from t-table.

Significance limits of t-probability of a value numerically greater than t are :

Degree of freedom	0·05	0·02	0·01
6	2·447	3·143	3·707
14	2·145	2·624	2·977
20	2·086	2·528	2·845

Solution. On the assumption that the two independent samples are taken from the populations with the same mean and same S. D. We have

$$\text{S. D.} = s = \sqrt{\left[\left(\frac{n_1 \sigma_1^2 + n_2 \sigma_2^2}{n_1 + n_2 - 2}\right)\right]}$$

$$= \sqrt{\left[\left\{\frac{15 \times (2 \cdot 40)^2 + 7 \times (2 \cdot 70)^2}{15 + 7 - 2}\right\}\right]}$$

$$= \sqrt{\left[\left(\frac{86 \cdot 4 + 51 \cdot 03}{20}\right)\right]}$$

$$= \sqrt{[(5 \cdot 8715)]} = 2 \cdot 62.$$

The test statistic t is :

$$t = \frac{\bar{x} - \bar{y}}{s} \sqrt{\left(\frac{n_1 n_2}{n_1 + n_2}\right)} = \left(\frac{68 \cdot 57 - 64 \cdot 14}{2 \cdot 62}\right) \sqrt{\left(\frac{15 \times 7}{15 + 7}\right)}$$

$$= \frac{4 \cdot 43}{2 \cdot 62} \times 2 \cdot 1846572 = 3 \cdot 6939.$$

The tabulated value of t for 20 degree of freedom at 5% level of significance (is $2 \cdot 086$) is less than the calculated value of t. This shows that the difference is significant.

Example 8. *For a random sample of 10 pigs fed on diet A, the increases in weight in pounds in a certain period were*

10, 6, 16, 17, 13, 12, 8, 14, 15, 9 kg.

For another random sample of 12 pigs fed on diet B, the increases in the same period were 7, 13, 22, 15, 12, 14, 18, 8, 21, 23, 10, 17 kg.

Test whether diet A and B differ significantly as regards the effect on increases in weight or test whether the mean increases in the two samples are significantly different. You may use the fact that 5% value of t for 20 degrees of freedom is 2·09.

Solution. Here $n_1 = 10, n_2 = 12.$

Mean of increases in weight on diet $A = \bar{x} = \frac{\Sigma x}{n_1} = \frac{120}{10} = 12$ kg.

Mean of increases in weight on diet $B = \bar{y} = \frac{\Sigma y}{n_2} = \frac{180}{12} = 15$ kg.

Diet A			Diet B		
x	$x-\bar{x}$	$(x-\bar{x})^2$	y	$y-\bar{y}$	$(y-\bar{y})^2$
10	−2	4	7	−8	64
6	−6	36	13	−2	4
16	4	16	22	7	49
17	5	25	15	0	0
13	1	1	12	−3	9
12	0	0	14	−1	1
8	−4	16	18	3	9
14	2	4	8	−7	49
15	3	9	21	6	36
9	−3	9	23	8	64
			10	−5	25
			17	2	4
120	0	120	180	0	314

$$s^2 = \frac{\Sigma(x-\bar{x})^2 + \Sigma(y-\bar{y})^2}{n_1+n_2-2}$$

$$= \frac{120+314}{10+12-2} = \frac{434}{20} = 21 \cdot 7.$$

$\therefore$ $s = \sqrt{[(21 \cdot 7)]} = 4 \cdot 65.$

Now the test statistic t is :

$$t = \frac{\bar{x} \sim \bar{y}}{s} \sqrt{\left[\left(\frac{n_1 n_2}{n_1+n_2}\right)\right]}$$

$$= \frac{12 \sim 15}{4 \cdot 65} \sqrt{\left[\left(\frac{12 \times 10}{12+10}\right)\right]} = 1 \cdot 5067721$$

$$v = \text{Degree of freedom} = n_1 + n_2 - 2$$
$$= 10 + 12 - 2 = 20.$$

The tabulated value of t (2·09) for 20 d. f. at 5% level of significance is greater than the calculated value of t. Hence the difference between the sample means is not significant.

EXERCISE 10 (C)

1. Calculate the value of t in the case of two characters A and B whose corresponding values are given below :

A:	16	10	8	9	9	8
B:	8	4	5	9	12	4

2. The figures below are for protein tests of the same variety of wheat grown in two districts. The average in District I is 12·74 and in District II is 13·03. Calculate t for testing the significance between the means of the two districts :

Protein results

District I :	12·6	13·4	11·9	12·8	13		
District II :	13·1	13·4	12·8	13·5	13·3	12·7	12·4

3. For a random sample of 12 boys fed on diet A, the increases in weight in pounds in a certain period were

25, 32, 30, 34, 24, 25, 14, 32, 24, 30, 31, 35

for another random sample of 15 boys fed on diet B the increases in weight in pounds in the same period were

44, 34, 22, 10, 47, 31, 40, 30, 32, 35, 18, 21, 35, 29, 22

Find whether diet B is superior to diet A, given that the 5% value of t for 25 d.f. is 2·06.

4. The means of two random samples of sizes 9 and 7 respectively, are 196·4 and 198·82. The sum of the squares of the deviations from the means are 26·94 and 18·73 respectively. Can the samples be considered to have been drawn from the same normal population, it being given that the value of t for 14 d. f. at 5% level of significance is 2·145 and at 1% level of significance is 2·977.

5. Two types of batteries A and B are tested for their length of life and following results are obtained :

	No. of sample	Mean	Variance
A	10	500 hours	100
B	10	500 hours	121

Is their a significant difference in the two means ? The value of t for 18 d.f. at 5% level of significance is 2·1.

ANSWERS

1. 1·66.
2. Not significant.
3. $t = 0 \cdot 61$ (nearly), no diet can be said to be superior to the other.
4. Samples can be considered to have been drawn from the same normal population.
5. Highly Significant.

§ 10.16. PAIRED DATA (t-TEST FOR PAIRED DATA)

To test the significance of Paired Samples when the size of the two samples is the same.

If the two samples are of the same size n and data are paired, then t is defined by the relation

$$t = \frac{\bar{d} - \mu}{s} \sqrt{n},$$

where $s^2 = \frac{1}{n-1} \Sigma (d_i - \bar{d})^2$

d_i = difference of the ith members of the samples

$\bar{d}$ = mean of the differences $= \Sigma d / n$

and μ = population mean of the differences.

If $\mu = 0$, then $t = \frac{\bar{d}}{s} \sqrt{n}$.

In this case, the number of degrees of freedom $= n - 1$.

ILLUSTRATIVE EXAMPLES

Example 1. ***Ten soldiers tour a rifle range once in a week for two successive weeks. Their scores in the first week were 67, 24, 57, 55, 63, 54, 56, 68, 33, 43. Their scores in the second week (in the same order) were 70, 38, 58, 58, 56, 67, 68, 75, 42, 38. Is there any significant improvement?***

Solution. Calculations For Mean and S. D.

S. No. of Soldiers	Score in Ist week I	Score in IInd week II	d = II − I	Deviation $d - \bar{d}$	$(d - \bar{d})^2$
1	67	70	3	−2	4
2	24	38	14	9	81
3	57	58	1	−4	16
4	55	58	3	−2	4
5	63	56	−7	−12	144
6	54	67	13	8	64
7	56	68	12	7	49
8	68	75	7	2	4
9	33	42	9	4	16
10	43	38	−5	−10	100
n = 10			Σd = 50		$\Sigma(d - \bar{d})^2 = 482$

$\therefore$ $\bar{d}$ = the mean of differences

$$= \frac{\Sigma d}{n} = \frac{50}{10} = 5.$$

s = standard deviation of differences

$$= \sqrt{\left[\left\{\frac{\Sigma (d - \bar{d})^2}{n-1}\right\}\right]} = \sqrt{\left[\left(\frac{482}{10-1}\right)\right]} = \sqrt{[(53 \cdot 76)]} = 7 \cdot 3.$$

Let null hypothesis H_0 : No improvement of scores in the second tour i.e., $\mu = 0$.

$$\therefore \quad t = \frac{\bar{d} - 0}{s} \sqrt{n} = \frac{5 - 0}{7 \cdot 3} \sqrt{(10)} = 2 \cdot 17.$$

Also d.f. $= n - 1 = 10 - 1 = 9$.

For 9 d.f. $t_{0.05} = 2 \cdot 262$ and $2 \cdot 17 < 2 \cdot 262$

i.e. calculated value of t < tabulated value of t. Thus it gives no indication against the hypothesis H_0. Consequently there is no significant improvement in the second week.

Example 2. ***Eleven school boys were given a test in Drawing. They were given a month's further tuition and a second test of equal difficulty was held at the end of it. Do the marks give evidence, that the students have been benefitted by the extra coaching?***

Boys :	1	2	3	4	5	6	7	8	9	10	11
Marks Ist test :	23	20	19	21	18	20	18	17	23	16	19
Marks IInd test :	24	19	22	18	20	22	20	20	23	20	17

Solution. Calculations For Mean And S. D.

S. No. of Boys	Marks x	Marks y	d = y − x	$d - \bar{d}$	$(d - \bar{d})^2$
1	23	24	1	0	0
2	20	19	−1	−2	4
3	19	22	3	2	4
4	21	18	−3	−4	16
5	18	20	2	1	1
6	20	22	2	1	1
7	18	20	2	1	1
8	17	20	3	2	4
9	23	23	0	−1	1
10	16	20	4	3	9
11	19	17	−2	−3	9
n = 11			Σd = 11		$\Sigma (d - \bar{d})^2 = 50$

$\therefore$ $\bar{d}$ = mean of differences

$= \Sigma d / n = 11 / 11 = 1$

s = standard deviation of differences

$$= \left[\left\{\frac{\Sigma (d - \bar{d})^2}{n-1}\right\}\right]^{1/2} = \sqrt{\left(\frac{50}{10}\right)} = \sqrt{5} = 2 \cdot 24.$$

Let Null hypothesis H_0 : Students have not been benefitted by the extra coaching, *i.e*, mean of the differences between the marks of two tests is zero *i.e.*, $\mu = 0$, then

$$t = \frac{d-0}{s}\sqrt{n} = \frac{1-0}{2\cdot 24}\sqrt{(11)} = 1\cdot 482.$$

Also d. f. $= n-1 = 11-1 = 10.$

For 10 d. f. $t_{0\cdot 05} = 2\cdot 228$

Since calculated value of $t = (1\cdot 482) <$ tabulated value of $t\ (= 2\cdot 228)$, therefore the difference is significant. Hence the test provides no evidence that the students have benefitted by the extra coaching.

EXERCISE 10 (E)

1. The yield of two 'type 17' and 'type 15' of grains in pounds per acre in 6 replications are given below. What comments would you make on the difference in the mean yields? You may assume that if there be 5 d. f. and $P = 0\cdot 02$, it is approximately $1\cdot 476$.

Replications	Yields in Pounds 'Type 17'	Yields in Pounds 'Type 15'
1	20·50	24·86
2	24·60	26·39
3	23·06	28·19
4	29·98	30·75
5	30·37	29·97
6	23·83	22·04

2. In certain experiment to compare two types of pig food A and B, the following results of increase in weight were observed in pigs.

Pig Number		1	2	3	4	5	6	7	8	Total
Increase in	Food A :	49	53	51	52	47	50	52	53	407
weight in lbs.	Food B :	52	55	52	53	50	54	54	53	423

Assuming that the two samples of pigs are independent, can we conclude that food B is better than food A ? Given that for 7 d. f. $t_{0\cdot 05} = 2\cdot 36$.

3. Two laboratories carry out independent estimates of fat content for ice-cream made by a certain firm. A sample is taken from each batch halved and the separate halves sent to the two laboratories. They obtain the following results :

Batch No. :	1	2	3	4	5	6	7	8	9	10
Lab. A :	7	8	7	3	8	6	9	4	7	8
Lab. B :	9	8	8	4	7	7	9	6	6	6

Is the test reliable. The value of t for 9 degrees of freedom at 5% level of significance is 2·57.

4. The sleep of 10 patients was measured for the effect of the sporofic drugs referred to in the following table as Drug A and Drug B from the data given below show that there is significant difference between the effects of two drugs, on the assumption that different random sample of patterns were used to test the two drugs A and B. You may assume that if there be 9 degrees of freedom and $P = 0\cdot 05$, $t = 2\cdot 26$.

Additional Hours of Sleep Gained B use of Sporofic Drugs

Patient	Drug A	Drug B	Patient	Drug A	Drug B
1	+0·7	1·9	6	3·4	4·4
2	−1·6	0·8	7	3·7	5·5
3	−0·2	1·1	8	0·8	1·6
4	−1·2	0·1	9	0	4·6
5	−0·1	−0·1	10	2·0	3·6

5. Write an essay on different uses of t-distribution in testing the significance.

ANSWERS

1. $t = 1\cdot 489 < t_{0\cdot 05}$ for 5 d.f.; so the difference is not significant at 5% level but is significant at 2% level.
2. Yes.
3. Have been reliable.
4. $t = 4\cdot 11 > t_{0\cdot 05}$ for 9. d.f., there is a significant difference between the effects of the two drugs which is in the favour of the second.

§ 10.17. NORMAL TEST (OR FISHER'S z-TEST)

To test the significance of the ratio of two independent estimates of the population variance.

Let two independent random samples of sizes n_1 and n_2 be drawn from two normal populations having standard deviations σ_1 and σ_2. Again let $x_1, x_2, \ldots, x_{n_1}$ and $y_1, y_2, \ldots, y_{n_2}$ be the values of these two random samples with estimated variance S_1^2 and S_2^2.

We are to test the hypothesis that $\sigma_1 = \sigma_2$. Under the null hypothesis $\sigma_1 = \sigma_2$. Fisher showed that the statistic

$$z = \frac{1}{2}\log_e \frac{S_1^2}{S_2^2} \quad \ldots(1)$$

or

$$z = \log_e \frac{S_1}{S_2} = \log_{10}\frac{S_1}{S_2} \times \log_e 10 \quad \ldots(2)$$

is distributed according to the law

$$y = y_0 \frac{e^{\nu_1 z}}{(\nu_1 e^{2z} + \nu_2)^{1/2}\,(\nu_1 + \nu_2)} \quad \ldots(3)$$

where ν_1 and ν_2 are the number of degrees of freedom, *i.e.*,

$$\nu_1 = n_1 - 1 \text{ and } \nu_2 = n_2 - 1,$$

$$S_1^2 = \frac{1}{n_1 - 1}\Sigma (x - \bar{x})^2,$$

$$S_2^2 = \frac{1}{n_2 - 1}\Sigma (y - \bar{y})^2.$$

Note that larger variance is always divided by the smaller, so that z is always positive.

This distribution is known as Fisher's z-distribution.

If y_0 is so chosen that the total area under the curve given by (3) is unity, then

$$y_0 = \frac{2\nu_1^{\frac{1}{2}\nu_1}\ \nu_2^{\frac{1}{2}\nu_2}}{\beta\left(\frac{1}{2}\nu_1, \frac{1}{2}\nu_2\right)}.$$

§ 10.18. FISHER'S z-TABLES AND THE SIGNIFICANCE TEST

We choose y_0 so that the total area under the curve given by (3) is unity. The probability (P_z) that a given value of $z < z_0$ is given by the area to the right of ordinate at z_0 (*i.e.*, by the area between the ordinates z_0 to ∞)

i.e., $$P_z = \int_{z_0}^{\infty} y\, dz.$$

Fisher has prepared tables showing 5 percent and 1 percent points of significance for z. The z-tables give only critical values corresponding to right tail areas. Therefore 5% points of z imply that the area to the right of the ordinate at the variable z is $0\cdot05$. Similarly 1% points of z are considered.

The hypothesis which is to be tested is that S_1^2 and S_2^2 are the estimates of the same population variance *i.e.*, $S_1^2 = S_2^2$.

Under this hypothesis $z = \frac{1}{2}\log\left[\frac{S_1^2}{S_2^2}\right] = 0.$

The basis of the test is the divergence of the value of z from 0. For $P_z = 0\cdot05$, (i) if $z < z_0$ then our hypothesis is correct; and (ii) if $z > z_0$ then our hypothesis is rejected and consequently the samples have been taken from population with different variances.

§ 10.19. F-DISTRIBUTION (VARIANCE RATIO TEST) OR TEST FOR RATIO OF VARIANCE

Distribution of the ratio of independent estimates of the population variance.

Let $x_1, x_2, \ldots, x_{n_1}$ and $y_1, y_2, \ldots, y_{n_2}$ be the values of two independent random samples drawn from two normal populations having equals variances *i.e.*, $\sigma_1 = \sigma_2$.

F-statistic is defined by the relation

$$F = \frac{S_1^2}{S_2^2} \text{ where } S_1^2 > S_2^2 \qquad \ldots(1)$$

$$= \frac{[\Sigma(x-\bar{x})^2].]/(n_1-1)}{[\Sigma(y-\bar{y})^2]/(n_2-1)}.$$

Where S_1^2 and S_2^2 are unbiased estimates of population variances. Note here that the larger of the two variances (S_1^2 and S_2^2) is always placed in the numerator.

The distribution of F is given by

$$y = y_0\, F^{(\nu_1-\nu_2)/2}\left(1 + \frac{\nu_1}{\nu_2}F\right)^{-(\nu_1+\nu_2)/2} \qquad \ldots(2)$$

Where y_0 is calculated under the condition that the total area under the curve is unity and

$$y_0 = \frac{(\nu_1/\nu_2)^{\nu_1/2}\ \Gamma\frac{1}{2}(\nu_1+\nu_2)}{\Gamma\left(\frac{1}{2}\nu_1\right)\Gamma\left(\frac{1}{2}\nu_2\right)}$$

$\nu_1 = n_1 - 1$ and $\nu_2 = n_2 - 1$ are the number of degrees of freedom.

§ 10.20. F-TABLES AND SIGNIFICANCE TEST

These tables give 5% and 1% points of significance for F (like z-tables). 5% points of F means that the area under the curve to the right of the ordinate at a value of F is $0\cdot05$ (like z-tables). Clearly the value of F at 5% level of significance will be less than the value of F at 1% or 2% level of significance. Therefore, these tables also give only one tail test. On the other hand, if we are testing the hypothesis that the population variances are same, both tail areas under the F-curve must be used. In these cases, these tables will provide 10% and 20% levels of significance. If however, a 5% or a 1% level of significance is required, a rough approx. for these may be obtained by interpolation.

Remark. Firstly the z-test should be applied to observe whether the two population variances are same. If a favourable verdict about the hypothesis $\sigma_1 = \sigma_2$ is obtained from z-test, then t-test should be applied for testing the significance of the difference between the two population means, since t-test depends on the equality of the population variances (*i.e.*, $\sigma_1 = \sigma_2$).

ILLUSTRATIVE EXAMPLES

Example 1. *Show how you would use Student's t-test and Fisher's z-test to decide whether the two sets of observations :*

	17	27	18	25	27	29	27	23	17
and	16	16	20	16	20	17	15	21	

indicate samples drawn from the same universe. The value of z at 5% points for 8 and 7 degree of freedom is 0·6575 and the value of z at 1% points for 8 and 7 degrees of freedom is 0·9614.

Solution. Calculations for Mean and S. D. of two series :

1st Deviation			2nd Deviation		
x	(x − 23)	$(x-23)^2$	y	(y − 16)	$(y-16)^2$
17	−6	36	16	0	0
27	4	16	16	0	0
18	−6	25	20	4	16
25	2	4	16	0	0
27	4	16	20	4	16
29	6	36	17	1	1
27	4	16	15	−1	1
23	0	0	21	5	29
17	−6	36			
Total 210	3	185	141	13	59

Taking assumed mean 23 and 16 for x and y series respectively.

$$\bar{x} = \frac{210}{9} = 23\cdot333,$$

$$\bar{y} = \frac{141}{8} = 17\cdot625,\ n_1 = 9,\ n_2 = 8,$$

and
$$S_1^2 = \frac{\Sigma(x-23)^2}{n_1-1} - \frac{[\Sigma(x-23)]^2}{n_1(n_1-1)}$$

$$= \frac{185}{8} - \frac{(3)^2}{9\times 8} = \frac{184}{8} = 23.$$

Similarly,
$$S_2^2 = \frac{\Sigma(y-16)^2}{n_2-1} - \frac{[\Sigma(y-16)]^2}{n_2(n_2-1)}$$

$$= \frac{59}{7} - \frac{(13)^2}{8\times 7} = \frac{303}{56} = 5\cdot4107.$$

Firstly we shall use z-test to find whether the population variances are same or not.

$$\therefore \quad z = \frac{1}{2}\log_e \frac{S_1^2}{S_2^2} = \frac{1}{2}\log_{10}\frac{S_1^2}{S_2^2} \times \log_e 10$$

$$= \frac{1}{2} \times 2\cdot3026 \log_{10}\frac{S_1^2}{S_2^2}$$

$$= \frac{1}{2} \times 2\cdot3026 \log_{10}\frac{23}{5\cdot4107}$$

$$= \frac{1}{2} \times 2\cdot30265 \times 0\cdot6284743$$

$$= 0\cdot7235624 = 0\cdot724 \text{ (approx.)}$$

Now for $\nu_1 = 8$ and $\nu_2 = 7$. The value of z at 5% level of significance $= 0\cdot6576$

Again for $\nu_1 = 8$ and $\nu_2 = 7$. The value of z at 1% level of significance $0\cdot9614$

Since the calculated value of z lies between $0\cdot6576$ and $0\cdot9614$ therefore the ratio of variance is significant for 5% points and is not significant at 1% points. Hence the two population variances are same at 1% point level.

Now we shall apply t-test to find the significance of the differences between the population mean. We assume the hypothesis that $M_1 = M_2$, then

$$t = \frac{\bar{x}-\bar{y}}{s}\left[\left(\frac{n_1 n_2}{n_1+n_2}\right)\right]$$

where
$$s^2 = \frac{(n_1-1)S_1^2 + (n_2-1)S_2^2}{n_1+n_2-2}$$

$$= \frac{8\times 23 + 7\times 5\cdot4107}{9+8-2}$$

$$= \frac{221\cdot8749}{15} = 14\cdot7916.$$

$$\therefore \quad s = 3\cdot846.$$

Hence
$$t = \frac{5\cdot708}{3\cdot846}\left[\left(\frac{8\times 9}{17}\right)\right] = 3\cdot05.$$

Here d. f. $= \nu = n_1 + n_2 - 2 = 9 + 8 - 2 = 15.$

Now for 15 d. f., $t_{0\cdot05} = 2\cdot131$ and $t_{0\cdot01} = 2\cdot947$.

Since the calculated value of t is greater than these both values, the difference between the population means is significant, *i.e.*, two samples are not drawn from the same population.

Example 2. ***Two samples of sizes 9 and 8 give the sum of squares of deviations from their respective means equal to 160 inches square and 91 inches squares respectively. Can they be regarded as drawn from the two normal populations with the same variance. Given that $F_{0\cdot05} = 3\cdot73$ for 8 and 7 d. f.***

Solution. Here, it is given that,

$$\Sigma(x-\bar{x})^2 = 160 \text{ and } \Sigma(y-\bar{y})^2 = 91.$$

$$\therefore \quad S_1^2 = \frac{\Sigma(x-\bar{x})^2}{n_1-1} = \frac{160}{9-1} = 20 \qquad [\because\ n_1 = 9]$$

and
$$S_2^2 = \frac{\Sigma(y-\bar{y})^2}{n_2-1} = \frac{91}{7} = 13. \qquad [\because\ n_2 = 8]$$

$$\therefore \quad F = S_1^2 / S_2^2 = 20/13 = 1\cdot54.$$

Here the calculated value of F is less than the value of $F_{0\cdot05}$ for 8 and 7 d. f. *i.e.*, $3\cdot73$. Thus the calculated value of F is not significant. Hence we may conclude that two samples are drawn from two normal populations with the same variance.

Example 3. ***In a sampling experiment $S_1 = 3\cdot6$, $S_2 = 2\cdot0$ and $\nu_1 = 5$, $\nu_2 = 10$. Is the difference between S_1 and S_2 significant at the 5% level ?***

Solution. Here
$$F = \frac{S_1^2}{S_2^2} = \frac{(3\cdot6)^2}{(2\cdot0)^2} = 3\cdot24.$$

The value of F for $v_1 = 5$ and $v_2 = 10$ at 5% level of significance is 3·33.

Since the calculated value of F is less than the tabulated value therefore the difference between the S_1 and S_2 at 5% level is not significant.

Note. Here z-test can also be applied.

Example 4. ***Two independent samples of 8 and 7 items respectively had the following values of the variables (weight in ounces) :***

Sample I	9	11	13	11	15	9	12	14
Sample II	10	12	10	14	9	8	10	

Do the estimates of population variance differ significantly ? Given that for 7 and 6 degrees of freedom the values of F at 5% level of significance is 4·20 nearly.

Solution. Calculation Tables is as follows :

Sample I			Sample II		
x	$(x-\bar{x})$	$(x-\bar{x})^2$	y	$(y-\bar{y})$	$(y-\bar{y})^2$
9	−2·75	7·5625	10	−0·43	0·1849
11	−0·75	0·5625	12	1·57	2·4649
13	1·25	1·5625	10	−0·43	0·1849
11	−0·75	0·5625	14	3·57	12·7449
15	3·25	10·5625	9	−1·43	2·0449
9	−2·75	7·5625	8	−2·43	5·9049
12	0·25	0·0625	10	−0·43	0·1849
14	2·25	5·0625			
$\Sigma x = 94$		$\Sigma(x-\bar{x})^2 = 33{\cdot}5$	$\Sigma y = 73$		$\Sigma(y-\bar{y})^2 = 23{\cdot}7143$

Here $\bar{x} = \Sigma x / n_1 = 94/8 = 11{\cdot}75$ Oz.

$\bar{y} = \Sigma y / n_2 = 73/7 = 10{\cdot}43$ Oz. (nearly)

$$\therefore \quad S_1^2 = \frac{\Sigma (x-\bar{x})^2}{n_1 - 1} = \frac{33{\cdot}5}{7}$$

and
$$S_2^2 = \frac{\Sigma (Y-\bar{y})^2}{n_2 - 1} = \frac{23{\cdot}1743}{6}.$$

$$\therefore \quad F = \frac{S_1^2}{S_2^2} = \frac{33{\cdot}5 \times 6}{23{\cdot}1743 \times 7} = 1{\cdot}24.$$

Here the calculated value of F (= 1·24) < the value (= 4·20) of F at 5% level of significance. Hence the difference is not significant, we may conclude that the samples are drawn from the same population.

Example 5. ***For a random sample of 10 pigs, fed on diet A, the increases in weight in pounds in a certain period were***

10, 6, 16, 17, 13, 12, 8, 14, 15, 9 lbs.

for another random sample of 12 pigs, fed on diet B, the increases in the same period were

7, 13, 22, 15, 12, 14, 18, 8, 21, 23, 10, 17 *lbs*

Show that the estimate of population variance in the two samples was not significantly different (for $v_1 = 11$, $v_2 = 9$, the 5% value of F = 3·112).

Solution. Calculation for variance :

Sample I			Sample II		
x	$(x-\bar{x})$	$(x-\bar{x})^2$	y	$(y-\bar{y})$	$(y-\bar{y})^2$
10	−2	4	7	−8	64
6	−6	36	13	−2	4
16	4	16	22	7	49
17	5	25	15	0	0
13	1	1	12	−3	9
12	0	0	14	−1	1
8	−4	16	18	3	9
14	2	4	8	−7	49
15	3	9	21	6	36
9	−3	9	23	8	64
			10	−5	25
			17	2	4
$\Sigma x = 120$		$\Sigma(x-\bar{x})^2 = 120$	$\Sigma y = 180$		$\Sigma(y-\bar{y})^2 = 314$

$$\bar{x} = \frac{\Sigma x}{n_1} = \frac{120}{10} = 12, \; \bar{y} = \frac{\Sigma y}{n_2} = \frac{180}{12} = 15 \qquad [\because\; n_1 = 10,\; n_2 = 12]$$

and
$$S_1^2 = \frac{\Sigma (x-\bar{x})^2}{n_1 - 1} = \frac{120}{10-1} = \frac{120}{9}$$

$$S_2^2 = \frac{\Sigma (y-\bar{y})^2}{n_2 - 1} = \frac{314}{12-1} = \frac{314}{11}.$$

$$\therefore \quad \text{Variance ratio or } F = \frac{\text{Larger estimate}}{\text{Smaller estimate}}$$

$$= \frac{S_2^2}{S_1^2} = \frac{314/11}{120/9} = \frac{314 \times 9}{120 \times 11} = 2{\cdot}141.$$

Since for 11 and 9 d. f. the calculated value of $F (= 2 \cdot 141) <$ the tabulated value of F at 5% level $(= 3 \cdot 112)$.

Hence the population variance from the two samples is not significantly different.

Example 6. ***Two gauge operators are tested for precision in making measurements. One operator completes a set of 26 readings with a standard deviation of 1·34 and the other does 34 readings with a standard deviation of 0·98. What is the level of significance of this difference? You are given that for $\nu_1 = 25$ and $\nu_2 = 33$, $z_{0.05} = 0 \cdot 306$, $z_{0.01} = 0 \cdot 432$.***

Solution. Here $n_1 = 26, \sigma_x = 1 \cdot 34, n_2 = 34, \sigma_y = 0 \cdot 98$

$$S_1 = \sqrt{\left(\frac{n_1}{n_1 - 1}\right)}, \sigma_x = \sqrt{\left(\frac{26}{25}\right)} \times 1 \cdot 34$$

$$S_2 = \sqrt{\left(\frac{n_2}{n_2 - 1}\right)}, \sigma_y = \sqrt{\left(\frac{34}{33}\right)} \times 0 \cdot 98$$

$$z = \log_e \frac{S_1}{S_2} = \log_{10} (S_1 / S_2) \times \log_e 10$$

$$= 2 \cdot 3026 \times \log_{10} (S_1 / S_2)$$

$$= 2 \cdot 3026 \times \log_{10} \left\{ \sqrt{\left(\frac{26}{25} \times \frac{33}{34}\right)} \times \frac{1 \cdot 34}{0 \cdot 98} \right\}$$

$$= 2 \cdot 3026 \times [\frac{1}{2} \{\log_{10} 13 + \log_{10} 33 - \log_{10} 25$$

$$- \log_{10} 17\} + \log_{10} 1 \cdot 34 - \log_{10} 0 \cdot 98]$$

$$= 1 \cdot 1514 \{1 \cdot 11394 + 1 \cdot 51851 - 1 \cdot 39794 - 1 \cdot 23045\} + 2 \cdot 3026 \{0 \cdot 12710 - 1 \cdot 99123\}$$

$$= 1 \cdot 1513 \times 0 \cdot 00406 + 2 \cdot 3026 \{0 \cdot 12710 + 0 \cdot 00877\}$$

$$= 1 \cdot 1513 [0 \cdot 000406 + 0 \cdot 27174]$$

$$= 1 \cdot 1513 \times 0 \cdot 27580 = 0 \cdot 3175.$$

Here for d. f. $\nu_1 = 25$ and $\nu_2 = 33$.

Since $z_{0.05} (= 0 \cdot 305)$

$<$ calculated value of $z (= 0 \cdot 3175)$

$< z_{0.01} (= 0 \cdot 432)$.

Hence the difference between the standard deviations is just significant at 5% level. But it is not significant at 2% level.

Remark. F test can also be used.

EXERCISE 10 (F)

1. In two groups of ten children each increase in weight due to two different diets in the same period, were in pounds :

   8, 5, 7, 8, 3, 2, 7, 6, 5 7

   3, 7, 5, 6, 5, 4, 4, 5, 3, 6.

   Find whether the variances are significantly different. The value of $F_{0.05}$ for 2 degree of freedom is approximately 3·2.

2. Two independent random samples each of 8 individuals provide the following data. Estimate the variance ratio and test the significance :

   63, 64, 65, 65, 66, 66, 67, 68

   69, 66, 67, 67, 66, 68, 69, 69.

   The value of F at 5% point for 7 degrees of freedom is 3·8 approximately.

3. Two samples are composed of 7 and 9 individuals respectively and have variances 9·6 and 4·8 respectively. Is the variance 9·6 significantly greater than the variance 4·8?

4. The monthly consumption of scrap by Bhilai Steel Plant for two sets of 12 months are shown below. Is there any significant difference in the variability of consumption in the two years ?

   Consumption of Scrap (1, 000 Tons)

   April 1986—March 1987

   153, 158, 143, 138, 134, 144, 154, 160, 144, 144, 123, 148.

   April 1987—March 1988

   145, 150, 158, 129, 139, 162, 166, 164, 145, 169, 174, 175.

5. Seven shells were fired from a Boffors gun and their velocities showed a variance of 150. The velocities of six shells fired from the same gun but with a different brand of powder showed a variance of 120. Test whether this difference in their variability is usual.

   [**Hint.** $S_1^2 = (7/6) \times 150, S_2^2 = (6/5) \times 120$

   $$\therefore \quad F = \frac{S_1^2}{S_2^2} = \frac{7 \times 150 \times 5}{6 \times 6 \times 150} = 1 \cdot 2153$$

   $F_{0.05} = 4 \cdot 95$ for $\nu_1 = 6$ and $\nu_2 = 5$.

   Thus the difference is not significant.]

6. Tests for breaking strength were carried out on two lots of 7 and 9 steel wires respectively. The variance of one lot was 230 and that of the other was 492. Is there a significant difference in their variability ?

7. Write a short note on F-distribution.

8. Show how you would use students t-test and Snedecor's F-test to decide whether the following two samples have been drawn from the same normal population. Which of the two tests would you apply first and why ?

	Size	Mean	Sum of the square of deviation from the mean
Sample I	9	68	36
Sample II	10	69	42

   Given that $t_{17} (50) = 2 \cdot 11, F_{02} (\cdot 05) = 3 \cdot 4$.

9. Write a short note on F-test.

ANSWERS

1. Not significant, $F = 2 \cdot 4, F_{0.05} = 3 \cdot 18$.
3. The first, variance cannot be regarded as significantly greater than the second
   $F = 2 \cdot 075, F_{0.05} = 3 \cdot 58, F_{0.01} = 6 \cdot 37$ or alternatively,
   $z = 0 \cdot 3648, z_{0.05} = 0 \cdot 6378, z_{0.01} = 0 \cdot 9229$
4. Not significant, $F = 1 \cdot 95, F_{0.05} = 2 \cdot 82, F_{0.01} = 4 \cdot 46$
6. Not significant, $F = 2 \cdot 06, F_{0.05} = 4 \cdot 15$.

§ 10.21. TEST OF SIGNIFICANCE OF CORRELATION COEFFICIENT (SMALL SAMPLES)

Assume that the variate x and y are distributed in a bivariate normal population with mean μ_1, μ_2 and standard deviations σ_1, σ_2 and have correlation coefficient ρ. Let a random sample $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ be drawn from this bivariate normal population. The n pairs of values being independent observations. We have to test the hypothesis that ρ of the population is zero.

We define the t-statistic by the relation

$$t = \frac{r\sqrt{(n-2)}}{\sqrt{(1-r^2)}}.$$

Here number of d. f. $= n - 2$ and r is the correlation coefficient in the sample given by

$$r = \frac{(1/n)\,\Sigma\,(x-\bar{x})(y-\bar{y})}{\left\{\frac{1}{n}\Sigma\,(x-\bar{x})^2\right\}^{1/2}\left\{\frac{1}{n}\Sigma\,(y-\bar{y})^2\right\}^{1/2}}.$$

In t-test, we test the hypothesis that the correlation coefficient ρ is zero or significant or meaning less.

ILLUSTRATIVE EXAMPLES

Example 1. ***A random sample of 15 from a normal population gives a correlation coefficient -0.5. Is this significant of the existence of correlation in the population ?***

Solution. We are given that $r = -0.5$ and $n = 15$.

$$\therefore \quad t = \frac{r\sqrt{[(n-2)]}}{\sqrt{[(1-r^2)]}} = \frac{-0.5\sqrt{[(15-2)]}}{\sqrt{\{[1-(0.5)^2]\}}}$$

or $$|t| = \frac{0.5\sqrt{(13)}}{\sqrt{(0.75)}} = 2.08$$

and degree of freedom $= \nu = 15 - 2 = 13$.

The tabulated value of $t_{0.05}$ for 13 d. f. is 2·16. Here the calculated value of $t\,(= 2.07) <$ the tabulated value of t at 5% level of significance. Consequently the sample correlation coefficient is not significant to warrant the existence of correlation in the population.

Example 2. ***Show that in the samples of 25 from an uncorrelated normal population the chance is 1 in 100 that r is greater than about 0·43.***

Solution. We are given that $n = 25$, $r = 0.43$.

$$\therefore \quad t = \frac{r\sqrt{[(n-2)]}}{\sqrt{[(1-r^2)]}} = \frac{0.43\times\sqrt{[(23)]}}{\sqrt{\{[1-(0.43)^2]\}}}$$

$$= \frac{0.43\times 4.7958}{\sqrt{(1-0.1849)}} = 2.3 \text{ (approx.)}$$

From tables $t_{0.01} = 2.492$.

Thus the chance (probability) that $t > 2.492$ is 0.01 *i.e.*, 1 in 100. We see that the calculated value of t is nearly equal to tabular value at ·01% *i.e.*, 1 in 100. Hence the result follows.

Example 3. ***Find the least value of r in a sample of 18 pairs from a bivariate normal population significant at 5% level.***

Solution. Here $n = 18$, d. f. $= n - 2 = 18 - 2 = 16$.

$$\therefore \quad t = \frac{r\sqrt{(n-2)}}{\sqrt{(1-r^2)}} = \frac{r\sqrt{(16)}}{\sqrt{(1-r^2)}} = \frac{4r}{\sqrt{(1-r^2)}}.$$

Now for 16 d. f. $t_{0.05} = 2.12$.

Since for 16 d. f. at 5% level, we must have

$$|t| > 2.12$$

$$\Rightarrow \quad \frac{4|r|}{\sqrt{(1-r^2)}} > 2.12 \Rightarrow \frac{16r^2}{1-r^2} > (2.12)^2$$

$$\Rightarrow \quad 16r^2 > 4.4944\,(1-r^2)$$

$$\Rightarrow \quad 16r^2 + 4.4944\,r^2 > 4.4944$$

$$\Rightarrow \quad 20.4944\,r^2 > 4.4944 \Rightarrow |r| > \left(\frac{4.4944}{20.4944}\right)^{1/2}$$

or $|r| > \sqrt{(0.219298)} = 0.468 = 0.47$ (nearly).

Hence the least value of $|r| = 0.47$.

Example 4. ***How many pairs of observations be included in a sample so that it has one r = 0·5 and one calculated value of t greater than 2·08.***

Solution. Given $r = 0.5$, $t = 2.08$, $n = ?$.

$$\therefore \quad t = \frac{r\sqrt{(n-2)}}{\sqrt{(1-r^2)}} \Rightarrow 2.08 = \frac{0.5\times\sqrt{(n-2)}}{\sqrt{[1-(0.5)^2]}}$$

$$\Rightarrow \quad 2.08\times\sqrt{(0.75)} = 0.5\times\sqrt{(n-2)}.$$

Squaring, we have

$$2.08\times 2.08\times 0.75 = 0.25\,(n-2)$$

$$\Rightarrow \quad n - 2 = 2.08\times 2.08\times 3$$

$$\Rightarrow \quad n - 2 = 12.98 \Rightarrow n = 14.98 \Rightarrow n = 15.$$

EXERCISE 10 (G)

1. Write a short note on Fisher's z-distribution.
2. Write a short note on the test of significance of correlation coefficient in case of small samples.
3. Test whether correlation is significant if $r = 0.6$ and $n = 38$.
4. Given : $n = 11$, $r = 0.5$. Test the significance of r by Student's t-test.

ANSWERS

3. Correlation is significant.
4. $t = 1.72 < t_{0.05} = 0.262$. Hence the given value of r is not significant.

§ 10.22. TEST OF SIGNIFICANCE OF CORRELATION COEFFICIENT BASED ON FISHER'S z-TRANSFORMATION (LARGE SAMPLES)

Professor Fisher gave a method to test the coefficient of correlation in large samples. In this method r is transformed into z and therefore it is termed as z-transformation.

Suppose r and ρ are the correlations in the sample and the population. Then it can be shown that the distribution of r is not normal and its probability curve is very skew in the neighbourhood of $\rho = \pm 1$ even for large values of n.

Fisher used the following transformation

$$r = \tanh z, \ \rho = \tanh \zeta$$

So
$$z = \frac{1}{2} \log_e \frac{1+r}{1-r} = 1 \cdot 1513 \log_{10} \frac{1+r}{1-r} \qquad \text{...(1)}$$

and
$$\zeta = \frac{1}{2} \log_e \frac{1+\rho}{1-\rho} = 1 \cdot 1513 \log_{10} \frac{1+\rho}{1-\rho}.$$

Also a table has been prepared which gives the value of z for different values of r.

Fisher showed that the distribution of z given by equation (1) is approximately normal with mean ζ and standard deviation $\frac{1}{\sqrt{(n-3)}}$. This approximation is more valid for large values of n, ($n > 50$, say). But it can be used for many practical purposes even for small samples.

Here $\frac{z-\zeta}{1/\sqrt{(n-3)}}$ is a standard normal variate with zero mean and unit standard deviation. $\frac{1}{\sqrt{(n-3)}}$ is the S. E. of z.

Hence if $|(z-\xi)\sqrt{(n-3)}| > 1 \cdot 96$, the difference between ρ and r is significant at 5 per cent level. If it is greater than $2 \cdot 58$, the difference is significant at 1 percent level.

Remark. It has been seen in the theory of variables for large samples that S. E. of r is $\frac{1-r^2}{\sqrt{n}}$. It is to be used with utmost reserve for values of r near unity, since the distribution in such a case is markedly skew unless n is very large ($n > 500$, say). When there is any doubt the alternative test discussed above based on Fisher's transformation must be used.

Further note that symbol z used here is different from Fisher's z-transformation.

ILLUSTRATIVE EXAMPLES

Example 1. ***What is the probability that a correlation coefficient of*** $+0 \cdot 75$ ***or less can arise in a sample of 30 from a normal population in which the true correlation is*** $0 \cdot 9$***?***

Solution. Here $r = 0 \cdot 75, \rho = 0 \cdot 9$ and $n = 30$.

$$\therefore \quad z = 1 \cdot 1513 \log_{10} \frac{1+r}{1-r}$$
$$= 1 \cdot 1513 \log_{10} \frac{1+0 \cdot 75}{1-0 \cdot 75}$$
$$= 1 \cdot 1513 \log_{10} \frac{1 \cdot 75}{0 \cdot 25}$$
$$= 1 \cdot 1513 [\log_{10} 1 \cdot 75 - \log_{10} 0 \cdot 25]$$
$$= 1 \cdot 1513 [0 \cdot 24304 - \bar{1} \cdot 39794]$$
$$= 0 \cdot 0973 \text{ (approx.)}$$
$$\zeta = 1 \cdot 1513 \log_{10} \frac{1+\rho}{1-\rho} = 1 \cdot 1513 \log_{10} \frac{1 \cdot 9}{0 \cdot 1}$$
$$= 1 \cdot 1513 [\log_{10} 1 \cdot 9 - \log_{10} 0 \cdot 1]$$
$$= 1 \cdot 1513 (0 \cdot 27875 + 1)$$
$$= 1 \cdot 1513 \times 1 \cdot 27875 = 1 \cdot 47 \text{ (approx.)}.$$

Thus
$$\left| \frac{z-\zeta}{1/\sqrt{(n-3)}} \right| = (1 \cdot 471 - 0 \cdot 973)\sqrt{27} \qquad [\because \ n = 30]$$
$$= 0 \cdot 498 \times 5 \cdot 196 = 2 \cdot 59 \text{ (approx.)}.$$

From the area to the right of normal curve, the area to the left of the ordinate of $x = 2 \cdot 59$ is $0 \cdot 9952$.

Hence the area to the right of $x = 2 \cdot 59$ is $1 - 0 \cdot 9952 = 0 \cdot 0048$, which is the required probability that $r \le 0 \cdot 75$ or that

$$|(z-\xi)\sqrt{(n-3)}| \ge 2 \cdot 59.$$

Example 2. ***A correlation coefficient of*** $0 \cdot 7$ ***is discovered in a sample of 28 pairs. Apply z-transformation to find out if this differs significantly***

(a) ***from 0,*** **(b)** ***from*** $0 \cdot 5$.

Solution. (a) Here $\rho = 0$, therefore

$$\zeta = \frac{1}{2} \log_e \frac{1+0}{1-0} = \frac{1}{2} \log_e 1 = 0$$

and
$$z = 1 \cdot 1513 \log_{10} \frac{1+r}{1-r}$$
$$= 1 \cdot 1513 \log_{10} \frac{1+0 \cdot 7}{1-0 \cdot 7} \qquad [\because \ r = 0 \cdot 7]$$
$$= 1 \cdot 1513 \log_{10} \frac{1 \cdot 7}{0 \cdot 3} = 0 \cdot 87 \qquad \text{[using log tables]}$$
$$\therefore \quad \frac{z-\zeta}{1/\sqrt{(n-3)}} = (0 \cdot 87 - 0)\sqrt{(25)} = 4 \cdot 35$$

which is greater than $1 \cdot 96$ and $2 \cdot 58$ both.

Hence $H_0 : \rho = 0$, is rejected at 5% and 1% level. It follows that the population is correlated.

(b) Now $\rho = 0 \cdot 5$, therefore

$$\zeta = 1 \cdot 1513 \log_{10} \frac{1 + 0 \cdot 5}{1 - 0 \cdot 5} = 0 \cdot 55 \quad \text{[using log tables]}$$

$$\therefore \quad \frac{z - \zeta}{1/\sqrt{(n-3)}} = (0 \cdot 87 - 0 \cdot 55)\sqrt{(25)} = 1 \cdot 6 < 1 \cdot 96 \text{ and } 2 \cdot 58$$

It implies that the difference between z and ζ is not significant and hence the hypothesis $\rho = 0 \cdot 5$ can not be rejected.

Example 3. ***The correlation between the price indices of animal feeding-stuffs and home-grown oats in a sample of 60 members is 0·68 Could the observed value have arisen :***

(a) ***From an uncorrelated population ?***

(b) ***From a population in which true correlation was 0·8?***

Solution. Here $r = 0 \cdot 68$, $n = 60$.

(a) Here
$$z = \frac{1}{2} \log_e \left(\frac{1+r}{1-r}\right)$$

$$= \frac{1}{2} \log_e \left(\frac{1 + 0 \cdot 68}{1 - 0 \cdot 068}\right) = 1 \cdot 1513 \log_{10} \frac{1 \cdot 68}{0 \cdot 32}$$

$$= 0 \cdot 829, \quad \text{[using log tables]}$$

S. E. of
$$z = \frac{1}{\sqrt{(n-3)}} = \frac{1}{\sqrt{(57)}} = 0 \cdot 13.$$

$$\therefore \quad \frac{z - \zeta}{1/\sqrt{(n-3)}} = \frac{0 \cdot 829 - 0}{0 \cdot 13} = 6 \cdot 38 \text{ (nearly)}$$

$$\Rightarrow \quad z - \zeta = (6 \cdot 38) \cdot \frac{1}{\sqrt{(n-3)}}$$

$$\Rightarrow \quad z - \zeta = (6 \cdot 38) \times (\text{S. E. of } z).$$

Since the deviation of z from ζ is more than six times the S. E., the hypothesis is not correct, *i.e.*, the population is correlated.

(b) Here
$$\rho = 0 \cdot 8.$$

$$\therefore \quad \zeta = 1 \cdot 1513 \log_{10} \frac{1 + \rho}{1 - \rho} = 1 \cdot 1513 \log \frac{1 \cdot 8}{0 \cdot 2}$$

$$= 1 \cdot 099 \quad \text{[using log tables]}$$

Hence
$$\left|\frac{z - \zeta}{1/\sqrt{(n-3)}}\right| = \left|\frac{0 \cdot 829 - 1 \cdot 099}{0 \cdot 13}\right| = \frac{0 \cdot 27}{0 \cdot 13} = 2 \cdot 08.$$

Here the deviation of z from ζ is nearly two times the S. E., which can be assumed by sampling fluctuations *i.e.*, ρ is likely to be less than $0 \cdot 8$.

◈ § 10.23. SIGNIFICANCE OF THE DIFFERENCE BETWEEN TWO INDEPENDENT CORRELATION COEFFICIENTS

Suppose two samples of sizes n_1 and n_2 give correlation coefficients r_1 and r_2. We are to test the hypothesis that the samples are taken from the same population or from two populations with the same correlation coefficient. Under the hypothesis be true, the t statistic defined by

$$t = \frac{z_1 - z_2}{\left(\dfrac{1}{n_1 - 3} + \dfrac{1}{n_2 - 3}\right)^{1/2}}$$

where $z_1 = \dfrac{1}{2} \log_e \dfrac{1 + r_1}{1 - r_1}$ and $z_2 = \dfrac{1}{2} \log_e \dfrac{1 + r_2}{1 - r_2}$ is approx. distributed normally with zero mean and unit S. D.

Note that S. E. of the difference of two values of $r = \sqrt{\left(\dfrac{1}{n_1 - 3} + \dfrac{1}{n_2 - 3}\right)}$.

If $|t| > 1 \cdot 96$, then the difference is significant at 5% level.

ILLUSTRATIVE EXAMPLES

Example 1. ***The first of two samples consists of 23 pairs and gives a correlation of 0·5 while the second of 28 pairs has a correlation of 0·8. Are these values significantly different ?***

Solution. Here $r_1 = 0 \cdot 5$, $r_2 = 0 \cdot 8$, $n_1 = 23$, $n_2 = 25$ using formulae of § 10.23, above, we have

$$z_1 = \frac{1}{2} \log_e \frac{1 + r_1}{1 - r_1}$$

$$= \frac{1}{2} \log_{10} \frac{1 + r_1}{1 - r_1} \times \log_e 10$$

$$= \frac{2 \cdot 3026}{2} \log_{10} \frac{1 + 0 \cdot 5}{1 - 0 \cdot 5}$$

$$= 1 \cdot 1513 \log 3 = 0 \cdot 55.$$

Similarly $z_2 = 1 \cdot 10.$

$$\therefore \quad t = \frac{z_1 - z_2}{\sqrt{\left(\dfrac{1}{n_1 - 3} + \dfrac{1}{n_2 - 3}\right)}} = \frac{0 \cdot 55}{\sqrt{\left(\dfrac{1}{20} + \dfrac{1}{25}\right)}} = 1 \cdot 8322$$

Here $t < 1 \cdot 96$. Hence the difference is not quite significant at 5% level. Thus the hypothesis cannot be rejected.

Example 2. ***For a sample of 15 boys, the correlation coefficient between the tastes in Mathematics and Language is 0·8. For a second sample of 12 girls r = 0·6. Is the difference significant ?***

Solution. Assume the hypothesis : that the difference in two values of r is not significant.

We have
$$n_1 = 15, n_2 = 12, r_1 = 0 \cdot 8, r_2 = 0 \cdot 6$$

$$z_1 = 1 \cdot 1513 \log_{10} \frac{1 + 0 \cdot 8}{1 - 0 \cdot 8} = 1 \cdot 098$$

$$z_2 = 1 \cdot 1513 \log \frac{1 + 0 \cdot 6}{1 - 0 \cdot 6} = 0 \cdot 693$$

$$t = \frac{z_1 - z_2}{\left(\frac{1}{n_1 - 3} + \frac{1}{n_2 - 3}\right)^{1/2}} = \frac{1 \cdot 098 - 0 \cdot 693}{\sqrt{\left(\frac{1}{12} + \frac{1}{9}\right)}} = 0 \cdot 920.$$

Clearly $|t|$ is sufficiently, smaller then $1 \cdot 96$ at 5% level. Hence the difference is not significant.

EXERCISE 10 (H)

1. Write a short note on Fisher's z-distribution.
2. Write a short note on the test of significance of correlation coefficient in case of small samples.
3. Test whether correlation is significant if $r = 0 \cdot 6$ and $n = 38$.
4. Given : $n = 11, r = 0 \cdot 5$. Test the significance of r by Student's t-test.

ANSWERS

3. Correlation is significant.
4. $t = 1 \cdot 72 < t_{0 \cdot 05} = 0 \cdot 262$. Hence the given value of r is not significant.

◈ § 10.24. SMALL SAMPLE TESTS (χ^2 DISTRIBUTION)

When the sample size is small (*i.e.*, $n < 30$) then we can not apply the same procedure for the testing of hypothesis which we have applied in large sample tests. Actually in large sample theory we have applied the central limit theorem on the test statistic, to find the statistical distribution of the test statistics which was normal distribution, but here we can not apply the central limit theorem to find the distribution of the test statistics. Actually normal distribution is not the basis of the small sample tests. So in the case of small samples we will apply the chi-square (χ^2), t, F and Z distributions which are also known as sampling distributions.

◈ § 10.25. THE CHI-SQUARE DISTRIBUTION (χ^2)

Let $X_1, X_2, \ldots, X_n$ be independent normal variates with mean μ and variance σ^2 then

$$Z_i = \frac{X_i - \mu}{\sigma}; \quad i = 1, 2, \ldots, n$$

are independent normal variates with mean 0 and variance 1.

Then the sum of squares of the variates $Z_1, Z_2, \ldots, Z_n$ *i.e.*,

$$\sum_{i=1}^{n} Z^2 = \sum_{i=1}^{n} \left(\frac{X_i - \mu}{\sigma}\right)^2$$

is distributed as χ^2 with n degrees of freedom

$$\chi^2 = \sum_{i=1}^{n} Z^2 = \sum_{i=1}^{n} \left(\frac{X_i - \mu}{\sigma}\right)^2$$

and the probability density function is given by

$$f(\chi^2) = \begin{cases} \dfrac{1}{2^{n/2}\,\Gamma(n/2)} \cdot e^{-\chi^2/2} \cdot (\chi^2)^{(n/2)-1} & ; \; 0 < x < \infty \\ 0 & ; \; \text{otherwise} \end{cases}$$

where n = degrees of freedom.

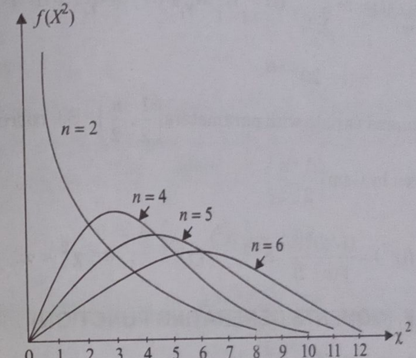


Fig. 10.4

The chi-square distribution is a continuous probability distribution and ranges from 0 to ∞ as shown in the figure 3.5. Because χ^2 is the sum of squares, it cannot be negative.

Note that, $\left(\frac{X_i - \mu}{\sigma}\right)^2 \sim \chi^2_{(1)}$ for all $i = 1, 2, \ldots, n$.

◈ § 10.26. DERIVATION OF χ^2 DISTRIBUTION

If X_i $(i = 1, 2, \ldots, n)$ are independent $N(\mu, \sigma^2)$ variates. Then, mgf of $Y_i^2 = \left(\frac{X_i - \mu}{\sigma}\right)^2$ is given by

$$M_{Y_i^2}(t) = E(e^{tY_i^2}) = \int_{-\infty}^{\infty} \frac{e^{ty_i^2}}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{X_i - \mu}{\sigma}\right)^2} dx_i \qquad \left[\text{put } y = \frac{x_i - \mu}{\sigma}\right]$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ty_i^2} \cdot e^{-\frac{1}{2}y_i^2} (\sigma\, dy_i) \qquad [\because dx_i = \sigma\, dy_i]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(\frac{1}{2} - t\right) y_i^2} dy_i = \sqrt{\frac{2}{\pi}} \int_{-0}^{\infty} e^{-\left(\frac{1-2t}{2}\right) y_i^2} dy.$$

Now, putting $y_i^2 = v$ i.e., $dy_i = \frac{dv}{2\sqrt{t}}$, we get

$$= \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-\left(\frac{1-2t}{2}\right)v} v^{1/2-1}\, dF \qquad \left[\because \int_0^\infty e^{-at} t^{\lambda-1} dt = \frac{\Gamma\lambda}{a^\lambda}\right]$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{\Gamma(1/2)}{\left(\frac{1-2t}{2}\right)^{1/2}} = (1-2t)^{-1/2}.$$

Thus, $M_{\chi^2_{(n)}}(t) = M_{\sum_{i=1}^n Y_i^2}(t) = \prod_{i=1}^n M_{Y_i^2}(t) = [M_{Y_i^2}(t)]^n$ [$\because$ Y_i's are i.i.d.]

$$= (1-2t)^{-n/2}$$

which is mgf of Gamma variate with parameters $\left(\frac{1}{2}, \frac{n}{2}\right)$. So, corresponding pdf of $\chi^2_{(n)}$ variate is given by Gam $\left(\frac{1}{2}, \frac{n}{2}\right)$

$$f(\chi^2) = \frac{(1/2)^{n/2}}{\Gamma(n/2)} \cdot e^{-\frac{1}{2}(\chi^2)} (\chi)^{2\frac{n}{2}-1} \; ; \; 0 \le \chi^2 < \infty.$$

◈ § 10.27. M.G.F. (MOMENTS GENERATING FUNCTION)

$$M_X(t) = E(e^{tX}) = \int_0^\infty e^{tx} f(x)\, dx$$

$$= \frac{1}{2^{n/2}\Gamma(n/2)} \int_0^\infty e^{tx} e^{-x/2} x^{(n/2)-1} dx$$

$$= \frac{1}{2^{n/2}\Gamma(n/2)} \int_0^\infty e^{-\left(\frac{1-2t}{2}\right)x} x^{n/2-1}\, dx$$

$$= \frac{1}{2^{n/2}\Gamma(n/2)} \cdot \frac{\Gamma(n/2)}{\left(\frac{1-2t}{2}\right)^{n/2}} = (1-2t)^{-n/2};\ |t| < 1.$$

Now, $\quad$ mean $= \mu_1' = \left[\frac{d}{dt} M_X(t)\right]_{t=0} = -\frac{n}{2}(-2) = n$

$$\mu_2' = \left[\frac{d^2}{dt^2} M_X(t)\right]_{t=0} = -\frac{n}{2}\left(-\frac{n}{2}-1\right)(-2)(-2) = n(n+2).$$

And, $\quad$ variance $= \mu_2 = \mu_2' - (\mu_1')^2 = n(n+2) - n^2 = 2n.$

◈ § 10.28. MODE AND SKEWNESS OF χ^2-DISTRIBUTION

Mode of χ^2-distribution will be given by $f'(x) = 0$ and $f''(x) < 0$. Now, $f'(x) = 0$

$$\Rightarrow \quad \frac{d}{dx}\left(\frac{1}{2^{n/2}\Gamma(n/2)} \cdot e^{-x/2} x^{n/2-1}\right) = 0$$

$$\Rightarrow \quad \left[-\frac{1}{2} e^{-x/2} x^{n/2-1} + e^{-x/2}\left(\frac{n}{2}-1\right) x^{\frac{n}{2}-2}\right] = 0$$

$$\Rightarrow \quad e^{-x/2} x^{\frac{n}{2}-2}\left[-\frac{x}{2} + \frac{n}{2} - 1\right] = 0$$

$$\Rightarrow \quad x = 0, \infty \text{ and } (n-2)$$

and $f''(x) = \frac{1}{2^{n/2}\Gamma(n/2)}\left[e^{-x/2}\left(-\frac{1}{2}\right)x^{n/2-2}\left(\frac{n-2-x}{2}\right) + e^{-x/2}x^{n/2-1}\left(-\frac{1}{2}\right)\right.$

$$\left. + e^{-x/2}\left(\frac{n}{2}-2\right)x^{n/2-3}\left(\frac{n-2-x}{2}\right)\right]$$

which is < 0 for only $x = n-2$.

Hence, the mode of χ^2-distribution is at $x = (n-2)$ and exists only if $n \ge 2$.

Also, (by definition) Karl Pearson's coefficient of skewness is given by

$$\text{Skewness} = \frac{\text{Mean} - \text{Mode}}{\sigma} = \frac{n-(n-2)}{\sqrt{2n}}$$

$$= \sqrt{\frac{2}{n}} > 0 \text{ i.e., positive skewed.}$$

◈ § 10.29. PROPERTIES OF χ^2-DISTRIBUTION

χ^2-distribution has the following useful properties :

(i) Since the value of χ^2 cannot be negative, so it occurs only in first quadrant.

(ii) Sampling distribution of χ^2 is positively skewed.

(iii) The χ^2-distribution does not contain any population parameter, so it may be referred to as non-parametric distribution.

(iv) The mean and variance of χ^2-distribution with n degrees of freedom are as follows :

$$\text{Mean} = n \quad \text{and} \quad \text{Variance} = 2n.$$

(v) **Additive property.** *Sum of independent χ^2-variates is also a χ^2-variate.*

Proof. Let X_i be independent $\chi^2_{(n_i)}$ variates ($i = 1, 2, \ldots, k$) and for each X_i, we have

$$M_{X_i}(t) = (1-2t)^{-n_i/2};\ i = 1, 2, \ldots, k.$$

Thus $\quad M_{\sum_{i=1}^n X_i(t)} = M_{X_1}(t) \cdot M_{X_2}(t) \ldots M_{X_k}(t)$

$$= (1-2t)^{-(n_1+n_2+\ldots+n_k)/2}$$

$$= (1-2t)^{-\sum_{i=1}^k n_i/2}$$

which is mgf of $\chi^2_{\left(\sum_{i=1}^k n_i\right)}$

Hence, distribution of sum of k independent χ^2-variates each with n_i d.f. $(i = 1, 2, \ldots, k)$ is χ^2 with $\left(\sum_{i=1}^{k} n_i\right)$ d.f.

§ 10.30. ASSUMPTIONS FOR χ^2-TEST

Following are the underlying assumptions for validity of χ^2-test

(i) Sample observations are independent

(ii) Total observed frequencies (ΣO_i) = Total expected frequencies (ΣE_i) = Total frequencies (N).

(iii) N is large (> 30)

(iv) No E_i should be < 5.

§ 10.31. APPLICATIONS OF χ^2-DISTRIBUTION

The χ^2-distribution is very helpful in testing of hypothesis. Some tests are listed below, which are based on χ^2-distribution :

(i) Test of goodness of fit.

(ii) Test of independence of attributes.

(iii) Test of significance for population variance.

§ 10.32. TEST OF GOODNESS OF FIT

A very powerful test for testing the significance of difference between theoretical distributions such as Binomial, Poisson, Normal etc. and empirical distributions *i.e.*, those obtained from sample data, was given by Prof. Karl Pearson in 1900 and is known as χ^2-test of goodness of fit. This test helps us to find whether the deviation from the theory is just by chance or is it really due to the inadequacy of the theory to fit the observed data. To apply this test, first of all we hypothesize a particular theoretical distribution it may be Normal, Binomial, Poisson or any other, then the test is carried out to find whether the sample data could have come from the hypothesized distribution or not.

To apply this test, the following steps are involved :

(a) First of all set up the null and alternative hypothesis such as :

H_0 : There is no difference between theoretical distribution and empirical distribution.

H_1 : Significant difference between them.

(b) After the setting of null and alternative hypothesis, select a random sample and find the observed frequencies (O values) for each category.

(c) Find the expected frequencies (E-values) for each category.

(d) Calculate the value of χ^2 statistic

$$\chi^2_{(n-1)} = \sum_{i=1}^{n} \frac{(O_i - E_i)^2}{E_i}.$$

(e) Decision Rule :

(i) Accept the null hypothesis at α% level of significance if the calculated value of χ^2 is less than the tabulated value of χ^2 with $(n-1)$ d.f. and α% level of significance.

(ii) Otherwise reject the null hypothesis.

SOME ILLUSTRATIVE EXAMPLES

Example 1. ***A die is tossed 120 times and each outcome is recorded as under :***

Faces	:	1	2	3	4	5	6
Frequency	:	20	22	17	18	19	24

Is the distribution of outcomes uniform ?

Solution. Here, (i) H_0 : Distribution is uniform *i.e.*, expected frequencies for each face are equal $(= n/6)$ v/s H_1 : Distribution is not uniform $(\neq n/6)$.

(ii) For this goodness of fit test a CR of size $\alpha = 0.05$ with $(n - 1 = 6 - 1 = 5)$ df will be given by

$$CR = \{\chi^2 : \chi^2_{(5)} > 11.1\} \qquad [\because \chi^2_{(5)}(0.05) = 11.1 \text{ (from } \chi^2 \text{ table)}]$$

(iii) Under H_0; each face should occur 20 times *i.e.*,

Faces	1	2	3	4	5	6	Total
O_i (observed frequency)	20	22	17	18	19	24	$120 = \sum_{i=1}^{6} O_i$
E_i (expected frequency)	20	20	20	20	20	20	$120 = \sum_{i=1}^{6} E_i$
$O_i - E_i$	0	2	-3	-2	-1	4	
$\frac{(O_i - E_i)^2}{E_i}$	0	$\frac{4}{20}$	$\frac{9}{20}$	$\frac{4}{20}$	$\frac{1}{20}$	$\frac{16}{20}$	$\frac{34}{20} = 1.7 = \chi^2_{(5)}$

Since, $\chi^2_{(5)} = 1.7 < 11.1$ *i.e.*, $\notin$ CR. Hence, our H_0 is accepted.

Example 2. ***The following table gives the number of aircraft accidents that occurred during various days of the week. Find whether the accidents are uniformly distributed over the week*** $(\chi^2_{6,\, 0.05} = 12.59)$

Days	:	Sun.	Mon.	Tue.	Wed.	Thu.	Fri.	Sat.
No. of Accidents	:	14	15	8	20	11	9	14

Solution. Here we set up the null hypothesis that the accidents are uniformly distributed over the week. That is

H_0 : Accidents are uniformly distributed over the week

Against H_1 : Accidents are not uniformly distributed over the week.

Now the expected number of accidents for each day will be

$$E = \frac{\text{Total number of accidents}}{7} = \frac{91}{7} = 13.$$

Now table for calculation of χ^2-test statistic

Days	O	E	$O - E$	$(O-E)^2$	$\frac{(O-E)^2}{E}$
Sun.	14	13	1	1	0·0769
Mon.	15	13	2	4	0·3077
Tue.	8	13	− 5	25	1·9231
Wed.	20	13	7	49	3·7692
Thu.	11	13	− 2	4	0·3077
Fri.	9	13	− 4	16	1·2308
Sat.	14	13	1	1	0·0769
Total	91	91			7·6923

Thus $$\chi^2 = \sum \frac{(O-E)^2}{E}$$

$$\chi^2 = 7 \cdot 6923.$$

From the table $\chi^2_6\,(0 \cdot 05) = 12 \cdot 60.$

Since calculated value of χ^2 is less than the tabulated value $\chi^2_6\,(0 \cdot 05)$. Thus we accept our null hypothesis H_0 and conclude that accidents are uniformly distributed over the week.

Example 3. ***A book has 700 pages. The number of pages with various numbers of misprints is recorded below:***

No. of misprints	:	0	1	2	3	4	5
No. of pages with misprints	:	616	70	10	2	1	1

Can a Poisson distribution be fitted to this set of data?

Solution. Here we set up the null hypothesis that the data are fitted with Poisson distribution. That is

H_0 : Data fitted with Poisson distribution

Again H_1 : Data are not fitted with Poisson distribution.

Since in the book, 700 pages and the maximum number of misprints are only 5. Thus we may apply Poisson distribution to calculate the expected number of misprints in each page of the book as follows :

Number of misprints (x)	Number of page y	$f \cdot x$
0	616	0
1	70	70
2	10	20
3	2	6
4	1	4
5	1	5
Total	700	105

Thus expected value $= \lambda = \frac{\Sigma fx}{\Sigma f}$

$$\lambda = \frac{105}{700} = 0 \cdot 15.$$

Now we know that, the Poisson distribution is given by

$$P(X = x) = \frac{e^{-\lambda}\,\lambda^x}{x!}.$$

Thus the expected frequencies are given below :

$$P(x = 0) = e^{-\lambda} = e^{-0 \cdot 15} = 0 \cdot 8607$$

$$N\ P(x=0) = 700 \times P(x=0) = 700 \times 0 \cdot 8607 = 602 \cdot 5$$

$$N\ P(x=1) = N \times P(x=0) \times \lambda = 602 \cdot 5 \times 0 \cdot 15 = 90 \cdot 38$$

$$N\ P(x=2) = N \times P(x=1) \times \frac{\lambda}{2} = 90 \cdot 38 \times \frac{0 \cdot 15}{2} = 6 \cdot 78$$

$$N\ P(x=3) = N \times P(x=2) \times \frac{\lambda}{3} = 6 \cdot 78 \times \frac{0 \cdot 15}{3} = 0 \cdot 34$$

$$N\ P(x=4) = N \times P(x=3) \times \frac{\lambda}{4} = 0 \cdot 34 \times \frac{0 \cdot 15}{4} = 0 \cdot 013$$

$$N\ P(x=5) = N \times P(x=4) \times \frac{\lambda}{5} = 0 \cdot 014 \times \frac{0 \cdot 15}{5} = 0.$$

Now we arrange the expected and observed frequencies in the table given below :

Misprints	O	E	$O - E$	$(O-E)^2$	$(O-E)^2/E$
0	616	606·50	13·50	182·25	0·302
1	70	90·38	− 20·38	415·34	4·595
2	10	6·78	3·22	10·37	1·529
3	2	0·034	3·65	13·32	37·733
4	1 } = 4	0·013 } = 0·353			
5	1	0			
Total					44·159

Thus $\chi^2_{cal} = 44 \cdot 159.$

Degree of freedom $= 6 - 1 - 3 = 2.$

Now from table, the $\chi^2_2\ (0 \cdot 05) = 5 \cdot 99.$

Since $\chi^2_{cal} = 44 \cdot 159 > \chi^2_2\ (0 \cdot 05) = 5 \cdot 99.$

Thus we reject our null hypothesis and conclude that data are not fitted with Poisson distribution.

Example 4. ***A die is thrown 132 times with the following results :***

Number of turned up	:	1	2	3	4	5	6
Frequency	:	16	20	25	14	29	28

Test the hypothesis that die is unbiased. Test it at 5% level of significance. Given, the table value of χ^2 for degrees of freedom 5 at 5% level of significance = 11·07.

Solution. We set up the null hypothesis that the die is unbiased.

That is, H_0 : Die is unbiased

Again H_1 : Die is biased.

Thus the expected frequency for each face will be $= \frac{132}{6} = 22.$

Number Turned up	O	E	(O – E)	$(O - E)^2$	$(O - E)^2 / E$
1	16	22	– 6	6	1·636
2	20	22	– 2	4	0·182
3	25	22	3	9	0·409
4	14	22	– 8	64	2·909
5	29	22	7	49	2·227
6	28	22	6	36	1·636
Total					8·999

We have $\chi^2 = \sum \frac{(O-E)^2}{E} = 8 \cdot 999.$

From table $\chi^{2\,(2)}_5\ (0 \cdot 05) = 11 \cdot 07.$

Now since calculated valued of χ^2 is less than the tabulated value of χ^2, so we accept our null hypothesis at 5% level of significance and conclude that the die is unbiased.

Example 5. ***100 students of a management institute obtained the following grade in statistics paper :***

Grade	:	A	B	C	D	E	*Total*
Frequency	:	15	17	30	22	16	100

Using χ^2-test, examine the hypothesis that the distribution of grades is uniform.

Solution. We set up the null hypothesis that the grades are uniformly distributed among the students. That is

H_0 : Distribution of grades is uniform

Against H_1 : Distribution of grades is not uniform.

Thus the expected frequency of grades will be $\frac{100}{5} = 20.$

Now

Grade	O	E	(O – E)	$(O - E)^2$	$(O - E)^2 / E$
A	15	20	– 5	25	1·25
B	17	20	– 3	9	0·45
C	30	20	10	100	5·00
D	22	20	2	6	0·20
E	16	20	– 4	16	0·80
Total					7·7

Thus $\chi^2 = \sum \frac{(O-E)^2}{E} = 7 \cdot 7.$

From the table $\chi^2_4\ (0 \cdot 05) = 9 \cdot 49.$

Since calculated value of χ^2 is (less) than, the tabulated value of χ^2 at 5% level of significance for 4 d.f., so we accept our null hypothesis at 5% level of significance and conclude that the grades are uniformly distributed among the students.

Example 6. ***A personnel manager is interested in trying to determine whether absenteeism is greater on one day of the week than on another. His records for the past year show the following sample distribution :***

Day of the week	:	*Mon.*	*Tue.*	*Wed.*	*Thu.*	*Fri.*
No. of Absentees	:	66	56	54	48	75

Test whether the absence is uniformly distributed over the week.

Solution. We set up the null hypothesis that absence is uniformly distributed over the week. That is

H_0 : Distribution of absence is uniform

Against H_1 : Distribution of absence is not uniform.

Thus the expected frequency of absentees will be

$$= \frac{\text{Total number of absentees}}{\text{Number of days}} = \frac{300}{5} = 60.$$

Days	O	E	$O-E$	$(O-E)^2$	$(O-E)^2/E$
Mon.	66	60	6	36	0·60
Tue.	56	60	– 4	16	0·27
Wed.	54	60	– 6	36	0·60
Thu.	48	60	– 12	144	2·40
Fri.	75	60	15	225	3·75
Total					7·62

Thus $\chi^2 = \sum \dfrac{(O-E)^2}{E} = 7 \cdot 62$

d.f. = 5 – 1 = 4.

From the table $\chi_4^2\,(0\cdot05) = 9\cdot49$. Since calculated value of χ^2 is less than the tabulated value of χ^2 at 5% level of significance for 4 d.f., so we accept our null hypothesis at 5% level of significance and conclude that absence is uniformly distributed over the week.

Example 7. ***A survey of 320 families with 5 children each revelated the following distribution :***

No. of boys	:	5	4	3	2	1	0
No. of girls	:	0	1	2	3	4	5
No. of families	:	14	56	110	88	40	12

Is this result consistent with the hypothesis that the male and female births are equally probable at 5% level of significance ?

Given $t_{10,005} = 2\cdot23$, $\chi^2_{0\cdot05}\,(5) = 11\cdot07$ ***and*** $P_r\,[Z > 1\cdot645] = 0\cdot05$.

Solution. Let us set up the null hypothesis that the data are consistent with the hypothesis of equal probability for male and female births

$$H_0 : p = 1/2$$

v/s

$$H_1 : p \neq \frac{1}{2}$$

$p(r)$ = Probability of r male births in a family of 5

$$= {}^5c_r \left(\frac{1}{2}\right)^r \left(\frac{1}{2}\right)^{5-r} = {}^5C_r \left(\frac{1}{2}\right)^5.$$

The frequency of r male births is given by :

$$f(r) = N\,.\,p(r) = 320 \times {}^5C_r \left(\frac{1}{2}\right)^5 = 10 \times {}^5C_r;\ r = 0, 1, 2, 3, 4, 5. \qquad \text{...(*)}$$

Substituting $r = 0, 1, 2, 3, 4, 5$ successively in (*) we get the expected frequencies as follows :

$f(0) = 10 \times 1 = 10$, $f(1) = 10 \times {}^5C_1 = 50$, $f(2) = 10 \times {}^5C_2 = 100$

$f(3) = 10 \times {}^5C_3 = 100$, $f(5) = 10 \times {}^5C_5 = 50$, $f(5) = 10 \times {}^5C_5 = 10$.

Calculation for x^2

No. of male births	O	E	$O-E$	$(O-E)^2$	$(O-E)^2/E$
0	12	10	2	4	0·4
1	40	50	– 10	100	2
2	88	100	– 12	144	1·44
3	110	100	10	100	1
4	56	50	6	36	0·72
5	14	10	4	16	1·6
Total	320	320			7·16

$$x^2 = \sum_{i=1}^{5} \frac{(O_i - E_i)^2}{E_i}$$

$$= 7\cdot16$$

$$x^2_{0\cdot05}\,(6-1) = x^2_{0\cdot05}\,(5) = 11\cdot07.$$

Since $x^2_{\text{cal}} = 7\cdot16$ is less than tabulated value $x^2_{0\cdot05}\,(5) = 11\cdot07$. So we accept H_0. It means male and female births are equally probable at 5% level of significance.

◈ § 10.33. CONTINGENCY TABLE

Let us consider a sample of size N in which observations are classified into two attributes A and B. Suppose attribute A consists of p mutually exclusive categories say $A_1, A_2, \ldots, A_p$ and the attribute B consists of q mutually exclusive categories say $B_1, B_2, \ldots, B_q$. Then these observations can be displayed in a table, which is called contingency table :

Attribute A \ Attribute B	B_1	B_2	...	B_j	...	B_{q-1}	B_q	Total
A_1	O_{11}	O_{12}	...	O_{1j}	...	$O_{1(q-1)}$	O_{1q}	(A_1)
A_2	O_{21}	O_{22}	...	O_{2j}	...	$O_{2(q-1)}$	O_{2q}	(A_2)
⋮	⋮	⋮		⋮		⋮	⋮	⋮
A_i	O_{i1}	O_{i2}		O_{ij}		$O_{i(q-1)}$	O_{iq}	(A_i)
⋮	⋮	⋮		⋮		⋮	⋮	
A_{p-1}	⋮	⋮		⋮		⋮	⋮	
A_p	O_{p1}	O_{p2}		O_{pj}		$O_{p(q-1)}$	O_{pq}	(A_p)
Total	(B_1)	(B_2)	...	(B_j)	...	(B_{q-1})	(B_q)	N

This table is called the contingency table of $(p \times q)$ order

where A_i $(i = 1, 2, \ldots, p)$ denotes the ith category of the attribute A

B_j $(j = 1, 2, \ldots, q)$ denotes the jth category of the attribute B

$O_{ij}\begin{pmatrix} i = 1, 2, \ldots, p \\ j = 1, 2, \ldots, q \end{pmatrix}$ denotes the observed frequency due to A_ith and B_jth attribute

(A_i) denotes the frequency of the attribute A_i and given by

$$(A_i) = O_{i1} + O_{i2} + \ldots + O_{iq} = \sum_{j=1}^{q} O_{ij}$$

(B_j) denotes the frequency of the attribute B_j and given by

$$(B_j) = O_{1j} + O_{2j} + \ldots O_{pj} = \sum_{i=1}^{p} O_{ij}$$

$$N = \sum_{i=1}^{p} \sum_{j=1}^{q} O_{ij} = \sum_{i=1}^{p} (A_i) = \sum_{j=1}^{q} (B_j) = \text{Total number of observations.}$$

Expected Frequency. When two attributes A and B are independent then expected frequency is given by

$$E = E\,[O_{ij}] = \frac{(A_i)\,.\,(B_j)}{N}.$$

§ 10.34. TEST OF INDEPENDENCE OF ATTRIBUTES

We discussed earlier contingency table. Now we shall consider the testing of independence of attributes which is based on contingency table. In testing of independence of attributes, we test whether the association exists or not between two variables where the sample data are presented in the form of a contingence table with p rows and q columns.

Now in the testing of independence of attributes. The following steps will be involved :

(a) First of all we set up the null and alternative hypotheses such as

H_0 : There is no association between two attributes *i.e.*, they are independent.

H_1 : There is association between them *i.e.*, they are dependent.

(b) After the setting of null and alternative hypothesis select a random sample and find the observed frequencies (O) in each cell of contingency table.

(c) Find the expected frequencies (E) for each cell.

(d) Find the value of χ^2-test statistic

$$\chi^2 = \sum \frac{(O-E)^2}{E}$$

(e) Find degrees of freedom

$$\text{d.f.} = (p-1)\,(q-1).$$

(f) Decision Rule :

(i) Accept the null hypothesis at α% level of significance if the calculated value of χ^2 is less than the tabulated value of χ^2 for $(p-1)(q-1)$ d.f. and α% level of significance.

(ii) Otherwise reject the null hypothesis.

SOME ILLUSTRATIVE EXAMPLES

Example 1. ***A certain drug is claimed to be effective in curing colds. In an experiment on 500 persons with cold, half of them were given the drug and half of them were given sugar pills. The patients reactions to the treatment are recorded in the following table :***

Treatment	Helped	Reaction	No effect	Total
Drug	150	30	70	250
Sugar pills	130	40	80	250
Total	280	70	150	500

On the basis of data, can it be concluded that there is a significance difference in the effect of the drug and sugar pills ?

Solution. We set up the null hypothesis that there is no difference between the effects of drug and sugar pills, *i.e.*,

H_0 : There is no difference between the effects of drug and sugar pills

Against H_1 : There is significant difference between the effects of drug and pills.

Now we find the expected frequencies with the help of given contingency table such as :

$$E_{11} = \frac{(A_1)\,(B_1)}{N} = \frac{250 \times 280}{500} = 140$$

$$E_{12} = \frac{(A_1)\,(B_2)}{N} = \frac{250 \times 70}{500} = 35$$

$$E_{13} = \frac{(A_1)\,(B_3)}{N} = \frac{250 \times 150}{500} = 75$$

$$E_{21} = \frac{(A_2)\,(B_1)}{N} = \frac{250 \times 280}{500} = 140$$

$$E_{22} = \frac{(A_2)\,(B_2)}{N} = \frac{250 \times 70}{500} = 35$$

$$E_{23} = \frac{(A_2)\,(B_3)}{N} = \frac{250 \times 150}{500} = 75$$

Since the order of contingency table is 2×3. Therefore the d.f. $= (2-1) \times (3-1) = 2$. So we need only to calculate two expected frequencies and others come out automatically.

Thus the contingency table of expected frequencies is given by

Treatment	Helped	Reaction	No effect	Total
Drug	140	35	75	250
Sugar pills	140	35	75	250
Total	280	70	150	500

Now the χ^2-test statistic is given by

$$\chi^2 = \sum \frac{(O-E)^2}{E}.$$

O	E	(O – E)	$(O-E)^2$	$(O-E)^2/E$
150	140	10	100	0·714
130	140	– 10	100	0·714
30	35	– 5	25	0·714
40	35	5	25	0·714
70	75	– 5	25	0·333
80	75	+ 5	25	0·333
Total				3·522

Thus $\chi^2 = 3\cdot 522$

d.f. = 2.

From table $\chi_2^2\ (0\cdot 05) = 5\cdot 99$. Since calculated value of χ^2 statistic at 5% level of significance for 2 d.f. is less than the tabulated value of χ^2, so we accept our null hypothesis and concluded that there is no significance difference between the effects of the drug and sugar pills.

Example 2. ***1600 families were selected at random in a city to test the belief that high income families usually send their children to public schools and low income families often send their children to government schools. The following results were obtained in the study conducted :***

	Income	Public School	Government School	Total
		(A)		
	Low	494	506	1000 (B_1)
(B)	*High*	162	438	600 (B_2)
	Total	656 (A_1)	944 (A_2)	1600

Examine by χ^2-test to ascertain if the income of the families and type of schools are independent.

Solution. We set the null hypothesis that there is no relation or association between income of families and types of schools, *i.e.*, income of families and type of schools are independent. Thus

H_0 : Income of families and type of schools are independent.

Against H_1 : Income of families and type of schools are dependent.

Now we find the expected frequencies with the help of given contingency table of observed frequencies

$$E_{11} = \frac{(A_1)(B_1)}{N} = \frac{656 \times 1000}{1600} = 410$$

$$E_{12} = \frac{(A_1)(B_2)}{N} = \frac{656 \times 600}{1600} = 246$$

$$E_{21} = \frac{(A_2)(B_1)}{N} = \frac{944 \times 1000}{1600} = 590$$

$$E_{22} = \frac{(A_2)(B_2)}{N} = \frac{944 \times 600}{1600} = 354.$$

Thus the contingency table for expected frequencies given by

Income	Public School	Government School	Total
Low	410	590	1000
High	246	354	600
Total	656	944	1600

Now the χ^2-test statistic is given by

$$\chi^2 = \sum \frac{(O-E)^2}{E}.$$

O	E	(O – E)	$(O-E)^2$	$(O-E)^2/E$
494	410	84	7056	17·2
162	246	– 84	7056	26·68
506	590	– 84	7056	11·96
438	354	84	7056	19·93
Total				75·77

$\chi^2 = 75\cdot 77$

d.f. = (2 – 1) (2 – 1) = 1.

From table $\chi_1^2\ (0\cdot 05) = 3\cdot 84$. Since calculated value of χ^2 statistic at 5% level of significance for 1 d.f. is greater than the tabulated value of χ^2. So we reject our null hypothesis and conclude that the income of families and the type of schools are dependent.

Example 3. ***From the following data, find out whether there is any relationship between sex and preference in colour.***

(A)

	Colour	Males	Females	Total
(B)	Red	10	40	50 (B_1)
	White	70	30	100 (B_2)
	Green	30	20	50 (B_3)
	Total	110 (A_1)	90 (A_2)	200

Solution. First of all we set up the null hypothesis that there is no relationship between sex and preference in colour. Thus

H_0: There is no relationship between sex and preference in colour.

Against H_1: There is relationship between sex and preference in colour.

Now we find the expected frequencies with the help of given contingency table of observed frequencies :

$$E_{11} = \frac{(A_1)(B_1)}{N} = \frac{110 \times 50}{200} = 27 \cdot 5$$

$$E_{12} = \frac{(A_1)(B_2)}{N} = \frac{110 \times 100}{200} = 55$$

$$E_{13} = \frac{(A_1)(B_3)}{N} = \frac{110 \times 50}{200} = 27 \cdot 5$$

$$E_{21} = \frac{(A_2)(B_1)}{N} = \frac{90 \times 50}{200} = 22 \cdot 5$$

$$E_{22} = \frac{(A_2)(B_2)}{N} = \frac{90 \times 100}{200} = 45$$

$$E_{23} = \frac{(A_2)(B_3)}{N} = \frac{90 \times 50}{200} = 22 \cdot 5.$$

Thus the contingency table for expected frequencies given by

Colour	Males	Females	Total
Red	27·5	22·5	50
White	55	45	100
Green	27·5	22·5	50
Total	110	90	200

Now the χ^2-test statistic is given by

$$\chi^2 = \sum \frac{(O-E)^2}{E}.$$

O	E	(O – E)	$(O-E)^2$	$(O-E)^2/E$
10	27·5	– 17·5	306·25	11·14
70	55	15	225	4·09
30	27·5	2·5	6·25	0·23
40	22·5	17·5	306·25	13·61
30	45	– 15	225	5
20	22·5	– 2·5	6·25	0·28
Total				34·35

Thus $\chi^2_{cal} = 34 \cdot 35$

d.f. $= (3-1) \times (2-1) = 2.$

From table $\chi^2_2\,(0 \cdot 05) = 5 \cdot 99$. Since calculated value of χ^2 statistic at 5% level of significance for 2 d.f. is greater than the tabulated value of χ^2. So we reject our null hypothesis and conclude that there is relation between sex (gender) and colour preference.

Example 4. ***A sample analysis of examination results of 200 MBA's was made. It was found that 46 students had failed, 68 second a third division, 62 secured a second division and the rest were placed in first division. Are these figures commensurate with the general examination result which is in the ratio of 4 : 3 : 2 : 1 for various categories respectively ?***

Solution. Set up the null hypothesis that the observed figures do not differ significantly from the hypothetical frequency which are in the ratio of 4 : 3 : 2 : 1.

Under the null hypothesis, the expected frequencies can be computed as shown in the following tables :

Category	O	E	(O – E)	$(O-E)^2$	$(O-E)^2/E$
Failed	46	$\frac{4}{10} \times 200 = 80$	– 34	1156	14·450
III Division	68	$\frac{3}{10} \times 200 = 60$	8	64	1·067
II Division	62	$\frac{2}{10} \times 200 = 40$	22	484	12·100
I Division	24	$\frac{1}{10} \times 200 = 20$	4	16	0·800
Total	200	200			28·417

$$\chi^2 = \sum_{i=1}^{4} \frac{(O_i - E_i)^2}{E_i} = 28\cdot 417$$

$$\text{d.f.} = 4 - 1 = 3$$

$$\chi^2_{0\cdot 05}\ (3) = 7\cdot 815.$$

Since $\chi^2_{cal} = 28\cdot 417$ is greater than tabulated value $\chi^2_{0\cdot 05}\ (3) = 7\cdot 815$. So we reject H_0. Hence data are not commensurate with the general examine result.

Example 5. ***Out of 8000 graduates in a town, 800 are females, and out of 1600 graduate employees in the town, 120 are females. Determine if any distinction is made in appointment on the basis of sex. [Note value of χ^2 at 5% level of significance for 1 degree of freedom is 3·84]***

Solution. Here

	Graduate employee	Graduate non-employee	Total
Female	120	800	920 B_1
Male	1480	7200	8680 B_2
Total	1600 A_1	8000 A_2	9600

H_0 : There is no distinction is made in appointment on the basis of sex.

H_1 : There is distinction is made in appointment on the basis of sex.

Now, we find the expected frequencies with the help of give contingency table of observed frequencies

$$E_{11} = \frac{A_1 \times B_1}{N} = \frac{1600 \times 920}{9600} = 153\cdot 33$$

$$E_{12} = \frac{A_1 \times B_1}{N} = \frac{1600 \times 8680}{9600} = 1446\cdot 67$$

$$E_{21} = \frac{A_2 \times B_1}{N} = \frac{8000 \times 920}{9600} = 766\cdot 67$$

$$E_{22} = \frac{A_2 \times B_2}{N} = \frac{8000 \times 8680}{9600} = 7233\cdot 33$$

Now the contigency table is :

	Graduate employee	Graduate non-employee	Total
Female	153·33	766·67	920
Male	1446·67	7233·33	8680
Total	1600	8000	9600

O	E	O − E	$(O-E)^2$	$(O-E)^2/E$
120	153·33	− 33·33	1110·889	7·25
7480	1446·67	33·33	1110·889	0·768
800	766·67	33·33	1110·889	1·45
7200	7233·33	− 33·33	1110·889	0·153
Total				9·621

$$\chi^2 = \sum_{i=1}^{n} \frac{(O_i - E_i)^2}{E_i}$$

$$\chi^2 = 9\cdot 621$$

$$\text{d.f.} = (2-1)(2-1) = 1.$$

From table $\chi^2_1\ (0\cdot 05) = 3\cdot 84$. Since calculated value of χ^2 statistic at 5% level of significance for 1 degree freedom is greater than the tabulated value of χ^2. So we reject our null hypothesis and conclude the distinction is made in appoint on the basis of sex.

EXERCISE 10 (I)

1. Define Chi-square.
2. Write a short note on degree of freedom.
3. Five coins are tossed 3200 times and the following results are obtained :

No. of heads	:	0	1	2	3	4	5
Frequency	:	80	570	1100	900	500	50

If χ^2 for 5 d.f. at level of significance be 11.070, test the hypothesis that the come are unbiased.

4. The following table gives the number of aircraft acceleration that occurred during the various days of the week. Find whether the accidents are uniformly distributed over the week :.

Days	:	Mon.	Tue.	Wed.	Thu.	Fri.	Sat.	Total
No. of accidents	:	14	16	12	19	9	14	84

[Given that χ^2 for 6 degrees of freedom at 3% level of significance = 12.592]

5. The following table occurs in member of Kurl Pearson

		Eye Colon in sons	
		Not light	Light
Eye colon in Fathers	Not light	230	148
	Light	151	471

Test whether the colour of the son's eye is associated with that of the father's ($\chi^2 = 3\cdot 84, \nu = 1$)

6. In an experiment on immunization of cattle from tuberculosis the following results were obtained :

	Affected	Uneffected
Innoculated	12	26
Not innoculated	16	6

Estimate the effect of vaccine in concluding susceptibility to tuberculosis.

7. The following table gives the result of series of controlled exponent. Discuss whether the treating may be considered to have any positive effect :

	Positive	No effect	Negative	Total
Treatment	9	2	1	12
Control	3	6	3	12
Total	12	8	4	24

The value of χ^2 for 2 degrees of freedom agt 5% level of significance is 5.99.

8. Write a short note on χ^2-distribution.
9. Research takes of the number of male and female births is 800 families having four children at follows :

No. of male births	No. of female births	No. of families
0	4	32
1	3	178
2	2	290
3	1	236
4	0	64

Test whether the data are consistent with the hypothesis that the binomial law holds and that the chance of a male birth to that of a female birth, that is $q = p = \frac{1}{2}$. You may use the table given below :

Degrees of freedom	:	1	2	3	4	5
5% value of χ^2	:	3·84	5·99	7·82	9·49	11·07

10. What is the purpose of chi-square distribution ?

ANSWERS

3. $\chi^2 = 58, v = 5$, coins are biased.
4. Accidents to be uniformly distributed over all days of week.
5. Appears to be associated.
6. Vaccine in effective.