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Detecting communities in space-time graphs

The inflated dynamic Laplacian for temporal networks

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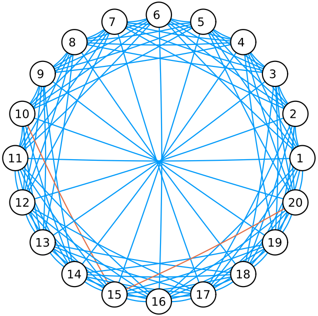
Overview

1. The problem: community detection in time-varying networks
2. Our approach: Spectral partitioning with $\mathcal{L}^{(a)}$
3. Spectrum of $\mathcal{L}^{(a)}$
4. Spectral partitioning algorithm and examples

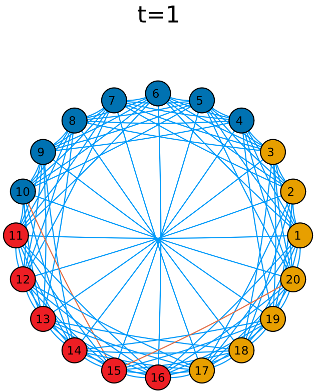
Community detection in temporal networks

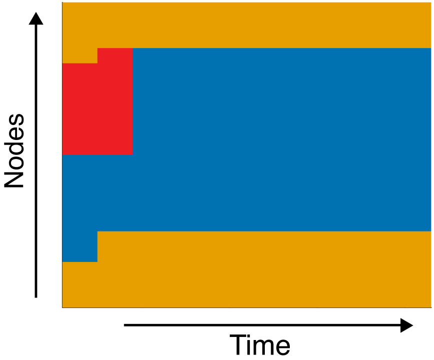
t=1



- Consider a temporal network $\boldsymbol{G} = (\boldsymbol{V}, \boldsymbol{E}, \boldsymbol{\mathcal{W}}^{(a)})$ with 20 nodes and edges that evolve in time $t = 1, \ldots, 21$.
- Network shows transition from 11-regular with no clear communities $\rightarrow$ two distinct d-regular communities.
- **Challenge:** How can we detect the single community computationally, without *a priori* information?

Community detection in temporal networks



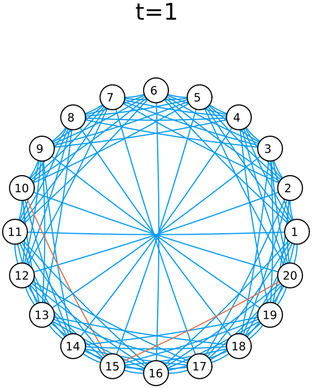


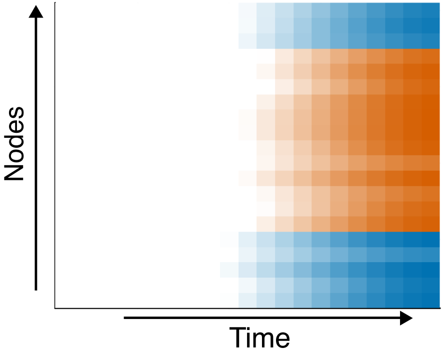
Slice-by-slice identification of communities using *Leiden* [1]

State-of-the-art allocates communities within regular graph!

[1] Traag, Waltman, & van Eck (2019). *Scientific Reports 9-1*.

Community detection in temporal networks





Community detection using the inflated dynamic Laplacian reveals better allocation!

Key idea: Construct the inflated dynamic Laplacian [2]

$$\Delta_{G_0,a}(F(t,x)) = a^2 \partial_{tt} F(t,x) + \Delta_{g_t} F(t,x)$$

for temporal networks, and analyze the eigenproblem to detect communities.

[2] Froyland & Koltai (2023). *Comm. Pure Appl. Math. 76*.

The inflated dynamic Laplacian on space-time graphs

- Consider a space-time graph $\boldsymbol{G} = (\boldsymbol{V}, \boldsymbol{E}, \boldsymbol{\mathcal{W}}^{(a)})$ where $\boldsymbol{E}$ is the edge set connecting vertices $\boldsymbol{V} \subset \mathbb{N} \times \mathbb{N}$.
- Define an edge-weight function $\boldsymbol{\mathcal{W}}^{(a)}$ such that $\boldsymbol{\mathcal{W}}^{(a)}_{(t,x),(s,y)}$ is the weight of the edge joining (t, x) and (s, y),
- The unnormalised graph *inflated dynamic Laplacian* $\mathcal{L}$ acts on functions $\boldsymbol{f} : \boldsymbol{V} \mapsto \mathbb{R}$,

$$\boldsymbol{\mathcal{L}}^{(a)} \boldsymbol{f}(t, x) = \sum_{(s,y) \in \boldsymbol{V}} \boldsymbol{\mathcal{W}}^{(a)}_{(t,x),(s,y)} (\boldsymbol{f}(t, x) - \boldsymbol{f}(s, y))$$

The inflated dynamic Laplacian on space-time graphs

- Analogous to the continuous setting, the spatial and temporal components of $\mathcal{W}^{(a)}$ and $\mathcal{L}^{(a)}$ can be split as follows,

$$\mathcal{W}^{(a)} = \mathcal{W}^{\mathrm{spat}} + a^2\mathcal{W}^{\mathrm{temp}}$$
$$\mathcal{L}^{(a)} = \mathcal{L}^{\mathrm{spat}} + a^2\mathcal{L}^{\mathrm{temp}},$$

- The splitting is achieved by defining

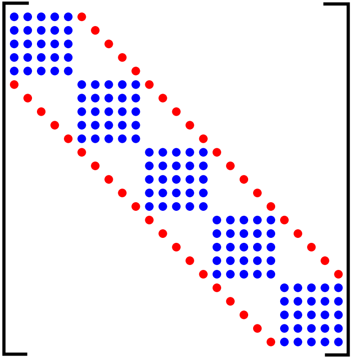
$$\mathcal{W}^{\mathrm{spat}}_{(t,x),(s,y)} = 0,\ t \neq s,$$
$$\mathcal{W}^{\mathrm{temp}}_{(t,x),(s,y)} = 0,\ x \neq y.$$

Matrix structure of $\boldsymbol{\mathcal{W}}^{(a)}$ and $\boldsymbol{\mathcal{L}}^{(a)}$

- **Spatial:** $\boldsymbol{\mathcal{W}}^{\text{spat}} = \bigoplus_t \boldsymbol{\mathcal{W}}^{(a)}_{(t,x),(t,y)}$
- **Temporal**: Given a $T \times T$ matrix W^{temp},

$$\boldsymbol{\mathcal{W}}^{\text{temp}} = W^{\text{temp}} \otimes I.$$

- Example: W^{temp} has nonzero terms on super- and subdiagonal only $\rightarrow$ temporal network!
- $\boldsymbol{\mathcal{L}}^{(a)}$ follows identically.



$$\boldsymbol{\mathcal{W}}^{(a)} = \boldsymbol{\mathcal{W}}^{\text{spat}} + a^2 \boldsymbol{\mathcal{W}}^{\text{temp}}$$

How does the inflated dynamic Laplacian $\mathcal{L}^{(a)}$ relate to graph partitioning? **Spectral partitioning and Cheeger constants.**

Balanced graph cuts and Cheeger constants

- Consider a pairwise disjoint space-time packing $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_K$ of $\boldsymbol{V}$. Cheeger constant h_K determines quality of partition,

$$\boldsymbol{h}_K = \min_{\boldsymbol{X}_1 \ldots \boldsymbol{X}_K} \max_k \frac{\mathrm{cut}(\boldsymbol{X}_k, \overline{\boldsymbol{X}_k})}{\min\{|\boldsymbol{X}_k|, |\overline{\boldsymbol{X}_k}|\}}.$$

- $\mathrm{cut}(X, Y) =$ sum of cut edge-weights between X and Y.
- Classical Cheeger inequality [1,2],

$$\boldsymbol{h}_2 \leq \sqrt{2\boldsymbol{\lambda}_2}$$
$$\boldsymbol{h}_K \leq 2^{3/2} K^2 \sqrt{\boldsymbol{\lambda}_K}$$

- $\boldsymbol{\lambda}_K$ is the K-th smallest eigenvalue of $\mathcal{L}$.
- **Extracting partition:** K-th eigenvector is good candidate.

[1] Chung (1996). Laplacians of graphs and cheeger's inequalities. *Combinatorics, Paul Erdos is Eighty*.

[2] Lee, Gharan, Trevisan (2014). *J. ACM.*

Cheeger constants are bounded by k-smallest eigenvalues $\boldsymbol{\lambda}_{k,a}$ of $\boldsymbol{\mathcal{L}}^{(a)}$.
How do $\boldsymbol{\lambda}_{k,a}$ behave?

Characterizing the spectrum of $\mathcal{L}^{(a)}$ as $a \to \infty$

Eigenvalues $\boldsymbol{\lambda}_{k,a}$ are either **(spatial)** $\boldsymbol{\lambda}_{k,a}^{\text{spat}}$ or **(temporal)** $\boldsymbol{\lambda}_{k,a}^{\text{temp}} = \mathcal{O}(a^2)$.

Theorem (Froyland, Koltai, K. '24)

Let N be the number of vertices per time fiber. Then for the first N spatial eigenvalues $\boldsymbol{\lambda}_{k,a}^{\text{spat}}$ we have,

$$\lim_{a\to\infty} \boldsymbol{\lambda}_{k,a}^{\text{spat}} = \lambda_k^D, \text{ where } \mathcal{L}^D f_k^D = \lambda_k^D f_k^D.$$

$\mathcal{L}^D$ is the dynamic Laplacian with respect to the adjacency $\mathcal{W}^D$,

$$\mathcal{W}_{x,y}^D = \frac{1}{T} \sum_t \boldsymbol{\mathcal{W}}_{(t,x),(t,y)}^{(a)}.$$

Algorithm for community detection in temporal networks

(1) Given a sequence $\{\mathcal{W}_t^{\mathrm{spat}}\}$, assemble $\boldsymbol{\mathcal{W}}^{(a)}, \boldsymbol{\mathcal{L}}^{(a)}$.

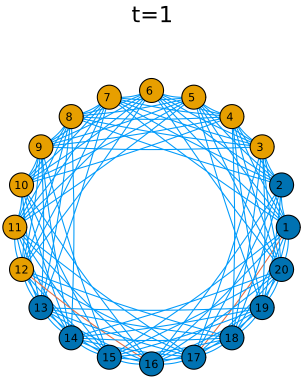
(2) Fix a at some critical value (usually where $\boldsymbol{\lambda}_{2,a} \sim \boldsymbol{\lambda}_{3,a}$).

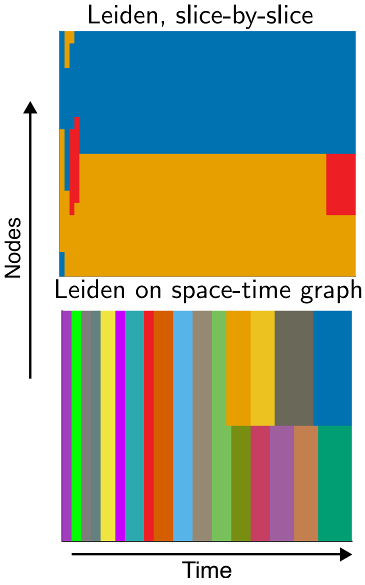
(3) Compute eigenvectors $\boldsymbol{f}_{k,a}$ from $\boldsymbol{\mathcal{L}}^{(a)}$.

(4) Use SEBA algorithm [4] to isolate partition elements from leading eigenvectors $\boldsymbol{f}_{k,a}$.

[4] Froyland, Rock & Sakellariou (2019). *Commun Nonlinear Sci Numer Simulat*.

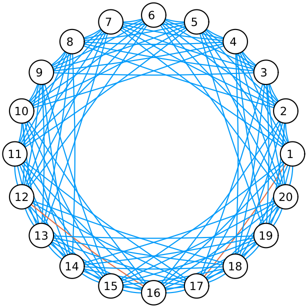
Example: d-regular → 2 clusters → 3 clusters

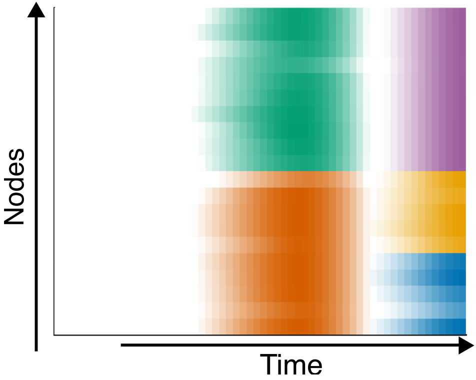




Example: d-regular → 2 clusters → 3 clusters

t=1

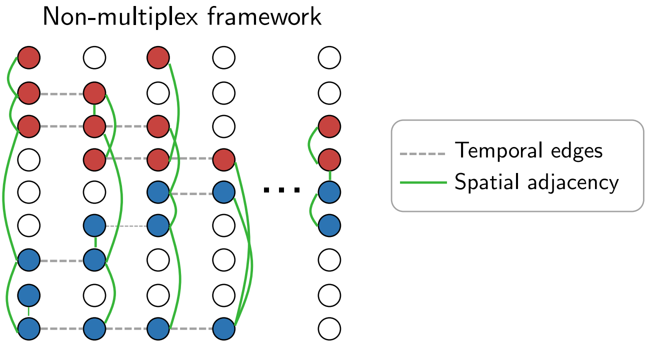




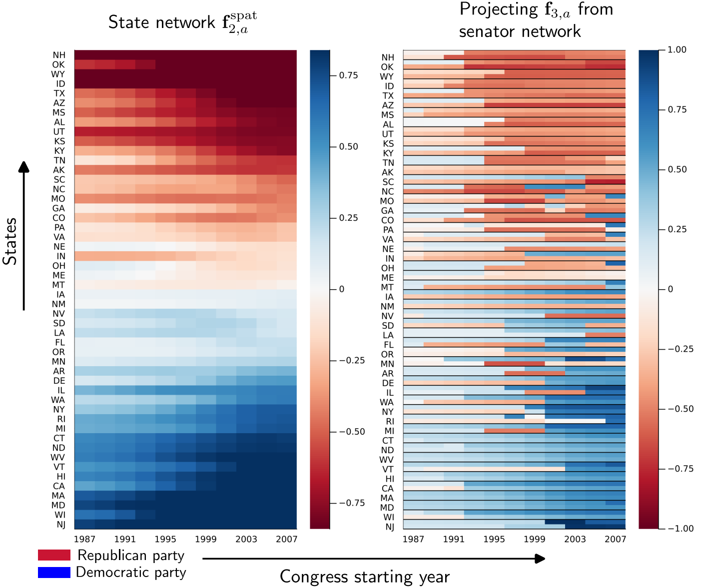
Clustering with spectral partitioning.

Networks with disappearing nodes (non-multiplex)

The algorithm also includes networks where nodes appear/disappear in time.



Example: Network of US senator voting similarities



Outlook

- **Summary:** Spectral clustering with the inflated dynamic Laplacian reveals better balanced cuts in temporal networks compared to state-of-the-art. Extendable to non-multiplex cases.
- **Challenges:** For now a is kept constant throughout space and time. Dynamics on edges and vertices?

Paper: arXiv:2409.11984
Julia package: `TemporalNetworks.jl`
(`https://github.com/mkalia94/TemporalNetworks.jl/`)

Questions?