A Useful Complex-Valued Logic

Michael D. Karayani
mkarajani@gmail.com

October 18, 2025

Abstract

A *useful complex-valued logic* based on three-valued logic (with balanced ternary representation) and assumption that the logical equivalence corresponds to the hypercomplex multiplication operation.

1 Balanced Ternary Algebra

1.1 Definitions

1.1.1 $\mathbb{N} \coloneqq \{1,\, 2,\, 3,\, \dots\}$

1.1.2 $\mathbb{Z}_3 \coloneqq \{-1,\, 0,\, 1\}$

1.1.3 $\forall x,\, y \in \mathbb{Z}_3\colon$

1.1.3.1 $x \oplus y \coloneqq (x + y + 4) \bmod 3 - 1$

1.1.3.2 $x \otimes y \coloneqq xy$

1.1.4 $\forall x \in \mathbb{Z}_3\colon$

1.1.4.1 $\ominus x \coloneqq -1 \otimes x$

1.1.4.2 $x^{\otimes 0} \coloneqq 1$

1.1.4.3 $\forall n \in \mathbb{N}\colon\ x^{\otimes n} \coloneqq x \otimes x^{\otimes (n-1)}$

1.2 Corollaries

$\forall x \in \mathbb{Z}_3\colon$

1.2.1 $x \oplus x = \ominus x$ (1.1.3.1, 1.1.4.1)

1.2.2 $\forall n \in \mathbb{N}\colon$

1.2.2.1 $x^{\otimes (2n-1)} = x$ (1.1.4.2, 1.1.4.3)

1.2.2.2 $x^{\otimes (2n)} = x^{\otimes 2}$ (1.1.4.2, 1.1.4.3)

2 Three-Valued Logic

2.1 Definitions

2.1.1 $\forall x \in \mathbb{Z}_3\colon\ \neg x \coloneqq \ominus x$

2.1.2 $\forall x,\, y \in \mathbb{Z}_3\colon$

2.1.2.1 $x \wedge y \coloneqq \min(x,\, y)$

2.1.2.2 $x \vee y \coloneqq \neg(\neg x \wedge \neg y)$

2.1.2.3 $x \Rightarrow y \coloneqq \neg x \vee y$

2.1.2.4 $x \Leftrightarrow y \coloneqq (x \Rightarrow y) \wedge (y \Rightarrow x)$

2.2 Corollaries

$\forall x,\, y \in \mathbb{Z}_3:$

2.2.1 $x \vee y = \max(x,\, y)$ (2.1.2.2)

2.2.2 $x \Leftrightarrow y = (x \vee \neg y) \wedge (\neg x \vee y)$ (2.1.2.3, 2.1.2.4)

2.2.3 $x \Leftrightarrow y = (x \wedge y) \vee (\neg x \wedge \neg y)$ (2.2.2)

2.3 Algebraic Normal Forms

2.3.1 Algorithm

2.3.1.1 Given:

2.3.1.1.1 D: k-vector of sorted values of algebra's domain.

2.3.1.1.2 T: k^2-vector of the truth table's values for binary operation $\circ$, where:

$$t_i = d_{\lceil \frac{i}{k} \rceil} \circ d_{(i \bmod k)}$$

2.3.1.2 Find:

2.3.1.2.1 A: k^2-vector of coefficients for algebraic normal form of binary operation $\circ$, where:

$$x \circ y = \bigoplus_{i=1}^{k^2} \left(a_i \otimes x^{\otimes \lfloor \frac{i-1}{k} \rfloor} \otimes y^{\otimes ((i-1) \bmod k)} \right)$$

2.3.1.3 Solution:

2.3.1.3.1 $V_{i,j} = d_i^{\otimes (j-1)}$ (Vandermonde matrix of size $k \times k$)

2.3.1.3.2 $P_1 = V^{-1}$ (matrix inversion)

2.3.1.3.3 $P_2 = P_1 \otimes P_1$ (Kronecker product)

2.3.1.3.4 $A = P_2 T$ (matrix multiplication)

2.3.2 Result

2.3.2.1 $D = \begin{pmatrix} \ominus 1 \\ 0 \\ 1 \end{pmatrix}$ (2.3.1.1.1)

2.3.2.2 $\forall x,\, y \in \mathbb{Z}_3:$

2.3.2.2.1 $x \wedge y = \ominus y \oplus y^{\otimes 2} \ominus x \ominus x \otimes y \oplus x^{\otimes 2} \ominus x^{\otimes 2} \otimes y^{\otimes 2}$ (2.1.2.1, 2.3.1.3.4)

2.3.2.2.2 $x \vee y = \ominus y \ominus y^{\otimes 2} \ominus x \oplus x \otimes y \ominus x^{\otimes 2} \oplus x^{\otimes 2} \otimes y^{\otimes 2}$ (2.2.1, 2.3.1.3.4)

2.3.2.2.3 $x \Rightarrow y = \ominus y \ominus y^{\otimes 2} \oplus x \ominus x \otimes y \ominus x^{\otimes 2} \oplus x^{\otimes 2} \otimes y^{\otimes 2}$ (2.1.2.3, 2.3.1.3.4)

2.3.2.2.4 $x \Leftrightarrow y = x \otimes y$ (2.1.2.4, 2.3.1.3.4)

3 Balanced Hypercomplex Algebras

3.1 Definitions

3.1.1 $\mathbb{C}_3 \coloneqq \left\{ a \oplus b \otimes i \mid a,\, b \in \mathbb{Z}_3,\ i^{\otimes 2} = -1 \right\}$

3.1.2 $\mathbb{D}_3 \coloneqq \left\{ a \oplus b \otimes \varepsilon \mid a,\, b \in \mathbb{Z}_3,\ \varepsilon^{\otimes 2} = 0 \wedge \varepsilon \neq 0 \right\}$

3.1.3 $\mathbb{S}_3 \coloneqq \left\{ a \oplus b \otimes j \mid a,\, b \in \mathbb{Z}_3,\ j^{\otimes 2} = 1 \wedge j \notin \{-1,\, 1\} \right\}$

3.1.4 $\forall u \in \{i,\, \varepsilon,\, j\}$, $\forall a,\, b,\, c,\, d \in \mathbb{Z}_3\colon\ \forall x = a \oplus b \otimes u,\ \forall y = c \oplus d \otimes u$:

- 3.1.4.1 $x \oplus y := a \oplus c \oplus (b \oplus d) \otimes u$
- 3.1.4.2 $x \otimes y := a \otimes c \oplus u^{\otimes 2} \otimes b \otimes d \oplus (a \otimes d \oplus b \otimes c) \otimes u$

3.1.5 $\forall \mathbb{K} \in \{\mathbb{C}_3,\, \mathbb{D}_3,\, \mathbb{S}_3\}\colon\ \forall x \in \mathbb{K}$:

- 3.1.5.1 $\ominus x := -1 \otimes x$
- 3.1.5.2 $x^{\otimes 0} := 1$
- 3.1.5.3 $\forall n \in \mathbb{N}\colon\ x^{\otimes n} := x \otimes x^{\otimes (n-1)}$

3.2 Corollaries

$\forall \mathbb{K} \in \{\mathbb{C}_3,\, \mathbb{D}_3,\, \mathbb{S}_3\}\colon\ \forall x \in \mathbb{K}$:

3.2.1 $x \oplus x = \ominus x$ (3.1.4.1, 3.1.5.1)

3.2.2 $\forall n \in \mathbb{N}$:

- 3.2.2.1 $x^{\otimes (4n-3)} = x$ (3.1.5.2, 3.1.5.3)
- 3.2.2.2 $x^{\otimes (4n-2)} = x^{\otimes 2}$ (3.1.5.2, 3.1.5.3)
- 3.2.2.3 $x^{\otimes (4n-1)} = x^{\otimes 3}$ (3.1.5.2, 3.1.5.3)
- 3.2.2.4 $x^{\otimes (4n)} = x^{\otimes 4}$ (3.1.5.2, 3.1.5.3)

3.3 Multiplication Tables (3.1.4.2)

$\mathbb{C}_3$	$\ominus 1 \ominus i$	$\ominus 1$	$\ominus 1 \oplus i$	$\ominus i$	0	i	$1 \ominus i$	1	$1 \oplus i$
$\ominus 1 \ominus i$	$\ominus i$	$1 \oplus i$	$\ominus 1$	$\ominus 1 \oplus i$	0	$1 \ominus i$	1	$\ominus 1 \ominus i$	i
$\ominus 1$	$1 \oplus i$	1	$1 \ominus i$	i	0	$\ominus i$	$\ominus 1 \oplus i$	$\ominus 1$	$\ominus 1 \ominus i$
$\ominus 1 \oplus i$	$\ominus 1$	$1 \ominus i$	i	$1 \oplus i$	0	$\ominus 1 \ominus i$	$\ominus i$	$\ominus 1 \oplus i$	1
$\ominus i$	$\ominus 1 \oplus i$	i	$1 \oplus i$	$\ominus 1$	0	1	$\ominus 1 \ominus i$	$\ominus i$	$1 \ominus i$
0	0	0	0	0	0	0	0	0	0
i	$1 \ominus i$	$\ominus i$	$\ominus 1 \ominus i$	1	0	$\ominus 1$	$1 \oplus i$	i	$\ominus 1 \oplus i$
$1 \ominus i$	1	$\ominus 1 \oplus i$	$\ominus i$	$\ominus 1 \ominus i$	0	$1 \oplus i$	i	$1 \ominus i$	$\ominus 1$
1	$\ominus 1 \ominus i$	$\ominus 1$	$\ominus 1 \oplus i$	$\ominus i$	0	i	$1 \ominus i$	1	$1 \oplus i$
$1 \oplus i$	i	$\ominus 1 \ominus i$	1	$1 \ominus i$	0	$\ominus 1 \oplus i$	$\ominus 1$	$1 \oplus i$	$\ominus i$

$\mathbb{D}_3$	$\ominus 1 \ominus \varepsilon$	$\ominus 1$	$\ominus 1 \oplus \varepsilon$	$\ominus \varepsilon$	0	ε	$1 \ominus \varepsilon$	1	$1 \oplus \varepsilon$
$\ominus 1 \ominus \varepsilon$	$1 \ominus \varepsilon$	$1 \oplus \varepsilon$	1	ε	0	$\ominus \varepsilon$	$\ominus 1$	$\ominus 1 \ominus \varepsilon$	$\ominus 1 \oplus \varepsilon$
$\ominus 1$	$1 \oplus \varepsilon$	1	$1 \ominus \varepsilon$	ε	0	$\ominus \varepsilon$	$\ominus 1 \oplus \varepsilon$	$\ominus 1$	$\ominus 1 \ominus \varepsilon$
$\ominus 1 \oplus \varepsilon$	1	$1 \ominus \varepsilon$	$1 \oplus \varepsilon$	ε	0	$\ominus \varepsilon$	$\ominus 1 \ominus \varepsilon$	$\ominus 1 \oplus \varepsilon$	$\ominus 1$
$\ominus \varepsilon$	ε	ε	ε	0	0	0	$\ominus \varepsilon$	$\ominus \varepsilon$	$\ominus \varepsilon$
0	0	0	0	0	0	0	0	0	0
ε	$\ominus \varepsilon$	$\ominus \varepsilon$	$\ominus \varepsilon$	0	0	0	ε	ε	ε
$1 \ominus \varepsilon$	$\ominus 1$	$\ominus 1 \oplus \varepsilon$	$\ominus 1 \ominus \varepsilon$	$\ominus \varepsilon$	0	ε	$1 \oplus \varepsilon$	$1 \ominus \varepsilon$	1
1	$\ominus 1 \ominus \varepsilon$	$\ominus 1$	$\ominus 1 \oplus \varepsilon$	$\ominus \varepsilon$	0	ε	$1 \ominus \varepsilon$	1	$1 \oplus \varepsilon$
$1 \oplus \varepsilon$	$\ominus 1 \oplus \varepsilon$	$\ominus 1 \ominus \varepsilon$	$\ominus 1$	$\ominus \varepsilon$	0	ε	1	$1 \oplus \varepsilon$	$1 \ominus \varepsilon$

$\mathbb{S}_3$	$\ominus 1 \ominus j$	$\ominus 1$	$\ominus 1 \oplus j$	$\ominus j$	0	j	$1 \ominus j$	1	$1 \oplus j$
$\ominus 1 \ominus j$	$\ominus 1 \ominus j$	$1 \oplus j$	0	$1 \oplus j$	0	$\ominus 1 \ominus j$	0	$\ominus 1 \ominus j$	$1 \oplus j$
$\ominus 1$	$1 \oplus j$	1	$1 \ominus j$	j	0	$\ominus j$	$\ominus 1 \oplus j$	$\ominus 1$	$\ominus 1 \ominus j$
$\ominus 1 \oplus j$	0	$1 \ominus j$	$\ominus 1 \oplus j$	$\ominus 1 \oplus j$	0	$1 \ominus j$	$1 \ominus j$	$\ominus 1 \oplus j$	0
$\ominus j$	$1 \oplus j$	j	$\ominus 1 \oplus j$	1	0	$\ominus 1$	$1 \ominus j$	$\ominus j$	$\ominus 1 \ominus j$
0	0	0	0	0	0	0	0	0	0
j	$\ominus 1 \ominus j$	$\ominus j$	$1 \ominus j$	$\ominus 1$	0	1	$\ominus 1 \oplus j$	j	$1 \oplus j$
$1 \ominus j$	0	$\ominus 1 \oplus j$	$1 \ominus j$	$1 \ominus j$	0	$\ominus 1 \oplus j$	$\ominus 1 \oplus j$	$1 \ominus j$	0
1	$\ominus 1 \ominus j$	$\ominus 1$	$\ominus 1 \oplus j$	$\ominus j$	0	j	$1 \ominus j$	1	$1 \oplus j$
$1 \oplus j$	$1 \oplus j$	$\ominus 1 \ominus j$	0	$\ominus 1 \ominus j$	0	$1 \oplus j$	0	$1 \oplus j$	$\ominus 1 \ominus j$

4 Hypercomplex-Valued Logics

4.1 Definitions

$\forall \mathbb{K} \in \{\mathbb{C}_3,\, \mathbb{D}_3,\, \mathbb{S}_3\}:$

4.1.1 $\forall x \in \mathbb{K}$:

4.1.1.1 $\neg x \coloneqq \ominus x$

4.1.1.2 $x \wedge \ominus 1 \coloneqq \ominus 1$

4.1.1.3 $x \wedge 1 \coloneqq x$

4.1.2 $\forall x,\, y \in \mathbb{K}$:

4.1.2.1 $x \vee y \coloneqq \neg\,(\neg x \wedge \neg y)$

4.1.2.2 $x \Rightarrow y \coloneqq \neg x \vee y$

4.1.2.3 $x \Leftrightarrow y \coloneqq (x \Rightarrow y) \wedge (y \Rightarrow x)$

4.2 Assumptions

$\forall \mathbb{K} \in \{\mathbb{C}_3,\, \mathbb{D}_3,\, \mathbb{S}_3\}:$

4.2.1 $\forall x,\, y \in \mathbb{K}:\ x \Leftrightarrow y = x \otimes y$

4.2.2 $\forall x,\, y,\, z \in \mathbb{K}:\ (x \wedge y) \wedge z = x \wedge (y \wedge z)$

4.2.3 $\forall x,\, y \in \mathbb{K} \setminus \{0\}:\ x \wedge y \neq 0$

4.3 Corollaries

$\forall \mathbb{K} \in \{\mathbb{C}_3,\, \mathbb{D}_3,\, \mathbb{S}_3\}:$

4.3.1 $\forall x \in \mathbb{K}$:

4.3.1.1 $x \vee \ominus 1 = x$ (4.1.1.3, 4.1.2.1)

4.3.1.2 $x \vee 1 = 1$ (4.1.1.2, 4.1.2.1)

4.3.2 $\forall x,\, y \in \mathbb{K}$:

4.3.2.1 $x \Leftrightarrow y = (x \vee \neg y) \wedge (\neg x \vee y)$ (4.1.2.2, 4.1.2.3)

4.3.2.2 $x \Leftrightarrow y = (x \wedge y) \vee (\neg x \wedge \neg y)$ (4.3.2.1)

4.3.2.3 $x \otimes y = (x \vee \neg y) \wedge (\neg x \vee y)$ (4.2.1, 4.3.2.1)

4.3.3 $\forall x,\, y,\, z \in \mathbb{K}:\ (x \vee y) \vee z = x \vee (y \vee z)$ (4.1.2.1, 4.2.2)

4.3.4 $\forall x,\, y \in \mathbb{K} \setminus \{0\}:\ x \vee y \neq 0$ (4.1.2.1, 4.2.3)

4.4 Conjunction Values Problem

4.4.1 Values by Definitions (2.1.2.1, 4.1.1.2, 4.1.1.3)

$\wedge$	$\ominus 1 \ominus u$	$\ominus 1$	$\ominus 1 \oplus u$	$\ominus u$	0	u	$1 \ominus u$	1	$1 \oplus u$
$\ominus 1 \ominus u$	x_1	$\ominus 1$	x_2	x_3	x_4	x_5	x_6	$\ominus 1 \ominus u$	x_7
$\ominus 1$	$\ominus 1$	$\ominus 1$	$\ominus 1$	$\ominus 1$	$\ominus 1$	$\ominus 1$	$\ominus 1$	$\ominus 1$	$\ominus 1$
$\ominus 1 \oplus u$	x_2	$\ominus 1$	x_8	x_9	x_{10}	x_{11}	x_{12}	$\ominus 1 \oplus u$	x_{13}
$\ominus u$	x_3	$\ominus 1$	x_9	x_{14}	x_{15}	x_{16}	x_{17}	$\ominus u$	x_{18}
0	x_4	$\ominus 1$	x_{10}	x_{15}	0	x_{19}	x_{20}	0	x_{21}
u	x_5	$\ominus 1$	x_{11}	x_{16}	x_{19}	x_{22}	x_{23}	u	x_{24}
$1 \ominus u$	x_6	$\ominus 1$	x_{12}	x_{17}	x_{20}	x_{23}	x_{25}	$1 \ominus u$	x_{26}
1	$\ominus 1 \ominus u$	$\ominus 1$	$\ominus 1 \oplus u$	$\ominus u$	0	u	$1 \ominus u$	1	$1 \oplus u$
$1 \oplus u$	x_7	$\ominus 1$	x_{13}	x_{18}	x_{21}	x_{24}	x_{26}	$1 \oplus u$	x_{27}

4.4.2 Searching Values by Assumptions

$\forall \mathbb{K} \in \{\mathbb{C}_3,\, \mathbb{D}_3,\, \mathbb{S}_3\}$:

4.4.2.1 $\mathcal{B}(\mathbb{K}) = \left\{X \in \mathbb{K}^{27} \mid X \text{ satisfies basic assumptions } (4.2.1,\ 4.2.2)\right\}$[1]

4.4.2.2 $\mathcal{C}(\mathbb{K}) = \{Y \in \mathcal{B}(\mathbb{K}) \mid Y \text{ satisfies crisp connectives assumption } (4.2.3)\}$ [2]

4.4.3 Search Results

4.4.3.1 By basic assumptions (4.4.2.1):

4.4.3.1.1 $|\mathcal{B}(\mathbb{C}_3)| = 2$ (see on github.io)

4.4.3.1.2 $|\mathcal{B}(\mathbb{D}_3)| = 1\ 492$ (see on github.io)

4.4.3.1.3 $|\mathcal{B}(\mathbb{S}_3)| = 15$ (see on github.io)

4.4.3.2 By crisp connectives assumption (4.4.2.2):

4.4.3.2.1 $\mathcal{C}(\mathbb{C}_3) = \left\{ \begin{pmatrix} \ominus 1 \oplus i \\ i \\ 1 \oplus i \\ 0 \\ \ominus i \\ 1 \\ 1 \ominus i \\ \ominus i \\ 1 \ominus i \\ 0 \\ 1 \oplus i \\ \ominus 1 \ominus i \\ 1 \\ \ominus 1 \ominus i \\ 0 \\ 1 \\ i \\ \ominus 1 \oplus i \\ 0 \\ 0 \\ 0 \\ 1 \ominus i \\ \ominus 1 \oplus i \\ \ominus 1 \ominus i \\ 1 \oplus i \\ \ominus i \\ i \end{pmatrix}, \begin{pmatrix} i \\ \ominus i \\ 1 \ominus i \\ 0 \\ 1 \oplus i \\ 1 \\ \ominus 1 \oplus i \\ \ominus 1 \ominus i \\ i \\ 0 \\ 1 \ominus i \\ 1 \oplus i \\ 1 \\ 1 \oplus i \\ 0 \\ 1 \\ \ominus 1 \oplus i \\ \ominus 1 \ominus i \\ 0 \\ 0 \\ 0 \\ \ominus 1 \oplus i \\ \ominus 1 \ominus i \\ \ominus i \\ \ominus i \\ i \\ 1 \ominus i \end{pmatrix} \right\}$

4.4.3.2.2 $\mathcal{C}(\mathbb{D}_3) = \varnothing$

4.4.3.2.3 $\mathcal{C}(\mathbb{S}_3) = \varnothing$

[1] $\forall \mathbb{K} \in \{\mathbb{C}_3,\, \mathbb{D}_3,\, \mathbb{S}_3\}$ there are $3 \times \left|\mathbb{K}^{27}\right| = 3 \times 9^{27} \approx 1{,}7445 \times 10^{26}$ variants to check.

[2] These are candidates for a *useful hypercomplex-valued logic*: their logical connectives are closed on a set of the crisp values $(\mathbb{K} \setminus \{0\})$.

5 Two Candidates

5.1 Truth Tables (4.1.2.1, 4.1.2.2, 4.1.2.3, 4.4.1, 4.4.3.2.1)

5.1.1 First candidate

$\wedge$	$\ominus 1 \ominus i$	$\ominus 1$	$\ominus 1 \oplus i$	$\ominus i$	0	i	$1 \ominus i$	1	$1 \oplus i$
$\ominus 1 \ominus i$	$\ominus 1 \oplus i$	$\ominus 1$	i	$1 \oplus i$	0	$\ominus i$	1	$\ominus 1 \ominus i$	$1 \ominus i$
$\ominus 1$	$\ominus 1$	$\ominus 1$	$\ominus 1$	$\ominus 1$	$\ominus 1$	$\ominus 1$	$\ominus 1$	$\ominus 1$	$\ominus 1$
$\ominus 1 \oplus i$	i	$\ominus 1$	$\ominus i$	$1 \ominus i$	0	$1 \oplus i$	$\ominus 1 \ominus i$	$\ominus 1 \oplus i$	1
$\ominus i$	$1 \oplus i$	$\ominus 1$	$1 \ominus i$	$\ominus 1 \ominus i$	0	1	i	$\ominus i$	$\ominus 1 \oplus i$
0	0	$\ominus 1$	0	0	0	0	0	0	0
i	$\ominus i$	$\ominus 1$	$1 \oplus i$	1	0	$1 \ominus i$	$\ominus 1 \oplus i$	i	$\ominus 1 \ominus i$
$1 \ominus i$	1	$\ominus 1$	$\ominus 1 \ominus i$	i	0	$\ominus 1 \oplus i$	$1 \oplus i$	$1 \ominus i$	$\ominus i$
1	$\ominus 1 \ominus i$	$\ominus 1$	$\ominus 1 \oplus i$	$\ominus i$	0	i	$1 \ominus i$	1	$1 \oplus i$
$1 \oplus i$	$1 \ominus i$	$\ominus 1$	1	$\ominus 1 \oplus i$	0	$\ominus 1 \ominus i$	$\ominus i$	$1 \oplus i$	i

$\vee$	$\ominus 1 \ominus i$	$\ominus 1$	$\ominus 1 \oplus i$	$\ominus i$	0	i	$1 \ominus i$	1	$1 \oplus i$
$\ominus 1 \ominus i$	$\ominus i$	$\ominus 1 \ominus i$	i	$1 \oplus i$	0	$1 \ominus i$	$\ominus 1$	1	$\ominus 1 \oplus i$
$\ominus 1$	$\ominus 1 \ominus i$	$\ominus 1$	$\ominus 1 \oplus i$	$\ominus i$	0	i	$1 \ominus i$	1	$1 \oplus i$
$\ominus 1 \oplus i$	i	$\ominus 1 \oplus i$	$\ominus 1 \ominus i$	$1 \ominus i$	0	$\ominus i$	$1 \oplus i$	1	$\ominus 1$
$\ominus i$	$1 \oplus i$	$\ominus i$	$1 \ominus i$	$\ominus 1 \oplus i$	0	$\ominus 1$	$\ominus 1 \ominus i$	1	i
0	0	0	0	0	0	0	0	1	0
i	$1 \ominus i$	i	$\ominus i$	$\ominus 1$	0	$1 \oplus i$	$\ominus 1 \oplus i$	1	$\ominus 1 \ominus i$
$1 \ominus i$	$\ominus 1$	$1 \ominus i$	$1 \oplus i$	$\ominus 1 \ominus i$	0	$\ominus 1 \oplus i$	i	1	$\ominus i$
1	1	1	1	1	1	1	1	1	1
$1 \oplus i$	$\ominus 1 \oplus i$	$1 \oplus i$	$\ominus 1$	i	0	$\ominus 1 \ominus i$	$\ominus i$	1	$1 \ominus i$

$\Rightarrow$	$\ominus 1 \ominus i$	$\ominus 1$	$\ominus 1 \oplus i$	$\ominus i$	0	i	$1 \ominus i$	1	$1 \oplus i$
$\ominus 1 \ominus i$	$\ominus 1 \oplus i$	$1 \oplus i$	$\ominus 1$	i	0	$\ominus 1 \ominus i$	$\ominus i$	1	$1 \ominus i$
$\ominus 1$	1	1	1	1	1	1	1	1	1
$\ominus 1 \oplus i$	$\ominus 1$	$1 \ominus i$	$1 \oplus i$	$\ominus 1 \ominus i$	0	$\ominus 1 \oplus i$	i	1	$\ominus i$
$\ominus i$	$1 \ominus i$	i	$\ominus i$	$\ominus 1$	0	$1 \oplus i$	$\ominus 1 \oplus i$	1	$\ominus 1 \ominus i$
0	0	0	0	0	0	0	0	1	0
i	$1 \oplus i$	$\ominus i$	$1 \ominus i$	$\ominus 1 \oplus i$	0	$\ominus 1$	$\ominus 1 \ominus i$	1	i
$1 \ominus i$	i	$\ominus 1 \oplus i$	$\ominus 1 \ominus i$	$1 \ominus i$	0	$\ominus i$	$1 \oplus i$	1	$\ominus 1$
1	$\ominus 1 \ominus i$	$\ominus 1$	$\ominus 1 \oplus i$	$\ominus i$	0	i	$1 \ominus i$	1	$1 \oplus i$
$1 \oplus i$	$\ominus i$	$\ominus 1 \ominus i$	i	$1 \oplus i$	0	$1 \ominus i$	$\ominus 1$	1	$\ominus 1 \oplus i$

$\Leftrightarrow$	$\ominus 1 \ominus i$	$\ominus 1$	$\ominus 1 \oplus i$	$\ominus i$	0	i	$1 \ominus i$	1	$1 \oplus i$
$\ominus 1 \ominus i$	$\ominus i$	$1 \oplus i$	$\ominus 1$	$\ominus 1 \oplus i$	0	$1 \ominus i$	1	$\ominus 1 \ominus i$	i
$\ominus 1$	$1 \oplus i$	1	$1 \ominus i$	i	0	$\ominus i$	$\ominus 1 \oplus i$	$\ominus 1$	$\ominus 1 \ominus i$
$\ominus 1 \oplus i$	$\ominus 1$	$1 \ominus i$	i	$1 \oplus i$	0	$\ominus 1 \ominus i$	$\ominus i$	$\ominus 1 \oplus i$	1
$\ominus i$	$\ominus 1 \oplus i$	i	$1 \oplus i$	$\ominus 1$	0	1	$\ominus 1 \ominus i$	$\ominus i$	$1 \ominus i$
0	0	0	0	0	0	0	0	0	0
i	$1 \ominus i$	$\ominus i$	$\ominus 1 \ominus i$	1	0	$\ominus 1$	$1 \oplus i$	i	$\ominus 1 \oplus i$
$1 \ominus i$	1	$\ominus 1 \oplus i$	$\ominus i$	$\ominus 1 \ominus i$	0	$1 \oplus i$	i	$1 \ominus i$	$\ominus 1$
1	$\ominus 1 \ominus i$	$\ominus 1$	$\ominus 1 \oplus i$	$\ominus i$	0	i	$1 \ominus i$	1	$1 \oplus i$
$1 \oplus i$	i	$\ominus 1 \ominus i$	1	$1 \ominus i$	0	$\ominus 1 \oplus i$	$\ominus 1$	$1 \oplus i$	$\ominus i$

5.1.2 Second candidate

$\wedge$	$\ominus 1 \ominus i$	$\ominus 1$	$\ominus 1 \oplus i$	$\ominus i$	0	i	$1 \ominus i$	1	$1 \oplus i$
$\ominus 1 \ominus i$	i	$\ominus 1$	$\ominus i$	$1 \ominus i$	0	$1 \oplus i$	1	$\ominus 1 \ominus i$	$\ominus 1 \oplus i$
$\ominus 1$	$\ominus 1$	$\ominus 1$	$\ominus 1$	$\ominus 1$	$\ominus 1$	$\ominus 1$	$\ominus 1$	$\ominus 1$	$\ominus 1$
$\ominus 1 \oplus i$	$\ominus i$	$\ominus 1$	$\ominus 1 \ominus i$	i	0	$1 \ominus i$	$1 \oplus i$	$\ominus 1 \oplus i$	1
$\ominus i$	$1 \ominus i$	$\ominus 1$	i	$1 \oplus i$	0	1	$\ominus 1 \oplus i$	$\ominus i$	$\ominus 1 \ominus i$
0	0	$\ominus 1$	0	0	0	0	0	0	0
i	$1 \oplus i$	$\ominus 1$	$1 \ominus i$	1	0	$\ominus 1 \oplus i$	$\ominus 1 \ominus i$	i	$\ominus i$
$1 \ominus i$	1	$\ominus 1$	$1 \oplus i$	$\ominus 1 \oplus i$	0	$\ominus 1 \ominus i$	$\ominus i$	$1 \ominus i$	i
1	$\ominus 1 \ominus i$	$\ominus 1$	$\ominus 1 \oplus i$	$\ominus i$	0	i	$1 \ominus i$	1	$1 \oplus i$
$1 \oplus i$	$\ominus 1 \oplus i$	$\ominus 1$	1	$\ominus 1 \ominus i$	0	$\ominus i$	i	$1 \oplus i$	$1 \ominus i$

$\vee$	$\ominus 1 \ominus i$	$\ominus 1$	$\ominus 1 \oplus i$	$\ominus i$	0	i	$1 \ominus i$	1	$1 \oplus i$
$\ominus 1 \ominus i$	$\ominus 1 \oplus i$	$\ominus 1 \ominus i$	$\ominus i$	i	0	$1 \oplus i$	$\ominus 1$	1	$1 \ominus i$
$\ominus 1$	$\ominus 1 \ominus i$	$\ominus 1$	$\ominus 1 \oplus i$	$\ominus i$	0	i	$1 \ominus i$	1	$1 \oplus i$
$\ominus 1 \oplus i$	$\ominus i$	$\ominus 1 \oplus i$	i	$1 \oplus i$	0	$1 \ominus i$	$\ominus 1 \ominus i$	1	$\ominus 1$
$\ominus i$	i	$\ominus i$	$1 \oplus i$	$1 \ominus i$	0	$\ominus 1$	$\ominus 1 \oplus i$	1	$\ominus 1 \ominus i$
0	0	0	0	0	0	0	0	1	0
i	$1 \oplus i$	i	$1 \ominus i$	$\ominus 1$	0	$\ominus 1 \ominus i$	$\ominus i$	1	$\ominus 1 \oplus i$
$1 \ominus i$	$\ominus 1$	$1 \ominus i$	$\ominus 1 \ominus i$	$\ominus 1 \oplus i$	0	$\ominus i$	$1 \oplus i$	1	i
1	1	1	1	1	1	1	1	1	1
$1 \oplus i$	$1 \ominus i$	$1 \oplus i$	$\ominus 1$	$\ominus 1 \ominus i$	0	$\ominus 1 \oplus i$	i	1	$\ominus i$

$\Rightarrow$	$\ominus 1 \ominus i$	$\ominus 1$	$\ominus 1 \oplus i$	$\ominus i$	0	i	$1 \ominus i$	1	$1 \oplus i$
$\ominus 1 \ominus i$	$1 \ominus i$	$1 \oplus i$	$\ominus 1$	$\ominus 1 \ominus i$	0	$\ominus 1 \oplus i$	i	1	$\ominus i$
$\ominus 1$	1	1	1	1	1	1	1	1	1
$\ominus 1 \oplus i$	$\ominus 1$	$1 \ominus i$	$\ominus 1 \ominus i$	$\ominus 1 \oplus i$	0	$\ominus i$	$1 \oplus i$	1	i
$\ominus i$	$1 \oplus i$	i	$1 \ominus i$	$\ominus 1$	0	$\ominus 1 \ominus i$	$\ominus i$	1	$\ominus 1 \oplus i$
0	0	0	0	0	0	0	0	1	0
i	i	$\ominus i$	$1 \oplus i$	$1 \ominus i$	0	$\ominus 1$	$\ominus 1 \oplus i$	1	$\ominus 1 \ominus i$
$1 \ominus i$	$\ominus i$	$\ominus 1 \oplus i$	i	$1 \oplus i$	0	$1 \ominus i$	$\ominus 1 \ominus i$	1	$\ominus 1$
1	$\ominus 1 \ominus i$	$\ominus 1$	$\ominus 1 \oplus i$	$\ominus i$	0	i	$1 \ominus i$	1	$1 \oplus i$
$1 \oplus i$	$\ominus 1 \oplus i$	$\ominus 1 \ominus i$	$\ominus i$	i	0	$1 \oplus i$	$\ominus 1$	1	$1 \ominus i$

$\Leftrightarrow$	$\ominus 1 \ominus i$	$\ominus 1$	$\ominus 1 \oplus i$	$\ominus i$	0	i	$1 \ominus i$	1	$1 \oplus i$
$\ominus 1 \ominus i$	$\ominus i$	$1 \oplus i$	$\ominus 1$	$\ominus 1 \oplus i$	0	$1 \ominus i$	1	$\ominus 1 \ominus i$	i
$\ominus 1$	$1 \oplus i$	1	$1 \ominus i$	i	0	$\ominus i$	$\ominus 1 \oplus i$	$\ominus 1$	$\ominus 1 \ominus i$
$\ominus 1 \oplus i$	$\ominus 1$	$1 \ominus i$	i	$1 \oplus i$	0	$\ominus 1 \ominus i$	$\ominus i$	$\ominus 1 \oplus i$	1
$\ominus i$	$\ominus 1 \oplus i$	i	$1 \oplus i$	$\ominus 1$	0	1	$\ominus 1 \ominus i$	$\ominus i$	$1 \ominus i$
0	0	0	0	0	0	0	0	0	0
i	$1 \ominus i$	$\ominus i$	$\ominus 1 \ominus i$	1	0	$\ominus 1$	$1 \oplus i$	i	$\ominus 1 \oplus i$
$1 \ominus i$	1	$\ominus 1 \oplus i$	$\ominus i$	$\ominus 1 \ominus i$	0	$1 \oplus i$	i	$1 \ominus i$	$\ominus 1$
1	$\ominus 1 \ominus i$	$\ominus 1$	$\ominus 1 \oplus i$	$\ominus i$	0	i	$1 \ominus i$	1	$1 \oplus i$
$1 \oplus i$	i	$\ominus 1 \ominus i$	1	$1 \ominus i$	0	$\ominus 1 \oplus i$	$\ominus 1$	$1 \oplus i$	$\ominus i$

5.2 Algebraic Normal Forms

$$D = \begin{pmatrix} \ominus 1 \ominus i \\ \ominus 1 \\ \ominus 1 \oplus i \\ \ominus i \\ 0 \\ i \\ 1 \ominus i \\ 1 \\ 1 \oplus i \end{pmatrix} \; (2.3.1.1.1)$$

5.2.1 First candidate (2.3.1.3.4, 5.1.1)

$\forall x,\, y \in \mathbb{C}_3$:

5.2.1.1 $$\begin{aligned} x \wedge y = {} & \ominus y \oplus y^{\otimes 2} \ominus y^{\otimes 3} \oplus y^{\otimes 4} \ominus y^{\otimes 5} \oplus y^{\otimes 6} \ominus y^{\otimes 7} \oplus y^{\otimes 8} \ominus x \ominus x \otimes y \oplus i \otimes x \otimes y^{\otimes 2} \oplus (1 \oplus i) \otimes x \otimes y^{\otimes 4} \ominus \\ & \ominus i \otimes x \otimes y^{\otimes 5} \ominus (1 \ominus i) \otimes x \otimes y^{\otimes 6} \oplus i \otimes x \otimes y^{\otimes 7} \oplus x^{\otimes 2} \oplus i \otimes x^{\otimes 2} \otimes y \oplus (1 \oplus i) \otimes x^{\otimes 2} \otimes y^{\otimes 2} \oplus x^{\otimes 2} \otimes y^{\otimes 4} \oplus \\ & \oplus x^{\otimes 2} \otimes y^{\otimes 5} \ominus i \otimes x^{\otimes 2} \otimes y^{\otimes 6} \ominus (1 \oplus i) \otimes x^{\otimes 2} \otimes y^{\otimes 7} \ominus x^{\otimes 3} \oplus x^{\otimes 3} \otimes y^{\otimes 4} \oplus i \otimes x^{\otimes 3} \otimes y^{\otimes 5} \ominus i \otimes x^{\otimes 3} \otimes y^{\otimes 7} \oplus x^{\otimes 4} \oplus \\ & \oplus (1 \oplus i) \otimes x^{\otimes 4} \otimes y \oplus x^{\otimes 4} \otimes y^{\otimes 2} \oplus x^{\otimes 4} \otimes y^{\otimes 3} \ominus (1 \ominus i) \otimes x^{\otimes 4} \otimes y^{\otimes 4} \oplus x^{\otimes 4} \otimes y^{\otimes 5} \ominus (1 \oplus i) \otimes x^{\otimes 4} \otimes y^{\otimes 6} \ominus \\ & \ominus i \otimes x^{\otimes 4} \otimes y^{\otimes 7} \ominus x^{\otimes 5} \ominus i \otimes x^{\otimes 5} \otimes y \oplus x^{\otimes 5} \otimes y^{\otimes 2} \oplus i \otimes x^{\otimes 5} \otimes y^{\otimes 3} \oplus x^{\otimes 5} \otimes y^{\otimes 4} \oplus i \otimes x^{\otimes 5} \otimes y^{\otimes 5} \ominus \\ & \ominus x^{\otimes 5} \otimes y^{\otimes 6} \ominus i \otimes x^{\otimes 5} \otimes y^{\otimes 7} \oplus x^{\otimes 6} \ominus (1 \ominus i) \otimes x^{\otimes 6} \otimes y \ominus i \otimes x^{\otimes 6} \otimes y^{\otimes 2} \ominus (1 \oplus i) \otimes x^{\otimes 6} \otimes y^{\otimes 4} \ominus \\ & \ominus x^{\otimes 6} \otimes y^{\otimes 5} \ominus i \otimes x^{\otimes 6} \otimes y^{\otimes 6} \ominus (1 \oplus i) \otimes x^{\otimes 6} \otimes y^{\otimes 7} \ominus x^{\otimes 7} \oplus i \otimes x^{\otimes 7} \otimes y \ominus (1 \oplus i) \otimes x^{\otimes 7} \otimes y^{\otimes 2} \ominus \\ & \ominus i \otimes x^{\otimes 7} \otimes y^{\otimes 3} \ominus i \otimes x^{\otimes 7} \otimes y^{\otimes 4} \ominus i \otimes x^{\otimes 7} \otimes y^{\otimes 5} \ominus (1 \oplus i) \otimes x^{\otimes 7} \otimes y^{\otimes 6} \oplus i \otimes x^{\otimes 7} \otimes y^{\otimes 7} \oplus x^{\otimes 8} \ominus x^{\otimes 8} \otimes y^{\otimes 8} \end{aligned}$$

5.2.1.2 $$\begin{aligned} x \vee y = {} & \ominus y \ominus y^{\otimes 2} \ominus y^{\otimes 3} \ominus y^{\otimes 4} \ominus y^{\otimes 5} \ominus y^{\otimes 6} \ominus y^{\otimes 7} \ominus y^{\otimes 8} \ominus x \oplus x \otimes y \oplus i \otimes x \otimes y^{\otimes 2} \oplus (1 \oplus i) \otimes x \otimes y^{\otimes 4} \oplus \\ & \oplus i \otimes x \otimes y^{\otimes 5} \ominus (1 \ominus i) \otimes x \otimes y^{\otimes 6} \ominus i \otimes x \otimes y^{\otimes 7} \ominus x^{\otimes 2} \oplus i \otimes x^{\otimes 2} \otimes y \ominus (1 \oplus i) \otimes x^{\otimes 2} \otimes y^{\otimes 2} \ominus x^{\otimes 2} \otimes y^{\otimes 4} \oplus \\ & \oplus x^{\otimes 2} \otimes y^{\otimes 5} \oplus i \otimes x^{\otimes 2} \otimes y^{\otimes 6} \ominus (1 \oplus i) \otimes x^{\otimes 2} \otimes y^{\otimes 7} \ominus x^{\otimes 3} \oplus x^{\otimes 3} \otimes y^{\otimes 4} \ominus i \otimes x^{\otimes 3} \otimes y^{\otimes 5} \oplus i \otimes x^{\otimes 3} \otimes y^{\otimes 7} \ominus x^{\otimes 4} \oplus \\ & \oplus (1 \oplus i) \otimes x^{\otimes 4} \otimes y \ominus x^{\otimes 4} \otimes y^{\otimes 2} \oplus x^{\otimes 4} \otimes y^{\otimes 3} \oplus (1 \ominus i) \otimes x^{\otimes 4} \otimes y^{\otimes 4} \oplus x^{\otimes 4} \otimes y^{\otimes 5} \oplus (1 \oplus i) \otimes x^{\otimes 4} \otimes y^{\otimes 6} \ominus \\ & \ominus i \otimes x^{\otimes 4} \otimes y^{\otimes 7} \ominus x^{\otimes 5} \oplus i \otimes x^{\otimes 5} \otimes y \oplus x^{\otimes 5} \otimes y^{\otimes 2} \ominus i \otimes x^{\otimes 5} \otimes y^{\otimes 3} \oplus x^{\otimes 5} \otimes y^{\otimes 4} \ominus i \otimes x^{\otimes 5} \otimes y^{\otimes 5} \ominus \\ & \ominus x^{\otimes 5} \otimes y^{\otimes 6} \oplus i \otimes x^{\otimes 5} \otimes y^{\otimes 7} \ominus x^{\otimes 6} \ominus (1 \ominus i) \otimes x^{\otimes 6} \otimes y \oplus i \otimes x^{\otimes 6} \otimes y^{\otimes 2} \oplus (1 \oplus i) \otimes x^{\otimes 6} \otimes y^{\otimes 4} \ominus \\ & \ominus x^{\otimes 6} \otimes y^{\otimes 5} \oplus i \otimes x^{\otimes 6} \otimes y^{\otimes 6} \ominus (1 \oplus i) \otimes x^{\otimes 6} \otimes y^{\otimes 7} \ominus x^{\otimes 7} \ominus i \otimes x^{\otimes 7} \otimes y \ominus (1 \oplus i) \otimes x^{\otimes 7} \otimes y^{\otimes 2} \oplus \\ & \oplus i \otimes x^{\otimes 7} \otimes y^{\otimes 3} \ominus i \otimes x^{\otimes 7} \otimes y^{\otimes 4} \oplus i \otimes x^{\otimes 7} \otimes y^{\otimes 5} \ominus (1 \oplus i) \otimes x^{\otimes 7} \otimes y^{\otimes 6} \ominus i \otimes x^{\otimes 7} \otimes y^{\otimes 7} \ominus x^{\otimes 8} \oplus x^{\otimes 8} \otimes y^{\otimes 8} \end{aligned}$$

5.2.1.3 $$\begin{aligned} x \Rightarrow y = {} & \ominus y \ominus y^{\otimes 2} \ominus y^{\otimes 3} \ominus y^{\otimes 4} \ominus y^{\otimes 5} \ominus y^{\otimes 6} \ominus y^{\otimes 7} \ominus y^{\otimes 8} \oplus x \ominus x \otimes y \ominus i \otimes x \otimes y^{\otimes 2} \ominus (1 \oplus i) \otimes x \otimes y^{\otimes 4} \ominus \\ & \ominus i \otimes x \otimes y^{\otimes 5} \oplus (1 \ominus i) \otimes x \otimes y^{\otimes 6} \oplus i \otimes x \otimes y^{\otimes 7} \ominus x^{\otimes 2} \oplus i \otimes x^{\otimes 2} \otimes y \ominus (1 \oplus i) \otimes x^{\otimes 2} \otimes y^{\otimes 2} \ominus x^{\otimes 2} \otimes y^{\otimes 4} \oplus \\ & \oplus x^{\otimes 2} \otimes y^{\otimes 5} \oplus i \otimes x^{\otimes 2} \otimes y^{\otimes 6} \ominus (1 \oplus i) \otimes x^{\otimes 2} \otimes y^{\otimes 7} \oplus x^{\otimes 3} \ominus x^{\otimes 3} \otimes y^{\otimes 4} \oplus i \otimes x^{\otimes 3} \otimes y^{\otimes 5} \ominus i \otimes x^{\otimes 3} \otimes y^{\otimes 7} \ominus x^{\otimes 4} \oplus \\ & \oplus (1 \oplus i) \otimes x^{\otimes 4} \otimes y \ominus x^{\otimes 4} \otimes y^{\otimes 2} \oplus x^{\otimes 4} \otimes y^{\otimes 3} \oplus (1 \ominus i) \otimes x^{\otimes 4} \otimes y^{\otimes 4} \oplus x^{\otimes 4} \otimes y^{\otimes 5} \oplus (1 \oplus i) \otimes x^{\otimes 4} \otimes y^{\otimes 6} \ominus \\ & \ominus i \otimes x^{\otimes 4} \otimes y^{\otimes 7} \oplus x^{\otimes 5} \ominus i \otimes x^{\otimes 5} \otimes y \ominus x^{\otimes 5} \otimes y^{\otimes 2} \oplus i \otimes x^{\otimes 5} \otimes y^{\otimes 3} \ominus x^{\otimes 5} \otimes y^{\otimes 4} \oplus i \otimes x^{\otimes 5} \otimes y^{\otimes 5} \oplus \\ & \oplus x^{\otimes 5} \otimes y^{\otimes 6} \ominus i \otimes x^{\otimes 5} \otimes y^{\otimes 7} \ominus x^{\otimes 6} \ominus (1 \ominus i) \otimes x^{\otimes 6} \otimes y \oplus i \otimes x^{\otimes 6} \otimes y^{\otimes 2} \oplus (1 \oplus i) \otimes x^{\otimes 6} \otimes y^{\otimes 4} \ominus \\ & \ominus x^{\otimes 6} \otimes y^{\otimes 5} \oplus i \otimes x^{\otimes 6} \otimes y^{\otimes 6} \ominus (1 \oplus i) \otimes x^{\otimes 6} \otimes y^{\otimes 7} \oplus x^{\otimes 7} \oplus i \otimes x^{\otimes 7} \otimes y \oplus (1 \oplus i) \otimes x^{\otimes 7} \otimes y^{\otimes 2} \ominus \\ & \ominus i \otimes x^{\otimes 7} \otimes y^{\otimes 3} \oplus i \otimes x^{\otimes 7} \otimes y^{\otimes 4} \ominus i \otimes x^{\otimes 7} \otimes y^{\otimes 5} \oplus (1 \oplus i) \otimes x^{\otimes 7} \otimes y^{\otimes 6} \oplus i \otimes x^{\otimes 7} \otimes y^{\otimes 7} \ominus x^{\otimes 8} \oplus x^{\otimes 8} \otimes y^{\otimes 8} \end{aligned}$$

5.2.1.4 $x \Leftrightarrow y = x \otimes y$

5.2.2 Second candidate (2.3.1.3.4, 5.1.2)

$\forall x,\, y \in \mathbb{C}_3:$

5.2.2.1
$$\begin{aligned} x \wedge y = {} & \ominus y \oplus y^{\otimes 2} \ominus y^{\otimes 3} \oplus y^{\otimes 4} \ominus y^{\otimes 5} \oplus y^{\otimes 6} \ominus y^{\otimes 7} \oplus y^{\otimes 8} \ominus x \ominus x \otimes y \ominus i \otimes x \otimes y^{\otimes 2} \oplus (1 \ominus i) \otimes x \otimes y^{\otimes 4} \oplus \\ & \oplus i \otimes x \otimes y^{\otimes 5} \ominus (1 \oplus i) \otimes x \otimes y^{\otimes 6} \ominus i \otimes x \otimes y^{\otimes 7} \oplus x^{\otimes 2} \ominus i \otimes x^{\otimes 2} \otimes y \oplus (1 \ominus i) \otimes x^{\otimes 2} \otimes y^{\otimes 2} \oplus x^{\otimes 2} \otimes y^{\otimes 4} \oplus \\ & \oplus x^{\otimes 2} \otimes y^{\otimes 5} \oplus i \otimes x^{\otimes 2} \otimes y^{\otimes 6} \ominus (1 \ominus i) \otimes x^{\otimes 2} \otimes y^{\otimes 7} \ominus x^{\otimes 3} \oplus x^{\otimes 3} \otimes y^{\otimes 4} \ominus i \otimes x^{\otimes 3} \otimes y^{\otimes 5} \oplus i \otimes x^{\otimes 3} \otimes y^{\otimes 7} \oplus x^{\otimes 4} \oplus \\ & \oplus (1 \ominus i) \otimes x^{\otimes 4} \otimes y \oplus x^{\otimes 4} \otimes y^{\otimes 2} \oplus x^{\otimes 4} \otimes y^{\otimes 3} \ominus (1 \oplus i) \otimes x^{\otimes 4} \otimes y^{\otimes 4} \oplus x^{\otimes 4} \otimes y^{\otimes 5} \ominus (1 \ominus i) \otimes x^{\otimes 4} \otimes y^{\otimes 6} \oplus \\ & \oplus i \otimes x^{\otimes 4} \otimes y^{\otimes 7} \ominus x^{\otimes 5} \oplus i \otimes x^{\otimes 5} \otimes y \oplus x^{\otimes 5} \otimes y^{\otimes 2} \ominus i \otimes x^{\otimes 5} \otimes y^{\otimes 3} \oplus x^{\otimes 5} \otimes y^{\otimes 4} \ominus i \otimes x^{\otimes 5} \otimes y^{\otimes 5} \ominus \\ & \ominus x^{\otimes 5} \otimes y^{\otimes 6} \oplus i \otimes x^{\otimes 5} \otimes y^{\otimes 7} \oplus x^{\otimes 6} \ominus (1 \oplus i) \otimes x^{\otimes 6} \otimes y \oplus i \otimes x^{\otimes 6} \otimes y^{\otimes 2} \ominus (1 \ominus i) \otimes x^{\otimes 6} \otimes y^{\otimes 4} \ominus \\ & \ominus x^{\otimes 6} \otimes y^{\otimes 5} \oplus i \otimes x^{\otimes 6} \otimes y^{\otimes 6} \ominus (1 \ominus i) \otimes x^{\otimes 6} \otimes y^{\otimes 7} \ominus x^{\otimes 7} \ominus i \otimes x^{\otimes 7} \otimes y \ominus (1 \ominus i) \otimes x^{\otimes 7} \otimes y^{\otimes 2} \oplus \\ & \oplus i \otimes x^{\otimes 7} \otimes y^{\otimes 3} \oplus i \otimes x^{\otimes 7} \otimes y^{\otimes 4} \oplus i \otimes x^{\otimes 7} \otimes y^{\otimes 5} \ominus (1 \ominus i) \otimes x^{\otimes 7} \otimes y^{\otimes 6} \ominus i \otimes x^{\otimes 7} \otimes y^{\otimes 7} \oplus x^{\otimes 8} \ominus x^{\otimes 8} \otimes y^{\otimes 8} \end{aligned}$$

5.2.2.2
$$\begin{aligned} x \vee y = {} & \ominus y \ominus y^{\otimes 2} \ominus y^{\otimes 3} \ominus y^{\otimes 4} \ominus y^{\otimes 5} \ominus y^{\otimes 6} \ominus y^{\otimes 7} \ominus y^{\otimes 8} \ominus x \oplus x \otimes y \ominus i \otimes x \otimes y^{\otimes 2} \oplus (1 \ominus i) \otimes x \otimes y^{\otimes 4} \ominus \\ & \ominus i \otimes x \otimes y^{\otimes 5} \ominus (1 \oplus i) \otimes x \otimes y^{\otimes 6} \oplus i \otimes x \otimes y^{\otimes 7} \ominus x^{\otimes 2} \ominus i \otimes x^{\otimes 2} \otimes y \ominus (1 \ominus i) \otimes x^{\otimes 2} \otimes y^{\otimes 2} \ominus x^{\otimes 2} \otimes y^{\otimes 4} \oplus \\ & \oplus x^{\otimes 2} \otimes y^{\otimes 5} \ominus i \otimes x^{\otimes 2} \otimes y^{\otimes 6} \ominus (1 \ominus i) \otimes x^{\otimes 2} \otimes y^{\otimes 7} \ominus x^{\otimes 3} \oplus x^{\otimes 3} \otimes y^{\otimes 4} \oplus i \otimes x^{\otimes 3} \otimes y^{\otimes 5} \ominus i \otimes x^{\otimes 3} \otimes y^{\otimes 7} \ominus x^{\otimes 4} \oplus \\ & \oplus (1 \ominus i) \otimes x^{\otimes 4} \otimes y \ominus x^{\otimes 4} \otimes y^{\otimes 2} \oplus x^{\otimes 4} \otimes y^{\otimes 3} \oplus (1 \oplus i) \otimes x^{\otimes 4} \otimes y^{\otimes 4} \oplus x^{\otimes 4} \otimes y^{\otimes 5} \oplus (1 \ominus i) \otimes x^{\otimes 4} \otimes y^{\otimes 6} \oplus \\ & \oplus i \otimes x^{\otimes 4} \otimes y^{\otimes 7} \ominus x^{\otimes 5} \ominus i \otimes x^{\otimes 5} \otimes y \oplus x^{\otimes 5} \otimes y^{\otimes 2} \oplus i \otimes x^{\otimes 5} \otimes y^{\otimes 3} \oplus x^{\otimes 5} \otimes y^{\otimes 4} \oplus i \otimes x^{\otimes 5} \otimes y^{\otimes 5} \ominus \\ & \ominus x^{\otimes 5} \otimes y^{\otimes 6} \ominus i \otimes x^{\otimes 5} \otimes y^{\otimes 7} \ominus x^{\otimes 6} \ominus (1 \oplus i) \otimes x^{\otimes 6} \otimes y \ominus i \otimes x^{\otimes 6} \otimes y^{\otimes 2} \oplus (1 \ominus i) \otimes x^{\otimes 6} \otimes y^{\otimes 4} \ominus \\ & \ominus x^{\otimes 6} \otimes y^{\otimes 5} \ominus i \otimes x^{\otimes 6} \otimes y^{\otimes 6} \ominus (1 \ominus i) \otimes x^{\otimes 6} \otimes y^{\otimes 7} \ominus x^{\otimes 7} \oplus i \otimes x^{\otimes 7} \otimes y \ominus (1 \ominus i) \otimes x^{\otimes 7} \otimes y^{\otimes 2} \ominus \\ & \ominus i \otimes x^{\otimes 7} \otimes y^{\otimes 3} \oplus i \otimes x^{\otimes 7} \otimes y^{\otimes 4} \ominus i \otimes x^{\otimes 7} \otimes y^{\otimes 5} \ominus (1 \ominus i) \otimes x^{\otimes 7} \otimes y^{\otimes 6} \oplus i \otimes x^{\otimes 7} \otimes y^{\otimes 7} \ominus x^{\otimes 8} \oplus x^{\otimes 8} \otimes y^{\otimes 8} \end{aligned}$$

5.2.2.3
$$\begin{aligned} x \Rightarrow y = {} & \ominus y \ominus y^{\otimes 2} \ominus y^{\otimes 3} \ominus y^{\otimes 4} \ominus y^{\otimes 5} \ominus y^{\otimes 6} \ominus y^{\otimes 7} \ominus y^{\otimes 8} \oplus x \ominus x \otimes y \oplus i \otimes x \otimes y^{\otimes 2} \ominus (1 \ominus i) \otimes x \otimes y^{\otimes 4} \oplus \\ & \oplus i \otimes x \otimes y^{\otimes 5} \oplus (1 \oplus i) \otimes x \otimes y^{\otimes 6} \ominus i \otimes x \otimes y^{\otimes 7} \ominus x^{\otimes 2} \ominus i \otimes x^{\otimes 2} \otimes y \ominus (1 \ominus i) \otimes x^{\otimes 2} \otimes y^{\otimes 2} \ominus x^{\otimes 2} \otimes y^{\otimes 4} \oplus \\ & \oplus x^{\otimes 2} \otimes y^{\otimes 5} \ominus i \otimes x^{\otimes 2} \otimes y^{\otimes 6} \ominus (1 \ominus i) \otimes x^{\otimes 2} \otimes y^{\otimes 7} \oplus x^{\otimes 3} \ominus x^{\otimes 3} \otimes y^{\otimes 4} \ominus i \otimes x^{\otimes 3} \otimes y^{\otimes 5} \oplus i \otimes x^{\otimes 3} \otimes y^{\otimes 7} \ominus x^{\otimes 4} \oplus \\ & \oplus (1 \ominus i) \otimes x^{\otimes 4} \otimes y \ominus x^{\otimes 4} \otimes y^{\otimes 2} \oplus x^{\otimes 4} \otimes y^{\otimes 3} \oplus (1 \oplus i) \otimes x^{\otimes 4} \otimes y^{\otimes 4} \oplus x^{\otimes 4} \otimes y^{\otimes 5} \oplus (1 \ominus i) \otimes x^{\otimes 4} \otimes y^{\otimes 6} \oplus \\ & \oplus i \otimes x^{\otimes 4} \otimes y^{\otimes 7} \oplus x^{\otimes 5} \oplus i \otimes x^{\otimes 5} \otimes y \ominus x^{\otimes 5} \otimes y^{\otimes 2} \ominus i \otimes x^{\otimes 5} \otimes y^{\otimes 3} \ominus x^{\otimes 5} \otimes y^{\otimes 4} \ominus i \otimes x^{\otimes 5} \otimes y^{\otimes 5} \oplus \\ & \oplus x^{\otimes 5} \otimes y^{\otimes 6} \oplus i \otimes x^{\otimes 5} \otimes y^{\otimes 7} \ominus x^{\otimes 6} \ominus (1 \oplus i) \otimes x^{\otimes 6} \otimes y \ominus i \otimes x^{\otimes 6} \otimes y^{\otimes 2} \oplus (1 \ominus i) \otimes x^{\otimes 6} \otimes y^{\otimes 4} \ominus \\ & \ominus x^{\otimes 6} \otimes y^{\otimes 5} \ominus i \otimes x^{\otimes 6} \otimes y^{\otimes 6} \ominus (1 \ominus i) \otimes x^{\otimes 6} \otimes y^{\otimes 7} \oplus x^{\otimes 7} \ominus i \otimes x^{\otimes 7} \otimes y \oplus (1 \ominus i) \otimes x^{\otimes 7} \otimes y^{\otimes 2} \oplus \\ & \oplus i \otimes x^{\otimes 7} \otimes y^{\otimes 3} \ominus i \otimes x^{\otimes 7} \otimes y^{\otimes 4} \oplus i \otimes x^{\otimes 7} \otimes y^{\otimes 5} \oplus (1 \ominus i) \otimes x^{\otimes 7} \otimes y^{\otimes 6} \ominus i \otimes x^{\otimes 7} \otimes y^{\otimes 7} \ominus x^{\otimes 8} \oplus x^{\otimes 8} \otimes y^{\otimes 8} \end{aligned}$$

5.2.2.4 $x \Leftrightarrow y = x \otimes y$

5.3 Functional Equivalence by Logical Connectives

5.3.1 Common values (5.1.1, 5.1.2)

5.3.1.1 By results:

5.3.1.1.1 $(\ominus 1 \ominus i) \wedge 0 = 0$

5.3.1.1.2 $(\ominus 1 \ominus i) \wedge (1 \ominus i) = 1$

5.3.1.1.3 $(\ominus 1 \oplus i) \wedge 0 = 0$

5.3.1.1.4 $(\ominus 1 \oplus i) \wedge (1 \oplus i) = 1$

5.3.1.1.5 $\ominus i \wedge 0 = 0$

5.3.1.1.6 $\ominus i \wedge i = 1$

5.3.1.1.7 $0 \wedge i = 0$

5.3.1.1.8 $0 \wedge (1 \ominus i) = 0$

5.3.1.1.9 $0 \wedge (1 \oplus i) = 0$

5.3.1.2 By equations:

5.3.1.2.1 $(\ominus 1 \oplus i) \wedge (1 \ominus i) = \ominus i \wedge \ominus i = (i \wedge i) \wedge (i \wedge i)$

5.3.1.2.2 $\ominus i \wedge (1 \oplus i) = i \wedge (1 \ominus i)$

5.3.2 Generalization

5.3.2.1 Definitions:

5.3.2.1.1 $\aleph \coloneqq i \wedge i$

5.3.2.1.2 $\beth \coloneqq \aleph \wedge \aleph$

5.3.2.2 Corollaries (5.1.1, 5.1.2, 5.3.2.1.1, 5.3.2.1.2):

5.3.2.2.1 $\begin{pmatrix} \aleph \\ \beth \end{pmatrix} \in \left\{ \begin{pmatrix} 1 \ominus i \\ 1 \oplus i \end{pmatrix}, \ \begin{pmatrix} \ominus 1 \oplus i \\ \ominus 1 \ominus i \end{pmatrix} \right\}$

5.3.2.2.2 $\ominus i \wedge \ominus i = \ominus \beth$

5.3.2.2.3 $\ominus i \wedge \ominus \beth = \beth$

5.3.2.2.4 $\ominus i \wedge \ominus \aleph = \aleph$

5.3.2.2.5 $\ominus i \wedge \aleph = i$

5.3.2.2.6 $\ominus i \wedge \beth = \ominus \aleph$

5.3.2.2.7 $\ominus \beth \wedge \ominus \beth = \ominus \aleph$

5.3.2.2.8 $\ominus \beth \wedge \ominus \aleph = i$

5.3.2.2.9 $\ominus \beth \wedge 0 = 0$

5.3.2.2.10 $\ominus \beth \wedge \aleph = 1$

5.3.2.2.11 $\ominus \beth \wedge \beth = \aleph$

5.3.2.2.12 $\ominus \beth \wedge i = \ominus i$

5.3.2.2.13 $\ominus \aleph \wedge \ominus \aleph = \ominus i$

5.3.2.2.14 $\ominus \aleph \wedge 0 = 0$

5.3.2.2.15 $\ominus \aleph \wedge \aleph = \ominus \beth$

5.3.2.2.16 $\ominus \aleph \wedge \beth = 1$

5.3.2.2.17 $\ominus \aleph \wedge i = \beth$

5.3.2.2.18 $0 \wedge \aleph = 0$

5.3.2.2.19 $0 \wedge \beth = 0$

5.3.2.2.20 $\aleph \wedge \beth = \ominus i$

5.3.2.2.21 $\aleph \wedge i = \ominus \aleph$

5.3.2.2.22 $\beth \wedge \beth = i$

5.3.2.2.23 $\beth \wedge i = \ominus \beth$

6 A Useful Complex-Valued Logic

6.1 Truth Tables (5.1.1, 5.1.2, 5.3.2.1.1, 5.3.2.1.2, 5.3.2.2)

$\wedge$	$\ominus 1$	$\ominus i$	$\ominus\beth$	$\ominus\aleph$	0	$\aleph$	$\beth$	i	1
$\ominus 1$	$\ominus 1$	$\ominus 1$	$\ominus 1$	$\ominus 1$	$\ominus 1$	$\ominus 1$	$\ominus 1$	$\ominus 1$	$\ominus 1$
$\ominus i$	$\ominus 1$	$\ominus\beth$	$\beth$	$\aleph$	0	i	$\ominus\aleph$	1	$\ominus i$
$\ominus\beth$	$\ominus 1$	$\beth$	$\ominus\aleph$	i	0	1	$\aleph$	$\ominus i$	$\ominus\beth$
$\ominus\aleph$	$\ominus 1$	$\aleph$	i	$\ominus i$	0	$\ominus\beth$	1	$\beth$	$\ominus\aleph$
0	$\ominus 1$	0	0	0	0	0	0	0	0
$\aleph$	$\ominus 1$	i	1	$\ominus\beth$	0	$\beth$	$\ominus i$	$\ominus\aleph$	$\aleph$
$\beth$	$\ominus 1$	$\ominus\aleph$	$\aleph$	1	0	$\ominus i$	i	$\ominus\beth$	$\beth$
i	$\ominus 1$	1	$\ominus i$	$\beth$	0	$\ominus\aleph$	$\ominus\beth$	$\aleph$	i
1	$\ominus 1$	$\ominus i$	$\ominus\beth$	$\ominus\aleph$	0	$\aleph$	$\beth$	i	1

$\vee$	$\ominus 1$	$\ominus i$	$\ominus\beth$	$\ominus\aleph$	0	$\aleph$	$\beth$	i	1
$\ominus 1$	$\ominus 1$	$\ominus i$	$\ominus\beth$	$\ominus\aleph$	0	$\aleph$	$\beth$	i	1
$\ominus i$	$\ominus i$	$\ominus\aleph$	$\beth$	$\aleph$	0	$\ominus\beth$	i	$\ominus 1$	1
$\ominus\beth$	$\ominus\beth$	$\beth$	$\ominus i$	i	0	$\ominus 1$	$\ominus\aleph$	$\aleph$	1
$\ominus\aleph$	$\ominus\aleph$	$\aleph$	i	$\ominus\beth$	0	$\beth$	$\ominus 1$	$\ominus i$	1
0	0	0	0	0	0	0	0	0	1
$\aleph$	$\aleph$	$\ominus\beth$	$\ominus 1$	$\beth$	0	i	$\ominus i$	$\ominus\aleph$	1
$\beth$	$\beth$	i	$\ominus\aleph$	$\ominus 1$	0	$\ominus i$	$\aleph$	$\ominus\beth$	1
i	i	$\ominus 1$	$\aleph$	$\ominus i$	0	$\ominus\aleph$	$\ominus\beth$	$\beth$	1
1	1	1	1	1	1	1	1	1	1

$\Rightarrow$	$\ominus 1$	$\ominus i$	$\ominus\beth$	$\ominus\aleph$	0	$\aleph$	$\beth$	i	1
$\ominus 1$	1	1	1	1	1	1	1	1	1
$\ominus i$	i	$\ominus 1$	$\aleph$	$\ominus i$	0	$\ominus\aleph$	$\ominus\beth$	$\beth$	1
$\ominus\beth$	$\beth$	i	$\ominus\aleph$	$\ominus 1$	0	$\ominus i$	$\aleph$	$\ominus\beth$	1
$\ominus\aleph$	$\aleph$	$\ominus\beth$	$\ominus 1$	$\beth$	0	i	$\ominus i$	$\ominus\aleph$	1
0	0	0	0	0	0	0	0	0	1
$\aleph$	$\ominus\aleph$	$\aleph$	i	$\ominus\beth$	0	$\beth$	$\ominus 1$	$\ominus i$	1
$\beth$	$\ominus\beth$	$\beth$	$\ominus i$	i	0	$\ominus 1$	$\ominus\aleph$	$\aleph$	1
i	$\ominus i$	$\ominus\aleph$	$\beth$	$\aleph$	0	$\ominus\beth$	i	$\ominus 1$	1
1	$\ominus 1$	$\ominus i$	$\ominus\beth$	$\ominus\aleph$	0	$\aleph$	$\beth$	i	1

$\Leftrightarrow$	$\ominus 1$	$\ominus i$	$\ominus\beth$	$\ominus\aleph$	0	$\aleph$	$\beth$	i	1
$\ominus 1$	1	i	$\beth$	$\aleph$	0	$\ominus\aleph$	$\ominus\beth$	$\ominus i$	$\ominus 1$
$\ominus i$	i	$\ominus 1$	$\ominus\aleph$	$\beth$	0	$\ominus\beth$	$\aleph$	1	$\ominus i$
$\ominus\beth$	$\beth$	$\ominus\aleph$	$\ominus i$	$\ominus 1$	0	1	i	$\aleph$	$\ominus\beth$
$\ominus\aleph$	$\aleph$	$\beth$	$\ominus 1$	i	0	$\ominus i$	1	$\ominus\beth$	$\ominus\aleph$
0	0	0	0	0	0	0	0	0	0
$\aleph$	$\ominus\aleph$	$\ominus\beth$	1	$\ominus i$	0	i	$\ominus 1$	$\beth$	$\aleph$
$\beth$	$\ominus\beth$	$\aleph$	i	1	0	$\ominus 1$	$\ominus i$	$\ominus\aleph$	$\beth$
i	$\ominus i$	1	$\aleph$	$\ominus\beth$	0	$\beth$	$\ominus\aleph$	$\ominus 1$	i
1	$\ominus 1$	$\ominus i$	$\ominus\beth$	$\ominus\aleph$	0	$\aleph$	$\beth$	i	1

6.2 Application

The values i and $\ominus i$ can be used as truth values of cross-world predicates in coherent worlds.