

Polynomial Functors in Lean4

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Chapter 1

Locally Cartesian Closed Categories

To be written

Chapter 2

Univaiate Polynomial Functors

In this section we develop some of the definitions and lemmas related to polynomial endofunctors that we will use in the rest of the notes.

Definition 2.0.1 (Polynomial endofunctor). UvPoly Let $\mathbb C$ be a locally Cartesian closed category (in our case, presheaves on the category of contexts). This means for each morphism $t: B \to A$ we have an adjoint triple

$$\begin{array}{c|c}
\mathbb{C}/B \\
t_! \left(\begin{array}{c} \uparrow \\ \neg t^* & \neg \end{array} \right) t_* \\
\mathbb{C}/A
\end{array}$$

where t^* is pullback, and $t_!$ is composition with t.

Let $t:B\to A$ be a morphism in $\mathbb C.$ Then define $P_t:\mathbb C\to\mathbb C$ be the composition

$$\begin{split} P_t &:= A_! \circ t_* \circ B^* \\ \mathbb{C} & \xrightarrow{B^*} \mathbb{C}/B \xrightarrow{t_*} \mathbb{C}/A \xrightarrow{A_!} \mathbb{C} \end{split}$$

Proposition 2.0.2 (Characterising property of Polynomial Endofunctors). The data of a map into the polynomial applied to an object in \mathbb{C}

$$\Gamma \longrightarrow P_t Y$$

corresponds to a pair of morphisms

$$\alpha:\Gamma \to A \qquad and \qquad \beta:\Gamma \cdot \alpha \to Y$$

and this correspondence is natural in both Γ and Y.

Given any such ϕ we can extract $\alpha:\Gamma\to A$ by composition

$$\Gamma \xrightarrow{\phi} P_t Y$$

$$A \xrightarrow{t_* B^* Y}$$

Applying the adjunction $A_! \dashv A^*$, and viewing $\phi : \alpha \to t_*B^*Y$ as a map in the slice, this corresponds to

$$\alpha: \Gamma \to A$$
 and
$$B_! t^* \alpha \xrightarrow[t^* \alpha]{\tilde{\phi}} B \times Y$$

Applying the adjunction $t^* \dashv t_*$, this corresponds to

$$\alpha: \Gamma \to A$$
 and
$$\Gamma \cdot \alpha := B_1 t^* \alpha \xrightarrow{----\beta} Y$$

Henceforth we will write

$$(\alpha, \beta): \Gamma \to P_t Y$$

for this map, since it is uniquely determined by this data.

This is natural in Γ . Precomposition by $\sigma: \Delta \to \Gamma$, acts on such a pair by

$$\begin{array}{c|c}
\Delta \\
\sigma \downarrow \\
\Gamma \xrightarrow[(\alpha,\beta)]{} P_t Y
\end{array}$$

It is also natural in $f: X \to Y$, meaning the morphism $P_t f$ acts on such a pair by

$$\Gamma \xrightarrow{(\alpha,\beta)} P_t X$$

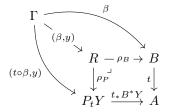
$$\downarrow_{P_t f}$$

$$P_t Y$$

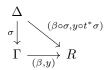
Lemma 2.0.3. prop:UvPoly.equiv Use R to denote the fiber product

$$R \xrightarrow{\rho_{\mathsf{Tm}}} B \\ \downarrow^{\rho_P} \downarrow \downarrow^{t} \\ P_t Y \xrightarrow[t*B^*Y]{} A$$

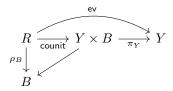
By the universal property of pullbacks and $\ref{eq:thm.pdf}$, The data of a map $\Gamma \to R$ corresponds to the data of $\beta: \Gamma \to B$ and $(t \circ \beta, y): \Gamma \to P_t Y$, or just $\beta: \Gamma \to B$ and $y: \Gamma \cdot t \circ \beta \to Y$



By uniqueness in the universal property of pullbacks and $\ref{eq:composition}$ by a map $\sigma: \Delta \to \Gamma$ acts on such a pair by



Definition 2.0.4 (Evaluation). defn:UvPoly Let counit : $\rho_B \to B \to B^*Y$ denote the counit of the adjunction $f^* \dashv f_*$ at the object B^*Y , recalling that $\rho_B = t^*t_*B^*Y$. Then viewing the object B^*Y in the slice as the object $Y \times B$ in the ambient category, we define $ev: R \to Y$ as the composition



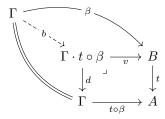
Lemma 2.0.5 (Evaluation Computation). *defn:UvPolyEv Suppose* $(\beta, y) : \Gamma \to R$, as in ??

$$\beta: \Gamma \to B$$
 and $y: \Gamma \cdot t \circ \beta \to Y$

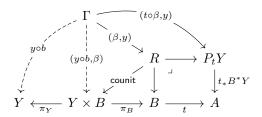
Then the evaluation of y at β can be computed as

$$\mathsf{ev} \, \circ (\beta, y) = y \circ b$$

where



and



Proof. It suffices to show (counit $\circ (\beta, y)) = (y \circ b, \beta)$ instead.

$$\begin{array}{lll} \operatorname{counit} \, \circ \, (\beta,y) \\ = & \operatorname{counit} \, \circ \, (v \circ b, y \circ t^* d \circ t^* b) & ?? \\ = & \operatorname{counit} \, \circ \, (v, y \circ t^* d) \circ b & ??,?? \\ = & \operatorname{counit} \, \circ \, t^* (t \circ \beta, y) \circ b & ?? \\ = & \overline{(t \circ \beta, y)} \circ b & ?? \\ = & (y, v) \circ b & ?? \\ = & (y \circ b, v \circ b) \\ = & (y \circ b, \beta) & \end{array}$$

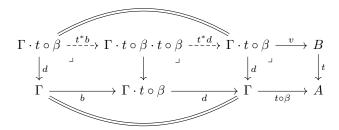


Figure 2.1: $t^*d \circ t^*b = \mathsf{id}_{\Gamma \cdot t \circ \beta}$

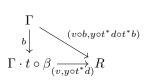
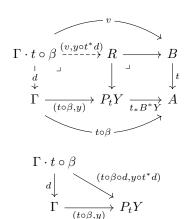


Figure 2.2: $(v, y \circ t^*d) \circ b = (v \circ b, y \circ t^*d \circ t^*b)$



using ??, ??

Figure 2.3: $t^*(t \circ \beta, y) = (v, y \circ t^*d)$

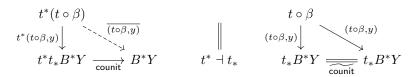
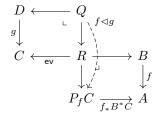


Figure 2.4: counit $\circ t^*(t \circ \beta, y) = \overline{(t \circ \beta, y)}$

Definition 2.0.6 (Polynomial composition). Let $f: B \to A$ and $g: D \to C$. Define the *polynomial composition* $f \triangleleft g: Q \to P_f C$ as the composition of the two vertical maps in the following



Then the two functors

$$P_{f \lhd g} \cong P_f \circ P_g$$

are naturally isomorphic.

Proof.
$$\Box$$

Definition 2.0.7 (Mate). Suppose

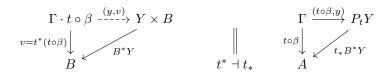
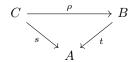


Figure 2.5: $\overline{(t \circ \beta, y)} = (y, v)$

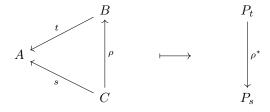


Then we have a mate $\mu_!: \rho_! \circ s^* \Rightarrow t^*$. This is given by the universal property of pullbacks: given $f: x \to y$ in the slice \mathbb{C}/A we have

By the calculus of mates we also have a reversed mate between the right adjoints $\mu^*: t_* \to s_* \circ \rho^*$. Explicitly μ^* is the composition

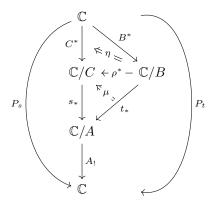
$$t_* \xrightarrow{\mathsf{unit}\, t_*} s_* \rho^* \rho_! s^* t_* \xrightarrow{s_* \rho^* \mu_! t_*} s_* \rho^* t^* t_* \xrightarrow{s_* \rho^*} \overset{counit}{\longrightarrow} s_* \rho^*$$

Definition 2.0.8 (Contravariant action of P_- on a slice). Let $P_-: (\mathbb{C}/A)^{\mathrm{op}} \to [\mathbb{C}, \mathbb{C}]$ be defined by taking $s \mapsto P_s$ on objects and act on a morphism by



where

$$\rho^{\star} := A_!(s_* \eta \circ \mu B^*) : P_t \to P_s$$



where $\mu = \mu^*$ is the mate from ??, and η is the natural isomorphism given by pullback pasting.

Pointwise, this natural transformation acts on a pair $(\alpha, \beta) : \Gamma \to P_t X$ by

$$\Gamma \xrightarrow{(\alpha,\beta)} P_t X$$

$$\downarrow^{\rho_X^*} \qquad \downarrow^{\rho_X^*}$$

$$P_s X$$

where $\alpha^* \rho$ is defined as

$$\begin{array}{cccc} \Gamma \cdot_s \alpha & \xrightarrow{s^* \alpha} & C \\ \alpha^* \rho \downarrow & & \downarrow \rho \\ \Gamma \cdot_t \alpha & \xrightarrow{t^* \alpha} & B \\ \downarrow & & \downarrow t \\ \Gamma & \xrightarrow{\alpha} & A \end{array}$$

We prove this now.

Proof. Firstly $\rho_X^* = A_!(s_*\eta_X \circ \mu_{B^*X})$, so the first component $\alpha: \Gamma \to A$ is preserved by ρ_X^* and it suffices to show, in \mathbb{C}/A

$$\alpha \xrightarrow{(\alpha,\beta)} t_* B^* X$$

$$\downarrow s_* \eta_X \circ \mu_{B^* X}$$

$$s_* C^* X$$

By the adjunction $s^* \dashv s_*$, it suffices to show, in \mathbb{C}/C

$$s^*\alpha \xrightarrow{s^*(\alpha,\beta)} s^*t_*B^*X$$

$$\downarrow \\ \downarrow \\ (\alpha,\beta\circ\alpha^*\rho) \\ \downarrow \\ C^*X$$

Now we calculate $\overline{s_*\eta_X\circ\mu_{B^*X}}=\eta_X\circ\overline{\mu_{B^*X}}$. So that our goal is to show

$$s^*\alpha \xrightarrow{s^*(\alpha,\beta)} s^*t_*B^*X \xrightarrow{\overline{\mu_{B^*X}}} \rho^*B^*X$$

$$\xrightarrow{(\alpha,\beta\circ\alpha^*\rho)} C^*X$$

Since η_X is an isomorphism between two limits of the same diagram, namely $\underline{X \times C} \cong \underline{C}_!C^*X \cong \underline{C}_!\rho^*B^*X$, it suffices to show that both $\overline{\mu_{B^*X}} \circ s^*(\alpha,\beta)$ and $(\alpha,\beta \circ \alpha^*\rho)$ are uniquely determined by the same two maps into X and C.

By the characterising property of polynomial endofunctors (??) we calculate

$$\overline{(\alpha, \beta \circ \alpha^* \rho)} = (\beta \circ \alpha^* \rho, s^* \alpha)$$

$$\alpha \xrightarrow{(\alpha,\beta \circ \alpha^* \rho)} s_* C^* X \qquad \qquad s^* \alpha \xrightarrow{(\alpha,\beta \circ \alpha^* \rho)} C^* X \qquad \qquad C_! s^* \alpha \xrightarrow{\beta \circ \alpha^* \rho} X$$

More formally, this means $\beta \circ \alpha^* \rho : C_! s^* \alpha \to X$ and $s^* \alpha : C_! s^* \alpha \to C$ are the two maps that uniquely determine the map $C_! \overline{\alpha, \beta \circ \alpha^* \rho} : C_! s^* \alpha \to X \times C$. On the other hand,

$$\alpha \xrightarrow{(\alpha,\beta)} t_*B^*X \xrightarrow{\text{unit}} t_*B^*X \xrightarrow{s_*\rho^*} \rho_! s^*t_*B^*X \xrightarrow{s_*\rho^*} t^*t_*B^*X \xrightarrow{s_*\rho^*} \text{counit}} s^*X \xrightarrow{s_*\rho^*} s^*P_*S^*X \xrightarrow{\mu_B^*X} s^*P_*X \xrightarrow{\mu_B^*X} s$$

The mate $\mu_!$ is calculated via the universal map into the pullback R (dotted below).

Using the characterization of maps into R from ?? we can calculate

$$\mu_! t_* B^* X \circ s^* (\alpha, \beta) = (\rho \circ s^* \alpha, \beta \circ t^* \alpha^* s)$$

since the first component is simply the map $\Gamma \cdot_s \alpha \to B$ and the second component is the second component of the map

$$(\alpha \circ \alpha^* s, \beta \circ t^* \alpha^* s) = (\alpha, \beta) \circ \alpha^* s : \Gamma \cdot_s \alpha \to P_t X$$

Then using ??

$$\overline{\overline{\mu_{B^*X}} \circ s^*(\alpha, \beta)} \tag{2.1}$$

$$= \operatorname{counit}_{B^*X} \circ \mu_! t_* B^*X \circ s^*(\alpha, \beta) \tag{2.2}$$

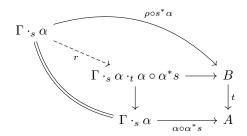
$$= \operatorname{counit}_{B^*X} \circ (\rho \circ s^* \alpha, \beta \circ t^* \alpha^* s) \tag{2.3}$$

$$= (\beta \circ t^* \alpha^* s \circ r, \rho \circ s^* \alpha) \tag{2.4}$$

$$= (\beta \circ \alpha^* \rho, \rho \circ s^* \alpha) \tag{2.5}$$

$$: \Gamma \cdot_s \alpha \to X \times B \tag{2.6}$$

where



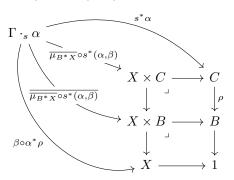
and

$$\Gamma \cdot_s \alpha = \Gamma \cdot_s \alpha \xrightarrow{s^* \alpha} C$$

$$\Gamma \cdot_s \alpha \cdot_t \alpha \circ \alpha^* s \xrightarrow[t^* \alpha^* s]{} \Gamma \cdot_t \alpha \xrightarrow{} B$$

$$\Gamma \cdot_s \alpha \xrightarrow[t^* \alpha^* s]{} \Gamma \cdot_t \alpha \xrightarrow{} A$$

Moving back along the adjunction $\rho_! \dashv \rho^*$?? tells us that



So that, as required, $\overline{\mu_{B^*X}} \circ s^*(\alpha, \beta)$ and $\overline{(\alpha, \beta \circ \alpha^* \rho)}$ are uniquely determined by the same two maps into X and C.

Definition 2.0.9 (Covariant action of P_{-} on a cartesian square). We can also view taking polynomial endofunctors as a covariant functor on the category of arrows with cartesian squares as morphisms

$$P_{-}: \mathsf{CartArr}(\mathbb{C}) \to [\mathbb{C}, \mathbb{C}]$$

where the action on a cartesian square is

$$\begin{array}{c|c}
\mathbb{C} & \mathbb{C} \\
D^* \downarrow & \eta^{-1} \downarrow B^* \\
\mathbb{C}/D \leftarrow \rho^* - \mathbb{C}/B \\
s_* \downarrow & \mu^{*-1} \downarrow t_* \\
\mathbb{C}/C \leftarrow \theta^* & \mathbb{C}/A \\
C_! \downarrow & \mu_! & \downarrow A_! \\
\mathbb{C} & \mathbb{C} & \mathbb{C}
\end{array}$$

given by the whiskered natural transformations

$$\begin{array}{c|c}
\mathbb{C} & \mathbb{C} \\
C^* \downarrow & \eta^{-1} \downarrow B^* \\
\mathbb{C}/D \leftarrow \rho^* - \mathbb{C}/B \\
s_* \downarrow & \mu^{*-1} \downarrow t_* \\
\mathbb{C}/C \leftarrow \theta^* & \mathbb{C}/A \\
C_! \downarrow & \mu_! & \downarrow A_! \\
\mathbb{C} & \mathbb{C} & \mathbb{C}
\end{array}$$

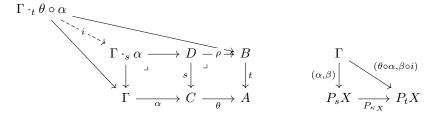
Furthermore, the natural transformation P_{κ} is cartesian. meaning each naturality square is a pullback square.

$$P_{s}X \xrightarrow{P_{\kappa Y}} P_{t}X$$

$$P_{s}f \downarrow \qquad \qquad \downarrow P_{s}f$$

$$P_{s}Y \xrightarrow{P_{\kappa Y}} P_{t}Y$$

The natural transformation P_{κ} computes in the following way



using the fact that $\Gamma \cdot_s \alpha$ and $\Gamma \cdot_t \theta \circ \alpha$ are limits of the same diagram.

Proof. We can use the computation of $P_{\kappa X}$ and $P_s f$ to show that the natural transformation P_{κ} is cartesian. Essentially, the first component of a map $\Gamma \to P_s X$ is determined by its composition with $P_s f$ and its second component is determined by its composition with $P_{\kappa X}$.

Corollary 2.0.10. If we have

$$D' \longrightarrow B'$$

$$\downarrow^{\rho_1} \qquad \rho_2 \downarrow$$

$$D \longrightarrow B$$

$$\downarrow^{q_1} \qquad q_2 \downarrow$$

$$C \longrightarrow A$$

then the two possible ways of obtaining composing the covariant and contravariant actions of P_{-} form a (strictly commuting) pullback square in $[\mathbb{C}, \mathbb{C}]$.

$$P_{q_{1}} \xrightarrow{P_{\kappa}} P_{q_{2}}$$

$$\downarrow^{\rho_{1}^{\star}} \qquad \downarrow^{\rho_{2}^{\star}}$$

$$P_{q'_{1}} \xrightarrow{P_{\kappa'}} P_{q'_{2}}$$

Proof. To check that it commutes and is a pullback, it suffices to do this pointwise, for some $X \in \mathbb{C}$. Then we simply unfold the computation for each of P_{κ} and ρ^{\star} .

Chapter 3

Multivariate Polynomial Functors

To be written