

# Math 342 Tutorial

June 18, 2025

Recall an algebraic monoid is a set  $G$  together with a binary operation  $G \times G \rightarrow G$  satisfying the following:

**M1** Associativity:  $a(bc) = (ab)c$  for all  $a, b, c \in G$ .

**M2** Identity: There is an element  $e \in G$  for which  $ea = ae = a$  for all  $a \in G$ .

An example of a monoid is the set of all  $n \times n$  matrices with entries over  $\mathbf{C}$ .

An algebraic group is a monoid  $G$  satisfying the additional axiom:

**G1** Inverse: For every  $a \in G$ , there is a  $b \in G$  for which  $ab = ba = e$ .

An example of a group is the set of all  $n \times n$  matrices over  $\mathbf{C}$  which are invertible.

The reader is also reminded of the following definition used previously. If  $f, g$  are two arithmetic functions, then their Dirichlet product  $f * g$  is defined as

$$(f * g)(n) = \sum_{d|n} f(d)g\left(\frac{n}{d}\right).$$

**Question 1.** For this question, you will need to use results we have proven in previous tutorials. Do the following. **(a)** Prove the set  $\mathcal{F}$  of all arithmetic functions is a monoid. What is the identity element of  $\mathcal{F}$ ? **(b)** What is the largest subset  $\mathcal{G}$  of  $\mathcal{F}$  such that  $\mathcal{G}$  is a group? **(c)** Name a second subset  $\mathcal{M} \neq \mathcal{G}$  of  $\mathcal{F}$  for which  $\mathcal{M}$  is also a group.

**Question 2.** Do the following. **(a)** Prove the Möbius inversion formula: Given  $f(n) = \sum_{d|n} g(d)$ , one has  $g(n) = \sum_{d|n} f(d)\mu\left(\frac{n}{d}\right)$ . **(b)** We previously established that  $\phi(n) = \sum_{d|n} d\mu\left(\frac{n}{d}\right)$ . Use part **(a)** to show that  $n = \sum_{d|n} \phi(d)$ .

**Question 3.** The Mangolt function  $\Lambda(n)$  is defined as

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^a \text{ for some prime } p, \\ 0 & \text{otherwise.} \end{cases}$$

Show the following. **(a)** If  $n \geq 1$ , then  $\log n = \sum_{d|n} \Lambda(d)$ . **(b)** Use part **(a)** to show that  $\Lambda(n) = \sum_{d|n} \mu(d) \log \frac{n}{d} = - \sum_{d|n} \mu(d) \log d$ .

An arithmetic function  $f$  is completely multiplicative if  $f(mn) = f(m)f(n)$  for all  $m, n \in \mathbf{Z}$  (we drop the condition that  $(m, n) = 1$ ).

**Question 4.** Let  $f$  be an arithmetic function. Show the following. **(a)** Let  $f$  be multiplicative. Then  $f$  is completely multiplicative if and only if  $f^{-1} = \mu f$ , where  $f^{-1}$  is the Dirichlet inverse of  $f$ . **(b)** If  $f$  is multiplicative, then  $\sum_{d|n} \mu(d)f(d) = \prod_{p|n} (1 - f(p))$ .

**Question 5.** If  $n = p_1^{e_1} \cdots p_k^{e_k}$ , then Liouville's function  $\lambda$  is defined as

$$\lambda(n) = (-1)^{e_1 + \cdots + e_k}.$$

Show the following. **(a)**  $\lambda$  is completely multiplicative. **(b)**  $\lambda(n) = 1$  if  $n$  is a square and 0 otherwise. Also,  $\lambda^{-1} = |\mu|$ .

**Question 6.** Show the following identities.

**(a)**

$$\frac{n}{\phi(n)} = \sum_{d|n} \frac{\mu^2(d)}{\phi(d)}.$$

[Hint: Show first that  $\frac{\mu^2(n)}{\phi(n)}$  is multiplicative.]

**(b)** Show for each  $k \geq 1$  that

$$\sum_{\substack{d|n \\ d^k|n}} \mu(d) = \begin{cases} 0 & \text{if } m^k \mid n \text{ for some } m > 1, \\ 1 & \text{otherwise.} \end{cases}$$