Math 342 Tutorial
June 18, 2025

Recall an algebraic monoid is a set G together with a binary operation $G \times G \to G$ satisfying the following:

M1 Associativity: $a(bc) = (ab)c$ for all $a,\, b,\, c \in G$.

M2 Identity: There is an element $e \in G$ for which $ea = ae = a$ for all $a \in G$.

An example of a monoid is the set of all $n \times n$ matrices with entries over $\mathbf{C}$.
An algebraic group is a monoid G satisfying the additional axiom:

G1 Inverse: For every $a \in G$, there is a $b \in G$ for which $ab = ba = e$.

An example of a group is the set of all $n \times n$ matrices over $\mathbf{C}$ which are invertible.
The reader is also reminded of the following definition used previously. If $f,\, g$ are two arithmetic functions, then their Dirchlet product $f * g$ is defined as

$$(f * g)(n) = \sum_{d|n} f(n) g\left(\frac{n}{d}\right).$$

Question 1. For this question, you will need to use results we have proven in previous tutorials. Do the following. **(a)** Prove the set $\mathscr{F}$ of all arithmetic functions is a monoid. What is the identity element of $\mathscr{F}$? **(b)** What is the largest subset $\mathscr{G}$ of $\mathscr{F}$ such that $\mathscr{G}$ is a group? **(c)** Name a second subset $\mathscr{M} \neq \mathscr{G}$ of $\mathscr{F}$ for which $\mathscr{M}$ is also a group.

Question 2. Do the following. **(a)** Prove the Möbius inversion formula: Given $f(n) = \sum_{d|n} g(d)$, one has $g(n) = \sum_{d|n} f(d)\mu(\frac{n}{d})$. **(b)** We previously established that $\phi(n) = \sum_{d|n} d\mu(\frac{n}{d})$. Use part **(a)** to show that $n = \sum_{d|n} \phi(d)$.

Question 3. The Mangolt function $\Lambda(n)$ is defined as

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^a \text{ for some prime } p, \\ 0 & \text{otherwise.} \end{cases}$$

Show the following. **(a)** If $n \geqq 1$, then $\log n = \sum_{d|n} \Lambda(d)$. **(b)** Use part **(a)** to show that $\Lambda(n) = \sum_{d|n} \mu(d) \log \frac{n}{d} = -\sum_{d|n} \mu(d) \log d$.

An arithmetic function f is completely multiplicative if $f(mn) = f(m)f(n)$ for all $m,\, n \in \mathbf{Z}$ (we drop the condition that $(m,\, n) = 1$).

Question 4. Let f be an arithmetic function. Show the following. **(a)** Let f be multiplicative. Then f is completely multiplicative if and only if $f^{-1} = \mu f$, where f^{-1} is the Dirichlet inverse of f. **(b)** If f is multiplicative, then $\sum_{d|n} \mu(d) f(d) = \prod_{p|n}(1 - f(p))$.

Question 5. If $n = p_1^{e_1} \cdots p_k^{e_k}$, then Liouville's function λ is defined as

$$\lambda(n) = (-1)^{e_1 + \cdots + e_k}.$$

Show the following. **(a)** λ is completely multiplicative. **(b)** $\sum_{d|n} \lambda(d) = 1$ if n is a square and 0 otherwise. Also, $\lambda^{-1} = |\mu|$.

Question 6. Show the following identities.

(a)

$$\frac{n}{\phi(n)} = \sum_{d|n} \frac{\mu^2(d)}{\phi(d)}.$$

[Hint: Show first that $\frac{\mu(n)}{\phi(n)}$ is multiplicative. Then use question 4.]

(b) Show for each $k \geqq 1$ that

$$\sum_{\substack{d|n \\ d^k|n}} \mu(d) = \begin{cases} 0 & \text{if } m^k \mid n \text{ for some } m > 1, \\ 1 & \text{otherwise.} \end{cases}$$