1- Introduction

Our goal is to find the Laplacian matrix A whose eigenvalues are given. If we can find the eigenvectors of matrix A, Eq. (1) can be used to generate matrix A.

For any arbitrary matrix A, the eigen decomposition is given by the following.

$$A = Q\Lambda Q^{-1} \tag{1}$$

Where the columns of Q are the eigenvectors of A and Λ is a diagonal matrix with the values of corresponding eigenvectors.

The Laplacian matrix has two main features:

1) It is symmetric
2) The summation of all rows is equal to zero.

for the case of a symmetric matrix, one can write:

$$A = A^T = (Q\Lambda Q^{-1})^T = (Q^{-1})^T \Lambda \mathrm{Q}^T \tag{2}$$

Comparing Eq. (1) and Eq. (2), it can be concluded that $\mathrm{Q}^T = Q^{-1}$ which means that the matrix Q is orthogonal matrix and the eigenvectors of matrix A form an orthogonal basis.

For all Laplacian matrices, the first eigenvalue is zero, which is corresponding to the eigenvector $\overrightarrow{q_1} = [1,1,\dots,1]^T$. This can be concluded from the second feature of the Laplacian matrix:

$$\lambda\,\overrightarrow{q_1} = A\,\overrightarrow{q_1} \tag{3}$$

Where for $\lambda = 0$ the right-hand side will be zero for $\overrightarrow{q_1} = [1,1,\dots,1]^T$ since $A\,\overrightarrow{q_1}$ gives the summation of the rows of A which is equal to zero when A is a Laplacian matrix.

Now, we need to create an orthogonal basis with dimension of n, with the first vector of $\overrightarrow{q_1}$. To aim this purpose, we use Gram-Schmidt's method as following.

2- Gram-Schmidt's method

Gram-Schmidt's algorithm provides a procedure to create an orthogonal basis from a given independent (but not orthogonal) basis. Suppose that $U = \{u_1, u_2, \dots, u_n\}$ is an independent basis. The orthogonal basis $v = \{v_1, v_2, \dots, v_n\}$ can be derived from U by using the following procedure.

$$\begin{aligned}
v_1 &= u_1 \\
v_2 &= u_2 - \frac{u_2.v_1}{v_1.v_1} v_1 \\
v_3 &= u_3 - \frac{u_3.v_1}{v_1.v_1} v_1 - \frac{u_3.v_2}{v_2.v_2} v_2 \\
&\vdots \\
v_n &= u_n - \sum_{k=1}^{n-1} \frac{u_n.v_k}{v_k.v_k} v_k
\end{aligned} \tag{4}$$

To use the Gram-Schmidt's method, an independent basis U with $u_1 = [1,1,\dots,1]^T$ is required. some examples of such sets are given below for $n = 3$.

$U = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$	(5)
$U = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$	(6)
$U = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$	(7)