0.1 Limiting Amplitude and Limiting Absorption Principle

In this section we perform several numerical experiments, with the normalization chosen as $\epsilon_0 = \mu_0 = 1$, $\omega = c = 1$. Also, we set $m_e = 1$ and $e = -1$. From this it follows that $w_c = -B_0$ and $w_p^2 = N_e$. We consider the following two cases:

- case $N_e \neq$ const, no resonance
- case $N_e \neq$ const, resonance

For every fixed absorption rate ν, in the time domain we choose the boundary conditions of the form

$$\begin{aligned} \partial_t H|_{x=-L} &= -\left.\partial_x E_y\right|_{x=-L} = G\sin(t),\ G \in \mathbb{R}, \\ \partial_t H|_{x=H} &= 0, \end{aligned} \tag{1}$$

and zero initial conditions, and in the frequency domain

$$\begin{aligned} \left.\partial_x \hat{E}_y\right|_{x=-L} &= G, \\ \left.\partial_x \hat{E}_y\right|_{x=H} &= 0. \end{aligned}$$

We compute the solution $\mathbf{E}^\nu(t)$ for large t in the time domain (the solution computed numerically at the time step n is denoted by $\mathbf{E}^\nu_n$), and the solution $\hat{\mathbf{E}}^\nu$ in the frequency domain. Our goal is to check whether

$$\lim_{t\to+\infty} \mathbf{E}^\nu(t) = \Im\left(\hat{\mathbf{E}}^\nu \exp(it)\right).$$

0.1.1 No-Resonance Case

We choose the parameters so that in the frequency domain, for the limiting amplitude problem, $\hat{E}_2$ satisfies the Airy equation (**??**).

We set $\omega_c = 0$ (thus $\delta(x) = 0$), $\omega = 1$ (hence $\alpha(x) = 1 - N_e(x)$), choose the domain as $[-0.5, 10]$ and set the electron density $N_e(x) = 1 + x$. Importantly, $N_e(x) > 0$ on the whole interval. The boundary conditions in (1) are chosen as $G = Ai'(0.5)$. In all the experiments in this section the CFL number was chosen to be equal to 0.5.

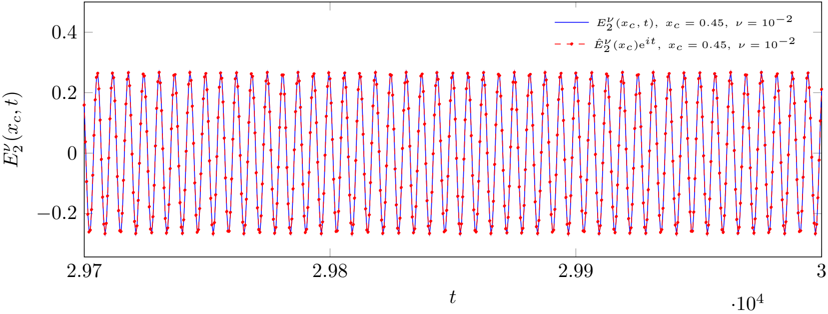
First let us fix $\nu = 1e-2$. To demonstrate that the limiting amplitude principle indeed holds, we fix a point $x = x_c$ inside the domain $(-L,\ H)$ and plot the dependence of the solution $E_2^\nu(x_c, t)$, $\hat{E}_2^\nu(x_c)\mathrm{e}^{it}$ on time t for a range of $t \gg 1$ in Figure **??**.

In Figure 1 (left figure) we compare this solution to the computed $\hat{E}_2^\nu \mathrm{e}^{it}$, for fixed values of t. Both solutions appear to be in close agreement. The computed L_2 error

$$\mathcal{E}(t) = \|\Im\left(\hat{E}_y^\nu \exp(it)\right) - E_y^\nu(t)\|_{L_2(-L;H)}. \tag{2}$$

for $\nu = 1e-2$ did not exceed $1.1e-3$ for values of $t \in (28501, 30000)$.

Figure show the solutions at fixed time steps for $\nu = 1e-4$. As before, to demonstrate that the limiting amplitude principle indeed holds, we fix a point



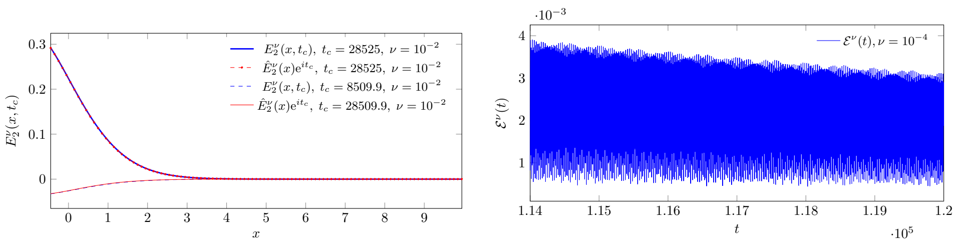


Figure 1: In the left figure we show the solution to (**??**) for $\nu = 10^{-2}$, for two fixed moment of times. In the right figure the dependence of the error (2) on time for $\nu = 10^{-4}$ is demonstrated. We can see that the error is decreasing.

$x = x_c$ inside the domain $(-L,\ H)$ and plot the dependence of the solution $E_2^\nu(x_c,t)$ on time t for a range of $t \gg 1$. The error (2) for $\nu = 1e-4$ at the time interval $[228000.05,\ 240000.05]$ does not exceed 2.8e-4. One of our observations was that for smaller ν one requires more time steps to achieve the limiting amplitude solution, c.f. e.g. Figure 1. We were not able to obtain the limiting amplitude solution for $t < 3 \cdot 10^4$, unlike in the case of $\nu = 10^{-2}$. For example, for $\nu = 1e-6$ we were not able to reach the limiting amplitude solution even on the time interval $t \leq 1.92e6$, see Figure 2.

0.1.2 Resonance Case

For the resonance case, we choose the parameters as given in Table **??**. Since

Parameter	Value
L	5
H	19
ω_c	$\sqrt{0.5}$
$N_e(x)$	$\begin{cases} 0.25, & x < -0.5, \\ \frac{1+x}{2}, & x \geq -0.5, x \leq 9 \\ 5, & x > 9. \end{cases}$
G as in (1)	0.11

Table 1: Parameters for numerical simulations in Section 0.1.2

$\alpha(x) = (1 - 2N_e(x))$, $\alpha(0) = 0$. Clearly, $\delta(0) \neq 0$ (resonance case).

The results for $\nu = 1e-2$ (chosen as a parameter both in frequency and

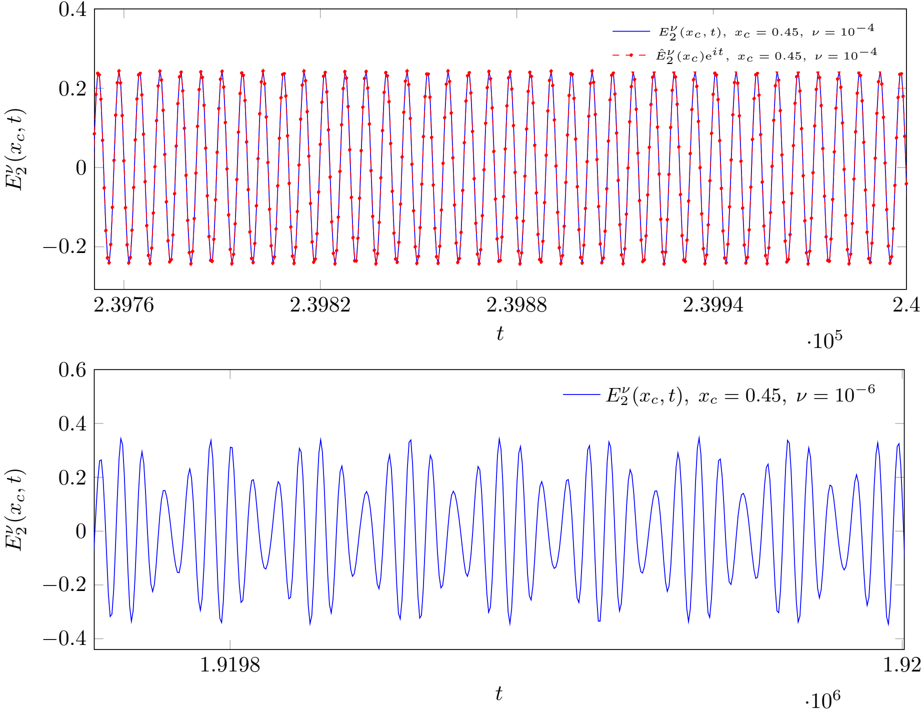


Figure 2: In the upper figure we plot the dependence of the solution $E_2^\nu(x_c,t)$ on time t, with $\nu = 10^{-4}$ and $x_c = 0.45$. In the lower figure we show the solution for $\nu = 10^{-6}$ at the same point x_c, for larger times. As we can see, for $\nu = 10^{-4}$ the limiting amplitude solution was reached for large t. For $\nu = 10^{-6}$ we were not able to obtain the limiting amplitude solution even for $t \approx 1.9e6$.

time domain) are shown in Figure **??**.