Thermal Transformation of Optical Elements

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Thermal expansion calculations:

I assume the optical axis is $\hat{z}$, for transverse axis I choose y.

Quadratic approximation for the surface of a sphere:

I will calculate everything for every facet of the lens independently.
Let us start by calculating the relation between the optical length $\xi\left(y,z\right),[\mathrm{m}]$ and R,n of one facet of the lens:
The facet of the lens is defined be:

$$z^2+y^2=R^2 \quad \backslash -y^2$$

$$z^2=R^2-y^2 \quad \backslash\sqrt{}$$

$$z=\sqrt{R^2-y^2} \quad \backslash\text{Simplify}$$

$$z=R\sqrt{1-\frac{y^2}{R^2}}$$

To leading order this equals:

$$z\left(y\right)\approx R\left(1-\frac{y^2}{2R^2}\right) \tag{1}$$

For a lens:

Multiplying (1) by n we get (all is up to an additive constant) the optical length ξ to be:

$$\xi=n\cdot z\left(y\right)=nR\left(1-\frac{y^2}{2R^2}\right)$$

And also it is useful to write:

$$\frac{d^2\xi}{dy^2}=\frac{d^2}{dy^2}\left[nR\left(1-\frac{y^2}{2R^2}\right)\right]$$

$$\frac{d^2\xi}{dy^2}=-\frac{n\cancel{R}}{R^{\cancel{2}}}=-\frac{n}{R}$$

$$\left[\frac{1}{\mathrm{m}}=\frac{1}{\mathrm{m}}\right]\checkmark$$

An important note is that $\frac{d^2\xi}{dy^2} =\propto n, R^{-1}$, and so if we later find that $\frac{d^2\xi}{dy^2}$ should be increased by a we can either transform $n \to n - aR$ (such that the new $\frac{d^2\xi}{dy^2}$ is

$$\frac{d^2\xi_{\text{new}}}{dy^2} = -\frac{n - aR}{R} = -\frac{n}{R} + a \tag{2}$$

) or transform $R \to R' = \frac{nR}{n-aR}$ (so that the new

$$\frac{d^2\xi_{\text{new}}}{dy^2} = -\frac{n}{R'} = -\frac{\cancel{n}}{\frac{\cancel{n}R}{n-aR}} = -\frac{n - aR}{R} = -\frac{n}{R} + a\checkmark \tag{3}$$

)
more generally, the total optical length is (up to a constant):

$$\xi(y) = \int\limits_0^{z(y)} n(y,z)\, dz$$

Where $z(y)$ is like before, the location of the end of the lens.
And so ξ can be modified by changing $n(y,z)$ or by changing $z(y)$.
As said before, this is the change due to the thermal effects only in one side of the lens:

The change ξ due to the change in n: This size can be expressed as:

$$\Delta\xi_n(y) = \int\limits_0^{z(y)} \Delta n(y,z)\, dz = \int\limits_0^{z(y)} \frac{\delta n(y,z)}{\delta T} \cdot \delta T(y,z)\, dz$$

For an approximately constant $\frac{\delta n}{\delta T}$ we can write:

$$\Delta\xi_n = \frac{\delta n}{\delta T} \int\limits_0^{z(y)} \delta T(y,z)\, dz \tag{4}$$

Assuming:

$$\int\limits_0^{z(y)} \delta T(y,z)\, dz \propto (T_{\max} - T_0) \cdot w \cdot F(y, \sigma = w) \tag{5}$$

For some dimensionless symmetrical function F with typical width of w, mean of 0, and maximum of 1 (like $e^{-\frac{y^2}{w^2}}$). The typical width of w is there because the wider the beam is, the wider the heating profile, and the typical height of w is because the wider it is, the deeper the heat will go into the lens before dissipating.
Moreover, $(T_{\max} - T_0)$ should go like

$$(T_{\max} - T_0) \propto \frac{\beta P_{\text{Laser}}}{\kappa w_{\text{on lens}}} \Rightarrow$$

$$\left(T_{\max}-T_0\right)=C\cdot\frac{\beta P_{\text{Laser}}}{\kappa w_{\text{on lens}}}$$

$$\left[\mathrm{K}=1\cdot\frac{1\cdot\cancel{\mathrm{J}}/\cancel{\mathrm{s}}}{\frac{\cancel{\mathrm{J}}/\cancel{\mathrm{s}}\cdot\cancel{\mathrm{m}}}{\cancel{\mathrm{m}^2}\cdot\mathrm{K}}\cdot\cancel{\mathrm{m}}}=\mathrm{K}\right]\checkmark$$

Where C is some dimensionless constant, β is the absorption coefficient on the face of the lens $[\beta]=1$, κ is the heat conductivity $[\kappa]=\frac{\mathrm{J}/\mathrm{s}}{\mathrm{m}\cdot\mathrm{K}}$, $w_{\text{on lens}}$ is the width of the laser $[w_{\text{on lens}}]=\mathrm{m}$ and P_{laser} is the laser power $[P_{\text{laser}}]=\frac{\mathrm{J}}{\mathrm{s}}$.
Plugging it back to (5):

$$\int_0^\infty \delta T\left(y,z\right)dz\approx C\cdot\frac{\beta P_{\text{Laser}}}{\kappa\cancel{w_{\text{on lens}}}}\cdot\cancel{w_{\text{on lens}}}\cdot F\left(y\right) \tag{5}$$

$$\boxed{\int_0^\infty \delta T\left(y,z\right)dz\approx C\cdot\frac{\beta P_{\text{Laser}}}{\kappa}F\left(y\right)} \tag{6}$$

$$\left[\mathrm{K}\cdot\mathrm{m}=\frac{1\cdot\cancel{\mathrm{W}}}{\cancel{\mathrm{W}}/\mathrm{m}\cdot\mathrm{K}}=\mathrm{m}\cdot\mathrm{K}\right]\checkmark$$

Plugging back to (4) we get:

$$\Delta\xi_n=C\cdot\frac{\delta n}{\delta T}\frac{\beta P_{\text{Laser}}}{\kappa}F\left(y\right)$$

$$\frac{d^2\xi_n}{dy^2}\approx -C\cdot\frac{\delta n}{\delta T}\frac{\beta P_{\text{Laser}}}{\kappa w^2}$$

Where we assumed $F\left(y\right)$ adds to the curvature a term which goes like w^2

The change in ξ due to the change in refractive index due to volumetric heating Treating only half of the lens, the optical path length will be the:

$$\Delta\xi_v=\frac{\delta n}{\delta T}\int_0^{z(y)}\delta T_{\text{volumetric}}\left(y,z\right)dz$$

$$\approx\frac{\delta n}{\delta T}\left(T_{\max}-T_0\right)\cdot z\left(y\right)\cdot F\left(y\right)$$

$$\approx\frac{\delta n}{\delta T}C_2\cdot\frac{\alpha_{\text{vol}}P_{\text{Laser}}}{\kappa}\cdot z\left(y\right)F\left(y\right)$$

Units check for the green temperature expression:

$$\left(T_{\max}-T_0\right)=C_2\cdot\frac{\alpha_{\text{vol}}P_{\text{Laser}}}{\kappa}\Rightarrow\left[\mathrm{K}=1\cdot\frac{\frac{1}{\mathrm{m}}\cdot\cancel{\mathrm{J}}/\cancel{\mathrm{s}}}{\frac{\cancel{\mathrm{J}}/\cancel{\mathrm{s}}\cdot\cancel{\mathrm{m}}}{\cancel{\mathrm{m}^2}\cdot\mathrm{K}}}\cdot\cancel{\mathrm{m}}=\mathrm{K}\right]\checkmark$$

The curvature at $y=0$ (set $y=0$ at the optical axis of the beam) will be:

$$\frac{d^2}{dy^2}\left(z\left(y\right)F\left(y\right)\right)_{y=0} = z''F + 2z'F' + zF''|_{y=0} \approx -\frac{1}{R} + \cancel{2\cdot 0\cdot 0} - \frac{z\left(0\right)}{w^2} =$$

Let us see if any of those two terms is negligible in the existing system:

$$R \approx 10^{-2}$$

$$z\left(0\right) \approx 5\cdot 10^{-4}$$

$$w^2 \approx \left(2\cdot 10^{-3}\right)^2 = 4\cdot 10^{-6}$$

$$\frac{1}{R} \approx 10^2$$

$$\frac{z\left(0\right)}{w^2} \approx \frac{10^{-4}}{10^{-6}} = 10^2$$

And so $\frac{1}{R}$ and $\frac{z(0)}{w^2}$ are comparable, **and we can not neglect any of them:**
And so in total we have:

$$\frac{d^2\Delta\xi_V}{dy^2} = -C_2\frac{\delta n}{\delta T}\frac{\alpha_{\text{vol}}P_{\text{Laser}}}{\kappa}\left(\frac{1}{R} + \frac{z\left(0\right)}{w^2}\right)$$

The change in ξ due to the change in the thickness of the lens We start by writing an expression for how much the thickness $z\left(y\right)$ changes due to the heating:

$$\Delta z\left(y\right) = \alpha\cdot\frac{1+\nu}{1-\nu}\cdot\int_0^{z(y)}\delta T\left(y,z\right)dz \tag{7}$$

$$\overset{(6)}{=} \alpha\cdot\frac{1+\nu}{1-\nu}\cdot C\cdot\frac{\beta P_{\text{Laser}}}{\kappa}F\left(y\right)$$

$$\left[\text{m} = \frac{\cancel{1}}{\cancel{K}}\cdot 1\cdot\cancel{K}\cdot\text{m} = \text{m}\right]\checkmark$$

$$\Delta\xi_R = n\cdot\Delta z\left(y\right) \quad \backslash(7)$$

$$\boxed{\Delta\xi_R = C\cdot\frac{1+\nu}{1-\nu}\cdot\frac{n\alpha\beta P_{\text{Laser}}}{\kappa}F\left(y\right)} \tag{8}$$

And so

$$\frac{d^2\Delta\xi}{dy^2} = \frac{d^2\Delta\xi_n}{dy^2} + \frac{d^2\Delta\xi_R}{dy^2} + \frac{d^2\Delta\xi_V}{dy^2}$$

$$\approx -\frac{P_{\text{Laser}}}{\kappa}\left(\underbrace{\frac{1}{w^2}C\beta\frac{\delta n}{\delta T}}_{\text{change in refractive index term}} + \underbrace{C_2\alpha_{\text{vol}}\frac{\delta n}{\delta T}\left(\cancel{\frac{1}{R}} + \frac{z(0)}{w^2}\right)}_{\substack{\text{change in refractive index}\\ \text{from volumetric absorbtion}}} + \underbrace{\frac{1}{w^2}Cn\cdot\alpha\cdot\beta\cdot\frac{1+\nu}{1-\nu}}_{\text{change in thickness term}}\right)$$

$$= -C\cdot\frac{\beta P_{\text{Laser}}}{\kappa w_{\text{on lens}}^2}\left(\frac{\delta n}{\delta T} + n\cdot\alpha\cdot\frac{1+\nu}{1-\nu}\right)$$

$$\frac{d^2\Delta\xi}{dy^2} = \frac{d^2\Delta\xi_n}{dy^2} + \frac{d^2\Delta\xi_R}{dy^2} \approx -\frac{P_{\text{Laser}}}{\kappa}\left(\frac{1}{w^2}C\beta\frac{\delta n}{\delta T} + C_2\alpha_{\text{vol}}\frac{\delta n}{\delta T}\left(\frac{1}{R} + \frac{z(0)}{w^2}\right) + \frac{1}{w^2}Cn\cdot\alpha\cdot\beta\cdot\frac{1+\nu}{1-\nu}\right)$$

$$\left[\frac{1}{\mathrm{m}} = \frac{\mathrm{m}}{\mathrm{m}^2} = 1\cdot\frac{1\cdot\cancel{\frac{J}{s}}}{\frac{\cancel{\frac{J}{s}}\cdot\mathrm{m}}{\cancel{m^2\cdot K}}\cdot\cancel{m^2}}\cdot\cancel{\left(\frac{1}{K}+\frac{1}{K}\right)} = \frac{1}{\mathrm{m}}\right]\checkmark$$

Where $\frac{\delta n}{\delta T}, \nu, \alpha, n_0, k, P_{\text{laser}}, \kappa, w_{\text{on lens}}, \beta$ are constants of the system.
Having this, we can substitute back to (2) and (3):

$$n \to n' = n - aR = n - \left(-C\cdot\frac{\beta P_{\text{Laser}}}{\kappa w_{\text{on lens}}^2}\cdot\frac{\delta n}{\delta T}\right)\cdot R$$

$$R \to R' = \frac{nR}{n-aR} = \frac{nR}{n - \left(-C\cdot\frac{\beta P_{\text{Laser}}}{\kappa w_{\text{on lens}}^2}n\cdot\alpha\cdot\frac{1+\nu}{1-\nu}\right)R}$$

The new origin of the sphere will be:

$$z'(0) - R'$$

where $z'(0)$ is the new position of the edge of the lens, which is

$$\begin{aligned} z'(0) &\equiv z(0) + \Delta z \\ &\overset{(7)}{=} z(0) + \alpha\cdot\frac{1+\nu}{1-\nu}\cdot\int_0^{z(y)}\delta T(y,z)\,dz \\ &\overset{(6)}{=} z(0) + \alpha\cdot\frac{1+\nu}{1-\nu}\cdot C\cdot\frac{\beta P_{\text{Laser}}}{\kappa} \end{aligned} \tag{9}$$

For a mirror:

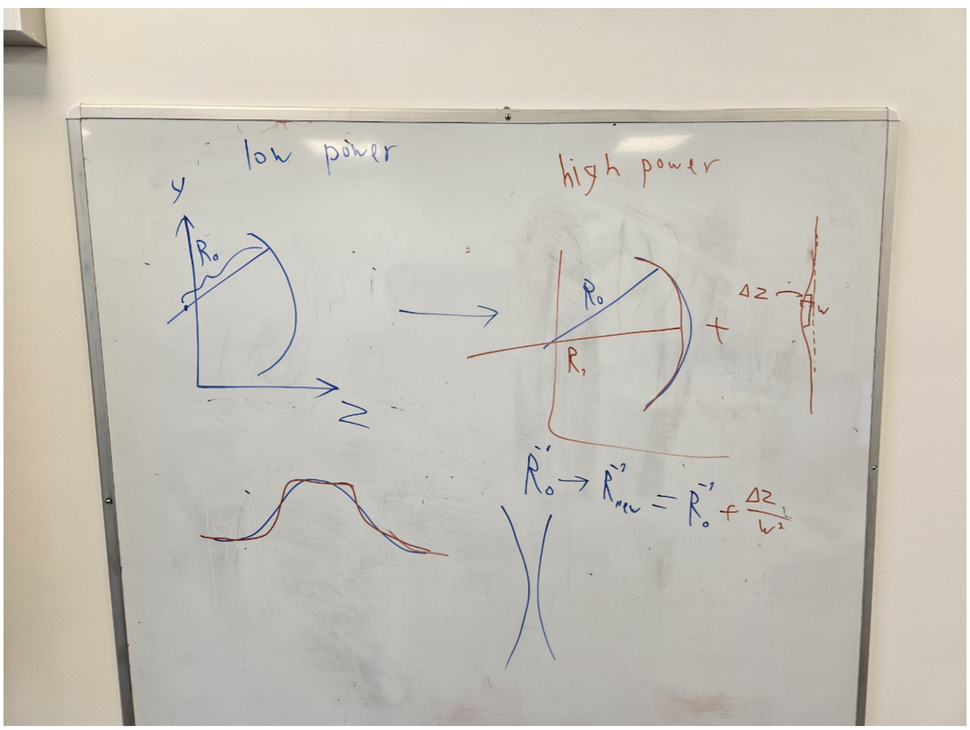
Taking the result from (9) we have:

$$\Delta z \overset{(9)}{=} \alpha \cdot \frac{1+\nu}{1-\nu} \cdot C \cdot \frac{\beta P_{\text{Laser}}}{\kappa}$$

A buldging of the form $\Delta z \cdot F\left(y, \sigma = w\right)$ with F with width of w, mean of 0 and maximum of 1like $e^{-\frac{x^2}{w^2}}$ will subtract a $\frac{\Delta z}{w^2}$ from the curvature:

$$\begin{aligned} R_{\text{new}}^{-1} &= R_0^{-1} - \frac{\Delta z}{w^2} \\ &= R_0^{-1} - \underbrace{C \cdot \frac{\alpha \beta P_{\text{Laser}}}{\kappa w^2} \cdot \frac{1+\nu}{1-\nu}}_{R_{\text{th}}^{-1}} \backslash^{-1} \end{aligned}$$

An image ilustration:



$$R_{\text{new}} = \frac{1}{R_0^{-1} + R_{\text{th}}^{-1}}$$

$$
\begin{aligned}
\Delta R &= R_{\text{new}} - R_0 \\
&= \frac{1}{R_0^{-1} + R_{\text{th}}^{-1}} - R_0 \\
&= \frac{1 - R_0\left(R_0^{-1} + R_{\text{th}}^{-1}\right)}{R_0^{-1} + R_{\text{th}}^{-1}} \\
&= \frac{\cancel{1 - R_0 R_0^{-1}} - R_0 R_{\text{th}}^{-1}}{R_0^{-1} + R_{\text{th}}^{-1}} \\
&= -\frac{R_0 R_{\text{th}}^{-1}}{R_0^{-1} + R_{\text{th}}^{-1}} \quad \backslash \cdot \frac{R_0}{R_0} \\
&= -R_0 \frac{\frac{R_0}{R_{\text{th}}}}{\cancel{1 + \frac{R_0}{R_{\text{th}}}}} \quad \backslash R_0 \ll R_{\text{th}} \\
&\approx -\frac{R_0^2}{R_{\text{th}}}
\end{aligned}
$$

Following (1), the optical path length of a mirror is twice the difference in $z\,(y)$ on the lens. denoting the path length with $\xi\,(y)$

$$
\xi\,(y) = 2 \cdot z\,(y) = 2 \cdot R\left(1 - \frac{y^2}{2R^2}\right)
$$

And the second derivative is:

$$
\frac{d^2\xi}{dy^2} = \frac{d^2}{dy^2}\left(2 \cdot R\left(\cancel{1} - \frac{y^2}{2R^2}\right)\right) = -\frac{2}{R} \tag{10}
$$

If we copy the results from the lens expansion we get:

$$
\frac{d^2\xi}{dy^2} \xrightarrow[\text{expansion}]{} \frac{d^2\xi}{dy^2} - \frac{\Delta z}{w^2}
$$

We get:

$$
\frac{d^2\xi}{dy^2} \xrightarrow[\text{expansion}]{} \frac{d^2\xi}{dy^2} - C \cdot \frac{1+\nu}{1-\nu} \cdot \frac{\alpha\beta P_{\text{Laser}}}{\kappa w^2} \overset{(10)}{=} \frac{2}{R} - C \cdot \frac{1+\nu}{1-\nu} \cdot \frac{\alpha\beta P_{\text{Laser}}}{\kappa w^2}
$$

And so the change in radius of curvature:

$$
2R'^{-1} = 2R^{-1} + C \cdot \frac{1+\nu}{1-\nu} \cdot \frac{\alpha\beta P_{\text{Laser}}}{\kappa w^2}
$$

$$
R'^{-1} = R^{-1} + \frac{C}{2} \cdot \frac{1+\nu}{1-\nu} \cdot \frac{\alpha\beta P_{\text{Laser}}}{\kappa w^2}
$$

For a mirror, we have from Jeremy's paper:

$$
R_{\text{th}}^{-1} = -N\frac{\alpha\beta P}{\kappa w^2}
$$

And

$$\left[\frac{1}{\mathrm{m}} = \frac{\cancel{\frac{1}{K}} \cdot \cancel{\frac{J}{s}}}{\frac{\cancel{\frac{J}{s}} \cdot \mathrm{m}}{\cancel{m^2} \cdot \cancel{K}} \cancel{m^2}} = \frac{1}{\mathrm{m}}\right] \checkmark$$

$$R^{-1} = R_0^{-1} + R_{\mathrm{th}}^{-1}$$