

#### Polynomial Functors in Lean4

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April 3, 2025

## Chapter 1

# Locally Cartesian Closed Categories

To be written

#### Chapter 2

#### Univaiate Polynomial Functors

In this section we develop some of the definitions and lemmas related to polynomial endofunctors that we will use in the rest of the notes.

**Definition 2.0.1** (Polynomial endofunctor). UvPoly Let  $\mathbb C$  be a locally Cartesian closed category (in our case, presheaves on the category of contexts). This means for each morphism  $t: B \to A$  we have an adjoint triple

$$\begin{array}{c|c}
\mathbb{C}/B \\
t_! & \uparrow \\
 & \downarrow \\$$

where  $t^*$  is pullback, and  $t_!$  is composition with t.

Let  $t:B\to A$  be a morphism in  $\mathbb C.$  Then define  $P_t:\mathbb C\to\mathbb C$  be the composition

$$\begin{split} P_t &:= A_! \circ t_* \circ B^* \\ \mathbb{C} & \xrightarrow{B^*} \mathbb{C}/B \xrightarrow{t_*} \mathbb{C}/A \xrightarrow{A_!} \mathbb{C} \end{split}$$

**Proposition 2.0.2** (Characterising property of Polynomial Endofunctors). UvPoly.equiv defn: UvPoly The data of a map into the polynomial applied to an object in  $\mathbb{C}$ 

$$\Gamma \longrightarrow P_t Y$$

corresponds to a pair of morphisms

$$\alpha: \Gamma \to A$$
 and  $\beta: \Gamma \cdot \alpha \to Y$ 

and this correspondence is natural in both  $\Gamma$  and Y.

Given any such  $\phi$  we can extract  $\alpha:\Gamma\to A$  by composition

$$\Gamma \xrightarrow{\phi} P_t Y$$

$$A \xrightarrow{t_* B^* Y}$$

Applying the adjunction  $A_! \dashv A^*$ , and viewing  $\phi : \alpha \to t_*B^*Y$  as a map in the slice, this corresponds to

$$\alpha: \Gamma \to A$$
 and 
$$B_! t^* \alpha \xrightarrow[t^* \alpha]{\tilde{\phi}} B \times Y$$

Applying the adjunction  $t^* \dashv t_*$ , this corresponds to

$$\alpha: \Gamma \to A$$
 and 
$$\Gamma \cdot \alpha := B_1 t^* \alpha \xrightarrow{----\beta} Y$$

Henceforth we will write

$$(\alpha, \beta): \Gamma \to P_t Y$$

for this map, since it is uniquely determined by this data.

This is natural in  $\Gamma$ . Precomposition by  $\sigma: \Delta \to \Gamma$ , acts on such a pair by

$$\begin{array}{c|c}
\Delta \\
\sigma \downarrow \\
\Gamma \xrightarrow[(\alpha,\beta)]{} P_t Y
\end{array}$$

It is also natural in  $f: X \to Y$ , meaning the morphism  $P_t f$  acts on such a pair by

$$\Gamma \xrightarrow{(\alpha,\beta)} P_t X$$

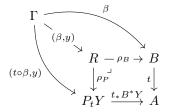
$$\downarrow_{P_t f}$$

$$P_t Y$$

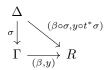
Lemma 2.0.3. prop:UvPoly.equiv Use R to denote the fiber product

$$R \xrightarrow{\rho_{\mathsf{Tm}}} B \\ \downarrow^{\rho_P} \downarrow \downarrow^{t} \\ P_t Y \xrightarrow[t*B^*Y]{} A$$

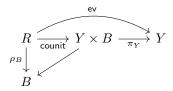
By the universal property of pullbacks and  $\ref{eq:thm.pdf}$ , The data of a map  $\Gamma \to R$  corresponds to the data of  $\beta: \Gamma \to B$  and  $(t \circ \beta, y): \Gamma \to P_t Y$ , or just  $\beta: \Gamma \to B$  and  $y: \Gamma \cdot t \circ \beta \to Y$ 



By uniqueness in the universal property of pullbacks and  $\ref{eq:composition}$  by a map  $\sigma: \Delta \to \Gamma$  acts on such a pair by



**Definition 2.0.4** (Evaluation). defn:UvPoly Let counit :  $\rho_B \to B \to B^*Y$  denote the counit of the adjunction  $f^* \dashv f_*$  at the object  $B^*Y$ , recalling that  $\rho_B = t^*t_*B^*Y$ . Then viewing the object  $B^*Y$  in the slice as the object  $Y \times B$  in the ambient category, we define  $ev: R \to Y$  as the composition



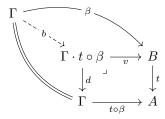
**Lemma 2.0.5** (Evaluation Computation). *defn:UvPolyEv Suppose*  $(\beta, y) : \Gamma \to R$ , as in ??

$$\beta: \Gamma \to B$$
 and  $y: \Gamma \cdot t \circ \beta \to Y$ 

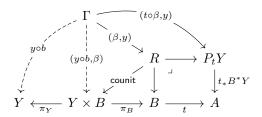
Then the evaluation of y at  $\beta$  can be computed as

$$\mathsf{ev} \, \circ (\beta, y) = y \circ b$$

where



and



*Proof.* It suffices to show ( counit  $\circ (\beta, y)) = (y \circ b, \beta)$  instead.

$$\begin{array}{lll} \operatorname{counit} \, \circ \, (\beta,y) \\ = & \operatorname{counit} \, \circ \, (v \circ b, y \circ t^* d \circ t^* b) & ?? \\ = & \operatorname{counit} \, \circ \, (v, y \circ t^* d) \circ b & ??,?? \\ = & \operatorname{counit} \, \circ \, t^* (t \circ \beta, y) \circ b & ?? \\ = & \overline{(t \circ \beta, y)} \circ b & ?? \\ = & (y, v) \circ b & ?? \\ = & (y \circ b, v \circ b) \\ = & (y \circ b, \beta) & \end{array}$$

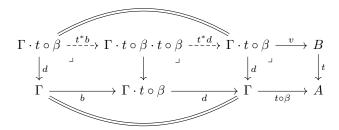


Figure 2.1:  $t^*d \circ t^*b = \mathsf{id}_{\Gamma \cdot t \circ \beta}$ 

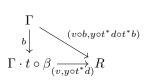
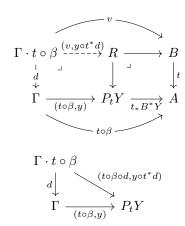


Figure 2.2:  $(v, y \circ t^*d) \circ b = (v \circ b, y \circ t^*d \circ t^*b)$ 



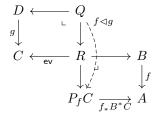
using ??, ??

Figure 2.3:  $t^*(t \circ \beta, y) = (v, y \circ t^*d)$ 



Figure 2.4: counit  $\circ t^*(t \circ \beta, y) = \overline{(t \circ \beta, y)}$ 

**Definition 2.0.6** (Polynomial composition). UvPoly.comp Let  $f: B \to A$  and  $g: D \to C$ . Define the *polynomial composition*  $f \lhd g: Q \to P_fC$  as the composition of the two vertical maps in the following



Then the two functors

$$P_{f \lhd g} \cong P_f \circ P_g$$

are naturally isomorphic.

Proof. 
$$\Box$$

Definition 2.0.7 (Mate). Suppose

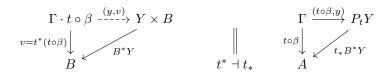
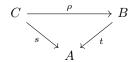


Figure 2.5:  $\overline{(t \circ \beta, y)} = (y, v)$ 

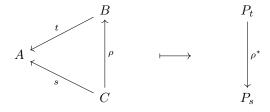


Then we have a mate  $\mu_!: \rho_! \circ s^* \Rightarrow t^*$ . This is given by the universal property of pullbacks: given  $f: x \to y$  in the slice  $\mathbb{C}/A$  we have

By the calculus of mates we also have a reversed mate between the right adjoints  $\mu^*: t_* \to s_* \circ \rho^*$ . Explicitly  $\mu^*$  is the composition

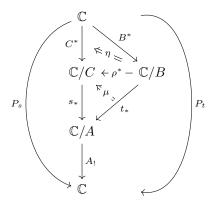
$$t_* \xrightarrow{\mathsf{unit}\, t_*} s_* \rho^* \rho_! s^* t_* \xrightarrow{s_* \rho^* \mu_! t_*} s_* \rho^* t^* t_* \xrightarrow{s_* \rho^*} \overset{counit}{\longrightarrow} s_* \rho^*$$

**Definition 2.0.8** (Contravariant action of  $P_-$  on a slice). Let  $P_-: (\mathbb{C}/A)^{\mathrm{op}} \to [\mathbb{C}, \mathbb{C}]$  be defined by taking  $s \mapsto P_s$  on objects and act on a morphism by



where

$$\rho^{\star} := A_!(s_* \eta \circ \mu B^*) : P_t \to P_s$$



where  $\mu = \mu^*$  is the mate from ??, and  $\eta$  is the natural isomorphism given by pullback pasting.

Pointwise, this natural transformation acts on a pair  $(\alpha, \beta) : \Gamma \to P_t X$  by

$$\Gamma \xrightarrow{(\alpha,\beta)} P_t X$$

$$\downarrow^{\rho_X^*} \qquad \downarrow^{\rho_X^*}$$

$$P_s X$$

where  $\alpha^* \rho$  is defined as

$$\begin{array}{cccc} \Gamma \cdot_s \alpha & \xrightarrow{s^* \alpha} & C \\ \alpha^* \rho \downarrow & & \downarrow \rho \\ \Gamma \cdot_t \alpha & \xrightarrow{t^* \alpha} & B \\ \downarrow & & \downarrow t \\ \Gamma & \xrightarrow{\alpha} & A \end{array}$$

We prove this now.

*Proof.* Firstly  $\rho_X^* = A_!(s_*\eta_X \circ \mu_{B^*X})$ , so the first component  $\alpha: \Gamma \to A$  is preserved by  $\rho_X^*$  and it suffices to show, in  $\mathbb{C}/A$ 

$$\alpha \xrightarrow{(\alpha,\beta)} t_* B^* X$$

$$\downarrow s_* \eta_X \circ \mu_{B^* X}$$

$$s_* C^* X$$

By the adjunction  $s^* \dashv s_*$ , it suffices to show, in  $\mathbb{C}/C$ 

$$s^*\alpha \xrightarrow{s^*(\alpha,\beta)} s^*t_*B^*X$$

$$\downarrow \\ \downarrow \\ (\alpha,\beta\circ\alpha^*\rho) \\ \downarrow \\ C^*X$$

Now we calculate  $\overline{s_*\eta_X\circ\mu_{B^*X}}=\eta_X\circ\overline{\mu_{B^*X}}$ . So that our goal is to show

$$s^*\alpha \xrightarrow{s^*(\alpha,\beta)} s^*t_*B^*X \xrightarrow{\overline{\mu_{B^*X}}} \rho^*B^*X$$

$$\xrightarrow{(\alpha,\beta\circ\alpha^*\rho)} C^*X$$

Since  $\eta_X$  is an isomorphism between two limits of the same diagram, namely  $\underline{X \times C} \cong \underline{C}_!C^*X \cong \underline{C}_!\rho^*B^*X$ , it suffices to show that both  $\overline{\mu_{B^*X}} \circ s^*(\alpha,\beta)$  and  $(\alpha,\beta \circ \alpha^*\rho)$  are uniquely determined by the same two maps into X and C.

By the characterising property of polynomial endofunctors (??) we calculate

$$\overline{(\alpha, \beta \circ \alpha^* \rho)} = (\beta \circ \alpha^* \rho, s^* \alpha)$$

$$\alpha \xrightarrow{(\alpha,\beta \circ \alpha^* \rho)} s_* C^* X \qquad \qquad s^* \alpha \xrightarrow{(\alpha,\beta \circ \alpha^* \rho)} C^* X \qquad \qquad C_! s^* \alpha \xrightarrow{\beta \circ \alpha^* \rho} X$$

More formally, this means  $\beta \circ \alpha^* \rho : C_! s^* \alpha \to X$  and  $s^* \alpha : C_! s^* \alpha \to C$  are the two maps that uniquely determine the map  $C_! \overline{\alpha, \beta \circ \alpha^* \rho} : C_! s^* \alpha \to X \times C$ . On the other hand,

$$\alpha \xrightarrow{(\alpha,\beta)} t_*B^*X \xrightarrow{\text{unit}} t_*B^*X \xrightarrow{s_*\rho^*} \rho_! s^*t_*B^*X \xrightarrow{s_*\rho^*} t^*t_*B^*X \xrightarrow{s_*\rho^*} \text{counit}} s^*X \xrightarrow{s_*\rho^*} s^*P_*S^*X \xrightarrow{\mu_B^*X} s^*P_*X \xrightarrow{\mu_B^*X} s$$

The mate  $\mu_!$  is calculated via the universal map into the pullback R (dotted below).

Using the characterization of maps into R from ?? we can calculate

$$\mu_! t_* B^* X \circ s^* (\alpha, \beta) = (\rho \circ s^* \alpha, \beta \circ t^* \alpha^* s)$$

since the first component is simply the map  $\Gamma \cdot_s \alpha \to B$  and the second component is the second component of the map

$$(\alpha \circ \alpha^* s, \beta \circ t^* \alpha^* s) = (\alpha, \beta) \circ \alpha^* s : \Gamma \cdot_s \alpha \to P_t X$$

Then using ??

$$\overline{\overline{\mu_{B^*X}} \circ s^*(\alpha, \beta)} \tag{2.1}$$

$$= \operatorname{counit}_{B^*X} \circ \mu_! t_* B^*X \circ s^*(\alpha, \beta) \tag{2.2}$$

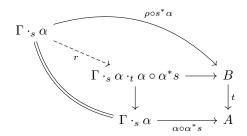
$$= \operatorname{counit}_{B^*X} \circ (\rho \circ s^* \alpha, \beta \circ t^* \alpha^* s) \tag{2.3}$$

$$= (\beta \circ t^* \alpha^* s \circ r, \rho \circ s^* \alpha) \tag{2.4}$$

$$= (\beta \circ \alpha^* \rho, \rho \circ s^* \alpha) \tag{2.5}$$

$$: \Gamma \cdot_s \alpha \to X \times B \tag{2.6}$$

where



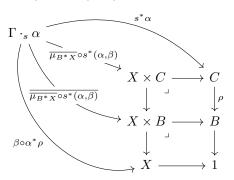
and

$$\Gamma \cdot_s \alpha = \Gamma \cdot_s \alpha \xrightarrow{s^* \alpha} C$$

$$\Gamma \cdot_s \alpha \cdot_t \alpha \circ \alpha^* s \xrightarrow[t^* \alpha^* s]{} \Gamma \cdot_t \alpha \xrightarrow{} B$$

$$\Gamma \cdot_s \alpha \xrightarrow[t^* \alpha^* s]{} \Gamma \cdot_t \alpha \xrightarrow{} A$$

Moving back along the adjunction  $\rho_! \dashv \rho^*$  ?? tells us that



So that, as required,  $\overline{\mu_{B^*X}} \circ s^*(\alpha, \beta)$  and  $\overline{(\alpha, \beta \circ \alpha^* \rho)}$  are uniquely determined by the same two maps into X and C.

**Definition 2.0.9** (Covariant action of  $P_{-}$  on a cartesian square). We can also view taking polynomial endofunctors as a covariant functor on the category of arrows with cartesian squares as morphisms

$$P_{-}: \mathsf{CartArr}(\mathbb{C}) \to [\mathbb{C}, \mathbb{C}]$$

where the action on a cartesian square is

$$\begin{array}{c|c}
\mathbb{C} & \mathbb{C} \\
D^* \downarrow & \eta^{-1} \downarrow B^* \\
\mathbb{C}/D \leftarrow \rho^* - \mathbb{C}/B \\
s_* \downarrow & \mu^{*-1} \downarrow t_* \\
\mathbb{C}/C \leftarrow \theta^* & \mathbb{C}/A \\
C_! \downarrow & \mu_! & \downarrow A_! \\
\mathbb{C} & \mathbb{C} & \mathbb{C}
\end{array}$$

given by the whiskered natural transformations

$$\begin{array}{c|c}
\mathbb{C} & \mathbb{C} \\
C^* \downarrow & \eta^{-1} \downarrow B^* \\
\mathbb{C}/D \leftarrow \rho^* - \mathbb{C}/B \\
s_* \downarrow & \mu^{*-1} \downarrow t_* \\
\mathbb{C}/C \leftarrow \theta^* & \mathbb{C}/A \\
C_! \downarrow & \mu_! & \downarrow A_! \\
\mathbb{C} & \mathbb{C} & \mathbb{C}
\end{array}$$

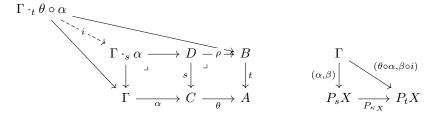
Furthermore, the natural transformation  $P_{\kappa}$  is cartesian. meaning each naturality square is a pullback square.

$$P_{s}X \xrightarrow{P_{\kappa Y}} P_{t}X$$

$$P_{s}f \downarrow \qquad \qquad \downarrow P_{s}f$$

$$P_{s}Y \xrightarrow{P_{\kappa Y}} P_{t}Y$$

The natural transformation  $P_{\kappa}$  computes in the following way



using the fact that  $\Gamma \cdot_s \alpha$  and  $\Gamma \cdot_t \theta \circ \alpha$  are limits of the same diagram.

*Proof.* We can use the computation of  $P_{\kappa X}$  and  $P_s f$  to show that the natural transformation  $P_{\kappa}$  is cartesian. Essentially, the first component of a map  $\Gamma \to P_s X$  is determined by its composition with  $P_s f$  and its second component is determined by its composition with  $P_{\kappa X}$ .

#### Corollary 2.0.10. If we have

$$D' \longrightarrow B'$$

$$\downarrow^{\rho_1} \qquad \rho_2 \downarrow$$

$$D \longrightarrow B$$

$$\downarrow^{q_1} \qquad q_2 \downarrow$$

$$C \longrightarrow A$$

then the two possible ways of obtaining composing the covariant and contravariant actions of  $P_{-}$  form a (strictly commuting) pullback square in  $[\mathbb{C}, \mathbb{C}]$ .

$$P_{q_{1}} \xrightarrow{P_{\kappa}} P_{q_{2}}$$

$$\downarrow^{\rho_{1}^{\star}} \qquad \downarrow^{\rho_{2}^{\star}}$$

$$P_{q'_{1}} \xrightarrow{P_{\kappa'}} P_{q'_{2}}$$

*Proof.* To check that it commutes and is a pullback, it suffices to do this pointwise, for some  $X \in \mathbb{C}$ . Then we simply unfold the computation for each of  $P_{\kappa}$  and  $\rho^{\star}$ .

## Chapter 3

## Multivariate Polynomial Functors

To be written