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Simplicity of Crossed Products of C^*-algebras

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Abstract

Write abstract

CONTENTS

Hilbert C^*-modules and the Multiplier Algebra

In this chapter we introduce some of the fundamental theory of Hilbert C^*-modules, which may be thought of as a generalization of Hilbert spaces. The primary source of inspiration and techniques are the excellent books on the subject: [19], [10] and [12]. After reviewing the basic definitions we will proceed to contruct and prove theorems regarding the Multiplier Algebra, which plays a crucial part in the development of the theory of the general crossed product, one of the basic constructions of this thesis.

1.1 Hilbert C^*-modules

Given a non-unital C^*-algebra A, we can extend a lot of the theory of unital C^*-algebras to A by considering the unitalization $A^\dagger = A + 1\mathbb{C}$ with the canonical operations. This is the 'smallest' unitalization in the sense that A the inclusion map is non-degenerate and maps A to an ideal of $A^\dagger$ such that $A^\dagger / A \cong \mathbb{C}$. We can also consider a maximal unitalization, called the *Multiplier Algebra* of A, denoted by $M(A)$. We will now construct it and some theory revolving it, in particular that of Hilbert-modules:

Definition 1.1.1. Let A be any C^*-algebra. An *inner-product A-module* is a complex vector space E which is also a right-A-module which is equipped with a map $\langle \cdot, \cdot \rangle_A \colon E \times A \to A$ such that for $x, y, z \in E$, $\alpha, \beta \in \mathbb{C}$ and $a \in A$ we have

$$\langle a, \alpha y + \beta z \rangle_A = \alpha \langle x, y \rangle_A + \beta x, z \rangle_A \tag{1.1}$$

$$\langle x, y \cdot a \rangle - A = \langle x, y \rangle_A \cdot a \tag{1.2}$$

$$\langle y, x \rangle_A = \langle x, y \rangle_A^* \tag{1.3}$$

$$\langle x, x \rangle_A \geqslant 0 \tag{1.4}$$

$$\langle x, x \rangle_A = 0 \implies x = 0. \tag{1.5}$$

If $\langle\cdot,\cdot\rangle$ satisfy (1.1)-(1.4) but not (1.5) above, we say that E is a *semi-inner-product A-module*. If there is no confusion, we simply write $\langle\cdot,\cdot\rangle$ to mean the A-valued inner-product on E. Similarly, we will often denote $x \cdot a$ by xa.

There is of course a corresponding version of left inner-product A-modules, with the above conditions adjusted suitably. However, we will unless otherwise specified always mean a right A-module in the following.

It is easy to see that this generalizes the notion of Hilbert spaces, for if $A = \mathbb{C}$, then any Hilbert space (over $\mathbb{C}$) is a inner-product $\mathbb{C}$-module. Even when E is only a semi-inner-product A-module, we still have a lot of the nice properties of regular inner-producte, e.g.:

Proposition 1.1.2 (Cauchy-Schwarz Inequality)**.** Let E be a semi-inner-product A-module. For $x, y \in E$, it holds that

$$\langle x,y\rangle^*\langle x,y\rangle \leqslant \|\langle x,x\rangle\|_A \langle y,y\rangle.$$

Proof.
Let $x, y \in E$ and let $a = \langle x,y\rangle \in A$ and $t > 0$. Then, since $(\langle x,y\rangle a)^* = a^*\langle y,x\rangle$, we have

$$0 \leqslant \langle xa - ty, xa - ty\rangle = t^2\langle y,y\rangle + a^*\langle x,x\rangle a - 2ta^*a.$$

For $c \geqslant 0$ in A, it holds that $a^*ca \leqslant \|c\|_A a^*a$: Let $A \subseteq \mathbb{B}(\mathcal{H})$ faithfully. The regular Cauchy-Schwarz implies that $\langle a^*ca\xi,\xi\rangle_{\mathcal{H}} \leqslant \|c\| \langle a^*a\xi,\xi\rangle_{\mathcal{H}}$ for all $\xi \in \mathcal{H}$. With this in mind, we then have

$$0 \leqslant a^*a \|\langle x,x\rangle\|_A - 2ta^*a + t^2\langle y,y\rangle.$$

If $\langle x,x\rangle = 0$, then $0 \leqslant 2a^*a \leqslant t\langle y,y\rangle \to 0$, as $t \to 0$ so that $a^*a = 0$ as wanted. If $\langle x,x\rangle \neq 0$, then $t = \|\langle x,x\rangle\|_A$ gives the wanted inequality. $\square$

Recall that for $0 \leqslant a \leqslant b \in A$, it holds that $\|a\|_A \leqslant \|b\|_A$. This and the above implies that $\|\langle x,y\rangle\|_A^2 \leqslant \|\langle x,x\rangle\|_A \|\langle y,y\rangle\|_A$.

Definition 1.1.3. If E is an (semi-)inner-product A-module, we define a (semi-)norm on E by

$$\|x\| := \|\langle x,x\rangle\|_A^{\frac{1}{2}}, \text{ for } x \in E.$$

If E is an inner-product A-module such that this norm is complete, we say that E is a *Hilbert C^*-module over A* or just a *Hilbert A-module*.

If the above defined function is not a proper norm but instead a semi-norm, then we may form the quotient space of E with respect to $N = \{x \in E \mid \langle x, x\rangle = 0\}$ to obtain an inner-product A-module with the obvious operations. Moreover, the usual ways of passing from pre-Hilbert spaces extend naturally to pre-Hilbert A-modules. For $x \in E$ and $a \in A$, we see that

$$\|xa\|^2 = \left\| a^* \underbrace{\langle x, x\rangle}_{\geqslant 0} a \right\| \leqslant \|a\|^2 \|x\|^2 ,$$

so that $\|xa\| \leqslant \|x\| \, \|a\|$, which shows that E is a normed A-module. For $a \in A$, let $R_a \colon x \mapsto xa \in E$. Then R_a is a linear map on E satisfying $\|R_a\| = \sup_{\|x\|=1} \|xa\| \leqslant \|a\|$, and the map $R \colon a \mapsto R_a$ is a contractive anti-homomorphism of A into the space of bounded linear maps on E (anti in the sense that $R_{ab} = R_b R_a$ for $a, b \in A$).

Example 1.1.4
The classic example of a Hilbert A-module is A itself with $\langle a, b\rangle = a^* b$, for $a, b \in A$. We will write A_A to specify that we consider A as a Hilbert A-module.

For a Hilbert A-module E, we let $\langle E, E\rangle = \mathrm{Span}\,\{\langle x, y\rangle \mid x, y \in E\}$. It is not hard to see that $\overline{\langle E, E\rangle}$ is a closed ideal in A, and moreover, we see that:

Should $\langle E, E\rangle$ be closed?

Lemma 1.1.5. The set $E\langle E, E\rangle$ is dense in E.

Proof.
Let (u_i) denote an approximate unit for $\langle E, E\rangle$. Then clearly $\langle x - xu_i, x - xu_i\rangle \to 0$ for $x \in E$ (seen by expanding the expression), which implies that $xu_i \to x$ in E □

In fact, the above proof also shows that EA is dense in E and that $x1 = x$ when $1 \in A$.

Definition 1.1.6. If $\langle E, E\rangle$ is dense in A we say that the Hilbert A-module is *full*.

Many classic constructions preserve Hilbert C^*-modules:

Example 1.1.7
If $E_1, \ldots, E_n$ are Hilbert A-modules, then $\bigoplus_{1 \leqslant i \leqslant n} E_i$ is a Hilbert A-module with the inner product $\langle (x_i), (y_i)\rangle = \sum_{1 \leqslant i \leqslant n} \langle x_i, y_i\rangle_{E_i}$. This is obvious.

Example 1.1.8
If $\{E_i\}$ is a set of Hilbert A-modules, and let $\bigoplus_i E_i$ denote the set

$$\bigoplus_i E_i := \left\{ (x_i) \in \prod_i E_i \mid \sum_i \langle x_i, x_i\rangle \text{ converges in } A \right\}.$$

Then $\bigoplus_i E_i$ is a Hilbert A-module. The details of the proof are nothing short of standard, so we refer to [10, p. 5].

Note 1.1.9
There are similarities and differences between Hilbert A-modules and regular Hilbert spaces. One major difference is that for closed submodules $F \subseteq E$, it does not hold that $F \oplus F^\perp = E$, where $F^\perp = \{y \in E \mid \langle x, y \rangle = 0,\ x \in F\}$: Let $A = C([0,1])$ and consider A_A. Let $F \subseteq A$ be the set $F = \{f \in A_A \mid f(0) = f(1) = 0\}$. Then F is clearly a closed sub A-module of A_A. Consider the element $f \in F$ given by

$$f(t) = \begin{cases} t & t \in [0, \frac{1}{2}] \\ 1 - t & t \in (\frac{1}{2}, 1] \end{cases}$$

For all $g \in F^\perp$ it holds that $\langle f, g \rangle = fg = 0$, so $g(t) = 0$ for $t \in (0,1)$ and by uniform continuity on all of $[0,1]$, from which we deduce that $F^\perp = \{0\}$ showing that $A \neq F \oplus F^\perp$. The number of important results in the theory of Hilbert spaces relying on the fact that $\mathcal{H} = F^\perp \oplus F$ for closed subspaces F of $\mathcal{H}$ is very significant, and hence we do not have the entirety of our usual toolbox available.

However, returning to the general case, we are pleased to see that it does hold for Hilbert A-modules E that for $x \in E$ we have

$$\|x\| = \sup \{\|\langle x, y \rangle\| \mid y \in E,\ \|y\| \leqslant 1\}. \tag{$\star$}$$

This is an obvious consequence of the Cauchy-Schwarz theorem for Hilbert A-modules.

Another huge difference is that adjoints don't automatically exist: Let $A = C([0,1])$ again and F as before. Let $\iota \colon F \to A$ be the inclusion map. Clearly ι is a linear map from F to A, however, if it had an adjoint $\iota^* \colon A \to F$, then for any self-adjoint $f \in F$, we have

$$f = \langle \iota(f), 1 \rangle_A = \langle f, \iota^*(1) \rangle = f\iota^*(1),$$

so $\iota^*(1) = 1$, but $1 \notin F$. Hence ι^* is not adjointable. We shall investigate properties of the adjointable operators in more detail further on.

1.2 The C^*-algebra of adjointable operators

The set of adjointable maps between Hilbert A-modules is of great importance and has many nice properties, and while not all results carry over from the theory of bounded operators on a Hilbert space, many do, as we shall see.

Definition 1.2.1. For Hilbert A-modules E_i with A-valued inner-products $\langle \cdot, \cdot \rangle_i$, $i = 1, 2$, we define $\mathcal{L}_A(E_1, E_2)$ to be the set of all maps $t \colon E_1 \to E_2$ for which there exists a map $t^* \colon E_2 \to E_1$ such that

$$\langle tx, y \rangle_2 = \langle x, t^* y \rangle_1,$$

for all $x \in E_1$ and $y \in E_2$, i.e., *adjointable maps*. When no confusion may occur, we will write $\mathcal{L}(E_1, E_2)$ and omit the subscript.

Proposition 1.2.2. If $t \in \mathcal{L}(E_1, E_2)$, then

1. t is A-linear and

2. t is a bounded map.

Proof.
Linearity is clear. To see A-linearity, note that for $x_1, x_2 \in E_i$ and $a \in A$ we have $a^* \langle x_1, x_2 \rangle_i = \langle x_1 a, x_2 \rangle_i$, and hence for $x \in E_1$, $y \in E_2$ and $a \in A$ we have $\langle t(xa), y \rangle_2 = a^* \langle x, t^*(y) \rangle_1 = \langle t(x)a, y \rangle_1$.

To see that t is bounded, we note that t has closed graph: Let $x_n \to x$ in E_1 and assume that $tx_n \to z \in E_2$. Then

$$\langle tx_n, y \rangle_2 = \langle x_n, t^* y \rangle_1 \to \langle x, t^* y \rangle_1 = \langle tx, y \rangle_2$$

so $tx = z$, and hence t is bounded, by the Closed Graph Theorem (CGT hereonforth). $\square$

In particular, similar to the way that the algebra $\mathbb{B}(\mathcal{H}) = \mathcal{L}_{\mathbb{C}}(\mathcal{H}_{\mathbb{C}}, \mathcal{H}_{\mathbb{C}})$ is of great importance, we examine for a fixed Hilbert A-module E, the $*$-subalgebra $\mathcal{L}(E) := \mathcal{L}(E, E)$ of bounded operators on E:

Proposition 1.2.3. The algebra $\mathcal{L}(E)$ is a closed $*$-subalgebra of $\mathbb{B}(E)$ such that for $t \in \mathcal{L}(E)$ we have

$$\|t^* t\| = \|t\|^2,$$

hence is it a C^*-algebra.

Proof.
Since $\mathbb{B}(E)$ is a Banach algebra, we have $\|t^* t\| \leqslant \|t^*\| \|t\|$ for all $t \in \mathcal{L}(E)$. Moreover, by Cauchy-Schwarz, we have

$$\|t^* t\| = \sup_{\|x\| \leqslant 1} \{\|t^* tx\|\} = \sup_{\|x\|, \|y\| \leqslant 1} \{\langle tx, ty \rangle\} \geqslant \sup_{\|x\| \leqslant 1} \{\|\langle tx, tx \rangle\|\} = \|t\|^2,$$

so $\|t^*t\| = \|t\|^2$. Assume that $t \neq 0$, then we see that $\|t\| = \|t^*t\| \, \|t\|^{-1} \leqslant \|t^*\|$ and similarly $\|t^*\| \leqslant \|t^{**}\| = \|t\|$, so $\|t\| = \|t^*\|$, in particular $t \mapsto t^*$ is continuous. It is clear that $\mathcal{L}(E)$ is a $*$-subalgebra. To see that is is closed, suppose that (t_α) is a Cauchy-net, and denote its limit by $t \in \mathbb{B}(E)$. By the above, we see that (t_α^*) is Cauchy with limit $f \in \mathbb{B}(E)$, since Then, we see that for all $x, y \in E$

$$\langle tx, y \rangle = \lim_\alpha \langle t_\alpha x, y \rangle = \lim_\alpha \langle x, t_\alpha^* y \rangle = \langle x, fy \rangle,$$

so t is adjointable, hence an element of $\mathcal{L}(E)$. $\square$

If we for $x \in E$ denote the element $\langle x, x \rangle^{\frac{1}{2}} \geqslant 0$ by $|x|_A$, then we obtain a variant of a classic and important inequality:

Lemma 1.2.4. Let $t \in \mathcal{L}(E)$, then $|tx|^2 \leqslant \|t\|^2 \, |x|^2$ for all $x \in E$.

Proof.
The operator $\|t\|^2 - t^*t \geqslant 0$, since $\mathcal{L}(E)$ is a C^*-algebra, and thus equal to s^*s for some $s \in \mathcal{L}(E)$. Hence

$$\|t\|^2 \langle x, x \rangle - \langle t^*tx, x \rangle = \langle (\|t\|^2 - t^*t)x, x \rangle = \langle sx, sx \rangle \geqslant 0,$$

for all $x \in E$, as wanted. $\square$

A quite useful result about positive operators on a Hilbert space carries over to the case of Hilbert C^*-modules, for we have:

Lemma 1.2.5. Let E be a Hilbert A-module and let T be a linear operator on E. Then T is a positive operator in $\mathcal{L}_A(E)$ if and only if for all $x \in E$ it holds that $\langle Tx, x \rangle \geqslant 0$ in A.

Proof.
If T is adjointable and positive, then $T = S^*S$ for some $S \in \mathcal{L}(E)$ and $\langle Tx, x \rangle = \langle Sx, Sx \rangle \geqslant 0$. For the converse, we note first that the usual polarization identity applies to Hilbert C^*-modules as well:

$$4\langle x, y \rangle = \sum_{k=0}^{3} i^k \langle u + i^k v, u + i^k v \rangle,$$

for all $u, v \in E$, since $\langle u, v \rangle = \langle v, u \rangle^*$. Hence, if $\langle Tx, x \rangle \geqslant 0$ for all $x \in E$, we see that $\langle Tx, x \rangle = \langle x, Tx \rangle$, so

$$4\langle Tx, y \rangle = \sum_{k=0}^{3} i^k \underbrace{\langle T(x + i^k y), x + i^k y \rangle}_{= \langle x + i^k y, T(x + i^k y) \rangle} = 4\langle x, Ty \rangle, \text{ for } x, y \in E.$$

Thus $T = T^*$ is an adjoint of T and $T \in \mathcal{L}_A(E)$. An application of the continuous functional calculus allows us to write $T = T^+ - T^-$ for positive operators $T^+, T^- \in \mathcal{L}_A(E)$ with $T^+T^- = T^-T^+ = 0$, see e.g., [24, Theorem 11.2]. Then, for all $x \in E$ we have

$$0 \leqslant \langle TT^- x, x\rangle = \langle -(T^-)^2 x, T^- x\rangle = -\underbrace{\langle (T^-)^3 x, x\rangle}_{\geqslant 0} \leqslant 0,$$

so $\langle (T^-)^3 x, x\rangle = 0$ for all x. The polarization identity shows that $(T^-)^3 = 0$ and hence $(T^-) = 0$, so $T = T^+ \geqslant 0$. $\square$

And, as in the Hilbert Space situation, we also have

Lemma 1.2.6. Let E be a Hilbert A-module and $T \geqslant 0$ in $\mathcal{L}_A(E)$. Then

$$\|T\| = \sup_{\|x\| \leqslant 1} \{\|\langle Tx, x\rangle\|\}$$

Proof.
Write $T = S^*S$ for $S \in \mathcal{L}_A(E)$ to obtain the identity. $\square$

It is well-known that the only ideal of $\mathbb{B}(\mathcal{H})$ is the set of compact operators, $\mathbb{K}(\mathcal{H})$, which is the norm closure of the finite-rank operators. We now examine the Hilbert C^*-module analogue of these two subsets:

Definition 1.2.7. Let E, F be Hilbert A-modules. Given $x \in E$ and $y \in F$, we let $\Theta_{x,y} \colon F \to E$ be the map

$$\Theta_{x,y}(z) := x\langle y, z\rangle,$$

for $z \in F$. It is easy to see that $\Theta_{x,y} \in \mathbb{B}(F, E)$.

Proposition 1.2.8. Let E, F be Hilbert A-modules. Then for every $x \in E$ and $y \in F$, it holds that $\Theta_{x,y} \in \mathcal{L}(F, E)$. Moreover, if F is another Hilbert A-module, then

1. $\Theta_{x,y}\Theta_{u,v} = \Theta_{x\langle y,u\rangle_F, v} = \Theta_{x, v\langle u,v\rangle_F}$, for $u \in F$ and $v \in G$,
2. $T\Theta_{x,y} = \Theta_{tx,y}$, for $T \in \mathcal{L}(E, G)$,
3. $\Theta_{x,y}S = \Theta_{x,S*y}$ for $S \in \mathcal{L}(G, F)$.

Proof.
Let $x, w \in E$ and $y, u \in F$ for Hilbert A-modules E, F as above. Then, we see that

$$\langle \theta_{x,y}(u), w\rangle_E = \langle x\langle y, u\rangle_F, w\rangle_E = \langle u, y\rangle_F\langle x, w\rangle_E = \langle u, y\langle x, w\rangle_E\rangle_F,$$

so $\Theta^*_{x,y} = \Theta_{y,x} \in \mathcal{L}(E, F)$. Suppose that G is another Hilbert A-module and $v \in G$ is arbitrary, then

$$\Theta_{x,y}\Theta_{u,v}(g) = x\langle y, u\langle v, g\rangle_G\rangle_F = x\langle y, u\rangle_F\langle v, g\rangle = \Theta_{x\langle y,u\rangle_F,v}(g),$$

for $g \in G$. The other identity in (1) follows analogously. Let $T \in \mathcal{L}(E, G)$, then

$$T\Theta_{x,y}(u) = Tx\langle y, u\rangle_F = \Theta_{Tx,y}(u),$$

so (2) holds and (3) follows analogously. □

Definition 1.2.9. Let E, F be Hilbert A-modules. We define the set of compact operators $F \to E$ to be the set

$$\mathbb{K}_A(F, E) := \overline{\mathrm{Span}\,\{\Theta_{x,y} \mid x \in E,\ y \in F\}} \subseteq \mathcal{L}_A(F, E).$$

It follows that for $E = F$, the $*$-subalgebra of $\mathcal{L}_A(E)$ obtained by taking the span of $\{\Theta_{x,y} \mid x, y \in E\}$ forms a two-sided algebraic ideal of $\mathcal{L}_A(E)$, and hence its closure is a C^*-algebra:

Definition 1.2.10. We define the *Compact Operators on a Hilbert A-module E* to be the C^*-algebra

$$\mathbb{K}_A(E) := \overline{\mathrm{Span}}\,\{\Theta_{x,y} \mid x, y \in E\} \subseteq \mathcal{L}_A(E).$$

Note 1.2.11
An operator $t \in \mathbb{K}_A(F, E)$ need not be a compact operator when E, F are considered as Banach spaces, i.e., a limit of finite-rank operators. This is easy to see by assuming that A is not-finite dimensional and unital: The operator $\mathrm{id}_A = \Theta_{1,1}$ is in $\mathbb{K}_A(A_A)$, but the identity operator is not a compact operator on an infinite-dimensional Banach space.

Lemma 1.2.12. There is an isomorphism $A \cong \mathbb{K}_A(A_A)$. If A is unital, then $\mathbb{K}_A(A_A) = \mathcal{L}_A(A_A)$.

Proof.
find all instances where $L_a(x)$ is used instead of $L_a x$

Fix $a \in A$ and define $L_a\colon A \to A$ by $L_a b = ab$. Then $\langle L_a x, y\rangle = x^*a^*y = \langle x, L_{a^*}y\rangle$ for all $x, y \in A$, hence L_a is adjointable with adjoint L_{a^*}. Note that $L_{ab^*}x = ab^*x = a\langle b, x\rangle = \Theta_{a,b}(x)$ for $a, b, x \in A$, hence if $L\colon a \mapsto L_a$, we have $\mathbb{K}_A(A_A) \subseteq \mathrm{Im}(L)$. It is clear that L

is a $*$-homomorphism of norm one, i.e., an isometric homomorphism, so it's image is a C^*-algebra. To see that it has image equal to $\mathbb{K}_A(A_A)$, let (u_i) be an approximate identity of A. Then for any $a \in A$ $L_{u_i a} \to L_a$, and since $L_{u_i a} \in \mathbb{K}_A(A_A)$ for all i, we see that $L_a \in \mathbb{K}_A(A_A)$.

If $1 \in A$, then for $t \in \mathcal{L}_A(A_A)$ we see that $t(a) = t(1 \cdot a) = t(1) \cdot a = L_{t(1)}a$, so $t \in \mathbb{K}_A(A_A)$, and $\mathcal{L}_A(A_A) \cong A$. □

Note 1.2.13
It does not hold in general that $A \cong \mathcal{L}_A(A_A)$, e.g., as we shall see soon for $A = C_0(X)$, where X is locally compact Hausdorff, for then $\mathcal{L}_A(A_A) = C_b(X)$.

In 1.1.5 we saw that for a Hilbert A-module E, the set $E\langle E, E\rangle$ was dense in E. We are going to improve this to say that every $x \in E$ is of the form $y\langle y, y\rangle$ for some unique $y \in E$. To do this, we need to examine the compact operators a bit more:

Proposition 1.2.14. Given a right Hilbert A-module E, then E is a full left Hilbert $\mathbb{K}_A(E)$-module with respect to the action $T \cdot x = T(x)$ for $x \in E$ and $T \in \mathbb{K}_A(E)$ and $\mathbb{K}_A(E)$-valued inner product ${}_{\mathbb{K}_A(E)}\langle x, y\rangle = \Theta_{x,y}$ for $x, y \in E$. Moreover, the corresponding norm $\|\cdot\|_{\mathbb{K}_A(E)} = \left\|{}_{\mathbb{K}_A(E)}\langle \cdot, \cdot\rangle\right\|^{\frac{1}{2}}$ coincides with the norm $\|\cdot\|_A$ on E.

Proof.
It is easy to see that the first, second and third condition of 1.1.1 are satisfied in light of 1.2.8. Note that

$$\langle {}_{\mathbb{K}_A(E)}\langle x, x\rangle \cdot y, y\rangle_A = \langle x\langle x, y\rangle, y\rangle_A = \langle x, y\rangle_A^*\langle x, y\rangle_A \geqslant 0, \tag{1.6}$$

for all $x, y \in E$, which by 1.2.5 implies ${}_{\mathbb{K}_A(E)}\langle x, x\rangle \geqslant 0$ for all $x \in E$. The above calculation also shows that if ${}_{\mathbb{K}_A(E)}\langle x, x\rangle = 0$ then $\langle x, y\rangle_A = 0$ for all $y \in E$ so $x = 0$.

To see the last condition, note that

$$\left\|\langle {}_{\mathbb{K}_A(E)}\langle x, x\rangle x, x\rangle_A\right\| = \|\langle x, x\rangle_A^*\langle x, x\rangle_A\| = \|x\|_A^2$$

so by 1.2.6, $\|x\|_{\mathbb{K}_A(E)} \geqslant \|x\|_A$ for all $x \in E$. For the other inequality, applying Cauchy-Schwarz to 1.6, we see that

$$\langle {}_{\mathbb{K}_A(E)}\langle x, x\rangle y, y\rangle_A \leqslant \|\langle x, x\rangle_A\| \langle y, y\rangle_A$$

so again by 1.2.6 we see that $\left\|{}_{\mathbb{K}_A(E)}\langle x, x\rangle\right\| \leqslant \|\langle x, x\rangle_A\|$ for all $x \in E$. □

When we considered $E = A_A$, we defined for $a \in A$ the operator $L_a\colon b \mapsto ab$, $b \in A$. Similarly, we do this in the general case:

Definition 1.2.15. Let E be a Hilbert A-module. We define for $x \in E$ the operators $L_x \colon A \to E$, $a \mapsto x \cdot a$, and $D_x \colon E \to A$, $y \mapsto \langle x, y \rangle_A$.

Remark 1.2.16
The calculation

$$\langle D_x(y), a \rangle_{A_A} = \langle \langle x, y \rangle_A^* a = \langle y, x \rangle_A a = \langle y, xa \rangle_A = \langle y, L_x(a) \rangle_{A_A},$$

for $x, y \in E$ and $a \in A$ shows that L_x and D_x are mutual adjoints, so $L_x \in \mathcal{L}_A(A_A, E)$ and $D_x \in \mathcal{L}_A(E, A_A)$.

In fact, the maps $D \colon x \mapsto D_x$ and $L \colon x \mapsto L_x$ are interesting on their own, which extends the result from the case $E = A_A$ seen in 1.2.12:

Lemma 1.2.17. Let E be a Hilbert A-module. Then D is a conjugate linear isometric isomorphism of E and $\mathbb{K}_A(E, A_A)$ and L is a linear isometric isomorphism of E and $\mathbb{K}_A(A_A, E)$.

Proof.
Conjugate linearity of D follows immediately from $\langle \cdot, y \rangle_A$ possessing said property. We see that D_x is an isometry via $\star$, hence it has closed image. Since $\Theta_{a,x} = D_{xa^*}$, it follows that $\mathbb{K}_A(E, A_A) \subseteq D_E$. For the converse, let (e_i) be an approximate unit of $\overline{\langle E, E \rangle}$. Then, since D_E is closed and D is continuous, we see that $D_x = \lim_i D_{xe_i} = \lim_i \Theta_{x,e_i} \in D_E$, hence $D_E \subseteq \mathbb{K}_A(E, A_A)$, since $E\langle E, E \rangle$ is dense in E.

The second statement follows since clearly $L \colon x \mapsto D_x^*$ is then a linear isometry and we saw in 1.2.8 that $\Theta_{a,x}^* = \Theta_{x,a}$, so the argument above applies to L as well. $\square$

Abusing notation, we will in the following use L_x to denote both the operator on E and A_A depending on the location of x. We now show a generalization of a standard trick regarding operators on direct sums of Hilbert spaces, the technique is from [4] and is based on the 2×2-matrix trick of Connes [6]: We may, given Hilbert spaces $\mathcal{H}, \mathcal{K}$ view $\mathbb{B}(\mathcal{H} \oplus \mathcal{K})$ as the set of matrices

$$\begin{pmatrix} \mathbb{B}(\mathcal{H}) & \mathbb{B}(\mathcal{K}, \mathcal{H}) \\ \mathbb{B}(\mathcal{H}, \mathcal{K}) & \mathbb{B}(\mathcal{K}) \end{pmatrix}$$

by compression with the orthogonal projection onto e.g. $\mathcal{H}$. This also holds for the compact operators on $\mathcal{H} \oplus \mathcal{K}$, and more generally:

Lemma 1.2.18. Let E be a Hilbert A-module. If $L_a \in \mathbb{K}_A(A_A)$, $S \in \mathbb{K}_A(A_A, E)$, $T \in \mathbb{K}_A(E, A_A)$ and $R \in \mathbb{K}_A(E)$, then

$$Q := \begin{pmatrix} L_a & T \\ S & R \end{pmatrix} \in \mathbb{K}_A(A_A \oplus E),$$

and its adjoint is

$$Q^* = \begin{pmatrix} L_a^* & S^* \\ T^* & R^* \end{pmatrix}$$

Moreover, every compact operator on $A_A \oplus E$ is of this form. If $Q \in \mathbb{K}_A(A_A \oplus E)$ is self-adjoint and anti-commutes with $1 \oplus -1$ then

$$Q = \begin{pmatrix} 0 & D_x \\ L_x & 0 \end{pmatrix},$$

for some $x \in E$ and L_x, D_x as in Lemma 1.2.17.

Proof.
It is clear that Q is adjointable with adjoint

$$Q^* = \begin{pmatrix} L_a^* & S^* \\ T^* & R^* \end{pmatrix}$$

To see that it is compact, note that each of the four corner embeddings

$$L_a \mapsto \begin{pmatrix} L_a & 0 \\ 0 & 0 \end{pmatrix}, \quad T \mapsto \begin{pmatrix} 0 & T \\ 0 & 0 \end{pmatrix}, \quad S \mapsto \begin{pmatrix} 0 & 0 \\ S & 0 \end{pmatrix}, \quad R \mapsto \begin{pmatrix} 0 & 0 \\ 0 & R \end{pmatrix}$$

are linear isometries and they map the module valued finite-rank maps to module valued finite-rank maps, e.g., we see that $\Theta_{a,x} \in \mathbb{K}_A(E, A)$ is mapped to $\Theta_{(a,0),(x,0)}$. By compressing with the orthogonal projection onto e.g., A_A, we see that we can decompose every finite rank operator in this way. Since compression by projections is continuous, this shows that every compact operator has this form.

The last statement follows readily from the isomorphism from Lemma 1.2.17, the anti-commutativity forces $L_a = 0$ and $R = 0$ and self-adjointess forces $D_x^* = L_x$. $\square$

We may finally prove the statement which we set out to show:

Proposition 1.2.19. Let E be a Hilbert A-module. Then for every $x \in E$ there is a unique $y \in E$ such that $x = y\langle y, y\rangle$.

Proof.
Let $x \in E$, and consider the self-adjoint operator

$$t = \begin{pmatrix} 0 & D_x \\ L_x & 0 \end{pmatrix}$$

on $A_A \oplus E$, which by Lemma 1.2.18 is self-adjoint and anti-commutes with $1 \oplus -1$. With t being self-adjoint, the function $f(x) = x^{\frac{1}{3}}$ is continuous on the spectrum $\sigma(t)$, so it defines a compact operator $f(t)$ on $A_A \oplus E$. This operator anti-commutes with $1 \oplus -1$ by the functional calculus applied to the map $\psi\colon \mathbb{K}_A(A_A \oplus E) \to \mathbb{K}_A(A_A \oplus E)$, $s \mapsto (1 \oplus -1)s + s(1 \oplus -1)$, which is a *-homomorphism, so there is $y \in E$ such that

$$f(t) = \begin{pmatrix} 0 & D_y \\ L_y & 0 \end{pmatrix}$$

And so $f(t)^3 = t$ implies that $L_x = L_y D_y L_y = L_{y\langle y,y\rangle}$, so $x = y\langle y, y\rangle$. □

We empose quite an assortment of topologies on $\mathbb{B}(\mathcal{H}, \mathcal{H}')$ for Hilbert spaces $\mathcal{H}$ and $\mathcal{H}'$ – this is not the case when we study Hilbert C^*-modules: Given Hilbert A-module E and F, we only consider two topologies – the norm topology and the *strict topology*:

Definition 1.2.20. The *strict topology* on $\mathcal{L}_A(E, F)$ is the locally convex topology on $\mathcal{L}_A(E, F)$ generated by the family of semi-norm pairs

$$t \mapsto \|tx\|\,, \text{ for } x \in E \text{ and } t \mapsto \|t^*y\|\,, \text{ for } y \in F.$$

This topology turns out to be very important, and there is many results of the classic theory which apply in this setting as well:

Proposition 1.2.21. Let E be a Hilbert A-module. Then the unit ball of $\mathbb{K}_A(E, F)$, $(\mathbb{K}_A(E, F))_1$, is strictly dense in the unit ball, $(\mathcal{L}_A(E, F))_1$, of $\mathcal{L}_A(E, F)$.

Proof.
Let (e_i) be an approximate identity of the C^*-algebra $\mathbb{K}_A(E)$. Then, for all $w\langle x, y\rangle \in E\langle E, E\rangle$ we see that

$$e_i w\langle x, y\rangle = e_i \Theta_{w,x}(y) \to \Theta_{w,x}(y) = w\langle x, y\rangle,$$

and hence $e_i x \to x$ for all $x \in E$, since $E\langle E, E\rangle \subseteq E$ is dense by 1.1.5. Then we clearly have, for $t \in (\mathcal{L}_A(E, F))_1$ that

$$\|t(e_i x - x)\| \to 0 \text{ and } \|e_i t^* y - t^* y\| \to 0,$$

for $x \in E$ and $y \in F$, so $te_i \to t$ strictly, and by 1.2.8 we see that $te_i \in (\mathbb{K}_A(E, F))_1$, finishing the proof. □

1.3 Unitalizations and the Multiplier Algebra

We are almost ready to introduce the Multiplier Algebra of a C^*-algebra A, however, we must introduce some new terminology first

Definition 1.3.1. An ideal I of a C^*-algebra A is an *essential ideal* if it is non-zero and satisfies $I \cap J \neq 0$ for all non-zero ideals J of A.

And there is another commonly used equivalent formulation of the definition of essential ideals:

Lemma 1.3.2. Let I, J be non-zero ideals of a C^*-algebra A. Then the following holds:

1. $I^2 = I$,
2. $I \cap J = IJ$,
3. I is essential if and only if $aI = 0$ implies $a = 0$ for all $a \in A$.

Proof.
1: It holds that $I^2 \subseteq I$, since I is an ideal. For the other inclusion, note that if $a \geqslant 0$ is an element of I, then $a = (a^{\frac{1}{2}})^2 \in I^2$, so $I_+ \subseteq I^2$ hence $I = I^2$.

2: Note that $I \cap J = (I \cap J)^2$ by (1), and $(I \cap J)(I \cap J) \subseteq IJ \subseteq I \cap J$.

3: Assume that I is essential and $aI = 0$. Then $(a)I = (a) \cap I = 0$, where (a) is the ideal generated by a in A, and hence $(a) = 0$ so $a = 0$. For the converse, let $J \neq 0$ be any ideal of A and let $0 \neq a \in J$. By assumption this implies that $aI \neq 0$, so $(a)I = (a) \cap I \neq 0$, which shows that $J \cap I \neq 0$. □

As mentioned earlier, it is a useful tool to be able to adjoin units to a non-unital C^*-algebra A. We formalize this process in the following definition:

Definition 1.3.3. A *unitalization* of a C^*-algebra A is a pair (B, ι) consisting of a unital C^*-algebra B and an injective homomorphism $\iota \colon A \hookrightarrow B$ such that $\iota(A)$ is an essential ideal of B.

The most common way of adjoining a unit to a non-unital C^*-algebra A is to consider the vector space $A \oplus \mathbb{C}$ and endow it with a proper $*$-algebra structure and give it a certain C^*-norm, see e.g., [24, Theorem 15.1]. This can also be done via the use of compact operators on A:

Example 1.3.4
Let $A^1 := A \oplus \mathbb{C}$ as a $*$-algebra with $(x,\alpha)(y,\beta) := xy + \alpha y + \beta x + \alpha\beta$ and entrywise addition and involution. Identify A with $\mathbb{K}_A(A_A)$ inside $\mathcal{L}_A(A_A)$ via the left-multipication operator, L, on A, and consider the algebra generated by 1 and $\mathbb{K}_A(A_A)$ in $\mathcal{L}_A(A_A)$.

It is a C^*-algebra, and we may extend L to an isomorphism of $*$-algebras between $\mathbb{K}_A(A_A)+\mathbb{C}$ and A^1, which we turn into a C^*-algebra by stealing the norm:

$$\|(a,\alpha)\| = \sup\{\|ax + \alpha x\| \mid \|x\| \leqslant 1,\ x \in A\}, \text{ for } (a,\alpha) \in A^1.$$

This gives us the pair (A^1, ι), where $\iota\colon A \to A^1$, $a \mapsto (a,0)$ is a unitalization of A: It is an isometry and hence injective: $\|\iota(a)\| = \|L_a\| = \|a\|$ for all $a \in A$. Moreover, $\iota(A)$ is an ideal since $\mathbb{K}_A(A_A)$ is an ideal. It remains to be shown that it is essential:

Suppose that $(x,\beta)\iota(A) = 0$ so for all $a \in A$ we have $xa + \beta a = 0$. If $\beta = 0$ then $\|(x,0)^*(x,0)\| = \|x\|^2 = 0$ so $x = 0$ and $(x,\beta) = (0,0)$.

Assume that $\beta \neq 0$, then $-\frac{x}{\beta}a = a$ for all $a \in A$ which implies that $a^* = a^*\left(-\frac{x}{\beta}\right)^*$. Similarly, with a^* instead we obtain $a = a\left(-\frac{x}{\beta}\right)^*$, since $a \in A$ is arbitrary. Letting $a = -\frac{x}{\beta}$, we see that $\frac{x^*}{\beta}$ is self-adjoint as well, since

$$-\frac{x}{\beta} = -\frac{x}{\beta}\left(-\frac{x}{\beta}\right)^* = \left(-\frac{x}{\beta}\right)^*,$$

and hence a unit of A, a contradiction. Therefore $\beta = 0$, and by the C^*-identity we obtain $(x,0) = (0,0)$ so $\iota(A)$ essential in A^1.

Example 1.3.5
The pair $(\mathcal{L}_A(A_A), L)$, $L\colon a \mapsto L_a$, is a unitalization of A - to see this we need only show that $\mathbb{K}_A(A_A)$ is essential in $\mathcal{L}_A(A_A)$: Suppose that $T\mathbb{K}_A(A_A) = 0$, so $TK = 0$ for all compact K on A, then with $\Theta_{x,y}\colon z \mapsto x\langle y,z\rangle = xy^*z$, we see by 1.2.8 that $T\Theta_{x,y} = \Theta_{Tx,y} = 0$ for all $x,y \in A$, which shows that for $x,y \in A$ we have $\Theta_{Tx,y}(y) = (Tx)(y^*y) = L_{Tx}(y^*y) = 0$, implying that $L_{Tx} = 0 \iff Tx = 0$ so $T = 0$.

And, it makes sense to talk about a 'largest' unitalization, in the sense that it has a certain universal property:

Definition 1.3.6. A unitalization (B,ι) of a C^*-algebra A is said to be *maximal* if to every other unitalization (C,ι'), there is a $*$-homomorphism $\varphi\colon C \to B$ such that diagram:

$$\begin{tikzcd} A \arrow[r, hook, "\iota"] \arrow[dr, hook, "\iota'"'] & B \\ & C \arrow[u, "\exists\varphi"'] \end{tikzcd} \tag{1.7}$$

commutes.

Definition 1.3.7. For a C^*-algebra A, we define the *multiplier algebra* of A to be $M(A) := \mathcal{L}_A(A_A)$.

It turns out that this is a maximal unitalization of A, as we shall see in a moment – but first we need a bit more work, for instance we need the notion of non-degenerate representations of a C^*-algebra on a Hilbert space to be extended to the case of Hilbert C^*-modules:

Definition 1.3.8. Let B be a C^*-algebra and E a Hilbert A-module. A homomorphism $\varphi\colon B \to \mathcal{L}_A(E)$ is *non-degenerate* if the set

$$\varphi(B)E := \operatorname{Span}\{\varphi(b)x \mid b \in B \text{ and } x \in E\},$$

is dense in E.

Example 1.3.9
In the proof of 1.2.21 we saw that the inclusion $\mathbb{K}_A(A_A) \subseteq \mathcal{L}_A(A_A)$ is non-degenerate for all C^*-algebras A.

Proposition 1.3.10. Let A and C be C^*-algebras, and let E denote a Hilbert A-module and let B be an ideal of C. If $\alpha\colon B \to \mathcal{L}_A(E)$ is a non-degenerate *-homomorphism, then it extends uniquely to a *homomorphism $\overline{\alpha}\colon C \to \mathcal{L}_A(E)$. If further $\iota(B)$ is an essential ideal of C, then the extension is injective.

Proof.
The last statement is clear, since $\ker\overline{\alpha} \cap B = \ker\alpha \cap B = 0$ which implies that $\ker\overline{\alpha} = 0$, since B is essential. Let (e_j) be an approximate identity of B. Define for $c \in C$ the operator $\overline{\alpha}(c)$ on $\alpha(B)X$ by

$$\overline{\alpha}(c)\left(\sum_{i=1}^n \alpha(b_i)(x_i)\right) := \sum_{i=1}^n \alpha(cb_i)(x_i).$$

This is well-defined and bounded, for we see that

$$\left\|\sum_{i=1}^n \alpha(cb_i)(x_i)\right\| = \lim_j \left\|\sum_{i=1}^n \alpha(\underbrace{ce_j}_{\in B})\alpha(b_i)(x_i)\right\| \leqslant \|c\| \left\|\sum_{i=1}^n \alpha(b_i)(x_i)\right\|.$$

This extends uniquely to a bounded operator on all of E, since $\alpha(B)E$ is dense. It is easy to see that $c \mapsto \overline{\alpha}(c)$ is a homomorphism. It remains to be shown that is is adjointable: Let (e_j) be an approximate identity for B, $c \in C$ and $x, y \in E$, then

$$\langle\overline{\alpha}(c)x, y\rangle = \lim_j\langle\underbrace{\alpha(ce_j)}_{\in\mathcal{L}_A(E)} x, y\rangle = \lim_j\langle x, \alpha(ce_j)^* y\rangle = \lim_j\langle x, \alpha(e_j^* c^*)y\rangle = \langle x, \overline{\alpha}(c)y\rangle,$$

this shows that $\overline{\alpha}(c^*)$ is an adjoint of $\overline{\alpha}(c)$, hence it is a *-homomorphism with values in $\mathcal{L}_A(E)$ □

Corollary 1.3.11. The inclusion of $\mathbb{K}_A(A_A) \subseteq \mathcal{L}_A(A_A)$ is a maximal unitalization.

Proof.
In 1.3.5 we saw that $\mathbb{K}_A(A_A)$ is an essential ideal of $\mathcal{L}_A(A_A)$ and nondegeneracy was seen in 1.3.9. By 1.3.10, the corollary follows. ☐

In particular, if $E = A_A$ and $C = M(B)$, then the above proposition guarantees an extension from B to $M(B)$ for all non-degenerate homomorphisms $\varphi\colon B \to M(A)$.

We shall now see that the homomorphism provided by the universal property of a maximal unitalization is unique, this is the final ingredient we shall need to show that $M(A)$ is a maximal unitalization of A:

Lemma 1.3.12. If $\iota\colon A \hookrightarrow B$ is a maximal unitalization and $j\colon A \hookrightarrow C$ is an essential embedding, then the homomorphism $\varphi\colon C \to B$ of 1.3.6 is unique and injective.

Proof.
Suppose that $\varphi, \psi\colon C \to B$ such that $\varphi \circ j = \psi \circ j = \iota$. Then, for $c \in C$, we see that

$$\varphi(c)\iota(a) - \psi(c)\iota(a) = \varphi(c)\varphi(cj(a)) - \psi(c)\psi(cj(a)) = \varphi(cj(a)) - \varphi(cj(a)) = 0,$$

for all $a \in A$, so essentiality of $\iota(A)$ in B and 1.3.2 implies that $\varphi = \psi$. For injectivity, note that $\ker\varphi \cap j(A) = 0$, hence $\ker\varphi = 0$, since $j(A)$ is essential in C. ☐

We may finally show that $(M(A), L)$ is a maximal unitalization for all C^*-algebras A:

Theorem 1.3.13. For any C^*-algebra A, the unitalization $(M(A), L)$ is maximal, and unique up to isomorphism

Proof.
For maximality, suppose that $j\colon A \hookrightarrow C$ is an essential embedding of A in a C^*-algebra C. We know that L is nondegenerate, hence by 1.3.10 we obtain and extension $\overline{L}\colon C \hookrightarrow M(A)$ such that $\overline{L} \circ j = L$. For uniqueness, suppose that (B, ι) is a maximal unitalization of A. By 1.3.10 and maximality of (B, ι) we obtain the following commutative diagram

$$\begin{tikzcd}
 & M(A) & \\
A \arrow[ur, hook, "L"'] \arrow[r, hook, "\iota"'] \arrow[dr, hook, "L"'] & B \arrow[u, hook, "\exists!\overline{L}"'] & \\
 & M(A) \arrow[u, hook, "\exists!\beta"'] \arrow[uu, bend right=60, "\mathrm{id}_{M(A)}"'] &
\end{tikzcd} \tag{1.8}$$

The identity operator on $M(A)$ is by 1.3.10 the unique operator satisfying $L = \mathrm{id}_{M(A)} \circ L$, but $\overline{L} \circ \beta \circ L = L$, so they are inverses of eachother hence isomorphisms. □

Often we want to be able to 'touch' the Multiplier Algebra of a C^*-algebra – and we can use the concretification as $\mathcal{L}_A(A_A)$, but there is (and other as well) another common way to decribe it:

Proposition 1.3.14. Let A, C be C^*-algebras and E be a Hilbert C^*-module. Suppose that $\alpha\colon A \to \mathcal{L}_C(E)$ is an injective nondegenerate homomorphism. Then α extends to an isomorphism of $M(A)$ with the idealizer $\mathcal{I}(A)$ of $\alpha(A)$ in $\mathcal{L}_C(E)$:

$$\mathcal{I}(A) = \{T \in \mathcal{L}_C(E) \mid T\alpha(A) \subseteq \alpha(A) \text{ and } \alpha(A)T \subseteq \alpha(T)\}.$$

Proof.
It is clear that $\alpha(A)$ is an ideal of the idealizer $\mathcal{I}(A)$, which is in fact essential by nondegeneracy of α: If $T\alpha(A) = 0$ then $T\alpha(A)E = 0$ hence $T = 0$.

By 1.3.13 it suffices to show that $(\mathcal{I}(A), \alpha)$ is a maximal unitalization of A. Let $\iota\colon A \hookrightarrow B$ be an essential embedding of A into a C^*-algebra B. Let $\overline{\alpha}\colon B \hookrightarrow \mathcal{L}_C(E)$ denote the injective extension of α from 1.3.10. We must show that $\overline{\alpha}(B) \subseteq \mathcal{I}(A)$: Let $b \in B$ and $a \in A$. Then, as $b\varphi(a) \in \varphi(A)$, we see that $\overline{\alpha}(b) \underbrace{\alpha(a)}_{=\overline{\alpha}(\iota(a))} = \overline{\alpha}(b\iota(a)) \in \alpha(A)$, as wanted. □

This allows us to, given any faithful nondegenerate representation of A on a Hilbert space $\mathcal{H}$, to think of $M(A)$ as the idealizer of A in $\mathbb{B}(\mathcal{H})$, and it immediately allows us to describe the Multiplier algebra of $\mathbb{K}_A(E)$:

Corollary 1.3.15. For all C^*-algebras A and Hilbert A-modules, we have $M(\mathbb{K}_A(E)) \cong \mathcal{L}_A(E)$.

Proof.
This follows from 1.3.14: As we have seen the inclusion $\mathbb{K}_A(E) \subseteq \mathcal{L}_A(E)$ is nondegenerate and $\mathcal{I}(\mathbb{K}_A(E)) = \mathcal{L}_A(E)$ since the compacts is an ideal. In particular $\mathcal{L}_A(E)$ has the strict topology: $T_i \to T$ strictly if and only if $T_iK \to TK$ and $KT_i \to KT$ for all $K \in \mathbb{K}_A(E)$. □

Using this, we can calculate both the adjointable operators and the Multiplier algebras in some settings:

Example 1.3.16
For the Hilbert $\mathbb{C}$-module $\mathcal{H}$, we obtain the identity that $M(\mathbb{K}(\mathcal{H})) = \mathbb{B}(\mathcal{H})$, as expected, since $\mathbb{B}(\mathcal{H})$ is the prototype of a 'large' abstract C^*-algebra.

A lot of the theory in functional analysis and the study of C^*-algebras is motivated by the study of abelian C^*-algebras of the form $C_0(\Omega)$ for a locally compact Hausdorff space. This construction is no exception: Unitalizations come naturally from compactifications, e.g., adjoining a unit to $C_0(\Omega)$ corresponds to the one-point compactification of Ω. It turns out that passing from $C_0(\Omega)$ to its multiplier algebra is connected to the largest compactification of Ω, namely the Stone-Čech compactification $\beta\Omega$ of Ω, which in fact is the protoexample of the multiplier construction.

Example 1.3.17
If Ω is a locally compact Hausdorff space, then $M(C_0(\Omega)) \cong C_b(\Omega) = C(\beta\Omega)$.

Proof.
The unital C^*-algebra $C_b(\Omega)$ is isomorphic to $C(Y)$ for some compact Hausdorff space Y, which can be realised as $\beta\Omega$ (see [16, p. 151]). Note that since $C_0(\Omega)$ separates points in Ω cf. Urysohn's lemma, it is an essential ideal of $C_b(\Omega)$, and the inclusion $C_0(\Omega) \hookrightarrow C_b(\Omega)$ is non-degenerate. Therefore 1.3.10 ensures a unique injection $*$-homomorphism $\varphi\colon C_b(\Omega) \to M(C_0(\Omega))$.

For surjectivity, suppose that $g \in M(C_0(\Omega))$ and let $(e_i) \subseteq C_0(\Omega)$ denote an approximate identity. Without loss of generality, assume that $g \geqslant 0$ in $M(C_0(\Omega))$. Then, for $x \in \Omega$, we have $\{ge_i(x)\}_i \subseteq [0, \|g\|]$ and $ge_i(x)$ is increasing, since e_i is and therefore it has a limit, which we denote by $h(x)$, which is bounded above by $\|g\|$, and it satisfies $hf = \lim ge_i f = gf \in C_0(X)$ for all $f \in C_0(\Omega)$.

We claim that $t \mapsto h(t)$ is a continuous map: Suppose that $t_\alpha \to t \in \Omega$, and without loss of generality assume that $t_\alpha \in K$ for all α, for some compact neighborhood of T. Use Urysohn's lemma to pick $f \in C_0(\Omega)$ such that $f\big|_K = 1$. Then, since $hf \in C_0(\Omega)$ we have

$$h(t) = f(t)h(t) = (hf)(t) = \lim_\alpha hf(t_\alpha) = \lim_\alpha h(t_\alpha),$$

so h is continuous. Let $f \in C_0(\Omega)$, then $\varphi(h)f = \varphi(h)\varphi(f) = \varphi(hf) = hf = gf$, so by non-degenerancy we see that $\varphi(h) = g$ so $g \in C_b(\Omega)$. □

In fact, one can show that when equipped with suitable orderings, then there is a correspondance between compactifications of locally compact spaces Ω and unitalizations of their associated C^*-algebras $C_0(\Omega)$.

One of the main interests in introducing the above discussed results will be clearer later on, but we will emphasize that it really is the strict topology on $M(A)$. This will be useful, since it coincides with another useful topology on bounded sets, called the $*$-strong topology, which we will define in a moment. This will be particularly useful when we recall that $\mathbb{B}(\mathcal{H}) = M(\mathbb{K}(\mathcal{H}))$, so that $\mathbb{B}(\mathcal{H})$ has the strict topology – this will allow us to expand and unify the representation theory of C^*-algebras and locally compact groups, which is the cornerstone of the setting which this thesis will cover.

Definition 1.3.18. Given a Hilbert A-module, the $*$-strong topology on $\mathcal{L}_A(E)$ is the topology for which $T_i \to T$ if and only if $T_i x \to Tx$ and $T_i^* x \to T^* x$ for all $x \in E$. It has a neighborhood basis consisting of sets of the form

$$V(T, x, \varepsilon) := \{S \in \mathcal{L}_A(E) \mid \|Sx - Tx\| + \|S^*x - T^*x\| < \varepsilon\},$$

for $T \in \mathcal{L}_A(E)$, $x \in E$ and $\varepsilon > 0$.

We will use $M_s(A)$ to mean $M(A)$ equipped with the strict topology, and we will write $\mathcal{L}_A(E)_{*-s}$ to mean we equip it with the $*$-strong topology. for $A = \mathbb{C}$, we see that $T_i \to T \in \mathbb{B}(\mathcal{H})$ in the $*$-strong topology if and only $T_i \to T$ and $T_i^* \to T$ in the strong operator topology. We said that we only have interest in the strict topology and the norm topology on $\mathcal{L}_A(E)$, and this is because the $*$-strong topology and strict topology coincides on bounded subsets:

Proposition 1.3.19. If E is a Hilbert A-module, if $T_i \to T \in \mathcal{L}_A(E)$ strictly, then $T_i \to T$ $*$-strongly. Moreover, on norm bounded subsets the topologies coincide.

Proof.
Suppose that $T_i \to T$ strictly. For $x \in E$, write $x = y\langle y, y\rangle$ for some unique $y \in E$ using Proposition 1.2.19. Let ${}_{\mathbb{K}_A(E)}\langle x, y\rangle = \Theta_{x,y} \in \mathbb{K}_A(E)$ be the associated $\mathbb{K}_A(E)$-valued inner-product on E from Proposition 1.2.14, so $x =_{\mathbb{K}_A(E)} \langle y, y\rangle(y)$. By definition of the strict topology on $\mathcal{L}_A(E) = M(\mathbb{K}_A(E))$, we see that

$$T_i(x) = T_i \circ_{\mathbb{K}_A(E)} \langle y, y\rangle(y) \to T \circ_{\mathbb{K}_A(E)} \langle y, y\rangle(y) = T(x).$$

Continuity of adjunction shows that $T_i^* \to T^*$ as well, hence also $T_i^* \to T^*$ strongly, so $T_i \to T$ $*$-strongly.

Suppose now (T_i) is bounded in norm by $K \geqslant 0$. For $x, y \in E$, if $T_i x \to Tx$ in norm, then $T_i\Theta_{x,y} \to T\Theta_{x,y}$ in norm:

$$\|T_i\Theta_{x,y} - T\Theta_{x,y}\| \leqslant \sup_{\|z\| \leqslant 1} \|T_i x - Tx\| \, \|\langle y, z\rangle\| \to 0,$$

and similarly $T_i^*\Theta_{x,y} \to T^*\Theta_{x,y}$. By approximating compact operators with finite linear combinations of $\Theta_{x,y}$'s, an easy $\frac{\varepsilon}{3K} > 0$ argument shows that $T_i K \to TK$ and $KT_i \to KT$ for $K \in \mathbb{K}_A(E)$ and similarly $T_i^* K \to T_i^* K$ and $KT_i^* \to KT$. $\square$

In particular, for a net of unitary operators on a Hilbert A-module, strict and $*$-strong convergence is the same. We shall utilize this fact repeatedly later on.

Crossed Products with Topological Groups

Given a C^*-dynamical system (A, G, α) with G discrete, it is not a difficult task to extend the representation theory of G and A to the $*$-algebra $C_c(G, A)$, and thus create new C^*-algebras through completion of it. Examples include the reduced- and full crossed product of A and G, $A \rtimes_{\alpha,r} G$ and $A \rtimes_\alpha G$.

However, when G is not discrete, we run into issues since we can not assume that a unitary representation of G is continuous in norm, which is required by classic constructions seen e.g., in [5]. In this chapter we will chapter develop the necessary theory to define the topological analogues of these techniques, i.e., for when G is a locally compact Hausdorff group.

We assume familiarity with basic theory of analysis in topological groups, basic representation theory of topological groups and Fourier analysis thereon and will not go lengths to explain results from this field, for a reference, we recommend the excellent books on the subject: [8], [2] and [9].

2.1 Integral Forms And Representation Theory

Throughout this chapter, we let G be a locally compact topological group and A a C^*-algebra, with no added assumptions unless otherwise specified. We will write $G \overset{\alpha}{\curvearrowright} A$ to specify that $\alpha\colon G \to \mathrm{Aut}(A)$ is an action of G on A, i.e., a strongly continuous group homomorphism. We will always fix a base left Haar measure μ on G, whose integrand we will denote by $\mathrm{d}t$, with modular function $\Delta\colon G \to (0, \infty)$.

define this

The main motivation of the theory we are about to develop is this: Given a covariant representation $(\pi, u, \mathcal{H})$ of (A, G, α), we wish to construct a representation $\pi \rtimes u$ of $C_c(G, \mathbb{B}(\mathcal{H}))$ satisfying

$$\pi \rtimes u(f) = \int \pi(f(t)) u_t \mathrm{d}t.$$

The problem at hand is, a priori u_t is not norm continuous, so we don't know if the integrand is integrable. We will however show that u_t is strictly convergent and use this to define our integral by applying it to $M(\mathbb{B}(\mathcal{H})) = \mathbb{B}(\mathcal{H})$.

For $f \in C_c(G, A)$, the map $s \mapsto \|f(s)\|$ is in $C_c(G)$, and we define the number $\|f\|_1 \in [0, \infty)$ to be

$$\|f\|_1 = \int_G \|f(t)\| \,\mathrm{d}t.$$

For $A = \mathbb{C}$, it is well known that given $f \in C_c(G, A)$ and $\varepsilon > 0$, there is a neighborhood V of $e \in G$ such that $sr^{-1} \in V$ or $s^{-1}r \in V$ implies $|f(s) - f(r)| < \varepsilon$, i.e., f is left- and right uniformly continuous. The statement holds for general A, and the proof is identical, see e.g. [8, Proposition 2.6] or [23, Lemma 1.88]

Lemma 2.1.1. For $f \in C_c(G, A)$ and $\varepsilon > 0$, there is a neighborhood V of $e \in G$ such that either $sr^{-1} \in V$ or $s^{-1}r \in V$ implies

$$\|f(s) - f(r)\| < \varepsilon.$$

We will use $\mathcal{D}$ to denote an arbitrary complex Banach space, and for $f \in C_c(G)$ and $x \in \mathcal{D}$, we denote by $f \otimes x$ the map $G \ni t \mapsto f(t)x \in \mathcal{D}$. It is then clear that $f \otimes x \in C_c(G, \mathcal{D})$. We introduce a 'topology' or a name for a special kind of convergence on $C_c(G, \mathcal{D})$, which will prove useful:

Definition 2.1.2. We say that a net $\{f_i\}_{i \in I} \subseteq C_c(G, \mathcal{D})$ converges to a function f on $C_c(G, \mathcal{D})$ in the *inductive limit topology* if $f_i \to f$ uniformly and there is a compact set $K \subseteq G$ and $i_0 \in I$ such that for $i \geqslant i_0$ we have $f_i\big|_{K^c} = 0$, i.e., $\mathrm{supp} f_i \subseteq K$, $i \geqslant i_0$.

Lemma 2.1.3. Let $\mathcal{D}_0 \subseteq \mathcal{D}$ be any dense subset. Then the set

$$C_c(G) \odot \mathcal{D}_0 := \mathrm{Span}\,\{f \otimes x \mid f \in C_c(G) \text{ and } x \in \mathcal{D}_0\}$$

is dense in the inductive limit topology on $C_c(G, \mathcal{D})$.

Proof.
Let $\varphi \in C_c(G, \mathcal{D})$ and $\varepsilon > 0$ be arbitrary. Let $K = \mathrm{supp}\varphi$ and let W denote an arbitrary compact neighborhood of $e \in G$, and pick a symmetric neighborhood $e \in V_\varepsilon \subseteq W$ such that $sr^{-1} \in V_\varepsilon$ implies $\|f(s) - f(r)\| < \frac{\varepsilon}{\mu(WK)}$. Since K is compact, there is a finite set $\{r_1, \ldots, r_n\} \subseteq K$ such that $K \subseteq \bigcup_{1 \leqslant i \leqslant n} V_\varepsilon r_i$. Recall that every locally compact Hausdorff group is paracompact and the cover $K^c \cup \left(\bigcup_{1 \leqslant i \leqslant n} V_\varepsilon r_i\right)$ admits a partition of unity $\{f_j\}_{j=0}^n$ such that $\mathrm{supp} f_0 \subseteq K^c$ and $\mathrm{supp} f_i \subseteq V_\varepsilon r_i$ for $i = 1, \ldots, n$.

Using this, we define

$$g_\varepsilon := \sum_{i=1}^{n} f_i \otimes x_i,$$

where $x_i \in \mathcal{D}_0$ such that $\|x_i - \varphi(r_i)\| < \frac{\varepsilon}{2\mu(WK)}$ for $i = 1, \ldots, n$. Then $g_\varepsilon \in C_c(G) \odot \mathcal{D}_0$ with $\mathrm{supp} g_\varepsilon \subseteq V_\varepsilon K \subseteq WK$. The partition of unity satisfies $\sum_{i=1}^{n} f_i(s) \leqslant 1$ for all $s \in G$ with equality for $s \in K$, so in particular $\sum_{1 \leqslant i \leqslant n} f_i(s)\varphi(s) = \varphi(s)$ for all $s \in G$. Hence we see that for $s \in G$:

$$\begin{aligned}\|\varphi(s) - g_\varepsilon(s)\| = \left\|\sum_{i=1}^{n} f_i(s)(\varphi(s) - \varphi(r_i) + \varphi(r_i) - x_i)\right\| &\leqslant \sum_{i=1}^{n} f_i(s)(\|\varphi(s) - \varphi(r_i)\| + \|\varphi(r_i) - x_i\|) \\ &\leqslant \frac{\varepsilon}{\mu(WK)},\end{aligned}$$

Since ε was arbitrary and W, K didn't depend on ε and $\mathrm{supp} g_\varepsilon \subseteq WK$, we see that $g_\varepsilon \to f$ in the inductive limit topology as $\varepsilon \to 0$. □

We will also need the following result, which can be found in [20, Theorem 3.20, part (b, c)]

Lemma 2.1.4. If $K \subseteq \mathcal{D}$ is compact, then the convex hull convex hull $\mathrm{conv}(K)$ is totally bounded, hence it has compact closure.

Definition 2.1.5. Let $f \in C_c(G, \mathcal{D})$ and $\varphi \in \mathcal{D}^*$. Then we can define a bounded linear functional L_f on $\mathcal{D}^*$ by

$$L_f(\varphi) := \int_G \varphi(f(s))\mathrm{d}s,$$

since $|L_f(\varphi)| \leqslant \int_G \|\varphi\| \, \|f(s)\| \, \mathrm{d}s = \|f\|_1 \, \|\varphi\|$.

Note 2.1.6
If $(f_i) \subseteq C_c(G, \mathcal{D})$ is a net which converges in the inductive limit topology to $f \in C_c(G, \mathcal{D})$, i.e., $f_i \to f$ uniformly on G and for some $K \subseteq G$ compact we have $\mathrm{supp}\, f_i \subseteq K$ eventually (and hence also $\mathrm{supp}\, f \subseteq K$), then

$$\|f_i - f\|_1 = \int_G \|f_i(s) - f(s)\| \, \mathrm{d}s \leqslant \|f_i - f\| \, \mu(K) \to 0,$$

so $f_i \to f$ in the L^1-norm, and hence by the above we see that $|L_{f_i}(\varphi) - L_f(\varphi)| \leqslant \|f_i - f\|_1 \, \|\varphi\| \to 0$ for all $\varphi \in \mathcal{D}^*$, i.e., $L_{f_i} \to L_f$ in the weak* topology of $\mathcal{D}^{**}$.

We will see that L_f is equal to $\widehat{a}\colon \varphi \to \varphi(a)$ for some unique $a \in \mathcal{D}$. Let $\iota\colon \mathcal{D} \to \mathcal{D}^{**}$ denote the inclusion $\iota(a) = \widehat{a}$, then we have

Lemma 2.1.7. If $f \in C_c(G, \mathcal{D})$, then $L_f \in \iota(\mathcal{D})$.

Proof.
Let W be a compact neighborhood of $\operatorname{supp} f$, and let $K := f(G) \cup \{0\} \subseteq \mathcal{D}$. Then K is compact, so by 2.1.4 the set $C := \overline{\operatorname{conv}}\{K\}$ is compact.

Fix $\varepsilon > 0$, and pick a neighborhood V of $e \in G$ such that $\|f(s) - f(r)\| < \varepsilon$ for $s^{-1}r \in V$. Assume without loss of generality that $\operatorname{supp} f V \subseteq W$, since we can make V arbitrarily small. Using compactness of $\operatorname{supp} f$, pick $s_1, \dots, s_n \in \operatorname{supp} f$ such that $\{s_i V\}$ covers $\operatorname{supp} f$ (recall that $e \in V$).

Since $\{s_i V\} \subseteq W$, it has compact closure, so we may pick a partition of unity $\{\varphi_i\}_{i=1}^n$ of $K \subseteq \bigcup_{i=1}^n s_i V$. Using this, we define

$$g_\varepsilon(s) := \sum_{i=1}^n f(s_i)\varphi(s),$$

so $\operatorname{supp} g_\varepsilon \subseteq W$. If $s \in \left(\bigcup_{i=1}^n s_i V\right)^c$ then $f(s) = g_\varepsilon(s) = 0$, and otherwise we see that if $s \in s_i V$, then $s_i^{-1} s \in V$ and $\|f(s) - f(s_i)\| < \varepsilon$, and hence

$$\|f(s) - g_\varepsilon\| \leqslant \sum_{i=1}^n \varphi(s) \|f(s) - f(s_i)\| < \varepsilon.$$

Since $s \mapsto \sum_{i=1}^n \varphi(s) \leqslant 1$ and has compact support and $\bigcup_{i=1}^n \operatorname{supp} \varphi_i \subseteq W$, we have $\sum_{i=1}^n \int_G \varphi_i(s) \mathrm{d}s \leqslant \mu(W) < \infty$, so it is equal to some number D. For $\psi \in \mathcal{D}^*$ we see that

$$L_{g_\varepsilon}(\psi) = \int_G \psi\left(\sum_{i=1}^n \varphi(s) f(s_i)\right) \mathrm{d}s = \sum_{i=1}^n \int_G \varphi(s) \mathrm{d}s \psi(f(s_i)) = \sum_{i=1}^n \int_G \varphi(s) \mathrm{d}s\, \underbrace{\iota(f(s_i))}_{\in \iota(C)}(\psi)$$

so $L_{g_\varepsilon} \in D\iota(C) \subseteq \mu(W)\iota(C)$, since $0 \in C$ so we can shrink the scalar $\mu(W)$ to D. For $\varepsilon \to 0$, we see that $g_\varepsilon \to f$ in the inductive limit topology and $L_{g_\varepsilon} \in \mu(W)\iota(C)$.

Since C is compact and convex, it is weakly compact which by definition is equivalent to $\iota(C)$ being compact in the weak*-topology of $\mathcal{D}^{**}$, and since $g_\varepsilon \to f$ in the inductive limit topology, we get from 2.1.6 that $L_{g_\varepsilon} \to L_f$ in the weak *-topology on $\mathcal{D}^{**}$. In particular, since $\iota(C)$ is weak*-compact, it is closed and hence $\mu(W)\iota(C)$ is closed in the weak*-topology so $L_f \in \mu(W)\iota(C) \subseteq \iota(\mathcal{D})$. $\square$

And we have a lemma emphasizing important properties of L_f:

Lemma 2.1.8. There exists a unique linear map $I \colon C_c(G, \mathcal{D}) \to \mathcal{D}$,

$$f \mapsto \int_G f(s) \mathrm{d}s$$

such that for all $f \in C_c(G, \mathcal{D})$ we have

$$\varphi\left(\int_G f(s\right) \mathrm{d}s) = \int_G \varphi(f(s))\mathrm{d}s, \quad \text{for all } \varphi \in \mathcal{D}^*, \tag{2.1}$$

$$\left\| \int_G f(s)\mathrm{d}s \right\| \leqslant \|f\|_1 \tag{2.2}$$

$$\int_G (g \otimes x)(s)\mathrm{d}s = x \int_G g(s)\mathrm{d}s, \text{ for all } g \otimes x \in C_c(G) \odot \mathcal{D}, \tag{2.3}$$

and if $T\colon \mathcal{D} \to D$ is a bounded linear operator between Banach spaces. Then

$$T\left(\int_G f(s)\mathrm{d}s\right) = \int_G T(f(s))\mathrm{d}s.$$

Proof.
Letting $I(f) := \iota^{-1}(L_f)$, with L_f as in 2.1.7, then $f \mapsto I(f)$ instantly satisfies the first properties, perhaps with exception of uniqueness, which follows from the fact that B^* separates points of B. For the last, suppose that $T\colon \mathcal{D} \to D$ is a bounded linear operator between Banach spaces. Then $T \circ f \in C_c(G, D)$ for all $f \in C_c(G, \mathcal{D})$, and $\varphi \circ T \in \mathcal{D}^*$ for all $\varphi \in D^*$. Hence

$$\varphi \circ T\left(\int_G f(s)\mathrm{d}s\right) = \int_G \varphi(T \circ f(s))\mathrm{d}s = \varphi\left(\int_G T(f(s))\mathrm{d}s\right),$$

for all $\varphi \in D^*$, finishing the proof. □

In light of the above, we may give the following definition:

Definition 2.1.9. For $f \in C_c(G, \mathcal{D})$, define the integral of f as $I(f) := \iota^{-1}(L_f)$, which we denote by $\int_G f(s)\mathrm{d}s := I(f)$.

In particular, if we let A be a C^*-algebra, then we obtain the following useful result:

Proposition 2.1.10. For $f \in C_c(G, A)$, if $\pi\colon A \to \mathbb{B}(\mathcal{H})$ is a representation of A, then for all $\xi, \eta \in \mathcal{H}$ the integral of 2.1.8 satisfies

$$\langle \pi\left(\int_G f(s)\mathrm{d}s)\right) \xi, \eta\rangle = \int_G \langle \pi(f(s))\xi, \eta\rangle \mathrm{d}s,$$

and

$$\left(\int_G f(s)\mathrm{d}s\right)^* = \int_G f(s)^*\mathrm{d}s$$

and for $a, b \in M(A)$ we have

$$a \int_G f(s)\mathrm{d}s b = \int_G a f(s) b \mathrm{d}s.$$

Proof.
Define the vector-state $\varphi_\pi \colon a \mapsto \langle \pi(a)\xi, \eta\rangle$. Then, $\varphi_\pi \in A^*$, and the first statement follows. Suppose now that π is a faithful representation, then since $\int_G \pi(f(s))\mathrm{d}s = \pi\left(\int_G f(s)\mathrm{d}s\right)$, we see that the second statement holds as well when applied to $\varphi_\pi(I(f)^*)$. Finally if $a, b \in M(A)$ then

$$\begin{aligned}
\langle \pi\left(a \int_G f(s)\mathrm{d}s b\right)\xi, \eta\rangle &= \langle \overline{\pi}(a)\pi\left(\int_G f(s)\mathrm{d}s\right)\overline{\pi}(b)\xi, \eta\rangle \\
&= \langle \pi\left(\int_G f(s)\mathrm{d}s\right)(\overline{\pi}(b)\xi), \overline{\pi}(a^*)\eta\rangle \\
&= \int_G \langle \pi(f(s))\overline{\pi}(b)\xi, \overline{\pi}(a^*)\eta\rangle \mathrm{d}s \\
&= \int_G \langle \pi(af(s)b)\xi, \eta\rangle \mathrm{d}s \\
&= \langle \pi\left(\int_G a f(s) b \mathrm{d}s\right)\xi, \eta\rangle.
\end{aligned}$$

□

As mentioned, we will need to consider maps of the form of the form $s \mapsto \pi(f(s))U_s$ where $f \in C_c(G, A)$, $\pi \colon A \to \mathbb{B}(\mathcal{H})$ and $U \colon G \to \mathcal{U}(\mathcal{H})$ is strongly continuous. In order to use the above theorem to make sense, we will want to change A out with $M_s(A)$, i.e., the multiplier algebra of A equipped with the strict topology, and combine it with the following lemma:

Lemma 2.1.11. Let E be a Hilbert A-module, and let $u \colon G \to \mathcal{U}(E)$ be a group homomorphism from G to the group of unitary operators on E, i.e., operators $T \in \mathcal{L}_A(E)$ such that $T^*T = TT^* = I_E$. Then u is strictly continous if and only if u is strongly continuous.

Proof.
We saw in proposition 1.3.19 that on bounded sets, the strict and the $*$-strong topology coincided. In particular, since $\mathcal{L}_A(E)$ is a C^*-algebra, the set of unitary operators is a subset of the unit sphere. Thus, it suffices to show that if $s \mapsto u_s x$ is continuous for x then $s \mapsto u_s^* x$ is continuous. Let $x \in E$, then

$$\|u_s^* x - u_t^* x\|_A^2 = \underbrace{\|u_s^* x\|_A^2}_{=\|x\|_A^2} + \underbrace{\|u_t^* x\|_A^2}_{=\|x\|_A^2} - \langle \underbrace{(u_t u_s^* + u_s u_t^*)}_{\xrightarrow{s \to t} 2I_E} x, x\rangle \xrightarrow{s \to t} 0,$$

finishing the proof, since $x \in E$ was arbitrary. □

We now generalize the result of proposition 2.1.10 to cover the class of functions which we shall deal with later on, and we recall that any non-degenerate representation π of a C^*-algebra A on a Hilbert space H extends to a representation $\overline{\pi}$ of $M(A)$ on H, which is faithful if π is.

Theorem 2.1.12. Given a C^*-algebra A, there is a unique linear map $I\colon C_c(G, M_s(A)) \to M(A)$, $f \mapsto \int_G f(s)\mathrm{d}s$, such that for any non-degenerate representation $\pi\colon A \to \mathbb{B}(\mathcal{H})$ and all $\xi, \eta \in \mathcal{H}$ the identity

$$\langle \overline{\pi}\left(\int_G f(s)\mathrm{d}s\right)\xi, \eta\rangle = \int_G \langle \overline{\pi}(f(s))\xi, \eta\rangle \mathrm{d}s, \tag{2.4}$$

and for all $f \in C_c(G, M_s(A))$

$$\|I(f)\mathrm{d}s\| \leqslant \|f\|_\infty \mu(\operatorname{supp} f). \tag{2.5}$$

Also, the identities

$$\int_G f(s)^*\mathrm{d}s = \left(\int_G f(s)\mathrm{d}s\right)^*, \tag{2.6}$$

and

$$\int_G af(s)b\mathrm{d}s = a\left(\int_G f(s)\mathrm{d}s\right)b \tag{2.7}$$

hold for all $f \in C_c(G, M_s(A))$ and $a, b \in M(A)$. If $T\colon A \to M(B)$ is a non-degenerate homomorphism of C^*-algebras, then

$$\overline{T}\left(\int_G f(s)\mathrm{d}s\right) = \int_G \overline{T}(f(s))\mathrm{d}s. \tag{2.8}$$

Proof.
If I exists, then letting π be a faithful representation of A we see that I must be unique by the first identity. For $f \in C_c(G, M_s(A))$ and $a \in A$, it is easy to see that if $f_a\colon s \mapsto f(s)a$, then $f_a \in C_c(G, A)$. This allows us to use proposition 2.1.10 to define the integral $\int_G f(s)a\mathrm{d}s$ for all $f \in C_c(G, M_s(A))$ and $a \in A$. Define $L_f\colon A \to A$ by

$$L_f(a) = \int_G f(s)a\mathrm{d}s, \quad \text{for } a \in A.$$

Then $L_f \in \mathcal{L}_A(A_A)$, for if $a, b \in A$ then proposition 2.1.10 implies that

$$\langle L_f a, b\rangle_{A_A} = (L_f a)^* b = \int_G (f(s)a)^*\mathrm{d}s b = \int_G a^* f(s)^*\mathrm{d}s = a^* \int_G f(s)^* b\mathrm{d}s = \langle a, L_{f^*} b\rangle_{A_A}.$$

So L_{f*} is an adjoint of L_f. Let $\int_G f(s)\mathrm{d}s := L_f$ for all $f \in C_c(G, M_s(A)$, then $L_f \in \mathcal{L}_A(A_A) = M(A)$, and it is not hard to see that the assignment $f \mapsto L_f$ is linear.

Again, by proposition 2.1.10, we see that a non-degenerate representation $\pi\colon A \to \mathbb{B}(\mathcal{H})$ with $a \in A$ and $\xi, \eta \in \mathcal{H}$ that

$$\begin{aligned}\left\langle \overline{\pi}\left(\int_G f(s)\mathrm{d}s\right)\pi(a)\xi, \eta\right\rangle = \langle \pi\left(f(s)a\mathrm{d}s\right)\xi, \eta\rangle &= \int_G \langle \pi(f(s)a)\xi, \eta\rangle \mathrm{d}s \\ &= \int_G \langle \overline{\pi}(f(s))\pi(a)\xi, \eta\rangle \mathrm{d}s,\end{aligned}$$

which shows eq. (2.4), since π is non-degenerate. The analogues of proposition 2.1.10 follows similarly. Let $a \in A$ with $\|a\| \leqslant 1$, then by proposition 2.1.10 applied to $f_a \in C_c(G, A)$, we see that

$$\left\| \int_G f(s)\mathrm{d}sa \right\| \leqslant \|f_a\|_1 = \int_G \|f(s)a\| \,\mathrm{d}s \leqslant \int_G 1_{\operatorname{supp} f} \|f\|_\infty \,\mathrm{d}s = \mu(\operatorname{supp} f)\, \|f\|_\infty \,,$$

Hence $\left\|\int_G f(s)\mathrm{d}s\right\| = \sup_{\|a\|\leqslant 1} \left\|\int_G f(s)\mathrm{d}sa\right\| \leqslant \|f\|_\infty \mu(\operatorname{supp} f)$. Let $T\colon A \to M(B)$ be a non-degenerate homomorphism, with unique extension $\overline{T}\colon M(A) \to M(B)$, and hence for all $a \in A$ and $b \in B$ we have

$$\begin{aligned}\overline{T}\left(\int_G f(s)\mathrm{d}s\right)T(a)b = T\left(\int_G f(s)a\mathrm{d}s\right)b = \int_G T(f(s)a)b\mathrm{d}s &= \int_G \overline{T}(f(s))T(a)b\mathrm{d}s \\ &= \int_G \overline{T}(f(s))\mathrm{d}sT(a)b,\end{aligned}$$

hence $\overline{T}\left(\int_G f(s)\mathrm{d}s\right) = \int_G \overline{T}(f(s))\mathrm{d}s$ by non-degeneracy. □

In the construction of the general crossed product and other C^*-algebras associated to a C^*-dynamical system (A, G, α) we will need to deal with the algebra $C_c(G, A)$ quite a lot. We will want to develop a sort of twisted-convolution incorporating the action α of G. However, in order to do this properly, we will need a form of Fubini's theorem for the above:

Lemma 2.1.13. For $f \in C_c(G \times G, B)$, the function

$$t \mapsto \int_G f(t, s)\mathrm{d}s$$

is an element of $C_c(G, B)$.

Proof.
It is easy to see that if $F\colon G \to B$ is the map $t \mapsto \int_G f(t, s)\mathrm{d}s$, then $\operatorname{supp} F$ is compact. We show continuity. Suppose that $x_i \to x \in G$ and let $\varepsilon > 0$. Pick compact $K_1, K_2 \subseteq G$ covering the support of f, i.e., such that $\operatorname{supp} f \subseteq K_1 \times K_2$. Note that $f(x_i, y) \to f(x, y)$ uniformly:

Assume towards a contradiction that $f(x_i, y)$ doesn't converge uniformly to $f(x, y)$. Then we can find a net $(r_j) \subseteq G$ and $\varepsilon_0 > 0$ such that

$$\|f(x_i, r_j) - f(x, r_j)\| \geqslant \varepsilon_0,$$

for all i, j. This shows that $r_j \in K_2$ for all j, hence we may assume that $r_j \to r \in K_2$ (up to passing to a subnet). But then for sufficiently large i, j we would have $\|f(x_i, r_j) - f(x, r_j)\| < \varepsilon_0$. Hence we can assume that $\|F(x_i) - F(x)\| < \frac{\varepsilon}{\mu(K_2)}$. Then,

$$\left\| \int_G f(x_i, s)\mathrm{d}s - \int_G f(x, s)\mathrm{d}s \right\| \leqslant \int_{K_2} \|f(x_i, s) - f(x, s)\| \ ds + \int_{K_2^c} \|f(x_i, s) - f(x, s)\| \,\mathrm{d}s \leqslant \varepsilon,$$

as $\operatorname{supp} f \subseteq K_1 \times K_2$, so $F \in C_c(G, B)$. $\square$

Corollary 2.1.14. Replacing the integral of lemma 2.1.13 with $\int_G f(t, s)\mathrm{d}t$ the statement still holds.

Theorem 2.1.15 (Fubini for C^*-valued integrals)**.** Suppose that $f \in C_c(G \times G, M_s(A))$, then

$$s \mapsto \int_G^{M_s(A)} f(s, t)\mathrm{d}t \text{ and } s \mapsto \int_G^{M_s(A)} f(t, s)\mathrm{d}t$$

are elements of $C_c(G, M_s(A))$ and moreover

$$\int_G^{M_s(A)} \int_G^{M_s(A)} f(s, t)\mathrm{d}s\mathrm{d}t = \int_G^{M_s(A)} \int_G^{M_s(A)} f(s, t)\mathrm{d}t\mathrm{d}s.$$

Proof.
Define for a fixed s, the map $t \mapsto f(s, t)$ is in $C_c(G, M_s(A)$, hence we can define $\int_G f(s, t)\mathrm{d}t$ and $\int_G f(t, s)\mathrm{d}t$ as in Theorem 2.1.12. Recall that $M(A) = M(\mathbb{K}_A(A_A))$, so the strict topology is induced by multiplication operators on A, so it suffices to show that $s \mapsto a\left(\int_G f(s, t)\mathrm{d}t\right)$ and $s \mapsto \left(\int_G f(s, t)\mathrm{d}t\right) a$ are in $C_c(G, A)$ for all $a \in A$ for the first claim. By Theorem 2.1.12, we have

$$a\left(\int_G f(s, t)\mathrm{d}t\right) = \int_G af(s, t)\mathrm{d}t,$$

for all $a \in A$ and $s \in G$, and similarly $\left(\int_G f(s, t)\mathrm{d}t\right) a = \int_G f(s, t)a\mathrm{d}t$. The maps

$$(s, t) \mapsto af(s, t) \text{ and } (s, t) \mapsto f(s, t)a,$$

are in $C_c(G \times G, A)$, clearly, so Lemma 2.1.13 ensures that $s \mapsto \int_G^{M_s(A)} f(s, t)\mathrm{d}t$ and $s \mapsto \int_G^{M_s(A)} f(t, s)\mathrm{d}t$ are in $C_c(G, M_s(A)$.

For the last claim, we now see that the double integrals are well-defined by Theorem 2.1.12, so by letting A be faithfully representted on $\mathbb{B}(\mathcal{H})$, we may obtain the equality using the regular Fubini's theorem for scalar valued integrals (see e.g., [21]) applied to an orthonormal basis through the inner-product, i.e., $\square$

2.2 Crossed Products

In analysis, an object of interest is the group algebra $L^1(G)$ for locally compact groups G. One can turn it into a $*$-algebra by showing that the convolution gives rise to a well-defined multiplication, and one can use the modular function to define an involution. Analogously to this, we do the same for A valued functions:

Definition 2.2.1. Suppose that $G \overset{\alpha}{\curvearrowright} A$. We let

$$C_c(G, A) := \{f\colon G \to A \mid f \text{ continuous and compactly supported}\}.$$

For $f, g \in C_c(G, A)$, we define the α-twisted convolution of f and g at $t \in G$ by

$$(f *_\alpha g)(t) := \int_G f(s)\alpha_s(g(s^{-1}t))\mathrm{d}s,$$

which is well defined since the map $(s,t) \mapsto f(s)\alpha_s g(s^{-1}t)$ is in $C_c(G\times G, A)$ and lemma 2.1.13 applies. Moreover, we define an α-twisted involution of $f \in C_c(G, A)$ at $t \in G$ by

$$f^*(t) := \Delta(t^{-1})\alpha_t(f(t^{-1})^*).$$

Straight forward calculations similar to the classic ones show that $C_c(G, A)$ becomes a $*$-algebra with multiplication given by the α-twisted convolution and involution given by the α-twisted involution. We denote the associated $*$-algebra by $C_c(G, A, \alpha)$.

To $f \in C_c(G, A, \alpha)$, we associate the number $\|f\|_1 \in [0, \infty)$ by

$$\|f\|_1 := \int_G \|f(t)\| \, \mathrm{d}t.$$

Straightforward calculations show that $f \mapsto \|f\|_1$ is a norm, and we denote the completion of $C_c(G, A, \alpha)$ in $\|\cdot\|_1$ by $L^1(G, A, \alpha)$.

For discrete and second-countable groups G acting on a C^*-algebra A via an action α, we may easily construct a $*$-representation of $C_c(G, A)$ from a covariant representation (π, u) by letting $\pi \rtimes u\colon C_c(A, G, \alpha) \to \mathbb{B}(\mathcal{H})$ be the map

$$\sum_{g\in G} a_g\delta_g \mapsto \pi(a_g)u_g,$$

however as discussed previously, this is not enough when G is locally compact, since we can't assume that u is continuous with respect to the norm on $\mathbb{B}(\mathcal{H})$. However, in the previous section we constructed the tools to construct the desired generalization:

Lemma 2.2.2. Suppose that $G \overset{\alpha}{\curvearrowright} A$ with G locally compact. If (π, u) is a covariant representation of the associated C^*-dynamical system, then for all $f \in C_c(G, A)$, the map

$$s \mapsto \pi(f(s))u_s, \text{ for } s \in G$$

is in $C_c(G, \mathbb{B}_s(\mathcal{H}))$, where $\mathbb{B}_s(\mathcal{H})$ indicates that we endow $\mathbb{B}(\mathcal{H})$ with the strict topology of $\mathbb{K}(\mathcal{H})$.

Proof.
By assumption, u is strongly continuous, which is equivalent to being strictly continuous by Lemma 2.1.11. Since $f \in C_c(G, A)$, we see that $\pi \circ f \in C_c(G, \mathbb{B}(\mathcal{H}))$, and norm continuity implies strict continuity, so $s \mapsto \pi(f(s))u_s$ is an element of $C_c(G, \mathbb{B}_s(\mathcal{H}))$. $\square$

The above and 2.1.12 shows that given a covariant representation (π, u) of $G \overset{\alpha}{\curvearrowright} A$, there is a well-defined linear map $\pi \rtimes u\colon C_c(G, A) \to \mathbb{B}(\mathcal{H})$ given by

$$\pi \rtimes u(f) = \int_G^{\mathbb{B}(\mathcal{H})} \pi(f(s))u_s \mathrm{d}s,$$

We shall see that this representation has a lot of nice properties:

Proposition 2.2.3. Suppose that (π, u) is a covariant representation of a dynamical system $G \overset{\alpha}{\curvearrowright} A$, with G locally compact. Then the map $\pi \rtimes u$ is a $*$-representation of $C_c(G, A, \alpha)$ on $\mathbb{B}(\mathcal{H})$ which is norm-decreasing. We call $\pi \rtimes u$ the *integrated form of π and u*. If π is non-degenerate, then $\pi \rtimes u$ is as well.

Proof.
Let $\xi, \eta \in \mathcal{H}$ and $f \in C_c(G, A)$, then

$$|\langle \pi \rtimes u(f)\xi, \eta\rangle| \leqslant \int_G |\langle \pi(f(s))u_s\xi, \eta\rangle| \mathrm{d}s \leqslant \int_G \|\pi(f(s))\| \, \|\xi\| \, \|\eta\| \, \mathrm{d}s \leqslant \|f\|_1 \, \|\xi\| \, \|\eta\|$$

Since ξ, η where arbitrary, we have $\|\pi \rtimes u(f)\| \leqslant \|f\|_1$ for all $f \in C_c(G, A)$.

Let $f \in C_c(G, A, \alpha)$, then

$$\begin{aligned}
\pi \rtimes u(f)^* = \int_G^{\mathbb{B}(\mathcal{H})} u_s^* \pi(f(s))^* \mathrm{d}s &= \int_G^{\mathbb{B}(\mathcal{H})} u_{s^{-1}} \pi(f(s)^*) \mathrm{d}s \\
&= \int_G^{\mathbb{B}(\mathcal{H})} \Delta(s)^{-1} u_s \pi(f(s^{-1})^*) \mathrm{d}s \\
&= \int_G^{\mathbb{B}(\mathcal{H})} \Delta(s)^{-1} \pi(\alpha_s(f(s^{-1})^*)) u_s \mathrm{d}s \\
&= \int_G^{\mathbb{B}(\mathcal{H})} \pi\left(\underbrace{\alpha_s\left(\Delta(s)^{-1} f(s^{-1})^*\right)}_{=f^*(s)}\right) u_s \mathrm{d}s \\
&= \int_G^{\mathbb{B}(\mathcal{H})} \pi(f^*(s)) u_s \mathrm{d}s \\
&= \pi \rtimes u(f^*)
\end{aligned}$$

and if $f, g \in C_c(G, A, \alpha)$ then using left-invariance of $\mathrm{d}s$, Proposition 2.1.10 and Theorem 2.1.15, we see that

$$\begin{aligned}
\pi \rtimes u(f *_\alpha g) &= \int_G^{\mathbb{B}(\mathcal{H})} \pi\left(f *_\alpha g(s)\right) u_s \mathrm{d}s \\
&= \int_G^{\mathbb{B}(\mathcal{H})} \pi\left(\int_G^A f(t) \alpha_t g(t^{-1}s) \mathrm{d}t\right) u_s \mathrm{d}s \\
&= \int_G^{\mathbb{B}(\mathcal{H})} \int_G^{\mathbb{B}(\mathcal{H})} \pi(f(t)) \pi(\alpha_t(g(t^{-1}s)) u_{tt^{-1}s} \mathrm{d}t \mathrm{d}s \\
&= \int_G^{\mathbb{B}(\mathcal{H})} \int_G^{\mathbb{B}(\mathcal{H})} \pi(f(t)) u_t \pi(g(t^{-1}s)) u_{t^{-1}s} \mathrm{d}s \mathrm{d}t \\
&= \int_G^{\mathbb{B}(\mathcal{H})} \pi(f(t)) u_t \int_G^{\mathbb{B}(\mathcal{H})} \pi(g(t^{-1}s)) u_{t^{-1}s} \mathrm{d}s \mathrm{d}t \\
&= \int_G^{\mathbb{B}(\mathcal{H})} \pi(f(t)) u_t \int_G^{\mathbb{B}(\mathcal{H})} \pi(g(s)) u_s \mathrm{d}s \mathrm{d}t \\
&= \int_G^{\mathbb{B}(\mathcal{H})} \pi(f(t)) u_t \mathrm{d}t \int_G^{\mathbb{B}(\mathcal{H})} \pi(g(s)) u_s \mathrm{d}s \\
&= \pi \rtimes u(f) \pi \rtimes u(g),
\end{aligned}$$

so $\pi \rtimes u$ is a $*$-homomorphism. Assuming non-degeneracy of π, for $h \in \mathcal{H}$ and $\varepsilon > 0$ we can pick $a \in A$, $\|a\| = 1$ such that $\|\pi(a)h - h\| < \varepsilon$. Being a unitary representation, we may choose a neighborhood $e \in V \subseteq G$ such that $\|u_s h - h\| < \varepsilon$ for $s \in V$. Pick $0 \leqslant \varphi \in C_c(G)$ such that $\int_G \varphi(s)\mathrm{d}s = 1$ and $\operatorname{supp} \varphi \subseteq V$. Then, $f := \varphi \otimes a \in C_c(G, A)$ satisfies for $k \in \mathcal{H}$,

$\|k\| = 1$, that

$$\begin{aligned}|\langle \pi \rtimes u(f)h - h, k\rangle| &= \left|\int_G \varphi(s)\langle \pi(a)u_s h - h, k\rangle\right| \\ &\leqslant \int_G \varphi(s)(|\langle \pi(a)(u_s h - h), k\rangle| + |\langle \pi(a)h - h, k\rangle|)\mathrm{d}s \\ &< 2\varepsilon,\end{aligned}$$

showing that $\|\pi \rtimes u(f)h - h\| \leqslant 2\varepsilon$. □

Now, given a locally compact topological group G, let $\lambda\colon G \to \mathcal{U}(L^2(G))$ be the left-regular representation $\langle \lambda_g \xi, \eta\rangle = \int_G \xi(g^{-1}s)\overline{\eta(s)}\mathrm{d}s$ for $g \in G$ and $\xi, \eta \in L^2(G)$. Even though $L^2(G)$ consists of equivalence classes (unless e.g., G is discrete), we will abbreviate $\lambda_g \xi(s) := \xi(g^{-1}s)$. Given a Hilbert space $\mathcal{H}$, we will use λ_g to denote the operator $I \otimes \lambda_g$ on $\mathcal{H} \otimes L^2(G) \cong L^2(G, \mathcal{H})$. With this, we have the following:

Lemma 2.2.4. Suppose that $G \overset{\alpha}{\curvearrowright} A$, and π is a representation of A on $\mathcal{H}$. Define $\tilde{\pi}(a)\colon L^2(G, H) \to L^2(G, H)$ by

$$\tilde{\pi}(a)\xi(r) = \pi(\alpha_{r^{-1}}(a))\xi(r), \text{ for } a \in A,\ r \in G \text{ and } \xi \in L^2(G, H).$$

Then $(\tilde{\pi}, \lambda)$ is a covariant representation of $G \overset{\alpha}{\curvearrowright} A$ on $L^2(G, \mathcal{H})$.

Proof.
Let $a \in A$ and $r \in G$. Then for all $\xi \in L^2(G, \mathcal{H})$ and $s \in G$:

$$\lambda_g(\tilde{\pi}(a)\lambda_{g^{-1}}\xi)(s) = \pi(\alpha^{-1}_{g^{-1}r}(a))\lambda_g^*\xi(g^{-1}s) = \pi(\alpha_{r^{-1}}\alpha_g(a))\xi(gg^{-1}s) = \tilde{\pi}(\alpha_g(a))\xi(s),$$

as wanted. □

Corollary 2.2.5. To every C^*-dynamical system $G \overset{\alpha}{\curvearrowright} A$, there exists a covariant representation.

In light of the above, we may define the following important class of covariant representations of a C^*-dynamical system:

Definition 2.2.6. Given $G \overset{\alpha}{\curvearrowright} A$, then for any representation π of A on $\mathcal{H}$, the above covariant representation is called the *induced regular representation*, and we will write $\mathrm{Ind}_e^G(\pi) := (\tilde{\pi}, \lambda)$. Any representation of the above form is called a *regular representation.*

As we shall see, the integrated form of the above plays a crucial role in the theory regarded in this thesis, but first, we shall need the following lemma:

Lemma 2.2.7. Suppose that $G \overset{\alpha}{\curvearrowright} A$, and π is a representation of A on $\mathcal{H}$. Then $\mathrm{Ind}_e^G \pi$ is non-degenerate if π is non-degenerate.

Proof.
Recall that the set of functions $f \otimes \eta \colon r \mapsto f(r)\eta$ with $f \in C_c(G)$ and $\eta \in \mathcal{H}$ has dense linear span in $L^2(G, \mathcal{H})$, and since π is non-degenerate, the set $\{f \otimes \pi(a)\eta \mid \eta \in \mathcal{H},\ f \in C_c(G) \text{ and } a \in A\}$ has dense linear span in $L^2(G, \mathcal{H})$ as well. Let $\varepsilon > 0$ and let (e_i) be an approximate identity of A. By the previous remark and the fact that $\tilde{\pi}$ is a linear contraction, it suffices to see that

$$\tilde{\pi}(e_i)(f \otimes \pi(a)\xi)(r) = f(r)\pi(\alpha_r^{-1}(e_i)a)\xi \xrightarrow{2} f(r)\pi(a)\xi,$$

for $f \in C_c(G)$, $\xi \in \mathcal{H}$ and $a \in A$ and $r \in \operatorname{supp} f$. The argument thus boils down to showing that $\alpha_r^{-1}(e_i)a \to a$, which an easy argument of contradiction using compactness of $\operatorname{supp} f$ and the approximate identity property passing to subnets shows that this is true. ☐

Remark 2.2.8
In particular, if π is a non-degenerate representation of A, then the integrated form, $\tilde{\pi} \rtimes \lambda$, of $\mathrm{Ind}_e^G \pi$ is non-degenerate by Proposition 2.2.3. This is not the only property which carries over from the map $\mathrm{Ind}_e^G \pi \mapsto \tilde{\pi} \rtimes \lambda$, but before we further comment on this, we must introduce some new maps

Definition 2.2.9. Suppose that $G \overset{\alpha}{\curvearrowright} A$. Let $f \otimes a \in C_c(G, A)$ be the map $r \mapsto f(r)a$. For $s \in G$, let $\iota_G(s)$ be the linear extension of the map $\lambda_s \otimes \alpha_s \colon f \otimes a \mapsto \lambda_s f \otimes \alpha_s(a)$ to all of $C_c(G, A)$, i.e., $\iota_G(s)h(t) = \alpha_s(h(s^{-1}t))$ for $h \in C_c(G, A)$.

And, we have the following handy lemma:

Lemma 2.2.10. Suppose that (π, u) is a covariant representation of a C^*-dynamical system $G \overset{\alpha}{\curvearrowright} A$, and let $\pi \rtimes u$ denote the representation of $C_c(G, A, \alpha)$. Then, for $f \in C_c(G, A)$ and $r \in G$ it holds that

$$u_r^* \circ \pi \rtimes u \circ \iota_G(r)(f) = \pi \rtimes u(f)$$

Proof.

It follows from the calculation

$$\begin{aligned}\pi \rtimes u(\iota_G(r)f) = \int_G \pi(\iota_G(r)f(s))u_s \mathrm{d}s &= \int_G \pi(\alpha_r(f(r^{-1}s)))u_r u_{r^{-1}s}\mathrm{d}s \\ &= \int_G u_r \pi(f(r^{-1}t))u_{r^{-1}s}\mathrm{d}s \\ &= u_r \int_G \pi(f(s))u_s \mathrm{d}s \\ &= u_r \circ \pi \rtimes u(f).\end{aligned}$$

□

Lemma 2.2.11. Suppose that $G \overset{\alpha}{\curvearrowright} A$ and π is a faitful representation of A on $\mathcal{H}$. Then the integrated form, $\tilde{\pi} \rtimes \lambda$, of $\mathrm{Ind}_e^G(\pi) = (\tilde{\pi}, \lambda)$ is a faithful representation of $C_c(G, A, \alpha)$.

Proof.
Let $0 \neq f \in C_c(G, A)$, and let $r \in G$ witness this, i.e, $f(r) \neq 0$. By Lemma 2.2.10 we can replace f by $\iota_G(r^{-1})f$ and assume that $r = e$. So without loss of generality we assume $r = e$, and let $\xi, \eta \in \mathcal{H}$ witness faithfulness of π, i.e., $\langle \pi(f(e))\xi, \eta\rangle \neq 0$. Pick an open neighborhood $e \in V$ such that for $s, r \in V$ we have

$$|\langle \pi(\alpha_r^{-1}(f(s)))\xi - \pi(f(e)))\xi, \eta\rangle| < \frac{|\langle \pi(f(e))\xi, \eta\rangle|}{3}.$$

Let $W \subseteq V$ be a symmetric neighborhood such that $W^2 \subseteq V$, and let $\varphi \in C_c(G)$ such that $\varphi \geqslant 0$ and $\operatorname{supp} \varphi \subseteq W$ and

$$\int_G \int_G \varphi(s^{-1}r)\varphi(r)\mathrm{d}r\mathrm{d}s = 1.$$

Let $\zeta = \varphi \otimes \xi$ and $\psi = \varphi \otimes \eta$ be the elements of $L^2(G, H)$ defined as in the proof of lemma 2.2.7 above. Then

$$\begin{aligned}\langle \tilde{\pi} \rtimes \lambda(f)\zeta, \psi\rangle = \int_G \langle \tilde{\pi}(f(s))\lambda_s\zeta, \psi\rangle \mathrm{d}s &= \int_G \int_G \langle \tilde{\pi} f(s)\lambda_s \zeta(r), \psi(r)\rangle \mathrm{d}r\mathrm{d}s \\ &= \int_G \int_G \langle \pi(\alpha_{r^{-1}} f(s))\zeta\left(r^{-1}s\right) \psi(s)\rangle \mathrm{d}r\mathrm{d}s \\ &= \int_G \int_G \varphi(r^{-1}s)\varphi(r)\langle \pi(\alpha_{r^{-1}} f(s))\xi, \eta\rangle \mathrm{d}r\mathrm{d}s\end{aligned}$$

and hence for all $r, s \in W$ we have

$$\begin{aligned}
|\langle \tilde{\pi} \rtimes \lambda(f)\zeta, \psi\rangle - \langle \pi(f(e))\zeta, \psi\rangle| &= \left| \int_G \int_G \varphi(r^{-1}s)\varphi(s)(\langle \pi(\alpha_{r^{-1}} f(s))\xi, \eta\rangle - \langle \pi(f(e))\xi, \eta\rangle \mathrm{d}r\mathrm{d}s \right| \\
&\leqslant \int_G \int_G \varphi(r^{-1}s)\varphi(s)|\langle \pi(\alpha_{r^{-1}}(f(s)))\xi, \eta\rangle - \langle \pi(f(e))\xi, \eta\rangle| \mathrm{d}r\mathrm{d}s \\
&< \int_G \int_G \varphi(r^{-1}s)\varphi(s) \frac{|\langle \pi(f(e))\xi, \eta\rangle|}{3} \mathrm{d}r\mathrm{d}s \\
&= \frac{|\langle \pi(f(e))\xi, \eta\rangle|}{2},
\end{aligned}$$

and in particular, it follows that $\tilde{\pi} \rtimes \lambda(f) \neq 0$. □

With the above result in hand, we can finally give meaning to a universal norm for the general convolution algebra $C_c(G, A, \alpha)$. Like in the discrete case, we define the *universal norm* of $f \in C_c(G, A, \alpha)$ to be non-negative real number

$$\begin{aligned}
\|f\|_u &:= \sup \{\|\pi \rtimes u(f)\| \mid \pi \rtimes u \text{ is a covariant represenation of } (A, G, \alpha)\} \\
&\leqslant \|f\|_1,
\end{aligned}$$

and it is easy to see that the assignment $C_c(G, A, \alpha) \ni f \mapsto \|f\|_u$ is a norm: It clearly satisfies $\|f + g\|_u \leqslant \|f\|_u + \|g\|_u$ and $\|cf\|_u = |c| \|f\|_u$ for $f, g \in C_c(G, A, \alpha)$ and $c \in \mathbb{C}$. If $\|f\|_u = 0$ then $\tilde{\pi} \rtimes \lambda(f) = 0$ for all faithful π, which implies by Lemma 2.2.11 that $f = 0$. Moreover, we see that $C_c(G, A, \alpha)$ is a pre C^*-algebra, since $\pi \rtimes u$ is a homomorphism for all covariant representations (π, u), and thus $\|\pi \rtimes u(f^* * f)\| = \|\pi \rtimes u(f)^* \circ \pi \rtimes u(f)\| = \|\pi \rtimes u(f)\|^2$, so that $\|f^* * f\|_u = \|f\|_u^2$.

Note 2.2.12
While it indeed is not possible to take the supremum over a class, it can be shown that it is enough to take the supremum over cyclic covariant representations, which is indeed a set - this is due to the fact that every non-degenerate representation of a *-algebra is a direct sum of cyclic representations, see e.g. [12, Theorem 5.1.3]. For now, we stick with the following:

Lemma 2.2.13. Let $G \overset{\alpha}{\curvearrowright} A$ be a C^*-dynamical system and let (π, u) be a covariant representation on $\mathcal{H}$. Let

$$V_\pi = \overline{\mathrm{Span}} \{\pi(a)\xi \mid a \in A \text{ and } \xi \in \mathcal{H}\} \subseteq \mathcal{H},$$

then V_π is invariant under both π and u by covariance, and if π^{V_π} and u^{V_π} denotes the corresponding subrepresentations on V_π, we have for all $f \in C_c(G, A, \alpha)$

$$\left\|\pi^{V_\pi} \rtimes u^{V_\pi}(f)\right\| = \|\pi \rtimes u(f)\|,$$

so that

$$\|f\|_u = \sup \left\{ \|\pi \rtimes u(f)\| \mid (\pi, u) \text{ is a non-degenerate covariant representation of } G \overset{\alpha}{\curvearrowright} A \right\}.$$

Proof.
This is a standard proof, and we refer to [23, p. 52] for the proof. □

This allows us to define the general crossed product as the completion of $C_c(G, A, \alpha)$:

Definition 2.2.14. For a C^*-dynamical system (A, G, α), we define the *full crossed product* (or simply the crossed product) of A and G to be norm closure of $C_c(G, A, \alpha)$ in the norm $\|\cdot\|_u$, and we will denote it by $A \rtimes_\alpha G$.

Remark 2.2.15
Another (equivalent) formulation of the crossed product is to consider the representation of $L^1(G, A)$ given by the direct sum of all non-degenerate covariant representations of $L^1(G, A)$. Again, one can reformulate the above to only consider cyclic covariant representations, and use the fact that every representation of a C^*-algebra is a sum of cyclic representations.

The perhaps most trivial example is in fact not a degenerate example, for it gives us a way to define the full group C^*-algebra of a locally compact group G:

Example 2.2.16
Recall the definition of the full group C^*-algebra $C^*(G)$, which is given as the closure of $C_c(G)$ (or $L^1(G)$) coming from the direct sum of all non-degenerate representations of the Banach *-algebra $L^1(G)$. This is equivalent to the construction above when we let $G \overset{\mathbb{C}}{\curvearrowright}$ be the trivial action 1.

Another fundamental example is the following:

Example 2.2.17
Let $G = \mathbb{Z}$ and let $\theta \in \mathbb{R}$. Define an action of $\mathbb{Z}$ on $C(S^1)$ by $\alpha_\theta(n)f(w) = f(e^{-2\pi\theta n}w)$. The corresponding crossed product $C(S^1) \rtimes_{\alpha_\theta} \mathbb{Z}$ is called the *(Ir)rational Rotation Algebra* (depending on wether θ is irrational or rational).

In fact, for θ irrational, we obtain one of the prototypes of simple crossed product algebras, which we recall is the topic of this thesis. Nice!

Inconvenient as it is, there is no guarantee that $A \rtimes_\alpha G$ contains a copy of A or G except under certain additional conditions on A and G. However, we can always embed A and G into the Multiplier Algebra $M(A \rtimes_\alpha G)$ of $A \rtimes_\alpha G$:

Lemma 2.2.18. Suppose that $G \overset{\alpha}{\curvearrowright} A$ is a C^*-dynamical system. For $a \in A$ define the map $\iota_A(a)\colon A \rtimes_\alpha G \to A \rtimes_\alpha G$ to be the extension of the map defined on $C_c(G, A, \alpha)$ by

$$\iota_A(a)f(s) = af(s), \text{ for } f \in C_c(G, A, \alpha) \text{ and } s \in G.$$

Then $\iota_A\colon A \to M(A \rtimes_\alpha G)$, $a \mapsto \iota_A(a)$, is a faithful non-degenerate *-representation, satisfying

$$\overline{\pi \rtimes u}\ \iota_A(a) = \pi(a),$$

for all covariant representations (π, u) of $G \overset{\alpha}{\curvearrowright} A$.

Proof.
First, for covariant representations (π, u) of $G \overset{\alpha}{\curvearrowright} A$, we see that

$$\pi \rtimes u(\iota_A(a)f) = \int_G \pi(a)\pi(f(s))u_s \mathrm{d}s = \pi(a)\pi \rtimes u(f),$$

for all $f \in C_c(G, A, \alpha)$ and $a \in A$, so $\|\iota_A(a)f\| \leqslant \|a\|\,\|f\|$. Hence $\iota_A(a)$ extends to an operator, which we also denote by $\iota_A(a)$. If $a \in A$ and $f \in C_c(G, A, \alpha)$, then

$$\left(\iota_A(a)f\right)^*(s) = \Delta(s^{-1})\alpha_s(f(s^{-1})^*)\alpha_s(a^*) = f^*(s)\alpha_s(a^*), \text{ for } s \in G,$$

implying that

$$\begin{aligned}(\iota_A(a)f)^* * g(s) = \int_G f^*(r)\alpha_r(a^*)\alpha_r(g(r^{-1}s))\mathrm{d}r &= \int_G f^*(r)\alpha_r\left((\iota_A(a^*)g)(r^{-1}s)\right)\mathrm{d}r \\ &= f^* * (\iota_A(a^*)g)(s).\end{aligned}$$

We conclude that $\iota_A(a^*)$ is an adjoint (with respect to the canonical $A \rtimes_\alpha G$)-valued inner product) on the dense sub$*$-algebra $C_c(G, A, \alpha) \subseteq A \rtimes_\alpha G$, hence everywhere by continuity, i.e., $\iota_A(a) \in M(A \rtimes_\alpha G)$.

For injectivity, let π be any faithful, non-degenerate representation of A, and let $\overline{\tilde{\pi} \rtimes \lambda}$ denote the extension of $\tilde{\pi} \rtimes \lambda$ to $M(A \rtimes_\alpha G)$. By 2.2.3 and the above, we see that $\tilde{\pi}(a) = \overline{\tilde{\pi} \rtimes \lambda} \circ \iota_A(a)$, hence $\iota_A(a) = 0 \implies \tilde{\pi}(a) = 0 \implies a = 0$, since $\tilde{\pi}$ is faithful.

For non-degeneracy, recall that the set of functions $\varphi \rtimes b$ for $\varphi \in C_c(G)$ and $b \in A$ span a dense subset of $C_c(G, A)$ in the inductive limit norm and hence in L^1-norm by lemma 2.1.3. This means that this set also spans a dense subset of $A \rtimes_\alpha G$ in the universal norm. Also, for $a \in A$, we see that $\iota_A(a)\varphi \rtimes b = \varphi \rtimes ab$, from which we conclude that the set $\iota_A(A)A \rtimes_\alpha G$ is dense, finishing the proof. ∎

Lemma 2.2.19. There is an injective and strictly continuous group homomorphism

$$\iota_G\colon G \to \mathcal{U}M(A \rtimes_\alpha G),$$

such that for $f \in C_c(G,A,\alpha) \subseteq A \rtimes_\alpha G$ we have

$$\iota_G(s)f(t) = \alpha_s(f(s^{-1}t)),$$

for all $s,t \in G$, and if (π,u) is a covariant representation of $G \overset{\alpha}{\curvearrowright} A$, then

$$\overline{\pi \rtimes u}\iota_G(s) = u_s,$$

for $s \in G$.

Before we commence the proof of the above, we want to point out that the above two lemmas ensure that for each covariant representation (π,u) of $G \overset{\alpha}{\curvearrowright} A$ we obtain the following commutative diagram

$$\begin{tikzcd} G \arrow[r, hook, "\iota_G"] \arrow[dr, "u"'] & M(A \rtimes_\alpha G) \arrow[d, "\overline{\pi \rtimes u}"] & A \arrow[l, hook', "\iota_A"'] \arrow[dl, "\pi"] \\ & \mathbb{B}(\mathcal{H}) & \end{tikzcd} \tag{2.9}$$

And we shall see that this pair is in fact a covariant pair. But first, the proof of the above lemma:

Proof.
Let $\iota_G(s)$ be defined on $C_c(G,A,\alpha)$ as earlier. By Lemma 2.2.10, we see that

$$\|\iota_G(r)f\|_u = \|f\|_u,$$

for all $f \in C_c(G,A,\alpha)$, and so it extends to $A \rtimes_\alpha G$. Let $s,r \in G$, then for $f \in C_c(G,A,\alpha)$, we see that

$$(\iota_G(r)f(s))^* = \Delta(s^{-1})\alpha_s((\alpha_r(f(r^{-1}s^{-1}))^*) = \Delta(s^{-1})\alpha_{sr}(f(r^{-1}s^{-1})^*),$$

which, combined with the fact that $\Delta(s^{-1})\mathrm{d}s$ is the integrand of a right Haar measure, can be used to show that $\iota_G(r)^* = \iota_G(r^{-1})$, so that $\iota_G(r) \in M(A \rtimes_\alpha G)$. In fact, since clearly $\iota_G(rs) = \iota_G(r) \circ \iota_G(s)$ for $s,r \in G$, it is a unitary element.

Let (π,u) be any non-degenerate covariant representation of $G \overset{\alpha}{\curvearrowright} A$. By Lemma 2.2.10, we see that $u_s = \overline{\pi \rtimes u}\iota_G(s)$ for all $s \in G$. Moreover, for faithful representations π (which also implies that $\tilde{\pi}$ is faithful) we have that the corresponding representation $\tilde{\pi} \rtimes \lambda$ is faithful on $C_c(G,A,\alpha)$. This implies that for $s \neq e$ we have $\mathrm{id} \neq \lambda_s = \overline{\tilde{\pi} \rtimes \lambda}\iota_G(s)$ for $s \neq e$, hence $\iota_G(s) \neq \mathrm{id}$ as well, so it is injective.

For continuity, let $f \in C_c(G,A,\alpha)$. We will see that ι_G is strongly continuous, which by Lemma 2.1.11 is equivalent to being strictly continuous: Let $(s_i) \subseteq G$, $s_i \to e$. Then we claim that $\iota_G(s_i)x \to x$ for $x \in A \rtimes_\alpha G$. To see this, note that we know that the set of functions $\varphi \otimes a$, $\varphi \in C_c(G)$ and $a \in A$, spans a dense subset of $C_c(G, A)$, hence it suffices to show that $\iota_G(s_i)\varphi \otimes a \xrightarrow{L^1} \varphi \otimes a$, but

$$\begin{aligned}\|\iota_G(r_i)\varphi \otimes a - \varphi \otimes a\|_1 &= \int_G \left\|\varphi(r_i^{-1}s)\alpha_{r_i}(a) - \varphi(r_i^{-1}s)a + \varphi(r_i^{-1}a)a - \varphi(s)a\right\|_A \mathrm{d}s \\ &\leqslant \|\alpha_{r_i}(a) - a\| \, \|\varphi\|_1 + \|\lambda_{r_i}\varphi - \varphi\|_1 \, \|a\|,\end{aligned}$$

which goes to 0, by uniform continuity of φ and the fact that $\alpha_{r_i}(a) \to a$ for all $a \in A$ as $r_i \to e$. □

And, we summarize the above in the following

Proposition 2.2.20. For a C^*-dynamical system $G \overset{\alpha}{\curvearrowright} A$, there exists a non-degenerate faithful *-homomorphsim

$$\iota_A \colon A \to M(A \rtimes_\alpha G),$$

and an injective strictly continuous unitary representation

$$\iota_G \colon G \to \mathcal{U}M(A \rtimes_\alpha G),$$

such that $\iota_G(r)f(s) = \alpha_r(f(r^{-1}s))$ and $\iota_A(a)f(s) = af(s)$ for $f \in C_c(G, A, \alpha)$ and $r, s \in G$ and $a \in A$. Moreover, the pair (ι_A, ι_G) is covariant with respect to α, i.e.,

$$\iota_A(\alpha_r(a)) = \iota_G(r)\iota_A(a)\iota_G(r)^*,$$

and for non-degenerate covariant representations (π, u) we have

$$\overline{\pi \rtimes u} \circ \iota_A(a) = \pi(a) \text{ and } \overline{\pi \rtimes u} \circ \iota_G(s) = u_s.$$

Proof.
The only thing left to be shown is the covariance relation, but this follows immediately from the following calculation:

$$\iota_G(r)\iota_A(a)\iota_G(r)^* f(s) = \alpha_r(\iota_A(a)\iota_G(r)^* f(r^{-1}s)) = \alpha_r(a)\alpha_r(\alpha_{r^{-1}} f(rr^{-1}s)) = \iota_A(\alpha_r(a))f(s),$$

for $f \in C_c(G, a, \alpha)$, $a \in A$ and $s, r \in G$. □

We now embark on our final task in our construction and picture painting of the general crossed product $A \rtimes_\alpha G$, namely we wish to properly characterize its representation theory.

We are going to see that $A \rtimes_\alpha G$ is the C^*-algebra whose representation theory is in a one-to-one correspondance with the covariant representations of (A, G, α) via the map $(\pi, u) \mapsto \pi \rtimes u$, but before we finalize the description of the representation theory of $A \rtimes_\alpha G$, we need to extend the map ι_G to $C^*(G)$, the full group C^*-algebra of G, since this will provide us with a useful characterization of a dense subset of $A \rtimes_\alpha G$, so without further ado:

Lemma 2.2.21. There exists a *-homomorphsim $\tilde{\iota}_G \colon C^*(G) \to M(A \rtimes_\alpha G)$ such that

$$\tilde{\iota}_G(z) = \int_G z(s)\iota_G(s)\mathrm{d}s,$$

for all $z \in C_c(G)$

Proof.
For all $z \in C_c(G)$, the map $s \mapsto z(s)\iota_G(s)$ is strictly continuous, and we may define the integral by Theorem 2.1.12. Let $z, w \in C_c(G)$, then

$$\begin{aligned}
\tilde{\iota}_G(z * w) &= \int_G \int_G z(r)w(r^{-1}s)\mathrm{d}r\iota_G(s)\mathrm{d}s \\
&= \int_G \int_G z(r)w(r^{-1}s)\iota_G(s)\mathrm{d}s\mathrm{d}r \\
&= \int_G \int_G z(r)w(s)\iota_G(rs)\mathrm{d}s\mathrm{d}r \\
&= \int_G z(r)\iota_G(r)\mathrm{d}r \int_G w(s)\iota_G(s)\mathrm{d}s \\
&= \tilde{\iota}_G(z)\tilde{\iota}_G(w),
\end{aligned}$$

so if it is well-defined and continuous on $C^*(G)$, it will be a *-homomorphism, since the integral also respects taking adjoints.

Let π be any faithful and non-degenerate representation of $A \rtimes_\alpha G$, so that its extension $\overline{\pi}$ to $M(A \rtimes_\alpha G)$ is faithful aswell.

Consider the composition $u \colon s \mapsto \overline{\pi}(\iota_G(s))$, so that u_s is unitary. To see it is strongly continuous: Suppose that $f \in C_c(G, A)$ and $\xi \in \mathcal{H}_\pi$, then for $s \in G$ we see that $\overline{\pi}(\iota_G(s))\pi(f)\xi = \pi(\iota_G(s)f)\xi$, and it follows that $s \mapsto u_s\xi$ is continuous for all $\xi \in \mathcal{H}$ by continuity of $s \mapsto \iota_G(s)f$ and non-degeneracy of π. Hence:

$$\begin{aligned}
\overline{\pi}(\tilde{\iota}_G(z)) = \overline{\pi}\left(\int_G z(s)\iota_G(s)\mathrm{d}s\right) &= \int_G z(s)\overline{\pi}(\iota_G(s))\mathrm{d}s \\
&= u(z),
\end{aligned}$$

Where the last line is the representation of $L^1(G)$ corresponding to the unitary representation $\overline{\pi} \circ \iota_G$ of G, which we shall discuss later on. In particular, this implies that

$$\|\tilde{\iota}_G(z)\| = \|u(z)\| \leqslant \|z\|, \tag{2.10}$$

for all $z \in C_c(G)$, so it extends to a *-homomorphism $C^*(G) \to M(A \rtimes_\alpha G)$, which we will also denote by ι_G. ☐

Corollary 2.2.22. Let $G \overset{\alpha}{\curvearrowright} A$ be a C^*-dynamical system and let $a \in A$, $f, h \in C_c(G, A)$ and $\varphi \in C_c(G)$. Then

$$\begin{aligned} \iota_A(a)\iota_G(\varphi) &= \varphi \otimes a, \\ \int_G \iota_A(f(s))\iota_G(s)h\mathrm{d}s &= f * h \text{ and} \\ \int_G \iota_A(f(s))\iota_G(s)\mathrm{d}s &= f, \end{aligned}$$

where we identified $C_c(G, A, \alpha) \subseteq A \rtimes_\alpha G \subseteq M(A \rtimes_\alpha G)$.

Proof.
Let (π, u) be a non-degenerate covariant representation of $G \overset{\alpha}{\curvearrowright} A$ and notice that

$$\overline{\pi \rtimes u}\left(\int_G \iota_a(f(s))\iota_G(s)\mathrm{d}s\right) = \pi \rtimes u(f),$$

which combined with lemma 2.2.13 shows the first and last identity. The second equality follows from the last and theorem 2.1.12. ☐

swap around the above and below

Remark 2.2.23
In particular, by the above and lemma 2.1.3 we see that the set of functions of the form $\iota_A(a)\iota_G(z)$ for $a \in A$ and $z \in C_c(G)$ span a dense subset of $A \rtimes_\alpha G$. This will be useful in the following:

Theorem 2.2.24. Let $G \overset{\alpha}{\curvearrowright} A$ be a C^*-dynamical system and let $\mathcal{H}$ be a Hilbert space. If $L \colon A \rtimes_\alpha G \to \mathbb{B}(\mathcal{H})$ is a non-degenerate $*$-representation, then there exists a non-degenerate covariant representation (π, u) of $G \overset{\alpha}{\curvearrowright} A$ into $\mathbb{B}(\mathcal{H})$ such that $L = \pi \rtimes u$ which can be realised by

$$u_s = \overline{L}(\iota_G(s)) \text{ and } \pi(a) = \overline{L}(\iota_A(a)),$$

for all $s \in G$ and $a \in A$. In fact, there is a bijective correspondance between covariant representations of $G \overset{\alpha}{\curvearrowright} A$ and $*$-representations of $A \rtimes_\alpha G$.

Proof.
We already have seen the assignment $(\pi, u) \mapsto \pi \rtimes u$. For the converse, suppose that L is a non-degenerate representation of $A \rtimes_\alpha G$. Let π, u be defined by the above. It is clear that

π is a $*$-representation of A on $\mathbb{B}(\mathcal{H})$ and similarly that u is a strongly continuous unitary-valued group representation on $\mathbb{B}(\mathcal{H})$, since ι_G is strictly and hence strongly continuous. To see that (π, u) is non-degenerate, let (e_i) be an approximate unit for A. Then $\iota_A(e_i) \to 1$ strictly in $M(A \rtimes_\alpha G)$, and so $\pi(e_i) = \overline{L}(\iota_A(e_i)) \to I$ strictly, so in particular $\pi(e_i)\xi \to \xi$ for all $\xi \in \mathcal{H}$ and π is non-degenerate. Now, let $s \in G$ and $a \in A$, then

$$u_s \pi(a) u_s^* = \overline{L}(\iota_G(s))\overline{L}(\iota_A(a))\overline{L}(\iota_G(s))^* = \overline{L}(\iota_A(\alpha_s(a))) = \pi(\alpha_s(a)),$$

so the pair is a proper non-degenerate covariant representation of $G \overset{\alpha}{\curvearrowright} A$. Finally, using theorem 2.1.12, we see that

$$\begin{aligned} \pi \rtimes u(\iota_A(a)\iota_G(z)) &= \pi(a) \circ \overline{\pi \rtimes u}(\iota_G(z)) = \overline{L}(\iota_A(a)) \int_G z(s)\overline{L}(\iota_G(s))\mathrm{d}s \\ &= \overline{L}(\iota_A(a))\overline{L}(\iota_G(z)) \\ &= \overline{L}(\iota_A(a)\iota_G(z)), \end{aligned}$$

for all $z \in C_c(G)$ and $a \in A$, which by the above remark ensures that $L = \pi \rtimes u$ on all of $A \rtimes_\alpha G$. $\square$

2.3 Properties of Crossed products

We want to examine some interesting properties of crossed products, such as what happens when G is amenable or how can we concretize the statespace $\mathcal{S}(A \rtimes_\alpha G)$. To do this, we will need to recall the usual definition of a positive definite function $G \to \mathbb{C}$ and extend it to the case of positive definite functions $G \to A^*$. The following is from [17]

Definition 2.3.1. Suppose that $G \overset{\alpha}{\curvearrowright} A$ is a dynamical system, then a bounded continuous (with respect to the norm on A^*) function $\varphi\colon G \to A^*$ is *positive definite* if any of the following equivalent statements hold:

1. There is a positive functional $\psi \in (A \rtimes_\alpha G)^*$ such that $\varphi(t)(a) = \psi(\iota_A(a)\iota_G(t))$ for all $t \in G$ and $a \in A$,
2. $\sum_{i,j} \varphi(s_i^{-1}s_j)(\alpha_{s_i^{-1}}(x_i^* x_j)) \geqslant 0$ for all finite sets $s_1, \dots, s_n \in G$ and $x_1, \dots, x_n \in A$,
3. $\sum_{i,j} \int_G \varphi(t)(x_i^* \alpha_t(x_j))(f_i^* \times f_j)(t)\mathrm{d}t \geqslant 0$ for all finite sets $f_1, \dots, f_n \in C_c(G, A, \alpha)$ and $x_1, \dots, x_n \in A$, and
4. $\int \varphi(t)(f^* \rtimes f)(t)\mathrm{d}t \geqslant 0$ for all $f \in C_c(G, A, \alpha)$.

Remark 2.3.2
Note that in the above, a priori without explanation the first definition does not make sense, since $\iota_A(a)\iota_G(t)$ might not be an element of $A \rtimes_\alpha G$, however, since states of $A \rtimes_\alpha G$ correspond to non-degenerate representations which extend to $M(A \rtimes_\alpha G)$, we may in fact extend each positive functional ψ on $A \rtimes_\alpha G$ to all of $M(A \rtimes_\alpha G)$. We will also denote this extension by ψ.

If G acts trivially on $\mathbb{C}$, then the above becomes the regular result about equivalent definitions of positive definite functions on G, as described in [8] or in [17].

The first definition is the general motivation behind the above for us, since we will be dealing with a class of functions arising this way, which leads to the following definition:

Definition 2.3.3. Suppose that $G \overset{\alpha}{\curvearrowright} A$. For each $\varphi \in (A \rtimes_\alpha G)^*$, define the map $\Gamma_\varphi\colon G \to A^*$, given by

$$\Gamma_\varphi(t)(a) := \varphi(\iota_A(a)\iota_G(t)) \in \mathbb{C},$$

for $t \in G$ and $a \in A$. The set of functions $G \to A^*$ defined in this way is denoted by $B(A \rtimes_\alpha G)$, and the set of $\Gamma_\varphi \in B(A \rtimes_\alpha G)$ for $\varphi \geqslant 0$, is denoted by $B_+(A \rtimes_\alpha G)$. We denote by $B_+^1(A \rtimes_\alpha G)$ the functions such that $\|\Gamma_\varphi(e)\| = 1$ and $\varphi \geqslant 0$.

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Remark 2.3.4
Note that we may use the fact that every non-degenerate representation of $A \rtimes_\alpha G$ is of the form $\pi \rtimes u$ for some covariant, non-degenerate representation (π, u) of $G \overset{\alpha}{\curvearrowright} A$ to show that for all $\varphi \in \mathcal{S}(A \rtimes_\alpha G)$ we have

$$\varphi(y) = \int_G \Gamma_\varphi(t)(y(t))\mathrm{d}t \tag{2.11}$$

for all $y \in C_c(G, A, \alpha) \subseteq A \rtimes_\alpha G$: To φ we associate a GNS triple $(\pi_\varphi, \xi, \mathcal{H})$ with π_φ non-degenerate. By theorem 2.2.24, we see that π_φ is of the form $\pi \rtimes u$ for a suitable covariant representation (π, u), and so

$$\begin{aligned}
\varphi(y) = \langle \pi_\varphi(y)\xi, \xi\rangle = \langle \pi \rtimes u(y)\xi, \xi\rangle &= \int_G \langle \pi(y(t))u_t\xi, \xi\rangle \mathrm{d}t \\
&= \int_G \langle \overline{\pi \rtimes u}(\iota_A(y(t)))\overline{\pi \rtimes u}(\iota_G(t))\xi, \xi\rangle \mathrm{d}t \\
&= \int_G \langle \overline{\pi \rtimes u}(\iota_A(y(t))\iota_G(t))\xi, \xi\rangle \mathrm{d}t \\
&= \int_G \varphi(\iota_A(y(t))\iota_G(t))\mathrm{d}t \\
&= \int_G \Gamma_\varphi(t)(y(t))\mathrm{d}t.
\end{aligned}$$

In particular, density of $C_c(G, A, \alpha) \subseteq A \rtimes_\alpha G$ implies by the above that the assignment $\mathcal{S}(A \rtimes_\alpha G) \ni \varphi \mapsto \Gamma_\varphi \in B^1_+(A \rtimes_\alpha G)$ is a bijection. Combining this with linearity of the assignment, i.e., that $\Gamma_{\alpha\varphi+\beta\psi} = \alpha\Gamma_\varphi + \beta\Gamma_\psi$ for $\alpha, \beta \in \mathbb{C}$ and $\varphi, \psi \in (A \rtimes_\alpha G)^*$, this shows that the assignment also induces a bijection between $(A \rtimes_\alpha G)^*$ and $B(A \rtimes_\alpha G)$.

Continuity of Γ_φ using cauchy-schwarz

For $A = \mathbb{C}$ again and $\alpha = 1$, we may represent G on $M(C^*G)$ via $t \mapsto \delta_t$, and then Γ_φ becomes the map $t \mapsto \varphi(\delta_t)$ for $\varphi \in C^*(G)^*$ which leads to the following result (see [17, p. 7.1.11]):

Proposition 2.3.5. For $A = \mathbb{C}$ and $\alpha = 1$, the isomorphism above is a homeomorphism from $\mathcal{S}(C^*(G))$ with the weak*-topology onto the set of positive-definite functions on G with $\Gamma_\varphi(e) = 1$ equipped with the topology of uniform convergence on compacta.

Generalizing the above, we introduce a topology on $B^1_+(A \rtimes_\alpha G)$, which extends the topology of uniform convergence on compacta of G to the general case of $G \overset{\alpha}{\curvearrowright} A$:

Definition 2.3.6. We define *the topology of weak* uniform convergence on compacta* on $B^1_+(A \rtimes_\alpha G)$ to be the topology satisfying that a net $\Gamma_{\varphi_i} \subseteq B^1_+(A \rtimes_\alpha G)$ converges to $\Gamma_\varphi \in B^1_+(A \rtimes_\alpha G)$ if and only if for each $a \in A$, the restriction of the map

$$t \mapsto \Gamma_{f_i}(t)(a)$$

to any compact subset $K \subseteq G$ is uniformly convergent to the map $t \mapsto \Gamma_\varphi(t)(a)$ restricted to K. A basis for the topology consists of the sets

$$V(\varepsilon, f, K, a) := \left\{ f' \in B^1_+(A \rtimes_\alpha G) \mid \sup_{t \in K} |f'(t)(a) - f(t)(a)| < \varepsilon \right\},$$

where $f \in B^1_+(A \rtimes_\alpha G)$, $\varepsilon > 0$, $a \in A$ and $K \subseteq G$ is compact.

And this allows us to generalize the above result to the general case:

Proposition 2.3.7. The map $\mathcal{S}(A \rtimes_\alpha G) \ni \varphi \mapsto \Gamma_\varphi \in B^1_+(A \rtimes_\alpha G)$ is a homeomorphism with respect to the weak*-topology on $\mathcal{S}(A \rtimes_\alpha G)$ and the topology of weak* convergence uniformly on compacta of G.

Proof.
We have already established the bijectivity. For the last part, suppose that we have a convergent net $(\varphi_i) \subseteq \mathcal{S}(A \rtimes_\alpha G)$ with limit $\varphi \in \mathcal{S}(A \rtimes_\alpha G)$. We will in the following also write Γ_φ to denote the extension from A to A^1. For $x, y \in A^1$, define the map $\Psi_\varphi(x,y) \colon G \to \mathbb{C}$ by $t \mapsto \Gamma_\varphi(x^* \alpha_t(y))$. The polarization identity and linearity ensures that if for all $x \in A^1$ we have $\Psi_{\varphi_i}(x,x) \to \Psi_\varphi(x,x)$ in the topology of convergence uniformly on compacta then $\Psi_{\varphi_i}(x,y) \to \Psi_\varphi(x,y)$ as well. By Proposition 2.3.5, we see that $\Psi_{\varphi_i}(x,x) \to \Psi_\varphi(x,x)$ in the topology of convergence uniformly on compacta of G, and setting $y = 1_{A^1}$, we see that $\Gamma_{\varphi_i} \to \Gamma_\varphi$.

Conversely, suppose that $\Gamma_{\varphi_i} \to \Gamma_\varphi$, then in particular $\varphi_i(f \otimes x) \to \varphi(f \otimes x)$ for all $x \in A$ and $f \in C_c(G)$ by Equation (2.11) and hence on all of $A \rtimes_\alpha G$, since the functions $f \otimes x$ span a dense subset of $A \rtimes_\alpha G$. □

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For discrete groups, it is well-known that if G is amenable (or more generally acts amenably on A), i.e., has a left-invariant mean $m \in \ell^\infty(G)$, then there is only one crossed product of G and any C^*-algebra A (if G is not amenable, but admits an amenable action on A, then the statement is true for crossed products with A as well). We will see that this is true for locally compact groups as well - in particular, it is true for abelian groups, c.f. [1, appendix G].

Lemma 2.3.8. Suppose that $G \overset{\alpha}{\curvearrowright} A$ and $(\pi, \mathcal{H})$ is a cyclic representation of A with respect to a set $(\xi_i) \subseteq \mathcal{H}$, and suppose that $(\eta_j) \subseteq L^2(G)$ is cyclic for the left-regular representation λ of G. Then the set of $\xi_i \otimes \eta_j$ is cyclic for the representation $(\tilde{\pi} \rtimes \lambda, L^2(G, \mathcal{H}))$.

Proof.
We will show that the orthogonal complement of $\sum_{i,j} (A \rtimes_\alpha G)(\xi_i \otimes \eta_j)$ is 0. Suppose that ξ

is orthogonal to $\tilde{\pi} \rtimes \lambda(x)\xi_i \otimes \eta_j$ for all i, j. Then for all i, j and $a \otimes f \in C_c(G, A) \subseteq A \rtimes_\alpha G$ we have

$$\begin{aligned}
0 &= \langle \tilde{\pi} \rtimes \lambda(a \otimes f)(\xi_i \otimes \eta_j), \xi \rangle_{L^2} \\
&= \int_G \langle \tilde{\pi} \rtimes \lambda(a \otimes f)(\xi_i \otimes \eta_j)(s), \xi(s) \rangle_{\mathcal{H}} \mathrm{d}s \\
&= \int_G \int_G \langle \pi(\alpha_{t^{-1}} \left((a \otimes f)(s)\right) \lambda_t \eta_j(s) \xi_i, \xi \rangle_{\mathcal{H}} \mathrm{d}t\mathrm{d}s \\
&= \int_G \int_G \langle \pi\left(\alpha_{t^{-1}}(a)\right) f(s) \eta_j(t^{-1}s) \xi_i, \xi \rangle_{\mathcal{H}} \mathrm{d}t\mathrm{d}s \\
&= \int_G \int_G \langle \pi(\alpha_{t^{-1}}(a)) \xi_i, \xi \rangle_H f(s) \eta_j(t^{-1}s) \mathrm{d}t\mathrm{d}s \\
&= \int_G \langle \pi(\alpha_{t^{-1}}(a)) \xi_i, \xi \rangle_{\mathcal{H}} (f * \eta_j)(s) \mathrm{d}s.
\end{aligned}$$

Now, let $E \subseteq G$ be any compact set. We will argue that $\xi = 0$ by showing that $\xi = 0$ on any arbitrarily large compact subset F of E: Let $\varepsilon > 0$, and use a Banach space-valued analogue of Lusin's theorem (e.g. as in [23, Appendix B]) to pick $F \subseteq E$ compact with $\mu(E \backslash F) < \varepsilon$ and such that $\xi|_F \in C(F, H)$. Cyclicity of $\{\eta_j\}_{j \in J}$ implies that the set $\{f * \eta_j \mid j \in J \text{ and } f \in C_c(G)\}$ spans a dense subset of $C_c(G)$ (ee e.g. [8, Proposition 3.33]), so

$$\int_F \langle \pi(\alpha_{t^{-1}}(x)) \xi_i, \xi(t) \rangle_H f(t) \mathrm{d}t = 0,$$

for all $f \in C(F)$, from which we conclude that $F \ni t \mapsto \langle \pi(\alpha_{t^{-1}}(x)) \xi_i, \xi(t) \rangle = 0$, and since ξ_i was cyclic for π, this means that $\xi|_F = 0$. In particular, since $\varepsilon > 0$ was arbitrary, this means that $\xi|_E = 0$ and since $E \subseteq G$ was arbitrary, this means that $\xi = 0$, finishing the proof. ☐

We now will go through a result which will allow us to describe regular representations pointwise through approximations of states of a certain form:

Proposition 2.3.9. Let (A, G, α) be a C^*-dynamical system. Given $\varphi \in \mathcal{S}(A)$, for each $z \in C_c(G, A, \alpha)$ such that $\varphi(z^* * z(e)) = 1$, there is a vector state $\tilde{\varphi}_z \in \mathcal{S}(A \rtimes_\alpha G)$ of the representation $(\tilde{\pi}_\varphi \rtimes \lambda, L^2(G, \mathcal{H}_\varphi))$ satisfying

$$\tilde{\varphi}_z(f) = \varphi\left((z^* * f * z)(e)\right), \quad \text{for all } f \in C_c(G, A, \alpha).$$

Moreover, the set of vector states contained in $(\tilde{\pi}_\varphi \rtimes \lambda, L^2(G, \mathcal{H}_\varphi))$ is a norm limit of vector states of this form.

Proof.
For $z \in C_c(G, A, \alpha)$with $t \mapsto \|z(t)\| \in L^2(G)$, define

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$$\xi_z(t) = \pi_\varphi\left(\alpha_{t^{-1}}(z(t)\right)\xi_\varphi,$$

so that $\xi_z \in L^2(G, \mathcal{H}_\varphi)$. A few calculations then show that

$$\begin{aligned}\langle \tilde{\pi}_\varphi \rtimes \lambda(y^* * y)\xi_z, \xi_z\rangle = \|\tilde{\pi}_\varphi \rtimes \lambda(y)\xi_z\|^2 &= \int_G \|\pi_\varphi(\alpha_{t^{-1}}(y * z)(t)\xi_\varphi\|^2 \,\mathrm{d}t \\ &= \int_G \varphi\left((y * z)^*(t)\alpha_t((y * z)(t^{-1}))\right)\mathrm{d}t \\ &= \varphi((\underbrace{(y * z)^*}_{=z^* * y^*} * (y * z)(e)) = \tilde{\varphi}_z(y^* * y),\end{aligned}$$

for all $y \in C_c(G, A, \alpha)$. By linearity, it follows that $\tilde{\varphi}_z$ is a vector state on $A \rtimes_\alpha G$ of $(\tilde{\pi} \rtimes \lambda, L^2(G, \mathcal{H}_\varphi))$.

For the last claim, we show that the set $\{\xi_z \mid z \in C_c(G, A, \alpha)\}$ is dense in $L^2(G, \mathcal{H}_\varphi)$: Note that for $z \in C_c(G, A, \alpha)$, $f \in C_c(G)$ and $t \in G$ we have

$$\tilde{\pi}_\varphi(z(s))(\lambda_s(f \otimes \xi_\varphi))(t) = \pi_\varphi(\alpha_t^{-1}(z(s)))f(s^{-1}t)\xi_\varphi,$$

which implies that

$$\tilde{\pi}_\varphi \rtimes \lambda(z)(f \otimes \xi_\varphi)(t) = \xi_{z*f}(t),$$

fix $*$'s

where $z * f(s) = \int_G z(s)f(s^{-1}t)\mathrm{d}s \in A$. By cyclicity of ξ_φ and Lemma 2.3.8, since $C_c(G) \subseteq L^2(G)$ is dense, we see that the set $\{f \otimes \xi_\varphi \mid f \in C_c(G)\}$ is cyclic for $\tilde{\pi}_\varphi \rtimes \lambda$, which finishes the proof. □

Recall the definition of the universal representation $(\pi_u, \mathcal{H}_u)$ of a C^*-algebra A. The induced representation $\mathrm{Ind}(\pi_u)$ gives rise to the reduced crossed product of a C^*-dynamical system (A, G, α):

Definition 2.3.10. For a C^*-dynamical system (A, G, α), we define the *reduced crossed product* of G and A to be the algebra

$$A \rtimes_{\alpha,r} G := \overline{\tilde{\pi}_u \rtimes \lambda(C_c(G, A, \alpha))},$$

where π_u is the universal representation of A. Equivalently, we may define $A \rtimes_{\alpha,r} G$ as the image of $A \rtimes_\alpha G$ of the extension of $\tilde{\pi}_u \rtimes \lambda$ to $A \rtimes_\alpha G$.

Remark 2.3.11

We will soon see that the above is actually independent of the choice of the representation π_u of A, and may just as well be replaced with any faithful representation π of A.

We need define yet another subset of functionals on $A \rtimes_\alpha G$, one which closely resembles vector states of $\tilde{\pi}_u \rtimes \lambda$:

Definition 2.3.12. The set of functionals $\psi \in (A \rtimes_\alpha G)^*$ of the form

$$\psi(x) = \sum_i \langle \tilde{\pi}_u \rtimes \lambda(x)\xi_i, \eta_i \rangle,$$

where $(\xi_i) \subseteq L^2(G, \mathcal{H}_u)$ and $(\eta_i) \subseteq L^2(G, \mathcal{H}_u)$ have finite ℓ^2-norm, i.e., $\sum_i \|\eta_i\|^2 < \infty$ and $\sum_i \|\xi_i\|^2 < \infty$, is easily seen to be a norm-closed subspace of $(A \rtimes_\alpha G)^*$. We define the set

$$A(A \rtimes_\alpha G) := \{\Gamma_\psi \in B(A \rtimes_\alpha G) \mid \psi(x) = \sum_i \langle \tilde{\pi}_u \rtimes \lambda(x)\xi_i, \eta_i \rangle,\ (\xi_i),\ (\eta_i) \text{ is } \ell^2 - \text{summable}\}.$$

Remark 2.3.13
It is an easy consequence of the commutativity of Equation (2.9) that if $\varphi \in \mathcal{S}(A)$ satisfies the condition of Proposition 2.3.9 with respect to some $z \in C_c(G, A, \alpha)$, then

$$\begin{aligned}\Gamma_{\tilde{\varphi}_z}(t)(x) = \varphi_z(\iota_A(x)\iota_G(t)) = \langle \overline{\pi \rtimes \lambda(\iota_A(x)\iota_G(t))}\xi_z, \xi_z \rangle &= \int_G \varphi(z^*(s)\alpha_s(x)\alpha_{st}(z\,(st)^{-1}))\mathrm{d}s \\ &= \varphi\left(\int_G z^*(s)\alpha_s(x)\alpha_{st}(z(st)^{-1})\mathrm{d}s\right),\end{aligned}$$

and in particular, we have $\Gamma_{\tilde{\varphi}_z} \in A_+(A \rtimes_\alpha G)$. Another consequence of Proposition 2.3.9 is that every function in $A_+(A \rtimes_\alpha G)$ is a norm limit of linear combinations of functions $\Gamma_{\tilde{\varphi}_z}$, for states $\varphi \in \mathcal{S}(A)$. This is because π_u is the direct sum of cyclic representations of A.

For a representation $(\pi, \mathcal{H})$ of A, we will also use $\tilde{\pi} \rtimes \lambda$ to denote the extension from $C_c(G, A, \alpha) \subseteq A \rtimes_{\alpha,r} G$ to all of $A \rtimes_{\alpha,r} G$ (doable, since the norm π_u is the universal representation of A, hence the norm will be bounded by it). With this, we may prove the following essential theorem in our description of the reduced crossed product:

rephrase

Theorem 2.3.14. For a C^*-dynamical system (A, G, α), given any faithful representation $(\pi, \mathcal{H})$ of A, it holds that the associated integrated form $\tilde{\pi} \rtimes \lambda$ will be faithful on $A \rtimes_{\alpha,r} G$.

Proof.
For this particular proof, we will let $\tilde{\pi} \rtimes_r \lambda$ denote the corresponding representation of $A \rtimes_{\alpha,r} G$.

Is this the proper way of describing states contained in $\tilde{\pi} \rtimes_r \lambda$, or should we use the identification using predual, instead of using composition with σ

We will show that $\tilde{\pi}_u \rtimes_r \lambda$ is *weakly contained* in $\tilde{\pi} \rtimes_r \lambda$ for all faithful representations π of A, i.e., that $\ker \tilde{\pi} \rtimes_r \lambda \subseteq \ker \pi_u \rtimes_r \lambda = 0$. By [7, p. 80], this is equivalent to showing that the set of vector states of $\tilde{\pi} \rtimes_r \lambda$ is weak* dense in the set of states in $\tilde{\pi}_u \rtimes_r \lambda$.

Let E be the set of states contained in $(\pi, \mathcal{H})$, which will be weak* dense in $\mathcal{S}(A)$ by faithfulness of π. For now, let σ denote the surjective $*$-representation $A \rtimes_\alpha G \to A \rtimes_{\alpha,r} G$. Let F denote set of functions

$$F = \left\{\Gamma_\psi \in A^1_+(A \rtimes_\alpha G) \mid \Gamma_\psi = \Gamma_{\varphi \circ \sigma} \text{ for some state } \varphi \text{ contained in } \tilde{\pi} \rtimes_r \lambda\right\}$$

Note that the set of vector states in $\tilde{\pi}_u \rtimes_r \lambda$ is exactly the set of functionals which give rise to $A^1_+(A \rtimes_\alpha G)$, by definition and the fact that we may view $A \rtimes_{\alpha,r} G$ as the image of $A \rtimes_\alpha G$ under $\tilde{\pi}_u \rtimes \lambda$.

Hence if $F \subseteq A^1_+(A \rtimes_\alpha G)$ is dense in the topology of weak* convergence uniformly on compacta, then, by the isomorphism between the statespace of $A \rtimes_\alpha G$ and $B(A \rtimes_\alpha G)$, we're done.

By the above remark and Proposition 2.3.9, it is enough to consider approximations of the functions of the form

$$\Gamma_{\tilde{\varphi}_z}(t)(x) = \varphi\left(\int_G z^*(s)\alpha_s(x)\alpha_{st}(z\left((st)^{-1}\right) \mathrm{d}s\right),$$

for suitable $\varphi \in \mathcal{S}(A)$ and $z \in C_c(G, A, \alpha)$, i.e., with $\varphi(z^* * z(e)) = 1$, since they span a dense subset of $A^1_+(A \rtimes_\alpha G)$: Let such φ and z be given. By density of E, pick a convergent net $\varphi_i \to \varphi$ in the weak*-topology. This implies that $\Gamma_{\tilde{\varphi}_{i_z}}(t) \to \Gamma_{\tilde{\varphi}_z}(t)$ weak*, uniformly on compacta of G. Since $\varphi_i \in E$, the corresponding representation $(\pi_{\varphi_i}, H_{\varphi_i})$ will be a sub-representation of (π, H), from which we conclude that $\Gamma_{\tilde{\varphi}_{i_z}} \in F$, finishing the proof. $\square$

In particular, this means that we may always consider $A \rtimes_{\alpha,r} G$ as the completion of $C_c(G, A, \alpha)$ under the regular representation induced by a faithful representation of A, which will be convenient, as we shall see, for in a large case of groups there is only one crossed product. The following useful lemma can be found in [17, Lemma 7.7.6]:

Lemma 2.3.15. If $\Gamma_\varphi \in B^1_+(A \rtimes_\alpha G)$ has compact support, then $\Gamma_\varphi \in A^1_+(A \rtimes_\alpha G)$

We recall one of the several equivalent definition of amenability for locally compact groups G:

Definition 2.3.16. A group G is *amenable* if there is a net of unit vectors $(f_i) \subseteq L^2(G)$ such that $(f_i * \tilde{f}_i)$ converges uniformly on compacta of G to 1, where $\tilde{f}(t) = \overline{f}(t^{-1})$ for $f \in L^2(G)$ and $t \in G$.

See e.g., [1, Appendix G], [17, Proposition 7.3.7 and 7.3.8] for the above and more characterizations.

With this, we obtain the following:

Theorem 2.3.17. If G is amenable, then whenever $G \overset{\alpha}{\curvearrowright} A$, we have $A \rtimes_\alpha G \cong A \rtimes_{\alpha,r} G$.

Proof.
By applying the above to $G \overset{1}{\curvearrowright} \mathbb{C}$ as the trivial action, we obtain a net $\Psi_i \in B_+(\mathbb{C} \rtimes_\alpha G)$ with compact support converging uniformly on compacta to 1, by amenability of G and

[17, Lemma 7.2.4]. It is easy to see that the pointwise product $\Psi_i\Gamma_\varphi \in B^1_+(A \rtimes_\alpha G)$ for all $\varphi \in \mathcal{S}(A \rtimes_\alpha G)$, and since Ψ_i has compact support, we see that $\Psi_i\Gamma_\varphi \in A_+(A \rtimes_\alpha G)$ and $\Psi_i\Gamma_\varphi \to \Gamma_\varphi$ in the topology of weak* uniformly convergence on compacta of G. Hence, $A_+(A \rtimes_\alpha G) \subseteq B_+(A \rtimes_\alpha G)$ is dense. In particular, this implies that if π denotes the universal faithful representation of $A \rtimes_\alpha G$, then $\ker \tilde{\pi}_u \rtimes \lambda \subseteq \ker \pi = 0$, by weak containment ([7, p. 80]), so $A \rtimes_\alpha G \cong \tilde{\pi}_u \rtimes \lambda(A \rtimes_\alpha G) = A \rtimes_{\alpha,r} G$. □

Combining the above , we obtain

Corollary 2.3.18. If $G \overset{\alpha}{\curvearrowright} A$ with G amenable, then $A \rtimes_\alpha G = \overline{\tilde{\pi} \rtimes \lambda(C_c(G, A, \alpha))}$ for any faithful representation π of A.

The above allows us to concretely work with the crossed products of a large class of groups which include finite groups, abelian groups, compact groups and others. This will be very useful.

For a fixed group G, the crossed product constructions described above are in fact functors from the category of G-algebras A to the category of C^*-algebras, where the objects in G-algebras are C^*-algebras A equipped with an action $G \overset{\alpha}{\curvearrowright} A$ and morphism given by G-equivariant $*$-morphism, i.e., $\varphi\colon A \to B$ for G-algebras is a morphism if the diagram

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "\alpha_g"'] & B \arrow[d, "\beta_g"] \\ A \arrow[r, "\varphi"'] & B \end{tikzcd} \tag{2.12}$$

commutes for every $g \in G$. This can bee seen by the following:

Proposition 2.3.19. Suppose that $G \overset{\alpha}{\curvearrowright} A$ and $G \overset{\beta}{\curvearrowright} B$ and $\varphi\colon A \to B$ is a G-equivariant morphism. Then the map $\tilde{\varphi}\colon C_c(G, A, \alpha) \to C_c(G, B, \beta)$, $\tilde{\varphi}(f)(s) := \varphi(f(s))$, extends to a $*$-morphism $A \rtimes_\alpha G \to B \rtimes_\beta G$.

Proof.
It is clear that $\tilde{\varphi}$ is a $*$-morphism. To see that it extends, we show that $\|\tilde{\varphi}(f)\| \leqslant \|f\|$, where the norms comes from the embeddings $C_c(G, A, \alpha) \subseteq A \rtimes_\alpha G$ and $C_c(G, B, \beta) \subseteq B \rtimes_\beta G$. For this, let (π, u) be a covariant representation of $G \overset{\beta}{\curvearrowright} B$. Then, clearly $(\pi \circ \varphi, u)$ is a covariant representation of $G \overset{\alpha}{\curvearrowright} A$, and moreover, if $f \in C_c(G, A, \alpha)$, then

$$(\pi \circ \varphi) \rtimes u(f) = \int_G \pi(\varphi(f(s)))u_s \mathrm{d}s = \int_G \pi(\tilde{\varphi}(f)(s))u_s \mathrm{d}s = \pi \rtimes u \circ \tilde{\varphi}(f),$$

so $\pi \circ \varphi \rtimes u = \pi \rtimes u \circ \tilde{\varphi}$, and hence

$$\|\pi \rtimes u \circ \tilde{\varphi}(f)\| = \|\pi \circ \varphi \rtimes u(f)\| \leqslant \|f\|,$$

since the integrated form of a covariant representation is norm decreasing. Hence, since the norm of $\tilde{\varphi}(f)$ is the supremum of integrated forms of covariant representations of $G \overset{\beta}{\curvearrowright} B$, we see that $\tilde{\varphi}$ extends to a morphism $A \rtimes_\alpha G \to B \rtimes_\beta G$. $\square$

The above proof applies to the case of the reduced crossed product. This is easily seen by applying the above proof to regular representations $(\tilde{\pi}, \lambda)$ rather than any covariant representation. The remaining properties for a functor are easily verified, i.e., preserves composition and sends the identity map to the identity map.

2.4 The case of Abelian groups and Takai Duality

In the 70's, Hiroshi Takai published a paper (see [22]) showing that the crossed product construction in a way captures the duality of locally compact abelian groups, in the sense of $\widehat{\widehat{G}} \cong G$, where $\widehat{G}$ denotes the dual group of G. He showed that taking the crossed product twice with G (in the sense that $\widehat{G} \overset{\widehat{\alpha}}{\curvearrowright} A \rtimes_\alpha G$), one obtains an isomorphism

$$A \otimes \mathbb{K}(L^2(G)) \cong (A \rtimes_\alpha G) \rtimes_{\widehat{\alpha}} \widehat{G},$$

and in fact that both of these are G-dynamical systems and the isomorphism is covariant. We will follow Raeburns take on this result, found in [18], since it is more aligned with the results we've developed so far.

revisit if really raeburn or if pedersen

Note 2.4.1
In the above, and in the rest, unless otherwise stated, we will use $A \otimes B$ between two C^*-algebras to mean the maximal tensor product, and we recall that for a commutative C^*-algebra, the maximal and minimal tensor products coincide, i.e., which are nuclear. For this, see e.g., [5].

For this, we first show that the left-hand side is a C^*-dynamical system for any (also non-abelian) locally compact group G:

Proposition 2.4.2. Let $G \overset{\alpha}{\curvearrowright} A$ and let ρ denote the right-regular representation of G on $L^2(G)$, $\rho_t f(s) = f(st)\Delta(t)^{\frac{1}{2}}$. Then $A \otimes \mathbb{K}(L^2(G)), G, \alpha \otimes \mathrm{Ad}\rho)$ is a C^*-dynamical system.

Proof.
The proof relies on the fact that in a Von Neumann algebra $\mathcal{W}$ (or Borel $*$-algebra B) equipped with a strongly continuous action $G \overset{\alpha}{\curvearrowright} \mathcal{W}$, the set of elements $x \in \mathcal{W}$ such that $t \mapsto \alpha_t(x)$ is norm-continuous forms a C^*-algebra which is invariant under G, see e.g., [17, Proof of Lemma 7.5.1] or [3, Proposition III.3.2.4]. .

easy proof for C^*-algebra as well, no need to cite

With this in mind, let (π, u) be a covariant representation of (A, G, α) on a Hilbert space H with π faithful. Define a representation σ of G on $H \otimes L^2(G)$ by

$$\sigma = u \otimes \rho,$$

so that σ is a strongly continuous unitary representation. For $x \in A \subseteq \mathbb{B}(H)$ and $y \in \mathbb{K}(L^2(G))$ we see that

$$\sigma_t(x \otimes y)\sigma_t^* = \alpha_t(x) \otimes \rho_t y \rho_t^* \in A \otimes \mathbb{K}(L^2(G)) \subseteq \mathbb{B}(H \otimes L^2(G)).$$

Hence $A \otimes \mathbb{K}(L^2(G))$ is invariant under Ad_σ. Moreover, for finite dimensional $y \in \mathbb{K}(L^2(G))$, the map $t \mapsto \mathrm{Ad}_{\sigma_t}(y)$ is norm continuous. The closure of $F = \mathrm{Span}\{a \otimes y \mid a \in A,\ y \text{ finite rank}\}$

equals $A \otimes \mathbb{K}(L^2(G))$. The C^*-algebra of elements $x \otimes y \in A \otimes \mathbb{K}(L^2(G))$ such that $t \mapsto \sigma_t x \otimes y\sigma_t^*$ is continuous and Ad_{σ_s} invariant for all s and contains F hence equals $A \otimes \mathbb{K}(L^2(G))$. Letting $\alpha \otimes \mathrm{Ad}_\rho := \mathrm{Ad}_\sigma$, we see that $(A \otimes \mathbb{K}(L^2(G)), G, \alpha \otimes \mathrm{Ad}_\rho)$ is a C^*-dynamical system. $\square$

The way we show Takai duality is that we will show that the C^*-dynamical system defined in the above proposition is covariantly isomorphic to another dynamical system, and then finally showing an isomorphism between this system and the dual system when G is abelian. We show the first step in this in the following:

Theorem 2.4.3. Let $G \overset{\alpha}{\curvearrowright} A$. Then there is a C^*-dynamical system $(C_0(G, A) \rtimes_{\gamma,r} G, G, \sigma)$ where we define the actions $G \overset{\gamma}{\curvearrowright} C_0(G, A)$ and $G \overset{\sigma}{\curvearrowright} C_0(G, A)$ by

$$\sigma_s(f)(t) = f(ts) \text{ and } \gamma_s f(t) = \alpha_s(f(s^{-1}t)),$$

for $f \in C_0(G, A)$ and $s, t \in G$. This system is covariantly isomorphic to $(A \otimes \mathbb{K}(L^2(G)), G, \alpha \otimes \mathrm{Ad}_\rho)$.

Proof.

Add as example the dynamical systems

Represent A faithfully on a Hilbert space $\mathcal{H}$, and define a representation of $C_0(G, A)$ on $L^2(G, A)$ as multiplication operators, i.e.,

$$\pi(f)\xi(s) = f(s)\xi(s),$$

for $f \in C_0(G, A)$, $\xi \in L^2(G, A)$ and $s \in G$. This is easily seen to be a faithful representation, so by Theorem 2.3.14 we obtain a faithful representation $\tilde{\pi} \rtimes \lambda$ of $C_0(G, A) \rtimes_{\gamma,r} G$ on $L^2(G, L^2(G, H)) \cong L^2(G \times G, H)$. Consider $C_c(G \times G, A) \subseteq C_c(G, C_0(G, A))$, via the map $(f\colon (x, y) \mapsto f(x, y)) \mapsto (x \mapsto (y \mapsto f(x, y)))$, and note that there is not necessarily equality (e.g., for the case $G = \mathbb{R}$ and $A = \mathbb{C}$). For $z \in C_c(G \times G, A)$, we see that

$$\tilde{\pi} \rtimes \lambda(z)\xi(s, t) = \int_G \alpha_{s^{-1}}(z(r, st))\xi(r^{-1}s, t)\mathrm{d}r.$$

Define the operator w on $L^2(G \times G, H)$ by

$$w\xi(s, t)\xi(st, t)\Delta(t)^{\frac{1}{2}},$$

then w is a unitary operator with adjoint $w^*\xi(s, t) = \xi(st^{-1}, t)\Delta(t)^{-\frac{1}{2}}$. We then calculate for $z \in C_c(G \times G, A)$

$$w^*(\tilde{\pi} \rtimes \lambda(z))w(\xi)(s, t) = \int_G \alpha_{ts^{-1}}(z(r, s))w\xi(r^{-1}st^{-1}, t)\mathrm{d}r\Delta(t)^{-\frac{1}{2}} = \int_G \alpha_{ts^{-1}}(z(r, s))\xi(r^{-1}s, t)\mathrm{d}r,$$

for all $\xi \in L^2(G\times G, H)$ and $s,t \in G$. It is not hard to see that any function $f \in C_c(G\times G, A)$ can be approximated by linear combinations of maps of the form $(s,t) \mapsto z(s)g(t)\alpha_t(x)$ for $z, g \in C_c(G)$ and $x \in A$, and hence also by functions of the form

$$f(s,t) = \alpha_t(x)z(s^{-1}t)g(t)\Delta(s^{-1}t),$$

for fixed $z, g \in C_c(G)$ and $x \in A$. Applying Ad_w to this we get

$$\begin{aligned} w^*(\tilde{\pi} \rtimes \lambda(f))w(\xi)(s,t) = \int_G \alpha_{ts^{-1}}(f(r,s))\xi(r^{-1}s,t)\mathrm{d}r &= \int_G \alpha_{ts^{-1}}(\alpha_s(x)z(r^{-1}s)g(s)\Delta(r^{-1}s))\xi(r^{-1}s,t)\mathrm{d}r \\ &= \int_G \alpha_t(x)z(r^{-1}s)g(s)\xi(r^{-1}s,t)\Delta(r^{-1}s)\mathrm{d}r \\ &= \alpha_t(x)g(s)\int_G z(r)\xi(r,t)\mathrm{d}r. \end{aligned}$$

Consider the faithful representation $\breve{\pi}$ of A on $L^2(G,H)$ by $\breve{\pi}(x)\xi(s) = \alpha_s(x)\xi(s)$. Given $K \in \mathbb{K}(H)$ and $x \in A$, if U denotes the unitary operator identifying $L^2(G)\otimes L^2(G,H)$ with $L^2(G, L^2(G,H)$, then for $h \in L^2(G)$ and $\xi \in L^2(G,H)$ we see that

$$U(\breve{\pi}(x)\otimes K(h\otimes\xi))(s)(t) = \alpha_s(x)h(s)K\xi(t),$$

so after identifying $L^2(G, L^2(G,H))$ with $L^2(G\times G, H)$, we see that for f as above:

$$w^*(\tilde{\pi} \rtimes \lambda(f))w = \breve{\pi}\otimes p_{zg} \in A \otimes \mathbb{K}(L^2(G)) \subseteq \mathbb{B}(L^2(G\times G, H)),$$

where is $p_{z,g}$ the rank-one operator $\eta \mapsto \langle \eta, \overline{z}\rangle g$, $\eta \in L^2(G)$. In particular, this implies that

$$A\otimes \mathbb{K}(L^2(G)) \subseteq w^*(\tilde{\pi} \rtimes \lambda(C_0(G,A) \rtimes_{\lambda,r} G))w,$$

since $\breve{\pi}$ is faithful and $\mathbb{K}(L^2(G))$ is the closure of the span of rank-one operators. Since we may approximate $C_c(G\times G, A)$ by linear combinations of forms $(s,t)\alpha_s(x)f(r^{-1}s)g(s)\Delta(r^{-1}s)$, and $C_c(G\times G, A)$ is dense in $C_0(G,A)\rtimes_{\lambda,r} G$, the above inclusion is an equality, so the two algebras are spatially isomorphic with respect to w.

Denote also by σ_t the representation of G on $C_c(G, C_0(G,A))$, $\sigma_t f(x)(y) = f(s)(yt)$. Then σ_t extends to all of $C_0(G,A)\rtimes_{\lambda,r} G$ by continuity and density. If f is as above, then

$$\begin{aligned} w^*(\tilde{\pi} \rtimes \lambda\sigma_r(f))w\xi(s,t) &= \alpha_{tr}(x)\underbrace{g(sr)\Delta(r)}_{=\rho_r g(s)\Delta(r)^{\frac{1}{2}}}\int_G \underbrace{z(qr)}_{=\rho_r z(q)\Delta(r)^{-\frac{1}{2}}}\xi(q,t)\mathrm{d}q \\ &= \alpha_t(\alpha_r(x))\rho_r g(s)\int_G \rho_r z(q)\xi(q,t)\mathrm{d}q \\ &= (\breve{\pi}(\alpha_r(x))\otimes \rho_r p_{z,g}\rho_r^*)\xi(s,t) \\ &= (\breve{\pi}(\alpha_r(x))\otimes \mathrm{Ad}_{\rho_r}p_{z,g})\xi(s,t). \end{aligned}$$

In particular, after identifying $\breve{\pi}(A)$ with A, we have

$$w^*(\tilde{\pi} \rtimes \lambda(\sigma_r(f)))w = (\alpha \otimes \mathrm{Ad}_\rho)_r(x \otimes p_{z,g}),$$

and hence by density, we see that $(C_0(G, A) \rtimes_{\lambda,r}, G, \sigma)$ is covariantly isomorphic to $(A \otimes \mathbb{K}(L^2(G)), G, \alpha \otimes \mathrm{Ad}_\sigma)$ via w, in the sense that w implements a G-equivariant spatial isomorphism between the two systems. □

define category of C^*-systems and covariant isomorphism

Applying the above to the case where $A = \mathbb{C}$ and $\alpha = 1$, we obtain the Stone-Von Neumann theorem for reduced crossed products:

Corollary 2.4.4. Let lt denote the representation of G on $C_0(G)$ given by left-translation, i.e., $\mathrm{lt}_s f(t) = f(s^{-1}t)$. Then lt corresponds to γ in the above case with $G \overset{1}{\curvearrowright} \mathbb{C}$, so

$$C_0(G) \rtimes_{\mathrm{lt},r} G \cong \mathbb{K}(L^2(G)) \otimes \mathbb{C} \cong \mathbb{K}(L^2(G)).$$

It is worth noting that the result also holds for the full crossed product, however, we mention it without proof. The inclined reader is refered to [23, Theorem 4.24] or [19, §C.6]. In particular, this implies that

$$C_0(G) \rtimes_{\mathrm{lt},r} G \cong C_0(G) \rtimes_{\mathrm{lt}} G,$$

for all groups G.

In the following, we will assume that G in addition to being locally compact is abelian, and we will use $+$ to denote the binary operation. We briefly recall the definition of the dual group $\widehat{G}$ and associated notions. For a better exposition on these topics, we again refer the reader to the excellent work [8].

Definition 2.4.5. For a locally compact group G, the dual group $\widehat{G}$ is the group of continuous homomorphisms $\xi\colon G \to S^1$, where S^1 denotes the circle. For $\xi \in \widehat{G}$, we will write $(t, \xi) := \xi(t)$. An alternative and equivalent definition of $\widehat{G}$ is as the set of irreducible unitary representations of G, which by commutativity of G are all one-dimensional, or we may identify $\widehat{G}$ with the spectrum, $\sigma(L^1(G))$, of the group algebra via the pairing

$$\xi(f) := \int_G (t, \xi) f(t) \mathrm{d}t, \tag{2.13}$$

for $f \in L^1(G)$ and $\xi \in \widehat{G}$.

Usually, one will examine the Fourier transformation of $M(G)$, where $M(G)$ is the space of bounded Radon Measures on G. It is useful to recall also that $L^1(G) \subseteq M(G)$ by assigning f to $f \cdot \mathrm{d}x$, where $\mathrm{d}x$ is the integrand of the fixed Haar measure on G. However, we will instead consider the inverse Fourier transformaton:

Definition 2.4.6. The inverse Fourier transformation of a measure $\mu \in M(G)$ is the bounded continuous map on $\widehat{G}$ given by

$$\widehat{\mu}(\xi) := \int_G (t, \xi) \mathrm{d}\mu(t).$$

For $f \in L^1(G) \subseteq M(G)$, we have

$$\widehat{f}(\xi) := \int_G (t, \xi) f(t) \mathrm{d}t.$$

Note that $\widehat{\mu} \in C_b(\widehat{G})$ and $\widehat{f} \in C_0(\widehat{G})$ for $\mu \in M(G)$ and $f \in L^1(G)$.

And it will be worth noting that there is a topological duality between G and $\widehat{G}$, in the following sense:

Proposition 2.4.7. A group G is discrete if and only if $\widehat{G}$ is compact and vice versa.

It will also be important to note that there is a one-to-one correspondance between unitary representations of G and representations of $M(G)$ as a Banach $*$-algebra:

Proposition 2.4.8. There is a bijective correspondance between unitary representations of G and representations of $M(G)$ which restricts to a non-degenerate representation of $L^1(G)$. If u is a unitary representation of G on $\mathcal{H}$, then the associated representation of $L^1(G)$ is given by

$$u_f(\xi) = \int_G u_s \xi f(s) \mathrm{d}s,$$

in the weak sense, i.e., that

$$\langle u_f(\xi), \eta \rangle = \int_G \langle u_s \xi, \eta \rangle f(s) \mathrm{d}s,$$

for $\xi, \eta \in \mathcal{H}$.

For the rest of this section, we will assume that G is any abelian locally compact group, and we shall think of $A \rtimes_\alpha G$ as operators on $L^2(G, \mathcal{H}_u)$, where $\mathcal{H}_u$ is the universal Hilbert space for A.

Definition 2.4.9. Suppose that $G \overset{\alpha}{\curvearrowright} A$. We define the dual action, $\widehat{\alpha}$, of $\widehat{G}$ on $A \rtimes_\alpha G$ on the dense subset $C_c(G, A, \alpha)$ by

$$\widehat{\alpha}_\xi y(t) = (t, \xi) y(t),$$

for $y \in C_c(G, A, \alpha)$, $\xi \in \widehat{G}$ and $t \in G$.

To see that this in fact is a well-defined action, we use the following:

Lemma 2.4.10. Let $G \overset{\alpha}{\curvearrowright} A$. Define a unitary representation u of $\widehat{G}$ on $L^2(G, H)$ by

$$u_\chi \xi(t) = (t, \chi)\xi(t).$$

Then, with $\widehat{\alpha}_\chi$ as above, it holds that

$$\widehat{\alpha}_\chi = \mathrm{Ad} u_\chi,$$

so that $(A \rtimes_\alpha G, \widehat{G}, \widehat{\alpha})$ becomes a C^*-dynamical system.

Proof.
Let $y \in C_c(G, A, \alpha)$, so that $y\xi(s) = \int_G \alpha_{-s}(y(t))\xi(t-s)\mathrm{d}s$. Then

$$\begin{aligned}
\mathrm{Ad} u_\chi(y)\xi(t) &= (t,\chi)\int_G \alpha_{-t}(y(s))(t,-\chi)(-s,-\chi)\xi(t-s)\mathrm{d}s \\
&= \int_G \alpha_{-t}((s,\chi)y(s))\xi(t-s)\mathrm{d}s \\
&= \widehat{\alpha}_\chi y\xi(s).
\end{aligned}$$

Hence, since $\widehat{\alpha}_\chi y \in C_c(G, A, \alpha)$, and conjugation by unitaries is an isometry, we may define

$$\widehat{\alpha}_\chi := (A \rtimes_\alpha G \ni x \mapsto \mathrm{Ad} u_\chi x \in A \rtimes_\alpha G) \in \mathrm{Aut}(A \rtimes_\alpha G),$$

and we will think of it as the automorphism satisfying $\widehat{\alpha}_\chi y(s) = (s,\chi)y(s)$ for $y \in C_c(G, A, \alpha)$. ☐

The above example is one of the motivations for the notion of a G-product:

Definition 2.4.11. Let G be a group with dual group $\widehat{G}$. We say that a C^*-algebra A is a *G-product* if:

1. There is a homomorphism $\lambda\colon G \to \mathcal{U}(M(A))$ such that the map $t \mapsto \alpha_t y$, $y \in A$ is continuous,
2. There is a homomorphism $\widehat{\alpha}\colon \widehat{G} \to \mathrm{Aut}(A)$ such that $(A, \widehat{G}, \widehat{\alpha})$ is a C^*-dynamical system satisfying

   $$\widehat{\alpha_\chi}\lambda_t = (t,\chi)\lambda_t,$$

   for all $t \in G$ and $\chi \in \widehat{G}$.

Moreover, we say that $x \in M(A)$ satisfies *Landstad's conditions* if:

1. $\widehat{\alpha}_\chi(x) = x$ for all $\chi \in \widehat{G}$,
2. $x\lambda_f \in A$ and $\lambda_f x \in A$ for all $f \in L^1(G)$,
3. the map $t \mapsto \lambda_t x \lambda_t^*$ is continuous.

Lemma 2.4.12. Suppose that $G \overset{\alpha}{\curvearrowright} A$. Then with the above action, it holds that $G \rtimes_\alpha A$ is a G-product, and each element of $A \subseteq M(A \rtimes_\alpha G)$ satisfies Landstad's condition.

Proof.
Let u and $\widehat{\alpha}$ be as above. Let λ_t be the operator on $L^2(G,H)$ given by

$$\lambda_t \xi(s) = \xi(s-t).$$

Then we may view λ as a map $G \to \mathcal{U}(M(A \rtimes_\alpha G))$, $\lambda_t y(s) = \alpha_t(y(s-t))$ for $y \in C_c(G,A) \subseteq A \rtimes_\alpha G$, by proposition 2.2.20 such that $t \mapsto \lambda_t y$ is continuous for all $y \in A \rtimes_\alpha G$. Combining this with the calculations in lemma 2.4.10, we see that $A \rtimes_\alpha G$ is a G-product, since

$$u_\chi \lambda_s u_{-\chi} \xi(t) = (t,\chi)(t-s,-\chi)\xi(t-s) = (s,\chi)\lambda_s \xi(t),$$

for all $\xi \in L^2(G,H)$, $s,t \in G$ and $\chi \in \widehat{G}$. For the last claim, we see that if $a \in A \subseteq M(A \rtimes_\alpha G)$, then $\widehat{\alpha}_\chi a = u_\chi a u_\chi^* = a$, since $ay(t) = \alpha_t(a)y(t)$ for all $y \in C_c(G,A,\alpha)$, so

$$\widehat{\alpha}_\chi(a)y(t) = (t,\chi)\alpha_{-t}(a)(t,-\chi)y(t) = \alpha_{-t}(a)y(t) = ay(t).$$

If $f \in C_c(G)$ and $x \in A$, then for all $\xi \in L^2(G,H)$ and $s \in G$ we see that

$$x\lambda_f \xi(s) = \int_G f(t)x\lambda_t\xi(s)\mathrm{d}t = \int_G f(t)\alpha_{-t}(x)\lambda_t\xi(s)\mathrm{d}t = \int_G (f \otimes x)(t)\lambda_t\xi(s)\mathrm{d}t = (f \otimes x)\xi(s),$$

where we have identified $f \otimes x \in C_c(G,A,\alpha) \subseteq A \rtimes_\alpha G$. In particular $x\lambda_f \in A \rtimes_\alpha G$ and likewise $\lambda_f x \in A \rtimes_\alpha G$ for $f \in C_c(G)$ and hence for $f \in L^1(G)$, by dominance of the norm on $A \rtimes_\alpha G$. □

Now, we are going to show the theorem of Takai by combining the previous results with the following: If $G \overset{\alpha}{\curvearrowright} A$, we let ι denote the trivial action of $\widehat{G}$ on A. Then

Lemma 2.4.13. For $G \overset{\alpha}{\curvearrowright} A$, there is an isomorphism $A \rtimes_\iota \widehat{G} \cong C_0(G,A)$. Moreover, the isomorphism is spatially implemented by a unitary u.

Proof.
Let $C_0(G,A)$ be faithfully represented on $L^2(G,H)$ as multiplication operators $f\xi(s) = f(s)\xi(s)$, where A is faithfully represented on H. By the Plancheral Theorem, we have

$L^2(G) \cong L^2(\widehat{G})$ via the (proper) Fourier transformation, and so we may define the isometry $u\colon L^2(G,H) \to L^2(\widehat{G},H)$ by

$$u\xi(\chi) = \int_G \xi(t)\overline{(t,\chi)}\mathrm{d}t, \text{ for } \xi \in L^2(G,H) \text{ and } \chi \in \widehat{G},$$

so that

$$u^*\eta(t) = \int_{\widehat{G}} \eta(\chi)(t,\chi)\mathrm{d}\chi, \text{ for } \eta \in L^2(\widehat{G},H) \text{ and } t \in G,$$

where the integrals are defined in the L^2-sense. Then, for all $f \in C_c(\widehat{G},A,\iota)$ we see that

$$\begin{aligned} fu\xi(\chi) = \int_{\widehat{G}} \iota_{-\sigma}(f(\chi))\lambda_\sigma(u\xi)(\chi)\mathrm{d}\sigma &= \int_{\widehat{G}} f(\chi)\int_G \xi(t)\overline{(t,\chi-\sigma)}\mathrm{d}t\mathrm{d}\sigma \\ &= \int_{\widehat{G}}\int_G f(\chi)(t,\sigma)\xi(t)\overline{(t,\chi)}\mathrm{d}t\mathrm{d}\sigma \\ &= \int_{\widehat{G}} \widehat{f}(t)\xi(t)\overline{(t,\chi)}\mathrm{d}t \\ &= \int_{\widehat{G}} (\widehat{f}\xi)(t)\overline{(t,\chi)}\mathrm{d}t \\ &= u\widehat{f}\xi(\chi) \end{aligned}$$

implying that $u^*fu = \widehat{f} \in C_0(G,A)$. By density of $C_c(\widehat{G},A,\iota) \subseteq A \rtimes_\iota \widehat{G}$, it follows that u implements the desired isomorphism. □

We are now ready to define the C^*-dynamical system which we will show is covariantly isomorphic to $(A\otimes\mathbb{K}(L^2(G)), G, \alpha\otimes\mathrm{Ad}\lambda)$ and $\left((A\rtimes_\alpha G)\rtimes_{\widehat{\alpha}}\widehat{G}, G, \widehat{\widehat{\alpha}}\right)$:

Proposition 2.4.14. Suppose that $G \overset{\alpha}{\curvearrowright} A$. Define for each $t \in G$ the map $\beta_t\colon C_c(\widehat{G},A,\iota) \to C_c(\widehat{G},A,\iota)$ by

$$\beta_t f(\chi) = \overline{(t,\chi)}\alpha_t(f(\chi)),$$

for $f \in C_c(\widehat{G},A,\iota)$ and $\chi \in \widehat{G}$. Then β_t extends to an automorphism of $A \rtimes_\iota \widehat{G}$ such $t \mapsto \beta_t$ is an action of G on $A \rtimes_\iota \widehat{G}$. With this β, we have a covariant isomorphism $(A\rtimes_\iota \widehat{G})\rtimes_\beta G \cong (A\rtimes_\alpha G)\rtimes_{\widehat{\alpha}}\widehat{G}$.

reformulate, you don't show covariant isomorphism

Proof.
Define $\gamma_t\colon C_0(G,A) \to C_0(G,A)$ to be the diagonal action

$$\gamma_t y(s) = \alpha_t(y(s-t)),$$

here diagonal comes from the fact that $\gamma_t = \alpha_t\otimes\mathrm{lt}_t$, when we identify $C_0(G,A) \cong C_0(G)\otimes A$.

add diagonal to earlier introduction

Let u be as in Lemma 2.4.13, $\widehat{\beta_s(y)}\eta(t) = u^*yu$. Then, if $\xi \in L^2(G,H)$ we see that

$$\widehat{\beta_s(y)}\xi(t) = \widehat{\beta_s(y)}(t)\xi(t) = \int_{\widehat{G}} (t,\sigma)\beta_s(y)(\sigma)\xi(t)\mathrm{d}\sigma = \int_{\widehat{G}} (t-s,\sigma)\alpha_s(y(\sigma))\xi(t)\mathrm{d}\sigma$$

and we also see that

$$\gamma_s(\widehat{y})\xi(t) = \alpha_s(\widehat{y}(t-s))\xi(t) = \int_{\widehat{G}} (t-s,\sigma)\alpha_s(y(\sigma))\xi(t)\mathrm{d}\sigma = \widehat{\beta_s(y)}\xi(t),$$

In particular, this allows us to extend β_t to an automorphism of $A \rtimes_\iota \widehat{G}$ such that the associated dynamical system is covariantly isomorphic to $(C_0(G,A),G,\gamma)$ via u.

We now show that the system $(\widehat{G} \rtimes_\iota A, G, \beta)$ is covariantly isomorphic to $(G \rtimes_\alpha A, \widehat{G}, \widehat{\alpha})$. For this, we note that an easy calculation shows for $x \in C_c(\widehat{G} \times G, A)$ and $\xi \in L^2(\widehat{G} \times G, H)$ that

$$x\xi(t,\chi) = \int_G \int_{\widehat{G}} \alpha_{-t}(x(t,\sigma))\xi(t-s,\chi-\sigma)(t,\sigma)\mathrm{d}\sigma\mathrm{d}s,$$

and similarly for $y \in C_c(G \times \widehat{G}, A)$ and $\eta \in L^2(G \times \widehat{G}, H)$ we have

$$y\eta(\chi,t) = \int_{\widehat{G}} \int_G \alpha_{-t}(y(\sigma,s))\xi(\chi-\sigma,t-s)\overline{(t,\sigma)}\mathrm{d}s\mathrm{d}\sigma,$$

for all $\chi \in \widehat{G}$ and $t \in G$. Define an isomorphism $\Phi\colon C_c(G \times \widehat{G}, A) \to C_c(\widehat{G} \times G, A)$ and the isometry $w\colon L^2(G \times \widehat{G}, H) \to L^2(\widehat{G} \times G, H)$ by

$$\Phi(y)(\chi,t) = (t,\chi)y(t,\chi) \text{ and } w\xi(\chi,t) = \overline{(t,\chi)}\xi(t,\chi)$$

so that

$$\Phi^{-1}(x)(t,\chi) = \overline{(t,\chi)}x(\chi,t) \text{ and } w^*\eta(t,\chi) = (t,\chi)\eta(\chi,t)$$

for $t \in G$, $\chi \in \widehat{G}$. Note now that for $\sigma,\chi \in \widehat{G}$ and $s,t \in G$ we have

$$(s,\sigma)\overline{(t-s,\chi-\sigma)(s,\sigma)} = \overline{(t,\chi)}(t,\sigma),$$

hence for $x \in C_c(G \times \widehat{G}, A)$ and $\xi \in L^2(G \times \widehat{G}, H)$ we get using the above identities:

$$\begin{aligned}
\Phi(x)w\xi(\chi,t) &= \int_{\widehat{G}} \int_G \alpha_{-t}(x(s,\sigma))\xi(t-s,\sigma-\chi)(s,\sigma)\overline{(t-s,\sigma-\chi)(s,\sigma)}\mathrm{d}s\mathrm{d}\sigma \\
&= \overline{(t,\chi)} \int_{\widehat{G}} \int_G \alpha_{-t}(x(s,\sigma))\xi(t-s,\sigma-\chi)(t,\sigma)\mathrm{d}s\mathrm{d}\sigma \\
&= wx\xi(t,\chi),
\end{aligned}$$

so that $w^*\Phi(x)w = x$ for all $x \in C_c(G \times \widehat{G}, A)$. By density, it follows that the two algebras $(G \rtimes_\alpha A) \rtimes_{\widehat{\alpha}} \widehat{G}$ and $(A \rtimes_\iota \widehat{G}) \rtimes_\beta G$ are isomorphic. $\square$

We are now ready to prove the main theorem by combining the above:

Theorem 2.4.15 (Takai Duality)**.** Suppose that $G \overset{\alpha}{\curvearrowright} A$. Then the double dual system $\left((A \rtimes_\alpha G) \rtimes_{\widehat{\alpha}} \widehat{G}, G, \widehat{\widehat{\alpha}}\right)$ is covariantly isomorphic to the C^*-dynamical system $(A \otimes \mathbb{K}(L^2(G)), G, \alpha \otimes \mathrm{Ad}\rho)$.

Proof.
We already have the following diagram of isomorphisms

$$A \otimes \mathbb{K}(L^2(G)) \xrightarrow{\cong} C_0(G, A) \rtimes_\gamma G \xrightarrow{\cong} (A \rtimes_\iota \widehat{G}) \rtimes_\beta G \xrightarrow{\cong} (A \rtimes_\alpha G) \rtimes_{\widehat{\alpha}} \widehat{G}$$

Moreover, the dynamical systems $(A \otimes \mathbb{K}(L^2(G), G, \alpha \otimes \mathrm{Ad}\rho)$ and $(C_0(G, A) \rtimes_\gamma G, G, \sigma)$ are covariantly isomorphic, c.f. theorem 2.4.3. We finish the proof by showing that $(C_0(G, A) \rtimes_\gamma G, G, \sigma)$ is covariantly isomorphic to $((A \rtimes_\alpha G) \rtimes_{\widehat{\alpha}} \widehat{G}, G, \widehat{\widehat{\alpha}})$, which is done by showing the relation on the generating set of functions $z \in C_c(G, C_0(G, A))$,

$$z(s,t) = xf(s)\widehat{g}(t),$$

for some $x \in A$, $f \in C_c(G)$ and $g \in C_c(\widehat{G})$. Denote by $\breve{z} \in C_c(G \times \widehat{G}, A)$ the image of z under the isomorphism above, so that

$$\breve{z}(s,\chi) = xf(s)g(\chi),$$

for $(s,\chi) \in G \times \widehat{G}$, and note that the map $g_r \colon \chi \mapsto g(\chi)(r,\chi)$ is mapped to the function $\widehat{g}_r \colon t \mapsto \widehat{g}(r+t)$. Denote by Φ the last isomorphism in the above, so that $\Phi(\breve{z}) \in C_c(\widehat{G} \times G, A)$ is the map

$$\Phi(\breve{z})(\chi, s) = (s,\chi)\breve{z}(s,\chi) = xf(s)g(\chi)(s,\chi).$$

In particular, for the action $G \overset{\sigma}{\curvearrowright} C_0(G, A) \rtimes_\gamma G$ is the action $\sigma_r y(s,t) = y(s, t+r)$ for $y \in C_c(G \times G, A)$, then we see that

$$\sigma_r z(s,t) = xf(s) + \widehat{g}(t+r) = xf(s)\widehat{g}_r(t),$$

whence we see that

$$\widetilde{\sigma_r z}(s,\chi) = xf(s)g(\chi)(r,\chi),$$

so that

$$\Phi(\widetilde{\sigma_r z})(\chi, s) = xf(s)g(\chi)(s,\chi)(r,\chi) = \Phi(\breve{z})(\chi,s)(r,\chi).$$

Note also, that if $G \overset{\hat{\hat{\alpha}}}{\curvearrowright} (A \rtimes_\alpha G) \rtimes_{\hat{\alpha}} \hat{G}$ is the dual of the dual action, then for $y \in C_c(\hat{G} \times G, A) \subseteq (A \rtimes_\alpha G) \rtimes_{\hat{\alpha}} \hat{G}$ then

$$\hat{\hat{\alpha}}_r y(\chi, t) = (r, \chi) y(\chi, t),$$

whence we see that $\hat{\hat{\alpha}}_r(\Phi(\breve{z})) = \Phi(\widetilde{\sigma_r(z)})$, and since z generates $C_0(G, A) \rtimes_\gamma G$, this shows the desired covariant isomorphism of the systems $(C_0(G, A) \rtimes_\gamma G, G, \sigma)$ and $((A \rtimes_\alpha G) \rtimes_{\hat{\alpha}} \hat{G}, G, \hat{\hat{\alpha}})$. □

Before we begin our discussion of simplicity in crossed product in the next chapter, we will actually go back to the concept of a G-product and properties thereof, since it will give us a useful characterization of crossed products. We follow the guidelines of [13] in doing this, but the results are due to Landstad ([11]).

Definition 2.4.16. An element $y \in M(A)_+$, for a G-product A, is said to be *$\hat{\alpha}$-integrable* if there is an element $I(y) \in M(A)_+$ such that

$$\varphi(I(y)) = \int_G \varphi(\hat{\alpha}_\chi(y)) \mathrm{d}\chi,$$

for all $\varphi \in A^*$. If $x \in M(A)$ is such that x may be decomposed into $\hat{\alpha}$-integrable positive parts, we then say that x is also $\hat{\alpha}$-integrable.

Note 2.4.17

If $0 \leqslant x \leqslant y \in M(A)_+$ and y is $\hat{\alpha}$-integrable, then both x is $\hat{\alpha}$-integrable. This can be seen from considering

finish easy argument using statespace theory

Lemma 2.4.18. Let A be a G-product. Assume that $x, y \in M(B)$ satisfy that x^*x and y^*y are $\hat{\alpha}$-integrable. Then y^*x is $\hat{\alpha}$-integrable with

$$\|I(y^*x)\|^2 \leqslant \|I(x^*x)\| \, \|I(y^*y)\| \, .$$

Proof.

It follows from the polarization identity that y^*x is $\hat{\alpha}$-integrable, since $0 \leqslant (x + i^k y)^*(x + i^k y) \leqslant 2(x^*x + y^*y)$. Assume that $M(B)$ is faithfully represented in $\mathbb{B}(H)$ for some Hilbert space H. Then for all $\xi, \eta \in H$ we have

$$\begin{aligned} |\langle I(y^*x)\xi, \eta\rangle| &= \left| \int_{\hat{G}} \langle \hat{\alpha}_\sigma(x)\xi, \hat{\alpha}_\sigma(y)\eta\rangle \mathrm{d}\sigma \right| \\ &\leqslant \left(\int_{\hat{G}} \|\hat{\alpha}_\sigma(x)\xi\|^2 \, \mathrm{d}\sigma \int_{\hat{G}} \|\hat{\alpha}_\sigma(y)\eta\|^2 \, \mathrm{d}\sigma \right)^{\frac{1}{2}} \\ &= (\langle I(x^*x)\xi, \xi\rangle \langle I(y^*y)\eta, \eta\rangle)^{\frac{1}{2}} \, , \end{aligned}$$

finishing the proof. □

Remark 2.4.19

For the following, recall that the Fourier transformation embeds $L^1(G)$ into a dense $*$-subalgebra of $C_0(\widehat{G})$, and recall that we saw that $M(C_0(\widehat{G})) \cong C_b(\widehat{G})$ in chapter 1, and hence given a G-product A, the (induced) $*$-representation $\lambda\colon L^1(G) \to M(A)$ extends by continuity to a $*$-representation of $\widehat{\lambda}\colon C_0(\widehat{G}) \to M(A)$. Taking the multiplier on each side, we obtain a representation $\widehat{\lambda}\colon M(C_0(\widehat{G})) = C_b(\widehat{G}) \to M(M(A)) = M(A)$.

Lemma 2.4.20. Let A be a G-product. If $f \in L^1(G) \cap L^2(G)$, then $\lambda_f^* \lambda_f$ is $\widehat{\alpha}$-integrable and

$$I(\lambda_f^* \lambda_f) = \|f\|_2^2 \, 1,$$

where 1 is the identity operator on $M(A)$.

Proof.

Note on dirac convolutions being measure convolutions and L1 functions

Recall that for $f \in L^1(G)$, the extension $\widehat{\lambda}$ to $C_0(\widehat{G})$ satisfies $\widehat{\lambda}(\widehat{f}) = \lambda_f$. Hence, if $\sigma \in \widehat{G}$ and $f \in L^1(G)$, then $\widehat{\sigma \cdot f} = \delta_{-\sigma} * \widehat{f}$, and so

$$\widehat{\lambda}(\delta_\sigma * \widehat{f}) = \lambda_{\sigma f} = \int_{\widehat{G}} \widehat{\alpha}_\sigma(\lambda_t) f(t) \mathrm{d}t = \widehat{\alpha}_\sigma(\lambda_f) = \widehat{\lambda}_\sigma(\widehat{\lambda}(\widehat{f})).$$

Continuity of the extension ensures that $\widehat{\lambda}(\delta_{-\sigma} * g) = \widehat{\alpha}_\sigma(\widehat{\lambda}(g))$ for all $g \in C_b(\widehat{G})$. Letting $f \in L^1(G) \cap L^2(G)$, we then obtain

$$\widehat{\alpha}_\sigma(\lambda_f^* \lambda_f) = \widehat{\alpha}_\sigma(\widehat{\lambda}(|\widehat{f}|^2)) = \widehat{\lambda}(\delta_\sigma * |\widehat{f}|^2).$$

Since $\|\delta_\sigma * f\|_2 = \|f\|_2$, we see that

$$I(\lambda_f^* \lambda_f) = I\left(\int \delta_\sigma * |\widehat{f}\|^2 \mathrm{d}\sigma\right) \widehat{\lambda}(\left\|\widehat{f}\right\|_2^2 1) = \|f\|_2^2 \, 1.$$

□

Lemma 2.4.21. Let A be a G-product and $f, g \in L^1(G) \cap L^2(G)$ and $x \in M(A)$. Then $\lambda_f^* x \lambda_g$ is $\widehat{\alpha}$-integrable. If we further assume that $x \in A$, then the function $t \mapsto I(\lambda_f^* x \lambda_g \lambda_t)$ is continuous with respect to the norm topology and $\lambda_f^* x \lambda_f - x \to 0$ whenever f ranges through an approximate identity of $L^1(G)$ which is also in $L^2(G)$.

Proof.

We first note that by lemma 2.4.18 and lemma 2.4.20, whenever $f \in L^1(G) \cap L^2(G)$ and $0 \leqslant x \in B$, we have $0 \leqslant \lambda_f^* x \lambda_f \leqslant \|x\| \, \lambda_f^* \lambda_f$ and similarly. Hence, replacing x with $x^{\frac{1}{2}}$

and letting $f, g \in L^1(G) \cap L^2(G)$ be arbitrary we see that $(x^{\frac{1}{2}}\lambda_f)^* x^{\frac{1}{2}} g = \lambda_f^* x \lambda_g$ will be $\widehat{\alpha}$-integrable. Hence it holds for every $x \in B$ as well.

For the last statement, we see using lemma 2.4.18 that

$$\begin{aligned}\left\|I(\lambda_f^* x \lambda_g \lambda_t) - I(\lambda_f^* x \lambda_g \lambda_s)\right\|^2 &= \left\|I(\lambda_f^* x \lambda_g(\lambda_t - \lambda_s)\right\|^2 \\ &\leqslant \left\|I(\lambda_f^* x x^* \lambda_f)\right\| \left\|I((\lambda_t - \lambda_s)^* \lambda_g^* \lambda_g(\lambda_t - \lambda_s))\right\| \\ &\leqslant \|f\|_2^2 \|x\|^2 \|g * (\delta_t - \delta_s)\|_2^2,\end{aligned}$$

which tends to 0 as $s \to t$, showing that the claim of continuity is true. For the final statment, if $f \geqslant 0$ with $\|f\|_1 = 1$, then

$$\begin{aligned}\left\|\lambda_f^* x \lambda_f - x\right\| &= \left\|\int\int(\lambda_t x \lambda_s - x) f(s) f^*(t) \mathrm{d}t \mathrm{d}s\right\| \\ &= \left\|\int\int(\lambda_t x - x\lambda_s^*) f^*(t) \mathrm{d}t \lambda_s f(s) \mathrm{d}s\right\| \\ &\leqslant \int\int \|(\lambda_t x - x + (x - x\lambda_s^*)\| \, |f^*(t)| \mathrm{d}t \, \|\lambda_s\| \, |f(s)| \mathrm{d}s \\ &\leqslant \int\int(\|\lambda_t x - x\| + \|x - x\lambda_s^*\|) f^*(t) f(s) \mathrm{d}t \mathrm{d}s \\ &= \int \|\lambda_t x - x\| \, f^*(t) \mathrm{d}t + \int \|x\lambda_s - x\lambda_s^* \lambda_s\| \, f(s) \mathrm{d}s,\end{aligned}$$

which, whenever $x \in A$ will go to zero whenever if f ranges over an approximate identity of $L^1(G)$ consisting of continuous functions with shrinking compact supports, since we assumed continuity of the maps $t \mapsto \lambda_t x$ and $s \mapsto x\lambda_s$ whenever $x \in A$ □

Proposition 2.4.22. Let A be a G-product. If $x = \lambda_f^* y \lambda_f \in A$ for $f \in L^1(G) \cap L^2(G)$ and $y \in A$, then $I(x)$ satisfies Landstad's condition and x is the norm limit of elements given by

$$\int I(x\lambda_{-t})\lambda_t f_i(t) \mathrm{d}t,$$

for some net $(f_i) \subseteq L^1(G)$ satisfying $(\widehat{f_i}) \subseteq L^1(\widehat{G})$ is an approximate identity.

Proof.
Note first that given $x \in A$ of the form given above, then by lemma 2.4.21, and x is $\widehat{\alpha}$-integrable and the map $t \mapsto I(x\lambda_t)$ is continuous in norm. We calculate

$$\begin{aligned}I(x\lambda_{-t})\lambda_t = \int_{\widehat{G}} \widehat{\alpha}_\sigma(x\lambda_{-t})\lambda_t \mathrm{d}\sigma &= \int_{\widehat{G}} \widehat{\alpha}_\sigma(x\lambda_{-t})\overline{(t,\sigma)}\widehat{\alpha}_\sigma(\lambda_t) \mathrm{d}\sigma \\ &= \int_{\widehat{G}} \widehat{\alpha}_\sigma(x)\overline{(t,\sigma)} \mathrm{d}\sigma,\end{aligned}$$

hence if $(f_i) \subseteq L^1(G)$ satisfies the conditions of the proposition, then

$$\int_G I(x\lambda_{-t})\lambda_t f_i(t)\mathrm{d}t = \int_{\widehat{G}}\int_G \widehat{\alpha}_\sigma(x)\overline{(t,\sigma)}f_i(t)\mathrm{d}t\mathrm{d}\sigma = \int_{\widehat{G}} \widehat{\alpha}_\sigma(x)\widehat{f}_i(-\sigma)\mathrm{d}\sigma \in A,$$

and in particular

$$\int_G I(x\lambda_{-t})\lambda_t f_i(t)\mathrm{d}t \to x.$$

add note on shrinking support and norm can be chosen to be 1

in norm, since $(\widehat{f}_i)$ is an approximate identity . To see that $I(x)$ satisfies Landstad's condition for each x of the above form, we first note that the first condition is immediately satisfied, since G is abelian and hence unimodular. For the second, et $g \in L^1(G)$ and $\varepsilon > 0$. Pick a symmetric neighborhood E of 0 satisfying

$$\|I(x\lambda_t) - I(x)\| < \varepsilon \text{ and } \|g - \delta_t \times g\|_1 < \varepsilon,$$

for all $t \in E$. Pick now positive $f \in L^1(G)$ satsifying $\|f\|_1 = 1$ with support contained in E such that $\widehat{f} \in L^1(\widehat{G})$. Then we see that

$$\begin{aligned}\|g - f \times g\|_1 = \int_G \left|g(s) - \int_G f(t)g(s-t)\mathrm{d}t\right|\mathrm{d}s &= \int_G\left|\int_G f(t)(g(s) - g(s-t))\mathrm{d}t\right|\mathrm{d}s \\ &\leqslant \int_G f(t)\int_G |g(s) - g(s-t)|\mathrm{d}s\mathrm{d}t \\ &= \int_G f(t)\,\|g - \delta_t * g\|\,\mathrm{d}t \\ &< \varepsilon.\end{aligned}$$

With this, we see that

$$\begin{aligned}\left\|I(x)\lambda_g - \int_G I(x\lambda_{-t})\lambda_t f(t)\mathrm{d}t\lambda_g\right\| &= \left\|I(x)\lambda_g - I(x)\lambda_f\lambda_g + I(x)\lambda_f\lambda_g - \int_G I(x\lambda_{-t})\lambda_t f(t)\mathrm{d}t\lambda_g\right\| \\ &\leqslant \|I(x)(\lambda_g - \lambda_f\lambda_g)\| + \left\|\int_G (I(x) - I(x\lambda_{-t}))\lambda_t f(t)\mathrm{d}t\lambda_g\right\| \\ &\leqslant \|I(x)\|\,\|g - f * g\|_1 + \int_G \|I(x) - I(x\lambda_{-t})\|\,f(t)\mathrm{d}t\,\|g\|_1 \\ &\leqslant (\|I(x)\| + \|g\|_1)\varepsilon,\end{aligned}$$

from which we conclude, by arbitrariness of ε, that $I(x)\lambda_g \in A$. Similarly $\lambda_g I(x) \in A$. For the third condition, note that $\lambda_t I(x)\lambda_t^* = I(\lambda_t I(x)\lambda_t^*)$, and the map $t \mapsto I(\lambda_t x\lambda_t^*)$ is continuous by lemma 2.4.21, finishing the proof. □

The above proposition was the final ingredient in the theorem of Landstad, which shows that every G-product B is also a crossed product $A \rtimes_\alpha G$ for some C^*-algebra A with $G \overset{\alpha}{\curvearrowright} A$:

Theorem 2.4.23. A C^*-algebra B is a G-product if and only if there exists a C^*-algebra A and an action $G \overset{\alpha}{\curvearrowright} A$ such that $B = A \rtimes_\alpha G$. The C^*-algebra A consists of exactly the elements of $M(B)$ satisfying Landstad's conditions, with $\alpha_t = \mathrm{Ad}\lambda_t|_A$, $t \in G$.

Proof.
We have already seen one implication earlier, i.e., that a crossed product $A \rtimes_\alpha G$ is a G-product. For the converse, assume that B is a G-product and $A \subseteq M(B)$ be the set of elements which satisfy Landstad's condition. The three conditions are all based on continuous operations in such a way that A is closed, this is immediate. It is easy to check that A is also a $*$-subalgebra, hence a C^*-subalgebra of $M(B)$.

The last of the Landstad conditions ensures that the map $t \mapsto \alpha_t := \mathrm{Ad}\lambda_t|_A$ is a strongly continuous action of G on A, so that (A, G, α) becomes a C^*-dynamical system.

Let $M(B)$ be faithfully represented on H for some Hilbert space. In particular, this means that $A \subseteq \mathbb{B}(H)$ so that we may identify $A \rtimes_\alpha G \subseteq \mathbb{B}(L^2(G, H))$. Note that by assumption that B is a G-product, the action $\widehat{G} \overset{\widehat{\alpha}}{\curvearrowright} B$ gives a faithful non-degenerate representation π of $B \rtimes_{\widehat{\alpha}} \widehat{G}$ on $\mathbb{B}(L^2(\widehat{G}, H))$ as well, given by

$$\pi(y)\eta(\sigma) = \widehat{\alpha}_{-\sigma}(y)\eta(\sigma),$$

for all $y \in B, \eta \in L^2(\widehat{G}, H)$ and $\sigma \in \widehat{G}$. This representation extends to a faithful representation $\overline{p}$ of $M(B)$ (by non-degeneracy) on $\mathbb{B}(L^2(\widehat{G}, H))$. We will also denote the extension by π. As in the proof of lemma 2.4.13, let $u\colon L^2(G, H) \to L^2(\widehat{G}, H)$ denote the isometry

$$u\xi(\sigma) = \int_G \lambda_t \xi(t)\overline{(t, \sigma)}\, d\sigma \text{ and}$$

$$u^*\eta(t) = \lambda_{-t} \int_{\widehat{G}} \eta(\sigma)(t, \sigma)\mathrm{d}\sigma.$$

If $f \otimes x \in C_c(G, A)$, $f \in C_c(G)$ and $x \in A$, then an easy calculation shows that $(f \otimes x)u^*\eta(\sigma) = (u^*\pi(x\lambda_f)\eta)(t)$, so that

$$u(f \otimes x)u^* = \pi(x\lambda_f) \in \pi(B), \tag{2.14}$$

since $x\lambda_f \in B$. Density of the span of functions $f \otimes x$ in $A \rtimes_\alpha G$ shows that

$$u(A \rtimes_\alpha G)u^* \subseteq \pi(B).$$

For the converse, let $x \in B$ be $\widehat{\alpha}$-integrable of the form $\lambda_g^* y \lambda_g$, for some $g \in L^1(G) \cap L^2(G)$ and $y \in B$. By proposition 2.4.22, we have $I(x) \in A$. For each $f \in C_c(G)$ and $t \in G$, we then calculate, using the identity of eq. (2.14):

$$u(I(x\lambda_{-t} \otimes \delta_t * f)u^* = \pi(I(x\lambda_{-t})\lambda_{\delta_t * f}) = \pi(I(x\lambda_{-t})\lambda_t\lambda_f).$$

Let $(f_i) \subseteq L^1(G)$ be a net satisfying the conditions of lemma 2.4.21. Then,

$$x_i := \int_G I(x\lambda_{-t})\lambda_t f_i(t)\mathrm{d}t \to x,$$

in norm. By proposition 2.4.22, we know that the map, $t \mapsto I(x\lambda_t) = I(\lambda_g^* y \lambda_g t)$, is a continuous map $G \to M(B)$. Continuity of $\mathrm{Ad}u$ and π plus the identity above gives that the map $t \mapsto I(x\lambda_{-t}) \otimes \delta_t * f$ is continuous from $G \to C_c(G, A) \subseteq A \rtimes_\alpha G$. In particular, density of $C_c(G) \subseteq L^1(G)$ ensures that the integral of $t \mapsto I(x\lambda_{-t} \otimes \delta_t * f) \cdot f_i(t)$ makes sense for all i, i.e., that

$$\begin{aligned} u^*\pi(x_i\lambda_f)u &= \int_G u^*(\pi(I(x_i\lambda_{-t}))\lambda_t\lambda_f)f_i(t))u\mathrm{d}t \\ &= \int_G (I(x_i\lambda_{-t}) \otimes \delta_t * f)f_i(t)\mathrm{d}t \in A \rtimes_\alpha G. \end{aligned}$$

In the limit of i, we get that $u^*\pi(x\lambda_f)u \in A \rtimes_\alpha G$, implying by proposition 2.4.22 that $u^*\pi(z\lambda_f)u \in A \rtimes_\alpha G$ for all $z \in B$. Letting f go through an approximate identity of $L^1(G)$ with shrinking compact support, we know that $z\lambda_f - z \to 0$ for all $z \in B$. Hence

$$u^*\pi(B)u \subseteq A \rtimes_\alpha G.$$

□

It is quite easy to show that the above C^*-dynamical system is unique up to covariant isomorphism, however we will not go into details and refer the reader to [13, Theorem 2.9]. The proof relies on the result, which we also wont go into detail with, but include.

Lemma 2.4.24 ([13, Lemma 2.10]). Let (A, G, α) be a C^*-dynamical system and let $B \subseteq A$ be a G-invariant C^*-subalgebra of A. Then there is a natural injection $B \rtimes_\alpha G \subseteq A \rtimes_\alpha G$ which is surjective if and only if $B = A$.

We end this chapter with the promise that the results describes here, in particular Takai Duality, has concrete utilizations in describing the ideal structure of crossed products, even if they are not apparent just yet. In particular, in a series papers of Olesen and Pedersen (we have covered some of the theory of one of them), see e.g., [13], [14] and [15], Takai duality is used to describe a necessary and sufficient condition for simplicity of $A \rtimes_\alpha G$ in terms of a certain subgroup of $\widehat{G}$ and certain properties of the action $G \overset{\alpha}{\curvearrowright} A$. These particular articles established the foundation of much of the theory which we will examine in the next chapter.

Simplicity Of Crossed Products

As previously mentioned, we will now begin describing the theory, both old and new, describing the ideal structure of crossed products $A \rtimes_\alpha G$ for topological groups G. We will begin by introducing some new structures and concepts associated to this theory.

Since we will describe theory which only describes dynamical systems (A, G, α) where G is abelian and also theory which describes the general case, we introduce the notation that a triple (A, G, α) is a C^*-dynamic a-system if $G \overset{\alpha}{\curvearrowright} A$ and G is abelian.

3.1 The Group Spectra

3.1.1 The Arveson and Connes spectra

For the time being, we will untill otherwise stated assume that G is abelian with dual group $\widehat{G}$ as in the previous chapter. Recall that whenever we have a C^*-dynamical a-system (A, G, α), we also obtain a $*$-representation of $L^1(G)$ via the map $f \mapsto \alpha_f$. In fact, for each bounded Radon measure $\mu \in M(G)$ we obtain an operator α_μ on A satisfying

$$\varphi(\alpha_f(a)) = \int_G \varphi(\alpha_t(x))\mathrm{d}\mu(t), \text{ for all } a \in A \text{ and } \varphi \in A^*.$$

We say that α_f and α_μ are the *weak*-integrals* of f and μ, respectively. Fore a more detailed exposition of vector-valued integration, see e.g., [17, Appendix 3, Lemma 7.4.4] or [8, Appendix A.3]. For a group G we define the subset $C_c^1(G)$ of $C_c(G)$ by

$$C_c^1(G) := \{f \in C_c(G) \mid \widehat{f} \in C_c(\widehat{G})\},$$

which is a dense ideal of $L^1(G)$, see [8, Theorem 4.60].

Definition 3.1.1. Let (A, G, α) be a C^*-dynamical system. For open sets $U \subseteq \widehat{G}$, we define the *spectral R-subspace* $R^\alpha(U)$ to be the subspace of A defined by

$$R^\alpha(U) := \overline{\operatorname{Span}\left\{\alpha_f(a) \mid a \in A, f \in C_c^1(G) \text{ with } \operatorname{supp} \widehat{f} \subseteq U\right\}}^w,$$

and similarly, for each closed subset $K \subseteq \widehat{G}$ we define the *spectral M-subspace* $M^\alpha(K)$ to be the subspace of A defined by

$$M^\alpha(K) := \overline{\operatorname{Span}\left\{x \in A \mid \alpha_f(x) = 0 \text{ for all } f \in C_c^1(G) \text{ with } \operatorname{supp} \widehat{f} \subseteq K\right\}},$$

i.e., as the annihilator of $R^{\alpha'}(K^c)$.

By [17, Theorem 8.1.4], it follows that the R- and M-spaces are G-invariant subsets, and

$$\bigcap_i M^\alpha(K_i) = M^\alpha(\bigcap_I K_i),$$

for all collections of subsets (K_i) of $\widehat{G}$. This allows us to give the following definition:

Definition 3.1.2. The *Arveson Spectrum* of an action $G \overset{\alpha}{\curvearrowright} A$ is defined to be the smallest closed subset of $\widehat{G}$ such that $M^\alpha(K) = A$, and we denote it by $\operatorname{Sp}(\alpha)$.

In his dissertation (see [6]), Alain Connes proved a series of equivalent characterizations of $\operatorname{Sp}(\alpha)$, which we will need. We include it without proof and instead refer the reader to [17, Proposition 8.1.9] for the proof.

Proposition 3.1.3. Let (A, G, α) be a C^*-dynamical a-system. Then for $\sigma \in \widehat{G}$ the following are equivalent:

1. $\sigma \in \operatorname{Sp}(\alpha)$,
2. For all neighborhoods U of σ: $R^\alpha(U) \neq 0$,
3. There is a net of unit vectors $(a_i) \subseteq A$ such that $\|\alpha_t(x_i) - (t, \sigma)x_i\| \to 0$ uniformly on compacta of G,
4. For every finite Radon measure μ on G: $|\widehat{\mu}(\sigma)| \leqslant \|\alpha_\mu\|$,
5. For every $f \in L^1(G)$: $|\widehat{f}(\sigma)| \leqslant \|\alpha_f\|$ and
6. If $f \in L^1(G)$, $\alpha_f = 0$, then $\widehat{f}(\sigma) = 0$.

One very common use of the above is to take the last condition as the definition:

$$\mathrm{Sp}(\alpha) := \left\{\chi \in \widehat{G} \mid \widehat{f}(\chi) = 0 \text{ whenever } f \in L^1(G) \text{ satisfies } \alpha_f(x) = 0 \text{ for all } x \in A\right\}.$$

We will frequently switch between the above equivalent conditions.

The main motivation (in our case) for introducing the Arveson spectrum is to define the Connes Spectrum. For this, we need to introduce the concept of hereditary sub-C^*-algebras:

Definition 3.1.4. A C^*-subalgebra $B \subseteq A$ is *hereditary* if for all positive $x \in A_+$ we have $x \in B_+$ whenever $y \in B_+$ with $0 \leqslant x \leqslant y$. We define $\mathscr{H}(A)$ to be the set of non-zero hereditary C^*-subalgebras of A.

If furthermore we assume that (A, G, α) is a C^*-dynamical system, we defin $\mathscr{H}^\alpha(A)$ to be the set

$$\mathscr{H}^\alpha(A) := \{D \in \mathscr{H}(A) \mid \alpha_g(D) = D \text{ for all } g \in G\}.$$

With this, we may define the Connes spectrum of an action $G \overset{\alpha}{\curvearrowright} A$.

Definition 3.1.5. The *Connes Spectrum* of an action $G \overset{\alpha}{\curvearrowright} A$ is the subset $\Gamma(\alpha)$ of $\widehat{G}$ given by

$$\Gamma(\alpha) := \bigcap_{D \in \mathscr{H}^\alpha(A)} \mathrm{Sp}(\alpha|_D).$$

As we shall see, this subset is very important in the theory of classification of crossed products, for instance it is an invariant under exterior equivalence (which we've yet to define).

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