SNM-GARCH: A Semi-Parametric Mixture Model to Account for Black Swans

Cheng-Jan Kao
ID: 3910587

1 logd.mgarch

1.1 log density

$$
\begin{aligned}
f(\mathbf{x};\theta_i,\sigma_t) &= \frac{1}{\sqrt{2\pi\theta_i^2\sigma_t^2}}\exp\left(-\frac{x^2}{2\theta_i^2\sigma_t^2}\right) \\
l(\mathbf{x};\theta_i,\sigma_t) = \log f(\mathbf{x};\theta_i,\sigma_t) &= -\frac{1}{2}\log 2\pi - \log\theta_i - \log\sigma_t - \frac{x_t^2}{2\theta_i^2\sigma_t^2} \\
\frac{\partial l}{\partial \sigma_t} &= \left(-\frac{1}{\sigma_t} + \frac{x_t^2}{\theta_i^2\sigma_t^3}\right)
\end{aligned}
$$

where

$$
\begin{aligned}
\sigma_t^2 &= \alpha_0 + \alpha_1\sigma_{t-1}^2 + \beta_1\sigma_{t-1}^2 \\
&= \alpha_0\sum_{i=0}^{t-2}\alpha_1^i + \alpha_1^{t-1}\sigma_1^2 + \beta_1\sum_{j=0}^{t-2}\alpha_1^j x_{t-j-1} \quad \text{for } t \geq 2 \\
\sigma_t &= \left(\alpha_0\sum_{i=0}^{t-2}\alpha_1^i + \alpha_1^{t-1}\sigma_1^2 + \beta_1\sum_{j=0}^{t-2}\alpha_1^j x_{t-j-1}\right)^{1/2}
\end{aligned}
$$

Then

$$
\begin{aligned}
\frac{\partial \sigma_t}{\partial \alpha_0} &= \frac{1}{2\sigma_t}\left(\sum_{i=0}^{t-2} \alpha_1^i\right) \\
\frac{\partial \sigma_t}{\partial \alpha_1} &= \frac{1}{2\sigma_t}\left(\alpha_0 \sum_{i=1}^{t-2} i\alpha_1^{i-1} + (t-1)\alpha_1^{t-2}\sigma_1^2 + \beta_1 \sum_{j=1}^{t-2} j\alpha_1^{j-1} x_{t-j-1}\right) \\
\frac{\partial \sigma_t}{\partial \beta_1} &= \frac{1}{2\sigma_t}\left(\sum_{j=0}^{t-2} \alpha_1^j x_{t-j-1}\right) \\
\frac{\partial \sigma_t}{\partial \sigma_1} &= \frac{1}{2\sigma_t}\left(2\alpha_1^{t-1}\sigma_1\right)
\end{aligned}
$$

1.2 Partial Beta

$$
\begin{aligned}
\frac{\partial l}{\partial \alpha_0} &= \frac{\partial l}{\partial \sigma} \times \frac{\partial \sigma}{\partial \alpha_0} \\
&= \left(\sum_{t=1}^{T}\sum_{i=1}^{M} -\frac{1}{\sigma_t} + \frac{x_t^2}{\theta_i^2\sigma_t^3}\right) \times \frac{1}{2\sigma_t}\left(\sum_{i=0}^{t-2} \alpha_1^i\right) \\
\frac{\partial l}{\partial \alpha_1} &= \frac{\partial l}{\partial \sigma} \times \frac{\partial \sigma}{\partial \alpha_1} \\
&= \left(\sum_{t=1}^{T}\sum_{i=1}^{M} -\frac{1}{\sigma_t} + \frac{x_t^2}{\theta_i^2\sigma_t^3}\right) \times \frac{1}{2\sigma_t}\left(\alpha_0 \sum_{i=1}^{t-2} i\alpha_1^{i-1} + (t-1)\alpha_1^{t-2}\sigma_1^2 + \beta_1 \sum_{j=1}^{t-2} j\alpha_1^{j-1} x_{t-j-1}\right) \\
\frac{\partial l}{\partial \beta_1} &= \frac{\partial l}{\partial \sigma} \times \frac{\partial \sigma}{\partial \beta_1} \\
&= \left(\sum_{t=1}^{T}\sum_{i=1}^{M} -\frac{1}{\sigma_t} + \frac{x_t^2}{\theta_i^2\sigma_t^3}\right) \times \frac{1}{2\sigma_t}\left(\sum_{j=0}^{t-2} \alpha_1^j x_{t-j-1}\right) \\
\frac{\partial l}{\partial \sigma_1} &= \frac{\partial l}{\partial \sigma} \times \frac{\partial \sigma}{\partial \sigma_1} \\
&= \left(\sum_{t=1}^{T}\sum_{i=1}^{M} -\frac{1}{\sigma_t} + \frac{x_t^2}{\theta_i^2\sigma_t^3}\right) \times \frac{\alpha^{t-1}\sigma_1}{\sigma_t}
\end{aligned}
$$

1.3 Partial Theta

$$
\begin{aligned}
\frac{\partial l}{\partial \boldsymbol{\theta}} &= \sum_{t=1}^{T}\sum_{i=1}^{M} -\frac{1}{\theta_i} + \frac{x_t^2}{\theta_i^3\sigma_t^2} \\
\frac{\partial^2 l}{\partial \boldsymbol{\theta}^2} &= \sum_{t=1}^{T}\sum_{i=1}^{M} \frac{1}{\theta_i^2} - \frac{3x_t^2}{\theta_i^4\sigma_t^2}
\end{aligned}
$$

2 Improvements and Further actions

1. (1) Get the model (With scaling) CORRECT!!!!!!!
2. Might remove beta[4], as it doesn't make much sense to estimate that.
3. Include skewness, and mean estimation (This might be very interesting and useful as we have see that commodity skews different direction to the normal commodities). The skewness seems slightly difficult to implement as it may not have analytical solution, will have a look at the implementation in the fGarch package.
4. survey all the GARCH/financial package.
5. Work on all sorts of financial data.
6. Get Matthieu and others at FAO to review it.
7. Improve the efficiency of the logd.mgarch function.
8. Test the difference between real data and simulated data.
9. Have a look at the FAO book and talk to Matthieu about how the model can work. How would I assess the model other than using likelihood/information criteria based method? (e.g. prediction)