FINAL PROJECT REPORT

Course Instructor

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SUBMITTED IN PARTIAL FULLFILLMENT OF THE REQUIREMENTS OF ME5130: FINITE ELEMENT METHOD

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FEM ANALYSIS OF CIRCULAR THIN PLATE

Problem description:

Finite element static analysis of a simply supported, uniformly loaded, isotropic, thin, circular plate using annular elements.

Assumptions:

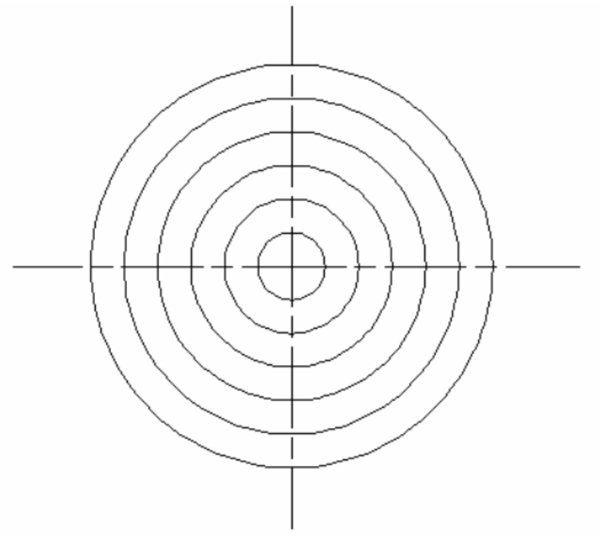
If deflections (w) of a plate are small in comparison to its thickness t, a very satisfactory approximate theory of bending of the plate by lateral loads can be developed by making the following assumptions:

- There is no deformation in the middle plane of the plate. This plane remains neutral during bending.
- Points of the plate lying initially on a normal-to the middle plane of the plate remain on the normal to the middle surface of the plate after bending.
- The normal stresses in the direction transverse to the plate can be disregarded.

Linear Analysis Assumptions:

- Infinitesimally small displacements
- Linearly elastic material
- No gaps or overlaps occurring during deformations – the displacement field is smooth

Element type:



Due to axial symmetry, the structure can be discretized into a number of concentric divisions. The bounding circumferential lines constitute a set of finite element nodal lines being an extension of the concept of nodes. Due to axial symmetry, the deflections along each of the nodal lines are constant. Each annular finite element has 2 nodes and each node has 2 degrees of freedom.

Governing Equation:

The governing equation of a thin plate subjected to axisymmetric load can be expressed as

$$\boldsymbol{D\nabla^2\nabla^2 w - 2\tau_0\nabla^2 w + q(r) = -(\rho h + 2\rho_0)\ddot{w} + \left(\frac{\rho h^3}{12} + \frac{\rho_0 h^2}{2} - \frac{\rho_0\tau_0\nu h^2}{6(1-\nu)}\right)\nabla^2\ddot{w}}$$

Where,

$D = \frac{Eh^3}{12(1-\nu^2)} + \frac{(2\mu_0+\lambda_0)h^2}{2} - \frac{\tau_0\nu h^2}{6(1-\nu)}$

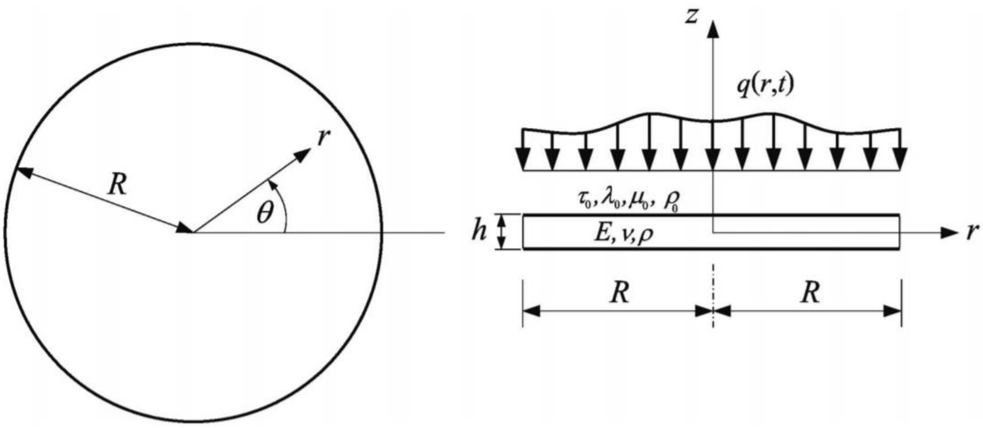
w = plate deflection; h = plate thickness;

μ_0 and λ_0 are surface lame constants

E and ν = Young's modulus and Poisson's ratio of bulk material, respectively;

τ_0 = surface residual stress;

ρ and ρ_0 = mass densities of the bulk and surface, respectively



For static loading,

$$D\nabla^2\nabla^2 w - 2\tau_0\nabla^2 w + q(r) = 0$$

Assuming there are no surface residual stresses we get,

$$D\nabla^2\nabla^2 w + q(r) = 0$$

$$\sigma_{rr} = -\frac{Ez}{1-\nu^2}\left(\frac{d^2w}{dr^2} + \frac{\nu}{r}\frac{dw}{dr}\right)$$

$$\sigma_{\theta\theta} = -\frac{Ez}{1-\nu^2}\left(\nu\frac{d^2w}{dr^2} + \frac{1}{r}\frac{dw}{dr}\right)$$

Applying Galerkin's weighted-residual method,

$$\oiiint_V \overline{\mathrm{w}}\left(\mathrm{D}\nabla^2\nabla^2\mathrm{w} + \mathrm{q(r)}\right)\mathrm{dV} = 0$$

$$\int_0^R \int_{-h/2}^{h/2} \int_0^{2\pi} \overline{\mathrm{w}}\left(\mathrm{D}\nabla^2\nabla^2\mathrm{w} + \mathrm{q(r)}\right)\mathrm{rd\theta dzdr} = 0$$

$$\Rightarrow \int_0^R \overline{\mathrm{w}}\, 2\pi\mathrm{h}\left(\mathrm{D}\nabla^2\nabla^2\mathrm{w} + \mathrm{q(r)}\right)\mathrm{rdr} = 0$$

$$\Rightarrow \int_0^R \overline{\mathrm{w}}\, 2\pi\left(\mathrm{D}\nabla^2\nabla^2\mathrm{w} + \mathrm{q(r)}\right)\mathrm{rdr} = 0$$

$$\Rightarrow \int_0^R 2\pi\mathrm{D}(\overline{\mathrm{w}}\nabla^2\nabla^2\mathrm{w})\mathrm{rdr} + \int_0^R \overline{\mathrm{w}}\, 2\pi\mathrm{q(r)rdr} = 0 \qquad \ldots\ldots\ldots\ldots\ldots (1)$$

$\int_0^R$ **$2\pi D(\overline{w}\nabla^2\nabla^2 w)rdr$:**

$$\nabla^2\nabla^2\mathrm{w} = \frac{1}{r^3}\frac{\mathrm{dw}}{\mathrm{dr}} - \frac{1}{r^2}\frac{\mathrm{d^2w}}{\mathrm{dr^2}} + \frac{2}{r}\frac{\mathrm{d^3w}}{\mathrm{dr^3}} + \frac{\mathrm{d^4w}}{\mathrm{dr^4}}$$

$$\int_0^R 2\pi\mathrm{D}(\overline{\mathrm{w}})\left(\frac{1}{\mathrm{r^3}}\frac{\mathrm{dw}}{\mathrm{dr}} - \frac{1}{\mathrm{r^2}}\frac{\mathrm{d^2w}}{\mathrm{dr^2}} + \frac{2}{\mathrm{r}}\frac{\mathrm{d^3w}}{\mathrm{dr^3}} + \frac{\mathrm{d^4w}}{\mathrm{dr^4}}\right)\mathrm{rdr}$$

$$\Rightarrow \int_0^R 2\pi\mathrm{D}(\overline{\mathrm{w}})\left(\frac{1}{\mathrm{r^3}}\frac{\mathrm{dw}}{\mathrm{dr}}\right)\mathrm{rdr} = 2\pi\mathrm{D}\int_0^R \left(\frac{\overline{\mathrm{w}}}{\mathrm{r^2}}\frac{\mathrm{dw}}{\mathrm{dr}}\right)\mathrm{dr} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

$$\int_0^R 2\pi\mathrm{D}(\overline{\mathrm{w}})\left(-\frac{1}{\mathrm{r^2}}\frac{\mathrm{d^2w}}{\mathrm{dr^2}}\right)\mathrm{rdr} = -2\pi\mathrm{D}\int_0^R \left(\frac{\overline{\mathrm{w}}}{\mathrm{r^2}}\frac{\mathrm{d^2w}}{\mathrm{dr^2}}\right)\mathrm{rdr}$$

$$\Rightarrow -2\pi\mathrm{D}\left(\frac{\overline{\mathrm{w}}}{\mathrm{r}}\frac{\mathrm{dw}}{\mathrm{dr}}\Big|_0^R - \int_0^R \frac{1}{\mathrm{r}}\frac{\mathrm{d\overline{w}}}{\mathrm{dr}}\frac{\mathrm{dw}}{\mathrm{dr}}\mathrm{dr} + \int_0^R \frac{\overline{\mathrm{w}}}{\mathrm{r^2}}\frac{\mathrm{dw}}{\mathrm{dr}}\mathrm{dr}\right) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

$$\int_0^R 2\pi\mathrm{D}(\overline{\mathrm{w}})\frac{2}{\mathrm{r}}\frac{\mathrm{d^3w}}{\mathrm{dr^3}}\mathrm{rdr} = \int_0^R 2\pi\mathrm{D}(\overline{\mathrm{w}})2\frac{\mathrm{d^3w}}{\mathrm{dr^3}}\mathrm{dr}$$

$$\Rightarrow 2\pi\mathrm{D}\times 2\int_0^R (\overline{\mathrm{w}})\frac{\mathrm{d^3w}}{\mathrm{dr^3}}\mathrm{dr} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

$$\int_0^R 2\pi\mathrm{D}(\overline{\mathrm{w}})\frac{\mathrm{d^4w}}{\mathrm{dr^4}}\mathrm{rdr} = 2\pi\mathrm{D}\int_0^R (\overline{\mathrm{w}}\mathrm{r})\frac{\mathrm{d^4w}}{\mathrm{dr^4}}\mathrm{dr}$$

$$\Rightarrow 2\pi D\left(\bar{w}r\frac{d^3w}{dr^3}\Big|_0^R - \int_0^R(\bar{w})\frac{d^3w}{dr^3}dr - \int_0^R r\frac{d\bar{w}}{dr}\frac{d^3w}{dr^3}dr\right)$$

$$2\pi D\left(\bar{w}r\frac{d^3w}{dr^3}\Big|_0^R - \int_0^R(\bar{w})\frac{d^3w}{dr^3}dr - r\frac{d\bar{w}}{dr}\frac{d^2w}{dr^2}\Big|_0^R + \int_0^R\frac{d\bar{w}}{dr}\frac{d^2w}{dr^2}dr + \int_0^R r\frac{d^2\bar{w}}{dr^2}\frac{d^2w}{dr^2}dr\right) \quad \ldots\ldots. (5)$$

(2) +(3):

$$\Rightarrow 2\pi D\int_0^R\left(\frac{\bar{w}}{r^2}\frac{dw}{dr}\right)dr - 2\pi D\left(\frac{\bar{w}}{r}\frac{dw}{dr}\Big|_0^R\right) + 2\pi D\int_0^R\frac{1}{r}\frac{d\bar{w}}{dr}\frac{dw}{dr}dr - 2\pi D\int_0^R\frac{\bar{w}}{r^2}\frac{dw}{dr}dr$$
$$\Rightarrow 2\pi D\int_0^R\frac{1}{r}\frac{d\bar{w}}{dr}\frac{dw}{dr}dr - 2\pi D\left(\frac{\bar{w}}{r}\frac{dw}{dr}\Big|_0^R\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots (6)$$

(4) + (5):

$$\Rightarrow 2\pi D\left(\bar{w}r\frac{d^3w}{dr^3}\Big|_0^R - \int_0^R(\bar{w})\frac{d^3w}{dr^3}dr - r\frac{d\bar{w}}{dr}\frac{d^2w}{dr^2}\Big|_0^R + \int_0^R\frac{d\bar{w}}{dr}\frac{d^2w}{dr^2}dr + \int_0^R r\frac{d^2\bar{w}}{dr^2}\frac{d^2w}{dr^2}dr\right)$$
$$+ 2\pi D\times 2\int_0^R(\bar{w})\frac{d^3w}{dr^3}dr$$

$$\Rightarrow 2\pi D\left(\int_0^R\bar{w}\frac{d^3w}{dr^3}dr + \bar{w}r\frac{d^3w}{dr^3}\Big|_0^R - r\frac{d\bar{w}}{dr}\frac{d^2w}{dr^2}\Big|_0^R + \int_0^R\frac{d\bar{w}}{dr}\frac{d^2w}{dr^2}dr + \int_0^R r\frac{d^2\bar{w}}{dr^2}\frac{d^2w}{dr^2}dr\right)$$

$$\int_0^R\bar{w}\frac{d^3w}{dr^3}dr = \bar{w}\frac{d^2w}{dr^2}\Big|_0^R - \frac{d\bar{w}}{dr}\frac{dw}{dr}\Big|_0^R + \int_0^R\frac{d^2\bar{w}}{dr^2}\frac{dw}{dr}dr$$

$$\Rightarrow 2\pi D\left(\bar{w}\frac{d^2w}{dr^2}\Big|_0^R - \frac{d\bar{w}}{dr}\frac{dw}{dr}\Big|_0^R + \int_0^R\frac{d^2\bar{w}}{dr^2}\frac{dw}{dr}dr + \bar{w}r\frac{d^3w}{dr^3}\Big|_0^R - r\frac{d\bar{w}}{dr}\frac{d^2w}{dr^2}\Big|_0^R + \int_0^R\frac{d\bar{w}}{dr}\frac{d^2w}{dr^2}dr\right.$$
$$\left. + \int_0^R r\frac{d^2\bar{w}}{dr^2}\frac{d^2w}{dr^2}dr\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots(7)$$

(6) + (7): (adding only integral terms)

$$\Rightarrow 2\pi D\left(\int_0^R\frac{d^2\bar{w}}{dr^2}\frac{dw}{dr}dr + \int_0^R\frac{d\bar{w}}{dr}\frac{d^2w}{dr^2}dr + \int_0^R r\frac{d^2\bar{w}}{dr^2}\frac{d^2w}{dr^2}dr + \int_0^R\frac{1}{r}\frac{d\bar{w}}{dr}\frac{dw}{dr}dr\right)$$

$$\Rightarrow 2\pi D\int_0^R\left(\frac{d^2\bar{w}}{dr^2}\frac{d^2w}{dr^2} + \frac{1}{r^2}\frac{d\bar{w}}{dr}\frac{dw}{dr} + \frac{1}{r}\left(\frac{d^2\bar{w}}{dr^2}\frac{dw}{dr} + \frac{d\bar{w}}{dr}\frac{d^2w}{dr^2}\right)\right)rdr$$

$$\Rightarrow 2\pi D\int_0^R\left(\frac{d^2\bar{w}}{dr^2}\frac{d^2w}{dr^2} + \frac{1}{r^2}\frac{d\bar{w}}{dr}\frac{dw}{dr}\right)dr + 2\pi D\int_0^R\frac{d}{dr}\left(\frac{d\bar{w}}{dr}\frac{dw}{dr}\right)dr$$
$$\Rightarrow 2\pi D\int_0^R\left(\frac{d^2\bar{w}}{dr^2}\frac{d^2w}{dr^2} + \frac{1}{r^2}\frac{d\bar{w}}{dr}\frac{dw}{dr}\right)dr + 2\pi D\left(\frac{d\bar{w}}{dr}\frac{dw}{dr}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots (8)$$

(6) + (7): (adding non-integral terms)

$$\Rightarrow -2\pi D\left(\frac{\overline{w}}{r}\frac{dw}{dr}\right)_0^R + 2\pi D\ \left(\overline{w}\frac{d^2w}{dr^2}\right)_0^R - 2\pi D\left(\frac{d\overline{w}}{dr}\frac{dw}{dr}\right)_0^R + 2\pi D\left(\overline{w}r\frac{d^3w}{dr^3}\right)_0^R - 2\pi Dr\frac{d\overline{w}}{dr}\frac{d^2w}{dr^2}\Big|_0^R$$

Adding and subtracting $\mathbf{2\pi Dv}\frac{\mathbf{d\overline{w}}}{\mathbf{dr}}\frac{\mathbf{dw}}{\mathbf{dr}}\Big|_{\mathbf{0}}^{\boldsymbol{R}}$ we get,

$$\Rightarrow 2\pi Dv\frac{d\overline{w}}{dr}\frac{dw}{dr}\Big|_0^R - 2\pi Dv\frac{d\overline{w}}{dr}\frac{dw}{dr}\Big|_0^R -2\pi D\ (\frac{\overline{w}}{r}\frac{dw}{dr}\Big|_0^R\ ) + 2\pi D\ \left(\overline{w}\frac{d^2w}{dr^2}\Big|_0^R\right) - 2\pi D\left(\frac{d\overline{w}}{dr}\frac{dw}{dr}\Big|_0^R\right) +2\pi D\left(\overline{w}r\ \frac{d^3w}{dr^3}\Big|_0^R\right) - 2\pi D\ (r\frac{d\overline{w}}{dr}\frac{d^2w}{dr^2}\Big|_0^R\ )$$

$$\Rightarrow 2\pi Dv\frac{d\overline{w}}{dr}\frac{dw}{dr}\Big|_0^R - 2\pi Dv\frac{d\overline{w}}{dr}\frac{dw}{dr}\Big|_0^R\ -2\pi D\left(\frac{d\overline{w}}{dr}\frac{dw}{dr}\Big|_0^R\right)\ -2\pi D\ (r\frac{d\overline{w}}{dr}\frac{d^2w}{dr^2}\Big|_0^R\ )$$

$$\Rightarrow -2\pi Dr\frac{d\overline{w}}{dr}\left(\frac{d^2w}{dr^2} - \frac{v}{r}\frac{dw}{dr}\right)_0^R - 2\pi D\overline{w}r\left(\frac{1}{r^2}\frac{dw}{dr} - \frac{1}{r}\frac{d^2w}{dr^2} - \frac{d^3w}{dr^3}\right)_0^R - 2\pi Dv\frac{d\overline{w}}{dr}\frac{dw}{dr}\Big|_0^R - 2\pi D\left(\frac{d\overline{w}}{dr}\frac{dw}{dr}\right)_0^R$$

$$\Rightarrow -2\pi Dr\frac{d\overline{w}}{dr}\left(\frac{d^2w}{dr^2} - \frac{v}{r}\frac{dw}{dr}\right)_0^R - 2\pi D\overline{w}r\left(-\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right)\right)_0^R - 2\pi Dv\frac{d\overline{w}}{dr}\frac{dw}{dr}\Big|_0^R - 2\pi D\left(\frac{d\overline{w}}{dr}\frac{dw}{dr}\right)_0^R$$

$$\Rightarrow \frac{d\overline{w}}{dr}\left[2\pi rM_r\right]_0^R - 2\pi\overline{w}rQ|_0^R - 2\pi Dv\frac{d\overline{w}}{dr}\frac{dw}{dr}\Big|_0^R - 2\pi D\left(\frac{d\overline{w}}{dr}\frac{dw}{dr}\right)_0^R \quad \dots\dots\dots\dots\dots\dots (9)$$

(8) + (9) :

$$\Rightarrow 2\pi D\int_0^R(\frac{d^2\overline{w}}{dr^2}\frac{d^2w}{dr^2} + \frac{1}{r^2}\frac{d\overline{w}}{dr}\frac{dw}{dr})dr + 2\pi D\left(\frac{d\overline{w}}{dr}\frac{dw}{dr}\right)_0^R + \frac{d\overline{w}}{dr}\left[2\pi rM_r\right]_0^R - 2\pi\overline{w}rQ|_0^R -2\pi Dv\left(\frac{d\overline{w}}{dr}\frac{dw}{dr}\right)_0^R - 2\pi D\left(\frac{d\overline{w}}{dr}\frac{dw}{dr}\right)_0^R$$

$$\Rightarrow 2\pi D\int_0^R(\frac{d^2\overline{w}}{dr^2}\frac{d^2w}{dr^2} + \frac{1}{r^2}\frac{d\overline{w}}{dr}\frac{dw}{dr})dr -2\pi Dv\left(\frac{d\overline{w}}{dr}\frac{dw}{dr}\right)_0^R + \frac{d\overline{w}}{dr}\left[2\pi rM_r\right]_0^R - 2\pi\overline{w}rQ|_0^R$$

$$\Rightarrow 2\pi D\int_0^R(\frac{d^2\overline{w}}{dr^2}\frac{d^2w}{dr^2} + \frac{1}{r^2}\frac{d\overline{w}}{dr}\frac{dw}{dr})dr - 2\pi Dv\int_0^R\frac{d}{dr}(\frac{d\overline{w}}{dr}\frac{dw}{dr})dr + \frac{d\overline{w}}{dr}\left[2\pi rM_r\right]_0^R -2\pi\overline{w}rQ|_0^R$$

Re-arranging and substituting terms in (1) we get weak form as,

$$\int_0^R 2\pi\left\{D\left[\frac{d^2\bar{w}}{dr^2}\frac{d^2w}{dr^2}+\frac{v}{r}\left(\frac{d^2w}{dr^2}\frac{d\bar{w}}{dr}+\frac{d^2\bar{w}}{dr^2}\frac{dw}{dr}\right)+\frac{1}{r^2}\frac{d\bar{w}}{dr}\frac{dw}{dr}\right]\right\}rdr + \int_0^R 2\pi q\bar{w}rdr + \frac{d\bar{w}}{dr}\,[2\pi r M_r]_0^R - 2\pi\bar{w}rQ|_0^R = 0$$

Where,

Plate bending moment $M_r = -D\left(\frac{d^2w}{dr^2}-\frac{v}{r}\frac{dw}{dr}\right)$,

Shear force $Q = -D\left[\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right)\right]$

Weak form and boundary conditions from the above equation:

$$\int_0^R 2\pi\left\{D\left[\frac{d^2\bar{w}}{dr^2}\frac{d^2w}{dr^2}+\frac{v}{r}\left(\frac{d^2w}{dr^2}\frac{d\bar{w}}{dr}+\frac{d^2\bar{w}}{dr^2}\frac{dw}{dr}\right)+\frac{1}{r^2}\frac{d\bar{w}}{dr}\frac{dw}{dr}\right]+2\pi q\bar{w}\right\}rdr = 0$$

$2\pi\bar{w}rQ|_0^R = 0$, $\frac{d\bar{w}}{dr}\,[2\pi r M_r]_0^R = 0$, (boundary conditions)

Simply Supported annular plate under uniformly distributed load of intensity q_0,

At r=b: $M_{rr}=0$, $(rQ_r)=0$

At r=a: w=0, $M_{rr}=0$

$$\Rightarrow \int_0^R\left\{D\left[\frac{d^2\bar{w}}{dr^2}\frac{d^2w}{dr^2}+\frac{v}{r}\left(\frac{d^2w}{dr^2}\frac{d\bar{w}}{dr}+\frac{d^2\bar{w}}{dr^2}\frac{dw}{dr}\right)+\frac{1}{r^2}\frac{d\bar{w}}{dr}\frac{dw}{dr}\right]+q\bar{w}\right\}rdr = 0$$

Substituting, $\bar{w} = \delta w$

$$\int_0^R\left\{D\left[\frac{d^2(\delta w)}{dr^2}\frac{d^2w}{dr^2}+\frac{v}{r}\left(\frac{d^2w}{dr^2}\frac{d(\delta w)}{dr}+\frac{d^2(\delta w)}{dr^2}\frac{dw}{dr}\right)+\frac{1}{r^2}\frac{d(\delta w)}{dr}\frac{dw}{dr}\right]\right\}rdr + \int_0^R q(\delta w)rdr = 0$$

$$\Rightarrow B(\delta w, w) = \int_0^R\left\{D\left[\frac{d^2(\delta w)}{dr^2}\frac{d^2w}{dr^2}+\frac{v}{r}\left(\frac{d^2w}{dr^2}\frac{d(\delta w)}{dr}+\frac{d^2(\delta w)}{dr^2}\frac{dw}{dr}\right)+\frac{1}{r^2}\frac{d(\delta w)}{dr}\frac{dw}{dr}\right]\right\}rdr$$

$$\Rightarrow l(\delta w) = \int_0^R q(\delta w)rdr$$

$$B(\delta w, w) = B(w, \delta w)$$

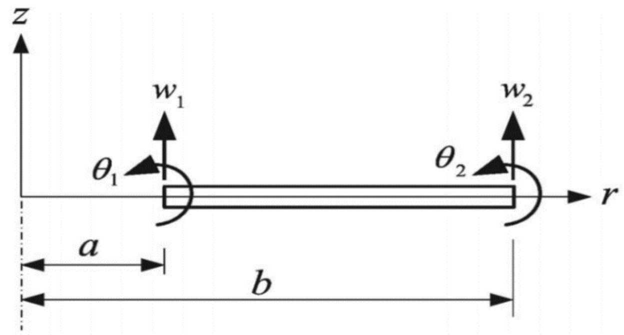
$\Rightarrow B(\delta w, w)$ is Bilinear and symmetric and $l(\delta w)$ is linear

Hence by Variational principle,

$$\pi = \frac{1}{2}B(w,w) - l(w)$$

$$\pi = \frac{1}{2}\int_0^R D\left[r\frac{d^2w}{dr^2}\frac{d^2w}{dr^2} + v\left(\frac{d^2w}{dr^2}\frac{dw}{dr} + \frac{dw}{dr}\frac{d^2w}{dr^2}\right) + \frac{1}{r}\frac{dw}{dr}\frac{dw}{dr}\right]dr - \int_0^R qwrdr$$

<u>Shape functions</u>:



Hermite Polynomials	N at r=a	$\frac{dN}{dr}$ at r=a	N at r=b	$\frac{dN}{dr}$ at r=b
$N_1(r) = \frac{(3a-b-2r)(b-r)^2}{(a-b)^3}$	1	0	0	0
$N_2(r) = -\frac{(a-r)(b-r)^2}{(a-b)^2}$	0	1	0	0
$N_3(r) = \frac{(a-r)^2(a-3b+2r)}{(a-b)^3}$	0	0	1	0
$N_4(r) = -\frac{(a-r)^2(b-r)}{(a-b)^2}$	0	0	0	1

Deflections:

$$\mathrm{w(r) = N(r)\ w_e} = N_1(r)w_1 + N_2(r)\theta_1 + N_3(r)w_2 + N_4(r)\theta_2$$

- ⇨ $\mathrm{w = N{w_e}^T}$
- ⇨ $\mathrm{\frac{dw}{dr} = N'{w_e}^T}$
- ⇨ $\mathrm{\frac{d^2w}{dr^2} = N''{w_e}^T}$

Substituting $w(r)$ in $B(w,w)$ and $l(w)$ is,

$$B(w,w) = \int_a^b \mathrm{D}\left[\mathrm{r}\frac{\mathrm{d^2w}}{\mathrm{dr^2}}\frac{\mathrm{d^2w}}{\mathrm{dr^2}} + \mathrm{v}\left(\frac{\mathrm{d^2w}}{\mathrm{dr^2}}\frac{\mathrm{dw}}{\mathrm{dr}} + \frac{\mathrm{dw}}{\mathrm{dr}}\frac{\mathrm{d^2w}}{\mathrm{dr^2}}\right) + \frac{1}{\mathrm{r}}\frac{\mathrm{dw}}{\mathrm{dr}}\frac{\mathrm{dw}}{\mathrm{dr}}\right]\mathrm{dr}$$

$$\Rightarrow \int_a^b \mathrm{D}\left[\mathrm{r}(\mathrm{N''{w_e}^T})(\mathrm{N''{w_e}^T}) + \mathrm{v}\big((\mathrm{N''{w_e}^T})(\mathrm{N'{w_e}^T}) + (\mathrm{N'{w_e}^T})(\mathrm{N''{w_e}^T})\big) + \frac{1}{\mathrm{r}}(\mathrm{N'{w_e}^T})(\mathrm{N'{w_e}^T})\right]\mathrm{dr}$$

$$\Rightarrow \int_a^b \mathrm{D}\left[\mathrm{r}(\mathrm{N''(N'')^T{w_e}^T{w_e}^T}) + \mathrm{v}(\mathrm{N''(N')^T{w_e}^T{w_e}^T + N'(N'')^T{w_e}^T{w_e}^T}) + \frac{1}{\mathrm{r}}(\mathrm{N'(N')^T{w_e}^T{w_e}^T})\right]\mathrm{dr}$$

$$\Rightarrow \int_a^b \mathrm{D}\left[\mathrm{r}(\mathrm{N''N''^T{w_e}^T{w_e}^T}) + \mathrm{v}(\mathrm{N''N'^T{w_e}^T{w_e}^T + N'N''^T{w_e}^T{w_e}^T}) + \frac{1}{\mathrm{r}}(\mathrm{N'N'^T{w_e}^T{w_e}^T})\right]\mathrm{dr}$$

$$B(w,w) = \int_a^b \mathrm{D}\left[\mathrm{rN''N''^T} + \mathrm{v}(\mathrm{N''N'^T + N'N''^T}) + \frac{1}{\mathrm{r}}(\mathrm{N'N'^T})\right]\mathrm{{w_e}^T{w_e}^T\ dr}$$

$$l(w) = \int_a^b \mathrm{q_0rwdr} = \int_a^b \mathrm{q_0r(N{w_e}^T)}dr = \int_a^b \mathrm{q_0rN{w_e}^T}dr$$

$$\pi = \sum_{e=1}^{N_e}\frac{1}{2}\int_a^b \mathrm{D}\left[\mathrm{rN''N''^T} + \mathrm{v}(\mathrm{N''N'^T + N'N''^T}) + \frac{1}{\mathrm{r}}(\mathrm{N'N'^T})\right]\mathrm{{w_e}^T{w_e}^Tdr} - \int_a^b \mathrm{q_0rN{w_e}^Tdr}$$

$$\pi = \sum_{e=1}^{N_e}\left(\frac{\mathrm{w_e}}{2}\int_a^b \mathrm{D}\left[\mathrm{rN''N''^T} + \mathrm{v}(\mathrm{N''N'^T + N'N''^T}) + \frac{1}{\mathrm{r}}(\mathrm{N'N'^T})\right]\mathrm{{w_e}^Tdr} - \int_a^b \mathrm{q_0rN{w_e}^Tdr}\right)$$

Principle of Minimum Potential Energy:

$$\delta\pi = 0 \qquad \Rightarrow \frac{d\pi}{dw_e} = 0\text{, for every element}$$

$$\frac{d\pi}{dw_e} = \frac{d\left[w_e\left(\frac{1}{2}\int_a^b D\left[rN''N''^T + v(N''N'^T + N'N''^T) + \frac{1}{r}(N'N'^T)\right]\right)w_e{}^T dr\right]}{dw_e} - \frac{d(\int_a^b q_0 rNdrw_e{}^T)}{dw_e}$$

$$\text{but,}\left(\frac{1}{2}\int_a^b D\left[rN''N''^T + v(N''N'^T + N'N''^T) + \frac{1}{r}(N'N'^T)\right]dr\right)^T$$

$$= \left(\frac{1}{2}\int_a^b D\left[rN''N''^T + v(N''N'^T + N'N''^T) + \frac{1}{r}(N'N'^T)\right]^T dr\right)$$

$$= \left(\frac{1}{2}\int_a^b D\left[r(N''N''^T)^T + v(N''N'^T + N'N''^T)^T + \frac{1}{r}(N'N'^T)^T\right]dr\right)$$

$$= \left(\frac{1}{2}\int_a^b D\left[rN''N''^T + v(N''N'^T + N'N''^T) + \frac{1}{r}(N'N'^T)\right]dr\right)$$

$$\Rightarrow \left(\frac{1}{2}\int_a^b D\left[rN''N''^T + v(N''N'^T + N'N''^T) + \frac{1}{r}(N'N'^T)\right]\right) \text{ is symmetric,}$$

and for $\alpha = x^T Ax$ when A is symmetric we have,

$$\frac{d\alpha}{dx} = 2x^T A$$

$$\Rightarrow \frac{d\pi}{dw_e} = 0 = 2w_e\left(\frac{1}{2}\int_a^b D\left[rN''N''^T + v(N''N'^T + N'N''^T) + \frac{1}{r}(N'N'^T)\right]dr\right) - \int_a^b q_0 rNdr\frac{d(w_e{}^T)}{dw_e}$$

$$\Rightarrow \int_a^b q_0 rNdr = w_e\int_a^b D\left[rN''N''^T + v(N''N'^T + N'N''^T) + \frac{1}{r}(N'N'^T)\right]dr$$

Applying transpose on both sides,

$$\Rightarrow \left(\int_a^b q_0 r N dr\right)^T = \left(w_e \int_a^b D\left[rN''N''^T + v(N''N'^T + N'N''^T) + \frac{1}{r}(N'N'^T)\right]^T dr\right)$$

$$\Rightarrow \int_a^b q_0 r N^T dr = \left(\int_a^b D\left[rN''N''^T + v(N''N'^T + N'N''^T) + \frac{1}{r}(N'N'^T)\right] dr\right) w_e{}^T$$

Comparing above equation with $F_e = K_e w_e$

$\Rightarrow F_e = \int_a^b q_0 r N^T dr$

$\Rightarrow K_e = \int_a^b D\left[rN''N''^T + v(N''N'^T + N'N''^T) + \frac{1}{r}(N'N'^T)\right] dr$

The centre of the plate has been removed by considering a small inner radius to avoid a singular stiffness matrix as the stiffness matrix for an annular element includes an undefined logarithmic term if considered for a circular element.

Analytical solution

The analytical solution for a annular circular plate subjected to uniform load q_o is taken from [2] is,

$$w(r) = \frac{q_0 R_o^4}{64D}\left\{-\left[1 - \left(\frac{r}{R_0}\right)^4\right] + \frac{2\alpha_1}{1+\nu}\left[1 - \left(\frac{r}{Ro}\right)^2\right] - \frac{4\alpha_2\beta^2}{1-\nu}\log\left(\frac{r}{R_o}\right)\right\}$$

Where,

R_i is the inner radius

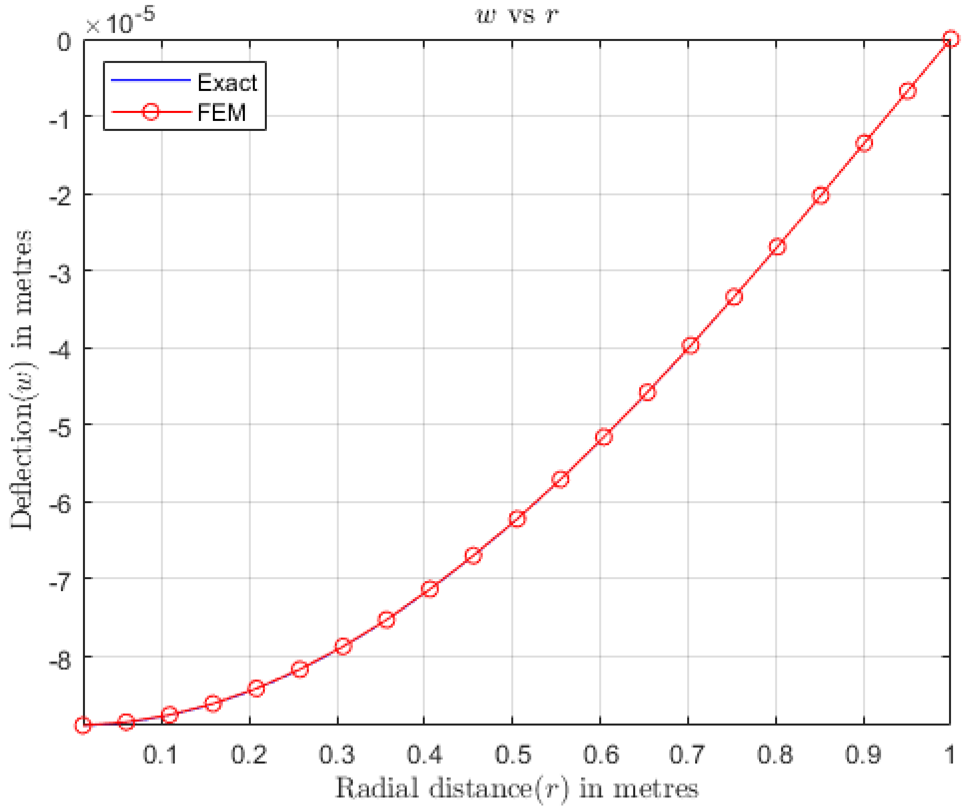
R_o is the outer radius

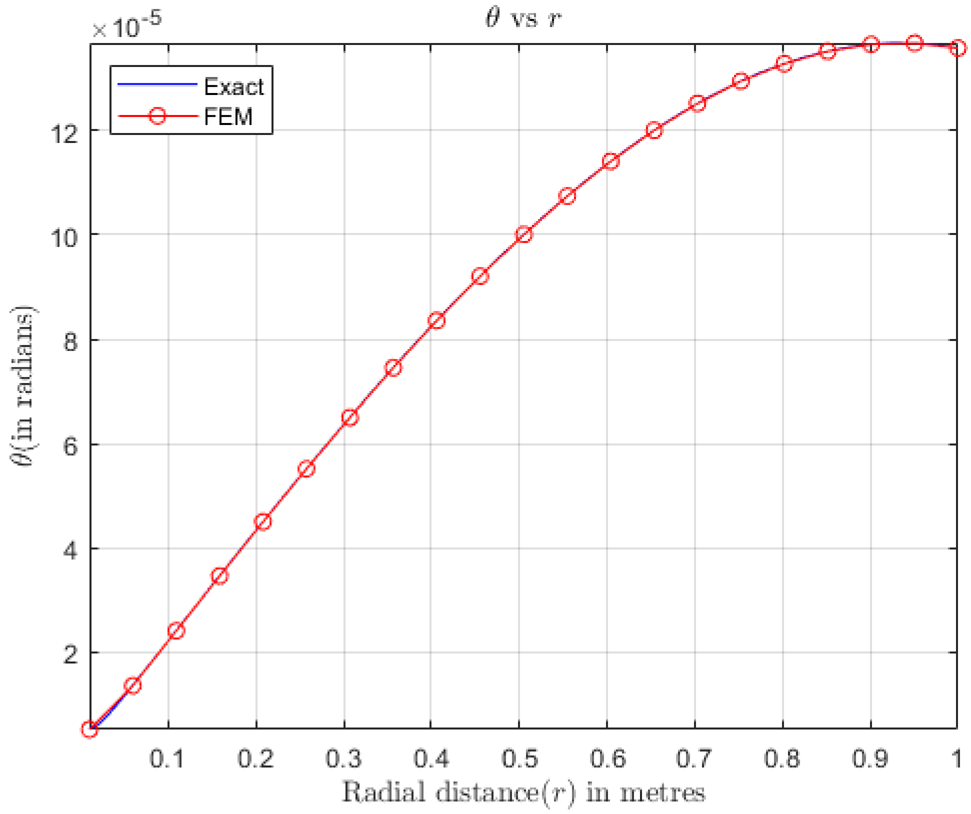
$$\alpha_1 = (3+\nu)(1-\beta^2) - 4(1+\nu)\beta^2\kappa$$

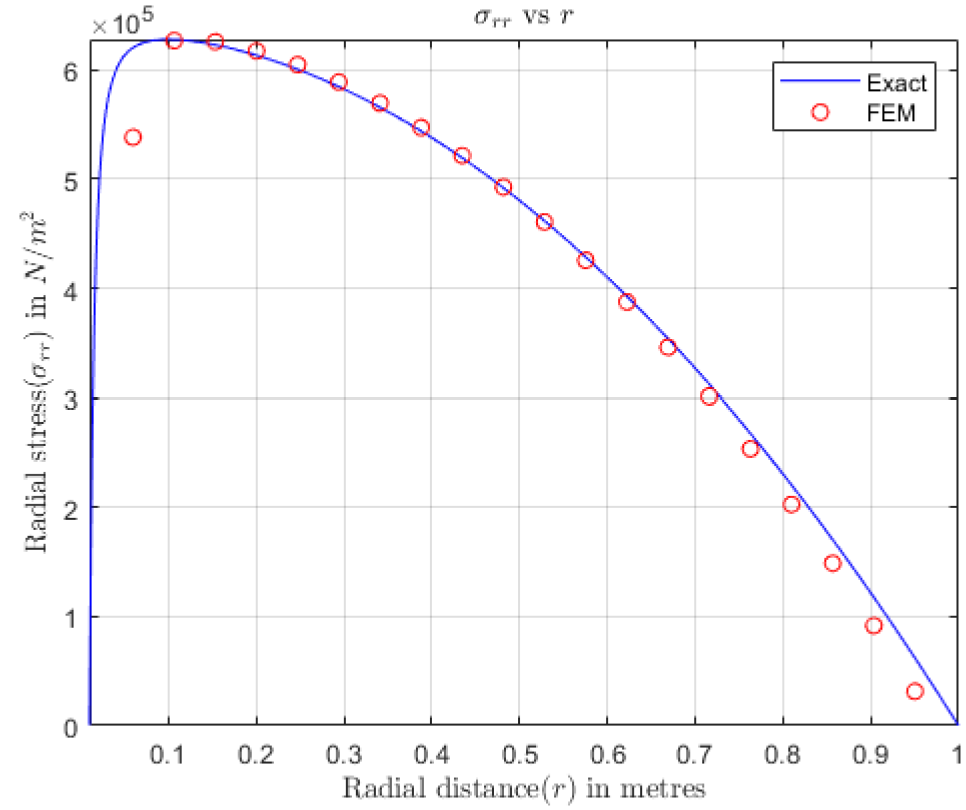
$$\alpha_2 = (3+\nu) + 4(1+\nu)$$

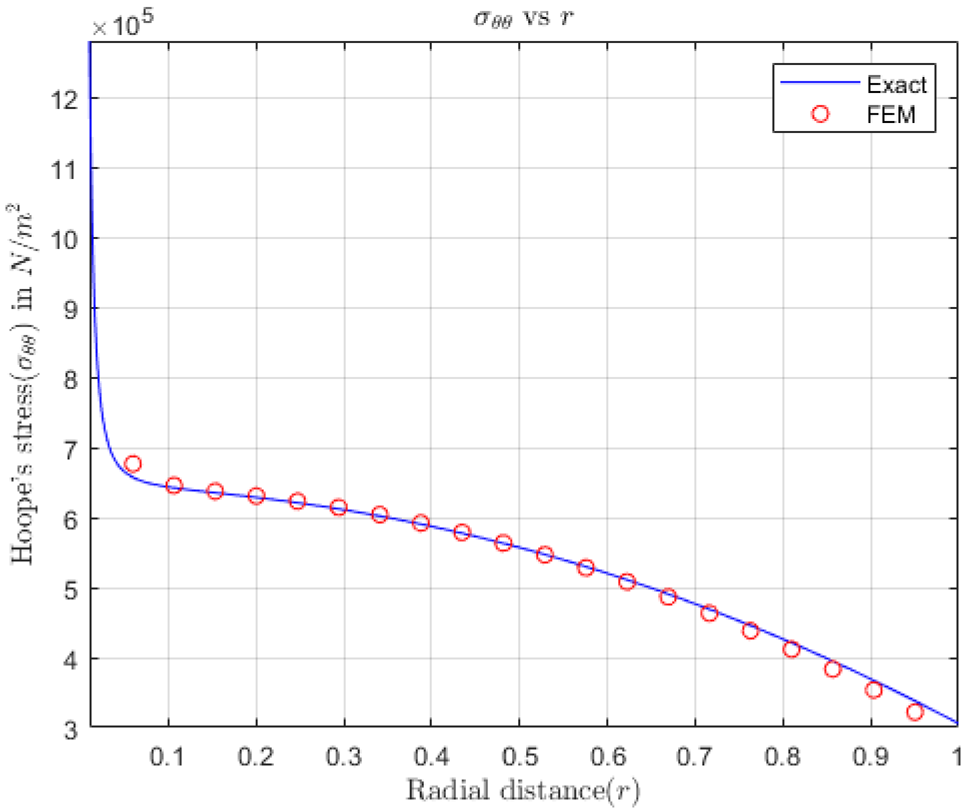
$$\kappa = \frac{\beta^2}{1-\beta^2}\log\beta, \quad \beta = \frac{R_i}{R_o}$$

Plots and Outputs





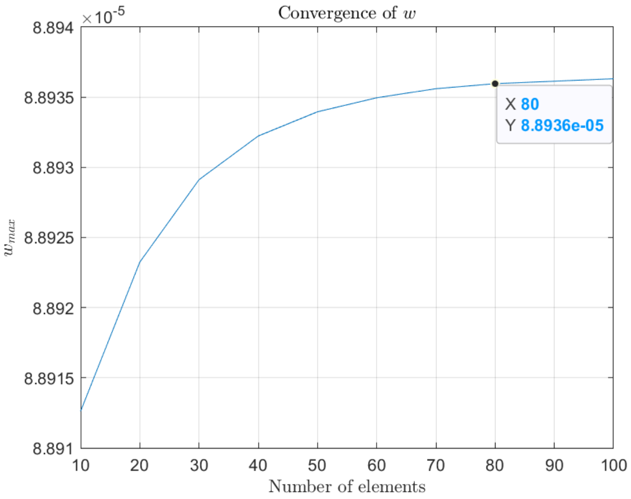

σ_{rr} vs r
$\times 10^5$
Exact
FEM
Radial stress(σ_{rr}) in N/m^2
Radial distance(r) in metres
0
1
2
3
4
5
6
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

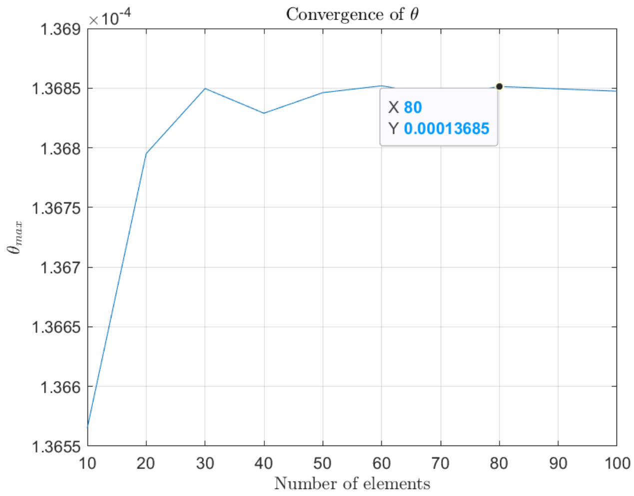

$\sigma_{\theta\theta}$ vs r
$\times 10^5$
Exact
FEM
Hoope's stress($\sigma_{\theta\theta}$) in N/m^2
Radial distance(r)
3
4
5
6
7
8
9
10
11
12
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

Outputs

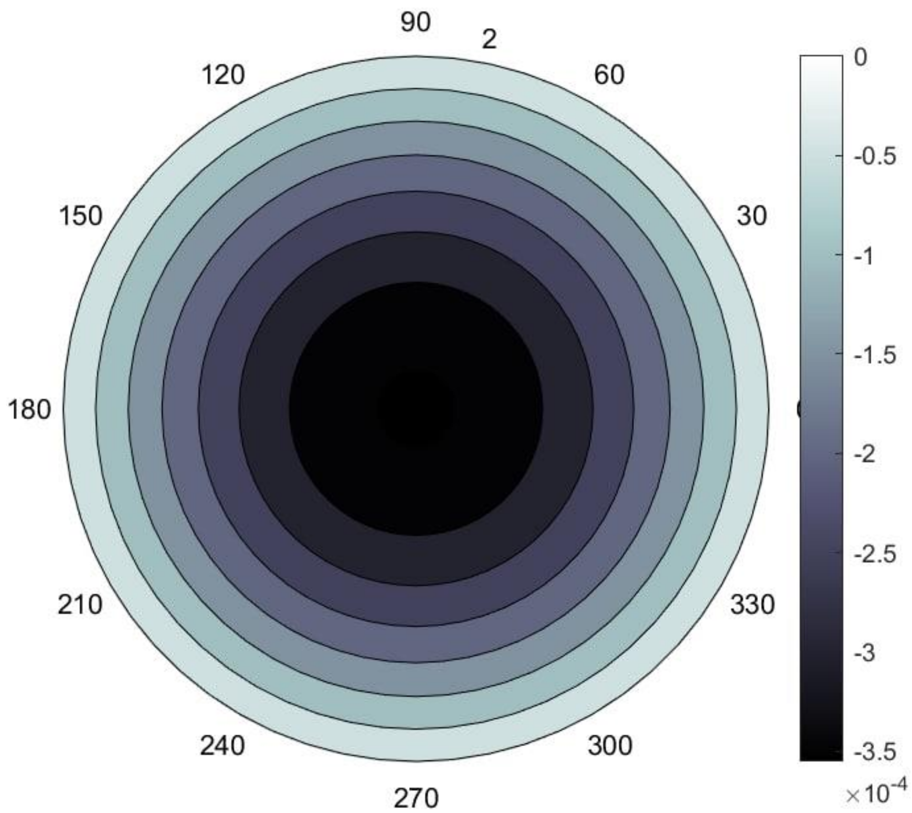
```
deflections =
     Dof        w
    1.0000   -0.0001
    3.0000   -0.0001
    5.0000   -0.0001
    7.0000   -0.0001
    9.0000   -0.0001
   11.0000   -0.0001
   13.0000   -0.0001
   15.0000   -0.0001
   17.0000   -0.0001
   19.0000   -0.0001
   21.0000   -0.0001
   23.0000   -0.0001
   25.0000   -0.0001
   27.0000   -0.0000
   29.0000   -0.0000
   31.0000   -0.0000
   33.0000   -0.0000
   35.0000   -0.0000
   37.0000   -0.0000
   39.0000   -0.0000
   41.0000         0

slopes =
      Dof        θ
    2.0000    0.0000
    4.0000    0.0000
    6.0000    0.0000
    8.0000    0.0000
   10.0000    0.0000
   12.0000    0.0001
   14.0000    0.0001
   16.0000    0.0001
   18.0000    0.0001
   20.0000    0.0001
   22.0000    0.0001
   24.0000    0.0001
   26.0000    0.0001
   28.0000    0.0001
   30.0000    0.0001
   32.0000    0.0001
   34.0000    0.0001
   36.0000    0.0001
   38.0000    0.0001
   40.0000    0.0001
   42.0000    0.0001

reactions =
    Dof        R
     41       661
```

Convergence study:





Contour plot of deflections:



<u>Results and Discussions</u>:

- Accuracy of the finite-element model is first verified by comparing with the analytical solutions for static deflections of a circular plate presented in J.N.Reddy, "Theory and Analysis of Elastic Plates and Shells, 2nd Edition" and it was found that they agreed satisfactorily.

- A convergence study was done, and it was found that completely converged solutions for static deflections can be obtained by using a plate with more than 80 finite elements. But beyond 20 elements, the change in the maximum values of the deflection and theta are of the order 10^{-7} and thus can be neglected. The number of elements required for satisfactory convergence will depend on the application and its demand for accuracy.

<u>References:</u>

- [1] Y. Sapsathiarn, R. K. N. D. Rajapakse " Finite Element Modelling of Circular Nano Plates" - Journal of "NANOMECHANICS AND MICROMECHANICS ASCE / SEPTEMBER 2013"
- [2] J. N. Reddy, "Theory and Analysis of Elastic Plates and Shells, 2nd Edition"
- [3] Saurabh Kumar, R. K. Srivatsava, (2006) - Journal of "STATIC ANALYSIS OF CIRCULAR PLATE USING CIRCULAR FINITE ELEMENTS"