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Categorical Syllogism as a Function

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Abstract: In this paper, categorical propositions are interpreted as the relations between two sets, including empty sets. The moods of a categorical syllogism are interpreted as the relations among three sets. Based on these, the *syllogistic function* is derived, which maps the relations of the premises to the corresponding subject-predicate relation in the conclusion.

Keywords: syllogistic function, relations between sets, existential property, algebraic normal form, Zhegalkin polynomial

1. Introduction

The purpose of this paper is to examine categorical syllogism from an alternative perspective: not as a collection of all possible relations among the minor, middle, and major terms, where only some combinations lead to valid conclusions, but rather as a function that maps the relations of the premises to the corresponding subject-predicate relation in the conclusion.

Categorical propositions are interpreted in the framework of *fundamental syllogistics*¹, “as many mathematicians note, this interpretation is motivated by the requirements of mathematical applications of logic.” [Bocharov, Markin, 2010, pp. 31-32].

2. Relations between Two Sets

2.1. Components of Relation

V. I. Markin [Markin, 2020] proposed interpreting *syllogistic constants* as signs of relations between two nonempty sets (the extensions of two general terms) and, based on this, developed an alphabet for the language of affirmative syllogistics. He introduced five *Eulerian* relations corresponding to Euler-Gergonne diagrams and showed that all possible relations can be expressed

¹This is how A. De Morgan [De Morgan, 1847] referred to the interpretation by G. Leibniz and F. Brentano.

as their combinations (including the *empty* combination), giving a total of 32 relations.

Representing categorical propositions as relations between two sets is instrumental in determining the subject-predicate relations in the conclusion. However, to carry out the computation arithmetically, a model with a fixed number of components is required. The most straightforward approach is to base it on the *existential proposition*² about the regions in Venn diagrams. The definition is introduced first.

Definition 1. The *components of the relation* between sets are all possible disjoint subsets derived from their intersections and differences.

The components of the relations between the sets S and P are the subsets $S \setminus P$, $S \cap P$, and $P \setminus S$ (see Figure 1)³.

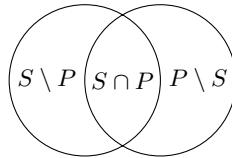


Fig. 1. Components of Relations between Two Sets

2.2. Existential Property

Following Leibniz and Brentano [Brandl, Textor, 2018], basic categorical propositions are translated into existential propositions as follows:

- “All S are P ” (SaP) \implies “An S that is not P does not exist” ($S \setminus P = \emptyset$);
- “No S is P ” (SeP) \implies “An S that is P does not exist” ($S \cap P = \emptyset$);
- “Some S are P ” (SiP) \implies “An S that is P exists” ($S \cap P \neq \emptyset$);
- “Some S are not P ” (SoP) \implies “An S that is not P exists” ($S \setminus P \neq \emptyset$).⁴

²In the original formulation, existential propositions are applied to individuals [Markin, 2021]; here, they are applied to sets (the extensions of general terms).

³There is a fourth component in the Venn diagram, represented by $S \downarrow P$, that is required for propositions u and q from *generalized affirmative syllogistics* by Markin [Markin, 1998], but not for basic categorical propositions a , e , i , and o .

⁴Also, for generalized affirmative syllogistics by Markin [Markin, 1998]:

- “Everything is either S or P ” (SuP) \implies “Neither S nor P does not exist” ($S \downarrow P = \emptyset$);
- “Something is neither S nor P ” (SqP) \implies “Neither S nor P exists” ($S \downarrow P \neq \emptyset$).

In this interpretation, the existential proposition is determined only about one component of relations, while the other two remain undetermined. This indeterminacy can be explained as follows: existential propositions about two of the three components have multiple truth values, or variants.

Example. The categorical proposition SaP is represented by the following triple of variants of the existential propositions:

$$\left(\{S \setminus P = \emptyset\}, \{S \cap P = \emptyset, S \cap P \neq \emptyset\}, \{P \setminus S = \emptyset, P \setminus S \neq \emptyset\} \right)$$

Definition 2. The *existential property* of a set is the generalized truth value of existential propositions concerning that set: *affirmative* if all propositions are affirmative, *negative* if all propositions are negative, and *indeterminate* if there are no propositions or there are affirmative and negative propositions. To map a set into an element of the three-valued ring $\mathbb{Z}_3 = \{0, 1, 2\}$, a function χ is used:

$$\chi(A) := \begin{cases} 0 & \Leftrightarrow V \neq \emptyset \wedge \forall a \in V : a = \emptyset \\ 1 & \Leftrightarrow V = \emptyset \vee \exists a \exists b \in V : a = \emptyset \wedge b \neq \emptyset \\ 2 & \Leftrightarrow V \neq \emptyset \wedge \forall a \in V : a \neq \emptyset \end{cases},$$

where V is the set of all variants of the set A ;

0, 1, 2 correspond to negative, indeterminate, and affirmative values of the existential properties, respectively.

Corollary 1. From Definition 2:

$$\chi(A \cup B) = \max\{\chi(A), \chi(B)\}$$

2.3. Component-wise Representation

Definition 3. The *component-wise representation* of the relation between two sets is an ordered triple⁵ of the existential properties of the components of the relation. To map sets into a component-wise representation, a two-variable function r is used:

$$r(S, P) := (\chi(S \setminus P), \chi(S \cap P), \chi(P \setminus S))$$

For basic categorical propositions in the Leibniz-Brentano interpretation, the following representations are obtained:

⁵Number of components depends on selected interpretation of syllogistics; e.g., the generalized affirmative syllogistics by Markin [Markin, 1998] requires at least quadruple of values.

- $SaP \implies r(S, P) = (0, 1, 1);$
- $SeP \implies r(S, P) = (1, 0, 1);$
- $SiP \implies r(S, P) = (1, 2, 1);$
- $SoP \implies r(S, P) = (2, 1, 1).$

Thus, $3^3 = 27$ *distinct* relations between two sets (including *empty* sets) can be represented like this. These relations differ from the relations given by Markin's syllogistic constants and, in general, are not reducible to them.

The generalization in the component-wise representation works as the *intersection* of the sets of categorical propositions; therefore, mutually exclusive propositions⁶ collapse into the empty set. In contrast, syllogistic constants describe relations that *unify* the sets of categorical propositions, including mutually exclusive propositions. To illustrate this, basic categorical propositions are extended by introducing the constants a' and o' :

$$\begin{aligned} Sa'P &:= PaS \\ So'P &:= PoS \end{aligned}$$

The distribution of the sets of categorical propositions over the components of the relations yields Table 1, which presents all aggregate truth values of the existential propositions about the components.

Table 1. Aggregate Truth Values of Existential Propositions

Aggregate Truth Value	$S \setminus P$	$S \cap P$	$P \setminus S$
Negative	$\{a\}$	$\{e\}$	$\{a'\}$
Affirmative	$\{o\}$	$\{i\}$	$\{o'\}$
Generalized	\emptyset	\emptyset	\emptyset
Composite	$\{a, o\}$	$\{e, i\}$	$\{a', o'\}$

The component-wise representation, by definition, does not involve a composite aggregation, and at first glance, it appears to describe fewer relations than Markin's syllogistic constants. However, a more detailed analysis reveals that some of the relations described by the syllogistic constants are equivalent to one another, so that the 32 relations reduce to 15 distinct ones (see Table 2).

⁶In syllogistic terms, this refers to categorical propositions standing in a *contradictory opposition* [Bocharov, Markin, 2010, p. 20].

3. Syllogistic Function

3.1. Function Definition

Definition 4. The *syllogistic function* is a six-variable function σ that takes the component-wise representations of the relations between terms of the major and minor premises of the first figure of the categorical syllogism⁷ as arguments and returns the component-wise representation of the subject-predicate relation in the conclusion:

$$\begin{aligned} \sigma\left(\chi(M \setminus P), \chi(M \cap P), \chi(P \setminus M), \right. \\ \left. \chi(S \setminus M), \chi(S \cap M), \chi(M \setminus S)\right) := \\ \left(\chi(S \setminus P), \chi(S \cap P), \chi(P \setminus S)\right) \end{aligned}$$

Equivalently, by Definition 3:

$$\sigma(r(M, P), r(S, M)) := r(S, P) \quad (1)$$

The arguments and values of the syllogistic function are simplified by combining them into an ordered sextuple x and an ordered triple y :

$$\begin{aligned} x = & \left(\chi(M \setminus P), \chi(M \cap P), \chi(P \setminus M), \right. \\ & \left. \chi(S \setminus M), \chi(S \cap M), \chi(M \setminus S)\right), \\ y = & \left(\chi(S \setminus P), \chi(S \cap P), \chi(P \setminus S)\right) \quad (2) \end{aligned}$$

The components of the ordered triple y are defined as functions of x :

$$y = (f_1(x), f_2(x), f_3(x)) \quad (3)$$

Thus, the problem of finding the function σ is reduced to the problem of determining the functions f_1 , f_2 , and f_3 . To this end, their truth table is constructed first and then the coefficients of the corresponding Zhegalkin polynomials are calculated from the values in the resulting table.

3.2. Truth Table Construction

To determine all possible mappings of the input existential properties into the output ones, all possible relations among the minor, middle, and major terms must be considered. All seven components of these relations are shown in Figure 2.

⁷Because of the presence of symmetrically equivalent relations in the component-wise representation, there is no need to use the remaining three figures of the syllogism.

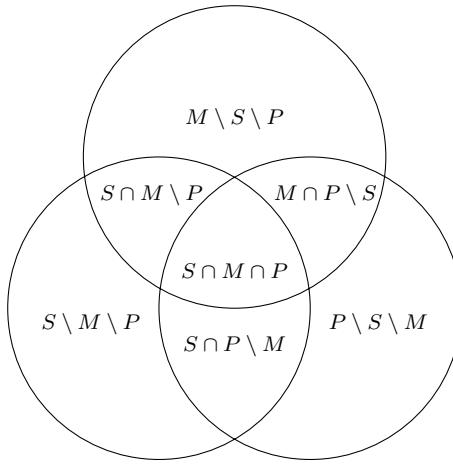


Fig. 2. Components of Relations among Three Sets

Considering the three values of the existential properties, the set B of all possible relations among three sets is obtained:

$$B = \left\{ b \in \{0, 1, 2\}^7 \mid b = (\chi(M \setminus S \setminus P), \chi(M \cap P \setminus S), \chi(P \setminus S \setminus M), \chi(S \cap P \setminus M), \chi(S \setminus M \setminus P), \chi(S \cap M \setminus P), \chi(S \cap M \cap P)) \right\} \quad (4)$$

From Equations 2 and 4 together with Corollary 1, the set C of all possible mappings of the arguments to the values of the syllogistic function is obtained:

$$\begin{aligned} C &= \left\{ c \in \{0, 1, 2\}^9 \mid b \in B \wedge c = (x, y) = \right. \\ &= \left(\chi(M \setminus P), \chi(M \cap P), \chi(P \setminus M), \chi(S \setminus M), \chi(S \cap M), \chi(M \setminus S), \right. \\ &\quad \left. \chi(S \setminus P), \chi(S \cap P), \chi(P \setminus S) \right) = \\ &= \left(\max\{b_1, b_6\}, \max\{b_2, b_7\}, \max\{b_3, b_4\}, \max\{b_4, b_5\}, \max\{b_6, b_7\}, \right. \\ &\quad \left. \max\{b_1, b_2\}, \max\{b_5, b_6\}, \max\{b_4, b_7\}, \max\{b_2, b_3\} \right) \} \quad (5) \end{aligned}$$

To construct the truth table, the resulting mappings are generalized to all possible combinations of the arguments of the syllogistic function. The truth table is defined as the block matrix T :

$$T = [X \ Y], \ X_{ij} = \left[\frac{i-1}{3^{6-j}} \right] \bmod 3, \quad (6)$$

where X is a 729×6 matrix containing all possible combinations of the arguments of the syllogistic function;

Y is a 729×3 matrix containing the target values of the syllogistic function.

From Equations 5 and 6 together with Definition 2, the values of the syllogistic function are obtained:

$$Y_{ij} = \chi(D_{ij})$$

where D_{ij} is the set generalized from the variants V_{ij} :

$$V_{ij} = \left\{ c_{j+6} \mid c \in C \wedge \left(\forall k \in \{1, 2, \dots, 6\} : X_{ik} \in \{1, c_k\} \right) \right\}.$$

As a result, the truth table is given on pp. 12-13.

3.3. Coefficient Calculation for Zhegalkin Polynomials

Yu. I. Bogdanov, N. A. Bogdanova, D. V. Fastovets, and V. F. Lukichev [Bogdanov et al., 2019] developed an algorithm for calculating the *algebraic normal form* of functions in k -valued logics, where k is a prime number. The algebraic normal form of an n -variable function f in a k -valued logic can be expressed as a Zhegalkin polynomial as follows:

$$f(x_1, \dots, x_n) = \left(\sum_{i=1}^{k^n} \left(a_i \prod_{j=1}^n x_j^{\lfloor \frac{i-1}{k^{n-j}} \rfloor \bmod k} \right) \right) \bmod k \quad (7)$$

From Definitions 2 and 4, only the case with $k = 3$ and $n = 6$ is needed. Applying the Kronecker product⁸ to P_1 (3×3 matrix corresponding to single-variable functions)⁹ yields P_6 (729×729 matrix corresponding to six-variable functions):

$$P_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 2 & 2 & 2 \end{pmatrix}, \quad P_6 = P_1^{\otimes 6}$$

The coefficients of the Zhegalkin polynomials corresponding to the functions f_1 , f_2 , and f_3 from Equation 3 can be obtained by multiplying the matrix P_6 with the values of the truth table of these functions:

$$A = P_6 Y$$

⁸The original article uses the tensor product, which in this case is equivalent.

⁹In the original article, the matrix is simply denoted P for $k = 3$.

Remark. Unlike the Zhegalkin polynomial described in Equation 7, which can be calculated using ordinary arithmetic, when calculating the matrices P_6 and A , modular arithmetic must be used. All additions and multiplications are performed modulo 3 (in \mathbb{Z}_3).

As a result, the 729×3 coefficient matrix A is given on pp. 14-15.

3.4. Calculation Result

Using the coefficients obtained, the syllogistic function can be represented as an ordered triple of Zhegalkin polynomials:

$$\sigma(x) = (f_1(x), f_2(x), f_3(x)),$$

$$f_i(x) = \left(\sum_{j=1}^{729} \left(A_{ji} \prod_{k=1}^6 x_k^{\lfloor \frac{j-1}{3^{6-k}} \rfloor \bmod 3} \right) \right) \bmod 3$$

The function is implemented in an online demonstration available at <https://codepen.io/mkarajani/pen/KKoQPYX>.

4. Conclusion

4.1. Correlations rather than Validity

Although the conclusions of the syllogistic moods can be classified as *valid* and *invalid*, those of the syllogistic function cannot, as they are always valid. Therefore, the outputs of the syllogistic function should be assessed from a different perspective: whether they exhibit any correlations between the subject and the predicate of the conclusion.

The absence of correlation is given by $r(S, P) = (1, 1, 1)$, when the existential properties of all components of the relation are indeterminate.

The syllogistic function establishes correlations for only 419 combinations of arguments. For 128 of these, the correlations involve no indeterminacy, i.e., the existential properties of all components of the subject-predicate relations are either affirmative or negative.

In addition, 64 combinations of syllogistic function arguments without indeterminacy in the premises can be examined: for 36 of these, the conclusions correspond to correlations, and for 27 of them, to correlations without indeterminacy.

4.2. Multiple Results via Partial Ordering

The set of component-wise represented values $\{0, 1, 2\}^3$ admits a partial order that induces a stratification by the number of indeterminate components:

$$(a_1, a_2, a_3) \leq (b_1, b_2, b_3) := a_1 \in \{b_1, 1\} \wedge a_2 \in \{b_2, 1\} \wedge a_3 \in \{b_3, 1\}$$

As illustrated in Figure 3, there are four layers that correspond to 3, 2, 1, and 0 occurrences of the value 1, respectively.

From this perspective, the output of the syllogistic function can be interpreted as an upper bound on the set of all admissible subject-predicate relations consistent with the premises. Thus, Equation 1 can be reformulated as an inequality:

$$r(S, P) \leq \sigma(r(M, P), r(S, M))$$

For example, if $\sigma(r(M, P), r(S, M)) = (0, 1, 0)$ then:

$$r(S, P) \leq (0, 1, 0) \implies r(S, P) \in \{(1, 1, 1), (0, 1, 1), (1, 1, 0), (0, 1, 0)\}$$

4.3. Using Balanced Ternary Representation

For the numerical representation of the values of the three-valued logic, either the *unbalanced* ternary $\{0, 1, 2\}$ or the *balanced* one $\{-1, 0, +1\}$ is used [Dubrova, 1999]. In Definition 2 of the function χ , the unbalanced representation is used.

Performing all calculations using the balanced ternary representation yields an ordered triple of polynomials with coefficients represented as a matrix of the same sizes (729×3) . However, the number of nonzero coefficients in this case is $284 + 424 + 284 = 992$, which considerably exceeds the number of nonzero coefficients obtained in this study, $242 + 106 + 242 = 590$.

Therefore, considering that the arguments and values of the syllogistic function can be converted from one representation to another by a straightforward transformation, it is more efficient to evaluate the syllogistic function using the unbalanced ternary representation.

4.4. Using Quantum Hardware

The article [Bogdanov et al., 2019] describes how to implement the computation of Zhegalkin polynomials in quantum logic circuits. This can significantly increase the efficiency of evaluating the syllogistic function.

4.5. Extension of the Approach to Alternative Interpretations

All calculations were carried out within the framework of the fundamental syllogistics. However, the algorithm used to compute the syllogistic function can also be applied to other interpretations of syllogistics if they can be formalized as mathematical models expressible in an algebraic normal form.

A different number of components for the component-wise representation can be required by another interpretation. The resulting coefficient matrix A will have size $3^{2c} \times c$, where c is the size of the component-wise representation.

Check the number of nonzero coefficients obtained for the balanced and unbalanced ternary representations to choose a more efficient way to evaluate the syllogistic function.

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Table 2. Representations of Relations between Two Sets

Generalized		Composite		No.
No.	Component-wise Representation	Set of Categorical Propositions	Markin's Syllogistic Constants	
1	(0, 0, 0)	{a, e, a'}		
2	(0, 0, 1)	{a, e}	{a, e, a', o'}	
3	(0, 0, 2)		{a, e, o'}	
4	(0, 1, 0)	{a, a'}	{a, e, i, a'}	
5	(0, 1, 1)	{a}	{a, e, i, a', o'}	
6	(0, 1, 2)	{a, o'}	{a, e, i, o'}	
7	(0, 2, 0)	{a, i, a'}		S1P
8	(0, 2, 1)	{a, i}	{a, i, a', o'}	S12P
9	(0, 2, 2)	{a, i, o'}		S2P
10	(1, 0, 0)	{e, a'}	{a, o, e, a'}	
11	(1, 0, 1)	{e}	{a, o, e, a', o'}	
12	(1, 0, 2)	{e, o'}	{a, o, e, o'}	
13	(1, 1, 0)	{a'}	{a, o, e, i, a'}	
14	(1, 1, 1)	\emptyset	\emptyset	SP
			{a, o, e, i, a', o'}	S125P
				S135P
				S145P
				S15P
				S235P
				S1235P
				S1245P
				S1345P
				S2345P
				S12345P
15	(1, 1, 2)	{o'}	{a, o, e, i, o'}	S25P
				S245P
16	(1, 2, 0)	{i, a'}	{a, o, i, a'}	S13P
17	(1, 2, 1)	{i}	{a, o, i, a', o'}	S123P
				S1234P
				S124P
				S134P
				S14P
				S23P
				S234P
18	(1, 2, 2)	{i, o'}	{a, o, i, o'}	S24P
19	(2, 0, 0)		{o, e, a'}	
20	(2, 0, 1)	{o, e}	{o, e, a', o'}	
21	(2, 0, 2)		{o, e, o'}	S5P
22	(2, 1, 0)	{o, a'}	{o, e, i, a'}	
23	(2, 1, 1)	{o}	{o, e, i, a', o'}	S35P
				S345P
24	(2, 1, 2)	{o, o'}	{o, e, i, o'}	S45P
25	(2, 2, 0)		{o, i, a'}	S3P
26	(2, 2, 1)	{o, i}	{o, i, a', o'}	S34P
27	(2, 2, 2)		{o, i, o'}	S4P

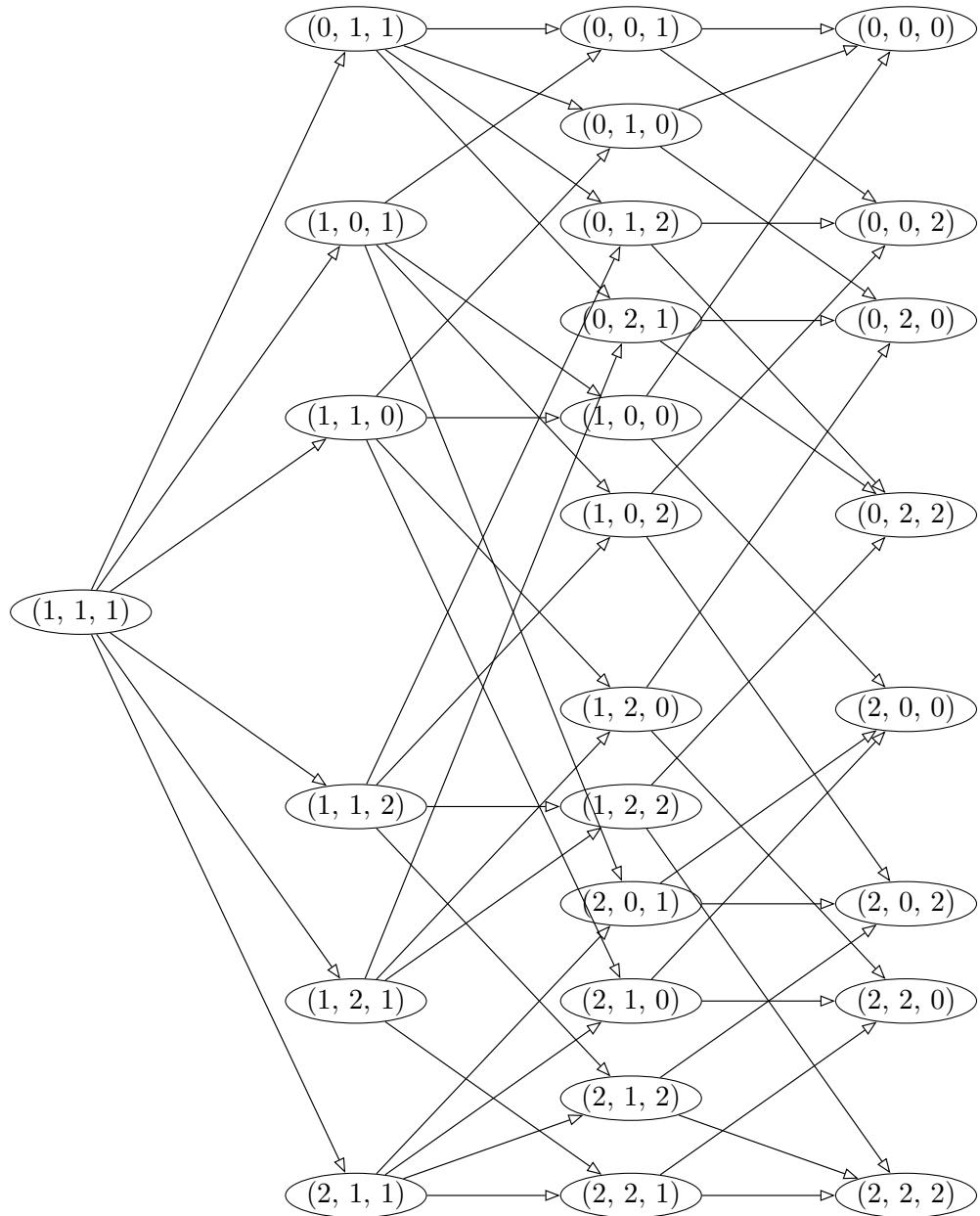


Fig. 3. Partial Ordering by Information Content