Readme File

Michael Kali - michael.kali@weizmann.ac.il

January 25, 2023

This file contains the derivations used in the `microscope.py` file for simulating a microscope with 2 frequencies cavity. the code is available here: Github link and is based mostly on the paper from here: relativistic ponderomotive potential paper.

The code structure

The simulation is meant to take an electron wave function, and propagate it through the different components of the microscope.
The wave function is always represented with the `WaveFunction` class, that contains the coordinates system on which it is evaluated, the values of it over all of the coordinates system's points, and it's energy:

```
class WaveFunction:
    def __init__(self,
                 psi: np.ndarray,  # The input wave function in one z=const plane
                 coordinates: CoordinateSystem,  # The coordinate system of the input wave
                 E0: float,  # Energy of the particle
                 ):
```

The base unit of the simulation is the `Propagator` class, which takes a `WaveFunction` as an input and returns a `WaveFunction` as an output. Examples for such Propagators are the lens, the cavity, the sample, etc...

```
class Propagator:
  def propagate(self, state: WaveFunction) -> WaveFunction
```

the main class is `Microscope`, which is defined by a list of `Propagators`, and knows how to take a `WaveFunction` as an input, pass it through all the `Propagators` one by one, save the results on the way, and after it finishes, it converts the last one to an image (by squaring and adding shot noise).
Here is a list of the current existing propagators (that, as said earlier - take a `WaveFunction` as an input and returns a `WaveFunction` as an output) followed by a detailed explanation of each one of them :

```
class LensPropagator(Propagator):
  ...
class SamplePropagator(Propagator):
  ...
class AberrationsPropagator(Propagator):
  ...
class CavityPropagator(Propagator):
  ...
class CavityAnalyticalPropagator(CavityPropagator):
  ...
class CavityNumericalPropagator(CavityPropagator):
```

`SamplePropagator`

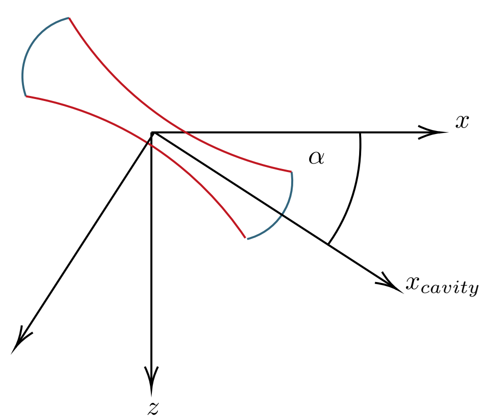
This class propagates a `WaveFunction` through a sample with a known potential $V(x,y,z)$.
It does so using the slice-by-slice method, as described in Kirland's book in section 6.4.
Simply speaking, if we can evaluate the potential in planes $z_n = n{\cdot}\Delta z$, and we denote the wavefuntion in some plane z_n as $\psi(x,y,z=z_n) \equiv \psi_n(x,y)$, we can propagate in steps each ψ_n to ψ_{n+1} by:

1. Propagate ψ_n in free space for a distance of Δz using angular spectrum of plane waves propagation (done in Fourier space): $\Psi_n(k_x,k_y) \rightarrow \Psi_n(k_x,k_y)\cdot e^{i\sqrt{k^2-k_x^2-k_y^2}\cdot\Delta z}$ (function `ASPW_propagation`)
2. Apply phase retardation to according to the potential $V(x,y,z_n)$ using the multiplication: $\psi(x,y) \rightarrow \psi(x,y)\cdot e^{i\sigma\Delta z\cdot V(x,y)}$ for $\sigma = \frac{\gamma me}{\hbar^2 k}$ (Function `propagate_through_potential_slice`)
3. Iterate alternately for $\{z_n\}_{n=0}^{N}$ until the wave comes out of the potential limits (out of the sample). (method `propagate`)

This class also have a useful method of loading dummy samples, such as letters or gaussians. (method `generate_dummy_potential`)

`CavityPropagator`

Cavity propagators are propagators that take the `WaveFunction` in the focal plane, and change it according to the electron-laser interaction with the laser in the cavity.
More formally, the cavity propagators propagate a `WaveFunction` through a cavity with two lasers, with wavelengths λ_1, λ_2 (denoted in the code as `l_1, l_2`) , with amplitudes E_1, E_2 (denoted in the code as `E_1, E_2`), numerical apertures $\mathrm{NA}_1, \mathrm{NA}_2$ (`NA_1, NA_2`), with polarization θ (`theta_polarization`) (which is shared for both lasers) and with tilt of the cavity of α (`alpha_cavity`).
This is how α looks like in a sketch, where the z axis is the axis of propagation on electrons:



The electron-laser interaction is derived in section Review of Osip's paper and for the case of a general electromagnetic field ends with equation (e_3) which expresses the total phase given to the electron:

$$\phi(x,y,t) = -\frac{1}{\hbar}\int_{-\infty}^{\infty}\frac{e^2}{2m\gamma\beta c}\left(\left(\boldsymbol{A}(\boldsymbol{x},T(z,t)) - \nabla G(\boldsymbol{x},T(z,t))\right)^2 - \beta^2\left(A_z(\boldsymbol{x},T(z,t)) - \partial_z G(\boldsymbol{x},T(z,t))\right)^2\right)dz \qquad \text{(e\_3)}$$

Where x, y are the center of the thin electron beam in the focal plane, t is the time at which it crosses the focal plane, and the function G is defined also in the same section in equation (8):

$$G(\boldsymbol{x}, t) = c\beta \int_{-\infty}^{t} A_z(z - c\beta(t - t'), t')\, dt' \tag{8}$$

When there are two frequencies in the cavity, the intensity A^2 will oscillate, which will result in an oscillating phase shift (oscillating in time t) of the form:

$$\phi(x, y, t) = \phi_{\text{const}}(x, y) + C\cos(\Delta\omega t) \tag{19}$$

With some ϕ_{const} and oscillation frequency C.
The effect of the oscillating phase shift is described in section Interaction with an oscillating laser and ends with the expression:

$$\boxed{\psi_{\text{output}}(x, y) = \psi_{\text{input}}(x, y) \cdot e^{i\phi_{\text{const}}(x,y)} \cdot J_0(C)} \tag{20}$$

There are two versions of this cavity propagator, and both inherit `CavityPropagator`: The first is `CavityAnalyticalPropagator` which calculates the mask $e^{i\phi_{\text{const}}(x,y)} \cdot J_0(C)$ of the last equation using a closed formula (with no integrals), and uses some simplifying approximations which will be described in the next subsection. The second is `CavityNumericalPropagator` that calculates the mask $e^{i\phi_{\text{const}}(x,y)} \cdot J_0(C)$ of the last equation directly with no simplifying assumptions (by doing a numeric integration of equation (e_3) that was quoted a few paragraphs before).
Both classes have the same propagate function: first calculate they the mask $e^{i\phi_{\text{const}}(x,y)} \cdot J_0(C)$ and then multiply the input `WaveFunction` by the mask to get the output `WaveFunction`
A few notes that are relevant for both cavity propagators (more elaborated explanation on each cavity class comes right afterwards):

1. The phase of the mask $e^{i\phi_{\text{const}}(x,y)}$ from the last equation (20), is the phase the electron acquires by passing through the ponderomotive potential. The derivation for this phase is given in section The cavity effect on one pencil beam centered at (x_0, y_0) , and in particular is given by equation (e_3) that was quoted a few paragraphs ago.

2. The absolute value of this complex number is the attenuation factor $J_0(C)$ that is given to this beam by being modulated in frequency by the cavity, and later passing through an energy filter that will keep only the zero'th energy.

3. When initializing the object with `E_1=-1` the object searches itself for the amplitude that will result in phase shift of $\frac{\pi}{2}$.

4. The ratio used in the simulation between E_1 and E_2 is derived in section Calculate the envelope $A_{\text{envelope}}(x, y, z)$ for two gaussian beams in t and in equation (Amplitudes Ratio).

   (a) The default value of `E_2` in the class is -1, and if this will indeed be the input, then the object will initiate `E_2` with the appropriate value according to equation (Amplitudes Ratio):

   $$E_2 = E_1 \cdot \frac{\lambda_1}{\lambda_2} \tag{Amplitudes Ratio}$$

   (b) If `E_2` is `None` then the cavity will have only one laser.

   (c) Those calculations happen in `__init__` method.

5. The ratio used in the simulation between NA_1 and NA_2 is derived in section Relation between wavelengths and numerical aperture in a fixed and in equation (NA ratios).

(a) Also here, when initiating the object and setting the value of NA_1, the values -1 and `None` behave the same wave as in `E_2`

(b) those calculations happen in `__init__` method.

6. The tilt of the cavity α_{cavity} used in the simulation is derived in section [The relation between $\beta_{\text{electron}}, \beta_{\text{lattice}}, \alpha, \lambda_1, \lambda_2$:] and appear in equation (41).

   (a) it depends on the velocity of the electron, which is not part of the object, but rather part of the input `WaveFunction`, and so it cannot happen in the `__init__` function. instead, it happens in the `beta_electron2alpha_cavity` function.

7. The propagator can work also with a single laser, by setting `E_2=None`.

`CavityAnalyticalPropagator`

As said before, this `Propagator` calculates the total phase and attenuation given to each pixel using simplifying approximations. all approximations are done in section [The cavity effect on one pencil beam centered at (x_0, y_0)] and end in the final equation (e_35):

$$\psi_{x_0,y_0}\begin{pmatrix}x\\y\\z\\t\end{pmatrix}\xrightarrow{\text{Passing through the cavity}}\psi_{x_0,y_0}\begin{pmatrix}x\\y\\z\\t\end{pmatrix}\cdot e^{i\phi_0\left(\frac{x_0}{\cos\alpha},y_0\right)\left(2+\underbrace{\left(\frac{\Gamma_+}{\Gamma_-}\right)^2+\left(\frac{\Gamma_-}{\Gamma_+}\right)^2}_{\text{Not in ring cavity}}\right)}\cdot J_0\left(\rho\left(\theta,\beta_e'\right)\phi_0\left(\frac{x_0}{\cos\alpha},y_0\right)\right) \tag{e\_35}$$

For:

1. J_0 is the Bessel function of first kind, θ is the polarization of light, x_0, y_0 is the location of the pixel in the focal plane (corresponding to the original transverse frequency), α is the tilt angle of the cavity

2. β_e' is defined to be

$$\beta_e' = \frac{\frac{1}{\gamma_c}\beta_e\cos\alpha}{1-\beta_e\sin\alpha\cdot\beta_c} \tag{37}$$

   Where β_e is the velocity of the electron in the lab's frame and β_c is the velocity of the standing wave created by the two laser frequency, which is given by:

$$\beta_c = \frac{\lambda_1-\lambda_2}{\lambda_1+\lambda_2} \tag{25}$$

3. ϕ_0 defined in (38):

$$\phi_0\left(\frac{x_0}{\cos\alpha},y_0\right) = \frac{e^2}{4\hbar mc\beta_e'\gamma_e'\omega^2}\sqrt{\frac{\pi}{2}}\frac{w_0^2E_0^2}{w^2\left(\frac{x_0}{\cos\alpha}\right)}e^{-\frac{2y_0^2}{w^2\left(\frac{x_0}{\cos\alpha}\right)}} \tag{38}$$

   (a) Note that two running waves in opposite directions $E_0e^{-i\omega t+kx}$, $E_0e^{-i\omega t-kx}$ will sum to a standing wave with a 2 factor before it, and a factor of 4 for the intensity:

$$\begin{aligned}E &= E_0e^{-i\omega t+kx}+E_0e^{-i\omega t-kx} = 2\cos(kx)\,e^{-i\omega t}\\ I &= 4\cos^2(kx)\end{aligned}$$

   In the original paper This 4 factor appears as part of ϕ_0 in the original paper [relativistic ponderomotive potential paper] but here because there are multiple waves that are not all standing, I took this 4 factor out of the definition of ϕ_0.

4. $\Gamma_\pm$ is defined in (21):

$$\gamma\left(1\mp\beta_c\right)=\frac{\left(1\mp\beta_c\right)}{\sqrt{1-\beta_c^2}}=\frac{\left(1\mp\beta_c\right)}{\sqrt{\left(1-\beta_c\right)\left(1+\beta_c\right)}}=\sqrt{\frac{1\mp\beta_c}{1\pm\beta_c}}\equiv\Gamma_\mp \tag{21}$$

5. $\rho\left(\theta,\beta\right)$ is defined in equation (17):

$$\rho\left(\theta,\beta\right)=1-2\beta^2\cos^2\left(\theta\right) \tag{17}$$

Here is a pseudo code of the propagation process, done in the method `propagate`:

- Create the total phase and amplitude mask which is the right hand side in (e_35) in method `phase_and_amplitude_mask`:
  - First, $\phi_0\left(\frac{x_0}{\cos\alpha},y_0\right)$ is calculated in the method `phi_0`.
  - Then, the total brown term in the exponent $\phi_0\left(\frac{x_0}{\cos\alpha},y_0\right)\left(2+\underbrace{\left(\frac{\Gamma_+}{\Gamma_-}\right)^2+\left(\frac{\Gamma_-}{\Gamma_+}\right)^2}_{\text{Not in ring cavity}}\right)$ is calculated in the method `phase_shift`.
  - Then, the teal attenuation factor $J_0\left(\rho\left(\theta,\beta_e\right)\phi_0\left(\frac{x_0}{\cos\alpha},y_0\right)\right)$ is calculated in `attenuation_factor`.
  - multiply the terms from (b) and (c) to get the total phase masks.
- Multiply the input `WaveFunction` and the `phase_and_amplitude_mask` to get the output `WaveFunction`.

`CavityNumericalPropagator`

This class does the same manipulation on a `WaveFunction` as `CavityAnalyticalPropagator` but instead of using the final expression for the phase shift in (e_35), it does the integral in (e_3) directly. The analytical expression in (e_35) that is used by the analytical propagator is the result of this integral after some approximation.
Here is the original equation (e_3) for your comfort:

$$\phi\left(x,y,t\right)=-\frac{1}{\hbar}\int_{-\infty}^{\infty}\frac{e^2}{2m\gamma\beta c}\left(\left(\boldsymbol{A}\left(\boldsymbol{x},T\left(z,t\right)\right)-\nabla G\left(\boldsymbol{x},T\left(z,t\right)\right)\right)^2-\beta^2\left(A_z\left(\boldsymbol{x},T\left(z,t\right)\right)-\partial_z G\left(\boldsymbol{x},T\left(z,t\right)\right)\right)^2\right)dz \tag{e\_3}$$

Note that this integral is very heavy computationally, and this problem is addressed in Addressing the complexity problem:. besides the numerical optimizations, the class knows to take the integral in (e_3) which needs to be done for every x,y,t triplets, and to divide it into segments such that no segments exceeds some arbitrary memory limit.
Also, once it computed the phase shift and amplitude attenuation for every pixel, it saves the result in a unique name, and then if the same setup is used again (frequencies, NA, amplitudes, polarization, resolution, etc...) then it just loads the result from the previously saved array.
A few notes:
Here is a pseudo code of the way the `propagate` method of this class calculates the `phase_and_amplitude_mask` (which later multiplies each pixel):

First check if the phase and amplitude mask was calculated for this set up in the past. if yes, load the results. if not - calculate proceed to calculate it for the first time
Suppose not - calculate $\phi\left(x,y,t\right)$ with the method `phi`:

- if the number of points in time for evaluation of ϕ is 3 (which is the default, and is stored as `self.n_t`), then set the `t` values on which ϕ will be evaluated to be $t_0,t_1,t_2=0,\frac{\pi}{2\Delta\omega},\frac{\pi}{\Delta\omega}$, from section Calculating the zero'th energy amplitude with just 3 points in time:.

- else, initiate a vector of evenly space t's with the number specified in `self.n_t`.
- evaluate ϕ on the x, y, t grids in batches using the method `phi_single_batch` (division to batches is done so that memory limit is not being exceeded)
- for every batch evaluated in `phi_single_batch`:
  - evaluate the integrand $\left(\boldsymbol{A}\left(\boldsymbol{x}, T\left(z,t\right)\right)-\nabla G\left(\boldsymbol{x}, T\left(z,t\right)\right)\right)^{2}-\beta^{2}\left(A_{z}\left(\boldsymbol{x}, T\left(z,t\right)\right)-\partial_{z} G\left(\boldsymbol{x}, T\left(z,t\right)\right)\right)^{2}$ using the function `phi_integr`
    - initiate lattices `X, Y, Z, T` of size $n_x \times n_y \times n_z \times n_t$ using the `generate_coordinates_lattice` method. (Here you can see it take a lot of memory - for an integration of 800 z points, for 3 points in time and for an image of `128X128` pixels, each of those lattices and A itself will have $4 \cdot 10^{7}$ elements)
      - explanation on the way they are initiated is in section Addressing the complexity problem: but in a nutshell:
      - the `Z` array is shifted such that for every x, the values of z in that x are centered around the optical axis of the cavity.
      - the `Z` axis is scaled so that for every x, the values of z range between $-5w\left(x\right)$ to $5w\left(x\right)$.
      - The `T` array is shifted according to z to make it $T\left(z,t\right)=t+\frac{z}{\beta c}$.
    - Calculate $A\left(x,y,z,t\left(z\right)\right)$ on the lattices using the method `rotated_gaussian_beam_A`. this method uses equation (e_37) for the evaluation of A.
    - Using the values calculated for A, calculate ∇G in the method `grad_G`:
      - Recall that G is defined to be:

$$G\left(x,y,z,t\right)=\int\limits_{Z} A_{z}\left(x,y,Z,t-\frac{z-Z}{\beta c}\right)dZ$$

      - The x component of ∇G is given (by just substituting into the definition) by:

$$\frac{G\left(x+dx,y,z,t\right)-G\left(x,y,z,t\right)}{dx}=\frac{1}{dx}\cdot\left[\int\limits_{Z} A_{z}\left(x+dx,y,Z,t-\frac{z-Z}{\beta c}\right)dZ-\int\limits_{Z} A_{z}\left(x,y,Z,t-\frac{z-Z}{\beta c}\right)dZ\right]$$

      - Same goes for the y component of ∇G
      - But the z component of G has this form, that is derived in (43):

$$\frac{\partial G}{\partial z}=-\frac{1}{\beta c}\frac{\partial G}{\partial t}+A_{z}\left(z,\left(t\left(z\right)-\frac{dz}{\beta c}\right)\right) \tag{43}$$

        Because of those we calculate A again (**Here is the heavy calculation!**) in small shifts in dx, dy, dt (and not dz!).
      - Calculate G again on those shifted A's
      - Numerically evaluate ∇G using the values of the shifted G's.
  - sum the values of $\boldsymbol{A}\left(\boldsymbol{x}, T\left(z,t\right)\right)$ and ∇G according to how they appear in the integrand above.
- integrate the values of `phi_integrand` to get ϕ.

using the resulted values of ϕ, calculate the amplitude attenuation factor using the method `extract_0th_energy_level_amplitude`:

- if ϕ was evaluated on 3 time points:

– then extract the values of the constant phase shift and the amplitude of the oscillating part using the derivation in ():

$$\phi_{\text{const}} = \frac{\phi(t_0) + \phi(t_2)}{2}$$
$$\varphi \equiv \tan^{-1}\left(\frac{\phi(t_0) - \phi_{\text{const}}}{\phi(t_1) - \phi_{\text{const}}}\right)$$
$$C_{\text{amplitude of oscilating phase}} = \frac{\phi(t_1) - \phi_{\text{const}}}{\cos(\varphi)} = \frac{\phi(t_0) - \phi_{\text{const}}}{\sin(\varphi)}$$

  * The phase mask is then given by the expression $e^{i\phi_{\text{const}}} \cdot J_0\left(C_{\text{amplitude of oscilating phase}}\right)$, as derived in (20) (and mentioned in the beginning of this document).

– If there are more then 3 points in time for the evaluated ϕ:

  * evaluate the amplitude of the middle band using `np.fft.fft`.

Finally, multiply the resulted phase mask with the input wave to get the output wave

`AberrationsPropagator`

This object adds the aberrations to the picture, and is meant to be used as a final `Propagator`.
It does so by multiplying the image in Fourier space by the factors:

$$\Psi(k_x, k_y) \to \Psi(k_x, k_y) \cdot e^{\frac{1}{2}\left[C_s \lambda_e^3 \cdot k^2 - f(k) k^2\right]}$$
$$k^2 = k_x^2 + k_y^2$$
$$f(k) = f_0 + f_a \cos(2(\varphi_k - \varphi_0))$$
$$\varphi_k = \tan^{-1}\left(\frac{k_y}{k_x}\right)$$

Where C_s is the spherical aberrations coefficient, λ_e is the wavelength of the electron, f_0 is the defocusing parameter, f_a is the astigmatism coefficient and φ_0 is the astigmatism orientation.
This formula is based on Berkleys notes in "Berkleys_Files\Simulator_Documentation.pdf" in this Git project.

Example code:

```
from microscope import *

l_1 = 1064e-9
l_2 = 532e-9
NA_1 = 0.05
N_POINTS = 128  # Resolution of image
pixel_size = 1e-10

input_coordinate_system = CoordinateSystem(dxdydz=(pixel_size, pixel_size), n_points=(N_POINTS, N_POINTS))
first_wave = WaveFunction(psi=np.ones((N_POINTS, N_POINTS)),
                          coordinates=input_coordinate_system,
                          E0=Joules_of_keV(300))

dummy_sample = SamplePropagator(dummy_potential=f'letters_{N_POINTS}',
```

```
                                coordinates_for_dummy_potential=CoordinateSystem(axes=(input_coordinate_system.x_axis,
                                                                                       input_coordinate_system.y_axis,
                                                                                       np.linspace(-5e-10, 5e-10, 2)
                                                                                       )))

first_lens = LensPropagator(focal_length=3.3e-3, fft_shift=True)
second_lens = LensPropagator(focal_length=3.3e-3, fft_shift=False)
cavity_2f_analytical = CavityAnalyticalPropagator(l_1=l_1, l_2=l_2, E_1=-1, NA_1=NA_1, ring_cavity=False,
                                                  starting_E_in_auto_E_search=1e6)
aberration_propagator = AberrationsPropagator(Cs=1e-8, defocus=1e-10, astigmatism_parameter=0,
                                              astigmatism_orientation=0)

M = Microscope([dummy_sample, first_lens, cavity_2f_analytical, second_lens, aberration_propagator],
               print_progress=True,
               n_electrons_per_square_angstrom=50)

pic = M.take_a_picture(first_wave)

plt.imshow(pic.values)
plt.title('image')
plt.show()
```

Analytical derivations

Review of Osip's paper

In this section I re-derive in a more elaborated way the same calculations already done in relativistic ponderomotive potential paper.

Find the phase shift using the field in the electron's rest frame S'

Start with the Schrodinger equation:

$$\frac{d}{dt}\psi \overset{\text{Schrodinger Eq.}}{=} \frac{i}{\hbar}\mathcal{H} \Big\backslash \int dt$$

$$\psi\left(t\right) \overset{\text{Schrodinger Eq.}}{=} e^{\frac{i}{\hbar}\int\mathcal{H}dt'}\psi\left(0\right) = e^{\frac{i}{\hbar}\int\left[\frac{1}{2}mv'^{2}-e\boldsymbol{A}'\left(\boldsymbol{x}'\left(t'\right),t'\right)\cdot\boldsymbol{v}'\left(t'\right)\right]dt'}\psi\left(0\right)$$

And so the total phase acquired is:

$$\phi\left(t\right) = \frac{1}{\hbar}\int\left[\frac{1}{2}mv'^{2} - e\boldsymbol{A}'\left(\boldsymbol{x}'\left(t'\right),t'\right)\cdot\boldsymbol{v}'\left(t'\right)\right]dt' \tag{1}$$

Since we are in Gibbs gauge, and $A_0 = 0$, the electric field is:

To first order the location of the electron in this frame is $\boldsymbol{x}'\left(t'\right) = \boldsymbol{x}_0$ (the initial position at rest) and so:

$$\boldsymbol{E}'\left(\boldsymbol{x}_0'\left(t'\right),t'\right) \approx \boldsymbol{E}'\left(\boldsymbol{x}_0',t'\right) = -\frac{\partial}{\partial t'}\boldsymbol{A}'\left(\boldsymbol{x}_0',t'\right)$$

And the force:

$$\boldsymbol{F}'(t') = -e\boldsymbol{E}'(\boldsymbol{x}_0', t') = e\frac{\partial}{\partial t}\boldsymbol{A}'(\boldsymbol{x}_0', t')$$

And so

$$\Delta\boldsymbol{p}(t') = \int_0^{t'} F'(t'')\,dt'' = e\int_0^{t'} \frac{\partial}{\partial t}\boldsymbol{A}'(\boldsymbol{x}_0', t'')\,dt'' = e\boldsymbol{A}'(\boldsymbol{x}_0', t')$$

And the velocity:

$$\boldsymbol{v}(t') = \frac{\Delta\boldsymbol{p}}{m} = \frac{e}{m}\boldsymbol{A}'(\boldsymbol{x}_0', t')$$

Plugging this velocity into (1), we get:

$$\begin{aligned}
\phi(\boldsymbol{x}_0', t) &= \frac{1}{\hbar}\int_0^t \left[\frac{1}{2}mv'^2 - e\boldsymbol{A}'(\boldsymbol{x}'(t'), t')\cdot\boldsymbol{v}'(t')\right]dt' \\
&= \frac{1}{\hbar}\int_0^t \left[\frac{1}{2}m\left(\frac{e}{m}A'(\boldsymbol{x}_0, t')\right)^2 - e\boldsymbol{A}'(\boldsymbol{x}_0, t')\cdot\left(\frac{e}{m}A'(\boldsymbol{x}_0, t')\right)\right]dt' \\
&= \frac{1}{\hbar}\int_0^t \left[\frac{1}{2}\frac{e^2}{m}A^2(\boldsymbol{x}_0, t') - \frac{e^2}{m}\boldsymbol{A}'^2(\boldsymbol{x}_0, t')\cdot\right]dt' \\
&= -\frac{1}{\hbar}\int_0^t \frac{e^2}{2m}\boldsymbol{A}'^2(\boldsymbol{x}_0, t')\,dt' \qquad (2)
\end{aligned}$$

Convert A' from electron's frame to A in lab's frame preserving Gibbs gauge:

Let us denote the field in coulomb gauge in the electron's frame with A', the field in the lab frame after Lorentz transformation from the electron's frame with $\tilde{A}$, the field in the lab's frame and in coulomb gauge with A.
Since A' and A are in coulomb gauge (Gibbs gauge, in particular), we have: $A_0 = 0 = \tilde{A}_0$.
Let us express everything in terms of $\left(\tilde{A}_0, \tilde{A}\right)$:

$$\begin{aligned}
\Phi' = 0 &= \frac{\gamma}{c}\tilde{A}_0 - \gamma\beta\tilde{A}_z \qquad (3) \\
\boldsymbol{A}'_\perp &= \tilde{\boldsymbol{A}}_\perp \\
A'_z &= -\frac{\gamma}{c}\beta\tilde{A}_0 + \gamma\tilde{A}_z = -\frac{\gamma}{\cancel{c}}\beta^2\cancel{c}\tilde{A}_z + \gamma\tilde{A}_z = \gamma\left(1-\beta^2\right)\tilde{A}_z = \frac{\left(1-\beta^2\right)}{\sqrt{1-\beta^2}}\tilde{A}_z = \sqrt{1-\beta^2}\tilde{A}_z = \frac{1}{\gamma}\tilde{A}_z \qquad (4) \\
\boldsymbol{A} &= \tilde{\boldsymbol{A}} + \nabla G \Rightarrow \\
A_z &= \tilde{A}_z + \partial_z G \qquad (5) \\
A_x &= \tilde{A}_x + \partial_x G \\
A_y &= \tilde{A}_y + \partial_y G \\
A_0 &= \tilde{A}_0 - \partial_t G = 0 \qquad (6)
\end{aligned}$$

Given an electromagnetic vector potential:

$$\left(\tilde{A}_0, \tilde{\boldsymbol{A}}\right)$$

Any gauge will be of the form:

$$\tilde{\boldsymbol{A}} \to \boldsymbol{A} = \tilde{\boldsymbol{A}} + \nabla G$$
$$\tilde{A}_0 \to A_0 = \tilde{A}_0 - \partial_t G$$

If we want to make $\tilde{A}_0$ vanish then we get an equation on G:

$$\partial_t G \overset{(6)}{=} \tilde{A}_0 \tag{S5}$$

If $\left(\tilde{A}_0, \tilde{\boldsymbol{A}}\right)$ was itself achieved by Lorentz transforming a coulomb gauged vector potential:

$$\tilde{A}_0 \overset{(3)}{=} \beta c \tilde{A}_z \tag{S1}$$

Substituting the result from (S5) to (S1)

$$A_z \overset{(5)}{=} \partial_z G + \tilde{A}_z \setminus -\partial_z G$$

$$(A_z - \partial_z G) \overset{(5)}{=} \tilde{A}_z$$

$$\partial_t G \overset{(S5)}{=} \tilde{A}_0 \overset{(S1)}{=} \beta c \tilde{A}_z \overset{(S4)}{=} \beta c \left(A_z - \partial_z G\right) \setminus + \beta c \partial_z G$$

$$\left[\partial_t + \beta c \partial_z\right] G = \beta c A_z \checkmark \tag{7}$$

Guess solution of the form:

$$G\left(\boldsymbol{x}, t\right) = c\beta \int_{-\infty}^{t} A_z\left(z - c\beta\left(t - t'\right), t'\right) dt' \tag{8}$$

Let's verify that this solution works: Define:

$$f_t\left(t'\right) = A_z\left(z - c\beta\left(t - t'\right), t'\right)$$

And get:

$$
\begin{aligned}
\frac{\partial}{\partial t}\left(c\beta\int\limits_{-\infty}^{t} A_z\left(z-c\beta\left(t-t'\right),t'\right)dt'\right) = & \\
&=\frac{\partial}{\partial t}\left(c\beta\int\limits_{-\infty}^{t} f_t\left(t'\right)dt'\right)\\
&=c\beta\frac{\partial}{\partial t}\left(\int\limits_{-\infty}^{t} f_t\left(t'\right)dt'\right)\\
&=c\beta\lim_{dt\to 0}\left(\int\limits_{-\infty}^{t+dt} f_{t+dt}\left(t'\right)dt'-\int\limits_{-\infty}^{t} f_t\left(t'\right)dt'\right):dt\\
&=c\beta\lim_{dt\to 0}\left(\int\limits_{t}^{t+dt} f_{t+dt}\left(t'\right)dt'+\int\limits_{-\infty}^{t} f_{t+dt}\left(t'\right)dt'-\int\limits_{-\infty}^{t} f_t\left(t'\right)dt'\right):dt\\
&=c\beta\lim_{dt\to 0}\left(\underbrace{\int\limits_{t}^{t+dt} f_{t+dt}\left(t'\right)dt'}_{f_{t+dt}(t')\cdot dt}+\int\limits_{-\infty}^{t}\underbrace{\left(f_{t+dt}\left(t'\right)-f_t\left(t'\right)\right)}_{\frac{d}{dt}f_t(t')\cdot dt}dt'\right):dt\\
&=c\beta\lim_{dt\to 0}\left(f_{t+dt}\left(t\right)\cdot\cancel{dt}+\int\limits_{-\infty}^{t}\frac{d}{dt}f_t\left(t'\right)\cdot\cancel{dt}dt'\right):\cancel{dt}\\
&=c\beta\lim_{dt\to 0}\left(f_{t+\cancel{dt}}\left(t\right)+\int\limits_{-\infty}^{t}\frac{d}{dt}f_t\left(t'\right)dt'\right)\\
&=c\beta\lim_{dt\to 0}\left(f_t\left(t\right)+\int\limits_{-\infty}^{t}\frac{d}{dt}A_z\left(z-c\beta\left(t-t'\right),t'\right)dt'\right)\\
&=c\beta\left(A_z\left(z,t\right)+\int\limits_{-\infty}^{t}\frac{\partial}{\partial z}A_z\left(z-c\beta\left(t-t'\right),t'\right)\cdot\underbrace{\frac{d}{dt}\left(z-c\beta\left(t-t'\right)\right)}_{-c\beta}dt'\right)\\
&=c\beta\left(A_z\left(z,t\right)+\int\limits_{-\infty}^{t}\frac{\partial}{\partial z}A_z\left(z-c\beta\left(t-t'\right),t'\right)\left(-c\beta\right)dt'\right)\\
&=c\beta\left(A_z\left(z,t\right)-\beta c\int\limits_{-\infty}^{t}\frac{\partial}{\partial z}A_z\left(z-c\beta\left(t-t'\right),t'\right)dt'\right)
\end{aligned}
\tag{9}
$$

$$
\frac{\partial}{\partial z}G=\frac{\partial}{\partial z}\left(c\beta\int\limits_{-\infty}^{t} A_z\left(z-c\beta\left(t-t'\right),t'\right)dt'\right)=c\beta\int\limits_{-\infty}^{t}\frac{\partial}{\partial z}A_z\left(z-c\beta\left(t-t'\right),t'\right)dt'
$$

Plugging in (7), we see indeed that:

$$\partial_t G + \beta c \frac{\partial}{\partial z} G = c\beta \left(A_z(z,t) - \cancel{\beta c \int_{-\infty}^{t} \frac{\partial}{\partial z} A_z(z - c\beta(t-t'), t')\, dt'} \right) + \cancel{(c\beta)^2 \int_{-\infty}^{t} \frac{\partial}{\partial z} A_z(z - c\beta(t-t'), t')\, dt'} =$$

$$\partial_t G + \beta c \frac{\partial}{\partial z} G = c\beta A_z(z,t) \checkmark$$

Proof's of Osip's approximation for G: First, let us rewrite the the integral of G as such:

$$G/\beta c = \int_{-\infty}^{t} A_z(z - \beta c(t-T), T)\, dT$$

Define:

$$Z = z - \beta c(t-T) \qquad \frac{1}{\beta c} dZ = dT \qquad T = t \iff Z = z \tag{10}$$

$$T = \frac{Z-z}{\beta c} + t$$

$$G/\beta c = \int_{-\infty}^{t} A_z(z - \beta c(t-T), T)\, dT = \frac{1}{\beta c} \int_{-\infty}^{z} A_z\left(Z, \frac{Z-z}{\beta c} + t\right) dZ$$

With the frequency:

$$\cos(\omega T) = \cos\left(\omega\left(\frac{Z-z}{\beta c} + t\right)\right) \Rightarrow \qquad \tilde{k} = \frac{\omega}{\beta c} = \frac{k_L}{\beta} \qquad \varphi_{z,t} = \omega\left(t - \frac{z}{\beta c}\right)$$

Now, let us show that $G \approx \beta c \int_0^t A(z,t')\, dt' = A_{\text{envelope}}(z) \cdot \sin(\omega t)$.
We will show it by looking at $\frac{\partial G/\beta c}{\partial t}$:

$$\frac{\partial G/\beta c}{\partial t} \overset{(9)}{=} A_z(z,t) - \beta c \int_{-\infty}^{t} \frac{\partial}{\partial z} A_z(z - c\beta(t-t'), t')\, dt' = A_z(z,t) - \beta c \int_{-\infty}^{t} A'(z - \beta c(t-t')) \cos(\omega t')\, dt' =$$

$$= A_z(z,t) - \frac{\beta \cancel{c}}{\beta c} \int_{-\infty}^{z} A'(Z) \cos\left(\frac{k}{\beta}(Z-z) + \omega t\right) dZ =$$

And so since $G(z,t) = \int_{-\infty}^{t} \frac{\partial}{\partial t'} G(z,t')\, dt'$, we want to show that just the second term is much smaller than the first term:

$$G(z,t) = \int_{-\infty}^{t} \frac{\partial}{\partial t'} G(z,t')\, dt' = \overset{\checkmark}{\int_{-\infty}^{t} A_z(z,t')\, dt'} - \int_{0}^{t} \int_{-\infty}^{z} A'(Z) \cos\left(\frac{k}{\beta}(Z-z) + \omega t'\right) dZ dt' =$$

Continuing only with the second element:

$$= -\int_{-\infty}^{z} A'(Z) \int_{0}^{t} \cos\left(\frac{k}{\beta}(Z-z)+\omega t'\right) dt' dZ$$

$$= -\frac{1}{\omega} \int_{-\infty}^{z} \underbrace{A'(Z)}_{=\frac{Z}{\sigma^2}A(Z)} \sin\left(\frac{k}{\beta}Z - \frac{k}{\beta}z + t\right) dZ$$

$$= -\frac{1}{\omega} \int_{-\infty}^{z} \frac{\overbrace{Z}^{\approx \cancel{w}}}{w^{\cancel{2}}} A(Z) \sin\left(\frac{k}{\beta}Z - \frac{k}{\beta}z + t\right) dZ$$

$$\approx -\frac{1}{\omega w} \underbrace{\int_{-\infty}^{z} A(Z) \sin\left(\frac{k}{\beta}Z - \frac{k}{\beta}z + t\right) dZ}$$

$$\beta \cancel{c} \overset{?}{\ll} -\omega w = \cancel{c} k w$$

$$\beta \overset{?}{\ll} kw = \frac{2\cancel{k}}{\cancel{k}\mathrm{NA}} = \frac{2}{\mathrm{NA}} \checkmark$$

Coming back to the main discussion: The field in the laboratory frame before gauging it back is:

$$\begin{aligned} \boldsymbol{A}'_{\perp}(\boldsymbol{x}',t') = & \begin{pmatrix} A'_x \\ A'_y \end{pmatrix}(\boldsymbol{x}',t') = \begin{pmatrix} \tilde{A}_x \\ \tilde{A}_y \end{pmatrix}(\boldsymbol{x},t) = \tilde{\boldsymbol{A}}_{\perp}(\boldsymbol{x},t) \\ A'_{\parallel}(\boldsymbol{x}',t') = & \quad A'_z(\boldsymbol{x}',t') = \quad \frac{1}{\gamma}\tilde{A}_z(\boldsymbol{x},t) = \frac{1}{\gamma}\tilde{A}_{\parallel}(\boldsymbol{x},t) \\ 0 = & \qquad\quad A'_0 = \quad \frac{\gamma}{c}\tilde{\Phi}(\boldsymbol{x},t) - \gamma\beta\tilde{A}_z \end{aligned}$$

And

$$\boldsymbol{A} = \tilde{\boldsymbol{A}} + \nabla G$$
$$\Phi = \tilde{\Phi} - \partial_t G$$

So we get:

$$
\begin{aligned}
\boldsymbol{A}'^2 &= A'^2_x + A'^2_z + A'^2_z \\
&= \tilde{A}^2_x + \tilde{A}^2_z + \frac{1}{\gamma^2}\tilde{A}^2_z \\
&= \tilde{A}^2_x + \tilde{A}^2_z + \tilde{A}^2_z + \left(\frac{1}{\gamma^2} - 1\right)\tilde{A}^2_z \\
&= \tilde{A}^2 + \left(\left(\cancel{1} - \beta^2\right) - \cancel{1}\right)\tilde{A}^2_z \\
&= \tilde{A}^2 - \beta^2\tilde{A}^2_z \\
\boldsymbol{A}'^2 &= \left(\boldsymbol{A} - \nabla G\right)^2 - \beta^2\left(A_z - \partial_z G\right)^2
\end{aligned}
\tag{11}
$$

And so substituting (2) into (11) we get:

$$
\phi\left(\boldsymbol{x}'_0, t\right) \overset{(2)}{=} -\frac{1}{\hbar}\int\limits_{-\infty}^{t}\frac{e^2}{2m}\boldsymbol{A}'^2\left(\boldsymbol{x}'_0, t'\right)dt'
$$

In the lab's frame $(\boldsymbol{x}'_0, t')$ become $(x_0, y_0, z_0 + \beta c T, T) = (x, y, z(T), T)$ and dt' becomes $\frac{1}{\gamma}dT$ and so we get:

$$
\phi\left(x_0, y_0, z_0, t\right) = -\frac{1}{\hbar}\int\limits_{-\infty}^{t}\frac{e^2}{2m}\left(\left(\boldsymbol{A}\left(x_0, y_0, z\left(T\right), T\right) - \nabla G\right)^2 - \beta^2\left(A_z - \partial_z G\right)^2\right)\underbrace{\left(\frac{1}{\gamma}dT\right)}_{dt'}
$$

(Also $\nabla G, A_z, \partial_z G$ are evaluated at $(x_0, y_0, z(T), T)$)
changing variables for comfort:

$$
z = z_0 + \beta c T \Rightarrow \qquad T\left(z, t_0\right) = t_0 + \frac{1}{\beta c}z \qquad dT = \frac{1}{\beta c}dz
$$

we get:

$$
\phi\left(z, t_0\right) = -\frac{1}{\hbar}\int\limits_{-\infty}^{z}\frac{e^2}{2m\gamma\beta c}\left(\left(\boldsymbol{A}\left(x, y, z, T\left(z, t_0\right)\right) - \nabla G\left(\boldsymbol{x}, T\left(z, t_0\right)\right)\right)^2 - \beta^2\left(A_z\left(\boldsymbol{x}, T\left(z, t_0\right)\right) - \partial_z G\left(\boldsymbol{x}, T\left(z, t_0\right)\right)\right)^2\right)dz
$$

And in total after passing through the whole potential (taking $z \to \infty$) we get:

$$
\boxed{\phi\left(t_0\right) = -\frac{1}{\hbar}\int\limits_{-\infty}^{\infty}\frac{e^2}{2m\gamma\beta c}\left(\left(\boldsymbol{A}\left(\boldsymbol{x}, T\left(z, t_0\right)\right) - \nabla G\left(\boldsymbol{x}, T\left(z, t_0\right)\right)\right)^2 - \beta^2\left(A_z\left(\boldsymbol{x}, T\left(z, t_0\right)\right) - \partial_z G\left(\boldsymbol{x}, T\left(z, t_0\right)\right)\right)^2\right)dz}
\tag{e\_3}
$$

$$
T\left(z, t_0\right) = t_0 + \frac{1}{\beta c}z
$$

And for the special case of G with gradient only in the $\hat{x}$ direction (which we will later find out is the case) and for $A_x = 0$:

$$\begin{aligned}\phi(t) &= -\frac{1}{\hbar}\int\limits_z \frac{e^2}{2m\gamma\beta c}\left(\cancel{A_x^2} + \cancel{2A_x\partial_x G} + (\partial_x G)^2 + A_y^2 + A_z^2 - \beta^2 A_z^2 + \cancel{2A_z\partial_z G} - \cancel{(\partial_z G)^2}\right)dz \\ &= -\frac{1}{\hbar}\int\limits_z \frac{e^2}{2m\gamma\beta c}\left(A_y^2 + \left(1-\beta^2\right)A_z^2 + (\partial_x G)^2\right)dz \\ &= -\frac{1}{\hbar}\int\limits_z \frac{e^2}{2m\gamma\beta c}\left(A_y^2(\boldsymbol{x}, t(z)) + \left(1-\beta^2\right)A_z^2(\boldsymbol{x}, t(z)) + (\partial_x G(\boldsymbol{x}, t(z)))^2\right)dz \end{aligned} \tag{12}$$

Fitting a vector potential $\boldsymbol{A}$ in the lab frame to the Gaussian beam standing wave:

$$\boldsymbol{A} = \underbrace{A\frac{w_0}{w(x)}e^{-\frac{\left(y^2+z^2\right)}{w^2(x)}}\cos\left(k_l x + k_l\frac{y^2+z^2}{2R(x)} - \psi(x)\right)}_{A_{\text{envelope}(x,y,z)}}\cdot\begin{pmatrix}0\\0\\\sin\theta\cos(\omega t-\varepsilon)\\\cos\theta\cos(\omega t)\end{pmatrix} \tag{13}$$

Let's make sure that this potential produces the electric field of a gaussian beam: differentiate it by time, we get:

$$\boldsymbol{E} = -\frac{\partial}{\partial t}\boldsymbol{A} =$$

$$\boldsymbol{E} = \omega A\frac{w_0}{w(x)}e^{-\frac{\left(y^2+z^2\right)}{w^2(x)}}\cos\left(k_l x + k_l\frac{y^2+z^2}{2R(x)} - \psi(x)\right)\cdot\begin{pmatrix}0\\\sin\theta\sin(\omega t-\varepsilon)\\\cos\theta\sin(\omega t)\end{pmatrix} \checkmark$$

Does it satisfies the gibbs gauge conditions? It satisfies the gibbs condition of $A_0 = 0$ by definition.
And the divergence:

$$\begin{aligned}\nabla\cdot\boldsymbol{A} &= \begin{pmatrix}\frac{\partial}{\partial x}\\\frac{\partial}{\partial y}\\\frac{\partial}{\partial z}\end{pmatrix}\begin{pmatrix}0\\A(y,z)\cos\left(k_l x + k_l\frac{y^2+z^2}{2R(x)} - \psi(x)\right)\cdot\sin\theta\cos(\omega t-\varepsilon)\\A(y,z)\cos\left(k_l x + k_l\frac{y^2+z^2}{2R(x)} - \psi(x)\right)\cdot\cos\theta\cos(\omega t)\end{pmatrix} = \\ &= [\partial_y A(y,z)]\cdot\cos\left(k_l x + k_l\frac{y^2+z^2}{2R(x)} - \psi(x)\right)\cdot\sin\theta\cos(\omega t-\varepsilon) + [\partial_z A(y,z)]\cos\left(k_l x + k_l\frac{y^2+z^2}{2R(x)} - \psi(x)\right)\cdot\cos\theta\cos(\omega t) \ll \boldsymbol{B}\end{aligned}$$

we can't say about anything which is not dimensionless that it is close to 0, but we can say that it is much closer to 0 comparing to another physical value with the same dimensions.
Since the sizes of the terms are in the order of magnitude of $\partial_y A(y,z)$ which is much smaller then $k_l \cdot A(y,z)$ (the derivative with respect to x), we get that this divergence is much smaller
then B (which has the same units), and therefore close to 0. (because B_z, for example, has the term $\partial_x A(y,z)\cdot\cos\ldots = k_l A(y,z)\sin\ldots$)

Going back to the main discussion: Given this A, the G function will be:

$$
\begin{aligned}
G &\equiv \beta c \int_{-\infty}^{t} A_z\left(\boldsymbol{x} - \beta c\,(t-T)\,\hat{z}, T\right) dT \\
&= \beta c \int_{-\infty}^{t} \left[\underbrace{A}_{\text{amplitude}}\left(x, y, z - \beta c\,(t-T)\right) \cos\left(k_l x + k_l \frac{y^2+z^2}{2R(x)} - \psi(x)\right) \cdot \cos\theta \cos(\omega T)\right] dT \\
&= A \frac{w_0}{w(x)} e^{-\frac{y^2}{w^2(x)}} \frac{1}{\omega} c\beta \cos\left(k_l x + k_l \frac{y^2+z^2}{2R(x)} - \psi(x)\right) \cdot \cos\theta \cdot \underbrace{\int_{-\infty}^{t} \left[e^{-\frac{(z-\beta c(t-T))^2}{w^2(x)}} \cos(\omega T)\right] dT}_{I(x,y,z,t)}
\end{aligned}
$$

$$
\text{Set}: Z = \beta c T + z - \beta c t \iff T = \frac{Z-z}{\beta c} + t \Rightarrow \frac{1}{\beta c} dZ = dT, T = t \iff Z = z
$$

$$
\begin{aligned}
&= A \frac{w_0}{w(x)} e^{-\frac{y^2}{w^2(x)}} \frac{1}{\omega} c\beta \cos\left(k_l x + k_l \frac{y^2+z^2}{2R(x)} - \psi(x)\right) \cdot \cos\theta \cdot \int_{-\infty}^{z} \left[e^{-\frac{Z^2}{w^2(x)}} \cos\left(\omega\left(\frac{Z-z}{\beta c} + t\right)\right)\right] \frac{1}{\beta c} dZ \\
&= A \frac{w_0}{w(x)} e^{-\frac{y^2}{w^2(x)}} \frac{1}{\omega} c\beta \cos\left(k_l x + k_l \frac{y^2+z^2}{2R(x)} - \psi(x)\right) \cdot \cos\theta \cdot \int_{-\infty}^{z} \left[e^{-\frac{Z^2}{w^2(x)}} \cos\left(\frac{\omega}{\beta c} Z + t - \frac{z}{\beta c}\right)\right] \frac{1}{\beta c} dZ \\
&= A \frac{w_0}{w(x)} e^{-\frac{y^2}{w^2(x)}} \frac{1}{\omega} c\beta \cos\left(k_l x + k_l \frac{y^2+z^2}{2R(x)} - \psi(x)\right) \cdot I(x,y,z,t) \cdot \cos\theta
\end{aligned}
$$

And the gradient (**Make sure the derivative with respect to z is negligible - remember that due to the fact that we integrate over z,t(z), there would be also a fast oscillating pattern in z**)

$$
\nabla G \approx \hat{x} \cdot A \frac{w_0}{w(x)} e^{-\frac{y^2}{w^2(x)}} \frac{1}{\omega} c\beta \sin\left(k_l x + k_l \frac{y^2+z^2}{2R(x)} - \psi(x)\right) \cdot I(x,y,z,t) \cdot \cos\theta \cdot \left(k_l + k_l \frac{y^2+z^2}{2} \underbrace{\frac{x_R^2 - x^2}{\left(x^2 + x_R^2\right)^2}}_{\frac{d}{dx}\frac{1}{R(x)} = \frac{d}{dx}\left(\frac{x}{x^2+x_R^2}\right)} + \underbrace{\frac{\frac{1}{x_R}}{1 + \left(\frac{x}{x_R}\right)^2}}_{\frac{d}{dx}\psi(x)}\right)
$$

(Also for a moving wave the result will be the same, as the cos will be replaced with complex exponent)

$$
= \hat{x} \cdot A \frac{w_0}{w(x)} e^{-\frac{y^2}{w^2(x)}} \cancel{\frac{1}{\omega}} c\beta \sin\left(k_l x + k_l \frac{y^2+z^2}{2R(x)} - \psi(x)\right) \cdot I(x,y,z,t) \cdot \cos\theta \cdot \underbrace{\left(\cancel{\frac{2\pi}{\lambda}} + \cancel{\frac{2\pi}{\lambda}} \frac{y^2+z^2}{2} \cdot \frac{\left(\frac{\lambda}{\pi \cdot (\mathrm{NA})^2}\right)^2 - x^2}{\left(x^2 + \left(\frac{\lambda}{\pi \cdot (\mathrm{NA})^2}\right)^2\right)^2} + \frac{\frac{\cancel{2\pi} \cdot (\mathrm{NA})^2}{2\cancel{\lambda}}}{1 + \left(\frac{\pi \cdot (\mathrm{NA})^2}{\lambda} \cdot x\right)^2}\right)}_{\text{Same but with } \lambda \text{ only and no } k_l, x_R}
$$

Recalling that $\frac{2\pi}{\lambda} = \frac{1}{c/\omega}$, we can write:

$$
= \hat{x} \cdot \beta A \frac{w_0}{w(x)} e^{-\frac{y^2}{w^2(x)}} \sin\left(k_l x + k_l \frac{y^2+z^2}{2R(x)} - \psi(x)\right) \cdot I(x,y,z,t) \cdot \cos\theta \cdot \left(1 + 1 \cdot \underbrace{\frac{y^2+z^2}{2\lambda^2}}_{\text{small correction}} \cdot \frac{\left(\frac{1}{\pi \cdot (\mathrm{NA})^2}\right)^2 - \left(\frac{x}{\lambda}\right)^2}{\left(\frac{x^2}{\lambda^2} + \frac{1}{\left(\pi \cdot (\mathrm{NA})^2\right)^2}\right)^2} + \underbrace{\frac{2\,(\mathrm{NA})^2}{1 + \left(\pi \cdot (\mathrm{NA})^2 \cdot \frac{x}{\lambda}\right)^2}}_{<6\% \text{ for NA=0.3}}\right) \approx
$$

$$
\approx \hat{x} \cdot \beta A \frac{w_0}{w(x)} e^{-\frac{y^2}{w^2(x)}} \sin\left(k_l x + k_l \frac{y^2+z^2}{2R(x)} - \psi(x)\right) \cdot I(x,y,z,t) \cdot \cos\theta
$$

Substituting all in (12) we get:

$$\phi(x,y) \overset{(12)}{=} -\frac{1}{\hbar}\int\limits_z \frac{e^2}{2m\gamma\beta c}\left(A_y^2(\boldsymbol{x},t(z)) + \left(1-\beta^2\right)A_z^2(\boldsymbol{x},t(z)) + \left(\partial_x G(\boldsymbol{x},t(z))\right)^2\right)dz$$

defining:

$$\varphi(x) = k_l x + k_l\frac{y^2+z^2}{2R(x)} - \psi(x) \tag{14}$$

Plugging in the wave of the form (13) we get:

$$\begin{aligned}\phi(x,y) = -\frac{e^2}{2m\gamma\beta c\hbar}\int\limits_z \Bigg[&\overbrace{\left(A_{\text{envelope}}^2(x,y,z)\cos^2(\varphi(x))\cdot\sin^2(\theta)\cos^2(\omega t(z)-\varepsilon)\right)}^{A_y} \\ &+\overbrace{\left(1-\beta^2\right)\cdot A_{\text{envelope}}^2(x,y,z)\cos^2(\varphi(x))\cdot\cos^2\theta\cos^2(\omega t(z))}^{A_z} \\ &+\underbrace{\beta^2 A_{\text{envelope}}^2(x,y,z)\sin^2(\varphi(x))\cdot\cos^2(\theta)\sin^2(\omega t(z))\Big]}_{A_x}dz\end{aligned}$$

$$\begin{aligned}= -\frac{e^2}{2m\gamma\beta c\hbar}\int\limits_z A_{\text{envelope}}^2(x,y,z)&\left[\left(\cos^2(\varphi(x))\cdot\sin^2(\theta)\cos^2(\omega t(z)-\varepsilon)\right)\right. \\ &\left.+\cos^2(\theta)\left[\left(\cos^2(\varphi(x))\cos^2(\omega t(z))\right)+\beta^2\left[-\cos^2(\varphi(x))\cos^2(\omega t(z))+\sin^2(\varphi(x))\sin^2(\omega t(z))\right]\right]\right]dz\end{aligned}$$

The integragtion over $\cos^2(\omega t(z)), \sin^2(\omega t(z)), \cos^2(\omega t(z)-\varepsilon)$ will just add a half factor and we get:

$$\begin{aligned}= -\frac{e^2}{2m\gamma\beta c\hbar}\int\limits_z \frac{1}{2}A_{\text{envelope}}^2(x,y,z)&\left[\left(\cos^2(\varphi(x))\cdot\sin^2(\theta)\right)\right. \\ &\left.+\cos^2(\theta)\left[\left(\cos^2(\varphi(x))\right)+\beta^2\left[-\cos^2(\varphi(x))+\sin^2(\varphi(x))\right]\right]\right]dz\end{aligned}$$

The red and the first blue parts are simply $\cos^2(\varphi(x))$ (because it is $\sin^2(\theta)+\cos^2(\theta)$, and the second blue part and the green part will be $-\cos^2(\varphi(x))+\sin^2(\varphi(x)) = -\cos(2\varphi(x))$):

$$= -\frac{e^2}{2m\gamma\beta c\hbar}\int\limits_z \frac{1}{2}A_{\text{envelope}}^2(x,y,z)\left[\left(\cos^2(\varphi(x))\right)-\cos^2(\theta)\beta^2\cos(2\varphi(x))\right]dz$$

$$= -\frac{e^2}{2m\gamma\beta c\hbar}\int\limits_z \frac{1}{2}A_{\text{envelope}}^2(x,y,z)\left[\frac{1}{2}+\frac{1}{2}\cos(2\varphi(x))-\cos^2(\theta)\beta^2\cos(2\varphi(x))\right]dz$$

$$= -\frac{e^2}{2m\gamma\beta c\hbar}\int\limits_z \frac{1}{2}A_{\text{envelope}}^2(x,y,z)\frac{1}{2}\left[1+\left(1-2\cos^2(\theta)\beta^2\right)\cos(2\varphi(x))\right]dz \tag{15}$$

If we ignore the fact that $\varphi(x)$ is z-dependent due to the curvature term, we can write:

$$\begin{aligned}\phi &= -\overbrace{\frac{e^2}{4m\gamma\beta c\hbar}\int\limits_z A^2_{\text{envelope}}(x,y,z)\,dz}^{\phi_0(x,y)}\cdot\frac{1}{2}\left[1+\overbrace{\left(1-2\cos^2(\theta)\beta^2\right)}^{\rho(\theta,\beta)}\cos(2\varphi(x))\right]\\ &= -\phi_0(x,y)\cdot\frac{1}{2}\left[1+\rho(\theta,\beta)\cos(2\varphi(x))\right]\end{aligned}$$

Where we defined:

$$\phi_0 = \frac{e^2}{4m\gamma\beta c\hbar}\int\limits_z A^2_{\text{envelope}}(x,y,z)\,dz \tag{16}$$

$$\rho(\theta,\beta) = 1-2\cos^2(\theta)\beta^2 \tag{17}$$

And for a moving wave (that does not have the $\cos(\varphi(x))$ grating in it, but only the slowly varying envelope):

$$\begin{aligned}\phi_{\text{running wave}}(x,y) &\overset{(12)}{=} -\frac{1}{\hbar}\int\limits_z\frac{e^2}{2m\gamma\beta c}\left(A_y^2(\boldsymbol{x},t(z))+\left(1-\beta^2\right)A_z^2(\boldsymbol{x},t(z))+\left(\partial_x G(\boldsymbol{x},t(z))\right)^2\right)dz\\ &= -\frac{1}{\hbar}\int\limits_z\frac{e^2A^2_{\text{envelope}}(x,y,z)}{2m\gamma\beta c}\left(\sin^2\theta\cancel{\cos^2(\omega t(z)-\varepsilon)}+\left(1-\beta^2\right)\cdot\left(\cdot\cos\theta\cancel{\cos(\omega t(z))}\right)^2+\left(\beta^2\cos^2(\theta)\cancel{\sin^2(\omega t(z))}\right)\right)dz\\ &= -\frac{1}{\hbar}\int\limits_z\frac{e^2A^2_{\text{envelope}}(x,y,z)}{4m\gamma\beta c}\left(\underbrace{\sin^2(\theta)+\cos^2(\theta)}_{=1}\cancel{-\beta^2\cos^2(\theta)+\left(\beta^2\cdot\cos^2(\theta)\right)}\right)dz\\ &= -\frac{1}{\hbar}\int\limits_z\frac{e^2A^2_{\text{envelope}}(x,y,z)}{4m\gamma\beta c}dz\end{aligned} \tag{18}$$

$$\phi_{\text{running wave}} = -\phi_0(x,y)$$

Interaction with an oscillating laser

Let us review the case where the intensity of A^2 oscillates as a function of time. since we are going to deal with the case of exactly two monochromatic laser beams, we can already now denote the oscillations frequency as $\Delta\omega$.
Since the laser intensity is oscillating $\phi(x,y,t)$ will have the form

$$\phi(x,y,t) = \phi_{\text{const}}(x,y)+C\cos(\Delta\omega t) \tag{19}$$

Therefore, the output wave as a function of frequency will look like so:

$$\psi_{\text{output}} = \psi_{\text{input}}\cdot e^{i\phi(x,y,t)} = \psi_{\text{input}}\cdot e^{i\phi_{\text{const}}(x,y)+C\cos(\Delta\omega t)} = \psi_{\text{input}}\cdot e^{i\phi_{\text{const}}(x,y)}\cdot\sum_{q\in\mathbb{Z}}J_q(C)\cdot e^{iq\Delta\omega t}$$

Adding an energy filter to filter out all the frequencies that are not the $q=0$ frequency, we get:

$$
\begin{aligned}
\psi_{\text{output of energy filter}} &= \psi_{\text{input}} \cdot e^{i\phi_{\text{const}}(x,y)} \cdot J_0(C) + \underbrace{\cancel{\sum_{q\in\mathbb{Z}\setminus\{1\}} J_q(C) \cdot e^{iq\Delta\omega t}}}_{\text{Filtered out}} \\
&= \psi_{\text{input}} \cdot e^{i\phi_{\text{const}}(x,y)} \cdot J_0(C)
\end{aligned} \tag{20}
$$

In practice, in the electron microscope, the energy filter does not sit in the focal plane but rather further down the microscope, but we can considered the filtered electrons as if as they were filtered on spot in the focal plane.

Calculate the envelope $A_{\textbf{envelope}}(x, y, z)$ for two gaussian beams in the moving frame:

Introduction

In the focal plane of the microscope we set a cavity there are two lasers with two different frequencies. This cavity is designed to interact with the passing electron through the ponderomotive potential.
There exists a moving reference frame where the blue-shifted red frequency and the red-shifted blue frequency become the same frequency. as we will see the problem is easier to describe in the reference frame. Here we will show how to find the amplitudes, the frequency, and the intensity in this frame, and this will later be used in the next section where we describe it's effect on the electrons.

Lorentz transform the field

Let's assume the laser propagates in the $\hat{x}$ direction.
The original electric and magnetic fields are:

$$
\boldsymbol{E}(\boldsymbol{x},t) = \left[E_1 e^{i\left(\omega_1 t - \frac{\omega_1}{c}x\right)} + E_1 e^{i\left(\omega_1 t + \frac{\omega_1}{c}x\right)} + E_2 e^{i\left(\omega_2 t - \frac{\omega_2}{c}x\right)} + E_2 e^{i\left(\omega_2 t + \frac{\omega_2}{c}x\right)}\right] \begin{pmatrix} 0 \\ \cos(\theta) \\ \sin(\theta) \end{pmatrix}
$$

The $\boldsymbol{B}$ vector depends on the direction of propagation:

$$
\begin{aligned}
\boldsymbol{B}(\boldsymbol{x},t) &= \frac{1}{c}\overbrace{\hat{k}}^{=\pm\hat{x}} \times \overbrace{\left[E_1 e^{i\left(\omega_1 t - \frac{\omega_1}{c}x\right)} + E_1 e^{i\left(\omega_1 t + \frac{\omega_1}{c}x\right)} + E_2 e^{i\left(\omega_2 t - \frac{\omega_2}{c}x\right)} + E_2 e^{i\left(\omega_2 t + \frac{\omega_2}{c}x\right)}\right] \begin{pmatrix} 0 \\ \cos(\theta) \\ \sin(\theta) \end{pmatrix}}^{\boldsymbol{E}} = \\
&= \pm\frac{1}{c}\left[E_1 e^{i\left(\omega_1 t - \frac{\omega_1}{c}x\right)} + E_1 e^{i\left(\omega_1 t + \frac{\omega_1}{c}x\right)} + E_2 e^{i\left(\omega_2 t - \frac{\omega_2}{c}x\right)} + E_2 e^{i\left(\omega_2 t + \frac{\omega_2}{c}x\right)}\right] \begin{pmatrix} 0 \\ -\sin\theta \\ \cos\theta \end{pmatrix}
\end{aligned}
$$

The field at point $\boldsymbol{E}'(\boldsymbol{x}', t')$ will be:

$$\boldsymbol{E}'(\boldsymbol{x}',t') = \begin{pmatrix} E'_x \\ E'_y \\ E'_z \end{pmatrix}(\boldsymbol{x}',ct')$$
$$= \begin{pmatrix} \cancel{E_x} \\ \gamma(E_y - vB_z) \\ \gamma(E_z + vB_y) \end{pmatrix}\left(\Lambda^{-1}(\boldsymbol{x}',ct')\right)$$

and for a general linear polarization orientation (in lab's frame) $\boldsymbol{E}(x,y) = E_0(x,y)\begin{pmatrix} 0 \\ \cos\theta \\ \sin\theta \end{pmatrix}$, $\boldsymbol{B}(x,y) = \pm\frac{E_0(x,y)}{c}\begin{pmatrix} 0 \\ -\sin\theta \\ \cos\theta \end{pmatrix}$ (the $+$ sign is for a wave propagating in the $+\hat{x}$ direction and the $-$ sign is for a field propagating in the $-\hat{x}$ direction) we get:

$$\boldsymbol{E}'(\boldsymbol{x}',t') = \begin{pmatrix} 0 \\ \gamma\left(\overbrace{E_0\left(\Lambda^{-1}(\boldsymbol{x}',ct')\right)\cos\theta}^{E_y} \mp v\overbrace{\frac{E_0\left(\Lambda^{-1}(\boldsymbol{x}',ct')\right)\cos(\theta)}{c}}^{-B_z}\right) \\ \gamma\left(\underbrace{E_0\left(\Lambda^{-1}(\boldsymbol{x}',ct')\right)\sin\theta}_{E_z} \mp v\underbrace{\frac{E_0\left(\Lambda^{-1}(\boldsymbol{x}',ct')\right)\sin(\theta)}{c}}_{+B_y}\right) \end{pmatrix}$$
$$= \begin{pmatrix} 0 \\ \cos\theta\gamma\left(1\mp\frac{v}{c}\right) \\ \sin\theta\gamma\left(1\mp\frac{v}{c}\right) \end{pmatrix} E_0\left(\Lambda^{-1}(\boldsymbol{x}',ct')\right)$$
$$= \begin{pmatrix} 0 \\ \cos\theta \\ \sin\theta \end{pmatrix}\gamma\left(1\mp\frac{v}{c}\right)E_0\left(\Lambda^{-1}(\boldsymbol{x}',ct')\right)$$

Denote:

$$\gamma(1\mp\beta) = \frac{(1\mp\beta)}{\sqrt{1-\beta^2}} = \frac{(1\mp\beta)}{\sqrt{(1-\beta)(1+\beta)}} = \sqrt{\frac{1\mp\beta}{1\pm\beta}} \equiv \Gamma_\mp \tag{21}$$

And get:

$$\boldsymbol{E}'(\boldsymbol{x}',t') = \Gamma_\mp\underbrace{E_0\left(\Lambda^{-1}(\boldsymbol{x}',ct')\right)}_{\text{Scalar}}\begin{pmatrix} 0 \\ \cos\theta \\ \sin\theta \end{pmatrix}$$

Looking at a coordinate $(\boldsymbol{x}', t')$, the corresponding space-time point will be:

$$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \Lambda^{-1} \begin{pmatrix} x' \\ y' \\ z' \\ t' \end{pmatrix} = \begin{pmatrix} \gamma\left(x' + \beta c t'\right) \\ y \\ z \\ \gamma\left(t' + \frac{\beta x'}{c}\right) \end{pmatrix}$$

$$\boldsymbol{E}'\left(\boldsymbol{x}', t'\right) = \Gamma_{\mp} \boldsymbol{E}\left(\gamma\left(x' + \beta c t'\right), 0, 0, \gamma\left(t + \frac{\beta x'}{c}\right)\right)$$

And get, after substituting for the relevant waves (before substituting $x \to \gamma\left(x' + \beta c t'\right)$ and $t \to \gamma\left(t + \frac{\beta x'}{c}\right)$):

$$= \left[\Gamma_{-} E_1 e^{i\left(\omega_1 t - \frac{\omega_1}{c} x\right)} + \Gamma_{+} E_1 e^{i\left(\omega_1 t + \frac{\omega_1}{c} x\right)} + \Gamma_{-} E_2 e^{i\left(\omega_2 t - \frac{\omega_2}{c} x\right)} + E_2 \Gamma_{+} e^{i\left(\omega_2 t + \frac{\omega_2}{c} x\right)}\right]_{(x,t) = \left(\gamma\left(x' + \beta c t'\right), \gamma\left(t' + \frac{\beta x'}{c}\right)\right)} \tag{22}$$

Performing the coordinates substitution, the argument in the exponent changes like

$$\begin{aligned}
\omega_i t \mp \frac{\omega_i}{c} x \to \omega_i \gamma\left(t' + \frac{\beta x'}{c}\right) &\mp \frac{\omega_i}{c} \gamma\left(x' + \beta c t'\right) \\
&= \omega_i \gamma t' + \omega_i \gamma \frac{\beta x'}{c} \mp \frac{\omega_i}{c} \gamma x' \mp \frac{\omega_i}{\cancel{c}} \gamma \beta \cancel{c} t' \\
&= \omega_i \left(1 \mp \beta\right) \gamma t' \mp \left(1 \mp \beta\right) \gamma \frac{\omega_i}{c} x' \\
&= \sqrt{\frac{1 \mp \beta}{1 \pm \beta}} \omega_i t' \mp \sqrt{\frac{1 \mp \beta}{1 \pm \beta}} \frac{\omega_i}{c} x' \\
&= \Gamma_{\mp} \omega_i t' \mp \Gamma_{\mp} \frac{\omega_i}{c} x' \\
&= \Gamma_{\mp} \left(\omega_i t' \mp \frac{\omega_i}{c} x'\right)
\end{aligned}$$

And if it is not a plane wave but a gaussian beam with also gouy phase

$$\begin{aligned}
\omega_i t \mp \frac{\omega_i}{c} x \mp k_i \frac{y^2 + z^2}{R(x)} \pm \psi_i(x) \to \omega_i \gamma\left(t' + \frac{\beta x'}{c}\right) &\mp \frac{\omega_i}{c} \gamma\left(x' + \beta c t'\right) \mp k_i \frac{y'^2 + z'^2}{R_i\left(\gamma\left(x' + \beta c t'\right)\right)} \pm \psi_i\left(\gamma\left(x' + \beta c t'\right)\right) \\
&= \omega_i \gamma t' + \omega_i \gamma \frac{\beta x'}{c} \mp \frac{\omega_i}{c} \gamma x' \mp \frac{\omega_i}{\cancel{c}} \gamma \beta \cancel{c} t' \mp k_i \frac{y'^2 + z'^2}{R_i\left(\gamma\left(x' + \beta c t'\right)\right)} \pm \psi_i\left(\gamma\left(x' + \beta c t'\right)\right) \\
&= \omega_i \left(1 \mp \beta\right) \gamma t' \mp \left(1 \mp \beta\right) \gamma \frac{\omega_i}{c} x' \mp k_i \frac{y'^2 + z'^2}{R_i\left(\gamma\left(x' + \beta c t'\right)\right)} \pm \psi_i\left(\gamma\left(x' + \beta c t'\right)\right) \\
&= \sqrt{\frac{1 \mp \beta}{1 \pm \beta}} \omega_i t' \mp \sqrt{\frac{1 \mp \beta}{1 \pm \beta}} \frac{\omega_i}{c} x' \mp k_i \frac{y'^2 + z'^2}{R_i\left(\gamma\left(x' + \beta c t'\right)\right)} \pm \psi_i\left(\gamma\left(x' + \beta c t'\right)\right) \\
&= \Gamma_{\mp} \omega_i t' \mp \Gamma_{\mp} \frac{\omega_i}{c} x' \mp k_i \frac{y'^2 + z'^2}{R_i\left(\gamma\left(x' + \beta c t'\right)\right)} \pm \psi_i\left(\gamma\left(x' + \beta c t'\right)\right) \\
&= \Gamma_{\mp} \left(\omega_i t' \mp \frac{\omega_i}{c} x'\right) \mp \underbrace{k_i \frac{y'^2 + z'^2}{R_i\left(\gamma\left(x' + \beta c t'\right)\right)} \pm \psi_i\left(\gamma\left(x' + \beta c t'\right)\right)}_{\Psi_i\left(\gamma\left(x' + \beta c t\right), y, z\right)}
\end{aligned} \tag{23}$$

Plugging it into the wave (22), we get:

$$E_y\left(\boldsymbol{x}',t'\right)=\Gamma_-E_1e^{i\Gamma_-\left(\omega_1t'-\frac{\omega_1}{c}x'\right)}+\Gamma_+E_1e^{\Gamma_+i\left(\omega_1t'+\frac{\omega_1}{c}x'\right)}+\Gamma_-E_2e^{\Gamma_-i\left(\omega_2t'-\frac{\omega_2}{c}x'\right)}+\Gamma_+E_2e^{\Gamma_+i\left(\omega_2t'+\frac{\omega_2}{c}x'\right)}$$

$$\begin{aligned}E_y\left(\boldsymbol{x}',t'\right)=&\Gamma_-E_1e^{i\Gamma_-\left(\omega_1t'-\frac{\omega_1}{c}x'\right)-\Psi_1\left(\gamma\left(x'+\beta ct\right),y,z\right)}+\Gamma_+E_1e^{\Gamma_+i\left(\omega_1t'+\frac{\omega_1}{c}x'\right)+\Psi_1\left(\gamma\left(x'+\beta ct\right),y,z\right)}\\&+\Gamma_-E_2e^{\Gamma_-i\left(\omega_2t'-\frac{\omega_2}{c}x'\right)-\Psi_2\left(\gamma\left(x'+\beta ct\right),y,z\right)}+\Gamma_+E_2e^{\Gamma_+i\left(\omega_2t'+\frac{\omega_2}{c}x'\right)+\Psi_2\left(\gamma\left(x'+\beta ct\right),y,z\right)}\end{aligned}$$

And so we see that if in one reference frame the gouy phases of the two laser will not match - then in no reference frame they will not match. this is reasonable because phase is Lorentz invariant.

Find the $\beta_{\textbf{lattice}}, E_1, E_2$ such that we have a standing wave :

Remark 1. In this subsection β_{lattice} and β represent both the velocity of the moving frame in which the blue-shifted red beam and the red-shifted blue beam have the same frequency and not the velocity of the electron.

Choose β_{lattice} such that

$$\Gamma_+\omega_1=\Gamma_-\omega_2 \ \backslash:\Gamma_+\omega_2$$

$$\frac{\omega_1}{\omega_2}=\frac{\Gamma_-}{\Gamma_+}\ \backslash\text{Substitute }\Gamma_\pm \tag{24}$$

$$\frac{\omega_1}{\omega_2}=\frac{\sqrt{\frac{1-\beta}{1+\beta}}}{\sqrt{\frac{1+\beta}{1-\beta}}}=\frac{1-\beta}{1+\beta}\ \backslash\left(1+\beta\right)$$

$$\frac{\omega_1}{\omega_2}\left(1+\beta\right)=1-\beta\ \backslash+\beta-\frac{\omega_1}{\omega_2}$$

$$\frac{\omega_1}{\omega_2}\beta+\beta=1-\frac{\omega_1}{\omega_2}$$

$$\left(1+\frac{\omega_1}{\omega_2}\right)\beta=1-\frac{\omega_1}{\omega_2}\ \backslash:\left(1+\frac{\omega_1}{\omega_2}\right)$$

$$\boxed{\beta_{\text{lattice}}=\frac{1-\frac{\omega_1}{\omega_2}}{1+\frac{\omega_1}{\omega_2}}=\frac{1-\frac{\lambda_2}{\lambda_1}}{1+\frac{\lambda_2}{\lambda_1}}=\frac{\lambda_1-\lambda_2}{\lambda_1+\lambda_2}} \tag{25}$$

Under β_{lattice}, we can denote:

$$\begin{aligned}&\Gamma_+\omega_1=\Gamma_-\omega_2\equiv\omega\\&\Gamma_-\omega_1\equiv\omega_-\\&\Gamma_+\omega_2\equiv\omega_+\end{aligned} \tag{26}$$

And get:

$$E_y\left(\boldsymbol{x}',t'\right)=\Gamma_-E_1e^{i\left(\omega_-t'-\frac{\omega_-}{c}x\right)}+\Gamma_+E_1e^{i\left(\omega t'+\frac{\omega}{c}x\right)}+\Gamma_-E_2e^{i\left(\omega t'-\frac{\omega}{c}x\right)}+\Gamma_+E_2e^{i\left(\omega_+t'+\frac{\omega_+}{c}x\right)} \tag{27}$$

Choosing equal amplitudes::

$$\Gamma_+E_1=\Gamma_-E_2 \setminus:\Gamma_+E_2$$

$$\frac{E_1}{E_2}=\frac{\Gamma_-}{\Gamma_+}=\frac{1-\beta}{1+\beta}\overset{(24)}{=}\frac{\omega_1}{\omega_2}=$$

$$\frac{E_1}{E_2}=\frac{\omega_1}{\omega_2}=\frac{\lambda_2}{\lambda_1}$$

$$\boxed{E_2=E_1\cdot\frac{\lambda_1}{\lambda_2}} \tag{Amplitudes Ratio}$$

And for the intensity and the power:

$$P_2\propto I_2=I_1\cdot\left(\frac{\lambda_1}{\lambda_2}\right)^2$$

Denoting

$$\Gamma_+E_1=\Gamma_-E_2=E_0 \tag{28}$$

and substituting again in (27), we get:

$$\begin{aligned}E_y\left(\boldsymbol{x}',t'\right)&=\frac{\Gamma_-}{\Gamma_+}E_0e^{i\left(\omega_-t'-\frac{\omega_-}{c}x'\right)}+E_0e^{i\left(\omega t'+\frac{\omega}{c}x'\right)}+E_0e^{i\left(\omega t'-\frac{\omega}{c}x'\right)}+\frac{\Gamma_+}{\Gamma_-}E_0e^{i\left(\omega_+t'+\frac{\omega_+}{c}x'\right)}\\&=E_0\left[\frac{\Gamma_-}{\Gamma_+}e^{i\left(\omega_-t'-\frac{\omega_-}{c}x'\right)}+e^{i\omega t'}\left[e^{i\frac{\omega}{c}x'}+e^{-i\frac{\omega}{c}x'}\right]+\frac{\Gamma_+}{\Gamma_-}e^{i\left(\omega_+t'+\frac{\omega_+}{c}x'\right)}\right]\\&=E_0\left[\frac{\Gamma_-}{\Gamma_+}e^{i\left(\omega_-t'-\frac{\omega_-}{c}x'\right)}+2e^{i\omega t'}\cos\left(\frac{\omega}{c}x'\right)+\frac{\Gamma_+}{\Gamma_-}e^{i\left(\omega_+t'+\frac{\omega_+}{c}x'\right)}\right]\end{aligned} \tag{29}$$

And if instead of a plane wave we had also the other factors of a gaussian beam $\varphi\left(x\right)=kx+k\frac{y^2+z^2}{R(x)}-\psi\left(x\right)$ then according to (23) we would have here:

$$E_{y,\text{gaussian beam}}=E_0\left[\frac{\Gamma_-}{\Gamma_+}e^{i\left(\omega_-t'+\varphi_-\left(\gamma\left(x'+\beta ct'\right)\right)\right)}+2e^{i\omega t'}\cos\left(\varphi\left(\gamma\left(x'+\beta ct'\right)\right)\right)+\frac{\Gamma_+}{\Gamma_-}e^{i\left(\omega_+t'+\varphi_+\left(\gamma\left(x'+\beta ct'\right)\right)\right)}\right]$$

Calculate the resulted intensity of the field

And the intensity:

$$E^2\left(\boldsymbol{x}',t'\right)=E_0^2\left|\frac{\Gamma_-}{\Gamma_+}e^{i\left(\omega_-t'-\frac{\omega_-}{c}x'\right)}+2e^{i\omega t'}\cos\left(\frac{\omega}{c}x'\right)+\frac{\Gamma_+}{\Gamma_-}e^{i\left(\omega_+t'+\frac{\omega_+}{c}x'\right)}\right|^2$$

$$= E_0^2 \cdot \left[\left(\frac{\Gamma_-}{\Gamma_+}\right)^2 + 4\cos^2\left(\frac{\omega}{c}x'\right) + \left(\frac{\Gamma_+}{\Gamma_-}\right)^2 + \frac{\cancel{\Gamma_-}}{\cancel{\Gamma_+}} e^{-i\left(\omega_- t' - \frac{\omega_-}{c}x'\right)} \frac{\cancel{\Gamma_+}}{\cancel{\Gamma_-}} e^{i\left(\omega_+ t' + \frac{\omega_+}{c}x'\right)} + \frac{\cancel{\Gamma_-}}{\cancel{\Gamma_+}} e^{i\left(\omega_- t' - \frac{\omega_-}{c}x'\right)} \frac{\cancel{\Gamma_+}}{\cancel{\Gamma_-}} e^{-i\left(\omega_+ t' + \frac{\omega_+}{c}x'\right)} + \right.$$
$$+ \frac{\Gamma_-}{\Gamma_+} e^{-i\left(\omega_- t' - \frac{\omega_-}{c}x'\right)} 2e^{i\omega t'} \cos\left(\frac{\omega}{c}x'\right) + \frac{\Gamma_-}{\Gamma_+} e^{i\left(\omega_- t' - \frac{\omega_-}{c}x'\right)} 2e^{-i\omega t'} \cos\left(\frac{\omega}{c}x'\right)$$
$$\left. + \frac{\Gamma_+}{\Gamma_-} e^{-i\left(\omega_+ t' + \frac{\omega_+}{c}x'\right)} 2e^{i\omega t'} \cos\left(\frac{\omega}{c}x'\right) + \frac{\Gamma_+}{\Gamma_-} e^{i\left(\omega_+ t' + \frac{\omega_+}{c}x'\right)} 2e^{-i\omega t'} \cos\left(\frac{\omega}{c}x'\right) \right]$$

$$= E_0^2 \left[\left(\frac{\Gamma_-}{\Gamma_+}\right)^2 + 4\cos^2\left(\frac{\omega}{c}x'\right) + \left(\frac{\Gamma_+}{\Gamma_-}\right)^2 + e^{i\left(-\omega_- t' + \frac{\omega_-}{c}x' + \omega_+ t' + \frac{\omega_+}{c}x'\right)} + e^{-i\left(-\omega_- t' + \frac{\omega_-}{c}x' + \omega_+ t' + \frac{\omega_+}{c}x'\right)} + \right.$$
$$+ 2\frac{\Gamma_-}{\Gamma_+} \left(e^{-i\left(\omega_- t' - \frac{\omega_-}{c}x' - \omega t'\right)} + e^{i\left(\omega_- t' - \frac{\omega_-}{c}x' - \omega t'\right)} \right) \cos\left(\frac{\omega}{c}x'\right)$$
$$\left. + 2\frac{\Gamma_+}{\Gamma_-} \left(e^{-i\left(\omega_+ t' + \frac{\omega_+}{c}x' - \omega t'\right)} + e^{i\left(\omega_+ t' + \frac{\omega_+}{c}x' - \omega t'\right)} \right) \cos\left(\frac{\omega}{c}x'\right) \right]$$

$$= E_0^2 \left[\left(\frac{\Gamma_-}{\Gamma_+}\right)^2 + 4\cos^2\left(\frac{\omega}{c}x'\right) + \left(\frac{\Gamma_+}{\Gamma_-}\right)^2 + \underbrace{2\cos\left(\left(\omega_+ - \omega_-\right)t' + \frac{\omega_- + \omega_+}{c}x'\right)}_{\text{Averages over time to } 0} \right.$$
$$\left. \underbrace{+4\cos\left(\frac{\omega}{c}x'\right)\left(\frac{\Gamma_-}{\Gamma_+}\cos\left(\left(\omega_- - \omega\right)t' - \frac{\omega_-}{c}x'\right) + \frac{\Gamma_+}{\Gamma_-}\cos\left(\frac{\omega}{c}x'\right)\cos\left(\left(\omega_+ - \omega\right)t' + \frac{\omega_-}{c}x'\right)\right)}_{\text{Averages over time to } 0} \right]$$

And the ponderomotive potential over one period (by definition we have $\omega_- < \omega < \omega_+$, so the frequency of the intensity oscillations will always be non 0):

$$\langle I \rangle = E_0^2 \left(\left(\frac{\Gamma_-}{\Gamma_+}\right)^2 + 4\cos^2\left(\frac{\omega}{c}x'\right) + \left(\frac{\Gamma_+}{\Gamma_-}\right)^2 \right) = \quad \text{(e\_10)}$$

Recalling the definition of $\Gamma_\mp$:

$$\langle I \rangle = E_0^2 \left(\frac{(1-\beta)^2}{(1+\beta)^2} + \frac{(1+\beta)^2}{(1-\beta)^2} + 2 + 2\cos\left(2kx'\right) \right)$$

And So we see that the intensity is a sum of constant terms ($\frac{(1-\beta)^2}{(1+\beta)^2} + \frac{(1+\beta)^2}{(1-\beta)^2} + 2$) and an oscillating term ($2\cos\left(2kx'\right)$).
And for some gaussian envelope $E_0(x, y, z)$ (neglecting the fact that near the reighley range different wavelengths have different spot-size):
Is it a reasonable assumption at all? I mean, in the waist where it is the most interesting their spots sizes differ by $1:2$!

$$\langle I \rangle (x', y', z') = E_0^2\left(\gamma x' + \gamma\beta ct, y', z'\right) \left(\frac{(1-\beta)^2}{(1+\beta)^2} + \frac{(1+\beta)^2}{(1-\beta)^2} + 2 + 2\cos\left(2\varphi\left(x', t'\right)\right) \right) \quad (30)$$

for $\varphi(x')$ defined in (14) as :

$$\varphi(x') = k_l x' + k_l \frac{y^2+z^2}{2R(x)} - \psi(x) \tag{14}$$

The cavity effect on one pencil beam centered at (x_0, y_0)

Assume we turn on the microscope and electrons are propagating through it. After passing through the sample, the electron wave will scatter, and each Fourier component of it will be concentrated to a thin column wave at some x_0, y_0 in the Fourier plane. we call such a thin columated wave a pencil beam.
We wish to find an expression for the effect of the cavity with two lasers on such a pencil beam.
Let $\psi_{x_0,y_0}(x,y,z,ct)$ be the wave function of the electron in the lab's frame, for a pencil beam centered around x_0, y_0.

Linear transformation of rotation and then lorentz transformation (required for the next stage):

In the previous section Calculate the envelope $A_{\text{envelope}}(x,y,z)$ for two gaussian beams in the moving frame: we found there is a moving reference frame where the laser intensity is composed of one standing wave and two constant terms. let us approach the problem from this reference frame.
Since this frame is moving with respect to some tilted axis (the cavity is tilted, according to The relation between $\beta_{\text{electron}}, \beta_{\text{lattice}}, \alpha, \lambda_1, \lambda_2$:), transforming the coordinates to this reference frame involve both rotation and Lorentz transform.
Let us denote with r the coordinates in the microscope's frame, with $\tilde{r}$ the coordinates in the rotated frame and with r' the coordinates in the rotated and moving frame. β and γ are the velocity and Lorentz factor of the moving frame. later the velocity and the lorentz factor of the electron will be denoted as β_e, γ_e.

The transformation:

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \\ c\tilde{t} \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\alpha & 0 & \sin\alpha & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\alpha & 0 & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{R} \begin{pmatrix} x \\ y \\ z \\ ct \end{pmatrix}$$

$$\begin{pmatrix} x' \\ y' \\ z' \\ ct' \end{pmatrix} = \underbrace{\begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & 1 \end{pmatrix}}_{L} \begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \\ c\tilde{t} \end{pmatrix}$$

And in total:

$$\begin{pmatrix} x' \\ y' \\ z' \\ ct' \end{pmatrix} = \underbrace{\begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & 1 \end{pmatrix}}_{R} \underbrace{\begin{pmatrix} \cos\alpha & 0 & \sin\alpha & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\alpha & 0 & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{L} \begin{pmatrix} x \\ y \\ z \\ ct \end{pmatrix} = \underbrace{\begin{pmatrix} \gamma\cos\alpha & 0 & \gamma\sin\alpha & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ -\sin\alpha & 0 & \cos\alpha & 0 \\ -\gamma\beta\cos\alpha & 0 & -\gamma\beta\sin\alpha & \gamma \end{pmatrix}}_{T} \begin{pmatrix} x \\ y \\ z \\ ct \end{pmatrix} =$$

$$= \begin{pmatrix} \gamma\left(x\cos\alpha + z\sin\alpha\right) - \gamma\beta ct \\ y \\ -\sin\alpha x + \cos\alpha z \\ -\gamma\beta\left(x\cos\alpha + z\sin\alpha\right) + \gamma ct \end{pmatrix}$$

The inverse transformation:

$$R^{-1} = \begin{pmatrix} \cos\alpha & 0 & -\sin\alpha & 0 \\ 0 & 1 & 0 & 0 \\ \sin\alpha & 0 & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$L^{-1} = \begin{pmatrix} \gamma & 0 & 0 & \gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma\beta & 0 & 0 & 1 \end{pmatrix}$$

$$T^{-1} = \left(R^{-1}L^{-1}\right) = \begin{pmatrix} \cos\alpha & 0 & -\sin\alpha & 0 \\ 0 & 1 & 0 & 0 \\ \sin\alpha & 0 & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma & 0 & 0 & \gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma\beta & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \gamma\cos\alpha & 0 & -\sin\alpha & \gamma\beta\cos\alpha \\ 0 & 1 & 0 & 0 \\ \gamma\sin\alpha & 0 & \cos\alpha & \gamma\beta\sin\alpha \\ \gamma\beta & 0 & 0 & \gamma \end{pmatrix}$$

And so:

$$\begin{pmatrix} y \\ z \\ x \\ ct \end{pmatrix} = \begin{pmatrix} \gamma x'\cos\alpha - z'\sin\alpha + \gamma\beta ct'\cos\alpha \\ y \\ \gamma x'\sin\alpha + z'\cos\alpha + \gamma\beta ct'\sin\alpha \\ \gamma\beta x' + \gamma ct' \end{pmatrix} \tag{31}$$

Describing the wave function and the laser field in the moving frame

In the rotated and moving lattice's frame this wave function will become (Using (31)):

$$\psi'\left(x', y', z', ct'\right) \overset{(31)}{=} \gamma\psi_{x_0,y_0}\left(T^{-1}\left(x', y', z', ct'\right)\right) = \gamma\psi_{x_0,y_0}\begin{pmatrix} \gamma x'\cos\alpha - z'\sin\alpha + \gamma\beta ct'\cos\alpha \\ y' \\ \gamma x'\sin\alpha + z'\cos\alpha + \gamma\beta ct'\sin\alpha \\ \gamma\beta x' + \gamma ct' \end{pmatrix}$$

If ψ_{x_0,y_0} had the form:

$$\psi_{x_0,y_0}(x,y,z,t) = e^{-\frac{\left((x-x_0)^2+(y-y_0)^2\right)}{2\sigma^2}} e^{ikz} e^{i\omega t} \approx$$
$$\approx \delta(x-x_0)\,\delta(y-y_0)\, e^{ikz} e^{i\omega t}$$

(Alternatively, we could also choose $\psi_{x_0,y_0}(x,y,z,t) = \delta(x-x_0, y-y_0)\, e^{ikz} e^{i\omega t}$)
Then ψ' has the form:

$$\psi'_{x_0,y_0}(x',y',z',ct') = e^{-\frac{\left(\overbrace{(\gamma x' \cos\alpha - z' \sin\alpha + \gamma\beta ct' \cos\alpha - x_0)}^{x-x_0}{}^2 + (y'-y_0)^2\right)}{2\sigma^2}} e^{ik\overbrace{(\gamma x' \sin\alpha + z' \cos\alpha + \gamma\beta ct' \sin\alpha)}^{z}} e^{i\omega\overbrace{(\gamma\beta x' + \gamma ct')}^{t}} \approx$$
$$\approx \delta(\gamma x' \cos\alpha - z' \sin\alpha + \gamma\beta ct' \cos\alpha - x_0)\,\delta(y-y_0)\, e^{ik\left(\gamma x' \sin\alpha + z' \cos\alpha + \gamma\beta ct' \sin\alpha\right)} e^{i\omega\left(\gamma\beta x' + \gamma ct'\right)} \tag{32}$$

And so we see that in the moving-rotated frame the wave function drifts to the left over time, and has a different frequency.
Let us choose the time $t=0$ as the time the gaussian envelope (The delta function) **of the wave function** is centered around $x'=y'=z'=0$.
At $z'=0$ the wave function will have the form:

$$\psi'_{x_0,y_0}(x',y',z'=0,ct') = \gamma\psi_{x_0,y_0}\begin{pmatrix} \gamma x' \cos\alpha + \gamma\beta ct' \cos\alpha \\ y' \\ \gamma x' \sin\alpha + \gamma\beta ct' \sin\alpha \\ \gamma\beta x' + \gamma ct' \end{pmatrix} = \gamma\psi_{x_0,y_0}\begin{pmatrix} \gamma\cos\alpha\,(x'+\beta ct') \\ y' \\ \gamma x' \sin\alpha + \gamma\beta ct' \sin\alpha \\ \gamma\beta x' + \gamma ct' \end{pmatrix} \tag{e\_32}$$

And so:

$$\psi_{x_0,y_0}\begin{pmatrix} \gamma\cos\alpha\,(x'+\beta ct') \\ y' \\ \gamma x' \sin\alpha + \gamma\beta ct' \sin\alpha \\ \gamma\beta x' + \gamma ct' \end{pmatrix} = e^{-\frac{\left((\gamma\cos\alpha(x'+\beta ct')-x_0)^2+(y'-y_0)^2\right)}{2\sigma^2}} e^{ik\left(\gamma x' \sin\alpha + \gamma\beta ct' \sin\alpha\right)} e^{i\omega\left(\gamma\beta x' + \gamma ct'\right)}$$
$$\approx \delta(\gamma\cos\alpha\,(x'+\beta ct') - x_0)\,\delta(y'-y_0)\, e^{ik\left(\gamma x' \sin\alpha + \gamma\beta ct' \sin\alpha\right)} e^{i\omega\left(\gamma\beta x' + \gamma ct'\right)} \tag{33}$$

For $t'=0$ we will have:

$$\psi_{x_0,y_0}\begin{pmatrix} \gamma x' \cos\alpha \\ y' \\ \gamma x' \sin\alpha \\ \gamma\beta x' \end{pmatrix} = e^{-\frac{\left((\gamma x' \cos\alpha - x_0)^2+(y'-y_0)^2\right)}{2\sigma^2}} e^{ik\gamma x' \sin\alpha} e^{i\omega\gamma\beta x'} = \underbrace{e^{-\frac{\left(x'-\frac{x_0}{\gamma\cos\alpha}\right)^2}{2\left(\frac{\sigma}{\gamma\cos\alpha}\right)^2}}}_{\substack{\text{Streched by a factor} \\ \text{of } \gamma\cos\alpha \text{ in } x}} e^{-\frac{(y'-y_0)^2}{2\sigma^2}} e^{ik\gamma x' \sin\alpha} e^{i\omega\gamma\beta x'}$$
$$\approx \underbrace{\gamma\cos\alpha}_{\text{Renormalization}}\ \delta\left(\gamma\cos\alpha\left(x' - \frac{x_0}{\gamma\cos\alpha}\right)\right)\delta(y'-y_0)\, e^{ik\gamma x' \sin\alpha} e^{i\omega\gamma\beta x'} \tag{34}$$

And so we see the pencil beam is a Gaussian delta function centered around $\left(\frac{x_0}{\gamma\cos\alpha}, y_0\right)$ (at $z'=0$ and $t'=0$). Taking the width of the pencil to be very small with respect to the features of the laser beam, we can approximate the electric field experienced by the electron

pencil beam to be the value at it's center (at (x_0, y_0, z)).
Now we recall that in our 2 frequencies cavity there are 2 frequencies. In particular, we saw that by choosing $\Gamma_+ E_1 = \Gamma_- E_2 = E_0$, we get for a standing wave cavity a wave of the form (all the expressions are explained after the equation):

$$E_{\text{laser,lattic'es frame}}(x', y', z', t') =$$

$$= E_{0,\text{envelope}}(\gamma x' + \gamma\beta c t', y', z') \cdot \left[\frac{\Gamma_-}{\Gamma_+} e^{i(\omega_- t' - \varphi_-(x'))} + 2e^{i\omega t'} \cos(\varphi(x')) + \frac{\Gamma_+}{\Gamma_-} e^{i(\omega_+ t' + \varphi_+(x'))}\right] \tag{29}$$

For:

$$E_0 \equiv \Gamma_+ E_1 = \Gamma_- E_2 \tag{28}$$

$$\underbrace{E_0(x', y', z', t')}_{\text{function}} \equiv \underbrace{E_0}_{\text{number}} \cdot \frac{w_0}{w(\gamma x' + \gamma c t')} \cdot e^{-\frac{y^2+z^2}{w^2(\gamma x' + \beta c t')}} \tag{35}$$

$$\Gamma_\pm \equiv \gamma(1 \pm \beta) \tag{21}$$

$$\omega \equiv \Gamma_+ \omega_1 = \Gamma_- \omega_2 \tag{26}$$

$$\omega_- \equiv \Gamma_- \omega_1$$

$$\omega_+ \equiv \Gamma_+ \omega_2$$

$$\varphi_\pm(x) = k_\pm x + k_\pm \frac{y^2 + z^2}{2R_\pm(x)} - \psi_\pm(x)$$

Remark 2. For a ring cavity we will not have the constant intensities marked in blue and red:

$$E_{\text{laser,lattic'es frame, ring cavity}}(x', y', z', t') = E_{0,\text{envelope}}(\gamma x' + \gamma\beta c t', y', z') \cdot 2e^{i\omega t} \cos(\varphi(x'))$$

And the intensity is given by equation (30):

$$\langle I_{\text{laser, lattic'es frame}}\rangle(x', y', z') = E_0^2(\gamma x' + \gamma\beta c t, y', z') \left(\left(\frac{\Gamma_-}{\Gamma_+}\right)^2 + \left(\frac{\Gamma_+}{\Gamma_-}\right)^2 + 2 + 2\cos(2\varphi(x'))\right) \tag{(30)}$$

And for a ring cavity:

$$\langle I\rangle_{\text{laser, lattic'es frame, ring cavity}} = E_0^2(\gamma x' + \gamma\beta c t, y', z') \cdot (2 + 2\cos(2\varphi(x')))$$

Now we are in a bit of a confusing situation - the envelope is running to the left with velocity β, so which value of the envelope will the electorn's pencil beam feel? We have to remember that also the electron beam crosses the $z' = 0$ plane at different x''s for different times t'. In particular - from the expression for the pencil beam in (33) we see that the pencil beam also drifts to the left with velocity βc and so **experiences a constant envelope value.**
Which value? Since it is constant in time, we can substitute the x' value of $t = 0$, which would be $\frac{x_0}{\gamma \cos\alpha}$ (according to equation (34)).

Substituting it into the envelope expression in (35) and the intensity from (30) we get:

$$\langle I\rangle\left(x'=\frac{x_0}{\gamma\cos\alpha},y',z',t'=0\right)=E_0^2\cdot\frac{w_0^2}{w^2\left(\cancel{\gamma}\frac{x_0}{\cancel{\gamma}\cos\alpha}+\cancel{\gamma\beta e\cdot 0}\right)}\cdot e^{-\frac{2\left(y^2+z^2\right)}{w^2\left(\cancel{\gamma}\frac{x_0}{\cancel{\gamma}\cos\alpha}+\cancel{\gamma\beta e\cdot 0}\right)}}\left(\left(\frac{\Gamma_-}{\Gamma_+}\right)^2+\left(\frac{\Gamma_+}{\Gamma_-}\right)^2+2+2\cos\left(2\varphi\left(x'\right)\right)\right)$$

$$=E_0^2\cdot\frac{w_0^2}{w^2\left(\frac{x_0}{\cos\alpha}\right)}\cdot e^{-\frac{2\left(y^2+z^2\right)}{w^2\left(\frac{x_0}{\cos\alpha}\right)}}\left(\left(\frac{\Gamma_-}{\Gamma_+}\right)^2+\left(\frac{\Gamma_+}{\Gamma_-}\right)^2+2+2\cos\left(2\varphi\left(x'\right)\right)\right) \tag{36}$$

Remark 3. And for a ring cavity the average intensity will not have the blue and red terms:

$$\langle I_{\text{ring cavity}}\rangle\left(x'=\frac{x_0}{\gamma\cos\alpha},y',z',t'=0\right)=E_0^2\cdot\frac{w_0^2}{w^2\left(\frac{x_0}{\cos\alpha}\right)}\cdot e^{-\frac{2\left(y^2+z^2\right)}{w^2\left(\frac{x_0}{\cos\alpha}\right)}}\left(2+2\cos\left(2\varphi\left(x'\right)\right)\right)$$

Calculating the interaction between the electron and the laser intensity

We saw in equations (15) and (18) how the different intensity factors contribute to the phase of the electron.
to implement those calculations here, we want to first find the velocity of the electron in the moving frame.
What are β_e',γ_e' in the moving frame?

$$\underbrace{\beta_e\hat{z}}_{\text{Lab's frame}}\xrightarrow[\text{rotate by }\alpha\text{ in }xy\text{ plane}]{}\underbrace{\beta_e\hat{z}\cos\alpha-\beta_e\sin\alpha\hat{x}}_{\text{rotated frame}}\xrightarrow[\text{lorentz transform in }\hat{x}]{}\frac{\frac{1}{\gamma_c}\beta_e\cos\alpha}{1-\beta_e\sin\alpha\cdot\beta_c}\hat{z}+\frac{-\beta_e\sin\alpha+\beta_c}{1-\beta_e\sin\alpha\cdot\beta_c}\hat{x}$$

If we choose now to set $-\beta_e\sin\alpha+\beta_c=0\iff\alpha=\sin^{-1}\left(\frac{\beta_c}{\beta_e}\right)$, then the electron will not move in the horizontal axis, and it's trajectory will be $\boldsymbol{r}'(t)=\begin{pmatrix}x_0'\\y_0'\\z_0'+\beta_e'ct'\end{pmatrix}$, which is assumed in the derivation of the interaction in section Review of Osip's paper , and in particular in equation Going back to the main discussion:. under this choice of α we can write:

$$\boldsymbol{v}_e'=\beta_e'\hat{z}=\frac{\frac{1}{\gamma_c}\beta_e\cos\alpha}{1-\beta_e\sin\alpha\cdot\beta_c}\hat{z} \tag{37}$$

In the paper Osip's paper, the envelope of the phase shift acquired is given by integrating the field's intensity envelope (only the envelope, without the green, blue, and red terms) of the electromagnetic wave over $z,t(z)$, (equation (9) in the paper) and so we get:

$$\phi_0\left(x'=\frac{x_0}{\cos\alpha},y'=y_0\right)=\frac{e^2}{4mc\beta_e'\gamma_e'\hbar}\cdot\int A_0^2(y,z)\,dz$$

$$=\frac{e^2}{4mc\beta_e'\gamma_e'\hbar}\cdot\int E_0^2\cdot\frac{w_0^2}{w^2\left(\frac{x_0}{\cos\alpha}\right)}\cdot e^{-\frac{2\left(y^2+z^2\right)}{w^2\left(\frac{x_0}{\cos\alpha}\right)}}dz$$

$$\overset{(e\_1)}{=}\frac{e^2}{\hbar 4mc\beta_e'\gamma_e'\omega^2}\sqrt{\frac{\pi}{2}}\frac{w_0^2E_0^2}{w^2\left(\frac{x_0}{\cos\alpha}\right)}e^{-\frac{2y_0^2}{w^2\left(\frac{x_0}{\cos\alpha}\right)}} \tag{38}$$

And with the relativistic correction from equation (15) we know that the $2+2\cos(2kx')$ term will end up as $2+2\rho\left(\theta,\beta_e'\right)\cos(2kx')$ but from equation (18) the two constant terms will just stay $\left(\frac{\Gamma_-}{\Gamma_+}\right)^2 + \left(\frac{\Gamma_+}{\Gamma_-}\right)^2$:

$$\begin{aligned}\phi(x_0,y_0) &= \overbrace{\phi_{\text{red term}}(x_0,y_0)+\phi_{\text{blue term}}(x_0,y_0)}^{\text{Do not appear in a ring cavity}}+\phi_{\text{green term}}(x_0,y_0)\\ &= \phi_0(x_0,y_0)\cdot\left[\left(\frac{\Gamma_-}{\Gamma_+}\right)^2+\left(\frac{\Gamma_+}{\Gamma_-}\right)^2+2+2\rho(\theta,\beta_e')\cos(2kx')\right]\end{aligned}$$

Remark 4. I was searching for some time where did we use the smart choice of the cavity tilt α, that appears in The relation between $\beta_{\text{electron}}, \beta_{\text{lattice}}, \alpha$ The answer is here on the last equation (38). Why? This equation assumes the electron propagates in the z direction (in particular, it comes from an integration over $t, z(t)$ where $z(t) = z_0 + \beta ct$).

We choose α such that in this frame this is indeed the path of the electron. Looking at the expression for the wave function from equation (32), it is not obvious that this is the direction of flow of the electron. However, examining a single wave-packet (instead of a full "pencil"), and using the derivation from The relation between $\beta_{\text{electron}}, \beta_{\text{lattice}}, \alpha, \lambda_1, \lambda_2$:, we see that a single packet of electron that moved in the z direction in the microscope coordinates, would move in the tilted-moving lattice also in the tilted $\hat{z}$ direction.

Adding it as a phase to the wave function ψ' from equation (e_32), we get:

$$\psi'(x',y',z'=0,ct'=0) \overset{(e\_32)}{=} \gamma\psi_{x_0,y_0}\begin{pmatrix}\gamma\cos\alpha(x'+\beta ct')\\ y'\\ \gamma x'\sin\alpha+\gamma\beta ct'\sin\alpha\\ \gamma\beta x'+\gamma ct'\end{pmatrix}\cdot e^{i\phi_0\left(\frac{x_0}{\cos\theta},y_0\right)\left(2+2\rho(\theta,\beta_e')\cos(2kx')+\underbrace{\left(\frac{\Gamma_-}{\Gamma_+}\right)^2+\left(\frac{\Gamma_+}{\Gamma_-}\right)^2}_{\text{Not in ring cavity}}\right)}$$

Going back to the lab's frame:

Doing the Lorentz transform and rotation back, we get:

$$x' \to \gamma(x\cos\alpha + z\sin\alpha) - \gamma\beta ct \Rightarrow$$

$$\cos(2\varphi(x')) = \cos(2\varphi(\gamma(x\cos\alpha+z\sin\alpha)-\gamma\beta ct))$$

And so after transforming back to the lab's frame the phase will be:

$$\overbrace{\gamma\psi_{x_0,y_0}\begin{pmatrix}\gamma x'\cos\alpha - z'\sin\alpha+\gamma\beta ct'\cos\alpha\\ y'\\ \gamma x'\sin\alpha+z'\cos\alpha+\gamma\beta ct'\sin\alpha\\ \gamma\beta x'+\gamma ct'\end{pmatrix}\cdot e^{i\phi_0\left(\frac{x_0}{\cos\alpha},y_0\right)\left(2+2\rho(\theta,\beta_e')\cos(2kx')+\left(\frac{\Gamma_+}{\Gamma_-}\right)^2+\left(\frac{\Gamma_-}{\Gamma_+}\right)^2\right)}}^{\text{Lattice's frame}} \xrightarrow[\text{Lorentz transform and rotation}]{}$$

$$\underbrace{\psi_{x_0,y_0}\begin{pmatrix}x\\ y\\ z\\ t\end{pmatrix}\cdot e^{i\phi_0\left(\frac{x_0}{\cos\alpha},y_0\right)\left(2+2\rho(\theta,\beta_e')\cos(2k(\gamma(x\cos\alpha+z\sin\alpha)-\gamma\beta ct))+\left(\frac{\Gamma_+}{\Gamma_-}\right)^2+\left(\frac{\Gamma_-}{\Gamma_+}\right)^2\right)}}_{\text{Lab's frame}}$$

Approximating again the wave function to be non-zero only at around $x = x_0$, we can substitute $x = x_0$ in the phase and get:

$$\psi_{x_0,y_0}\begin{pmatrix}x\\y\\z\\t\end{pmatrix}\cdot e^{i\phi_0\left(\frac{x_0}{\cos\alpha},y_0\right)\left(2+2\rho\left(\theta,\beta'_e\right)\cos(2\gamma k_l[(x_0\cos\alpha+z\sin\alpha)-\beta ct])+\underbrace{\left(\frac{\Gamma_+}{\Gamma_-}\right)^2+\left(\frac{\Gamma_-}{\Gamma_+}\right)^2}_{\text{Not in ring cavity}}\right)} \tag{39}$$

And the middle band will be:

$$e^{i\phi_0\left(\frac{x_0}{\cos\alpha},y_0\right)(\cos(2\gamma k_l[(x_0\cos\alpha+z\sin\alpha)-\beta ct])+\text{Const})} = \underbrace{e^{i\phi_0\left(\frac{x_0}{\cos\alpha},y_0\right)\left(2+\left(\frac{\Gamma_+}{\Gamma_-}\right)^2+\left(\frac{\Gamma_-}{\Gamma_+}\right)^2\right)}}_{\text{No t dependance}}\cdot e^{i\rho\left(\theta,\beta'_e\right)\phi_0\left(\frac{x_0}{\cos\alpha},y_0\right)\left(2\sin\left(2\gamma k_l\beta c\left(-t+\frac{\pi}{2\gamma k_l\beta c}\right)+2\gamma k_l(x_0\cos\alpha+z\sin\alpha\right.\right.}$$

Using the identity:

$$e^{i[A_0+A\sin(\omega t)]} = e^{iA_0}\cdot\sum_q J_q(A)\,e^{iq\omega t}$$

We get:

$$= e^{i\phi_0\left(\frac{x_0}{\cos\alpha},y_0\right)\left(2+\underbrace{\left(\frac{\Gamma_+}{\Gamma_-}\right)^2+\left(\frac{\Gamma_-}{\Gamma_+}\right)^2}_{\text{Not in ring cavity}}\right)}\cdot\sum_{q=-\infty}^{\infty}J_q\left(\rho\left(\theta,\beta'_e\right)\phi_0\left(\frac{x_0}{\cos\alpha},y_0\right)\right)e^{iq\left(2\gamma k_l\beta c\left(-t+\frac{\pi}{4\gamma k_l\beta c}\right)+2\gamma k_l(x_0\cos\alpha+z\sin\alpha)\right)} \Rightarrow$$

And after the energy filter that will keep only the 0'th order:

$$\psi_{x_0,y_0}\begin{pmatrix}x\\y\\z\\t\end{pmatrix}\xrightarrow{\text{Passing through the cavity}}\psi_{x_0,y_0}\begin{pmatrix}x\\y\\z\\t\end{pmatrix}\cdot e^{i\phi_0\left(\frac{x_0}{\cos\alpha},y_0\right)\left(2+\underbrace{\left(\frac{\Gamma_+}{\Gamma_-}\right)^2+\left(\frac{\Gamma_-}{\Gamma_+}\right)^2}_{\text{Not in ring cavity}}\right)}\cdot J_0\left(\rho\left(\theta,\beta'_e\right)\phi_0\left(\frac{x_0}{\cos\alpha},y_0\right)\right) \tag{e\_35}$$

And for a ring cavity:

$$\psi_{x_0,y_0}\begin{pmatrix}x\\y\\z\\t\end{pmatrix}\xrightarrow{\text{Passing through the cavity}}\psi_{x_0,y_0}\begin{pmatrix}x\\y\\z\\t\end{pmatrix}\cdot e^{i2\phi_0\left(\frac{x_0}{\cos\alpha},y_0\right)}\cdot J_0\left(\rho\left(\theta,\beta'_e\right)\phi_0\left(\frac{x_0}{\cos\alpha},y_0\right)\right) \tag{e\_36}$$

for

$$\phi_0\left(x'=\frac{x_0}{\cos\alpha},y'=y_0\right)=\frac{e^2}{\hbar 4mc\beta_e\gamma_e}\sqrt{\frac{\pi}{2}}\frac{E_0^2w_0^2}{w\left(\frac{x_0}{\cos\alpha}\right)}e^{-\frac{2y_0^2}{w^2\left(\frac{x_0}{\cos\alpha}\right)}} \tag{e\_11}$$

Side calculations

The integral over the intensity envelope of the gaussian beam:

The intensity envelope:

$$I\left(x,y,z\right)=E_0^2\frac{w_0^2}{w^2\left(x\right)}e^{-\frac{2\left(y^2+z^2\right)}{w^2\left(x\right)}}=\frac{\pi}{2}E_0^2w_0^2\cdot\frac{1}{\sqrt{2\pi}\cdot\frac{1}{2}w\left(x\right)}e^{-\frac{z^2}{2\left(\frac{w\left(x\right)}{2}\right)^2}}\cdot\frac{1}{\sqrt{2\pi}\cdot\frac{1}{2}w\left(x\right)}e^{-\frac{y^2}{2\left(\frac{w\left(x\right)}{2}\right)^2}}$$

The integral over the z axis will be:

$$\int\limits_{-\infty}^{\infty}\frac{\pi}{2}E_0^2w_0^2\cdot\frac{1}{\sqrt{2\pi}\cdot\frac{1}{2}w\left(x\right)}e^{-\frac{y^2}{2\left(\frac{w\left(x\right)}{2}\right)^2}}\cdot\frac{1}{\sqrt{2\pi}\cdot\frac{1}{2}w\left(x\right)}e^{-\frac{z^2}{2\left(\frac{w\left(x\right)}{2}\right)^2}}dz=\frac{\pi}{2}E_0^2w_0^2\frac{1}{\sqrt{2\pi}\cdot\frac{1}{2}w\left(x\right)}e^{-\frac{y^2}{2\left(\frac{w\left(x\right)}{2}\right)^2}}\underbrace{\cancel{\int\limits_{-\infty}^{\infty}\cdot\frac{1}{\sqrt{2\pi}\cdot\frac{1}{2}w\left(x\right)}e^{-\frac{z^2}{2\left(\frac{w\left(x\right)}{2}\right)^2}}\cdot dz}}_{1}=$$

$$\begin{aligned}\int\limits_{z}I\left(y,z\right)dz&=\frac{\pi}{2}E_0^2w_0^2\frac{1}{\sqrt{2\pi}\cdot\frac{1}{2}w\left(x\right)}e^{-\frac{y^2}{2\left(\frac{w\left(x\right)}{2}\right)^2}}\\&=\sqrt{\frac{\pi}{2}}\frac{E_0^2w_0^2}{w\left(x\right)}e^{-\frac{2y^2}{w^2\left(x\right)}}\end{aligned}$$

Adding the constants of the ponderomotive potential from ponderomotive potential article for the final phase (and also adding $\frac{1}{\omega^2}$ factor because ϕ is integration of A^2 which is $\frac{E^2}{\omega^2}$):

$$\boxed{\phi_0\left(x,y\right)=\frac{e^2}{\hbar4mc\beta\gamma\omega^2}\sqrt{\frac{\pi}{2}}\frac{E_0^2w_0^2}{w\left(x\right)}e^{-\frac{2y^2}{w^2\left(x\right)}}}\qquad\text{(e\_1)}$$

Using the relation $\boldsymbol{p}=\gamma mv=\gamma m\beta c=\hbar k$ we can also write:

$$\frac{e^2}{\hbar4mc\beta\gamma}=\frac{e^2}{4\hbar^2k_{\text{electron}}}$$

Which is more like Kirkland’s notation.

Relation between wavelengths and numerical aperture in a fixed cavity

Suppose we have two wavelengths, λ_1 and λ_2 in a symmetric cavity, with radius of curvature R, length L, and unconcentricity $u=R-\frac{L}{2}$.

We know that:

$$
\begin{aligned}
u &\equiv R - \frac{L}{2} \\
&= \frac{L}{2}\left(\cancel{1} + \frac{4z^2}{L^2}\right) - \cancel{\frac{L}{2}} \\
&= \frac{2z_R^2}{L} \\
&= \frac{2\pi^2 \cdot w_0^2}{\lambda^2 \cdot L} \\
&= \frac{2\pi^2 \cdot w_0^2}{\lambda^2 \cdot L} \cdot \frac{\pi^2}{\lambda^2} \cdot \frac{\lambda^2}{\pi^2} \\
&= \frac{2\pi^4 \cdot w_0^4}{\lambda^2 \cdot L} \cdot \frac{\lambda^2}{\pi^2} \\
&= \frac{2\lambda^2}{\mathrm{NA}^4 \cdot L \cdot \pi^2} \quad \backslash \cdot \frac{\mathrm{NA}^4}{u} \\
\\
\mathrm{NA}^4 &= \frac{2\lambda^2}{L\pi^2 u} \Rightarrow \\
\\
\mathrm{NA}^4 &\propto \lambda^2 \quad \backslash \sqrt[4]{} \\
\\
\mathrm{NA} &\propto \sqrt{\lambda}
\end{aligned}
\tag{40}
$$

$$
\mathrm{NA}_2 = \mathrm{NA}_1 \cdot \sqrt{\frac{\lambda_2}{\lambda_1}} \tag{NA ratios}
$$

Thus if we have $\lambda_1 = 1064\text{nm}$ and $\lambda_2 = 532\text{nm}$, and $\mathrm{NA}_1 = 0.1$ then $\mathrm{NA}_2 = 0.1 \cdot \frac{1}{\sqrt{2}}$

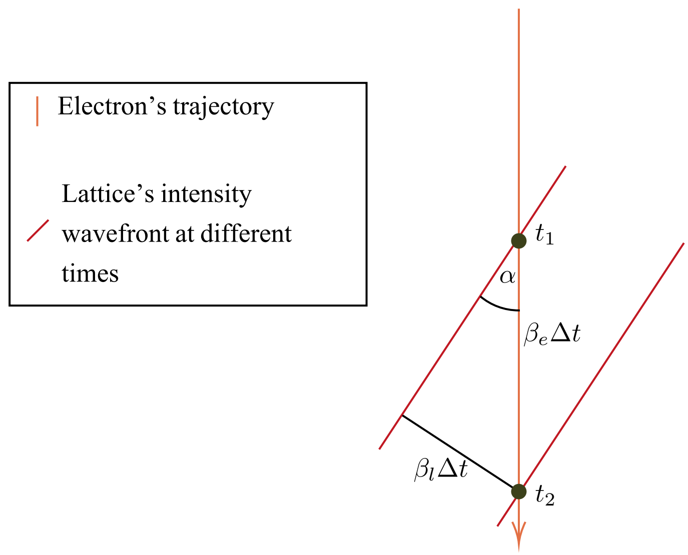
The relation between $\beta_{\textbf{electron}}, \beta_{\textbf{lattice}}, \alpha, \lambda_1, \lambda_2$:

In equation (36) and in the paragraph before, we demand that

We saw that the velocity of the standing lattice, calculated in (25) is $\beta_{\text{lattice}} = \frac{1-\frac{\omega_1}{\omega_2}}{1+\frac{\omega_1}{\omega_2}} = \frac{1-\frac{\lambda_2}{\lambda_1}}{1+\frac{\lambda_2}{\lambda_1}} = \frac{\lambda_1-\lambda_2}{\lambda_1+\lambda_2}$.

In the next figure we can see that for the electron to experience a constant intensity α satisfies the relation:

$$
\sin\alpha = \frac{\beta_{\text{lattice}}}{\beta_{\text{electron}}} \tag{41}
$$



Numerics

In this part I will present small derivations that are required mainly when one comes to implement the calculations in real code, such as complexity and the derivation of small mathematical expressions.

The integral to calculate G:

Let us calculate the exact expression we want to calculate when calculating the gauge function G:
Assume a gaussian beam's electromagnetic potential that is aligned with $\tilde{x}$ axis is $\tilde{A}(x, y, z, t)$.
If $\tilde{x}$ is tilted by an angle α with the cavity (such that the positive $\tilde{x}$ is tilted towards the positive $\tilde{z}$ direction) then the potential in the lab's frame will be:

$$\boldsymbol{A}(x, y, z, t) = \begin{pmatrix} \tilde{A}_x \cos\alpha + \tilde{A}_z \sin\alpha \\ A_y \\ -\tilde{A}_x \sin\alpha + \tilde{A}_z \cos\alpha \end{pmatrix} (x\cos\alpha - z\sin\alpha, y, x\sin\alpha + z\cos\alpha) \tag{e\_37}$$

Therefore the integrand to calculate G will be:

$$\text{Integrand}(x, y, z, t, T) = A_z(x, y, z - \beta c(t-T), T) = \left[-\tilde{A}_x \sin\alpha + A_z \cos\alpha\right] \underbrace{(x\cos\alpha - (z - \beta c(t-T))\sin\alpha, y, x\sin\alpha + (z - \beta c(t-T)}_{\text{This is the argument of the function A, not a multiplication factor}}$$

Since we choose a gauge where $\tilde{A}_x = 0$ then we are left with:

$$\cos\alpha \cdot \tilde{A}_z(x\cos\alpha - (z - \beta c(t-T))\sin\alpha, y, x\sin\alpha + (z - \beta c(t-T))\cos\alpha, T) =$$

integral over T will be:

$$G = \cos\alpha\beta c \int_{-\infty}^{T} \tilde{A}_z(x\cos\alpha - (z - \beta c(t-T))\sin\alpha, y, x\sin\alpha + (z - \beta c(t-T))\cos\alpha, T)\, dT \tag{42}$$

$$Z = z - \beta c\,(t - T) \qquad \frac{1}{\beta c} dZ = dT \qquad T = t \iff Z = z$$

$$T = \frac{Z - z}{\beta c} + t$$

And so the integral becomes:

$$\boxed{G = \cos\alpha \int_{-\infty}^{z} \tilde{A}_z \left(x\cos\alpha - Z\sin\alpha, y, x\sin\alpha + Z\cos\alpha, \frac{Z - z}{\beta c} + t \right) dZ} \tag{e\_25}$$

Which is more comfortable in my eyes then integration over t.
If you search for the polarization term $\cos\theta$ - it is inside $\tilde{A}_z$.
This function appears in

The discrete interval dz required for an accurate integration of $A_z\,(z, t\,(z))$:

Let us calculate the typical rate of change (as a function of z) for a standing wave A of the form:

$$A\,(z,t) = \cos\,(kx)\cos\,(\omega t)$$

And for and electron passing in it in a velocity βc
The time as a function of z, $t\,(z)$ is given by $t\,(z) = \frac{z}{\beta c}$.
And so $A\,(z, t\,(z))$ is:

$$\begin{aligned} A\,(z, t\,(z)) &= \cos\,(kz)\cos\,(\omega t\,(z)) \\ &= \cos\,(kz)\cos\left(\omega \frac{z}{\beta c}\right) \\ &= \cos\,(kz)\cos\left(\frac{k}{\beta} z\right) \\ &= \frac{1}{2}\left[\cos\left(\left(1 + \frac{1}{\beta}\right) kz\right) + \cos\left(\left(1 - \frac{1}{\beta}\right) kz\right)\right] \end{aligned}$$

And so we see that for a monochromatic light the greatest wavelength experienced by the electron is $k_{\text{eff}} = \left(1 + \frac{1}{\beta}\right) k \Rightarrow \lambda_{\text{eff}} = \frac{\lambda}{1 + \frac{1}{\beta}}$.
Given this, we know dz should be around:

$$dz < \lambda_{\text{eff}} < \frac{\lambda}{1 + \frac{1}{\beta}} \tag{e\_40}$$

in practice, playing with it, I found that $dz = \frac{\lambda}{3\left(1 + \frac{1}{\beta}\right)}$ is enough, and smaller dz does not produce much different results.

The derivative of G with respect to z:

The definition of G:

$$G(z,t) = \int_{-\infty}^{z} A_z\left(Z, t - \frac{z-Z}{\beta c}\right) dZ$$

We have this blue term in the integral of the phase:

$$\int_{-\infty}^{\infty} \left(A_z(z,t(z)) - \nabla G(z,t(z))\right)^2 \ldots dz$$

Where $t(z) = \underbrace{t_0}_{t \text{ at } z=0} + \frac{z}{\beta c}$

note how the z'th component of the gradient is:

$$\lim_{dz \to 0} \frac{G(z+dz, t(z)) - G(z,t(z))}{dz}$$

And not:

$$\lim_{dz \to 0} \frac{G(z+dz, t(z+dz)) - G(z,t(z))}{dz}$$

Therefore:

$$\begin{aligned}
G(z+dz,t(z)) &= \int_{-\infty}^{z+dz} A_z\left(Z, t(z) - \frac{z+dz-Z}{\beta c}\right) dZ \\
&= \int_{-\infty}^{z+dz} A_z\left(Z, \left(t(z) - \frac{dz}{\beta c}\right) - \frac{z-Z}{\beta c}\right) \\
&= \int_{-\infty}^{z} A_z\left(Z, \left(t(z) - \frac{dz}{\beta c}\right) - \frac{z-Z}{\beta c}\right) dZ + \underbrace{\int_{z}^{z+dz} A_z\left(Z, \left(t(z) - \frac{dz}{\beta c}\right) - \frac{z-Z}{\beta c}\right)}_{\xrightarrow[dz\to 0]{} A_z\left(z, \left(t(z) - \frac{dz}{\beta c}\right) - \cancel{\frac{z-z}{\beta c}}\right) \cdot dz} \\
&= \int_{-\infty}^{z} A_z\left(Z, \left(t(z) - \frac{dz}{\beta c}\right) - \frac{z-Z}{\beta c}\right) dZ + A_z\left(z, \left(t(z) - \frac{dz}{\beta c}\right)\right) \cdot dz \\
&= G\left(z, t(z) - \frac{dz}{\beta c}\right) + A_z\left(z, \left(t(z) - \frac{dz}{\beta c}\right)\right) \cdot dz
\end{aligned} \tag{e_26}$$

And the derivative is:

$$\begin{aligned}
\frac{\partial G}{\partial z} &= \lim_{dz\to 0}\left[\frac{G\left(z,t\left(z\right)-\frac{dz}{\beta c}\right)+A_z\left(z,\left(t\left(z\right)-\frac{dz}{\beta c}\right)\right)\cdot dz-G\left(z,t\left(z\right)\right)}{dz}\right] \\
&= \lim_{dz\to 0}\left[\frac{G\left(z,t\left(z\right)-\frac{dz}{\beta c}\right)-G\left(z,t\left(z\right)\right)}{dz}\right]+A_z\left(z,\left(t\left(z\right)-\frac{dz}{\beta c}\right)\right) \\
\frac{\partial G}{\partial z} &= -\frac{1}{\beta c}\frac{\partial G}{\partial t}+A_z\left(z,\left(t\left(z\right)-\frac{dz}{\beta c}\right)\right)
\end{aligned} \tag{43}$$

Addressing the complexity problem:

Problem 5. We note that in the total phase acquired term (12) we have $\int_z\ldots\partial_x G\left(x,y,z,t\right)$ where G itself is an integral over $\int_Z A_z\left(x,y,Z,\frac{Z-z}{\beta c}+\right.$ It means that we have an integral inside a derivative inside an integral - which we need to calculate to a grid of $n_x\times n_y$ for many t's. since every integral calls it's integrand many times - it means that the inner most function $A_z\left(x,y,z,t\right)$ might be called (say, 1000 times per integral and $(1+3)$ times for gradient): $1000\cdot 1000\cdot n_x\cdot n_y\cdot n_t\cdot 4$ which is a lot (for $n_x\times n_y=1000\times 1000$).

Solution 6. However, we notice that when calculating G, the term $A_z\left(x,y,z_1,t_1\right)$ is calculated many times: once in $G\left(x,y,z_2,t_1-\frac{z_1-z_2}{\beta c}\right)$ (because when we run the integral over Z and we get to $Z=z_1$ then the integrand is

$$A_z\left(x,y,Z,t-\frac{z-Z}{\beta c}\right)=A_z\left(x,y,z_1,t_1-\cancel{\frac{z_1-z_2}{\beta c}}-\cancel{\frac{z_2-z_1}{\beta c}}\right)=A_z\left(x,y,z_1,t_1\right)$$

) and once again in $G\left(x,y,z_3,t_1-\frac{z_1-z_3}{\beta c}\right)$, because also then for $Z=z_1$ we get::

$$A_z\left(x,y,Z,t-\frac{z-Z}{\beta c}\right)=A_z\left(x,y,z_1,t_1-\cancel{\frac{z_1-z_3}{\beta c}}-\cancel{\frac{z_3-z_1}{\beta c}}\right)=A_z\left(x,y,z_1,t_1\right)$$

So, in principle, we can calculate it once, and then use the same value repetitively for all integrals of the form $G\left(x,y,z,t-\frac{z_1-z}{\beta c}\right)$. We can initiate a large array of $A_z\left(x,y,z_j,t_k\right)$, and estimate the value of G at:

$$G\left(x,y,z_j,t_k\right)=\sum_{i=0}^{j}A_z\left(x,y,z_i,t_k-\frac{z_j-\overbrace{z_i}^{\equiv\text{"}Z\text{"}}}{\beta c}\right)$$

Defining a time-dimensional position $\tilde{z}=\frac{z}{\beta c}$, we can rewrite it as:

$$G\left(x,y,z_j,t_k\right)=\sum_{i=0}^{j}A_z\left(x,y,z_i,t_k-\left(\tilde{z}_j-\tilde{z}_i\right)\right)=\sum_{i=0}^{j}A_z\left(x,y,z_i,\tilde{z}_i+t_k-\tilde{z}_j\right)$$

If we choose the spacing in z of the grid to be dz and the spacing in t of t to be $\beta c\cdot dz$ then this translates to:

$$G\left(x,y,z_j,t_k\right)=\sum_{i=0}^{j}A_z\left(x,y,z_i,t_{i+(k-j)}\right)$$

In matrix form, this equals summing the $(k-j)$-offset diagonal from the first row and until the j'th row:
For example, in $j=4$ and $k=6$ (and in particular $(k-j=2)$) this is equal to summing all the blue elements:

$$G(z_3, t_4), G(z_4 t_6) \Rightarrow$$

$$\begin{array}{c|cccccccc}
 & t_0 & t_1 & t_2 & t_3 & t_4 & t_5 & t_6 & t_7 \\
\hline
z_0 & A_{00} & A_{01} & A_{02} & A_{03} & & & & \\
z_1 & A_{10} & A_{11} & A_{12} & A_{13} & A_{14} & & & \\
z_2 & & & & A_{23} & A_{24} & A_{25} & & \\
z_3 & & & & & A_{34} & A_{35} & A_{36} & \\
z_4 & & & & & & A_{45} & A_{46} & A_{47} \\
z_5 & & & & & & & A_{56} & A_{57} \\
z_6 & & & & & & & & \\
z_7 & & & & & & & &
\end{array}$$

If the last array is achieved by evaluation A on the arrays:

$$A(Z,T)\text{for:} \qquad Z = \begin{array}{cccc} z_0 & z_0 & z_0 & z_0 \\ z_1 & z_1 & z_1 & z_1 \\ z_2 & z_2 & z_2 & z_2 \\ z_3 & z_3 & z_3 & z_3 \end{array} \qquad T = \begin{array}{cccc} t_0 & t_1 & t_2 & t_3 \\ t_0 & t_1 & t_2 & t_3 \\ t_0 & t_1 & t_2 & t_3 \\ t_0 & t_1 & t_2 & t_3 \end{array}$$

Therefore, we can "skew" this A array by replacing the T matrix like so:

$$T \to T' = T + \frac{Z}{\beta c}$$

$$\begin{array}{cccc} t_0 & t_1 & t_2 & t_3 \\ t_0 & t_1 & t_2 & t_3 \\ t_0 & t_1 & t_2 & t_3 \\ t_0 & t_1 & t_2 & t_3 \end{array} \to \begin{array}{cccc} t_0 + \frac{z_0}{\beta c} & t_1 + \frac{z_0}{\beta c} & t_2 + \frac{z_0}{\beta c} & t_3 + \frac{z_0}{\beta c} \\ t_0 + \frac{z_1}{\beta c} & t_1 + \frac{z_1}{\beta c} & t_2 + \frac{z_1}{\beta c} & t_3 + \frac{z_1}{\beta c} \\ t_0 + \frac{z_2}{\beta c} & t_1 + \frac{z_2}{\beta c} & t_2 + \frac{z_2}{\beta c} & t_3 + \frac{z_2}{\beta c} \\ t_0 + \frac{z_3}{\beta c} & t_1 + \frac{z_3}{\beta c} & t_2 + \frac{z_3}{\beta c} & t_3 + \frac{z_3}{\beta c} \end{array} = \begin{array}{cccc} t_0 & t_1 & t_2 & t_3 \\ t_0 & t_2 & t_3 & t_4 \\ t_0 & t_3 & t_4 & t_5 \\ t_0 & t_4 & t_5 & t_6 \end{array}$$

While keeping the z matrix as:

$$Z \to Z$$

Then evaluating A on the new array T like so:

$$A(Z, T')$$

will result in a skewed A like so:

$$A\left(Z,T'\right)=\begin{array}{c|ccccc} & t_0+\frac{Z}{\beta c} & t_1+\frac{Z}{\beta c} & t_2+\frac{Z}{\beta c} & t_3+\frac{Z}{\beta c} & t_4+\frac{Z}{\beta c}\\ \hline z_0 & A_{00} & A_{01} & A_{02} & A_{03} & \\ z_1 & A_{11} & A_{12} & A_{13} & A_{14} & \\ z_2 & A_{22} & A_{23} & A_{24} & A_{25} & \\ z_3 & A_{33} & A_{34} & A_{35} & A_{36} & \\ z_4 & A_{44} & A_{45} & A_{46} & A_{47} & \\ z_5 & & & & & \\ z_6 & & & & & \\ z_7 & & & & & \end{array}$$

And now the value of G can be achieved by simply taking the cumulative integral:

$$G=\texttt{np.cumsum(A(Z,T'),axis=0)}=\begin{array}{c|ccccc} & t_0+\frac{Z}{\beta c} & t_1+\frac{Z}{\beta c} & t_2+\frac{Z}{\beta c} & t_3+\frac{Z}{\beta c} & t_4+\frac{Z}{\beta c}\\ \hline z_0 & G_{00} & G_{01} & G_{02} & G_{03} & \\ z_1 & G_{11} & G_{12} & G_{13} & G_{14} & \\ z_2 & G_{22} & G_{23} & G_{24} & G_{25} & \\ z_3 & G_{33} & G_{34} & G_{35} & G_{36} & \\ z_4 & G_{44} & G_{45} & G_{46} & G_{47} & \\ z_5 & & & & & \\ z_6 & & & & & \\ z_7 & & & & & \end{array}$$

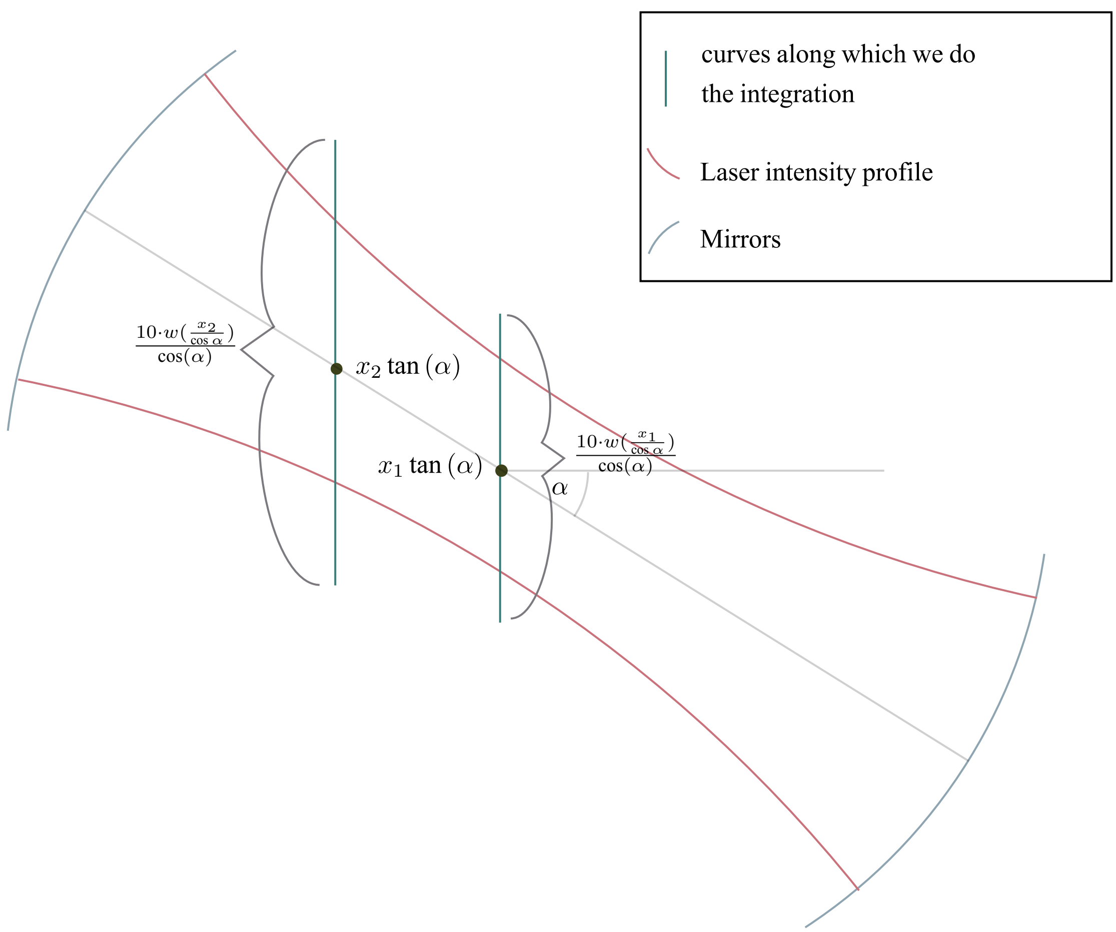
In particular the `np.cumsum` command will give us in each column the values of $G\left(z,t\left(z\right)\right)$ as required for the final integral of ϕ.

Shift and scale of the z axis along which we integrate:

Since the laser has a gaussian intensity profile, we wish to integrate mostly around the center of the intensity. Since the width of the intensity varies for different x values, we wish to adjust the interval $[z_{\text{min}},z_{\text{max}}]$ for different x values.
Additionaly, we want to shift is to always be centered around the center of the spot, which is not always at $z=0$, since the cavity is tilted. The intensity width goes like $w\left(x\right)$ in the cavity's axis, and therefore in the lab's axis (with respect to which the cavity's axes are rotated by an angle α), the width will be approximately $\frac{w(x)}{\cos(\alpha)}$.
If we wish to integrate along $\pm5\sigma$ of the intensity, then we get in total that the formula is (and a drawing to emphasize it right afterwards):

$$[z_{\text{min}},z_{\text{max}}]\left(x_{\text{lab}}\right)=\left[z_{\text{center}}\left(x\right)-5w_{\text{lab}}\left(x\right),z_{\text{center}}\left(x\right)+5w_{\text{lab}}\left(x\right)\right]=\left[x\tan\left(\alpha\right)-5\frac{w\left(\frac{x}{\cos\alpha}\right)}{\cos\alpha},x\tan\left(\alpha\right)+5\frac{w\left(\frac{x}{\cos\alpha}\right)}{\cos\alpha}\right]$$

Those claculation take part in `CavityNumericalPropagator.generate_z_vector` and `CavityNumericalPropagator.generate_coordinates_lattice`.



curves along which we do
the integration
Laser intensity profile
Mirrors
$\frac{10 \cdot w(\frac{x_2}{\cos\alpha})}{\cos(\alpha)}$
$x_2 \tan(\alpha)$
$x_1 \tan(\alpha)$
$\frac{10 \cdot w(\frac{x_1}{\cos\alpha})}{\cos(\alpha)}$
α

Calculating the zero'th energy amplitude with just 3 points in time:

We have a phase of the form:

$$\phi(x,y,t) = \phi_{\text{const}}(x,y) + C(x,y) \cdot \sin(\Delta\omega t + \varphi(x,y))$$

Since x, y do not play a role in this equation, and we are going to do the calculation to all pairs of x, y separately, then we can omit them for abbreviation: $C(x,y) \to C$, $\phi_{\text{const}}(x,y) \to \phi_{\text{const}}$

$$\phi(t) = \phi_{\text{const}} + C \cdot \sin(\Delta\omega t + \varphi)$$

We can evaluate the function in three different points in time:

$$\begin{aligned} t_0 &= 0 \\ t_1 &= \frac{\pi}{2\Delta\omega} \\ t_2 &= \frac{\pi}{\Delta\omega} \end{aligned}$$

So that:

$$\begin{aligned} \phi(t_0) &= \phi_{\text{const}} + C\sin(0+\varphi) &&= \phi_{\text{const}} + C\sin(\varphi) \\ \phi(t_1) &= \phi_{\text{const}} + C\sin\left(\frac{\pi}{2\cancel{\Delta\omega}}\cancel{\Delta\omega} + \varphi\right) &&= \phi_{\text{const}} + C\cos(\varphi) \\ \phi(t_2) &= \phi_{\text{const}} + C\sin\left(\frac{\pi}{\cancel{\Delta\omega}}\cancel{\Delta\omega} + \varphi\right) &&= \phi_{\text{const}} - C\sin(\varphi) \end{aligned}$$

$$\begin{aligned} \phi(t_0) &= \phi_{\text{const}} + \cancel{C\sin(\varphi)} \\ \phi(t_2) &= \phi_{\text{const}} - \cancel{C\sin(\varphi)} \ \backslash + \end{aligned}$$

$$\boxed{\phi_{\text{const}} = \frac{\phi(t_0) + \phi(t_2)}{2}}$$

$$\begin{aligned} \phi(t_0) &= \phi_{\text{const}} + C\sin(\varphi) \ \backslash - \phi_{\text{const}} \\ \phi(t_1) &= \phi_{\text{const}} + C\cos(\varphi) \ \backslash - \phi_{\text{const}} \end{aligned}$$

$$\begin{aligned} \phi(t_0) - \phi_{\text{const}} &= \cancel{C}\sin(\varphi) \\ \phi(t_1) - \phi_{\text{const}} &= \cancel{C}\cos(\varphi) \ \backslash : \end{aligned}$$

$$\tan\left(\varphi\right) = \frac{\phi\left(t_0\right) - \phi_{\text{const}}}{\phi\left(t_1\right) - \phi_{\text{const}}} \setminus \tan^{-1}$$

$$\boxed{\varphi = \tan^{-1}\left(\frac{\phi\left(t_0\right) - \phi_{\text{const}}}{\phi\left(t_1\right) - \phi_{\text{const}}}\right)}$$

And now if φ is very close to 0 or π we substitute it in the equation for $\phi\left(t_1\right)$:

$$\boxed{C = \underbrace{\frac{\phi\left(t_1\right) - \phi_{\text{const}}}{\cos\left(\varphi\right)}}_{\text{All are known by this point}}} \tag{e\_27}$$

And otherwise we can substitute it in the equation for$\phi\left(t_0\right)$ and get:

$$\boxed{C = \underbrace{\frac{\phi\left(t_0\right) - \phi_{\text{const}}}{\sin\left(\varphi\right)}}_{\text{All are known by this point}}}$$

Extracting C and ϕ_{const} of another amplitude from previous calculation

Suppose we once calculated ϕ_{const} and C from the previous section for some E_1, and now we wish to do the same calculation for the same setup but with a different E_1.
From equation (e_3) we see that the total phase shift is given by an expression of the form:

$$\phi\left(x, y, z, t_0\right) = \int \left|I\left(x, y, z, t_0\right)\right|^2 dx$$

Since this term is a sum of two lasers, each one contributing to the integrand an expression of the form $I_{1,2}\left(t_0, z\right) e^{i\omega_{1,2} t_0}$, we get:

$$\begin{aligned} \int \left|I\left(t_0, z\right)\right|^2 dz &= \int \left|I_1\left(t_0, z\right) e^{i\omega_1 t_0} + I_2\left(t_0, z\right) e^{i\omega_1 t_0}\right|^2 dz \\ &= \int \left|I_1^2\right|\left(t_0, z\right) + \left|I_2^2\right|\left(t_0, z\right) + 2\cos\left(\Delta\omega t_0\right) \cdot \left|I_1\left(t_0, z\right) I_2\left(t_0, z\right)\right| dz \end{aligned}$$

And if $I_{1,2}^2$ go like some E_0^2 (for both $_1$ and $_2$, as in (28)) so it is of the form:

$$I_i^2 = \hat{I}_i^2 \cdot E_0^2$$

then we can express the integral as:

$$\int \left|I\left(t_0 + \Delta t, z\right)\right|^2 dz = E_0^2 \int_z \left|I_1^2\right|\left(t_0, z\right) + \left|I_2^2\right|\left(t_0, z\right) dz + E_0^2 2\cos\left(\Delta\omega t_0\right) \int_z \left|I_1\left(t_0, z\right) I_2\left(t_0, z\right)\right| dz$$

And therefore we see that under increase of $E_0 \to E_0'$ we will have separate increase of:

$$\left\{\phi_{\text{const}}\left(E\right),C\left(E_{0}\right)\right\}\xrightarrow[E_{0}\rightarrow E_{0}']{}\left\{\phi_{\text{const}}\left(E_{0}\right)\cdot\left(\frac{E_{0}'}{E_{0}}\right)^{2},C\left(E_{0}\right)\cdot\left(\frac{E_{0}'}{E_{0}}\right)^{2}\right\} \tag{e_42}$$

Checking the difference between the extrapolated value according to (e_42) and the actual value achieved by calculating everything again, for amplitudes save in a `:.5g` accuracy, resulted in relative errors of around 10^{-4} (between the value calculated again and the value achieved by extrapolation).

Extracting the amplitude required for a given phase shift based on one sample

Suppose the phase shift given to the electron as a function of the amplitude is:

$$\phi=aE^{2}$$

For some unknown a.
If we evaluate the function once, we get:

$$\phi_{1}=aE_{1}^{2}\ \backslash:E_{1}^{2}$$

Isolating a, we get:

$$a=\frac{\phi_{1}}{E_{1}^{2}}$$

Now, we are interesting in the amplitude that will result in $\phi_{2}=\frac{\pi}{2}$ phase shift:

$$\frac{\pi}{2}=\phi_{2}=aE_{2}^{2}=\frac{\phi_{1}}{E_{1}^{2}}E_{2}^{2}\ \backslash:\frac{\phi_{1}}{E_{1}^{2}}$$

Isolate E_{2}:

$$\frac{E_{1}^{2}}{\phi_{1}}\phi_{2}=E_{2}^{2}\ \backslash\sqrt{}$$

$$\boxed{E_{2}=E_{1}\sqrt{\frac{\phi_{2}}{\phi_{1}}}} \tag{e_41}$$

Power to amplitude relation

If the gaussian beam has the shape:

$$E=E_{0}\cdot\frac{w_{0}}{w\left(x\right)}e^{-\frac{y^{2}+z^{2}}{w^{2}\left(x\right)}}\cdot e^{i\varphi\left(x,y,z\right)}$$

Then the intensity has the form:

$$I=E_{0}^{2}\cdot\frac{w_{0}^{2}}{w^{2}\left(x\right)}e^{-2\frac{y^{2}+z^{2}}{w^{2}\left(x\right)}}=E_{0}^{2}\cdot\frac{w_{0}^{2}}{w^{2}\left(x\right)}e^{-\frac{y^{2}+z^{2}}{2\left(\frac{w\left(x\right)}{2}\right)^{2}}}$$

And for $x = 0$ we have: (using the Intensity definition from wikipedia)

$$I(0,y,z) = \frac{c\varepsilon_0}{2} \cdot |E|^2 = \frac{c\varepsilon_0}{2} E_0^2 \frac{\cancel{w_0^2}}{\cancel{w_0^2}} e^{-\frac{y^2+z^2}{2\left(\frac{w_0}{2}\right)^2}} = \frac{c\varepsilon_0}{2} E_0^2 e^{-\frac{y^2+z^2}{2\left(\frac{w_0}{2}\right)^2}} = \left(\sqrt{2\pi} \cdot \frac{w_0}{\sqrt{2}}\right)^2 \frac{c\varepsilon_0}{2} E_0^2 \frac{1}{\sqrt{2\pi} \cdot \frac{w_0}{\sqrt{2}}} e^{-\frac{y^2}{2\left(\frac{w_0}{2}\right)^2}} \cdot \frac{1}{\sqrt{2\pi} \cdot \frac{w_0}{\sqrt{2}}} e^{-\frac{z^2}{2\left(\frac{w_0}{2}\right)^2}} =$$

Integrating over dy, dz we get:

$$\begin{aligned}
P &= \int\limits_{y,z} I(0,y,z)\, dydz \\
&= \int\limits_{y,z} \frac{c\varepsilon_0}{2} E_0^2 e^{-\frac{y^2+z^2}{\frac{1}{2} w_0^2}} dydz \\
&= \int\limits_{\theta} \int\limits_{r} \frac{c\varepsilon_0}{2} E_0^2 e^{-\frac{r^2}{\left(\frac{w_0}{\sqrt{2}}\right)^2}} drrd\theta \\
&= 2\pi \frac{c\varepsilon_0}{2} E_0^2 \int\limits_{\tilde{r}} re^{-\left(\frac{r}{\left(\frac{w_0}{\sqrt{2}}\right)}\right)^2} dr \\
&\overset{r \to \tilde{r} = \frac{r}{\left(\frac{w_0}{\sqrt{2}}\right)}}{=} 2\pi \left(\frac{w_0}{\sqrt{2}}\right)^2 \frac{c\varepsilon_0}{2} E_0^2 \int\limits_{\tilde{r}} \tilde{r} e^{-\tilde{r}^2} d\tilde{r} \\
&\overset{\tilde{r}^2 \to s, ds = \frac{1}{2}\tilde{r} d\tilde{r}}{=} 2\pi \left(\frac{w_0}{\sqrt{2}}\right)^2 \frac{c\varepsilon_0}{2} E_0^2 \int\limits_{s} \frac{1}{2} e^{-s} ds \\
&= 2\pi \left(\frac{w_0}{\sqrt{2}}\right)^2 \frac{c\varepsilon_0}{2} E_0^2 \left. \frac{1}{2} e^{-s} \right|_{\infty}^{0} \\
&= 2\pi \frac{w_0^2}{4} \frac{c\varepsilon_0}{2} E_0^2 \\
&= \\
P &= \frac{\pi c\varepsilon_0 w_0^2}{4} E_0^2
\end{aligned}$$

Or also:

$$P = \int\limits_{y,z} I(0,y,z)\, dydz = \int\limits_{y,z} \left(\sqrt{2\pi} \cdot \frac{w_0}{\sqrt{2}}\right)^2 \frac{c\varepsilon_0}{2} E_0^2 \frac{1}{\sqrt{2\pi} \cdot \frac{w_0}{2}} e^{-\frac{y^2}{2\left(\frac{w_0}{2}\right)^2}} \cdot \frac{1}{\sqrt{2\pi} \cdot \frac{w_0}{2}} e^{-\frac{z^2}{2\left(\frac{w_0}{2}\right)^2}} dydz =$$

$$= \left(\sqrt{2\pi} \cdot \frac{w_0}{2}\right)^2 E_0^2 \frac{c\varepsilon_0}{2} \underbrace{\int\limits_{y} \frac{1}{\sqrt{2\pi} \cdot \frac{w_0}{2}} e^{-\frac{y^2}{2\left(\frac{w_0}{2}\right)^2}} dy}_{=1} \cdot \underbrace{\int\limits_{z} \frac{1}{\sqrt{2\pi} \cdot \frac{w_0}{2}} e^{-\frac{z^2}{2\left(\frac{w_0}{2}\right)^2}} dz}_{=1} =$$

$$= \left(\sqrt{2\pi} \cdot \frac{w_0}{2}\right)^2 \frac{c\varepsilon_0}{2} E_0^2 =$$

$$P = \frac{\pi c\varepsilon_0 w_0^2}{4} E_0^2$$

Using $w_0 = \frac{\lambda}{\pi \cdot \mathrm{NA}}$, we can also write:

$$P = \frac{\pi c \varepsilon_0 w_0^2}{4} E_0^2 = P = \frac{\pi c \varepsilon_0}{4} \left(\frac{\lambda}{\pi \cdot \mathrm{NA}} \right)^2 E_0^2 = \frac{c \varepsilon_0 \lambda^2}{4\pi \cdot \mathrm{NA}^2} E_0^2$$

Or alternatively:

$$\boxed{E_0 = \frac{2\mathrm{NA}}{\lambda} \sqrt{\frac{\pi P}{c \varepsilon_0}}} \tag{e\_43}$$

And the two dimen