Probability

CSCI 2400 "Part 5"

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Disclaimer: Lecture notes can't and won't cover everything I say in class. You should attend class each day and use these for review or reinforcement.

8 Probability

There are several ways to frame the idea of probability. We use the "many worlds" framing:

- There exists a universe U containing all possible worlds.
- A "world" is a combination of all variables.
- *Inference* is the process of figuring out what world we live in.

For example, given a tossed coin not yet observed, there are possible worlds in which it is heads, and those in which it is tails. Those worlds contain many other variables with many values, but this is the only variable we care about. We ask: what are the odds we live in a world where the coin is heads side up?

8.1 Random Variables

To formally consider this problem, we require the concept of a *Random Variable* (RV): a symbol that can take on values, which can be different each time the symbol is checked, with an associated distribution that determines the likelihood of each value appearing. An RV is customarily written in all capital letters, with its list of possible values written in angle brackets. For example:

$$HEADS =< T, F >$$
$$BEARD =< T, F >$$

RVs are often written with a matching probability vector showing the likelihood of each result:

$$P(HEADS) =< 0.5, 0.5 >$$
$$P(BEARD) =< 0.8, 0.2 >$$

In this case, we have an RV representing a fair coin flip, with a 50% chance to land on heads, and an RV representing the 80% chance that a particular person is sporting a beard on a given day. If we did not know the likelihood distribution, we could estimate it by *sampling*: observing the RV many different times and recording the results. For example, if we observed our mystery person once each day for 30 days, and noted that they had a beard on 23 of those days, we might estimate the likelihood distribution $P(BEARD)$ to be $< 0.77, 0.23 >$, because $23/30 = 0.7666$.

When discussing logic, we used "atoms" to represent specific facts. We use those for probability as well, generally representing an *event* or *observation*, and write them in lowercase letters. We can use these interchangeably with the result of checking an RV's value:

$$BEARD = TRUE \iff beard$$
$$BEARD = FALSE \iff \neg beard$$
$$P(BEARD = TRUE) = 0.8$$
$$P(beard) = 0.8$$

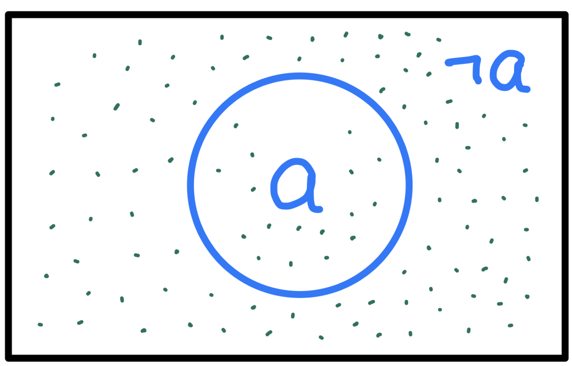
The RV $BEARD$ taking on the value $TRUE$ is the same as observing the event $beard$, and the RV $BEARD$ taking on the value $FALSE$ is the same as observing the event $\neg beard$. The probability of RV $BEARD$ taking on the value $TRUE$ is the same as the probability of observing the event $beard$.

An RV may have many value choices, not only TRUE or FALSE. For example:

$$WEATHER =< rain, snow, hot, mild, spontaneous\ combustion >$$
$$P(WEATHER) =< 0.3, 0.01, 0.25, 0.4, 0.04 >$$

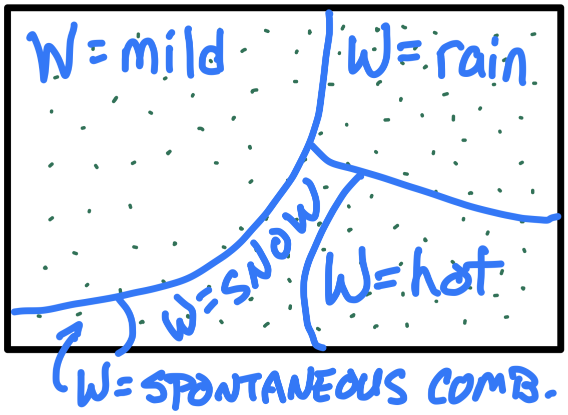
The probability of all possible values for an RV must sum to one, because probability is simply the proportion of all possible worlds (U) in which the variable takes this value.

If we are considering only a single RV, it is easy to visualize the universe U:



Each small dot in the universe represents a possible world. Dots within the circle represent worlds in which some event a occurred (or a is true), and dots outside the circle represent worlds in which event a did not occur (or a is false). Since the area of the universe is defined to be exactly 1, the area of the circle shown is the proportion of worlds in which a occurred. That is, the area of the circle is $P(a)$! It easily follows that $P(\neg a) = 1 - P(a)$.

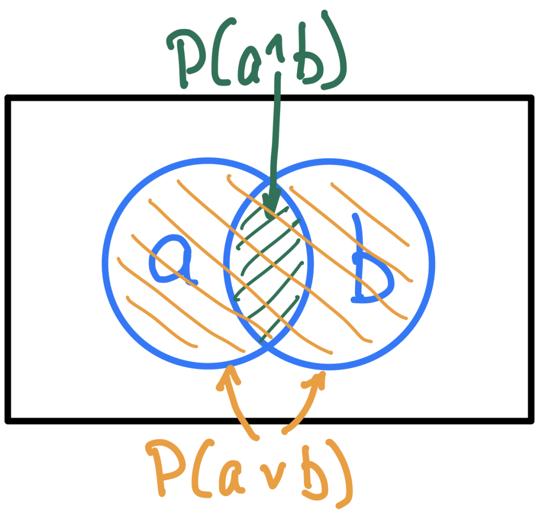
We can do the same visualization for a single RV with many values, simply dividing up the available area proportional to the likelihood of a particular value:



Since the total area of our universe U (all possible worlds) is defined to be 1, all probabilities must be between zero and one, inclusive. (The area associated with an event cannot be more than the area of the universe, and it is not clear what a negative area would represent, since a zero probability already means the event does not occur in *any* world.)

8.2 Probabilistic Inference

What happens when two random variables meet? This is where probability really becomes useful for our purposes. We can consider two possible events a and b, arising from two different RVs, as two different circles in our universe U. There are worlds in which both events occurred, worlds in which only one occurred, and worlds in which neither occurred. Again, it is easy to visualize:



As when considering logic, we use the symbol $\wedge$ to represent conjunction ("and") and symbol $\vee$ to represent disjunction ("or"). In a world where both a and b occurred, the event $a \wedge b$ has occurred, and $P(a \wedge b)$ is the area of the intersection of the two circles. In a world where at least one of a and b occurred, the event $a \vee b$ has occurred, and $P(a \vee b)$ is the area inside either (or both) of the circles.

If $P(a)$ and $P(b)$ are the areas of circles a and b, respectively, and $P(a \wedge b)$ is the area of their intersection, how may we compute $P(a \vee b)$?

$$P(a \vee b) = P(a) + P(b) - P(a \wedge b)$$

Why did we have to subtract $P(a \wedge b)$? Area $P(a)$ includes the area that is $P(a \wedge b)$. Area $P(b)$ also includes that same area. Thus by adding $P(a) + P(b)$, we double counted the area of $P(a \wedge b)$! We had to subtract it (once) to ensure it was only counted one time, not twice.

In class, we considered an example drawn from our textbook: that exciting experience of going to the dentist and having the technician scrape that horrible metal tool across your teeth to see if it catches on any small cavities. This scenario gives us three binary RVs to play with: CAVITY (do you have a cavity?), TOOTHACHE (did you go to the dentist because of a toothache?), and CATCH (does the instrument catch in your teeth?).

The *full joint distribution* (FJD) of two or more RVs captures the probability of every combination of possible events: $P(CAVITY, TOOTHACHE, CATCH)$. (We write a comma as an alternative to $\wedge$ when expressing the joint distribution of multiple RVs.). For example:

	toothache		¬ toothache	
	catch	¬ catch	catch	¬ catch
cavity	0.108	0.012	0.072	0.008
¬ cavity	0.016	0.064	0.144	0.576

Table 1: Full joint distribution of three random variables.

Note that the sum of all entries in the FJD must equal one, because the table includes all possible combinations, and thus includes every "world" in the universe U.

With the FJD, we can compute any probability that we might need, simply by summing up the appropriate joint events. For a conjunction of all three RVs, we need only look up the number in a single cell:

$$P(toothache \wedge cavity \wedge catch) = 0.108$$

This is the proportion of all possible worlds in which all three positive events occur: you have a toothache, and you have a cavity, and the instrument catches in your teeth.

More interestingly, how can we compute $P(cavity)$? That is, the probability that you have a cavity, irrespective of toothaches and instruments getting caught? It is just the sum of all conjunctive events that include having a cavity:

$$\begin{aligned} P(cavity) = &P(cavity \wedge toothache \wedge catch) + P(cavity \wedge toothache \wedge \neg catch) + \\ &P(cavity \wedge \neg toothache \wedge catch) + P(cavity \wedge \neg toothache \wedge \neg catch) \end{aligned}$$

We can easily enough plug in those values from the FJD to see that:

$$P(cavity) = 0.108 + 0.012 + 0.072 + 0.008 = 0.2$$

To put it in simpler terms, we simply summed the entire *cavity* row of the table. This process is called *summing out* or *marginalization*, because when such calculations were done for two RVs on paper, the total probabilities for each event would be written at the edge of the table, in the margins of the paper!

We can compute partial combinations as well. What is the probability that you have a toothache and a cavity, without regard for the instrument catching or not?

$$\begin{aligned} P(cavity \land toothache) &= P(cavity \land toothache \land catch) + P(cavity \land toothache \land \neg catch) \\ &= 0.108 + 0.012 = 0.12 \end{aligned}$$

What is the probability of having a toothache or a cavity?

$$\begin{aligned} P(cavity \lor toothache) &= P(cavity) + P(toothache) - P(cavity \land toothache) \\ &= 0.2 + 0.2 - 0.12 = 0.28 \end{aligned}$$

8.3 Conditional Probability

The full joint distribution (FJD) has all the information to calculate the probability of any single event (e.g. $P(a)$), or any conjunction of events (e.g. $P(a \land b)$), or any disjunction of events (e.g. $P(a \lor b)$). Given how our intelligent agents work, however, we will usually have information from our sensors, and want to draw conclusions *based on* that information. That is to say, we will primarily want to compute *conditional* probabilities. "Given that I know a, now how likely is it that b is true?" or "Given that I observed B_{21} (breeze at 2,1), how likely is P_{22} (pit at 2,2)?"

We write conditional probabilities with a pipe symbol "|". For example, $P(y|z)$ means "given that z is true, how likely is y to be true?" Remember that event y is shorthand for Random Variable Y having the value $TRUE$, so:

$$\begin{aligned} P(y|z) &= P(Y = TRUE|Z = TRUE) \\ P(y|\neg z) &= P(Y = TRUE|Z = FALSE) \end{aligned}$$

We define conditional probability as:

$$P(y|z) = \frac{P(y \land z)}{P(z)}$$

You should not need to memorize this, but rather understand why it must be true. For these two events y and z, there are only four possibilities in the FJD: $y \land z$, $y \land \neg z$, $\neg y \land z$, and $\neg y \land \neg z$. This is my "universe" of possibilities in this case. But *given* z (i.e. $Z = TRUE$), only two possibilities remain in my universe; I do not care about the cases where $Z = FALSE$. My universe has "shrunk" by knowing $Z = TRUE$, and I am asking for one of the two remaining cases *as a proportion of that smaller universe*. So more expansively, we are looking at this:

$$P(y|z) = \frac{P(y \land z)}{P(y \land z) + P(\neg y \land z)} = \frac{P(y \land z)}{P(z)}$$

In other words, the probability I care about ($P(y \wedge z)$) out of the subset of worlds I care about ($P(z)$). A picture can also help make this clear:

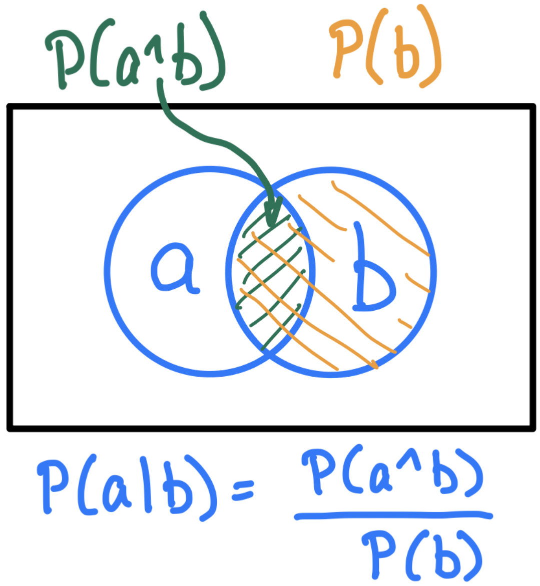


Figure 1: Of all the possible worlds in b, what proportion of them are also in a?

Consider the dentist example again. We can now answer question that we previously could not. "If I have a toothache, what is the likelihood that I have a cavity?" This is an important consideration, because having a toothache probably makes it *more likely* that we have a cavity. We can compute it like this:

$$\begin{aligned} P(cavity|toothache) &= \frac{P(cavity \wedge toothache)}{P(toothache)} \\ &= \frac{0.12}{0.108 + 0.012 + 0.016 + 0.064} \\ &= \frac{0.12}{0.2} = 0.6 \end{aligned}$$

So *given* that I have a toothache, there is now a 60% chance that I have a cavity, compared with a 20% chance of a cavity when we did not know that I have a toothache. Out of all worlds where I *do* have a toothache, the proportion of those worlds in which I *also* have a cavity is 0.6.

The denominator of that calculation is called the *normalization constant*. It is just the sum of all probabilities in the universe that we care about (i.e. where the given condition is true). Without it, we get answers that are *proportionally correct* relative to each other:

$$\begin{aligned} P(cavity \wedge toothache) &= 0.12 \\ P(\neg cavity \wedge toothache) &= 0.08 \end{aligned}$$

But these are not easy to interpret, because they are not *probabilities*. They do not sum to one. Dividing by the sum of all the possibilities ensures they will sum to one:

$$\frac{P(cavity \wedge toothache)}{P(cavity \wedge toothache) + P(\neg cavity \wedge toothache)} = \frac{0.12}{0.2} = 0.6$$
$$\frac{P(\neg cavity \wedge toothache)}{P(cavity \wedge toothache) + P(\neg cavity \wedge toothache)} = \frac{0.08}{0.2} = 0.4$$

Now that our answers are scaled to sum to one again, we can easily compare them (60% and 40% chances) with other results.

8.4 Product, Chain, and Total Probability Rules

What our agents will usually want to ask themselves are questions like $P(x|e)$, or "what is the probability of some query x given some evidence e?" For example: "Given the various evidence collected by our sensors, what are the chances there is a pit at 2,2?" We can now easily lay out this equation in the general sense:

$$P(x|e) = \frac{x \wedge e}{P(e)} = \alpha P(x \wedge e)$$

where again, x is some query we care about ("is 2,2 a pit?") and e is all the evidence collected by our sensors (e.g. breezes in locations). This is just the definition of conditional probability again: out of all the worlds in which I saw this exact evidence, in how many of them is query x also true?

There is a huge problem here. To calculate $P(x \wedge e)$, the probability of me seeing exactly the sensor readings I have seen ($P(\neg W_{11} \wedge P(\neg P_{11}) \wedge \neg B_{11} \wedge \neg W_{12} \wedge \dots$) **and** that my query is true (P_{22}?), assumes that I have the FJD for my entire universe (e.g. the Wumpus game). Of course, for a partially-observable environment that I have only *partly* explored (like the Wumpus game or, frankly, most interesting problems), I *don't* have the FJD. So we need some additional tools to cope with *hidden variables*, things I care about or need to know, but that I do not or cannot know!

Earlier we discussed *marginalization*, a way to eliminate an unwanted variable by summing across all of its possible values in the FJD. For example:

$$P(cavity) = \sum_z P(cavity \wedge z)$$

where z takes on each of the four possible combinations of $CATCH, TOOTHACHE$.

We can use the *product rule* to split a joint probability into a conditional and an unconditional probability:

$$P(Y, Z) = P(Y|Z)P(Z)$$

where joint probability $P(Y, Z)$ could also be written $P(Y \wedge Z)$. For example: $P(Y, Z)$ could be the chance that I have both a cavity and a toothache. $P(Y|Z)$ could be the chance that I have a cavity given that I *do* have a toothache. $P(Z)$ could be the chance that I have a cavity in general. "What's the probability that *if* Z happens, then Y *also* happens, times the probability that Z happens in the first place?"

If we apply the product rule multiple times, we get the *chain rule*:

$$\begin{aligned} P(X,Y,Z) &= P(X|Y,Z)P(Y,Z) \\ &= P(X|Y,Z)P(Y|Z)P(Z) \\ P(W,X,Y,Z) &= P(W|X,Y,Z)P(X,Y,Z) \end{aligned}$$

In the first example, we used the product rule to split the joint probability of X, Y, and Z into the probability of X conditioned on both Y and Z, times the joint probability of Y and Z. Then we used the product rule again to further split the joint probability of Y and Z. In the second example, we used the product rule to split the joint probability of four RVs into a conditional probability times the three RV joint probability we calculated previously.

So the product rule (and the chain rule) allow us to change between conditional and joint distributions as needed. This will be an important part of answering questions for our intelligent agent when we need the FJD, but only have some conditional probabilities. We can also formulate the *total probability rule* via conditioning:

$$P(Y) = \sum_{z \in Z} P(Y|Z=z)P(Z=z)$$

where Y and Z are random variables, and z is one particular possible value for RV Z. If the two RVs are both boolean, then we have:

$$P(Y) = P(Y \wedge Z) + P(Y \wedge \neg Z) = P(Y|z)P(z) + P(Y|\neg z)P(\neg z)$$

which covers all of the possibilities. Either z occurs or it does not, and for both cases, we then ask "now what is the probability that y also occurs?" This is very similar to obtaining total probability via marginalization, but now we are using the product rule to get conditional probabilities instead of joint probabilities.

So the total probability rule (or conditioning) allows us to change between conditional probabilities and the unconditional probability of a single random variable. With this, the product rule, and the chain rule, we can now get from basically any probability we have to any probability we need, without requiring the FJD. Did this solve our problem of hidden information?

8.5 Bayes' Rule

Imagine that you are a doctor with a patient. The patient has symptoms, which we will treat as a boolean S, true for this patient's exact symptoms and false otherwise. You want to predict if the symptoms are being caused by some specific disease D.

It is easy to find the overall prevalence of a disease in the general population (e.g. about 10% of U.S. residents have diabetes). It is also easy to find the prevalence of symptoms (e.g. about 15% of U.S. residents have a headache at any given time). Government agencies publish this information in many forms, so $P(D)$ and $P(S)$ can be obtained.

In the academic medical literature, you often see lists of symptom rates among people with a disease. For example, about 30% of migraine sufferers have *aura* (a visual hallucination of scintillating light shortly before onset of pain). From this, we can also obtain $P(S|D)$. "Of patients with this disease, how many have this symptom?"

What we really wanted to know was $P(D|S)$: "given these symptoms, how likely is it that your patient has this particular disease?" We don't have a good way to get that information. To directly calculate it, we would need the FJD of these symptoms with all possible diseases that could cause them, even the potential hidden causes we don't know about! But maybe we could find a way to relate $P(S|D)$, which we have, to $P(D|S)$, which we want:

$$\begin{aligned} P(S \wedge D) &= P(D \wedge S) \\ P(S|D)P(D) &= P(D|S)P(S) \\ \frac{P(S|D)P(D)}{P(S)} &= P(D|S) \end{aligned}$$

The first assertion is obviously true, because the boolean operator **and** is commutative; $\forall a, b : a \wedge b = b \wedge a$. We applied the product rule to both sides to obtain the second statement. We divided both sides by $P(S)$ (just some real number) to obtain the third statement. This final result is called *Bayes' Rule* (or sometimes Bayes' Law), and allows us to "flip" a conditional probability around to the other direction. In this case, it allows us to use the three pieces of information we have: $P(S)$, $P(D)$, $P(S|D)$, to obtain the missing information we want: $P(D|S)$.

8.5.1 Practical Example

Let's apply Bayes' rule to a realistic example. ***Disclaimer: All of the numbers in this section are absolutely made up!***

Imagine that a new, cost-effective Covid-19 test has been developed and is under consideration for public distribution. In preliminary trials, the test returns a negative result for 1% of people who actually do have Covid-19, and it returns a positive result for 10% of people who do not actually have Covid-19. That is, the test has a 10% false positive rate (false alarm) and a 1% false negative rate (misses a real Covid case). Assume 3% of the general population currently have Covid-19. A person takes this new test, and the result is positive. What is the probability that the person actually has Covid-19?

Carefully considering the above, and letting C represent the person actually having Covid-19, and T represent a positive test result, we can say that we know:

$$\begin{aligned} P(C) &= 0.03 \\ P(\neg C) &= 0.97 \\ P(\neg T|C) &= 0.01 \\ P(T|\neg C) &= 0.1 \\ P(T|C) &= 0.99 \\ P(\neg T|\neg C) &= 0.9 \end{aligned}$$

The first two statements represent the prevalence of Covid-19, the third line is the false negative rate of the test, the fourth line is the false positive rate of the test, and the final two lines represent the cases where the test is accurate. What we would like to know is $P(C|T)$. We can start by using Bayes' Rule:

$$P(C|T) = \frac{P(T|C)P(C)}{P(T)}$$

There is a problem. We don't know $P(T)$, the positivity rate of the test among the general population. This is a hidden variable that we cannot directly observe given our data and knowledge. However, we can calculate it using conditioning and the product rule:

$$P(T) = P(T|C)P(C) + P(T|\neg C)P(\neg C)$$

We are summing all possible combinations that include a positive test, of which there are only two: the person has Covid-19 or not. We then use the product rule to further separate each joint probability. Plugging in our information:

$$P(T) = 0.99(0.3) + 0.1(0.97) = 0.0297 + 0.097 = 0.1267$$

This says, given the prevalence of the disease and the accuracy of the test, if we grab a random person off the street (with no idea if they have Covid-19), we expect a 12.67% chance the test would be positive. *Now* we can apply Bayes' Rule to get the result we wanted:

$$P(C|T) = \frac{0.99(0.3)}{0.1267} = 0.234$$

So only 23.4% of people who test positive will actually turn out to have Covid-19. The test is actually reasonably accurate, though! It only gives false positives 10% of the time and false negatives 1% of the time. So what's the problem? Well, at least according to the numbers we used in this example, Covid-19 was *rare to begin with* (only 3% of people having it). This doesn't necessarily mean the test is awful, but it does show why doctors tend to run *multiple tests* to confirm whether or not there is a problem.

8.6 Independence and Conditional Independence

For our purposes in AI, where we are trying to use probability to improve performance in partially-observable worlds, we will often apply Bayes' Rule in this way:

$$P(cause|effect) = \frac{P(effect|cause)P(cause)}{P(effect)}$$

There are *hidden causes* of the *effects we can observe*. In our Wumpus game, we cannot directly observe the Wumpus or the pits. We don't have sensors for that, and when we walk into one, our character dies. However, the Wumpus causes nearby smells, and the pits cause nearby breezes, which we *can* observe. So our task is to use our observed evidence (the effects) to infer the presence of the hidden causes. "Given that I have observed these breezes, and those smells, how likely is there to be a pit or a Wumpus at 2,2?"

Before we can apply this to our problems, there is one more important property we must understand. Some random variables (RVs) affect the distribution of other RVs, and some do not. Whether a person has a cavity does (at least partly) depend on whether they have a toothache, and vice versa. Whether a person has a cavity does *not depend* on the phase of the moon! We can illustrate this numerically:

$$\begin{aligned} P(cavity) &= 0.2 \\ P(cavity|toothache) &= 0.6 \\ P(cavity|full{-}moon) &= 0.2 \end{aligned}$$

The first numbers are the real values we calculated during the dentist example. Having a toothache should make us realize we are much more likely to have a cavity. Looking outside and seeing that the moon is full *does not change our estimate* of the likelihood we have a cavity. Why is the first result useful? Having a cavity is not directly observable (to us), but having a toothache certainly is! We can use the information that we *can* observe to gain some knowledge about the thing we *cannot* observe.

Two RVs are *independent* if and only if:

$$P(A|B) = P(A) \qquad P(B|A)P(B) \qquad P(A,B) = P(A)P(B)$$

Otherwise, they are dependent. The three statements above are equivalent, so if any of them are true, they all are (and the RVs are independent).

How does that result help us? Recall the random variable:

$$WEATHER =< rain, snow, hot, mild, combust >$$

What if we obtained the FJD of this RV with our three dental RVs?

$$P(\text{CAVITY}, \text{TOOTHACHE}, \text{CATCH}, \text{WEATHER}) \leftarrow \text{this has 40 entries } (2 \times 2 \times 2 \times 5)$$

But we just discovered that if two RVs are independent, then their joint probability is just their separate probabilities multiplied together. Are our dental variables independent of the weather? Sure they are! So we can separate them out like so:

$$P(\text{CAVITY}, \text{TOOTHACHE}, \text{CATCH})P(\text{WEATHER}) \leftarrow \text{this has 13 entries } (2 \times 2 \times 2 + 5)$$

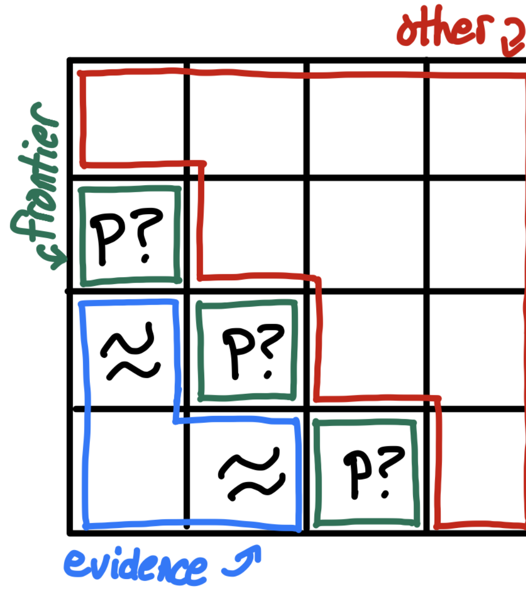
So because of independence, we now need only 13 pieces of information to reconstruct any of the 40 possible values of the FJD. Given that learning things from our sensors is the hard part of AI, not needing to sense as many things is a big plus! However, the downside of two RVs being independent is that they do not contain any information about each other. (Learning that I have a cavity does not let me predict the weather.)

Similarly, two RVs can be *conditionally independent* if, when given the value of a third RV, the two then contain no information about each other. This is written $A \perp B|C$. "A is independent of B given C." We define this as:

$$A \perp B|C \iff P(A,B|C) = P(A|C)P(B|C)$$

Note that this is the same as regular independence, except everything is conditioned on C. Also note that A and B may *or may not* be unconditionally independent.

As we discussed in class, one example of this is height, vocabulary, and age. Do you gain information about a person's potential vocabulary by observing their height? Yes, children tend to be short and have smaller vocabularies. Given a person's age, do you gain information about a person's vocabulary by observing their height? No, there is no reason to think that a shorter or taller adult will have a smaller or larger vocabulary.



8.7 Wumpus World

We can now use probability to answer the question that our propositional logic agent could not. Remember the Wumpus World example in which the logical agent was stumped: Near the beginning of the game, having started in the lower left corner (1,1), the agent moved up to (1,2) and felt a breeze, then moved down and right to (2,1) and felt a breeze. Given this evidence, it was impossible to declare any adjacent (frontier) square definitely safe, and so we gave up.

Let's describe the relevant information about the game. We know that there is an independent 20% chance to generate a pit in each square, and that a pit always generates a breeze in every adjacent square. We have visited (1,1), (2,1), and (1,2), fallen in no pits, and detected two breezes. We would like to infer the position of some hidden causes of breezes (the pits).

$$\begin{aligned} \forall i,j : P(P_{ij}) &= 0.2 \\ \text{evidence} &: \neg P_{11}, \neg P_{12}, \neg P_{21}, \neg B_{11}, B_{12}, B_{21} \\ \text{hidden causes} &: P_{11}, P_{12}, P_{13}, P_{14}, \ldots \end{aligned}$$

We would especially like to know P_{13}, P_{22}, and P_{31}, as those are the nearby places we need to explore next. Together, we can call our evidence about pits *pits* and our evidence about breezes *breezes*. We call the area we need to explore next the $frontier$ and the remaining area *other*.

To answer questions about the joint likelihood of our evidence and particular configurations of pits, we would need the FJD, and could separate it at least somewhat with the product rule:

$$P(P_{11}, P_{12}, P_{13}, P_{14}, \ldots, B_{11}, B_{12}, B_{21}) = P(B_{11}, B_{12}, B_{21} | P_{11}, \ldots, P_{44}) P(P_{11}, \ldots, P_{44})$$

The first (conditional) part of our equation looks complicated, but is quite easy. Given a particular configuration of pits, what is the likelihood that we would observe a particular combination of breezes? Well, the probability will be exactly one if the breezes are in the locations required by our breezes-next-to-pits rule, and exactly zero otherwise!

What can we do with the second part of our equation: the unconditional probability of generating a particular configuration of pits? Since we know that pits are independent of one another, this is also easy! The probability of generating any particular configuration of n pits is $0.2^n(0.8)^{16-n}$. That's the chance to generate a pit n times, and the chance to generate a non-pit $16-n$ times, because we have a 4x4 grid of 16 squares.

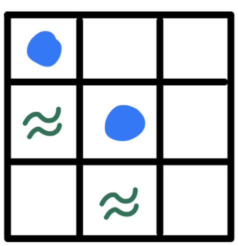
Now we can ask our question for a particular square. What is the chance there is a pit in 1,3, or rather $P(P_{13})$? We do not need to account for breezes outside our current "evidence" area, as they will have no effect. (It is irrelevant whether there is a breeze at 4,4.). We do account for all pits and our breeze evidence.

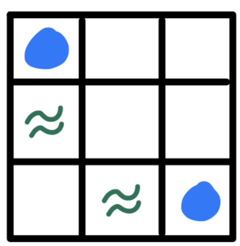
In class, we worked through the derivation to obtain:

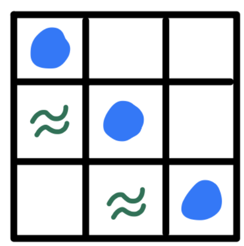
$$P(P_{13}|evidence) = \alpha P(P_{13}) \sum_{frontier} P(breezes|P_{13}, pits, frontier)P(frontier)$$

and observed that *breezes* and *pits* are constant terms (evidence we know for sure), P_{13} has two possibilities, and *frontier* has four possibilities (because it is just two pits with two possibilities each). Thus there are only eight cases for us to consider, where the FJD we started with was at least $2^{19} = 4096$ cases. Our independence assumptions dramatically reduced the problem again!

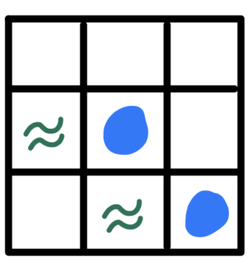
There are three cases with a pit at (1,3) that are also consistent with our breezes-next-to-pits rule. For these cases, the conditional probability in our equation above will equal one, and so we need to compute them.

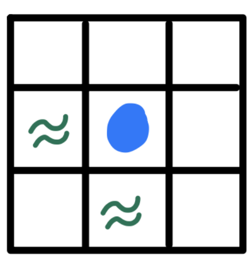




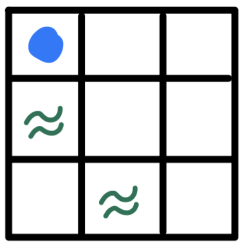
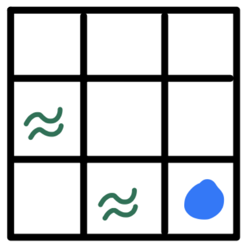
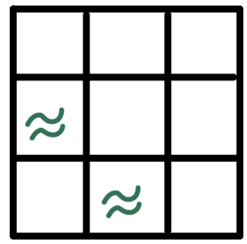


There are two cases without a pit at (1,3) that are also consistent with the breezes-next-to-pits rule. We do need to calculate these. Since we are working with a subset of the probability universe U, we need to normalize our results to sum to one, and we can only do that if we compute both the true and false cases for P_{13}.





There are three cases that are *not* consistent with the breezes-next-to-pits rule. We do not need to calculate these, because the conditional probability term in our equation will be zero, eliminating these possibilities.

The first three cases will be computed as part of $P_{13} = TRUE$ and the second two cases as part of $P_{13} = FALSE$, giving us:

$$\begin{aligned}
P(P_{13}|evidence) &= \alpha < 0.2(0.2 \times 0.8 + 0.8 \times 0.2 + 0.2 \times 0.2, 0.8(0.2 \times 0.2 + 0.2 \times 0.8) > \\
&= \alpha < 0.2(0.16 + 0.16 + 0.04), 0.8(0.04 + 0.16) > \\
&= \alpha < 0.2(0.36), 0.8(0.2) > \\
&= \alpha < 0.072, 0.16 > \\
&=< \frac{0.072}{0.072 + 0.16}, \frac{0.16}{0.072 + 0.16} > \\
&=< 0.3103, 0.6897 >
\end{aligned}$$

So *given our evidence*, there is a 31% chance of a pit at 1,3. Since the game situation we found ourselves in is diagonally symmetric, the probability of a pit at 3,1 will be exactly the same. We can easily compute the math for the case of a pit at 2,2:

$$\begin{aligned}
P(P_{22}|evidence) &= \alpha < 0.2(0.16 + 0.16 + 0.04 + 0.64), 0.8(0.04) > \\
&= \alpha < 0.2(1.0), 0.8(0.04) > \\
&= \alpha < 0.2, 0.032 > \\
&=< \frac{0.2}{0.2 + 0.032}, \frac{0.032}{0.2 + 0.032} > \\
&=< 0.8621, 0.1379 >
\end{aligned}$$

So *given our evidence*, there is an 86% chance of a pit at 2,2! Where the agent based on propositional logic could only respond "don't know where to go next", the probability-based agent knows that exploring 2,2 next would be a near-certain death sentence, while exploring either 1,3 or 3,1 would be the far safer option.

Acknowledgements and thanks to Professors Mark Riedl and Jim Rehg of the Georgia Tech School of Interactive Computing.