SNM-GARCH:
A Semi-Parametric Mixture Model

Supervisor: Dr Yong Wang
Michael Kao

Dept. of Statistics, University of Auckland

Outline

History

- The introduction of ARCH/GARCH by Engle(1982) and Bollerslev(1986), made major contribution to the advancement of volatility modelling.
- A paper by Tim Bollerslev (2008) documented more than 100 generalisation/extension of the ARCH/GARCH type model.
- Its great, but we can do better!

Motivation

- Problem: The standard normal assumption is often violated in practice due to the skewness and the heavy-tail nature of the financial data.
- Solution: Relax the standard normal assumption to a Scale Normal Mixture (SNM).
- Result: Improved fit and decomposition of the source of volatility.

The general GARCH(p, q) model

The GARCH model proposed by Tim Bollerslev (1986).

GARCH(p, q)

$$x_t = \mu + \sigma_t \epsilon_t \qquad \epsilon_t \sim N(0,1)$$

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{p} \beta_i \sigma_{t-i}^2 + \sum_{j=1}^{q} \alpha_j x_{t-j}^2$$

Where the log return x_t is defined as $\nabla \log(P_t)$.

Semi-Parametric Mixture Model

- Non-parametric $\leftarrow$ Semi-parametric $\rightarrow$ Parametric.
- In this context, the SNM-GARCH model compose of parametric estimation of volatility movement and non-parametric estimation of the Scale Nomal Mixture (SNM) error distribution.

Scale Normal Mixture Distribution

A scale normal mixture in its simplest term is just a distribution consists of a mixture of normal distributions.

Scale Normal Mixture Distribution

$$f(x; G) = \int_{\Omega} \phi(x; \theta) \mathrm{d}G(\theta)$$

where the $\phi(x; \theta)$ is the Normal component density and $G(\theta)$ the mixing distribution function.

Scale Normal Mixture Distribution Cont.

Since our estimation is based on maximum likelihood with finite data points, the MLE of $G(\theta)$ must be discrete and we can rewrite the density as the following:

Scale Normal Mixture Distribution

$$G(\theta) = \sum_{i=1}^{m} \pi_i \delta_{\theta_i}$$

$$f(x; \mu, \boldsymbol{\pi}, \boldsymbol{\theta}) = \sum_{i=1}^{m} \pi_i \; \phi(\mu, \theta_i),$$

Where θ_i's are the support points of the mixing distribution.

Scale Normal Mixture GARCH (p, q)

A SNM-GARCH model is simply a GARCH model where the assumption of standard normal error distribution is replaced with a scale normal mixture (SNM) distribution.

SNM-GARCH

$$x_t = \mu + \sigma_t \epsilon_t \qquad \epsilon_t \sim SNM(0, G)$$

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{p} \beta_i \sigma_{t-i}^2 + \sum_{j=1}^{q} \alpha_j x_{t-j}^2$$

Estimation

The estimation is carried out under the semi-parametric mixture framework in Wang (2010) via maximum likelihood estimation.

CNM-MS algorithm

1. Add support point via the CNM algorithm.
2. Use BFGS to optimise the likelihood with respect to all the parameters $(\alpha_0, \alpha_1, \beta_1, \mu, \boldsymbol{\pi}, \boldsymbol{\theta})$ and update.
3. Scale to ensure the Scale Normal Mixture Distribution has variance 1.

Likelihood Function

Likelihood Function

$$l(x;\alpha_0,\alpha_1,\beta_1,\mu,\sigma_t,\boldsymbol{\theta})$$

$$=\prod_{t=1}^{T}\prod_{i=1}^{m}\frac{1}{\sqrt{2\pi\theta_i^2(\alpha_0+\sum_{i=1}^{p}\alpha_1\sigma_{t-1}^2+\sum_{j=1}^{q}\beta_1x_{t-1}^2)}}$$

$$\exp\left(\frac{-(x_t-\mu)^2}{2\theta_i^2(\alpha_0+\sum_{i=1}^{p}\alpha_1\sigma_{t-1}^2+\sum_{j=1}^{q}\beta_1x_{t-1}^2)}\right),$$

$$\alpha_0>0,\forall\alpha_i\geq 0,\forall\beta_i\geq 0$$

NYSE

This example is taken from Shumway and Stoffer (2010). The data consists of 2000 daily log return of the New York Stock Exchange time series over the period starting from February 1st, 1984 and ends on the December 31st, 1991.

NYSE

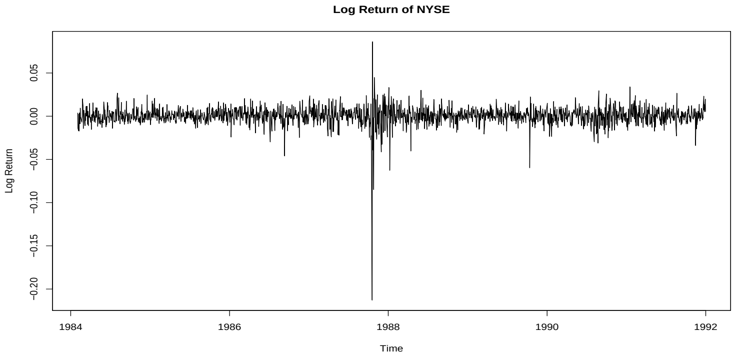
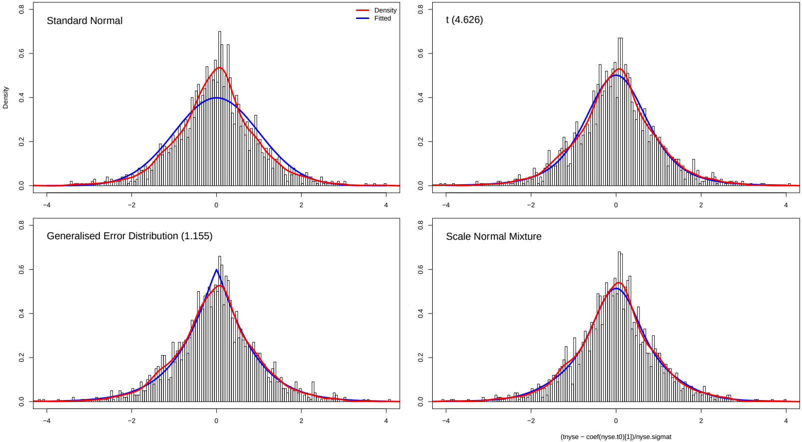
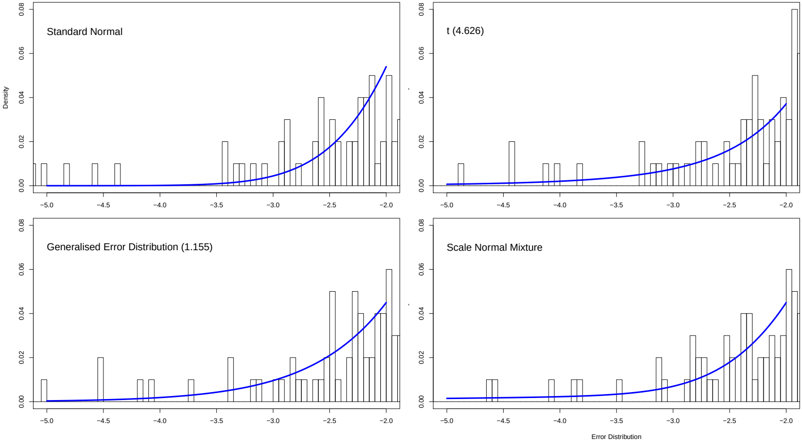


Figure: The plot of the daily log return. The large negative spike corresponds to the market crash on October 19th, 1987 also known as the Black Monday

Fit


Standard Normal
Density
Fitted
t (4.626)
Generalised Error Distribution (1.155)
Scale Normal Mixture
Density
(tnyse − coef(nyse.t0)[1])/nyse.sigmat

Zoomed Fit


Standard Normal
t (4.626)
Generalised Error Distribution (1.155)
Scale Normal Mixture
Density
Error Distribution
0.00
0.02
0.04
0.06
0.08
−5.0
−4.5
−4.0
−3.5
−3.0
−2.5
−2.0

S&P 500

This dataset contains the daily log return of the S&P 500 since Janurary 1st, 2005 to current date.

S&P 500

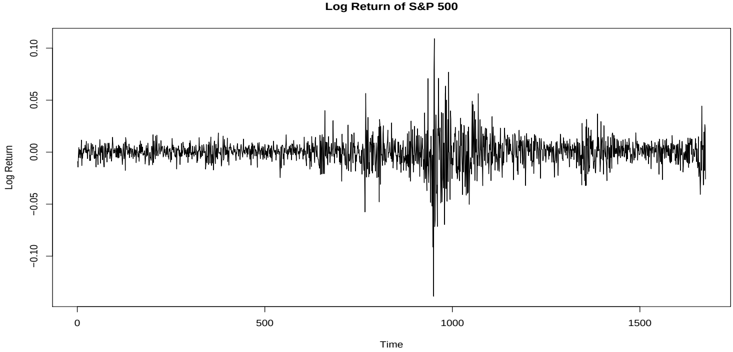
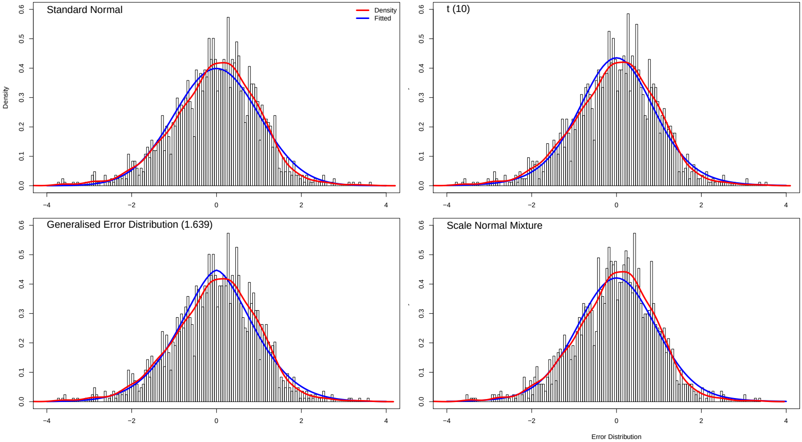
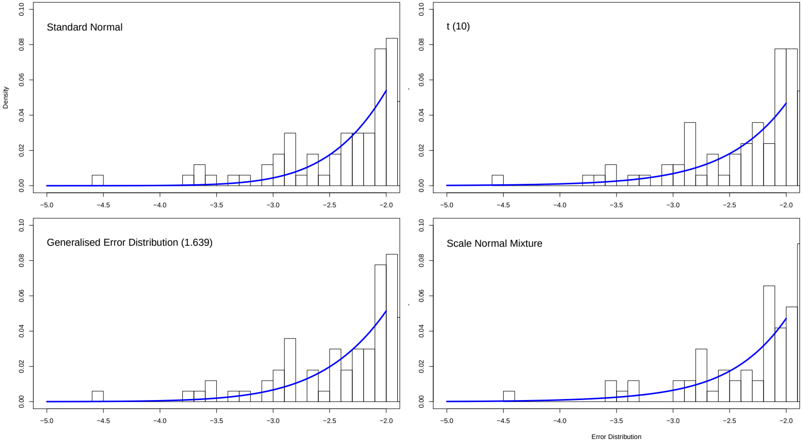


Figure: The S&P 500 data, we can see the volatility structure is quite different to that of the NYSE.

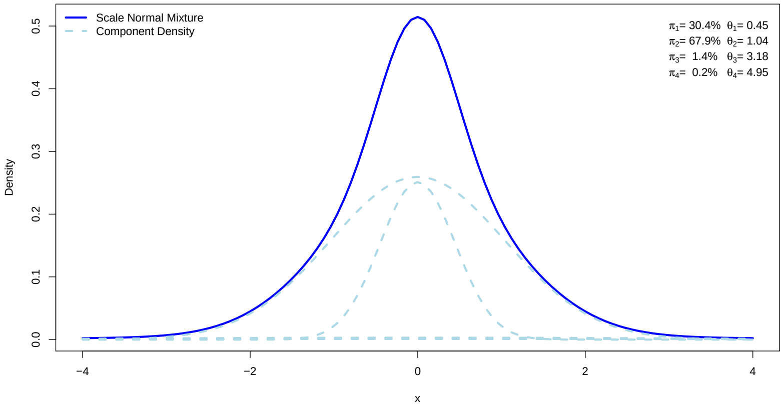
Fit


Standard Normal
Density
Fitted
t (10)
Generalised Error Distribution (1.639)
Scale Normal Mixture
Density
Error Distribution
0.0
0.1
0.2
0.3
0.4
0.5
0.6
−4
−2
0
2
4

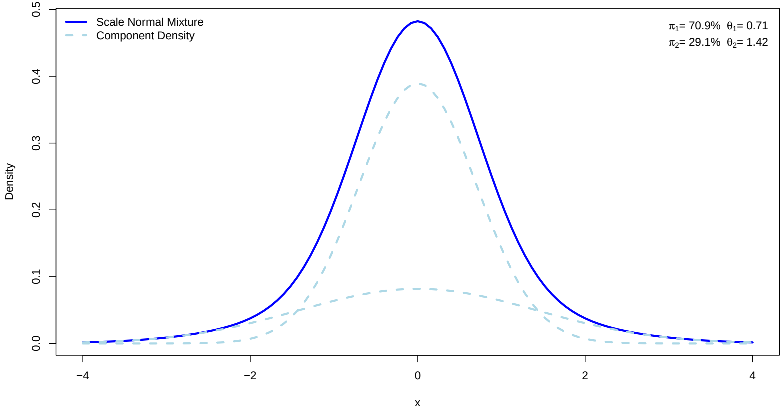
Fit Zoomed


Density
Standard Normal
t (10)
Generalised Error Distribution (1.639)
Scale Normal Mixture
Error Distribution
0.00
0.02
0.04
0.06
0.08
0.10
−5.0
−4.5
−4.0
−3.5
−3.0
−2.5
−2.0

NYSE


Scale Normal Mixture
Component Density
π1= 30.4% θ1= 0.45
π2= 67.9% θ2= 1.04
π3= 1.4% θ3= 3.18
π4= 0.2% θ4= 4.95
Density
0.0
0.1
0.2
0.3
0.4
0.5
−4
−2
0
2
4
x

S&P 500


Scale Normal Mixture
Component Density
π1= 70.9% θ1= 0.71
π2= 29.1% θ2= 1.42
Density
x
0.0
0.1
0.2
0.3
0.4
0.5
−4
−2
0
2
4

Conclusion

- A flexible frame work for GARCH modelling while preserving the theoretical advantage of the normal distribution.
- Further understand the source of volatility.

Further Development

- Extend to GARCH(p, q).
- Incorporate Skewness parameter in the model.
- Formal Tests.

Thank you!!!!!!!!!!