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Polynomial Functors in Lean4

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Chapter 1

Locally Cartesian Closed Categories

To be written

Chapter 2

Univaiate Polynomial Functors

In this section we develop some of the definitions and lemmas related to polynomial endofunctors that we will use in the rest of the notes.

Definition 2.0.1 (Polynomial endofunctor)**.** UvPoly Let $\mathbb{C}$ be a locally Cartesian closed category (in our case, presheaves on the category of contexts). This means for each morphism $t : B \to A$ we have an adjoint triple

$$\begin{tikzcd} \mathbb{C}/B \arrow[d, bend right=60, "t_!"'] \arrow[d, bend left=60, "t_*"] \arrow[d, phantom, "\dashv \; t^* \; \dashv"] \\ \mathbb{C}/A \arrow[u] \end{tikzcd}$$

where t^* is pullback, and $t_!$ is composition with t.

Let $t : B \to A$ be a morphism in $\mathbb{C}$. Then define $P_t : \mathbb{C} \to \mathbb{C}$ be the composition

$$P_t := A_! \circ t_* \circ B^*$$

$$\mathbb{C} \xrightarrow{B^*} \mathbb{C}/B \xrightarrow{t_*} \mathbb{C}/A \xrightarrow{A_!} \mathbb{C}$$

Proposition 2.0.2 (Characterising property of Polynomial Endofunctors)**.** *UvPoly.equiv defn:UvPoly The data of a map into the polynomial applied to an object in* $\mathbb{C}$

$$\Gamma \longrightarrow P_t Y$$

corresponds to a pair of morphisms

$$\alpha : \Gamma \to A \qquad \textit{and} \qquad \beta : \Gamma \cdot \alpha \to Y$$

and this correspondance is natural in both Γ *and* Y*.*

Given any such ϕ *we can extract* $\alpha : \Gamma \to A$ *by composition*

$$\begin{tikzcd}
\Gamma \arrow[rr, "\phi"] \arrow[dr, dashed, "\alpha"'] & & P_t Y \arrow[dl, "t_* B^* Y"] \\
& A &
\end{tikzcd}$$

Applying the adjunction $A_! \dashv A^*$, and viewing $\phi : \alpha \to t_* B^* Y$ as a map in the slice, this corresponds to

$$\alpha : \Gamma \to A \qquad \textit{and}$$

$$\begin{tikzcd}
B_! t^* \alpha \arrow[rr, dashed, "\tilde{\phi}"] \arrow[dr, "t^* \alpha"'] & & B \times Y \arrow[dl, "B^* Y"] \\
& B &
\end{tikzcd}$$

Applying the adjunction $t^* \dashv t_*$, this corresponds to

$$\alpha : \Gamma \to A \qquad \textit{and}$$

$$\Gamma \cdot \alpha := B_! t^* \alpha \overset{\beta}{\dashrightarrow} Y$$

Henceforth we will write

$$(\alpha, \beta) : \Gamma \to P_t Y$$

for this map, since it is uniquely determined by this data.

This is natural in Γ. Precomposition by $\sigma : \Delta \to \Gamma$, acts on such a pair by

$$\begin{tikzcd}
\Delta \arrow[d, "\sigma"'] \arrow[dr, "(\alpha \circ \sigma, \beta \circ t^* \sigma)"] & \\
\Gamma \arrow[r, "(\alpha, \beta)"'] & P_t Y
\end{tikzcd}$$

It is also natural in $f : X \to Y$, meaning the morphism $P_t f$ acts on such a pair by

$$\begin{tikzcd}
\Gamma \arrow[r, "(\alpha, \beta)"] \arrow[dr, "(\alpha, f \circ \beta)"'] & P_t X \arrow[d, "P_t f"] \\
& P_t Y
\end{tikzcd}$$

Lemma 2.0.3. *prop:UvPoly.equiv Use R to denote the fiber product*

$$\begin{tikzcd}
R \arrow[r, "\rho_{\mathsf{Tm}}"] \arrow[d, "\rho_P"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "t"] \\
P_t Y \arrow[r, "t_* B^* Y"'] & A
\end{tikzcd}$$

By the universal property of pullbacks and **??**, *The data of a map $\Gamma \to R$ corresponds to the data of $\beta : \Gamma \to B$ and $(t \circ \beta, y) : \Gamma \to P_t Y$, or just $\beta : \Gamma \to B$ and $y : \Gamma \cdot t \circ \beta \to Y$*

$$
\begin{tikzcd}
\Gamma \arrow[rrd, bend left, "\beta"] \arrow[rd, dashed, "(\beta,y)"] \arrow[rdd, bend right, "(t\circ\beta,y)"'] & & \\
& R \arrow[r, dashed, "\rho_B"] \arrow[d, "\rho_P"] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "t"'] \\
& P_tY \arrow[r, "t_*B^*Y"] & A
\end{tikzcd}
$$

By uniqueness in the universal property of pullbacks and **??**, *Precomposition by a map $\sigma : \Delta \to \Gamma$ acts on such a pair by*

$$
\begin{tikzcd}
\Delta \arrow[d, "\sigma"'] \arrow[dr, "(\beta\circ\sigma, y\circ t^*\sigma)"] & \\
\Gamma \arrow[r, "(\beta,y)"'] & R
\end{tikzcd}
$$

Definition 2.0.4 (Evaluation)**.** defn:UvPoly Let $\mathsf{counit} : \rho_B \to B \to B^*Y$ denote the counit of the adjunction $f^* \dashv f_*$ at the object B^*Y, recalling that $\rho_B = t^*t_*B^*Y$. Then viewing the object B^*Y in the slice as the object $Y \times B$ in the ambient category, we define $\mathsf{ev} : R \to Y$ as the composition

$$
\begin{tikzcd}
R \arrow[r, "\mathsf{counit}"'] \arrow[rr, bend left, "\mathsf{ev}"] \arrow[d, "\rho_B"'] & Y \times B \arrow[r, "\pi_Y"'] \arrow[dl] & Y \\
B & &
\end{tikzcd}
$$

Lemma 2.0.5 (Evaluation Computation)**.** *defn:UvPolyEv Suppose $(\beta, y) : \Gamma \to R$, as in* **??**

$$\beta : \Gamma \to B \quad \text{and} \quad y : \Gamma \cdot t \circ \beta \to Y$$

Then the evaluation of y at β can be computed as

$$\mathsf{ev} \circ (\beta, y) = y \circ b$$

where

$$
\begin{tikzcd}
\Gamma \arrow[rrd, bend left, "\beta"] \arrow[rd, dashed, "b"] \arrow[rdd, bend right, equal] & & \\
& \Gamma \cdot t \circ \beta \arrow[r, "v"'] \arrow[d, "d"] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "t"] \\
& \Gamma \arrow[r, "t\circ\beta"'] & A
\end{tikzcd}
$$

and

$$\begin{tikzcd}
& \Gamma \arrow[dd, dashed, "(y\circ b,\beta)"] \arrow[dr, "(\beta,y)"] \arrow[drr, bend left, "(t\circ\beta,y)"] \arrow[ddl, dashed, bend right, "y\circ b"'] & & \\
& & R \arrow[r] \arrow[d] \arrow[dl, "\mathsf{counit}"'] \arrow[dr, phantom, "\lrcorner"] & P_tY \arrow[d, "t_*B^*Y"] \\
Y & Y\times B \arrow[l, "\pi_Y"] \arrow[r, "\pi_B"'] & B \arrow[r, "t"'] & A
\end{tikzcd}$$

Proof. It suffices to show $(\mathsf{counit} \circ (\beta, y)) = (y \circ b, \beta)$ instead.

$$\begin{aligned}
&\mathsf{counit} \circ (\beta, y) && \\
&= \mathsf{counit} \circ (v \circ b, y \circ t^* d \circ t^* b) && \textbf{??} \\
&= \mathsf{counit} \circ (v, y \circ t^* d) \circ b && \textbf{??}, \textbf{??} \\
&= \mathsf{counit} \circ t^*(t \circ \beta, y) \circ b && \textbf{??} \\
&= \overline{(t \circ \beta, y)} \circ b && \textbf{??} \\
&= (y, v) \circ b && \textbf{??} \\
&= (y \circ b, v \circ b) && \\
&= (y \circ b, \beta) &&
\end{aligned}$$

$$\begin{tikzcd}
\Gamma\cdot t\circ\beta \arrow[r, dashed, "t^*b"] \arrow[d, "d"] \arrow[dr, phantom, "\lrcorner"] \arrow[rr, bend left, equal] & \Gamma\cdot t\circ\beta\cdot t\circ\beta \arrow[r, dashed, "t^*d"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \Gamma\cdot t\circ\beta \arrow[r, "v"] \arrow[d, "d"] \arrow[dr, phantom, "\lrcorner"] & B \arrow[d, "t"] \\
\Gamma \arrow[r, "b"'] \arrow[rr, bend right, equal] & \Gamma\cdot t\circ\beta \arrow[r, "d"'] & \Gamma \arrow[r, "t\circ\beta"'] & A
\end{tikzcd}$$

Figure 2.1: $t^* d \circ t^* b = \mathsf{id}_{\Gamma\cdot t\circ\beta}$

$$\begin{tikzcd}
\Gamma \arrow[d, "b"'] \arrow[dr, "(v\circ b, y\circ t^*d\circ t^*b)"] & \\
\Gamma\cdot t\circ\beta \arrow[r, "(v, y\circ t^*d)"'] & R
\end{tikzcd}$$

Figure 2.2: $(v, y \circ t^* d) \circ b = (v \circ b, y \circ t^* d \circ t^* b)$

$\square$

$$
\begin{tikzcd}
\Gamma \cdot t \circ \beta \arrow[r, dashed, "{(v, y \circ t^* d)}"] \arrow[d, dashed, "d"'] \arrow[rr, bend left, "v"] \arrow[dr, phantom, "\lrcorner", very near start] & R \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "t"] \\
\Gamma \arrow[r, "{(t \circ \beta, y)}"'] \arrow[rr, bend right, "t \circ \beta"'] & P_t Y \arrow[r, "t_* B^* Y"'] & A
\end{tikzcd}
$$

$$
\begin{tikzcd}
\Gamma \cdot t \circ \beta \arrow[d, "d"'] \arrow[dr, "{(t \circ \beta \circ d, y \circ t^* d)}"] & \\
\Gamma \arrow[r, "{(t \circ \beta, y)}"'] & P_t Y
\end{tikzcd}
$$

using **??**, **??**

Figure 2.3: $t^*(t \circ \beta, y) = (v, y \circ t^* d)$

$$
\begin{tikzcd}
t^*(t \circ \beta) \arrow[d, "{t^*(t \circ \beta, y)}"'] \arrow[dr, dashed, "\overline{(t \circ \beta, y)}"] & \\
t^* t_* B^* Y \arrow[r, "\mathsf{counit}"'] & B^* Y
\end{tikzcd}
\qquad \Big\| \qquad t^* \dashv t_* \qquad
\begin{tikzcd}
t \circ \beta \arrow[d, "{(t \circ \beta, y)}"'] \arrow[dr, "{(t \circ \beta, y)}"] & \\
t_* B^* Y \arrow[r, equal, "\widetilde{\mathsf{counit}}"'] & t_* B^* Y
\end{tikzcd}
$$

Figure 2.4: $\mathsf{counit} \circ t^*(t \circ \beta, y) = \overline{(t \circ \beta, y)}$

Definition 2.0.6 (Polynomial composition)**.** UvPoly.comp Let $f : B \to A$ and $g : D \to C$. Define the *polynomial composition* $f \lhd g : Q \to P_f C$ as the composition of the two vertical maps in the following

$$
\begin{tikzcd}
D \arrow[d, "g"'] & Q \arrow[l] \arrow[d] \arrow[dl, phantom, "\llcorner", very near start] \arrow[dd, dashed, bend left, "f \lhd g"] & \\
C & R \arrow[l, "\mathsf{ev}"] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "f"] \\
& P_f C \arrow[r, "f_* B^* C"'] & A
\end{tikzcd}
$$

Then the two functors

$$P_{f \lhd g} \cong P_f \circ P_g$$

are naturally isomorphic.

Proof. $\square$

Definition 2.0.7 (Mate)**.** Suppose

$$
\begin{tikzcd}
\Gamma \cdot t \circ \beta \arrow[r, dashed, "{(y,v)}"] \arrow[d, "{v=t^*(t\circ\beta)}"'] & Y \times B \arrow[dl, "B^*Y"] \\
B &
\end{tikzcd}
\qquad
\begin{array}{c} \Big\Vert \\ t^* \dashv t_* \end{array}
\qquad
\begin{tikzcd}
\Gamma \arrow[r, "{(t\circ\beta,y)}"] \arrow[d, "t\circ\beta"'] & P_t Y \arrow[dl, "t_* B^* Y"] \\
A &
\end{tikzcd}
$$

Figure 2.5: $\overline{(t \circ \beta, y)} = (y, v)$

$$
\begin{tikzcd}
C \arrow[rr, "\rho"] \arrow[dr, "s"'] & & B \arrow[dl, "t"] \\
& A &
\end{tikzcd}
$$

Then we have a mate $\mu_! : \rho_! \circ s^* \Rightarrow t^*$. This is given by the universal property of pullbacks: given $f : x \to y$ in the slice $\mathbb{C}/A$ we have

$$
\begin{tikzcd}
\bullet \arrow[r, "\mu_{!x}"] \arrow[d, "s^*f"'] \arrow[dr, phantom, "= \mu_! \Rightarrow"] & \bullet \arrow[r] \arrow[d, "t^*f"] & X \arrow[d, "f"] \arrow[dd, bend left=60, "x"] \\
\bullet \arrow[r, "\mu_{!y}"'] \arrow[d, "s^*y"'] & \bullet \arrow[r] \arrow[d, "t^*y"'] & Y \arrow[d, "y"] \\
C \arrow[r, "\rho"'] & B \arrow[r, "t"'] & A
\end{tikzcd}
$$

By the calculus of mates we also have a reversed mate between the right adjoints $\mu^* : t_* \to s_* \circ \rho^*$. Explicitly μ^* is the composition

$$
t_* \xrightarrow{\text{unit } t_*} s_* \rho^* \rho_! s^* t_* \xrightarrow{s_* \rho^* \mu_! t_*} s_* \rho^* t^* t_* \xrightarrow{s_* \rho^* \text{ counit}} s_* \rho^*
$$

Definition 2.0.8 (Contravariant action of P_{-} on a slice)**.** Let $P_{-} : (\mathbb{C}/A)^{\text{op}} \to [\mathbb{C}, \mathbb{C}]$ be defined by taking $s \mapsto P_s$ on objects and act on a morphism by

$$
\begin{tikzcd}
& B \arrow[dl, "t"'] \\
A & \\
& C \arrow[ul, "s"] \arrow[uu, "\rho"']
\end{tikzcd}
\quad \longmapsto \quad
\begin{tikzcd}
P_t \arrow[dd, "\rho^\star"] \\
\\
P_s
\end{tikzcd}
$$

where

$$
\rho^\star := A_!(s_* \eta \circ \mu B^*) : P_t \to P_s
$$

$$
\begin{tikzcd}
\mathbb{C} \arrow[d, "C^*"'] \arrow[dr, "B^*"] \arrow[ddd, bend right=60, "P_s"'] \arrow[ddd, bend left=90, "P_t"] & \\
\mathbb{C}/C \arrow[d, "s_*"'] \arrow[ur, phantom, "\Leftarrow \eta \cong"] & \mathbb{C}/B \arrow[l, "\rho^*"'] \arrow[dl, "t_*"] \\
\mathbb{C}/A \arrow[d, "A_!"] \arrow[ur, phantom, "\Leftarrow \mu \cong"] & \\
\mathbb{C} &
\end{tikzcd}
$$

where $\mu = \mu^*$ is the mate from **??**, and η is the natural isomorphism given by pullback pasting.

Pointwise, this natural transformation acts on a pair $(\alpha, \beta) : \Gamma \to P_t X$ by

$$
\begin{tikzcd}
\Gamma \arrow[r, "{(\alpha,\beta)}"] \arrow[dr, "{(\alpha,\beta\circ\alpha^*\rho)}"'] & P_t X \arrow[d, "\rho^\star_X"] \\
& P_s X
\end{tikzcd}
$$

where $\alpha^*\rho$ is defined as

$$
\begin{tikzcd}
\Gamma \cdot_s \alpha \arrow[r, "s^*\alpha"] \arrow[d, dashed, "\alpha^*\rho"'] \arrow[dr, phantom, "\lrcorner" very near start] & C \arrow[d, "\rho"] \\
\Gamma \cdot_t \alpha \arrow[r, "t^*\alpha"] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & B \arrow[d, "t"] \\
\Gamma \arrow[r, "\alpha"'] & A
\end{tikzcd}
$$

We prove this now.

Proof. Firstly $\rho^\star_X = A_!(s_*\eta_X \circ \mu_{B^*X})$, so the first component $\alpha : \Gamma \to A$ is preserved by $\rho^\star_X$ and it suffices to show, in $\mathbb{C}/A$

$$
\begin{tikzcd}
\alpha \arrow[r, "{(\alpha,\beta)}"] \arrow[dr, "{(\alpha,\beta\circ\alpha^*\rho)}"'] & t_*B^*X \arrow[d, "s_*\eta_X\circ\mu_{B^*X}"] \\
& s_*C^*X
\end{tikzcd}
$$

By the adjunction $s^* \dashv s_*$, it suffices to show, in $\mathbb{C}/C$

$$
\begin{tikzcd}
s^*\alpha \arrow[r, "{s^*(\alpha,\beta)}"] \arrow[dr, "\overline{(\alpha,\beta\circ\alpha^*\rho)}"'] & s^*t_*B^*X \arrow[d, "\overline{s_*\eta_X\circ\mu_{B^*X}}"] \\
& C^*X
\end{tikzcd}
$$

Now we calculate $\overline{s_* \eta_X \circ \mu_{B^*X}} = \eta_X \circ \overline{\mu_{B^*X}}$. So that our goal is to show

$$\begin{tikzcd}
s^*\alpha \arrow[r, "s^*(\alpha,\beta)"] \arrow[dr, "\overline{(\alpha,\beta\circ\alpha^*\rho)}"'] & s^*t_*B^*X \arrow[r, "\overline{\mu_{B^*X}}"] & \rho^*B^*X \arrow[dl, "\sim"', "\eta_X"] \\
 & C^*X &
\end{tikzcd}$$

Since η_X is an isomorphism between two limits of the same diagram, namely $X \times C \cong C_! C^* X \cong C_! \rho^* B^* X$, it suffices to show that both $\overline{\mu_{B^*X}} \circ s^*(\alpha, \beta)$ and $\overline{(\alpha, \beta \circ \alpha^*\rho)}$ are uniquely determined by the same two maps into X and C.

By the characterising property of polynomial endofunctors (**??**) we calculate

$$\overline{(\alpha, \beta \circ \alpha^*\rho)} = (\beta \circ \alpha^*\rho, s^*\alpha)$$

$$\begin{array}{c||c||c}
\alpha \xrightarrow{(\alpha,\beta\circ\alpha^*\rho)} s_*C^*X & s^*\alpha \xrightarrow[(\beta\circ\alpha^*\rho,s^*\alpha)]{\overline{(\alpha,\beta\circ\alpha^*\rho)}} C^*X & C_!s^*\alpha \xrightarrow[\beta\circ\alpha^*\rho]{} X
\end{array}$$

More formally, this means $\beta \circ \alpha^*\rho : C_! s^*\alpha \to X$ and $s^*\alpha : C_! s^*\alpha \to C$ are the two maps that uniquely determine the map $C_! \overline{\alpha, \beta \circ \alpha^*\rho} : C_! s^*\alpha \to X \times C$.

On the other hand,

$$\begin{tikzcd}
\alpha \arrow[r, "(\alpha,\beta)"] & t_*B^*X \arrow[r, "\mathsf{unit}_{t_*B^*X}"] \arrow[rrr, bend right, "\mu_{B^*X}"'] & s_*\rho^*\rho_!s^*t_*B^*X \arrow[r, "s_*\rho^*\mu_!t_*B^*X"] & s_*\rho^*t^*t_*B^*X \arrow[r, "s_*\rho^*\,\mathsf{counit}_{B^*X}"] & s_*\rho * B^*X
\end{tikzcd}$$

$$\overline{} \; s^* \dashv s_*$$

$$\begin{tikzcd}
s^*\alpha \arrow[r, "s^*(\alpha,\beta)"] & s^*t_*B^*X \arrow[r, "\overline{\mathsf{unit}_{t_*B^*X}}"] \arrow[rrr, bend right, "\overline{\mu_{B^*X}}"'] & \rho^*\rho_!s^*t_*B^*X \arrow[r, "\rho^*\mu_!t_*B^*X"] & \rho^*t^*t_*B^*X \arrow[r, "\rho^*\,\mathsf{counit}_{B^*X}"] & \rho^*B^*X
\end{tikzcd}$$

$$\overline{} \; \rho_! \dashv \rho^*$$

$$\begin{tikzcd}
\rho_!s^*\alpha \arrow[r, "\rho_!s^*(\alpha,\beta)"] \arrow[d, equal] & \rho_!s^*t_*B^*X \arrow[r, equal, "\overline{\overline{\mathsf{unit}_{t_*B^*X}}}"] \arrow[d, equal] & \rho_!s^*t_*B^*X \arrow[r, "\mu_!t_*B^*X"] & t^*t_*B^*X \arrow[r, "\mathsf{counit}_{B^*X}"] \arrow[d, equal] & B^*X \arrow[d, equal] \\
\Gamma \cdot_s \alpha & S & & R & X \times B
\end{tikzcd}$$

The mate $\mu_!$ is calculated via the universal map into the pullback R (dotted below).

$$\begin{tikzcd}
\Gamma \cdot_s \alpha \arrow[r] \arrow[d, "s^*(\alpha,\beta)"'] \arrow[dd, bend right=60, "s^*\alpha"'] & \Gamma \cdot_t \alpha \arrow[r] \arrow[d] & \Gamma \arrow[d, "(\alpha,\beta)"] \arrow[dd, bend left=60, "\alpha"] \\
S \arrow[r, dashed, "\mu_! t_* B^* X"'] \arrow[d, "s^* t_* B^* X"'] & R \arrow[r] \arrow[d, "t^* t_* B^* X"] \arrow[dr, phantom, "\lrcorner", very near start] & P_t X \arrow[d, "t_* B^* X"] \\
C \arrow[r, "\rho"'] & B \arrow[r, "t"'] & A
\end{tikzcd}$$

Using the characterization of maps into R from **??** we can calculate

$$\mu_! t_* B^* X \circ s^*(\alpha, \beta) = (\rho \circ s^*\alpha, \beta \circ t^*\alpha^* s)$$

since the first component is simply the map $\Gamma \cdot_s \alpha \to B$ and the second component is the second component of the map

$$(\alpha \circ \alpha^* s, \beta \circ t^*\alpha^* s) = (\alpha, \beta) \circ \alpha^* s : \Gamma \cdot_s \alpha \to P_t X$$

Then using **??**

$$\begin{aligned}
&\overline{\overline{\mu_{B^*X}} \circ s^*(\alpha, \beta)} && (2.1)\\
&= \mathsf{counit}_{B^*X} \circ \mu_! t_* B^* X \circ s^*(\alpha, \beta) && (2.2)\\
&= \mathsf{counit}_{B^*X} \circ (\rho \circ s^*\alpha, \beta \circ t^*\alpha^* s) && (2.3)\\
&= (\beta \circ t^*\alpha^* s \circ r, \rho \circ s^*\alpha) && (2.4)\\
&= (\beta \circ \alpha^*\rho, \rho \circ s^*\alpha) && (2.5)\\
&: \Gamma \cdot_s \alpha \to X \times B && (2.6)
\end{aligned}$$

where

$$\begin{tikzcd}
\Gamma \cdot_s \alpha \arrow[dr, dashed, "r"'] \arrow[drr, bend left, "\rho \circ s^*\alpha"] \arrow[ddr, bend right, equal] & & \\
& \Gamma \cdot_s \alpha \cdot_t \alpha \circ \alpha^* s \arrow[r] \arrow[d] & B \arrow[d, "t"] \\
& \Gamma \cdot_s \alpha \arrow[r, "\alpha \circ \alpha^* s"'] & A
\end{tikzcd}$$

and

$$\begin{tikzcd}
\Gamma \cdot_s \alpha \arrow[r, equal] \arrow[d, dashed, "r"'] \arrow[dd, bend right=60, equal] & \Gamma \cdot_s \alpha \arrow[r, "s^*\alpha"] \arrow[d, "\alpha^*\rho"] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "\rho"] \arrow[dd, bend left=60, "s"] \\
\Gamma \cdot_s \alpha \cdot_t \alpha \circ \alpha^* s \arrow[r, dashed, "t^*\alpha^* s"'] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma \cdot_t \alpha \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "t"] \\
\Gamma \cdot_s \alpha \arrow[r, "\alpha * s"'] & \Gamma \arrow[r, "\alpha"'] & A
\end{tikzcd}$$

Moving back along the adjunction $\rho_! \dashv \rho^*$ **??** tells us that

$$
\begin{tikzcd}
\Gamma \cdot_s \alpha \arrow[dr, "\overline{\mu_{B^*X}}\circ s^*(\alpha,\beta)"] \arrow[drr, bend left, "s^*\alpha"] \arrow[ddr, "\overline{\overline{\mu_{B^*X}}\circ s^*(\alpha,\beta)}"'] \arrow[dddr, bend right, "\beta\circ\alpha^*\rho"'] & & \\
& X \times C \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "\rho"] \\
& X \times B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d] \\
& X \arrow[r] & 1
\end{tikzcd}
$$

So that, as required, $\overline{\mu_{B^*X}}\circ s^*(\alpha,\beta)$ and $\overline{(\alpha, \beta \circ \alpha^*\rho)}$ are uniquely determined by the same two maps into X and C. □

Definition 2.0.9 (Covariant action of P_- on a cartesian square)**.** We can also view taking polynomial endofunctors as a covariant functor on the category of arrows with cartesian squares as morphisms

$$P_- : \mathsf{CartArr}(\mathbb{C}) \to [\mathbb{C}, \mathbb{C}]$$

where the action on a cartesian square is

$$
\begin{tikzcd}
\mathbb{C} \arrow[r, equal] \arrow[d, "D^*"'] \arrow[ddd, bend right=60, "P_s"'] \arrow[dr, phantom, "\overset{\eta^{-1}}{\Rightarrow}"] & \mathbb{C} \arrow[d, "B^*"] \arrow[ddd, bend left=60, "P_t"] \\
\mathbb{C}/D \arrow[d, "s_*"'] \arrow[dr, phantom, "\overset{\mu^{*-1}}{\Rightarrow}"] & \mathbb{C}/B \arrow[l, "\rho^*"'] \arrow[d, "t_*"] \\
\mathbb{C}/C \arrow[d, "C_!"'] \arrow[dr, phantom, "\overset{\mu_!}{\Rightarrow}"] & \mathbb{C}/A \arrow[l, "\theta^*"] \arrow[d, "A_!"] \\
\mathbb{C} \arrow[r, equal] & \mathbb{C}
\end{tikzcd}
$$

given by the whiskered natural transformations

$$
\begin{tikzcd}
\mathbb{C} \arrow[r, equal] \arrow[d, "C^*"'] \arrow[ddd, bend right=60, "P_s"'] \arrow[dr, phantom, "\overset{\eta^{-1}}{\Rightarrow}"] & \mathbb{C} \arrow[d, "B^*"] \arrow[ddd, bend left=60, "P_t"] \\
\mathbb{C}/D \arrow[d, "s_*"'] \arrow[dr, phantom, "\overset{\mu^{*-1}}{\Rightarrow}"] & \mathbb{C}/B \arrow[l, "\rho^*"'] \arrow[d, "t_*"] \\
\mathbb{C}/C \arrow[d, "C_!"'] \arrow[dr, phantom, "\overset{\mu_!}{\Rightarrow}"] & \mathbb{C}/A \arrow[l, "\theta^*"] \arrow[d, "A_!"] \\
\mathbb{C} \arrow[r, equal] & \mathbb{C}
\end{tikzcd}
$$

Furthermore, the natural transformation P_κ is cartesian. meaning each naturality square is a pullback square.

$$\begin{tikzcd}
P_s X \arrow[r, "P_{\kappa Y}"] \arrow[d, "P_s f"'] \arrow[dr, phantom, "\lrcorner", very near start] & P_t X \arrow[d, "P_s f"] \\
P_s Y \arrow[r, "P_{\kappa Y}"'] & P_t Y
\end{tikzcd}$$

The natural transformation P_κ computes in the following way

$$\begin{tikzcd}
\Gamma \cdot_t \theta \circ \alpha \arrow[dr, dashed, "i"] \arrow[drrr, bend left=10] \arrow[ddr] & & & \\
& \Gamma \cdot_s \alpha \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & D \arrow[r, "\rho"] \arrow[d, "s"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "t"] \\
& \Gamma \arrow[r, "\alpha"'] & C \arrow[r, "\theta"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
\Gamma \arrow[d, "(\alpha, \beta)"'] \arrow[dr, "(\theta\circ\alpha, \beta\circ i)"] & \\
P_s X \arrow[r, "P_{\kappa X}"'] & P_t X
\end{tikzcd}$$

using the fact that $\Gamma \cdot_s \alpha$ and $\Gamma \cdot_t \theta \circ \alpha$ are limits of the same diagram.

Proof. We can use the computation of $P_{\kappa X}$ and $P_s f$ to show that the natural transformation P_κ is cartesian. Essentially, the first component of a map $\Gamma \to P_s X$ is determined by its composition with $P_s f$ and its second component is determined by its composition with $P_{\kappa X}$. $\square$

Corollary 2.0.10. *If we have*

$$\begin{tikzcd}
D' \arrow[r] \arrow[d, "\rho_1"] \arrow[dd, bend right=40, "q_1'"'] \arrow[dr, phantom, "\lrcorner", very near start] & B' \arrow[d, "\rho_2"'] \arrow[dd, bend left=40, "q_2'"] \\
D \arrow[r] \arrow[d, "q_1"] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "q_2"'] \\
C \arrow[r, "\theta"'] & A
\end{tikzcd}$$

then the two possible ways of obtaining composing the covariant and contravariant actions of P_{-} *form a (strictly commuting) pullback square in* $[\mathbb{C}, \mathbb{C}]$.

$$\begin{tikzcd}
P_{q_1} \arrow[r, "P_\kappa"] \arrow[d, "{\rho_1}^\star"'] \arrow[dr, phantom, "\lrcorner", very near start] & P_{q_2} \arrow[d, "{\rho_2}^\star"] \\
P_{q_1'} \arrow[r, "P_{\kappa'}"'] & P_{q_2'}
\end{tikzcd}$$

Proof. To check that it commutes and is a pullback, it suffices to do this pointwise, for some $X \in \mathbb{C}$. Then we simply unfold the computation for each of P_κ and $\rho^\star$. $\square$

Chapter 3

Multivariate Polynomial Functors

To be written