A Useful Complex-Valued Logic

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Abstract

A useful complex-valued logic based on three-valued logic (with balanced ternary representation) and assumption that the logical equivalence corresponds to the hypercomplex multiplication operation.

1 Balanced Ternary Algebra

1.1 Definitions

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1.1.1 \ \mathbb{N} := \{1, 2, 3, \dots\}
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$$1.1.2 \ \mathbb{Z}_3 \coloneqq \{-1, 0, 1\}$$

1.1.3
$$\forall x, y \in \mathbb{Z}_3$$
:

$$1.1.3.1 \ x \oplus y \coloneqq (x+y+4) \bmod 3 - 1$$

$$1.1.3.2 \ x \otimes y \coloneqq xy$$

1.1.4 $\forall x \in \mathbb{Z}_3$:

$$1.1.4.1 \ \ominus x := -1 \otimes x$$

$$1.1.4.2 \ x^{\otimes 0} \coloneqq 1$$

1.1.4.3
$$\forall n \in \mathbb{N} \colon x^{\otimes n} \coloneqq x \otimes x^{\otimes (n-1)}$$

1.2 Corollaries

 $\forall x \in \mathbb{Z}_3$:

$$1.2.1 \ x \oplus x = \ominus x \ (1.1.3.1, \ 1.1.4.1)$$

 $1.2.2 \ \forall n \in \mathbb{N}$:

$$1.2.2.1 \ x^{\otimes (2n-1)} = x \ (1.1.4.2, \ 1.1.4.3)$$

$$1.2.2.2 \ x^{\otimes (2n)} = x^{\otimes 2} \ (1.1.4.2, 1.1.4.3)$$

2 Three-Valued Logic

2.1 Definitions

$$2.1.1 \ \forall x \in \mathbb{Z}_3 \colon \ \neg x \coloneqq \ominus x$$

 $2.1.2 \ \forall x, y \in \mathbb{Z}_3$:

$$2.1.2.1 \ x \wedge y := \min(x, y)$$

$$2.1.2.2 \ x \lor y \coloneqq \neg (\neg x \land \neg y)$$

$$2.1.2.3 \ x \Rightarrow y \coloneqq \neg x \lor y$$

$$2.1.2.4 \ x \Leftrightarrow y := (x \Rightarrow y) \land (y \Rightarrow x)$$

2.2 Corollaries

 $\forall x, y \in \mathbb{Z}_3$:

$$2.2.1 \ x \lor y = \max(x, y) \ (2.1.2.2)$$

$$2.2.2 \ x \Leftrightarrow y = (x \lor \neg y) \land (\neg x \lor y) \ (2.1.2.3, \ 2.1.2.4)$$

$$2.2.3 \ x \Leftrightarrow y = (x \land y) \lor (\neg x \land \neg y) \ (2.2.2)$$

2.3 Algebraic Normal Forms

2.3.1 Algorithm

2.3.1.1 Given:

- $2.3.1.1.1\ D.$ $k\mbox{-vector}$ of sorted values of algebra's domain.
- 2.3.1.1.2 T: k^2 -vector of the truth table's values for binary operation \circ , where:

$$t_i = d_{\left\lceil \frac{i}{k} \right\rceil} \circ d_{(i \bmod k)}$$

2.3.1.2 Find:

2.3.1.2.1 A: k^2 -vector of coefficients for algebraic normal form of binary operation \circ , where:

$$x \circ y = \bigoplus_{i=1}^{k^2} \left(a_i \otimes x^{\otimes \left\lfloor \frac{i-1}{k} \right\rfloor} \otimes y^{\otimes ((i-1) \bmod k)} \right)$$

2.3.1.3 Solution:

2.3.1.3.1
$$V_{i,j} = d_i^{\otimes (j-1)}$$
 (Vandermonde matrix of size $k \times k$)

2.3.1.3.2
$$P_1 = V^{-1}$$
 (matrix inversion)

2.3.1.3.3
$$P_2 = P_1 \otimes P_1$$
 (Kronecker product)

2.3.1.3.4
$$A = P_2T$$
 (matrix multiplication)

2.3.2 Result

2.3.2.1
$$D = \begin{pmatrix} \ominus 1 \\ 0 \\ 1 \end{pmatrix}$$
 (2.3.1.1.1)

 $2.3.2.2 \ \forall x, y \in \mathbb{Z}_3$:

$$2.3.2.2.1 \ x \wedge y = \ominus y \oplus y^{\otimes 2} \ominus x \ominus x \otimes y \oplus x^{\otimes 2} \ominus x^{\otimes 2} \otimes y^{\otimes 2} \ (2.1.2.1, \ 2.3.1.3.4)$$

$$2.3.2.2.2 \quad x \lor y = \ominus y \ominus y^{\otimes 2} \ominus x \oplus x \otimes y \ominus x^{\otimes 2} \oplus x^{\otimes 2} \otimes y^{\otimes 2} \quad (2.2.1, 2.3.1.3.4)$$

$$2.3.2.2.3 \quad x \Rightarrow y = \ominus y \ominus y^{\otimes 2} \oplus x \ominus x \otimes y \ominus x^{\otimes 2} \oplus x^{\otimes 2} \otimes y^{\otimes 2} \quad (2.1.2.3, 2.3.1.3.4)$$

$$2.3.2.2.4 \ x \Leftrightarrow y = x \otimes y \ (2.1.2.4, \ 2.3.1.3.4)$$

3 Balanced Hypercomplex Algebras

3.1 Definitions

$$3.1.1 \ \mathbb{C}_3 := \left\{ a \oplus b \otimes i \mid a, b \in \mathbb{Z}_3, \ i^{\otimes 2} = -1 \right\}$$

3.1.2
$$\mathbb{D}_3 := \{ a \oplus b \otimes \varepsilon \mid a, b \in \mathbb{Z}_3, \ \varepsilon^{\otimes 2} = 0 \wedge \varepsilon \neq 0 \}$$

$$3.1.3 \, \mathbb{S}_3 := \left\{ a \oplus b \otimes j \mid a, \, b \in \mathbb{Z}_3, \, j^{\otimes 2} = 1 \wedge j \notin \{-1, \, 1\} \right\}$$

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\begin{aligned} 3.1.4 \ \forall u \in \{i, \, \varepsilon, \, j\} \,, \ \forall a, \, b, \, c, \, d \in \mathbb{Z}_3 \colon \ \forall x = a \oplus b \otimes u, \ \forall y = c \oplus d \otimes u \colon \\ 3.1.4.1 \ x \oplus y &\coloneqq a \oplus c \oplus (b \oplus d) \otimes u \\ 3.1.4.2 \ x \otimes y &\coloneqq a \otimes c \oplus u^{\otimes 2} \otimes b \otimes d \oplus (a \otimes d \oplus b \otimes c) \otimes u \\ 3.1.5 \ \forall \mathbb{K} \in \{\mathbb{C}_3, \, \mathbb{D}_3, \, \mathbb{S}_3\} \colon \ \forall x \in \mathbb{K} \colon \\ 3.1.5.1 \ \ominus x &\coloneqq -1 \otimes x \\ 3.1.5.2 \ x^{\otimes 0} &\coloneqq 1 \\ 3.1.5.3 \ \forall n \in \mathbb{N} \colon \ x^{\otimes n} &\coloneqq x \otimes x^{\otimes (n-1)} \end{aligned}
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3.2 Corollaries

 $\forall \mathbb{K} \in \{\mathbb{C}_3, \mathbb{D}_3, \mathbb{S}_3\} : \forall x \in \mathbb{K} :$ $3.2.1 \ x \oplus x = \ominus x \ (3.1.4.1, \ 3.1.5.1)$ $3.2.2 \ \forall n \in \mathbb{N} :$ $3.2.2.1 \ x^{\otimes (4n-3)} = x \ (3.1.5.2, \ 3.1.5.3)$ $3.2.2.2 \ x^{\otimes (4n-2)} = x^{\otimes 2} \ (3.1.5.2, \ 3.1.5.3)$ $3.2.2.3 \ x^{\otimes (4n-1)} = x^{\otimes 3} \ (3.1.5.2, \ 3.1.5.3)$ $3.2.2.4 \ x^{\otimes (4n)} = x^{\otimes 4} \ (3.1.5.2, \ 3.1.5.3)$

3.3 Multiplication Tables (3.1.4.2)

\mathbb{C}_3	$\ominus 1 \ominus i$	⊖1	$\ominus 1 \oplus i$	$\ominus i$	0	i	$1\ominus i$	1	$1 \oplus i$
$\ominus 1 \ominus i$	$\ominus i$	$1 \oplus i$	⊖1	$\ominus 1 \oplus i$	0	$1\ominus i$	1	$\ominus 1 \ominus i$	i
$\ominus 1$	$1 \oplus i$	1	$1\ominus i$	i	0	$\ominus i$	$\ominus 1 \oplus i$	⊖1	$\ominus 1 \ominus i$
$\ominus 1 \oplus i$	$\ominus 1$	$1\ominus i$	i	$1 \oplus i$	0	$\ominus 1 \ominus i$	$\ominus i$	$\ominus 1 \oplus i$	1
$\ominus i$	$\ominus 1 \oplus i$	i	$1 \oplus i$	⊖1	0	1	$\ominus 1 \ominus i$	$\ominus i$	$1\ominus i$
0	0	0	0	0	0	0	0	0	0
i	$1\ominus i$	$\ominus i$	$\ominus 1 \ominus i$	1	0	⊖1	$1 \oplus i$	i	$\ominus 1 \oplus i$
$1\ominus i$	1	$\ominus 1 \oplus i$	$\ominus i$	$\ominus 1 \ominus i$	0	$1 \oplus i$	i	$1\ominus i$	$\ominus 1$
1	$\ominus 1 \ominus i$	$\ominus 1$	$\ominus 1 \oplus i$	$\ominus i$	0	i	$1\ominus i$	1	$1 \oplus i$
$1 \oplus i$	i	$\ominus 1 \ominus i$	1	$1\ominus i$	0	$\ominus 1 \oplus i$	$\ominus 1$	$1 \oplus i$	$\ominus i$
\mathbb{D}_3	$\ominus 1 \ominus \varepsilon$	⊖1	$\ominus 1 \oplus \varepsilon$	$\ominus \varepsilon$	0	ε	$1\ominus \varepsilon$	1	$1 \oplus \varepsilon$
$\ominus 1 \ominus \varepsilon$	$1\ominus \varepsilon$	$1 \oplus \varepsilon$	1	ε	0	$\ominus \varepsilon$	⊖1	$\ominus 1 \ominus \varepsilon$	$\ominus 1 \oplus \varepsilon$
$\ominus 1$	$1 \oplus \varepsilon$	1	$1\ominus \varepsilon$	ε	0	$\ominus \varepsilon$	$\ominus 1 \oplus \varepsilon$	⊖1	$\ominus 1 \ominus \varepsilon$
$\ominus 1 \oplus \varepsilon$	1	$1\ominus \varepsilon$	$1 \oplus \varepsilon$	ε	0	$\ominus \varepsilon$	$\ominus 1 \ominus \varepsilon$	$\ominus 1 \oplus \varepsilon$	$\ominus 1$
06	ε	ε	ε	0	0	0	$\ominus \varepsilon$	$\ominus \varepsilon$	$\ominus \varepsilon$
$\ominus \varepsilon$	C	ح ا	ے ا	0	U	0	00	06	06
0	0	0	0	0	0	0	0	0	0
$\frac{0}{\varepsilon}$			$0 \\ \ominus \varepsilon$				$\frac{0}{\varepsilon}$	ϵ	$\frac{0}{\varepsilon}$
$\begin{array}{c} 0 \\ \varepsilon \\ 1\ominus\varepsilon \end{array}$	0	0	0	0	0	0	0	$\begin{array}{c} 0 \\ \varepsilon \\ 1\ominus\varepsilon \end{array}$	$\begin{array}{c} 0 \\ \varepsilon \\ 1 \end{array}$
$\begin{array}{c} 0 \\ \varepsilon \\ 1 \ominus \varepsilon \\ 1 \end{array}$	$0 \\ \ominus \varepsilon$	$0 \\ \ominus \varepsilon$	$\begin{array}{c} 0 \\ \ominus \varepsilon \\ \ominus 1 \ominus \varepsilon \\ \ominus 1 \oplus \varepsilon \end{array}$	0	0 0 0	$0 \\ 0 \\ \varepsilon \\ \varepsilon$	$\begin{array}{c} 0 \\ \varepsilon \\ 1 \oplus \varepsilon \\ 1 \ominus \varepsilon \end{array}$	$\begin{array}{c} 0 \\ \varepsilon \\ 1\ominus\varepsilon \\ 1 \end{array}$	$\begin{array}{c} 0 \\ \varepsilon \\ 1 \\ 1 \oplus \varepsilon \end{array}$
$\begin{array}{c} 0 \\ \varepsilon \\ 1\ominus\varepsilon \end{array}$	$\begin{array}{c} 0 \\ \ominus \varepsilon \\ \ominus 1 \end{array}$	$\begin{array}{c} 0 \\ \ominus \varepsilon \\ \ominus 1 \oplus \varepsilon \end{array}$	$\begin{array}{c} 0 \\ \ominus \varepsilon \\ \ominus 1 \ominus \varepsilon \end{array}$	$\begin{array}{c} 0 \\ 0 \\ \ominus \varepsilon \end{array}$	0 0 0	$0 \\ 0 \\ \varepsilon$	$\begin{array}{c} 0 \\ \varepsilon \\ 1 \oplus \varepsilon \end{array}$	$\begin{array}{c} 0 \\ \varepsilon \\ 1\ominus\varepsilon \end{array}$	$\begin{array}{c} 0 \\ \varepsilon \\ 1 \end{array}$
$\begin{array}{c} 0 \\ \varepsilon \\ 1 \ominus \varepsilon \\ 1 \end{array}$	$\begin{array}{c} 0 \\ \ominus \varepsilon \\ \ominus 1 \\ \ominus 1 \ominus \varepsilon \end{array}$	$\begin{array}{c} 0 \\ \ominus \varepsilon \\ \ominus 1 \oplus \varepsilon \\ \ominus 1 \end{array}$	$\begin{array}{c} 0 \\ \ominus \varepsilon \\ \ominus 1 \ominus \varepsilon \\ \ominus 1 \oplus \varepsilon \end{array}$	$\begin{array}{c} 0 \\ 0 \\ \ominus \varepsilon \\ \ominus \varepsilon \end{array}$	0 0 0	$0 \\ 0 \\ \varepsilon \\ \varepsilon$	$\begin{array}{c} 0 \\ \varepsilon \\ 1 \oplus \varepsilon \\ 1 \ominus \varepsilon \end{array}$	$\begin{array}{c} 0 \\ \varepsilon \\ 1\ominus\varepsilon \\ 1 \end{array}$	$\begin{array}{c} 0 \\ \varepsilon \\ 1 \\ 1 \oplus \varepsilon \end{array}$
$\begin{array}{c} 0 \\ \varepsilon \\ 1\ominus\varepsilon \\ 1 \\ 1\oplus\varepsilon\end{array}$	$\begin{array}{c c} 0 \\ \ominus \varepsilon \\ \ominus 1 \\ \ominus 1 \ominus \varepsilon \\ \ominus 1 \oplus \varepsilon \end{array}$	$\begin{array}{c} 0 \\ \ominus \varepsilon \\ \ominus 1 \oplus \varepsilon \\ \ominus 1 \\ \ominus 1 \\ \ominus \varepsilon \end{array}$	$\begin{array}{c} 0 \\ \ominus \varepsilon \\ \ominus 1 \ominus \varepsilon \\ \ominus 1 \oplus \varepsilon \\ \ominus 1 \end{array}$	$\begin{array}{c} 0 \\ 0 \\ \ominus \varepsilon \\ \ominus \varepsilon \\ \ominus \varepsilon \end{array}$	0 0 0 0	$0 \\ 0 \\ \varepsilon \\ \varepsilon \\ \varepsilon$	$\begin{array}{c} 0 \\ \varepsilon \\ 1 \oplus \varepsilon \\ 1 \ominus \varepsilon \\ 1 \end{array}$	$\begin{array}{c} 0 \\ \varepsilon \\ 1\ominus\varepsilon \\ 1 \\ 1\oplus\varepsilon \end{array}$	$\begin{array}{c} 0 \\ \varepsilon \\ 1 \\ 1 \oplus \varepsilon \\ 1 \ominus \varepsilon \end{array}$
$\begin{array}{c} 0 \\ \varepsilon \\ 1\ominus\varepsilon \\ 1 \\ 1\oplus\varepsilon \end{array}$ $\begin{array}{c} 1 \\ 0 \\ 0 \\ 0 \end{array}$	$\begin{array}{c c} 0 \\ \ominus \varepsilon \\ \ominus 1 \\ \ominus 1 \ominus \varepsilon \\ \ominus 1 \ominus \varepsilon \\ \hline \ominus 1 \ominus j \end{array}$	$\begin{array}{c} 0 \\ \ominus \varepsilon \\ \ominus 1 \oplus \varepsilon \\ \ominus 1 \\ \ominus 1 \ominus \varepsilon \\ \end{array}$ $\begin{array}{c} \ominus 1 \\ \ominus 1 \\ \vdots \\ 1 \\ \end{array}$	$\begin{array}{c} 0 \\ \ominus \varepsilon \\ \ominus 1 \ominus \varepsilon \\ \ominus 1 \oplus \varepsilon \\ \ominus 1 \\ \hline \ominus 1 \\ \hline \end{array}$	$\begin{matrix} 0 \\ 0 \\ \ominus \varepsilon \\ \ominus \varepsilon \\ \ominus \varepsilon \end{matrix}$	0 0 0 0 0	$egin{array}{c} 0 \ 0 \ arepsilon \ \ arepsilon \ ar$	$\begin{matrix} 0 \\ \varepsilon \\ 1 \oplus \varepsilon \\ 1 \ominus \varepsilon \\ 1 \end{matrix}$	$\begin{array}{c} 0 \\ \varepsilon \\ 1 \ominus \varepsilon \\ 1 \\ 1 \oplus \varepsilon \end{array}$	$\begin{array}{c} 0 \\ \varepsilon \\ 1 \\ 1 \oplus \varepsilon \\ 1 \ominus \varepsilon \end{array}$ $1 \oplus \varepsilon$ $1 \oplus j$
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$\begin{array}{c} 0 \\ \varepsilon \\ 1\ominus\varepsilon \\ 1 \\ 1\oplus\varepsilon \end{array}$ $\begin{array}{c} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array}$	$\begin{array}{c c} 0 \\ \ominus \varepsilon \\ \ominus 1 \\ \ominus 1 \ominus \varepsilon \\ \hline \ominus 1 \ominus \varepsilon \\ \hline \ominus 1 \ominus j \\ \hline \ominus 1 \ominus j \\ \hline 0 \\ 1 \ominus j \\ \hline 0 \\ \hline 0 \\ \hline 0 \\ \hline \ominus 1 \ominus j \\ \hline \end{array}$	$\begin{matrix} 0 \\ \ominus \varepsilon \\ \ominus 1 \oplus \varepsilon \\ \ominus 1 \\ \hline \ominus 1 \ominus \varepsilon \\ \hline \\ \ominus 1 \\ \hline \\ 1 \\ \hline \\ 1 \\ \hline \\ 1 \\ \hline \\ j \\ \hline \\ 0 \\ \ominus j \\ \end{matrix}$	$\begin{array}{c} 0 \\ \ominus \varepsilon \\ \ominus 1 \ominus \varepsilon \\ \ominus 1 \oplus \varepsilon \\ \ominus 1 \end{array}$ $\begin{array}{c} \ominus 1 \oplus j \\ \hline 0 \\ 1 \ominus j \\ \ominus 1 \oplus j \\ \hline 0 \\ 1 \oplus j \\ \hline 0 \\ 1 \oplus j \\ \hline \end{array}$	$\begin{matrix} 0 \\ 0 \\ \vdots \\ \Theta \varepsilon \\ \Theta \varepsilon \\ \Theta \varepsilon \end{matrix}$ $\begin{matrix} \Theta j \\ 1 \oplus j \\ \vdots \\ 0 1 \oplus j \\ 1 \end{matrix}$	0 0 0 0 0 0 0 0 0 0	$\begin{matrix} 0 \\ 0 \\ \varepsilon \\ \varepsilon \\ \end{matrix}$ $\begin{matrix} \varepsilon \\ \vdots \\ \varepsilon \end{matrix}$ $\begin{matrix} 0 \\ 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ 0 \\ \end{matrix}$	$\begin{matrix} 0 \\ \varepsilon \\ 1 \oplus \varepsilon \\ 1 \ominus \varepsilon \\ 1 \end{matrix}$ $\begin{matrix} 1 \ominus j \\ 0 \\ \ominus 1 \oplus j \\ 1 \ominus j \\ 1 \ominus j \\ 0 \\ \ominus 1 \oplus j \end{matrix}$	$\begin{matrix} 0 \\ \varepsilon \\ 1 \ominus \varepsilon \\ 1 \\ 1 \oplus \varepsilon \end{matrix}$ $\begin{matrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{matrix}$	$\begin{matrix} 0 \\ \varepsilon \\ 1 \\ 1 \oplus \varepsilon \\ 1 \ominus \varepsilon \end{matrix}$ $\begin{matrix} 1 \oplus j \\ 0 \\ 0 \\ 0 \\ 1 \oplus j \\ 0 \\ 0 \\ 1 \oplus j \end{matrix}$
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$\begin{array}{c} 0 \\ \varepsilon \\ 1\ominus\varepsilon \\ 1 \\ 1\oplus\varepsilon \end{array}$ $\begin{array}{c} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array}$	$\begin{array}{c c} 0 \\ \ominus \varepsilon \\ \ominus 1 \\ \ominus 1 \ominus \varepsilon \\ \hline \ominus 1 \ominus \varepsilon \\ \hline \ominus 1 \ominus j \\ \hline \ominus 1 \ominus j \\ \hline 0 \\ 1 \ominus j \\ \hline 0 \\ \hline 0 \\ \hline 0 \\ \hline \ominus 1 \ominus j \\ \hline \end{array}$	$\begin{matrix} 0 \\ \ominus \varepsilon \\ \ominus 1 \oplus \varepsilon \\ \ominus 1 \\ \hline \ominus 1 \ominus \varepsilon \\ \hline \\ \ominus 1 \\ \hline \\ 1 \\ \hline \\ 1 \\ \hline \\ 1 \\ \hline \\ j \\ \hline \\ 0 \\ \ominus j \\ \end{matrix}$	$\begin{array}{c} 0 \\ \ominus \varepsilon \\ \ominus 1 \ominus \varepsilon \\ \ominus 1 \oplus \varepsilon \\ \ominus 1 \end{array}$ $\begin{array}{c} \ominus 1 \oplus j \\ \hline 0 \\ 1 \ominus j \\ \ominus 1 \oplus j \\ \hline 0 \\ 1 \oplus j \\ \hline 0 \\ 1 \oplus j \\ \hline \end{array}$	$\begin{matrix} 0 \\ 0 \\ \\ \ominus \varepsilon \\ \\ \ominus \varepsilon \end{matrix}$ $\begin{matrix} \ominus \varepsilon \\ \\ \ominus \varepsilon \end{matrix}$ $\begin{matrix} \ominus j \\ \\ 1 \oplus j \\ \\ \\ 0 \\ \\ \end{bmatrix}$ $\begin{matrix} 1 \\ \\ 0 \\ \\ \\ \end{bmatrix}$ $\begin{matrix} 1 \\ \\ 0 \\ \\ \end{bmatrix}$	0 0 0 0 0 0 0 0 0 0	$\begin{matrix} 0 \\ 0 \\ \varepsilon \\ \varepsilon \\ \end{matrix}$ $\begin{matrix} \varepsilon \\ \vdots \\ \varepsilon \end{matrix}$ $\begin{matrix} 0 \\ 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ 0 \\ \end{matrix}$	$\begin{matrix} 0 \\ \varepsilon \\ 1 \oplus \varepsilon \\ 1 \ominus \varepsilon \\ 1 \end{matrix}$ $\begin{matrix} 1 \ominus j \\ 0 \\ \ominus 1 \oplus j \\ 1 \ominus j \\ 1 \ominus j \\ 0 \\ \ominus 1 \oplus j \end{matrix}$	$\begin{matrix} 0 \\ \varepsilon \\ 1 \ominus \varepsilon \\ 1 \\ 1 \oplus \varepsilon \end{matrix}$ $\begin{matrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{matrix}$	$\begin{matrix} 0 \\ \varepsilon \\ 1 \\ 1 \oplus \varepsilon \\ 1 \ominus \varepsilon \end{matrix}$ $\begin{matrix} 1 \oplus j \\ 0 \\ 0 \\ 0 \\ 1 \oplus j \\ 0 \\ 0 \\ 1 \oplus j \end{matrix}$

4 Hypercomplex-Valued Logics

4.1 Definitions

```
\forall \mathbb{K} \in \{\mathbb{C}_3, \mathbb{D}_3, \mathbb{S}_3\} :
4.1.1 \ \forall x \in \mathbb{K} :
4.1.1.1 \ \neg x \coloneqq \ominus x
4.1.1.2 \ x \land \ominus 1 \coloneqq \ominus 1
4.1.1.3 \ x \land 1 \coloneqq x
4.1.2 \ \forall x, y \in \mathbb{K} :
4.1.2.1 \ x \lor y \coloneqq \neg (\neg x \land \neg y)
4.1.2.2 \ x \Rightarrow y \coloneqq \neg x \lor y
4.1.2.3 \ x \Leftrightarrow y \coloneqq (x \Rightarrow y) \land (y \Rightarrow x)
```

4.2 Assumptions

```
\forall \mathbb{K} \in {\mathbb{C}_3, \mathbb{D}_3, \mathbb{S}_3}:
4.2.1 \ \forall x, y \in \mathbb{K}: \ x \Leftrightarrow y = x \otimes y
4.2.2 \ \forall x, y, z \in \mathbb{K}: \ (x \wedge y) \wedge z = x \wedge (y \wedge z)
4.2.3 \ \forall x, y \in \mathbb{K} \setminus {0}: \ x \wedge y \neq 0
```

4.3 Corollaries

```
\forall \mathbb{K} \in \{\mathbb{C}_3, \mathbb{D}_3, \mathbb{S}_3\} :
4.3.1 \ \forall x \in \mathbb{K}:
4.3.1.1 \ x \lor \ominus 1 = x \ (4.1.1.3, 4.1.2.1)
4.3.1.2 \ x \lor 1 = 1 \ (4.1.1.2, 4.1.2.1)
4.3.2 \ \forall x, y \in \mathbb{K}:
4.3.2.1 \ x \Leftrightarrow y = (x \lor \neg y) \land (\neg x \lor y) \ (4.1.2.2, 4.1.2.3)
4.3.2.2 \ x \Leftrightarrow y = (x \land y) \lor (\neg x \land \neg y) \ (4.3.2.1)
4.3.2.3 \ x \otimes y = (x \lor \neg y) \land (\neg x \lor y) \ (4.2.1, 4.3.2.1)
4.3.3 \ \forall x, y, z \in \mathbb{K}: \ (x \lor y) \lor z = x \lor (y \lor z) \ (4.1.2.1, 4.2.2)
4.3.4 \ \forall x, y \in \mathbb{K} \setminus \{0\}: \ x \lor y \neq 0 \ (4.1.2.1, 4.2.3)
```

4.4 Conjunction Values Problem

4.4.1 Values by Definitions (2.1.2.1, 4.1.1.2, 4.1.1.3)

\land	$\ominus 1 \ominus u$	$\ominus 1$	$\ominus 1 \oplus u$	$\ominus u$	0	u	$1\ominus u$	1	$1 \oplus u$
$\ominus 1 \ominus u$	x_1	$\ominus 1$	x_2	x_3	x_4	x_5	x_6	$\ominus 1 \ominus u$	x_7
⊖1	⊖1	$\ominus 1$	⊖1	$\ominus 1$	$\ominus 1$	$\ominus 1$	⊖1	⊖1	$\ominus 1$
$\ominus 1 \oplus u$	x_2	$\ominus 1$	x_8	x_9	x_{10}	x_{11}	x_{12}	$\ominus 1 \oplus u$	x_{13}
$\ominus u$	x_3	$\ominus 1$	x_9	x_{14}	x_{15}	x_{16}	x_{17}	$\ominus u$	x_{18}
0	x_4	$\ominus 1$	x_{10}	x_{15}	0	x_{19}	x_{20}	0	x_{21}
u	x_5	$\ominus 1$	x_{11}	x_{16}	x_{19}	x_{22}	x_{23}	u	x_{24}
$1\ominus u$	x_6	$\ominus 1$	x_{12}	x_{17}	x_{20}	x_{23}	x_{25}	$1\ominus u$	x_{26}
1	$\ominus 1 \ominus u$	$\ominus 1$	$\ominus 1 \oplus u$	$\ominus u$	0	u	$1\ominus u$	1	$1 \oplus u$
$1 \oplus u$	x_7	$\ominus 1$	x_{13}	x_{18}	x_{21}	x_{24}	x_{26}	$1 \oplus u$	x_{27}

4.4.2 Searching Values by Assumptions

 $\forall \mathbb{K} \in {\mathbb{C}_3, \mathbb{D}_3, \mathbb{S}_3} :$

4.4.2.1 $\mathcal{B}(\mathbb{K}) = \{X \in \mathbb{K}^{27} \mid X \text{ satisfies basic assumptions } (4.2.1, 4.2.2)\}^1$

4.4.2.2 $\mathcal{C}(\mathbb{K}) = \{Y \in \mathcal{B}(\mathbb{K}) \mid Y \text{ satisfies crisp connectives assumption (4.2.3)} \}^2$

4.4.3 Search Results

4.4.3.1 By basic assumptions (4.4.2.1):

$$4.4.3.1.1 \mid \mathcal{B}(\mathbb{C}_3) \mid = 2 \text{ (see on github.io)}$$

$$4.4.3.1.2 |\mathcal{B}(\mathbb{D}_3)| = 1 492 \text{ (see on github.io)}$$

 $4.4.3.1.3 |\mathcal{B}(\mathbb{S}_3)| = 15 \text{ (see on github.io)}$

$$4.4.3.2.2 \ \mathcal{C}(\mathbb{D}_3) = \varnothing$$

 $^{4.4.3.2.3 \ \}mathcal{C}(\mathbb{S}_3) = \varnothing$

 $^{1 \}forall \mathbb{K} \in \{\mathbb{C}_3, \mathbb{D}_3, \mathbb{S}_3\}$ there are $3 \times |\mathbb{K}^{27}| = 3 \times 9^{27} \approx 1,7445 \times 10^{26}$ variants to check.

²These are candidates for a useful hypercomplex-valued logic: their logical connectives are closed on a set of the crisp values $(\mathbb{K} \setminus \{0\})$.

5 Two Candidates

5.1 Algebraic Normal Forms

$$D = \begin{pmatrix} \ominus 1 \ominus i \\ \ominus 1 \\ \ominus 1 \oplus i \\ \ominus i \\ 0 \\ i \\ 1 \ominus i \\ 1 \\ 1 \oplus i \end{pmatrix}$$
 (2.3.1.1.1)

5.1.1 First candidate

 $\forall x, y \in \mathbb{C}_3$:

- $\begin{array}{lll} 5.1.1.1 & x \wedge y = \ominus y \oplus y^{\otimes 2} \ominus y^{\otimes 3} \oplus y^{\otimes 4} \ominus y^{\otimes 5} \oplus y^{\otimes 6} \ominus y^{\otimes 7} \oplus y^{\otimes 8} \ominus x \ominus x \otimes y \oplus i \otimes x \otimes y^{\otimes 2} \oplus (1 \oplus i) \otimes x \otimes y^{\otimes 4} \ominus y^{\otimes 4} \ominus y^{\otimes 8} \oplus x \ominus x \otimes y \oplus i \otimes x \otimes y^{\otimes 2} \oplus (1 \oplus i) \otimes x \otimes y^{\otimes 4} \ominus y^{\otimes 4} \oplus y^{\otimes 8} \oplus x \ominus x \otimes y \oplus i \otimes x \otimes y^{\otimes 2} \oplus (1 \oplus i) \otimes x \otimes y^{\otimes 4} \oplus y^{\otimes 4} \oplus y^{\otimes 4} \oplus y^{\otimes 4} \oplus y^{\otimes 5} \oplus (1 \oplus i) \otimes x^{\otimes 2} \otimes y^{\otimes 6} \oplus (1 \oplus i) \otimes x^{\otimes 2} \otimes y^{\otimes 7} \ominus x^{\otimes 3} \oplus x^{\otimes 3} \otimes y^{\otimes 4} \oplus i \otimes x^{\otimes 3} \otimes y^{\otimes 5} \ominus i \otimes x^{\otimes 3} \otimes y^{\otimes 7} \oplus x^{\otimes 4} \oplus y^{\otimes 6} \oplus y^{$
- $5.1.1.2 \ x \lor y = \ominus y \ominus y^{\otimes 2} \ominus y^{\otimes 3} \ominus y^{\otimes 4} \ominus y^{\otimes 5} \ominus y^{\otimes 6} \ominus y^{\otimes 7} \ominus y^{\otimes 8} \ominus x \oplus x \otimes y \oplus i \otimes x \otimes y^{\otimes 2} \oplus (1 \oplus i) \otimes x \otimes y^{\otimes 4} \oplus i \otimes x \otimes y^{\otimes 5} \ominus (1 \ominus i) \otimes x \otimes y^{\otimes 6} \ominus i \otimes x \otimes y^{\otimes 7} \ominus x^{\otimes 2} \oplus i \otimes x^{\otimes 2} \otimes y \ominus (1 \oplus i) \otimes x^{\otimes 2} \otimes y^{\otimes 2} \ominus x^{\otimes 2} \otimes y^{\otimes 4} \oplus i \otimes x^{\otimes 2} \otimes y^{\otimes 5} \oplus i \otimes x^{\otimes 2} \otimes y^{\otimes 6} \ominus (1 \oplus i) \otimes x^{\otimes 2} \otimes y^{\otimes 7} \ominus x^{\otimes 3} \oplus x^{\otimes 3} \otimes y^{\otimes 4} \oplus i \otimes x^{\otimes 3} \otimes y^{\otimes 5} \oplus i \otimes x^{\otimes 3} \otimes y^{\otimes 7} \ominus x^{\otimes 4} \oplus i \otimes x^{\otimes 4} \otimes y^{\otimes 5} \oplus (1 \oplus i) \otimes x^{\otimes 4} \otimes y^{\otimes 7} \ominus x^{\otimes 4} \oplus i \otimes x^{\otimes 4} \otimes y^{\otimes 4} \oplus x^{\otimes 4} \otimes y^{\otimes 5} \oplus (1 \oplus i) \otimes x^{\otimes 4} \otimes y^{\otimes 6} \ominus i \otimes x^{\otimes 4} \otimes y^{\otimes 7} \ominus x^{\otimes 5} \otimes y \oplus x^{\otimes 5} \otimes y^{\otimes 2} \ominus i \otimes x^{\otimes 5} \otimes y^{\otimes 3} \oplus x^{\otimes 5} \otimes y^{\otimes 4} \ominus i \otimes x^{\otimes 5} \otimes y^{\otimes 5} \ominus i \otimes x^{\otimes 5} \otimes y^{\otimes 5} \ominus i \otimes x^{\otimes 5} \otimes y^{\otimes 7} \ominus x^{\otimes 6} \ominus (1 \ominus i) \otimes x^{\otimes 6} \otimes y \oplus i \otimes x^{\otimes 6} \otimes y^{\otimes 2} \oplus (1 \oplus i) \otimes x^{\otimes 6} \otimes y^{\otimes 4} \ominus i \otimes x^{\otimes 7} \otimes y^{\otimes 7} \ominus x^{\otimes 8} \oplus x^{\otimes 8} \otimes y^{\otimes 8} \ominus i \otimes x^{\otimes 7} \otimes y^{\otimes 4} \oplus i \otimes x^{\otimes 7} \otimes y^{\otimes 6} \ominus i \otimes x^{\otimes 7} \otimes y^{\otimes 6} \ominus i \otimes x^{\otimes 7} \otimes y^{\otimes 7} \ominus x^{\otimes 8} \oplus x^{\otimes 8} \otimes y^{\otimes 8} \otimes x^{\otimes 8} \otimes y^{\otimes 8} \otimes x^{\otimes 8} \otimes x^{\otimes 8} \otimes y^{\otimes 8} \otimes x^{\otimes 8} \otimes$
- $\begin{array}{l} 5.1.1.3 \ \ x \Rightarrow y = \ominus y \ominus y^{\otimes 2} \ominus y^{\otimes 3} \ominus y^{\otimes 4} \ominus y^{\otimes 5} \ominus y^{\otimes 6} \ominus y^{\otimes 7} \ominus y^{\otimes 8} \oplus x \ominus x \otimes y \ominus i \otimes x \otimes y^{\otimes 2} \ominus (1 \oplus i) \otimes x \otimes y^{\otimes 4} \ominus i \otimes x \otimes y^{\otimes 5} \oplus (1 \ominus i) \otimes x \otimes y^{\otimes 6} \oplus i \otimes x \otimes y^{\otimes 7} \ominus x^{\otimes 2} \oplus i \otimes x^{\otimes 2} \otimes y \ominus (1 \oplus i) \otimes x^{\otimes 2} \otimes y^{\otimes 2} \ominus x^{\otimes 2} \otimes y^{\otimes 4} \oplus i \otimes x^{\otimes 2} \otimes y^{\otimes 5} \oplus i \otimes x^{\otimes 2} \otimes y^{\otimes 6} \ominus (1 \oplus i) \otimes x^{\otimes 2} \otimes y^{\otimes 7} \oplus x^{\otimes 3} \ominus x^{\otimes 3} \otimes y^{\otimes 4} \oplus i \otimes x^{\otimes 3} \otimes y^{\otimes 5} \ominus i \otimes x^{\otimes 3} \otimes y^{\otimes 7} \ominus x^{\otimes 4} \oplus i \otimes x^{\otimes 3} \otimes y^{\otimes 5} \oplus i \otimes x^{\otimes 3} \otimes y^{\otimes 7} \ominus x^{\otimes 4} \oplus i \otimes x^{\otimes 4} \otimes y^{\otimes 7} \oplus x^{\otimes 4} \otimes y^{\otimes 2} \oplus x^{\otimes 4} \otimes y^{\otimes 3} \oplus (1 \ominus i) \otimes x^{\otimes 4} \otimes y^{\otimes 4} \oplus x^{\otimes 4} \otimes y^{\otimes 5} \oplus (1 \oplus i) \otimes x^{\otimes 4} \otimes y^{\otimes 6} \ominus i \otimes x^{\otimes 4} \otimes y^{\otimes 7} \oplus x^{\otimes 5} \otimes y^{\otimes 4} \oplus x^{\otimes 5} \otimes y^{\otimes 3} \ominus x^{\otimes 5} \otimes y^{\otimes 4} \oplus x^{\otimes 5} \otimes y^{\otimes 5} \oplus i \otimes x^{\otimes 5} \otimes y^{\otimes 5} \oplus i \otimes x^{\otimes 5} \otimes y^{\otimes 7} \ominus x^{\otimes 6} \ominus (1 \ominus i) \otimes x^{\otimes 6} \otimes y \oplus i \otimes x^{\otimes 6} \otimes y^{\otimes 2} \oplus (1 \oplus i) \otimes x^{\otimes 6} \otimes y^{\otimes 4} \ominus x^{\otimes 6} \otimes y^{\otimes 4} \oplus x^{\otimes 6} \otimes y^{\otimes 6} \otimes x^{\otimes 7} \otimes y^{\otimes 6} \oplus x^{\otimes 7} \otimes y^{\otimes 6} \oplus x^{\otimes 7} \otimes y^{\otimes 7} \otimes x^{\otimes 8} \otimes y^{\otimes 8} \otimes x^{\otimes 8} \otimes x^{\otimes 8} \otimes y^{\otimes 8} \otimes x^{\otimes 8} \otimes x^{\otimes 8} \otimes y^{\otimes 8} \otimes x^{\otimes 8} \otimes x^{$
- $5.1.1.4 \ x \Leftrightarrow y = x \otimes y \ (2.3.1.3.4, \ 4.1.2.3)$

5.1.2 Second candidate

 $\forall x, y \in \mathbb{C}_3$:

- $5.1.2.1 \quad x \wedge y = \ominus y \oplus y^{\otimes 2} \ominus y^{\otimes 3} \oplus y^{\otimes 4} \ominus y^{\otimes 5} \oplus y^{\otimes 6} \ominus y^{\otimes 7} \oplus y^{\otimes 8} \ominus x \ominus x \otimes y \ominus i \otimes x \otimes y^{\otimes 2} \oplus (1 \ominus i) \otimes x \otimes y^{\otimes 4} \oplus \oplus i \otimes x \otimes y^{\otimes 5} \ominus (1 \oplus i) \otimes x \otimes y^{\otimes 6} \ominus i \otimes x \otimes y^{\otimes 7} \oplus x^{\otimes 2} \ominus i \otimes x^{\otimes 2} \otimes y \oplus (1 \ominus i) \otimes x^{\otimes 2} \otimes y^{\otimes 2} \oplus x^{\otimes 2} \otimes y^{\otimes 4} \oplus \oplus x^{\otimes 2} \otimes y^{\otimes 5} \oplus i \otimes x^{\otimes 2} \otimes y^{\otimes 6} \ominus (1 \ominus i) \otimes x^{\otimes 2} \otimes y^{\otimes 7} \ominus x^{\otimes 3} \oplus x^{\otimes 3} \otimes y^{\otimes 4} \ominus i \otimes x^{\otimes 3} \otimes y^{\otimes 5} \oplus i \otimes x^{\otimes 3} \otimes y^{\otimes 7} \oplus x^{\otimes 4} \oplus \oplus (1 \ominus i) \otimes x^{\otimes 4} \otimes y \oplus x^{\otimes 4} \otimes y^{\otimes 2} \oplus x^{\otimes 4} \otimes y^{\otimes 3} \ominus (1 \ominus i) \otimes x^{\otimes 4} \otimes y^{\otimes 6} \oplus i \otimes x^{\otimes 4} \otimes y^{\otimes 5} \ominus (1 \ominus i) \otimes x^{\otimes 4} \otimes y^{\otimes 6} \oplus i \otimes x^{\otimes 4} \otimes y^{\otimes 5} \ominus (1 \ominus i) \otimes x^{\otimes 4} \otimes y^{\otimes 6} \oplus i \otimes x^{\otimes 4} \otimes y^{\otimes 5} \oplus i \otimes x^{\otimes 5} \otimes y^{\otimes 5} \ominus (1 \ominus i) \otimes x^{\otimes 6} \otimes y^{\otimes 5} \ominus (1 \ominus i) \otimes x^{\otimes 6} \otimes y^{\otimes 5} \ominus (1 \ominus i) \otimes x^{\otimes 6} \otimes y^{\otimes 5} \ominus (1 \ominus i) \otimes x^{\otimes 6} \otimes y^{\otimes 4} \ominus i \otimes x^{\otimes 5} \otimes y^{\otimes 4} \ominus i \otimes x^{\otimes 6} \otimes y^{\otimes 6} \ominus i \otimes x^{\otimes 7} \otimes y^{\otimes 4} \ominus i \otimes x^{\otimes 7} \otimes y^{\otimes 6} \ominus i \otimes x^{\otimes 7} \otimes y^{\otimes 7} \oplus x^{\otimes 8} \otimes y^{\otimes 8} \ominus (1 \ominus i) \otimes x^{\otimes 6} \otimes y^{\otimes 6} \ominus i \otimes x^{\otimes 7} \otimes y^{\otimes 7} \oplus x^{\otimes 8} \otimes y^{\otimes 8} \ominus (1 \ominus i) \otimes x^{\otimes 7} \otimes y^{\otimes 6} \ominus i \otimes x^{\otimes 7} \otimes y^{\otimes 7} \oplus x^{\otimes 8} \otimes y^{\otimes 8} \ominus (1 \ominus i) \otimes x^{\otimes 7} \otimes y^{\otimes 6} \ominus i \otimes x^{\otimes 7} \otimes y^{\otimes 7} \oplus x^{\otimes 8} \otimes y^{\otimes 8} \ominus (1 \ominus i) \otimes x^{\otimes 7} \otimes y^{\otimes 6} \ominus i \otimes x^{\otimes 7} \otimes y^{\otimes 7} \oplus x^{\otimes 8} \otimes y^{\otimes 8} \ominus (1 \ominus i) \otimes x^{\otimes 7} \otimes y^{\otimes 6} \ominus i \otimes x^{\otimes 7} \otimes y^{\otimes 7} \oplus x^{\otimes 8} \otimes y^{\otimes 8} \ominus (1 \ominus i) \otimes x^{\otimes 7} \otimes y^{\otimes 7} \oplus x^{\otimes 8} \otimes y^{\otimes 8} \ominus (1 \ominus i) \otimes x^{\otimes 7} \otimes y^{\otimes 6} \ominus i \otimes x^{\otimes 7} \otimes y^{\otimes 7} \oplus x^{\otimes 8} \otimes y^{\otimes 8} \otimes y^{\otimes 8} \ominus (1 \ominus i) \otimes x^{\otimes 7} \otimes y^{\otimes 7} \oplus x^{\otimes 8} \otimes y^{\otimes 8} \otimes$
- $5.1.2.2 \ x \lor y = \ominus y \ominus y^{\otimes 2} \ominus y^{\otimes 3} \ominus y^{\otimes 4} \ominus y^{\otimes 5} \ominus y^{\otimes 6} \ominus y^{\otimes 7} \ominus y^{\otimes 8} \ominus x \oplus x \otimes y \ominus i \otimes x \otimes y^{\otimes 2} \oplus (1 \ominus i) \otimes x \otimes y^{\otimes 4} \ominus i \otimes x \otimes y^{\otimes 5} \ominus (1 \oplus i) \otimes x \otimes y^{\otimes 6} \oplus i \otimes x \otimes y^{\otimes 7} \ominus x^{\otimes 2} \ominus i \otimes x^{\otimes 2} \otimes y \ominus (1 \ominus i) \otimes x^{\otimes 2} \otimes y^{\otimes 2} \ominus x^{\otimes 2} \otimes y^{\otimes 4} \oplus i \otimes x^{\otimes 2} \otimes y^{\otimes 5} \ominus i \otimes x^{\otimes 2} \otimes y^{\otimes 6} \ominus (1 \ominus i) \otimes x^{\otimes 2} \otimes y^{\otimes 7} \ominus x^{\otimes 3} \oplus x^{\otimes 3} \otimes y^{\otimes 4} \oplus i \otimes x^{\otimes 3} \otimes y^{\otimes 5} \ominus i \otimes x^{\otimes 3} \otimes y^{\otimes 7} \ominus x^{\otimes 4} \oplus i \otimes x^{\otimes 4} \otimes y^{\otimes 5} \ominus i \otimes x^{\otimes 4} \otimes y^{\otimes 6} \ominus x^{\otimes 4} \otimes y^{\otimes 6} \oplus i \otimes x^{\otimes 4} \otimes y^{\otimes 6} \oplus i \otimes x^{\otimes 4} \otimes y^{\otimes 5} \oplus (1 \ominus i) \otimes x^{\otimes 4} \otimes y^{\otimes 6} \oplus i \otimes x^{\otimes 4} \otimes y^{\otimes 7} \ominus x^{\otimes 6} \ominus (1 \oplus i) \otimes x^{\otimes 6} \otimes y^{\otimes 2} \oplus i \otimes x^{\otimes 5} \otimes y^{\otimes 3} \oplus x^{\otimes 5} \otimes y^{\otimes 5} \ominus (1 \ominus i) \otimes x^{\otimes 6} \otimes y^{\otimes 5} \ominus i \otimes x^{\otimes 5} \otimes y^{\otimes 7} \ominus x^{\otimes 6} \ominus (1 \oplus i) \otimes x^{\otimes 6} \otimes y \ominus i \otimes x^{\otimes 6} \otimes y^{\otimes 2} \oplus (1 \ominus i) \otimes x^{\otimes 6} \otimes y^{\otimes 4} \ominus i \otimes x^{\otimes 7} \otimes y^{\otimes 4} \ominus i \otimes x^{\otimes 7} \otimes y^{\otimes 4} \ominus i \otimes x^{\otimes 7} \otimes y^{\otimes 6} \ominus i \otimes x^{\otimes 7} \otimes y^{\otimes 8} \otimes y^{\otimes$
- $5.1.2.3 \quad x \Rightarrow y = \ominus y \ominus y^{\otimes 2} \ominus y^{\otimes 3} \ominus y^{\otimes 4} \ominus y^{\otimes 5} \ominus y^{\otimes 6} \ominus y^{\otimes 7} \ominus y^{\otimes 8} \oplus x \ominus x \otimes y \oplus i \otimes x \otimes y^{\otimes 2} \ominus (1 \ominus i) \otimes x \otimes y^{\otimes 4} \oplus i \otimes x \otimes y^{\otimes 5} \oplus (1 \oplus i) \otimes x \otimes y^{\otimes 6} \ominus i \otimes x \otimes y^{\otimes 7} \ominus x^{\otimes 2} \ominus i \otimes x^{\otimes 2} \otimes y \ominus (1 \ominus i) \otimes x^{\otimes 2} \otimes y^{\otimes 2} \ominus x^{\otimes 2} \otimes y^{\otimes 4} \oplus i \otimes x^{\otimes 2} \otimes y^{\otimes 5} \ominus i \otimes x^{\otimes 2} \otimes y^{\otimes 6} \ominus (1 \ominus i) \otimes x^{\otimes 2} \otimes y^{\otimes 7} \oplus x^{\otimes 3} \ominus x^{\otimes 3} \otimes y^{\otimes 4} \ominus i \otimes x^{\otimes 3} \otimes y^{\otimes 5} \oplus i \otimes x^{\otimes 3} \otimes y^{\otimes 7} \ominus x^{\otimes 4} \oplus i \otimes x^{\otimes 4} \otimes y^{\otimes 5} \oplus (1 \ominus i) \otimes x^{\otimes 4} \otimes y^{\otimes 7} \ominus x^{\otimes 4} \oplus i \otimes x^{\otimes 4} \otimes y^{\otimes 4} \oplus x^{\otimes 4} \otimes y^{\otimes 5} \oplus (1 \ominus i) \otimes x^{\otimes 4} \otimes y^{\otimes 6} \oplus i \otimes x^{\otimes 4} \otimes y^{\otimes 7} \oplus x^{\otimes 5} \otimes y^{\otimes 3} \oplus (1 \oplus i) \otimes x^{\otimes 4} \otimes y^{\otimes 5} \oplus (1 \ominus i) \otimes x^{\otimes 4} \otimes y^{\otimes 6} \oplus i \otimes x^{\otimes 4} \otimes y^{\otimes 7} \oplus x^{\otimes 5} \otimes y^{\otimes 2} \ominus i \otimes x^{\otimes 5} \otimes y^{\otimes 3} \ominus x^{\otimes 5} \otimes y^{\otimes 4} \ominus i \otimes x^{\otimes 5} \otimes y^{\otimes 5} \oplus i \otimes x^{\otimes 5} \otimes y^{\otimes 5} \oplus i \otimes x^{\otimes 5} \otimes y^{\otimes 7} \ominus x^{\otimes 6} \ominus (1 \oplus i) \otimes x^{\otimes 6} \otimes y \ominus i \otimes x^{\otimes 6} \otimes y^{\otimes 2} \oplus (1 \ominus i) \otimes x^{\otimes 6} \otimes y^{\otimes 4} \ominus i \otimes x^{\otimes 7} \otimes y^{\otimes 4} \ominus i \otimes x^{\otimes 7} \otimes y^{\otimes 4} \oplus i \otimes x^{\otimes 7} \otimes y^{\otimes 7} \ominus x^{\otimes 8} \oplus x^{\otimes 8} \otimes y^{\otimes 8} \oplus i \otimes x^{\otimes 7} \otimes y^{\otimes 3} \ominus i \otimes x^{\otimes 7} \otimes y^{\otimes 4} \oplus i \otimes x^{\otimes 7} \otimes y^{\otimes 5} \oplus (1 \ominus i) \otimes x^{\otimes 7} \otimes y^{\otimes 6} \ominus i \otimes x^{\otimes 7} \otimes y^{\otimes 7} \ominus x^{\otimes 8} \oplus x^{\otimes 8} \otimes y^{\otimes 8} \oplus i \otimes x^{\otimes 7} \otimes y^{\otimes 7} \ominus x^{\otimes 8} \oplus x^{\otimes 8} \otimes y^{\otimes 8} \oplus x^{\otimes 8} \otimes y^{\otimes 8} \oplus x^{\otimes 8} \otimes y^{\otimes 8} \otimes x^{\otimes 8} \otimes x^{\otimes 8} \otimes y^{\otimes 8} \otimes x^{\otimes 8} \otimes$
- $5.1.2.4 \ x \Leftrightarrow y = x \otimes y \ (2.3.1.3.4, \ 4.1.2.3)$

5.2Functional Equivalence by Logical Connectives

Common values (4.4.1, 4.4.3.2.1)

5.2.1.1 By results:

$$5.2.1.1.1 \ (\ominus 1 \ominus i) \land 0 = 0$$

$$5.2.1.1.2 \ (\ominus 1 \ominus i) \land (1 \ominus i) = 1$$

$$5.2.1.1.3 \ (\ominus 1 \oplus i) \land 0 = 0$$

$$5.2.1.1.4 \ (\ominus 1 \oplus i) \land (1 \oplus i) = 1$$

$$5.2.1.1.5 \ \ominus i \land 0 = 0$$

$$5.2.1.1.6 \ \ominus i \land i = 1$$

$$5.2.1.1.7 \ 0 \land i = 0$$

$$5.2.1.1.8 \ 0 \land (1 \ominus i) = 0$$

$$5.2.1.1.9 \ 0 \land (1 \oplus i) = 0$$

5.2.1.2 By equations:

$$5.2.1.2.1 \ (\ominus 1 \oplus i) \land (1 \ominus i) = \ominus i \land \ominus i = (i \land i) \land (i \land i)$$

$$5.2.1.2.2 \ \ominus i \land (1 \oplus i) = i \land (1 \ominus i)$$

5.2.2 Generalization

5.2.2.1 Definitions:

$$5.2.2.1.1 \ \aleph := i \wedge i$$

$$5.2.2.1.2 \ \exists := \aleph \wedge \aleph$$

5.2.2.2 Corollaries (4.4.1, 4.4.3.2.1, 5.2.2.1.1, 5.2.2.1.2):

$$5.2.2.2.1 \ \begin{pmatrix} \aleph \\ \beth \end{pmatrix} \in \left\{ \begin{pmatrix} 1 \ominus i \\ 1 \oplus i \end{pmatrix}, \ \begin{pmatrix} \ominus 1 \oplus i \\ \ominus 1 \ominus i \end{pmatrix} \right\}$$

$$5.2.2.2.2 \ominus i \land \ominus i = \ominus \Box$$

$$5.2.2.2.3 \ominus i \land \ominus \beth = \beth$$

$$5.2.2.2.4 \ominus i \land \ominus \aleph = \aleph$$

$$5.2.2.2.5 \ominus i \land \aleph = i$$

$$5.2.2.2.6 \ \ominus i \land \beth = \ominus \aleph$$

$$5.2.2.2.7 \ \ominus \Box \land \ominus \Box = \ominus \aleph$$

$$5.2.2.2.8 \ominus \Box \land \ominus \aleph = i$$

$$5.2.2.2.9 \ \ominus \Box \land 0 = 0$$

$$5.2.2.2.10 \ominus \square \land \aleph = 1$$

$$5.2.2.2.11 \ \ominus \Box \land \Box = \aleph$$

$$5.2.2.2.12 \ \ominus \beth \wedge i = \ominus i$$

$$5.2.2.2.13 \ominus \aleph \wedge \ominus \aleph = \ominus i$$

$$5.2.2.2.14 \ \ominus \aleph \wedge 0 = 0$$

$$5.2.2.2.15 \ominus \aleph \wedge \aleph = \ominus \beth$$

$$5.2.2.2.16 \ominus \aleph \wedge \beth = 1$$

$$5.2.2.2.17 \ \ominus \aleph \wedge i = \beth$$

$$5.2.2.2.18 \ 0 \land \aleph = 0$$

 $5.2.2.2.19 \ 0 \land \beth = 0$

$$5.2.2.2.20 \ \aleph \wedge \square = \ominus i$$

 $5.2.2.2.21 \ \aleph \wedge i = \ominus \aleph$

$$5.2.2.2.22 \ \Box \land \Box = i$$

$$5.2.2.2.23 \ \square \wedge i = \ominus \square$$

6 A Useful Complex-Valued Logic

6.1 Truth Tables

\land	$\ominus 1$	$\ominus i$		$\bowtie \ominus \bowtie$	0	×	コ	i	1
$\ominus 1$	$\ominus 1$	$\ominus 1$	$\ominus 1$	$\ominus 1$	$\ominus 1$	$\ominus 1$	$\ominus 1$	$\ominus 1$	$\ominus 1$
$\ominus i$	$\ominus 1$		コ	×	0	i	\ominus \aleph	1	$\ominus i$
\ominus	⊖1		\ominus \aleph	i	0	1	X	$\ominus i$	⊖⊐
Θ	$\ominus 1$	×	i	$\ominus i$	0		1		\ominus \aleph
0	$\ominus 1$	0	0	0	0	0	0	0	0
×	$\ominus 1$	i	1		0		$\ominus i$	\ominus \aleph	×
コ	⊖1	\ominus \aleph	×	1	0	$\ominus i$	i	\ominus	
i	$\ominus 1$	1	$\ominus i$		0	$\forall \Theta$	$\sqsubseteq \exists$	×	i
1	$\ominus 1$	$\ominus i$	\ominus	\ominus \aleph	0	X	コ	i	1
V	⊖1	$\ominus i$		\ominus \aleph	0	×	コ	i	1
$\ominus 1$	$\ominus 1$	$\ominus i$	\ominus	\ominus \aleph	0	X	コ	i	1
$\ominus i$	$\ominus i$	\ominus \aleph	コ	×	0	⊖⊐	i	$\ominus 1$	1
⊖⊐		コ	$\ominus i$	i	0	$\ominus 1$	⊖⋈	×	1
68	⊖⋈	Х	i	⊖⊐	0	コ	$\ominus 1$	$\ominus i$	1
0	0	0	0	0	0	0	0	0	1
14	Х	⊖⊐	$\ominus 1$	コ	0	i	$\ominus i$	⊖⋈	1
コ		i	⊖⋈	⊖1	0	$\ominus i$	×	⊖⊐	1
i	i	⊖1	×	$\ominus i$	0	⊖⋈	\ominus	コ	1
1	1	1	1	1	1	1	1	1	1
		_							
\Rightarrow	⊖1	$\ominus i$		⊖⋈	0	×	٦	i	1
								1	
\Rightarrow	⊖1	$\ominus i$	⊖⊐	⊖⋈	0	×	٦		1
\Rightarrow $\ominus 1$	⊖1 1	<i>⊖i</i>	⊖ ⊐ 1	⊖\\\\ 1 ⊖i ⊖1	0	\\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\	1	1	1 1
$\begin{array}{c c} \Rightarrow \\ \hline \ominus 1 \\ \hline \ominus i \\ \end{array}$	⊖1 1	$\begin{array}{ c c }\hline \ominus i\\\hline 1\\\hline \ominus 1\\\hline \end{array}$	⊖⊐ 1 ℵ	⊖\\\\ 1 ⊖i	0 1 0	\text{\chi} 1 ⊕\text{\chi}	1 ⊖⊐	1	1 1 1
⇒ ⊖1 ⊖i ⊖⊐	⊖1 1 i ⊐	$\begin{array}{ c c }\hline \ominus i\\\hline 1\\\hline \ominus 1\\\hline i\\\hline \end{array}$	⊖⊐ 1 № ⊖%	⊖\\\\ 1 ⊖i ⊖1	0 1 0 0	\text{\tiny{\text{\tiny{\text{\tiny{\tinit}\\ \text{\ti}}\\ \ti}}\\ \tinttitex{\text{\text{\text{\text{\text{\text{\text{\texi}}\text{\text{\text{\text{\text{\text{\text{\text{\texi}\tint{\text{\text{\texi}\text{\text{\texi}\tilitt{\text{\texit{\texi{\texi}\titt{\text{\texi}\text{\text{\texi}\text{\text{	コ 1 ⊖コ ※	1 □ ⊖⊐	1 1 1 1
⇒ ⊖1 ⊖i ⊖3 ⊖% 0 ℵ	⊖1 1 i ⊐ ℵ	⊖i 1 ⊖1 i ⊖⊒ 0 ℵ	⊖⊐ 1 № ⊖№ ⊖1	 ⊖ℵ 1 ⊖i ⊖1 ⊐ 	0 1 0 0 0	$\begin{array}{ c c } \hline \aleph \\ \hline 1 \\ \hline \ominus \aleph \\ \hline \ominus i \\ \hline i \\ \hline \end{array}$	1 ⊝⊐ ℵ ⊖ <i>i</i>	1	1 1 1 1 1
⇒	⊕1 1 i ⊃ ℵ 0	⊖i	⊖⊐ 1 N ⊖N ⊖N ⊖1 0	$ \begin{array}{c c} \ominus\aleph \\ \hline 1 \\ \ominus i \\ \ominus 1 \\ \hline $	0 1 0 0 0	$\begin{array}{c c} \aleph & \\ \hline 1 & \\ \ominus \aleph & \\ \hline 0 & \\ \hline 3 & \\ \ominus 1 & \\ \hline \end{array}$	1 ⊖⊐ ℵ ⊖i 0 ⊖1 ⊖%	1 ⇒⊐ ⇒ℵ 0	1 1 1 1 1 1 1 1
	$ \begin{array}{c c} \ominus 1 \\ \hline 1 \\ i \\ \hline \\ \aleph \\ \hline 0 \\ \ominus \aleph \\ \ominus 1 \\ \hline \\ \ominus i \\ \end{array} $	⊖i	⊖⊐ 1	$\begin{array}{c c} \ominus\aleph \\ \hline 1 \\ \ominus i \\ \ominus 1 \\ \hline \Box \\ \hline 0 \\ \ominus \Box \\ i \\ \aleph \\ \end{array}$	0 1 0 0 0 0	№ 1 ⊕ ⊕ ⊕ ⊕	$\begin{array}{c c} \square & \\ 1 & \\ \ominus\square & \aleph & \\ \hline \ominus i & \\ 0 & \\ \ominus 1 & \\ \hline \ominus \aleph & \\ i & \\ \end{array}$	1	1 1 1 1 1 1 1 1
⇒	⊕1	⊖i	$\begin{array}{c c} \ominus \beth \\ \hline 1 \\ \aleph \\ \ominus \aleph \\ \ominus 1 \\ \hline 0 \\ i \\ \ominus i \\ \hline \end{array}$	$ \begin{array}{c c} \ominus\aleph \\ \hline 1 \\ \ominus i \\ \ominus 1 \\ \hline $	0 1 0 0 0 0	$\begin{array}{c c} \aleph & \\ \hline 1 & \\ \ominus \aleph & \\ \hline 0 & \\ \hline 3 & \\ \ominus 1 & \\ \hline \end{array}$	1 ⊖⊐ ℵ ⊖i 0 ⊖1 ⊖%	1 ⇒⊐ ⇒ℵ 0 ⇒i	1 1 1 1 1 1 1 1
	$ \begin{array}{c c} \ominus 1 \\ \hline 1 \\ i \\ \hline \\ \aleph \\ \hline 0 \\ \ominus \aleph \\ \ominus 1 \\ \hline \\ \ominus i \\ \end{array} $	⊖i	$\begin{array}{c c} \ominus \beth \\ \hline 1 \\ \aleph \\ \ominus \aleph \\ \ominus 1 \\ \hline 0 \\ i \\ \ominus i \\ \hline \end{bmatrix}$	$\begin{array}{c c} \ominus\aleph \\ \hline 1 \\ \ominus i \\ \ominus 1 \\ \hline \Box \\ \hline 0 \\ \ominus \Box \\ i \\ \aleph \\ \end{array}$	0 1 0 0 0 0 0	№ 1 ⊕ ⊕ ⊕ ⊕	$\begin{array}{c c} \square & \\ 1 & \\ \ominus\square & \aleph & \\ \hline \ominus i & \\ 0 & \\ \ominus 1 & \\ \hline \ominus \aleph & \\ i & \\ \end{array}$	1	1 1 1 1 1 1 1 1
⇒	$ \begin{array}{c c} \ominus 1 \\ \hline 1 \\ i \\ \hline \\ N \\ \hline \\ \Theta \\ \hline \\ \ominus 1 \\ \hline \\ \ominus 1 \\ \hline \end{array} $	$\begin{array}{c c} \ominus i \\ \hline 1 \\ \ominus 1 \\ i \\ \ominus \Box \\ \hline 0 \\ \aleph \\ \hline \Box \\ \ominus \aleph \\ \ominus i \\ \end{array}$	⊕ □ 1	$\begin{array}{c} \ominus \aleph \\ \hline \\ 1 \\ \ominus i \\ \ominus 1 \\ \hline \\ 0 \\ \hline \\ 0 \\ \hline \\ 0 \\ \hline \\ \vdots \\ \\ \aleph \\ \\ \ominus \aleph \\ \end{array}$	0 1 0 0 0 0 0 0 0	$\begin{array}{c} \aleph \\ 1 \\ \ominus \aleph \\ \ominus i \\ i \\ 0 \\ \hline $	$\begin{array}{c c} \square & \\ 1 & \\ \ominus\square & \\ \aleph & \\ \hline \ominus i & \\ 0 & \\ \ominus 1 & \\ \hline \ominus \aleph & \\ i & \\ \hline \end{array}$	$\begin{array}{c} 1 \\ $	1 1 1 1 1 1 1 1 1
⇒ ⊖1 ⊖i ⊖i ⊖i ⊖i ⊖i ⊖i ⊖i	$ \begin{array}{c c} \ominus 1 \\ \hline 1 \\ i \\ \hline \\ 0 \\ \hline \ominus \\ \ominus \\ \hline \\ \ominus \\ \hline \\ \hline \\ \hline \\ \hline \\ \hline \\$	$ \begin{array}{c c} \ominus i \\ \hline 1 \\ \ominus 1 \\ \hline i \\ \ominus \Box \\ \hline 0 \\ \hline \aleph \\ \hline \Box \\ \hline \ominus \aleph \\ \hline \ominus i \\ \hline \end{array} $	⊕ □ 1 N ⊕ N ⊕ N ⊕ 1 0 i ⊕ i □ □ □ □ □ □ □ □ □ □ □ □ □ □ □ □ □ □	⊕\\ 1	0 1 0 0 0 0 0 0 0 0	№ 1 ⊕	$ \begin{array}{c c} \square \\ 1 \\ \ominus \square \\ \aleph \\ \ominus i \\ 0 \\ \ominus 1 \\ \ominus \aleph \\ i \\ \square \\ \end{array} $	$\begin{array}{c c} 1 \\ $	1 1 1 1 1 1 1 1 1 1
⇒ ⊖1 ⊖i ⊖i ⊖i ⊖i ⊖i ⊖i ⊖i	⊝1	$ \begin{array}{c c} \ominus i \\ \hline 1 \\ \ominus 1 \\ \hline i \\ \ominus 2 \\ \hline 0 \\ \aleph \\ \hline \Box \\ \hline \ominus \aleph \\ \hline \ominus i \\ \hline \vdots \\ \hline i \\ \hline \end{array} $	⊕ □ 1	⊕\\ 1	0 1 0 0 0 0 0 0 0 0 0	№ 1 ⊕ ⊕ ⊕ ⊕ ⊕ ⊕ ⊕ ⊕ ⊕ ⊕ ⊕ ⊕ ⊕ ⊕ ⊕ ⊕ ⊕ ⊕ ⊕ ⊕ ⊕ ⊕ ⊕ ⊕ ⊕ ⊕ ⊕ ⊕ ⊕ ⊕ ⊕ ⊕ ⊕ ⊕ ⊕	1 ⊕3 ℵ ⊕i 0 ⊕1 ⊕ℵ i 3	$ \begin{array}{ccc} 1 & & \\ & & $	1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
⇒ ⊖1 ⊖i ⊖i 0 0 0 0 0 0 0 0 0	$ \begin{array}{c c} \ominus 1 \\ \hline 1 \\ i \\ \hline \\ 0 \\ \hline \ominus \\ \ominus \\ \hline \\ \ominus \\ \hline \\ \hline \\ \hline \\ \hline \\ \hline \\$	$ \begin{array}{c c} \ominus i \\ \hline 1 \\ \ominus 1 \\ i \\ \hline 0 \\ \aleph \\ \hline \ominus i \\ \hline \ominus \aleph \\ \hline \ominus i \\ \hline \vdots \\ \hline \end{bmatrix} $	⊕ □ 1	⊕\\ 1	0 1 0 0 0 0 0 0 0 0 0	№ 1	1 ⊕□ ⋈ ⊕i 0 ⊕1 ⊕⋈ i □ □	1 □□ □□ □N 0 □i N □1 i □i 1	$\begin{array}{c c} 1 & \\ 1 & \\ 1 & \\ 1 & \\ 1 & \\ 1 & \\ 1 & \\ 1 & \\ 1 & \\ 1 & \\ 1 & \\ 0 & \\ 1 & \\ \ominus i & \\ \end{array}$
⇒	$ \begin{array}{c c} \ominus 1 \\ \hline i \\ \hline i \\ \hline \rangle \\ \hline \rangle \\ \ominus \\ \hline \\ \ominus 1 \\ \hline \\ \\ \hline \\ \hline \\ \\ \\ \hline \\$	$ \begin{array}{c c} \ominus i \\ \hline 1 \\ \ominus 1 \\ i \\ \hline 0 \\ \aleph \\ \hline \ominus i \\ \hline \ominus \aleph \\ \hline \ominus i \\ \hline \vdots \\ \hline \end{bmatrix} $	⊕ □ 1 № ⊕ № ⊕ 1 ⊕ 0 i ⊕ i □ □ □ ⊕ □ □ □ □ □ □ □ □ □ □ □ □ □ □	⊕\\ 1	0 1 0 0 0 0 0 0 0 0 0	№ 1	□ □ □ □ □ □ □ □ □ □ □ □ □ □ □ □ □ □ □	$ \begin{array}{ccc} 1 & & \\ & & $	1 1 1 1 1 1 1 1 1 1 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0
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6.2 Application

The values i and $\ominus i$ can be used as truth values of cross-world predicates in coherent worlds.