Department of Mathematical Sciences
UNIVERSITY OF COPENHAGEN

Malthe Munk Karbo

Simplicity of Crossed Products of C^*-algebras

Masters thesis in mathematics

Department of Mathematical Sciences
University of Copenhagen

Advisor
Søren Eilers

January 25th, 2019

Abstract

Write abstract

CONTENTS

Hilbert C^*-modules and the Multiplier Algebra

In this chapter we introduce some of the fundamental theory of Hilbert C^*-modules, which may be thought of as a generalization of Hilbert spaces. The primary source of inspiration and techniques are the excellent books on the subject: [RW98], [Lan95] and [Mur14]. After reviewing the basic definitions we will proceed to contruct and prove theorems regarding the Multiplier Algebra, which plays a crucial part in the development of the theory of the general crossed product, one of the basic constructions of this thesis.

1.1 Hilbert C^*-modules

Given a non-unital C^*-algebra A, we can extend a lot of the theory of unital C^*-algebras to A by considering the unitization $A^\dagger = A + 1\mathbb{C}$ with the canonical operations. This is the 'smallest' unitization in the sense that A the inclusion map is non-degenerate and maps A to an ideal of $A^\dagger$ such that $A^\dagger/A \cong \mathbb{C}$. We can also consider a maximal unitization, called the *Multiplier Algebra* of A, denoted by $M(A)$. We will now construct it and some theory revolving it, in particular that of Hilbert-modules:

Definition 1.1.1. Let A be any C^*-algebra. An *inner-product A-module* is a complex vector space E which is also a right-A-module which is equipped with a map $\langle \cdot, \cdot \rangle_A \colon E \times E \to A$ such that for $x, y, z \in E$, $\alpha, \beta \in \mathbb{C}$ and $a \in A$ we have

$$\langle a, \alpha y + \beta z \rangle_A = \alpha \langle x, y \rangle_A + \beta \langle x, z \rangle_A \tag{1.1}$$

$$\langle x, y \cdot a \rangle_A = \langle x, y \rangle_A \cdot a \tag{1.2}$$

$$\langle y, x \rangle_A = \langle x, y \rangle_A^* \tag{1.3}$$

$$\langle x, x \rangle_A \geq 0 \tag{1.4}$$

$$\langle x, x \rangle_A = 0 \implies x = 0. \tag{1.5}$$

If $\langle\cdot,\cdot\rangle$ satisfy (1.1)-(1.4) but not (1.5) above, we say that E is a *semi-inner-product A-module*. If there is no confusion, we simply write $\langle\cdot,\cdot\rangle$ to mean the A-valued inner-product on E. Similarly, we will often denote $x \cdot a$ by xa.

There is of course a corresponding version of left inner-product A-modules, with the above conditions adjusted suitably. However, we will unless otherwise specified always mean a right A-module in the following.

It is easy to see that this generalizes the notion of Hilbert spaces, for if $A = \mathbb{C}$, then any Hilbert space (over $\mathbb{C}$) is a inner-product $\mathbb{C}$-module. Even when E is only a semi-inner-product A-module, we still have a lot of the nice properties of regular inner-producte, e.g.:

Proposition 1.1.2 (Cauchy-Schwarz Inequality)**.** Let E be a semi-inner-product A-module. For $x, y \in E$, it holds that

$$\langle x,y\rangle^*\langle x,y\rangle \leq \|\langle x,x\rangle\|_A \langle y,y\rangle.$$

Proof.
Let $x, y \in E$ and let $a = \langle x,y\rangle \in A$ and $t > 0$. Then, since $(\langle x,y\rangle a)^* = a^*\langle y,x\rangle$, we have

$$0 \leq \langle xa - ty, xa - ty\rangle = t^2\langle y,y\rangle + a^*\langle x,x\rangle a - 2ta^*a.$$

For $c \geq 0$ in A, it holds that $a^*ca \leq \|c\|_A a^*a$: Let $A \subseteq \mathbb{B}(\mathcal{H})$ faithfully. The regular Cauchy-Schwarz implies that $\langle a^*ca\xi, \xi\rangle_{\mathcal{H}} \leq \|c\| \langle a^*a\xi, \xi\rangle_{\mathcal{H}}$ for all $\xi \in \mathcal{H}$. With this in mind, we then have

$$0 \leq a^*a \|\langle x,x\rangle\|_A - 2ta^*a + t^2\langle y,y\rangle.$$

If $\langle x,x\rangle = 0$, then $0 \leq 2a^*a \leq t\langle y,y\rangle \to 0$, as $t \to 0$ so that $a^*a = 0$ as wanted. If $\langle x,x\rangle \neq 0$, then $t = \|\langle x,x\rangle\|_A$ gives the wanted inequality. $\square$

Recall that for $0 \leq a \leq b \in A$, it holds that $\|a\|_A \leq \|b\|_A$. This and the above implies that $\|\langle x,y\rangle\|_A^2 \leq \|\langle x,x\rangle\|_A \|\langle y,y\rangle\|_A$.

Definition 1.1.3. If E is an (semi-)inner-product A-module, we define a (semi-)norm on E by

$$\|x\| := \|\langle x,x\rangle\|_A^{\frac{1}{2}}, \text{ for } x \in E.$$

If E is an inner-product A-module such that this norm is complete, we say that E is a *Hilbert C^*-module over A* or just a *Hilbert A-module*.

If the above defined function is not a proper norm but instead a semi-norm, then we may form the quotient space of E with respect to $N = \{x \in E \mid \langle x, x\rangle = 0\}$ to obtain an inner-product A-module with the obvious operations. Moreover, the usual ways of passing from pre-Hilbert spaces extend naturally to pre-Hilbert A-modules. For $x \in E$ and $a \in A$, we see that

$$\|xa\|^2 = \left\| a^* \underbrace{\langle x, x\rangle}_{\geq 0} a \right\| \leq \|a\|^2 \|x\|^2 ,$$

so that $\|xa\| \leq \|x\| \|a\|$, which shows that E is a normed A-module. For $a \in A$, let $R_a \colon x \mapsto xa \in E$. Then R_a is a linear map on E satisfying $\|R_a\| = \sup_{\|x\|=1} \|xa\| \leq \|a\|$, and the map $R \colon a \mapsto R_a$ is a contractive anti-homomorphism of A into the space of bounded linear maps on E (anti in the sense that $R_{ab} = R_b R_a$ for $a, b \in A$).

Example 1.1.4
The classic example of a Hilbert A-module is A itself with $\langle a, b\rangle = a^* b$, for $a, b \in A$. We will write A_A to specify that we consider A as a Hilbert A-module.

For an inner product A-module E, we let $\langle E, E\rangle = \operatorname{span}\{\langle x, y\rangle \mid x, y \in E\}$. It is not hard to see that $\overline{\langle E, E\rangle}$ is a closed ideal in A, and moreover, we see that:

Lemma 1.1.5. The set $E\langle E, E\rangle$ is dense in E.

Proof.
Let (u_i) denote an approximate unit for the ideal $\overline{\langle E, E\rangle}$. Then clearly $\langle x - xu_i, x - xu_i\rangle \to 0$ for $x \in E$ (seen by expanding the expression), which implies that $xu_i \to x$ in E □

In fact, the above proof also shows that EA is dense in E and that $x1 = x$ when $1 \in A$.

Definition 1.1.6. Let E be a Hilbert A-module. If $\langle E, E\rangle$ is dense in A we say that the Hilbert A-module is *full*.

Many classic constructions preserve Hilbert C^*-modules:

Example 1.1.7
If $E_1, \ldots, E_n$ are Hilbert A-modules, then $\bigoplus_{1\leq i\leq n} E_i$ is a Hilbert A-module with the inner product $\langle (x_i), (y_i)\rangle = \sum_{1\leq i\leq n} \langle x_i, y_i\rangle_{E_i}$.

Moreover, if $\{E_i\}$ is a set of Hilbert A-modules, and let $\bigoplus_i E_i$ denote the set

$$\bigoplus_i E_i := \left\{ (x_i) \in \prod_i E_i \mid \sum_i \langle x_i, x_i\rangle \text{ converges in } A \right\}.$$

Then $\bigoplus_i E_i$ is a Hilbert A-module.

For the details in the above, see [Lan95, p. 5].

Remark 1.1.8
There are similarities and differences between Hilbert A-modules and regular Hilbert spaces. One major difference is that for closed submodules $F \subseteq E$, it does not hold that $F \oplus F^\perp = E$, where $F^\perp = \{y \in E \mid \langle x, y \rangle = 0,\ x \in F\}$: Let $A = C([0,1])$ and consider A_A. Let $F \subseteq A$ be the set $F = \{f \in A_A \mid f(0) = f(1) = 0\}$. Then F is clearly a closed sub A-module of A_A. Consider the element $f \in F$ given by

$$f(t) = \begin{cases} t & t \in [0, \frac{1}{2}] \\ 1 - t & t \in (\frac{1}{2}, 1] \end{cases}$$

For all $g \in F^\perp$ it holds that $\langle f, g \rangle = fg = 0$, so $g(t) = 0$ for $t \in (0,1)$ and by uniform continuity on all of $[0,1]$, from which we deduce that $F^\perp = \{0\}$ showing that $A \neq F \oplus F^\perp$. The number of important results in the theory of Hilbert spaces relying on the fact that $\mathcal{H} = F^\perp \oplus F$ for closed subspaces F of $\mathcal{H}$ is very significant, and hence we do not have the entirety of our usual toolbox available.

However, returning to the general case, we are pleased to see that it does hold for Hilbert A-modules E that for $x \in E$ we have

$$\|x\| = \sup \{\|\langle x, y \rangle\| \mid y \in E,\ \|y\| \leq 1\}. \tag{1.6}$$

This is an obvious consequence of the Cauchy-Schwarz theorem for Hilbert A-modules.

Another huge difference is that adjoints don't automatically exist: Let $A = C([0,1])$ again and F as before. Let $\iota\colon F \to A$ be the inclusion map. Clearly ι is a linear map from F to A, however, if it had an adjoint $\iota^*\colon A \to F$, then for any self-adjoint $f \in F$, we have

$$f = \langle \iota(f), 1 \rangle_A = \langle f, \iota^*(1) \rangle = f\iota^*(1),$$

so $\iota^*(1) = 1$, but $1 \notin F$. Hence ι^* is not adjointable. We shall investigate properties of the adjointable operators in more detail further on.

1.2 The C^*-algebra of adjointable operators

The set of adjointable maps between Hilbert A-modules is of great importance and has many nice properties, and while not all results carry over from the theory of bounded operators on a Hilbert space, enough results carry over to ensure a rich and useful theory.

Definition 1.2.1. For Hilbert A-modules E_i with A-valued inner-products $\langle \cdot, \cdot \rangle_i$, $i = 1, 2$, we define $\mathcal{L}_A(E_1, E_2)$ to be the set of all maps $t\colon E_1 \to E_2$ for which there exists a map

$t^*\colon E_2 \to E_1$ such that

$$\langle tx, y\rangle_2 = \langle x, t^* y\rangle_1,$$

for all $x \in E_1$ and $y \in E_2$, i.e., *adjointable maps.* When no confusion may occur, we will write $\mathcal{L}(E_1, E_2)$ and omit the subscript.

Proposition 1.2.2. If $t \in \mathcal{L}(E_1, E_2)$, then

1. t is A-linear and

2. t is a bounded map.

Proof.
Linearity is clear. To see A-linearity and if $x_1, x_2 \in E_i$ and $a \in A$ then $a^* \langle x_1, x_2\rangle_i = \langle x_1 a, x_2\rangle_i$ so that

$$\langle t(xa), y\rangle_2 = a^* \langle x, t^*(y)\rangle_1 = \langle t(x)a, y\rangle_1,$$

for all $x \in E_1$, $y \in E_2$ and $a \in A$. To see that t is bounded, we note that t has closed graph: Let $x_n \to x$ in E_1 and assume that $tx_n \to z \in E_2$. Then

$$\langle tx_n, y\rangle_2 = \langle x_n, t^* y\rangle_1 \to \langle x, t^* y\rangle_1 = \langle tx, y\rangle_2$$

so $tx = z$, and hence t is bounded, by the Closed Graph Theorem. □

In particular, similar to the way that the algebra $\mathbb{B}(\mathcal{H}) = \mathcal{L}_{\mathbb{C}}(\mathcal{H}_{\mathbb{C}}, \mathcal{H}_{\mathbb{C}})$ is of great importance, we examine for a fixed Hilbert A-module E, the $*$-subalgebra $\mathcal{L}(E) := \mathcal{L}(E, E)$ of $\mathbb{B}(E)$, i.e., the bounded operators on E.

Proposition 1.2.3. The algebra $\mathcal{L}(E)$ is a closed $*$-subalgebra of $\mathbb{B}(E)$ such that for $t \in \mathcal{L}(E)$ we have

$$\|t^* t\| = \|t\|^2,$$

hence is it a C^*-algebra.

Proof.
Since $\mathbb{B}(E)$ is a Banach algebra, we have $\|t^* t\| \leq \|t^*\|\,\|t\|$ for all $t \in \mathcal{L}(E)$. Moreover, by Cauchy-Schwarz, we have

$$\|t^* t\| = \sup_{\|x\| \leq 1} \{\|t^* tx\|\} = \sup_{\|x\|, \|y\| \leq 1} \{\langle tx, ty\rangle\} \geq \sup_{\|x\| \leq 1} \{\|\langle tx, tx\rangle\|\} = \|t\|^2,$$

so $\|t^*t\| = \|t\|^2$. Assume that $t \neq 0$, then we see that $\|t\| = \|t^*t\| \, \|t\|^{-1} \leq \|t^*\|$ and similarly $\|t^*\| \leq \|t^{**}\| = \|t\|$, so $\|t\| = \|t^*\|$, in particular $t \mapsto t^*$ is continuous. It is clear that $\mathcal{L}(E)$ is a $*$-subalgebra. To see that is is closed, suppose that $(t_n et)$ is a Cauchy sequence, and denote its limit by $t \in \mathbb{B}(E)$. By the above, we see that (t_n^*) is Cauchy with limit $f \in \mathbb{B}(E)$, since Then, we see that for all $x, y \in E$

$$\langle tx, y \rangle = \lim_n \langle t_n x, y \rangle = \lim_n \langle x, t_n^* y \rangle = \langle x, fy \rangle,$$

so t is adjointable, hence an element of $\mathcal{L}(E)$. □

If we for $x \in E$ denote the element $\langle x, x \rangle^{\frac{1}{2}} \geq 0$ by $|x|_A$, then we obtain a variant of a classic and important inequality:

Lemma 1.2.4. Let $t \in \mathcal{L}(E)$, then $|tx|^2 \leq \|t\|^2 \, |x|^2$ for all $x \in E$.

Proof.
The operator $\|t\|^2 - t^*t \geq 0$, since $\mathcal{L}(E)$ is a C^*-algebra, and thus equal to s^*s for some $s \in \mathcal{L}(E)$. Hence

$$\|t\|^2 \langle x, x \rangle - \langle t^*tx, x \rangle = \langle (\|t\|^2 - t^*t)x, x \rangle = \langle sx, sx \rangle \geq 0,$$

for all $x \in E$, as wanted. □

A quite useful characterization of positive operators on a Hilbert space carries over to the case of Hilbert C^*-modules, for we have:

Lemma 1.2.5. Let E be a Hilbert A-module and let T be a linear operator on E. Then T is a positive operator in $\mathcal{L}_A(E)$ if and only if for all $x \in E$ it holds that $\langle Tx, x \rangle \geq 0$ in A.

Proof.
If T is adjointable and positive, then $T = S^*S$ for some $S \in \mathcal{L}(E)$ and $\langle Tx, x \rangle = \langle Sx, Sx \rangle \geq 0$. For the converse, we note first that the usual *polarization identity* applies to Hilbert C^*-modules as well:

$$4\langle u, v \rangle = \sum_{k=0}^{3} i^k \langle u + i^k v, u + i^k v \rangle,$$

for all $u, v \in E$, since $\langle u, v \rangle = \langle v, u \rangle^*$. Hence, if $\langle Tx, x \rangle \geq 0$ for all $x \in E$, we see that $\langle Tx, x \rangle = \langle x, Tx \rangle$, so

$$4\langle Tx, y \rangle = \sum_{k=0}^{3} i^k \underbrace{\langle T(x + i^k y), x + i^k y \rangle}_{= \langle x + i^k y, T(x + i^k y) \rangle} = 4\langle x, Ty \rangle, \text{ for } x, y \in E.$$

Thus $T = T^*$ is an adjoint of T and $T \in \mathcal{L}_A(E)$. An application of the continuous functional calculus allows us to write $T = T^+ - T^-$ for positive operators $T^+, T^- \in \mathcal{L}_A(E)$ with $T^+T^- = T^-T^+ = 0$, see e.g., [Zhu93, Theorem 11.2]. Then, for all $x \in E$ we have

$$0 \leq \langle T(T^- x), T^- x \rangle = \langle -(T^-)^2 x, T^- x \rangle = -\underbrace{\langle (T^-)^3 x, x \rangle}_{\geq 0} \leq 0,$$

so $\langle (T^-)^3 x, x \rangle = 0$ for all x. The polarization identity shows that $(T^-)^3 = 0$ and hence $(T^-) = 0$, so $T = T^+ \geq 0$. □

Corollary 1.2.6. Let E be a Hilbert A-module and $T \geq 0$ in $\mathcal{L}_A(E)$. Then

$$\|T\| = \sup_{\|x\| \leq 1} \{\|\langle Tx, x \rangle\|\}$$

Proof.
Write $T = S^*S$ for $S \in \mathcal{L}_A(E)$ to obtain the identity. □

It is well-known that the only ideal of $\mathbb{B}(\mathcal{H})$, $\mathcal{H}$ separable, is the set of compact operators $\mathbb{K}(\mathcal{H})$, which is the norm closure of the finite-rank operators. We now examine the Hilbert C^*-module analogue of these two subsets:

Definition 1.2.7. Let E, F be Hilbert A-modules. Given $x \in E$ and $y \in F$, we define $\Theta_{x,y} \colon F \to E$ by

$$\Theta_{x,y}(z) := x\langle y, z \rangle,$$

for $z \in F$. It is easy to see that $\Theta_{x,y} \in \mathbb{B}(F, E)$.

Proposition 1.2.8. Let E, F be Hilbert A-modules. Then for every $x \in E$ and $y \in F$, it holds that $\Theta_{x,y} \in \mathcal{L}(F, E)$. Moreover, if F is another Hilbert A-module, then

1. $\Theta_{x,y}\Theta_{u,v} = \Theta_{x\langle y,u \rangle_F, v} = \Theta_{x, v\langle u,y \rangle_F}$, for $u \in F$ and $v \in G$,
2. $T\Theta_{x,y} = \Theta_{Tx,y}$, for $T \in \mathcal{L}(E, G)$,
3. $\Theta_{x,y}S = \Theta_{x,S^*y}$ for $S \in \mathcal{L}(G, F)$.

Proof.
Let $x, w \in E$ and $y, u \in F$ for Hilbert A-modules E, F as above. Then, we see that

$$\langle \Theta_{x,y}(u), w \rangle_E = \langle x\langle y, u \rangle_F, w \rangle_E = \langle y, u \rangle_F^* \langle x, w \rangle_E = \langle u, y \rangle_F \langle x, w \rangle_E = \langle u, y\langle x, w \rangle_E \rangle_F,$$

so $\Theta_{x,y}^* = \Theta_{y,x} \in \mathcal{L}(E,F)$. Suppose that G is another Hilbert A-module and $v \in G$ is arbitrary, then

$$\Theta_{x,y}\Theta_{u,v}(g) = x\langle y, u\langle v,g\rangle_G\rangle_F = x\langle y,u\rangle_F\langle v,g\rangle = \Theta_{x\langle y,u\rangle_F,v}(g),$$

for $g \in G$. The other identity in (1) follows analogously. Let $T \in \mathcal{L}(E,G)$, then

$$T\Theta_{x,y}(u) = Tx\langle y,u\rangle_F = \Theta_{Tx,y}(u),$$

so (2) holds and (3) follows analogously. □

Definition 1.2.9. Let E, F be Hilbert A-modules. We define the set of compact operators $F \to E$ to be the set

$$\mathbb{K}_A(F,E) := \overline{\operatorname{span}\{\Theta_{x,y} \mid x \in E,\ y \in F\}} \subseteq \mathcal{L}_A(F,E).$$

It follows that for $E = F$, the $*$-subalgebra of $\mathcal{L}_A(E)$ obtained by taking the span of $\{\Theta_{x,y} \mid x,y \in E\}$ forms a two-sided algebraic ideal of $\mathcal{L}_A(E)$, and hence its closure is a C^*-algebra:

Definition 1.2.10. We define the *Compact Operators on a Hilbert A-module E* to be the C^*-algebra

$$\mathbb{K}_A(E) := \overline{\operatorname{span}\{\Theta_{x,y} \mid x,y \in E\}} \subseteq \mathcal{L}_A(E).$$

Note 1.2.11
An operator $t \in \mathbb{K}_A(F,E)$ need not be a compact operator when E, F are considered as Banach spaces, i.e., a limit of finite-rank operators. This is easy to see by assuming that A is not-finite dimensional and unital: The operator $\mathrm{id}_A = \Theta_{1,1}$ is in $\mathbb{K}_A(A_A)$, but the identity operator is not a compact operator on an infinite-dimensional Banach space.

Lemma 1.2.12. There is an isomorphism $A \cong \mathbb{K}_A(A_A)$. If A is unital, then $A \cong \mathbb{K}_A(A_A) = \mathcal{L}_A(A_A)$.

Proof.
Fix $a \in A$ and define $L_a\colon A \to A$ by $L_a b = ab$. Then $\langle L_a x, y\rangle = x^*a^*y = \langle x, L_{a^*}y\rangle$ for all $x, y \in A$, hence L_a is adjointable with adjoint L_{a^*}. Note that $L_{ab^*}x = ab^*x = a\langle b,x\rangle = \Theta_{a,b}(x)$ for $a,b,x \in A$, hence if $L\colon a \mapsto L_a$, we have $\mathbb{K}_A(A_A) \subseteq \operatorname{Im}(L)$. It is clear that L is a $*$-homomorphism of norm one, i.e., an isometric homomorphism, so it's image is a C^*-algebra. To see that it has image equal to $\mathbb{K}_A(A_A)$, let (u_i) be an approximate identity of A. Then for any $a \in A$ $L_{u_i a} \to L_a$, and since $L_{u_i a} \in \mathbb{K}_A(A_A)$ for all i, we see that $L_a \in \mathbb{K}_A(A_A)$.

If $1 \in A$, then for $t \in \mathcal{L}_A(A_A)$ we see that $t(a) = t(1 \cdot a) = t(1)\cdot a = L_{t(1)}a$, so $t \in \mathbb{K}_A(A_A)$, and $\mathcal{L}_A(A_A) \cong A$. □

Note 1.2.13
It does not hold in general that $A \cong \mathcal{L}_A(A_A)$, e.g., as we shall see soon for $A = C_0(X)$, where X is locally compact Hausdorff, for then $\mathcal{L}_A(A_A) = C_b(X) \cong C(\beta X)$, where βX is the Stone-Čech compactification of X.

In Lemma 1.1.5 we saw that for a Hilbert A-module E, the set $E\langle E, E\rangle$ was dense in E. We are going to improve this to say that every $x \in E$ is of the form $y\langle y, y\rangle$ for some unique $y \in E$. To do this, we need to examine the compact operators a bit more:

Proposition 1.2.14. Given a right Hilbert A-module E, then E is a full left Hilbert $\mathbb{K}_A(E)$-module with respect to the action $T \cdot x = T(x)$ for $x \in E$ and $T \in \mathbb{K}_A(E)$ and $\mathbb{K}_A(E)$-valued inner product ${}_{\mathbb{K}_A(E)}\langle x, y\rangle = \Theta_{x,y}$ for $x, y \in E$. Moreover, the corresponding norm $\|\cdot\|_{\mathbb{K}_A(E)} = \left\|{}_{\mathbb{K}_A(E)}\langle \cdot, \cdot\rangle\right\|^{\frac{1}{2}}$ coincides with the norm $\|\cdot\|_A$ on E.

Proof.
It is easy to see that the first, second and third condition of Definition 1.1.1 are satisfied in light of Proposition 1.2.8. Note that

$$\langle {}_{\mathbb{K}_A(E)}\langle x, x\rangle \cdot y, y\rangle_A = \langle x\langle x, y\rangle, y\rangle_A = \langle x, y\rangle_A^* \langle x, y\rangle_A \geq 0, \tag{1.7}$$

for all $x, y \in E$, which by Lemma 1.2.5 implies ${}_{\mathbb{K}_A(E)}\langle x, x\rangle \geq 0$ for all $x \in E$. The above calculation also shows that if ${}_{\mathbb{K}_A(E)}\langle x, x\rangle = 0$ then $\langle x, y\rangle_A = 0$ for all $y \in E$ so $x = 0$.

To see the last condition, note that

$$\left\|\langle {}_{\mathbb{K}_A(E)}\langle x, x\rangle x, x\rangle_A\right\| = \|\langle x, x\rangle_A^* \langle x, x\rangle_A\| = \|x\|_A^2$$

so by Corollary 1.2.6, $\|x\|_{\mathbb{K}_A(E)} \geq \|x\|_A$ for all $x \in E$. For the other inequality, applying Cauchy-Schwarz to Equation (1.7), we see that

$$\langle {}_{\mathbb{K}_A(E)}\langle x, x\rangle y, y\rangle_A \leq \|\langle x, x\rangle_A\| \, \langle y, y\rangle_A$$

so again by Corollary 1.2.6 we see that $\left\|{}_{\mathbb{K}_A(E)}\langle x, x\rangle\right\| \leq \|\langle x, x\rangle_A\|$ for all $x \in E$. ☐

When we considered $E = A_A$, we defined for $a \in A$ the operator $L_a\colon b \mapsto ab$, $b \in A$. Similarly, we do this in the general case:

Definition 1.2.15. Let E be a Hilbert A-module. We define for $x \in E$ the operators $L_x\colon A \to E$, $a \mapsto x \cdot a$, and $D_x\colon E \to A$, $y \mapsto \langle x, y\rangle_A$.

Remark 1.2.16
The calculation

$$\langle D_x(y), a\rangle_{A_A} = \langle\langle x, y\rangle_A^* a = \langle y, x\rangle_A a = \langle y, xa\rangle_A = \langle y, L_x(a)\rangle_{A_A},$$

for $x, y \in E$ and $a \in A$ shows that L_x and D_x are mutual adjoints, so $L_x \in \mathcal{L}_A(A_A, E)$ and $D_x \in \mathcal{L}_A(E, A_A)$.

In fact, the maps $D\colon x \mapsto D_x$ and $L\colon x \mapsto L_x$ are interesting on their own, which extends the result from the case $E = A_A$ seen in Lemma 1.2.12:

Lemma 1.2.17. Let E be a Hilbert A-module. Then $E \ni x \mapsto D_x$ is a conjugate linear isometric isomorphism of E and $\mathbb{K}_A(E, A_A)$ and $E \ni x \mapsto L_x$ is a linear isometric isomorphism of E and $\mathbb{K}_A(A_A, E)$.

Proof.
Conjugate linearity of D follows immediately from $\langle \cdot, y\rangle_A$ possessing said property. We see that D_x is an isometry via eq. (1.6), hence it has closed image. Since $\Theta_{a,x} = D_{xa^*}$, it follows that $\mathbb{K}_A(E, A_A) \subseteq D_E$, where D_E denotes the image of the identification.

For the converse inclusion, let (e_i) be an approximate unit of $\overline{\langle E, E\rangle}$. Then, since D_E is closed and D is continuous, we see that $D_x = \lim_i D_{xe_i} = \lim_i \Theta_{x,e_i} \in D_E$, hence $D_E \subseteq \mathbb{K}_A(E, A_A)$, since $E\langle E, E\rangle$ is dense in E.

The second statement follows since clearly $L\colon x \mapsto D_x^*$ is then a linear isometry and we saw in Proposition 1.2.8 that $\Theta_{a,x}^* = \Theta_{x,a}$, so the argument above applies to L as well. $\square$

Note 1.2.18
Abusing notation, we will in the following use L_x to denote both the operator on E and A_A depending on the location of x.

We now show a generalization of a standard trick regarding operators on direct sums of Hilbert spaces, the technique is from [Bla96] and is based on the 2×2-matrix trick of Connes [Con73]: We may, given Hilbert spaces $\mathcal{H}, \mathcal{K}$ view $\mathbb{B}(\mathcal{H} \oplus \mathcal{K})$ as the set of matrices

$$\left\{ \begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix} \right\} \in \begin{pmatrix} \mathbb{B}(\mathcal{H}) & \mathbb{B}(\mathcal{K}, \mathcal{H}) \\ \mathbb{B}(\mathcal{H}, \mathcal{K}) & \mathbb{B}(\mathcal{K}) \end{pmatrix}$$

by compression with the orthogonal projections onto the pairwise coordinates. This also holds for the compact operators on $\mathcal{H} \oplus \mathcal{K}$, and more generally:

Lemma 1.2.19. Let E be a Hilbert A-module. If $L_a \in \mathbb{K}_A(A_A)$, $S \in \mathbb{K}_A(A_A, E)$, $T \in \mathbb{K}_A(E, A_A)$ and $R \in \mathbb{K}_A(E)$, then

$$Q := \begin{pmatrix} L_a & T \\ S & R \end{pmatrix} \in \mathbb{K}_A(A_A \oplus E),$$

and its adjoint is

$$Q^* = \begin{pmatrix} L_a^* & S^* \\ T^* & R^* \end{pmatrix}$$

Moreover, every compact operator on $A_A \oplus E$ is of this form. If $Q \in \mathbb{K}_A(A_A \oplus E)$ is self-adjoint and anti-commutes with $1 \oplus -1$ then

$$Q = \begin{pmatrix} 0 & D_x \\ L_x & 0 \end{pmatrix},$$

for some $x \in E$ and L_x, D_x as in Lemma 1.2.17.

Proof.
It is clear that Q is adjointable with adjoint

$$Q^* = \begin{pmatrix} L_a^* & S^* \\ T^* & R^* \end{pmatrix}$$

To see that it is compact, note that each of the four corner embeddings

$$L_a \mapsto \begin{pmatrix} L_a & 0 \\ 0 & 0 \end{pmatrix}, \quad T \mapsto \begin{pmatrix} 0 & T \\ 0 & 0 \end{pmatrix}, \quad S \mapsto \begin{pmatrix} 0 & 0 \\ S & 0 \end{pmatrix}, \quad R \mapsto \begin{pmatrix} 0 & 0 \\ 0 & R \end{pmatrix}$$

are linear isometries and they map the module valued finite-rank maps to module valued finite-rank maps, e.g., we see that $\Theta_{a,x} \in \mathbb{K}_A(E, A)$ is mapped to $\Theta_{(a,0),(x,0)}$. By compressing with the orthogonal projection onto e.g., A_A, we see that we can decompose every finite rank operator in this way. Since compression by projections is continuous, this shows that every compact operator has this form.

The last statement follows readily from the isomorphism from Lemma 1.2.17, the anti-commutativity forces $L_a = 0$ and $R = 0$ and self-adjointess forces $D_x^* = L_x$. □

We may finally prove the statement which we set out to show:

Proposition 1.2.20. Let E be a Hilbert A-module. Then for every $x \in E$ there is a unique $y \in E$ such that $x = y\langle y, y\rangle$.

Proof.
Let $x \in E$, and consider the self-adjoint operator

$$T = \begin{pmatrix} 0 & D_x \\ L_x & 0 \end{pmatrix}$$

on $A_A \oplus E$, which by Lemma 1.2.19 is self-adjoint and anti-commutes with $1 \oplus -1$. With t being self-adjoint, the function $f(x) = x^{\frac{1}{3}}$ is continuous on the spectrum $\sigma(T)$, so it defines a compact operator $f(T)$ on $A_A \oplus E$. This operator anti-commutes with $1 \oplus -1$ by the functional calculus applied to the map $\psi\colon \mathbb{K}_A(A_A \oplus E) \to \mathbb{K}_A(A_A \oplus E)$, $s \mapsto (1 \oplus -1)s + s(1 \oplus -1)$, which is a *-homomorphism, so there is $y \in E$ such that

$$f(T) = \begin{pmatrix} 0 & D_y \\ L_y & 0 \end{pmatrix}$$

And so $f(T)^3 = T$ implies that $L_x = L_y D_y L_y = L_{y\langle y,y\rangle}$, so $x = y\langle y, y\rangle$. □

We empose quite an assortment of topologies on $\mathbb{B}(\mathcal{H}, \mathcal{H}')$ for Hilbert spaces $\mathcal{H}$ and $\mathcal{H}'$ – this is not the case when we study Hilbert C^*-modules: Given Hilbert A-module E and F, we only consider two topologies – the norm topology and the *strict topology*:

Definition 1.2.21. The *strict topology* on $\mathcal{L}_A(E, F)$ is the locally convex topology on $\mathcal{L}_A(E, F)$ generated by the family of semi-norm pairs

$$t \mapsto \|tx\|, \text{ for } x \in E \text{ and } t \mapsto \|t^* y\|, \text{ for } y \in F.$$

This topology turns out to be very important, and there is many results of the classic theory which apply in this setting as well:

Proposition 1.2.22. Let E be a Hilbert A-module. Then the unit ball of $\mathbb{K}_A(E, F)$, $(\mathbb{K}_A(E, F))_1$, is strictly dense in the unit ball, denoted $(\mathcal{L}_A(E, F))_1$, of $\mathcal{L}_A(E, F)$.

Proof.
Let (e_i) be an approximate identity of the C^*-algebra $\mathbb{K}_A(E)$. Then, for all $w\langle x, y\rangle \in E\langle E, E\rangle$ we see that

$$e_i w\langle x, y\rangle = e_i \Theta_{w,x}(y) \to \Theta_{w,x}(y) = w\langle x, y\rangle,$$

and hence $e_i x \to x$ for all $x \in E$, since $E\langle E, E\rangle \subseteq E$ is dense by Lemma 1.1.5. Then we clearly have, for $t \in (\mathcal{L}_A(E, F))_1$ that

$$\|t(e_i x - x)\| \to 0 \text{ and } \|e_i t^* y - t^* y\| \to 0,$$

for $x \in E$ and $y \in F$, so $te_i \to t$ strictly, and by Proposition 1.2.8 we see that $te_i \in (\mathbb{K}_A(E, F))_1$, finishing the proof. □

1.3 Unitizations and the Multiplier Algebra

We are almost ready to introduce the Multiplier Algebra of a C^*-algebra A, however, we must introduce some new terminology first.

Definition 1.3.1. An ideal I of a C^*-algebra A is an *essential ideal* if it is non-zero and satisfies $I \cap J \neq 0$ for all non-zero ideals J of A.

And there is another commonly used equivalent formulation of the definition of essential ideals:

Lemma 1.3.2. Let I, J be non-zero ideals of a C^*-algebra A. Then the following holds:

1. $I^2 = I$,
2. $I \cap J = IJ$,
3. I is essential if and only if $aI = 0$ implies $a = 0$ for all $a \in A$.

Proof.
1: It holds that $I^2 \subseteq I$, since I is an ideal. For the other inclusion, note that if $a \geq 0$ is an element of I, then $a = (a^{\frac{1}{2}})^2 \in I^2$, so $I_+ \subseteq I^2$ hence $I = I^2$.

2: Note that $I \cap J = (I \cap J)^2$ by (1), and $(I \cap J)(I \cap J) \subseteq IJ \subseteq I \cap J$.

3: Assume that I is essential and $aI = 0$. Then $(a)I = (a) \cap I = 0$, where (a) is the ideal generated by a in A, and hence $(a) = 0$ so $a = 0$. For the converse, let $J \neq 0$ be any ideal of A and let $0 \neq a \in J$. By assumption this implies that $aI \neq 0$, so $(a)I = (a) \cap I \neq 0$, which shows that $J \cap I \neq 0$. □

As mentioned earlier, it is a useful tool to be able to adjoin units to a non-unital C^*-algebra A. We formalize this process in the following definition:

Definition 1.3.3. A *unitization* of a C^*-algebra A is a pair (B, ι) consisting of a unital C^*-algebra B and an injective $*$-homomorphism $\iota \colon A \hookrightarrow B$ such that $\iota(A)$ is an essential ideal of B.

The most common way of adjoining a unit to a non-unital C^*-algebra A is to consider the vector space $A \oplus \mathbb{C}$ and endow it with a proper $*$-algebra structure and give it a certain C^*-norm, see e.g., [Zhu93, Theorem 15.1]. This can also be done via the use of compact operators on A.

Example 1.3.4
Let $A^1 := A \oplus \mathbb{C}$ as a $*$-algebra with $(x,\alpha)(y,\beta) := xy + \alpha y + \beta x + \alpha\beta$ and entrywise addition and involution. Identify A with $\mathbb{K}_A(A_A)$ inside $\mathcal{L}_A(A_A)$ via the left-multipication operator, L, on A, and consider the algebra generated by 1 and $\mathbb{K}_A(A_A)$ in $\mathcal{L}_A(A_A)$.

It is a C^*-algebra, and we may extend L to an isomorphism of $*$-algebras between $\mathbb{K}_A(A_A) + \mathbb{C}$ and A^1, which we turn into a C^*-algebra by stealing the norm:

$$\|(a,\alpha)\| = \sup\{\|ax + \alpha x\| \mid \|x\| \leq 1,\ x \in A\},\ \text{for } (a,\alpha) \in A^1.$$

This gives us the pair (A^1, ι), where $\iota\colon A \to A^1$, $a \mapsto (a, 0)$.

We claim that the pair (A^1, ι) is a unitization of A: ι is an isometry and hence injective as $\|\iota(a)\| = \|L_a\| = \|a\|$ for all $a \in A$ and $\iota(A)$ is an ideal since $\mathbb{K}_A(A_A)$ is an ideal. It remains to be shown that it is essential, so suppose that $(x,\beta)\iota(A) = 0$ implying that all $a \in A$ we have $xa + \beta a = 0$. If $\beta = 0$ then $\|(x,0)^*(x,0)\| = \|x\|^2 = 0$ so $x = 0$ and $(x,\beta) = (0,0)$.

Assume that $\beta \neq 0$, then $-\frac{x}{\beta}a = a$ for all $a \in A$ which implies that $a^* = a^*\left(-\frac{x}{\beta}\right)^*$. Similarly, with a^* instead, we obtain $a = a\left(-\frac{x}{\beta}\right)^*$, since $a \in A$ is arbitrary. Letting $a = -\frac{x}{\beta}$, we see that $\frac{x^*}{\beta}$ is self-adjoint as well, since

$$-\frac{x}{\beta} = -\frac{x}{\beta}\left(-\frac{x}{\beta}\right)^* = \left(-\frac{x}{\beta}\right)^*,$$

and hence a unit of A, a contradiction. Therefore $\beta = 0$, implying that $\iota(A)$ is essential in A^1.

Example 1.3.5
The pair $(\mathcal{L}_A(A_A), L)$, $L\colon a \mapsto L_a$, is a unitization of A - to see this we need only show that $\mathbb{K}_A(A_A)$ is essential in $\mathcal{L}_A(A_A)$: Suppose that $T\mathbb{K}_A(A_A) = 0$, so $TK = 0$ for all compact K on A, then with $\Theta_{x,y}\colon z \mapsto x\langle y, z\rangle = xy^*z$, we see by Proposition 1.2.8 that $T\Theta_{x,y} = \Theta_{Tx,y} = 0$ for all $x, y \in A$, which shows that for $x, y \in A$ we have $\Theta_{Tx,y}(y) = (Tx)(y^*y) = L_{Tx}(y^*y) = 0$, implying that $L_{Tx} = 0 \iff Tx = 0$ so $T = 0$.

We are interested in a non-commutative analogue of a largest compactification, namely a universal unitization in which every other unitization embeds into:

Definition 1.3.6. A unitization (B, ι) of a C^*-algebra A is said to be *maximal* if to every other unitization (C, ι'), there is a $*$-homomorphism $\pi\colon C \to B$ such that the diagram:

$$\begin{tikzcd} A \arrow[r, hook, "\iota"] \arrow[dr, hook, "\iota'"'] & B \\ & C \arrow[u, "\exists\pi"'] \end{tikzcd} \tag{1.8}$$

commutes.

Definition 1.3.7. For a C^*-algebra A, we define the *multiplier algebra* of A to be $M(A) := \mathcal{L}_A(A_A)$.

It turns out that this is a maximal unitization of A, as we shall see in a moment – but first we need a bit more work, for instance we need the notion of non-degenerate representations of a C^*-algebra on a Hilbert space to be extended to the case of Hilbert C^*-modules:

Definition 1.3.8. Let B be a C^*-algebra and E a Hilbert A-module. A $*$-homomorphism $\varphi\colon B \to \mathcal{L}_A(E)$ is *non-degenerate* if the set

$$\varphi(B)E := \operatorname{span}\left\{\varphi(b)x \mid b \in B \text{ and } x \in E\right\},$$

is dense in E.

Example 1.3.9
In the proof of Proposition 1.2.22 we saw that the inclusion $\mathbb{K}_A(A_A) \subseteq \mathcal{L}_A(A_A)$ is non-degenerate for all C^*-algebras A.

Proposition 1.3.10. Let A and C be C^*-algebras, and let E denote a Hilbert A-module and let B be an ideal of C. If $\varphi\colon B \to \mathcal{L}_A(E)$ is a non-degenerate $*$-homomorphism, then it extends uniquely to a *-homomorphism $\overline{\varphi}\colon C \to \mathcal{L}_A(E)$ resulting in a commutative diagram

$$\begin{tikzcd} B \arrow[r, "\varphi"] \arrow[d, hook] & \mathcal{L}_A(E) \\ C \arrow[ur, "\exists!\overline{\varphi}"'] & \end{tikzcd}$$

If further we assume that φ is injective and B is an essential ideal of C, then the extension is injective.

Proof.
The last statement is clear, since $\ker\overline{\varphi} \cap B = \ker\varphi \cap B = 0$ which implies that $\ker\overline{\varphi} = 0$, since B is essential. Let (e_j) be an approximate identity of B. Define for $c \in C$ the operator $\overline{\varphi}(c)$ on $\varphi(B)E$ by

$$\overline{\varphi}(c)\left(\sum_{i=1}^{n} \varphi(b_i)(x_i)\right) := \sum_{i=1}^{n} \varphi(cb_i)(x_i).$$

This is well-defined and bounded, for we see that

$$\left\|\sum_{i=1}^{n} \varphi(cb_i)(x_i)\right\| = \lim_j \left\|\sum_{i=1}^{n} \varphi(\underbrace{ce_j}_{\in B})\varphi(b_i)(x_i)\right\| \leq \|c\| \left\|\sum_{i=1}^{n} \varphi(b_i)(x_i)\right\|.$$

This extends uniquely to a bounded operator on all of E, since $\varphi(B)E$ is dense. It is easy to see that $c \mapsto \overline{\varphi}(c)$ is a $*$-homomorphism. It remains to be shown that $\overline{\varphi}(c)$ is adjointable: Let (e_j) be an approximate identity for B and $x, y \in E$. Then

$$\langle \overline{\varphi}(c)x, y\rangle = \lim_j \langle \underbrace{\varphi(ce_j)}_{\in \mathcal{L}_A(E)} x, y\rangle = \lim_j \langle x, \varphi(ce_j)^* y\rangle = \lim_j \langle x, \varphi(e_j^* c^*) y\rangle = \langle x, \overline{\varphi}(c) y\rangle,$$

showing that $\overline{\varphi}(c^*)$ is an adjoint of $\overline{\varphi}(c)$, hence it is a *-homomorphism with values in $\mathcal{L}_A(E)$ □

Corollary 1.3.11. The inclusion of $\mathbb{K}_A(A_A) \subseteq \mathcal{L}_A(A_A)$ is a maximal unitization.

Proof.
In Example 1.3.5 we saw that $\mathbb{K}_A(A_A)$ is an essential ideal of $\mathcal{L}_A(A_A)$ and nondegeneracy was seen in Example 1.3.9. By Proposition 1.3.10, the corollary follows. □

In particular, if $E = A_A$ and $C = M(B)$, then the above proposition guarantees an extension from B to $M(B)$ for all non-degenerate $*$-homomorphisms $\varphi\colon B \to M(A)$. This will be particularly useful whenever we consider the following case: Let π be a non-degenerate representation of a C^*-algebra A on a Hilbert space $\mathcal{H}$, i.e., $\pi\colon A \to \mathbb{B}(\mathcal{H})$. Then π extends to a representation of $M(A)$ on $\mathbb{B}(\mathcal{H})$. These two cases can be visualized by the following commutative diagrams:

$$\begin{tikzcd} B \arrow[r, "\varphi"] \arrow[d, hook] & M(A) \\ M(B) \arrow[ur, dashed, "\exists! \overline{\varphi}"'] & \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\pi"'] \arrow[d, hook] & \mathbb{B}(\mathcal{H}) \\ M(A) \arrow[ur, dashed, "\exists! \overline{\pi}"'] & \end{tikzcd}$$

We shall now see that the homomorphism provided by the universal property of a maximal unitization is unique, this is the final ingredient we shall need to show that $M(A)$ is a maximal unitization of A:

Lemma 1.3.12. If $\iota\colon A \hookrightarrow B$ is a maximal unitization and $j\colon A \hookrightarrow C$ is an essential embedding, then the $*$-homomorphism $\varphi\colon C \to B$ of Definition 1.3.6 is unique and injective.

Proof.
Any $*$-homomorphism φ comming from Definition 1.3.6 must be injective, for $\ker\varphi \cap j(A) = 0$, hence $\ker\varphi = 0$, since $j(A)$ is essential in C. Suppose now that we are given two $*$-homomorphisms $\varphi, \psi\colon C \to B$ such that $\varphi \circ j = \psi \circ j = \iota$. Then, for $c \in C$, we see that

$$0 = \varphi(c)\iota(a) - \psi(c)\iota(a) = \varphi(c)\varphi(j(a)) - \psi(c)\psi(j(a)) = \varphi(cj(a)) - \psi(cj(a))$$

for all $a \in A$, so essentiality of $\iota(A)$ in B and Lemma 1.3.2 implies that $\varphi = \psi$. $\square$

We may finally show that $(M(A), L)$ is a maximal unitization for all C^*-algebras A:

Theorem 1.3.13. For any C^*-algebra A, the unitization $(M(A), L)$ is maximal, and unique up to isomorphism

Proof.
For maximality, suppose that $j\colon A \hookrightarrow C$ is an essential embedding of A in a C^*-algebra C. We know that L is nondegenerate, so it extends to $\overline{L}\colon C \hookrightarrow M(A)$ satisfying $\overline{L} \circ j = L$. For uniqueness, suppose that (B, ι) is a maximal unitization of A. By Proposition 1.3.10 and maximality of (B, ι) we obtain the following commutative diagram

$$\begin{tikzcd}
 & M(A) & \\
A \arrow[ur, hook, "L"'] \arrow[r, hook, "\iota"'] \arrow[dr, hook, "L"'] & B \arrow[u, "\exists!\overline{L}"'] & \\
 & M(A) \arrow[u, "\exists!\beta"'] \arrow[uu, bend right=60, "\mathrm{id}_{M(A)}"'] &
\end{tikzcd} \tag{1.9}$$

The identity map on $M(A)$ is by Proposition 1.3.10 the unique $*$-homomorphism satisfying $L = \mathrm{id}_{M(A)} \circ L$, but $\overline{L} \circ \beta \circ L = L$, implying that $\widehat{L} = \beta^{-1}$ and $\overline{L}^{-1} = \beta$, finishing the proof. $\square$

It is useful to be able to concretely represent the Multiplier Algebra of a C^*-algebra – and while we can use the concretification as $\mathcal{L}_A(A_A)$, there are other useful descriptions as well. We cover a commonly used way in the following:

Proposition 1.3.14. Let A and C be C^*-algebras and E be a Hilbert C^*-module. Suppose that $\alpha\colon A \to \mathcal{L}_C(E)$ is an injective nondegenerate $*$-homomorphism. Then α extends to an isomorphism of $M(A)$ with the *idealizer* $\mathbb{I}(A)$ of $\alpha(A)$ in $\mathcal{L}_C(E)$:

$$\mathbb{I}(A) = \{T \in \mathcal{L}_C(E) \mid T\alpha(A) \subseteq \alpha(A) \text{ and } \alpha(A)T \subseteq \alpha(T)\}.$$

Proof.
It is clear that $\alpha(A)$ is an ideal of the idealizer $\mathbb{I}(A)$, which is in fact essential by nondegeneracy of α: If $T\alpha(A) = 0$ then $T\alpha(A)E = 0$ hence $T = 0$.

By Theorem 1.3.13 it suffices to show that $(\mathbb{I}(A), \alpha)$ is a maximal unitization of A. Let $\iota\colon A \hookrightarrow B$ be an essential embedding of A into a C^*-algebra B. Let $\overline{\alpha}\colon B \hookrightarrow \mathcal{L}_C(E)$ denote the injective extension of α from Proposition 1.3.10. We must show that $\overline{\alpha}(B) \subseteq \mathbb{I}(A)$: Let $b \in B$ and $a \in A$. Then, as $b\varphi(a) \in \varphi(A)$, we see that $\overline{\alpha}(b)\underbrace{\alpha(a)}_{=\overline{\alpha}(\iota(a))} = \overline{\alpha}(b\iota(a)) \in \alpha(A)$, finish the argument by universality of $M(A)$. ☐

This allows us to, given any faithful nondegenerate representation of A on a Hilbert space $\mathcal{H}$, to think of $M(A)$ as the idealizer of A in $\mathbb{B}(\mathcal{H})$, and it immediately allows us to describe the Multiplier algebra of $\mathbb{K}_A(E)$:

Corollary 1.3.15. For all C^*-algebras A and Hilbert A-modules, we have $M(\mathbb{K}_A(E)) \cong \mathcal{L}_A(E)$.

Proof.
This follows from Proposition 1.3.14: As we have seen the inclusion $\mathbb{K}_A(E) \subseteq \mathcal{L}_A(E)$ is nondegenerate and $\mathbb{I}(\mathbb{K}_A(E)) = \mathcal{L}_A(E)$ since the compacts is an ideal. In particular $\mathcal{L}_A(E)$ has the strict topology: $T_i \to T$ strictly if and only if $T_iK \to TK$ and $KT_i \to KT$ for all $K \in \mathbb{K}_A(E)$. ☐

Using this, we can calculate both the adjointable operators and the Multiplier algebras in some settings:

Example 1.3.16
For the Hilbert $\mathbb{C}$-module $\mathcal{H}$, we obtain the identity that $M(\mathbb{K}(\mathcal{H})) = \mathbb{B}(\mathcal{H})$, as expected, since $\mathbb{B}(\mathcal{H})$ is the prototype of a 'large' abstract C^*-algebra.

A lot of the theory in functional analysis and the study of C^*-algebras is motivated by the study of abelian C^*-algebras of the form $C_0(\Omega)$ for a locally compact Hausdorff space. This construction is no exception: Unitizations come naturally from compactifications, e.g., adjoining a unit to $C_0(\Omega)$ corresponds to the one-point compactification of Ω. It turns out that passing from $C_0(\Omega)$ to its multiplier algebra is connected to the largest compactification of Ω, namely the Stone-Čech compactification $\beta\Omega$ of Ω, which in fact is the protoexample of the multiplier construction.

Example 1.3.17
If Ω is a locally compact Hausdorff space, then $M(C_0(\Omega)) \cong C_b(\Omega) = C(\beta\Omega)$.

Proof.
The unital C^*-algebra $C_b(\Omega)$ is isomorphic to $C(Y)$ for some compact Hausdorff space Y, which can be realised as $\beta\Omega$ (see [Ped89, p. 151]). Note that since $C_0(\Omega)$ separates points in Ω cf. Urysohn's lemma, it is an essential ideal of $C_b(\Omega)$, and the inclusion $C_0(\Omega) \hookrightarrow$

$C_b(\Omega)$ is non-degenerate (as unital C^*-algebras equals their multiplier algebra). Therefore Proposition 1.3.10 ensures a unique injective $*$-homomorphism $\varphi\colon C_b(\Omega) \to M(C_0(\Omega))$.

For surjectivity, suppose that $g \in M(C_0(\Omega))$ and let $(e_i) \subseteq C_0(\Omega)$ denote an approximate identity. Without loss of generality, assume that $g \geq 0$ in $M(C_0(\Omega))$. Then $\{ge_i(x)\}_i \subseteq [0, \|g\|]$ for all $x \in \Omega$. Also $ge_i(x)$ is increasing, since e_i is and, therefore it has a limit for each $x \in \Omega$. Denote this limit by $h(x)$, which is bounded above by $\|g\|$. The function $h\colon x \mapsto h(x)$ satisfies $hf = \lim ge_i f = gf \in C_0(\Omega)$ for all $f \in C_0(\Omega)$.

We claim that $t \mapsto h(t)$ is a continuous map: Suppose that $t_\alpha \to t \in \Omega$, and without loss of generality assume that $t_\alpha \in K$ for all α, for some compact neighborhood of t. Use Urysohn's lemma to pick $f \in C_0(\Omega)$ such that $f\big|_K = 1$. Then, since $hf \in C_0(\Omega)$ we have

$$h(t) = f(t)h(t) = (hf)(t) = \lim_\alpha hf(t_\alpha) = \lim_\alpha h(t_\alpha),$$

so h is continuous. Let $f \in C_0(\Omega)$, then $\varphi(h)f = \varphi(h)\varphi(f) = \varphi(hf) = hf = gf$, so by non-degeneracy we see that $\varphi(h) = g$ so $g \in C_b(\Omega)$. $\square$

In fact, one can show that when equipped with suitable orderings, then there is a correspondance between compactifications of locally compact spaces Ω and unitizations of their associated C^*-algebras $C_0(\Omega)$.

We will now cover an important conerstone for developing the theory of the general crossed product. It begins by defining a new topology

Definition 1.3.18. Given a Hilbert A-module, the *$*$-strong topology* on $\mathcal{L}_A(E)$ is the topology for which $T_i \to T$ if and only if $T_i x \to Tx$ and $T_i^* x \to T^* x$ for all $x \in E$. It has a neighborhood basis consisting of sets of the form

$$V(T, x, \varepsilon) := \{S \in \mathcal{L}_A(E) \mid \|Sx - Tx\| + \|S^*x - T^*x\| < \varepsilon\},$$

for $T \in \mathcal{L}_A(E)$, $x \in E$ and $\varepsilon > 0$.

We will use $M_s(A)$ to mean $M(A)$ equipped with the strict topology, and we will write $\mathcal{L}_A(E)_{*-s}$ to mean we equip it with the $*$-strong topology. for $A = \mathbb{C}$, we see that $T_i \to T \in \mathbb{B}(\mathcal{H})$ in the $*$-strong topology if and only $T_i \to T$ and $T_i^* \to T$ in the strong operator topology. We said that we only have interest in the strict topology and the norm topology on $\mathcal{L}_A(E)$, and this is because the $*$-strong topology and strict topology coincides on bounded subsets:

Proposition 1.3.19. If E is a Hilbert A-module, if $T_i \to T \in \mathcal{L}_A(E)$ strictly, then $T_i \to T$ $*$-strongly. Moreover, on norm bounded subsets the topologies coincide.

Proof.
Suppose that $T_i \to T$ strictly. For $x \in E$, write $x = y\langle y, y\rangle$ for some unique $y \in E$ using Proposition 1.2.20. Let ${}_{\mathbb{K}_A(E)}\langle x, y\rangle = \Theta_{x,y} \in \mathbb{K}_A(E)$ be the associated $\mathbb{K}_A(E)$-valued inner-product on E from Proposition 1.2.14, so $x =_{\mathbb{K}_A(E)} \langle y, y\rangle(y)$. By definition of the strict topology on $\mathcal{L}_A(E) = M(\mathbb{K}_A(E))$, we see that

$$T_i(x) = T_i \circ_{\mathbb{K}_A(E)} \langle y, y\rangle(y) \to T \circ_{\mathbb{K}_A(E)} \langle y, y\rangle(y) = T(x).$$

Continuity of adjunction shows that $T_i^* \to T^*$ as well, hence also $T_i^* \to T^*$ strongly, so $T_i \to T$ $*$-strongly.

Suppose now (T_i) is bounded in norm by $K \geq 0$. For $x, y \in E$, if $T_i x \to Tx$ in norm, then $T_i\Theta_{x,y} \to T\Theta_{x,y}$ in norm:

$$\|T_i\Theta_{x,y} - T\Theta_{x,y}\| \leq \sup_{\|z\|\leq 1} \|T_i x - Tx\| \, \|\langle y, z\rangle\| \to 0,$$

and similarly $T_i^*\Theta_{x,y} \to T^*\Theta_{x,y}$. By approximating compact operators with finite linear combinations of $\Theta_{x,y}$'s, an easy $\frac{\varepsilon}{3K} > 0$ argument shows that $T_i K \to TK$ and $KT_i \to KT$ for $K \in \mathbb{K}_A(E)$ and similarly $T_i^* K \to T_i^* K$ and $KT_i^* \to KT$. □

In particular, for a net of unitary operators on a Hilbert A-module, strict and $*$-strong convergence is the same. We shall utilize this fact repeatedly later on when we develop the theory of general crossed products. We will also need the following results regarding tensor products of C^*-algebras and their multipliers:

Theorem 1.3.20. Let A, B be C^*-algebras. Then there is a faithful non-degenerate $*$-homomorphism

$$\psi\colon M(A) \otimes_{\min} M(B) \hookrightarrow M(A \otimes_{\min} B).$$

Proof.
Let A, B be represented by faithful non-degenerate representations π_A and π_B on $\mathcal{H}_A$ and $\mathcal{H}_B$. Identify $M(A)$ and $M(B)$ with the idealizers in $\mathbb{B}(\mathcal{H}_A)$ and $\mathbb{B}(\mathcal{H}_B)$, respectively. Let π be the corresponding non-degenerate and faithful $*$-homomorphism $A \otimes_{\min} B \to \mathbb{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$ given $\pi(a \otimes b)(\xi \otimes \eta) = \pi_A(a)\xi \otimes \pi_B(b)\eta$, so that $M(A \otimes_{\min} B)$ is the idealizer of $A \otimes_{\min} B$ in $\mathbb{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$.

If $t \in M(A)$ and $s \in M(B)$, so $t \otimes s \in M(A) \otimes M(B)$, then for all $a \otimes b \in A \otimes_{\min} B$, we have

$$t \otimes s(\pi(a \otimes b)) = t\pi_A(a) \otimes s\pi_B(b) \in \pi(A \otimes_{\min} B)$$

and similarly $(\pi(a\otimes b))t\otimes s \in \pi(A\otimes_{\min} B)$, so $t\otimes s \in M(A\otimes_{\min} B)$ by density of the span of elements of the form $a\otimes b$ in $A\otimes_{\min} B$, showing that $M(A)\otimes_{\min} M(B) \subseteq M(A\otimes_{\min} B)$. $\square$

There is not equiality in the above, for instance if we consider the compact operators in the spot of both A and B. However, if B is finite-dimensional, the inclusion is an equality:

Lemma 1.3.21. For all C^*-algebras A and $n \geq 1$: $M(A\otimes M_n(\mathbb{C})) = M_n(M(A))$.

Proof.
For this, suppose that $M_n(M(A)) \subseteq M_n(\mathbb{B}(H))$. It suffices to show that given any $t = (t_{i,j})$ in the idealizer of $M_n(M(A))$, each $t_{i,j}$ is in the idealizer of $M(A)$. Let $\{e_{l,k}\}$ be the canonical matrix units of $M_n(\mathbb{C})$, then for all fixed $k, l = 1, \ldots, n$ we have

$$\sum_{i=1}^{n} t_{i,k}a\otimes e_{i,l} = ta\otimes e_{k,l} \in M_n(A),$$

and similarly from the right. This shows that $a\otimes t_{i,j}, t_{i,j}a \in M(A)$ for all $a \in A$ and $i, j = 1, \ldots, n$, that $t_{i,j} \in M(A)$ and hence $t \in M_n(M(A))$. $\square$

Crossed Products with Topological Groups

Given a C^*-dynamical system (A, G, α) with G discrete, it is not a difficult task to extend the representation theory of G and A to the $*$-algebra $C_c(G, A)$, and thus create new C^*-algebras through completion of it. Examples include the reduced- and full crossed product of A and G, $A \rtimes_{\alpha,r} G$ and $A \rtimes_\alpha G$.

However, when G is not discrete, we run into issues as we a prior can not assume that a unitary representation of G is continuous in norm, which is required by classic constructions seen e.g., in [BO08]. In this chapter we will develop the necessary theory to define the topological analogues of these techniques, i.e., for when G is a locally compact Hausdorff group.

We assume familiarity with basic theory of analysis in topological groups, basic representation theory of topological groups and Fourier analysis thereon and will not go to lengths to explain results from this field, for a reference, we recommend the excellent books on the subject: [Fol16], [BCR84] and [Fol13]. After defining and studying the crossed product construction we examine some important cases including the case where G is abelian and the case where A is.

2.1 Integral Forms And Representation Theory

Throughout this chapter, we let G be a locally compact topological group and A a C^*-algebra, with no added assumptions unless otherwise specified. We will write $G \overset{\alpha}{\curvearrowright} A$ to specify that $\alpha\colon G \to \operatorname{Aut}(A)$ is an action of G on A, i.e., a strongly continuous group homomorphism, and we adopt the notation $\alpha_t := \alpha(t)$, $t \in G$, whenever it is sensible to do so. We will always fix a base left Haar measure μ on G, whose integrand we will denote by $\mathrm{d}t$, with modular function $\Delta\colon G \to (0, \infty)$.

The main motivation of the theory we are about to develop is this: Given a covariant representation $(\pi, u, \mathcal{H})$ of (A, G, α), we wish to construct a representation $\pi \rtimes u$ of $C_c(G, \mathbb{B}(\mathcal{H}))$

satisfying

$$\pi \rtimes u(f) = \int \pi(f(t))u_t \mathrm{d}t.$$

The problem at hand is that a priori the map $t \mapsto u_t$ is not norm continuous, so we don't know if the term $\pi(f(t))u_t$ is integrable. We will however show that the assignment $t \mapsto u_t$ is strictly continuous and use this to define our integral by applying it to $M(\mathbb{B}(\mathcal{H})) = \mathbb{B}(\mathcal{H})$.

For $f \in C_c(G, A)$, the map $s \mapsto \|f(s)\|$ is in $C_c(G)$, and we define the number $\|f\|_1 \in [0, \infty)$ to be

$$\|f\|_1 = \int_G \|f(t)\| \, \mathrm{d}t.$$

For $A = \mathbb{C}$, it is well known that given $f \in C_c(G, A)$ and $\varepsilon > 0$, there is a neighborhood V of $e \in G$ such that $sr^{-1} \in V$ or $s^{-1}r \in V$ implies $|f(s) - f(r)| < \varepsilon$, i.e., f is left- and right uniformly continuous. The statement holds for general A, and the proof is identical, see e.g. [Fol16, Proposition 2.6] or [Wil07, Lemma 1.88].

Lemma 2.1.1. For $f \in C_c(G, A)$ and $\varepsilon > 0$, there is a neighborhood V of $e \in G$ such that either $sr^{-1} \in V$ or $s^{-1}r \in V$ implies

$$\|f(s) - f(r)\| < \varepsilon.$$

We will repeatedly use the Partitions of Unity result. For a proof, see e.g. [Wil07, Lemma 1.43].

Lemma 2.1.2 (Partitions of Unity)**.** Suppose that K is a compact subset of a locally compact Hausdorff group G with a finite open cover of pre-compact sets $U_1, \ldots, U_n$. Then, for $i = 1, \ldots, n$ there exists $\varphi_i \in C_c(G)$ satisfying

1. $\varphi_i(G) \subseteq [0, 1]$ for all i,
2. $\operatorname{supp} \varphi_i \subseteq U_i$ for all i,
3. $\sum_{i=1}^n \varphi_i(x) = 1$ for all $x \in K$ and
4. $\sum_{i=1}^n \varphi_i(x) \leq 1$ for all $x \notin K$.

We will use $\mathcal{D}$ to denote an arbitrary complex Banach space, and for a pair (f, x) with $f \in C_c(G)$ and $x \in \mathcal{D}$, we denote by $f \otimes x$ the map $G \ni t \mapsto f(t)x \in \mathcal{D}$. It is then clear that $f \otimes x \in C_c(G, \mathcal{D})$. We introduce a 'topology' or a name for a special kind of convergence on $C_c(G, \mathcal{D})$, which will prove useful:

Definition 2.1.3. We say that a net $\{f_i\}_{i\in I} \subseteq C_c(G,\mathcal{D})$ converges to a function f on $C_c(G,\mathcal{D})$ in the *inductive limit topology* if $f_i \to f$ uniformly and there is a compact set $K \subseteq G$ and $i_0 \in I$ such that for $i \geq i_0$ we have $f_i\big|_{K^c} = 0$, i.e., $\mathrm{supp} f_i \subseteq K$, $i \geq i_0$.

Lemma 2.1.4. Let $\mathcal{D}_0 \subseteq \mathcal{D}$ be any dense subset. Then the set

$$C_c(G) \odot \mathcal{D}_0 := \mathrm{span}\,\{f \otimes x \mid f \in C_c(G) \text{ and } x \in \mathcal{D}_0\}$$

is dense in the inductive limit topology on $C_c(G,\mathcal{D})$.

Proof.
Let $\varphi \in C_c(G,\mathcal{D})$ and $\varepsilon > 0$ be arbitrary. Let $K = \mathrm{supp}\varphi$ and let W denote an arbitrary compact neighborhood of $e \in G$, and pick a symmetric neighborhood $e \in V_\varepsilon \subseteq W$ such that $sr^{-1} \in V_\varepsilon$ implies $\|\varphi(s) - \varphi(r)\| < \frac{\varepsilon}{\mu(WK)}$. Since K is compact, there is a finite set $\{r_1, \ldots, r_n\} \subseteq K$ such that $K \subseteq \bigcup_{1\leq i\leq n} V_\varepsilon r_i$. The cover $K^c \cup \left(\bigcup_{1\leq i\leq n} V_\varepsilon r_i\right)$ of K admits a partition of unity $\{f_j\}_{j=0}^n$ such that $\mathrm{supp} f_0 \subseteq K^c$ and $\mathrm{supp} f_i \subseteq V_\varepsilon r_i$ for $i = 1, \ldots, n$.

Using this, we define

$$g_\varepsilon := \sum_{i=1}^n f_i \otimes x_i,$$

where $x_i \in \mathcal{D}_0$ with $\|x_i - \varphi(r_i)\| < \frac{\varepsilon}{2\mu(WK)}$ for $i = 1, \ldots, n$. Then $g_\varepsilon \in C_c(G) \odot \mathcal{D}_0$ with $\mathrm{supp} g_\varepsilon \subseteq V_\varepsilon K \subseteq WK$. The partition of unity satisfies $\sum_{i=1}^n f_i(s) \leq 1$ for all $s \in G$ with equality for $s \in K$, so in particular $\sum_{1\leq i\leq n} f_i(s)\varphi(s) = \varphi(s)$ for all $s \in G$. Hence we see that for $s \in G$:

$$\begin{aligned}\|\varphi(s) - g_\varepsilon(s)\| = \left\|\sum_{i=1}^n f_i(s)(\varphi(s) - \varphi(r_i) + \varphi(r_i) - x_i)\right\| &\leq \sum_{i=1}^n f_i(s)(\|\varphi(s) - \varphi(r_i)\| + \|\varphi(r_i) - x_i\|) \\ &\leq \frac{\varepsilon}{\mu(WK)},\end{aligned}$$

Since ε was arbitrary and W, K didn't depend on ε and $\mathrm{supp} g_\varepsilon \subseteq WK$, we see that $g_\varepsilon \to \varphi$ in the inductive limit topology as $\varepsilon \to 0$. $\square$

We will also need the following result, which can be found in [Rud91, Theorem 3.20, part (b, c)]

Lemma 2.1.5. If $K \subseteq \mathcal{D}$ is compact, then the convex hull $\mathrm{conv}(K)$ is totally bounded, hence it has compact closure.

Definition 2.1.6. Let $f \in C_c(G,\mathcal{D})$ and $\varphi \in \mathcal{D}^*$. Then we can define a bounded linear functional L_f on $\mathcal{D}^*$ by

$$L_f(\varphi) := \int_G \varphi(f(s))\mathrm{d}s,$$

since $|L_f(\varphi)| \leq \int_G \|\varphi\| \|f(s)\| \,\mathrm{d}s = \|f\|_1 \|\varphi\|$.

Note 2.1.7
If $(f_i) \subseteq C_c(G, \mathcal{D})$ is a net which converges in the inductive limit topology to $f \in C_c(G, \mathcal{D})$ with respect to some compact set $K \subseteq G$. Then

$$\|f_i - f\|_1 = \int_G \|f_i(s) - f(s)\| \,\mathrm{d}s \leq \|f_i - f\| \mu(K) \to 0,$$

so $f_i \to f$ in the L^1-norm, and hence by the above we see that $|L_{f_i}(\varphi) - L_f(\varphi)| \leq \|f_i - f\|_1 \|\varphi\| \to 0$ for all $\varphi \in \mathcal{D}^*$, i.e., $L_{f_i} \to L_f$ in the weak* topology of $\mathcal{D}^{**}$.

We will see that L_f is equal to $\mathrm{ev}_a \colon \varphi \mapsto \varphi(a)$ for some unique $a \in \mathcal{D}$. Let $\iota \colon \mathcal{D} \to \mathcal{D}^{**}$ denote the inclusion $\iota(a) = \mathrm{ev}_a$, then we have

Lemma 2.1.8. If $f \in C_c(G, \mathcal{D})$, then $L_f \in \iota(\mathcal{D})$.

Proof.
Let W be a compact neighborhood of supp$f$, and let $K := f(G) \cup \{0\} \subseteq \mathcal{D}$. Then K is compact, so by Lemma 2.1.5 the set $C := \overline{\mathrm{conv}}\,\{K\}$ is compact.

Fix $\varepsilon > 0$, and pick a neighborhood V of $e \in G$ such that $\|f(s) - f(r)\| < \varepsilon$ for $s^{-1}r \in V$. Assume without loss of generality that supp$fV \subseteq W$, since we can make V arbitrarily small. Using compactness of $\operatorname{supp} f$, pick $s_1, \dots, s_n \in \operatorname{supp} f$ such that $\{s_i V\}$ covers $\operatorname{supp} f$ (recall that $e \in V$). Since $\bigcup_{i=1}^n s_i V \subseteq W$, it has compact closure and we may pick a partition of unity $\{\varphi_i\}_{i=1}^n$ of $\operatorname{supp} f \subseteq \bigcup_{i=1}^n s_i V$. Using this, we define

$$g_\varepsilon(s) := \sum_{i=1}^n f(s_i)\varphi_i(s),$$

so $\operatorname{supp} g_\varepsilon \subseteq W$. If $s \in \left(\bigcup_{i=1}^n s_i V\right)^c$ then $f(s) = g_\varepsilon(s) = 0$. If $s \in s_i V$, then

$$\|f(s) - g_\varepsilon\| \leq \sum_{i=1}^n \varphi(s) \|f(s) - f(s_i)\| < \varepsilon,$$

as $\|f(s) - f(s_i)\| < \varepsilon$ since $s_i^{-1}s \in V$.

As the map $s \mapsto \sum_{i=1}^n \varphi_i(s)$ is bounded by above by 1 and is compactly supported, the fact that $\bigcup_{i=1}^n \operatorname{supp} \varphi_i \subseteq W$ implies that

$$\delta := \sum_{i=1}^n \int_G \varphi_i(s) \mathrm{d}s \leq \mu(W) < \infty.$$

For $\psi \in \mathcal{D}^*$ we see that

$$L_{g_\varepsilon}(\psi) = \int_G \psi\left(\sum_{i=1}^n \varphi_i(s) f(s_i)\right) \mathrm{d}s = \sum_{i=1}^n \int_G \varphi_i(s)\mathrm{d}s \psi(f(s_i)) = \sum_{i=1}^n \int_G \varphi_i(s)\mathrm{d}s\, \underbrace{\iota(f(s_i))}_{\in \iota(C)}(\psi)$$

so $L_{g_\varepsilon} \in \delta\iota(C) \subseteq \mu(W)\iota(C)$ as $0 \in C$. For $\varepsilon \to 0$, we see that $g_\varepsilon \to f$ in the inductive limit topology and $L_{g_\varepsilon} \in \mu(W)\iota(C)$.

Since C is compact and convex, it is weakly compact which by definition is equivalent to $\iota(C)$ being compact in the weak*-topology of $\mathcal{D}^{**}$. Moreover, since $g_\varepsilon \to f$ in the inductive limit topology, we get from Note 2.1.7 that $L_{g_\varepsilon} \to L_f$ in the weak *-topology on $\mathcal{D}^{**}$. In particular, since $\iota(C)$ is weak*-compact, it is closed implying that $\mu(W)\iota(C)$ is closed in the weak*-topology so $L_f \in \mu(W)\iota(C) \subseteq \iota(\mathcal{D})$. □

And we have a lemma emphasizing important properties of L_f:

Lemma 2.1.9. There exists a unique linear map $I\colon C_c(G, \mathcal{D}) \to \mathcal{D}$,

$$f \mapsto \int_G f(s)\mathrm{d}s$$

such that for all $f \in C_c(G, \mathcal{D})$ we have

$$\varphi\left(\int_G f(s)\mathrm{d}s\right) = \int_G \varphi(f(s))\mathrm{d}s, \quad \text{for all } \varphi \in \mathcal{D}^*, \tag{2.1}$$

$$\left\|\int_G f(s)\mathrm{d}s\right\| \le \|f\|_1, \tag{2.2}$$

$$\int_G (g \otimes x)(s)\mathrm{d}s = x \int_G g(s)\mathrm{d}s, \text{ for all } g \otimes x \in C_c(G) \odot \mathcal{D}, \tag{2.3}$$

$$T\left(\int_G f(s)\mathrm{d}s\right) = \int_G T(f(s))\mathrm{d}s. \tag{2.4}$$

for all bounded linear operators T between Banach spaces.

Proof.
Letting $I(f) := \iota^{-1}(L_f)$, with L_f as in Lemma 2.1.8, then $f \mapsto I(f)$ instantly satisfies the first properties, perhaps with exception of uniqueness, which follows from the fact that B^* separates points of B. For the last, suppose that $T\colon \mathcal{D} \to D$ is a bounded linear operator between Banach spaces. Then $T \circ f \in C_c(G, D)$ for all $f \in C_c(G, \mathcal{D})$, and $\varphi \circ T \in \mathcal{D}^*$ for all $\varphi \in D^*$. Hence

$$\varphi \circ T\left(\int_G f(s)\mathrm{d}s\right) = \int_G \varphi(T \circ f(s))\mathrm{d}s = \varphi\left(\int_G T(f(s))\mathrm{d}s\right),$$

for all $\varphi \in D^*$, finishing the proof. □

In light of the above, we may give the following definition:

Definition 2.1.10. For $f \in C_c(G, \mathcal{D})$, define the integral of f as $I(f) := \iota^{-1}(L_f)$, which we denote by $\int_G f(s)\mathrm{d}s := I(f)$.

We recall that any non-degenerate representation π of a C^*-algebra A on a Hilbert space H extends to a representation $\overline{\pi}$ of $M(A)$ on H, which is faithful if π is. In particular, if we let A be a C^*-algebra, then we obtain the following useful result:

Proposition 2.1.11. For $f \in C_c(G, A)$, if $\pi\colon A \to \mathbb{B}(\mathcal{H})$ is a representation of A, then for all $\xi, \eta \in \mathcal{H}$ the integral of Lemma 2.1.9 satisfies

$$\langle \pi \left(\int_G f(s)\mathrm{d}s \right) \xi, \eta \rangle = \int_G \langle \pi(f(s))\xi, \eta\rangle \mathrm{d}s,$$
$$\left(\int_G f(s)\mathrm{d}s \right)^* = \int_G f(s)^* \mathrm{d}s,$$

and for $a, b \in M(A)$

$$a \int_G f(s)\mathrm{d}s b = \int_G a f(s) b \mathrm{d}s.$$

Proof.
Define the vector-state $\varphi_\pi\colon a \mapsto \langle \pi(a)\xi, \eta\rangle$. Then, $\varphi_\pi \in A^*$, and the first statement follows. Suppose now that π is a faithful representation, then as $\int_G \pi(f(s))\mathrm{d}s = \pi\left(\int_G f(s)\mathrm{d}s\right)$, we see that the second statement holds as well when applied to $\varphi_\pi(I(f)^*)$. Finally if $a, b \in M(A)$ then

$$\begin{aligned}
\langle \pi \left(a \int_G f(s)\mathrm{d}s b \right) \xi, \eta \rangle &= \langle \overline{\pi}(a) \pi \left(\int_G f(s)\mathrm{d}s \right) \overline{\pi}(b)\xi, \eta \rangle \\
&= \langle \pi \left(\int_G f(s)\mathrm{d}s \right) (\overline{\pi}(b)\xi), \overline{\pi}(a^*)\eta \rangle \\
&= \int_G \langle \pi(f(s))\overline{\pi}(b)\xi, \overline{\pi}(a^*)\eta\rangle \mathrm{d}s \\
&= \int_G \langle \pi(a f(s) b)\xi, \eta\rangle \mathrm{d}s \\
&= \langle \pi \left(\int_G a f(s) b \mathrm{d}s \right) \xi, \eta \rangle.
\end{aligned}$$

, ∎

As mentioned, we will need to consider maps of the form of the form $s \mapsto \pi(f(s))u_s$ where $f \in C_c(G, A)$, $\pi\colon A \to \mathbb{B}(\mathcal{H})$ and $u\colon G \to \mathcal{U}(\mathcal{H})$ is strongly continuous. In order to use the above theorem to make sense, we will want to change A out with $M_s(A)$, i.e., the multiplier algebra of A equipped with the strict topology, and combine it with the following lemma:

Lemma 2.1.12. Let E be a Hilbert A-module, and let $u\colon G \to \mathcal{U}(E)$ be a group homomorphism from G to the group of unitary operators on E. Then u is strictly continous if and only if u is strongly continuous.

Proof.
We saw in Proposition 1.3.19 that on bounded sets, the strict and the $*$-strong topology coincided. In particular, since $\mathcal{L}_A(E)$ is a C^*-algebra, the set of unitary operators is a subset of the unit sphere. Thus, it suffices to show that if $s \mapsto u_s x$ is continuous for x then $s \mapsto u_s^* x$ is continuous. Let $x \in E$, then

$$\|u_s^* x - u_t^* x\|_A^2 = \underbrace{\|u_s^* x\|_A^2}_{=\|x\|_A^2} + \underbrace{\|u_t^* x\|_A^2}_{=\|x\|_A^2} - \langle \underbrace{(u_t u_s^* + u_s u_t^*)}_{\xrightarrow{s\to t} 2I_E} x, x\rangle \xrightarrow{s\to t} 0,$$

finishing the proof, since $x \in E$ was arbitrary. $\square$

We now generalize the result of Proposition 2.1.11 to cover the class of functions which we shall deal with later on.

Theorem 2.1.13. Given a C^*-algebra A, there is a unique linear map $I\colon C_c(G, M_s(A)) \to M(A)$, $f \mapsto \int_G f(s)\mathrm{d}s$, such that for any non-degenerate representation $\pi\colon A \to \mathbb{B}(\mathcal{H})$ and all $\xi, \eta \in \mathcal{H}$ identity

$$\langle \overline{\pi}\left(\int_G f(s)\mathrm{d}s\right)\xi, \eta\rangle = \int_G \langle \overline{\pi}(f(s))\xi, \eta\rangle \mathrm{d}s, \tag{2.5}$$

holds and and for all $f \in C_c(G, M_s(A))$

$$\|I(f)\mathrm{d}s\| \leq \|f\|_\infty \mu(\operatorname{supp} f). \tag{2.6}$$

Also, the identities

$$\int_G f(s)^* \mathrm{d}s = \left(\int_G f(s)\mathrm{d}s\right)^*, \tag{2.7}$$

and

$$\int_G a f(s) b \mathrm{d}s = a\left(\int_G f(s)\mathrm{d}s\right) b \tag{2.8}$$

hold for all $f \in C_c(G, M_s(A))$ and $a, b \in M(A)$. If $T\colon A \to M(B)$ is a non-degenerate $*$-homomorphism of C^*-algebras, then

$$\overline{T}\left(\int_G f(s)\mathrm{d}s\right) = \int_G \overline{T}(f(s))\mathrm{d}s. \tag{2.9}$$

Proof.
If I exists, then letting π be a faithful representation of A we see that I must be unique by the first identity. For $f \in C_c(G, M_s(A))$ and $a \in A$, it is easy to see that if $f_a \colon s \mapsto f(s)a$, then $f_a \in C_c(G, A)$. This allows us to use Proposition 2.1.11 to define the integral $\int_G f(s)a\mathrm{d}s$ for all $f \in C_c(G, M_s(A))$ and $a \in A$. Define $L_f \colon A \to A$ by

$$L_f(a) = \int_G f(s)a\mathrm{d}s, \quad \text{for } a \in A.$$

Then $L_f \in \mathcal{L}_A(A_A)$, for if $a, b \in A$ then Proposition 2.1.11 implies that

$$\langle L_f a, b\rangle_{A_A} = (L_f a)^* b = \int_G (f(s)a)^* \mathrm{d}s b = \int_G a^* f(s)^* \mathrm{d}s = a^* \int_G f(s)^* b \mathrm{d}s = \langle a, L_{f^*} b\rangle_{A_A}.$$

So L_{f^*} is an adjoint of L_f. Let $\int_G f(s)\mathrm{d}s := L_f$ for all $f \in C_c(G, M_s(A)$, then $L_f \in \mathcal{L}_A(A_A) = M(A)$, and it is not hard to see that the assignment $f \mapsto L_f$ is linear.

Again, by Proposition 2.1.11, we see that a non-degenerate representation $\pi \colon A \to \mathbb{B}(\mathcal{H})$ with $a \in A$ and $\xi, \eta \in \mathcal{H}$ that

$$\begin{aligned}\langle \overline{\pi}\left(\int_G f(s)\mathrm{d}s\right)\pi(a)\xi, \eta\rangle = \langle \pi\left(f(s)a\mathrm{d}s\right)\xi, \eta\rangle &= \int_G \langle \pi(f(s)a)\xi, \eta\rangle \mathrm{d}s \\ &= \int_G \langle \overline{\pi}(f(s))\pi(a)\xi, \eta\rangle \mathrm{d}s,\end{aligned}$$

which shows Equation (2.5), since π is non-degenerate. The analogues of Proposition 2.1.11 follows similarly. Let $a \in A$ with $\|a\| \leq 1$, then by Proposition 2.1.11 applied to $f_a \in C_c(G, A)$, we see that

$$\left\|\int_G f(s)\mathrm{d}s a\right\| \leq \|f_a\|_1 = \int_G \|f(s)a\| \,\mathrm{d}s \leq \int_G 1_{\operatorname{supp} f} \|f\|_\infty \,\mathrm{d}s = \mu(\operatorname{supp} f)\, \|f\|_\infty\,,$$

Hence $\left\|\int_G f(s)\mathrm{d}s\right\| = \sup_{\|a\|\leq 1} \left\|\int_G f(s)\mathrm{d}s a\right\| \leq \|f\|_\infty\, \mu(\operatorname{supp} f)\infty$. Let now $T \colon A \to M(B)$ be a non-degenerate homomorphism, with unique extension $\overline{T} \colon M(A) \to M(B)$. Then all $a \in A$ and $b \in B$ we have

$$\begin{aligned}\overline{T}\left(\int_G f(s)\mathrm{d}s\right)T(a)b = T\left(\int_G f(s)a\mathrm{d}s\right)b = \int_G T(f(s)a)b\mathrm{d}s &= \int_G \overline{T}(f(s))T(a)b\mathrm{d}s \\ &= \int_G \overline{T}(f(s))\mathrm{d}s T(a)b,\end{aligned}$$

hence $\overline{T}\left(\int_G f(s)\mathrm{d}s\right) = \int_G \overline{T}(f(s))\mathrm{d}s$ by non-degeneracy. □

It might happen that we consider e.g., multiple integrals of the form above with distinct domains. Therefore, whenever needed we will add the domain to the notation in the following way: If $I\colon C_c(G, M_s(A)) \to M(A)$, then we write

$$I(f) := \int_G^{M_s(A)} f(s)\mathrm{d}s.$$

In the construction of the general crossed product and other C^*-algebras associated to a C^*-dynamical system (A, G, α) we will need to deal with the algebra $C_c(G, A)$ quite a lot. We will want to develop a sort of twisted-convolution incorporating the action α of G. However, in order to do this properly, we will need a form of Fubini's theorem for the above:

Lemma 2.1.14. For $f \in C_c(G \times G, B)$, the function

$$t \mapsto \int_G f(t, s)\mathrm{d}s$$

is an element of $C_c(G, B)$.

Proof.
It is easy to see that if $F\colon G \to B$ is the map $t \mapsto \int_G f(t, s)\mathrm{d}s$, then $\operatorname{supp} F$ is compact. We show continuity. Suppose that $x_i \to x \in G$ and let $\varepsilon > 0$. Pick compact $K_1, K_2 \subseteq G$ covering the support of f, i.e., such that $\operatorname{supp} f \subseteq K_1 \times K_2$. Note that $f(x_i, y) \to f(x, y)$ uniformly: Assume towards a contradiction that $f(x_i, y)$ doesn't converge uniformly to $f(x, y)$. Then we can find a net $(r_j) \subseteq G$ and $\varepsilon_0 > 0$ such that

$$\|f(x_i, r_j) - f(x, r_j)\| \geq \varepsilon_0,$$

for all i, j. This shows that $r_j \in K_2$ for all j, hence we may assume that $r_j \to r \in K_2$ (up to passing to a subnet). But then for sufficiently large i, j we would have $\|f(x_i, r_j) - f(x, r_j)\| < \varepsilon_0$. Hence we can assume that $\|F(x_i) - F(x)\| < \frac{\varepsilon}{\mu(K_2)}$. Then,

$$\left\| \int_G f(x_i, s)\mathrm{d}s - \int_G f(x, s)\mathrm{d}s \right\| \leq \int_{K_2} \|f(x_i, s) - f(x, s)\| \ ds + \int_{K_2^c} \|f(x_i, s) - f(x, s)\| \,\mathrm{d}s \leq \varepsilon,$$

as $\operatorname{supp} f \subseteq K_1 \times K_2$, so $F \in C_c(G, B)$. $\square$

Corollary 2.1.15. Replacing the integral of Lemma 2.1.14 with $\int_G f(t, s)\mathrm{d}t$ the statement still holds.

Theorem 2.1.16 (Fubini for C^*-valued integrals)**.** Suppose that $f \in C_c(G \times G, M_s(A))$, then

$$g \mapsto \int_G^{M_s(A)} f(g, t)\mathrm{d}t \text{ and } g \mapsto \int_G^{M_s(A)} f(t, g)\mathrm{d}t$$

are elements of $C_c(G, M_s(A))$ and moreover

$$\int_G^{M_s(A)} \int_G^{M_s(A)} f(g,t)\mathrm{d}g\mathrm{d}t = \int_G^{M_s(A)} \int_G^{M_s(A)} f(g,t)\mathrm{d}t\mathrm{d}g.$$

Proof.
Define for a fixed g, the map $t \mapsto f(g,t)$ is in $C_c(G, M_s(A)$, hence we can define $\int_G f(s,t)\mathrm{d}t$ and $\int_G f(t,s)\mathrm{d}t$ as in Theorem 2.1.13. Recall that $M(A) = M(\mathbb{K}_A(A_A))$, so the strict topology is induced by multiplication operators on A, so it suffices to show that $g \mapsto a\left(\int_G f(g,t)\mathrm{d}t\right)$ and $g \mapsto \left(\int_G f(g,t)\mathrm{d}t\right)a$ are in $C_c(G, A)$ for all $a \in A$ for the first claim. By Theorem 2.1.13, we have

$$a\left(\int_G f(g,t)\mathrm{d}t\right) = \int_G af(g,t)\mathrm{d}t,$$

for all $a \in A$ and $g \in G$, and similarly $\left(\int_G f(g,t)\mathrm{d}t\right)a = \int_G f(g,t)a\mathrm{d}t$. The maps

$$(g,t) \mapsto af(g,t) \text{ and } (g,t) \mapsto f(g,t)a,$$

are in $C_c(G \times G, A)$, clearly, so Lemma 2.1.14 ensures that $g \mapsto \int_G^{M_s(A)} f(g,t)\mathrm{d}t$ and $g \mapsto \int_G^{M_s(A)} f(t,g)\mathrm{d}t$ are in $C_c(G, M_s(A)$.

For the last claim, we now see that the double integrals are well-defined by Theorem 2.1.13, so by letting A be faithfully representted on $\mathbb{B}(\mathcal{H})$, we may obtain the equality using the regular Fubini's theorem for scalar valued integrals (see e.g., [Sch17]) to the inner-product, i.e., let $\xi, \eta \in H$ and see that

$$\int_G \int_G \langle f(g,t)\xi, \eta\rangle \mathrm{d}g\mathrm{d}t = \int_G \int_G \langle f(g,t)\xi_i, \xi_j\rangle \mathrm{d}t\mathrm{d}g,$$

finishing the proof. $\square$

2.2 Crossed Products

In analysis, an object of interest is the group algebra $L^1(G)$ for locally compact groups G. One can turn it into a $*$-algebra by showing that the convolution gives rise to a well-defined multiplication, and one can use the modular function to define an involution. Analogously to this, we do the same for A valued functions:

Definition 2.2.1. Suppose that $G \overset{\alpha}{\curvearrowright} A$. We turn the algebra

$$C_c(G, A) = \{f\colon G \to A \mid f \text{ continuous and compactly supported}\},$$

into a $*$-algebra in the following way: For $f, g \in C_c(G, A)$, we define the α-twisted convolution of f and g at $t \in G$ by

$$(f *_\alpha g)(t) := \int_G f(s)\alpha_s(g(s^{-1}t))\mathrm{d}s,$$

which is well defined since the map $(s, t) \mapsto f(s)\alpha_s g(s^{-1}t)$ is in $C_c(G\times G, A)$ and Lemma 2.1.14 applies. Moreover, we define an α-twisted involution of $f \in C_c(G, A)$ at $t \in G$ by

$$f^*(t) := \Delta(t^{-1})\alpha_t(f(t^{-1})^*).$$

Straight forward calculations similar to the classic ones show that $C_c(G, A)$ becomes a $*$-algebra with multiplication given by the α-twisted convolution and involution given by the α-twisted involution. We denote the associated $*$-algebra by $C_c(G, A, \alpha)$.

We associate to $f \in C_c(G, A, \alpha)$ the number $\|f\|_1 \in [0, \infty)$, defined by

$$\|f\|_1 := \int_G \|f(t)\|\,\mathrm{d}t.$$

Straightforward calculations show that $f \mapsto \|f\|_1$ is a norm, and we denote the completion of $C_c(G, A, \alpha)$ in $\|\cdot\|_1$ by $L^1(G, A, \alpha)$.

Remark 2.2.2
If there is no danger of confusion, we simply write $C_c(G, A)$ and $L^1(G, A)$. If we want to emphasize the action, we do so.

Definition 2.2.3. Let G be a C^*-algebra and $G \overset{\alpha}{\curvearrowright} A$ and action of G on a C^*-algebra. We call the associated triple (A, G, α) a *C^*-dynamical system.* We will sometimes just write $G \overset{\alpha}{\curvearrowright} A$ to indicate the associated C^*-dynamical system.

Definition 2.2.4. Suppose that $G \overset{\alpha}{\curvearrowright} A$. A *covariant representation* of (A, G, α) is a triple $(\pi, u\mathcal{H})$ consisting of a Hilbert space $\mathcal{H}$, $*$-representation π of A on $\mathcal{H}$ and a unitary representation of u on $\mathcal{H}$ such that for all $t \in G$ and $a \in A$ we have

$$u_t\pi(a)u_t^* = \pi(\alpha_s(a)).$$

If no confusion can be obtained, we will use the shorthan $(\pi, u) := (\pi, u, \mathcal{H})$. We say that a covariant representation (π, u) is a *non-degenerate covaraint representation* if π is, and degenerate otherwise.

For discrete and countable groups G acting on a C^*-algebra A via an action α, we may easily construct a $*$-representation of $C_c(G, A)$ from a covariant representation (π, u) by letting $\pi \rtimes u \colon C_c(A, G, \alpha) \to \mathbb{B}(\mathcal{H})$ be the map

$$\sum_{g \in G} a_g \delta_g \mapsto \pi(a_g) u_g,$$

however as discussed previously, this is not enough when G is not discrete Luckily, our efforts in the previous section will prove fruitful, for it turns out that we developed the tools to construct the desired generalization:

Lemma 2.2.5. Suppose that $G \overset{\alpha}{\curvearrowright} A$. If (π, u) is a covariant representation of the associated C^*-dynamical system, then for all $f \in C_c(G, A)$, the map

$$s \mapsto \pi(f(s)) u_s, \text{ for } s \in G$$

is in $C_c(G, \mathbb{B}_s(\mathcal{H}))$, where $\mathbb{B}_s(\mathcal{H})$ indicates that we endow $\mathbb{B}(\mathcal{H})$ with the strict topology.

Proof.
By assumption, u is strongly continuous, which is equivalent to being strictly continuous by Lemma 2.1.12. Since $f \in C_c(G, A)$, we see that $\pi \circ f \in C_c(G, \mathbb{B}(\mathcal{H}))$, and norm continuity implies strict continuity, so $s \mapsto \pi(f(s)) u_s$ is an element of $C_c(G, \mathbb{B}_s(\mathcal{H}))$. □

The above and Theorem 2.1.13 shows that given a covariant representation (π, u) of $G \overset{\alpha}{\curvearrowright} A$, there is a well-defined linear map $\pi \rtimes u \colon C_c(G, A) \to \mathbb{B}(\mathcal{H})$ given by

$$\pi \rtimes u(f) = \int_G^{\mathbb{B}(\mathcal{H})} \pi(f(s)) u_s \mathrm{d}s,$$

We shall see that this representation has a lot of nice properties:

Proposition 2.2.6. Suppose that (π, u) is a covariant representation of a dynamical system $G \overset{\alpha}{\curvearrowright} A$, with G locally compact. Then the map $\pi \rtimes u$ is a $*$-representation of $C_c(G, A, \alpha)$ on $\mathbb{B}(\mathcal{H})$ which is norm-decreasing. We call $\pi \rtimes u$ the *integrated form of π and u*. If π is non-degenerate, then $\pi \rtimes u$ is as well.

Proof.
Let $\xi, \eta \in \mathcal{H}$ and $f \in C_c(G, A)$, then

$$|\langle \pi \rtimes u(f)\xi, \eta\rangle| \leq \int_G |\langle \pi(f(s)) u_s \xi, \eta\rangle| \mathrm{d}s \leq \int_G \|\pi(f(s))\| \, \|\xi\| \, \|\eta\| \, \mathrm{d}s \leq \|f\|_1 \|\xi\| \, \|\eta\|$$

Since ξ, η where arbitrary, we have $\|\pi \rtimes u(f)\| \leq \|f\|_1$ for all $f \in C_c(G, A)$.

Let $f \in C_c(G, A, \alpha)$, then

$$\begin{aligned}
\pi \rtimes u(f)^* = \int_G^{\mathbb{B}(\mathcal{H})} u_s^* \pi(f(s))^* \mathrm{d}s &= \int_G^{\mathbb{B}(\mathcal{H})} u_{s^{-1}} \pi(f(s)^*) \mathrm{d}s \\
&= \int_G^{\mathbb{B}(\mathcal{H})} \Delta(s)^{-1} u_s \pi(f(s^{-1})^*) \mathrm{d}s \\
&= \int_G^{\mathbb{B}(\mathcal{H})} \Delta(s)^{-1} \pi(\alpha_s(f(s^{-1})^*)) u_s \mathrm{d}s \\
&= \int_G^{\mathbb{B}(\mathcal{H})} \pi \left(\underbrace{\alpha_s \left(\Delta(s)^{-1} f(s^{-1})^* \right)}_{=f^*(s)} \right) u_s \mathrm{d}s \\
&= \int_G^{\mathbb{B}(\mathcal{H})} \pi(f^*(s)) u_s \mathrm{d}s \\
&= \pi \rtimes u(f^*).
\end{aligned}$$

If $f, g \in C_c(G, A, \alpha)$ then using left-invariance of $\mathrm{d}s$, Proposition 2.1.11 and Theorem 2.1.16 yields

$$\begin{aligned}
\pi \rtimes u(f *_\alpha g) &= \int_G^{\mathbb{B}(\mathcal{H})} \pi\left(f *_\alpha g(s)\right) u_s \mathrm{d}s \\
&= \int_G^{\mathbb{B}(\mathcal{H})} \pi \left(\int_G^A f(t) \alpha_t g(t^{-1}s) \mathrm{d}t \right) u_s \mathrm{d}s \\
&= \int_G^{\mathbb{B}(\mathcal{H})} \int_G^{\mathbb{B}(\mathcal{H})} \pi(f(t)) \pi(\alpha_t(g(t^{-1}s)) u_{tt^{-1}s} \mathrm{d}t \mathrm{d}s \\
&= \int_G^{\mathbb{B}(\mathcal{H})} \int_G^{\mathbb{B}(\mathcal{H})} \pi(f(t)) u_t \pi(g(t^{-1}s)) u_{t^{-1}s} \mathrm{d}s \mathrm{d}t \\
&= \int_G^{\mathbb{B}(\mathcal{H})} \pi(f(t)) u_t \int_G^{\mathbb{B}(\mathcal{H})} \pi(g(t^{-1}s)) u_{t^{-1}s} \mathrm{d}s \mathrm{d}t \\
&= \int_G^{\mathbb{B}(\mathcal{H})} \pi(f(t)) u_t \int_G^{\mathbb{B}(\mathcal{H})} \pi(g(s)) u_s \mathrm{d}s \mathrm{d}t \\
&= \int_G^{\mathbb{B}(\mathcal{H})} \pi(f(t)) u_t \mathrm{d}t \int_G^{\mathbb{B}(\mathcal{H})} \pi(g(s)) u_s \mathrm{d}s \\
&= \pi \rtimes u(f) \pi \rtimes u(g),
\end{aligned}$$

implying that $\pi \rtimes u$ is a $*$-homomorphism. Assuming non-degeneracy of π, for $h \in \mathcal{H}$ and $\varepsilon > 0$ we can pick $a \in A$, $\|a\| = 1$ such that $\|\pi(a)h - h\| < \varepsilon$. Being a unitary representation, we may choose a neighborhood $e \in V \subseteq G$ such that $\|u_s h - h\| < \varepsilon$ for $s \in V$. Pick

$0 \leq \varphi \in C_c(G)$ such that $\int_G \varphi(s)\mathrm{d}s = 1$ and $\operatorname{supp}\varphi \subseteq V$. Then, $f := \varphi \otimes a \in C_c(G, A)$ satisfies for $k \in \mathcal{H}$, $\|k\| = 1$, that

$$\begin{aligned} |\langle \pi \rtimes u(f)h - h, k\rangle| &= \left| \int_G \varphi(s)\langle \pi(a)u_s h - h, k\rangle \right| \\ &\leq \int_G \varphi(s)(|\langle \pi(a)(u_s h - h), k\rangle| + |\langle \pi(a)h - h, k\rangle|)\mathrm{d}s \\ &< 2\varepsilon, \end{aligned}$$

showing that $\|\pi \rtimes u(f)h - h\| \leq 2\varepsilon$. $\square$

Now, given a locally compact topological group G, let $\lambda\colon G \to \mathcal{U}(L^2(G))$ be the left-regular representation $\langle \lambda_g \xi, \eta\rangle = \int_G \xi(g^{-1}s)\overline{\eta(s)}\mathrm{d}s$ for $g \in G$ and $\xi, \eta \in L^2(G)$. Even though $L^2(G)$ consists of equivalence classes (unless e.g., G is discrete), we will abbreviate $\lambda_g\xi(s) := \xi(g^{-1}s)$. Given a Hilbert space $\mathcal{H}$, we will also λ_g to denote the operator $I \otimes \lambda_g$ on $\mathcal{H} \otimes L^2(G) \cong L^2(G, \mathcal{H})$. With this, we have the following:

Lemma 2.2.7. Suppose that $G \overset{\alpha}{\curvearrowright} A$, and π is a representation of A on $\mathcal{H}$. Define $\tilde{\pi}(a)\colon L^2(G, \mathcal{H}) \to L^2(G, \mathcal{H})$ by

$$\tilde{\pi}(a)\xi(r) = \pi(\alpha_{r^{-1}}(a))\xi(r), \text{ for } a \in A,\ r \in G \text{ and } \xi \in L^2(G, \mathcal{H}).$$

Then $(\tilde{\pi}, \lambda)$ is a covariant representation of $G \overset{\alpha}{\curvearrowright} A$ on $L^2(G, \mathcal{H})$.

Proof.
Let $a \in A$ and $g \in G$. Then for all $\xi \in L^2(G, \mathcal{H})$ and $s \in G$:

$$\lambda_g(\tilde{\pi}(a)\lambda_{g^{-1}}\xi)(s) = \pi(\alpha^{-1}_{g^{-1}s}(a))\lambda_g^*\xi(g^{-1}s) = \pi(\alpha_{s^{-1}}\alpha_g(a))\xi(gg^{-1}s) = \tilde{\pi}(\alpha_g(a))\xi(s),$$

as wanted. $\square$

Corollary 2.2.8. To every C^*-dynamical system $G \overset{\alpha}{\curvearrowright} A$, there exists a covariant representation.

In light of the above, we may define the following important class of covariant representations of a C^*-dynamical system:

Definition 2.2.9. Given $G \overset{\alpha}{\curvearrowright} A$, then for any representation π of A on $\mathcal{H}$, the above covariant representation is called the *induced regular representation*, and we will write $\operatorname{Ind}_e^G(\pi) := (\tilde{\pi}, \lambda)$. Any representation of the above form is called a *regular representation*.

Remark 2.2.10
The notation $\mathrm{Ind}_e^G(\pi)$ is the common notation, and the subscript and superscript are due to a more general case where one induced representations, denoted by $\mathrm{Ind}_H^G(\pi)$, for normal subgroups H of G. However, we will not be pursuing the case other than $H = e$, but we keep the notation both as a tribute to the original literature as well as future reference.

As we shall see, the integrated form of the above plays a crucial role in the theory regarded in this thesis, but first, we shall need the following lemma:

Lemma 2.2.11. Suppose that $G \overset{\alpha}{\curvearrowright} A$, and π is a representation of A on $\mathcal{H}$. Then $\mathrm{Ind}_e^G \pi$ is non-degenerate if π is non-degenerate.

Proof.
Recall that the set of functions $f \otimes \eta \colon r \mapsto f(r)\eta$ with $f \in C_c(G)$ and $\eta \in \mathcal{H}$ has dense linear span in $L^2(G, \mathcal{H})$, and since π is non-degenerate, the set $\{f \otimes \pi(a)\eta \mid \eta \in \mathcal{H},\ f \in C_c(G) \text{ and } a \in A\}$ has dense linear span in $L^2(G, \mathcal{H})$ as well. Let $\varepsilon > 0$ and let (e_i) be an approximate identity of A. By the previous remark and the fact that $\tilde{\pi}$ is a linear contraction, it suffices to see that

$$\tilde{\pi}(e_i)(f \otimes \pi(a)\xi)(r) = f(r)\pi(\alpha_r^{-1}(e_i)a)\xi \xrightarrow{2} f(r)\pi(a)\xi,$$

for $f \in C_c(G)$, $\xi \in \mathcal{H}$ and $a \in A$ and $r \in \operatorname{supp} f$. Assume towards contradiction that this is not the case, and let $\varepsilon > 0$ and some set subnet $\{e_j\}_{j \in J}$ and some set $\{r_j\}_{j \in J} \subseteq G$ we have

$$\left\| \alpha_{r_j}^{-1}(e_j)a - a \right\| \geq \varepsilon,$$

for all $j \in J$. Up to passing to a subnet we may assume $r_j \to r$ by compactness of $\operatorname{supp} f$. This yields a contradiction as

$$\left\| \alpha_{r_j}^{-1}(e_j)a - a \right\| = \left\| e_j \alpha_{r_j}(a) - \alpha_{r_j}(a) \right\|$$

which tends to 0 as it is dominated by

$$\left\| e_j(\alpha_{r_j}(a)) - \alpha_r(a) \right\| + \left\| e_j \alpha_r(a) - \alpha_r(a) \right\| + \left\| \alpha_r(a) - \alpha_{r_j}(a) \right\| \to 0,$$

by repeated use of the triangle inequality and the fact that (e_j) also defines an approximate unit for A. $\square$

Remark 2.2.12
In particular, if π is a non-degenerate representation of A, then the integrated form, $\tilde{\pi} \rtimes \lambda$, of $\mathrm{Ind}_e^G \pi$ is non-degenerate by Proposition 2.2.6. This is not the only property which carries over from the map $\mathrm{Ind}_e^G \pi \mapsto \tilde{\pi} \rtimes \lambda$, but before we further comment on this, we must introduce some new maps

Definition 2.2.13. Suppose that $G \overset{\alpha}{\curvearrowright} A$. Let $f \otimes a \in C_c(G, A)$ be the map $r \mapsto f(r)a$. For $s \in G$, let $\iota_G(s)$ be the linear extension of the map $\lambda_s \otimes \alpha_s \colon f \otimes a \mapsto \lambda_s f \otimes \alpha_s(a)$ to all of $C_c(G, A)$, i.e., $\iota_G(s)h(t) = \alpha_s(h(s^{-1}t))$ for $h \in C_c(G, A)$.

And, we have the following handy lemma:

Lemma 2.2.14. Suppose that (π, u) is a covariant representation of a C^*-dynamical system $G \overset{\alpha}{\curvearrowright} A$, and let $\pi \rtimes u$ denote the integrated representation of $C_c(G, A, \alpha)$. Then, for $f \in C_c(G, A)$ and $r \in G$ it holds that

$$u_r^* \circ \pi \rtimes u \circ \iota_G(r)(f) = \pi \rtimes u(f)$$

Proof.
It follows from the calculation

$$\begin{aligned} \pi \rtimes u(\iota_G(r)f) = \int_G \pi(\iota_G(r)f(s))u_s \mathrm{d}s &= \int_G \pi(\alpha_r(f(r^{-1}s)))u_r u_{r^{-1}s} \mathrm{d}s \\ &= \int_G u_r \pi(f(r^{-1}t))u_{r^{-1}s} \mathrm{d}s \\ &= u_r \int_G \pi(f(s))u_s \mathrm{d}s \\ &= u_r \circ \pi \rtimes u(f). \end{aligned}$$

□

Lemma 2.2.15. Suppose that $G \overset{\alpha}{\curvearrowright} A$ and π is a faitful representation of A on $\mathcal{H}$. Then the integrated form, $\tilde{\pi} \rtimes \lambda$, of $\operatorname{Ind}_e^G(\pi) = (\tilde{\pi}, \lambda)$ is a faithful representation of $C_c(G, A, \alpha)$.

Proof.
Let $0 \neq f \in C_c(G, A)$, and let $r \in G$ witness this, i.e, $f(r) \neq 0$. By Lemma 2.2.14 we can replace f by $\iota_G(r^{-1})f$ and assume that $r = e$. So without loss of generality we assume $r = e$, and let $\xi, \eta \in \mathcal{H}$ witness faithfulness of π, i.e., $\langle \pi(f(e))\xi, \eta \rangle \neq 0$. Pick an open neighborhood $e \in V$ such that for $s, r \in V$ we have

$$|\langle \pi(\alpha_r^{-1}(f(s)))\xi - \pi(f(e)))\xi, \eta \rangle| < \frac{|\langle \pi(f(e))\xi, \eta \rangle|}{3}.$$

Let $W \subseteq V$ be a symmetric neighborhood such that $W^2 \subseteq V$, and let $\varphi \in C_c(G)$ such that $\varphi \geq 0$ and $\operatorname{supp} \varphi \subseteq W$ and

$$\int_G \int_G \varphi(s^{-1}r)\varphi(r) \mathrm{d}r \mathrm{d}s = 1.$$

If $\zeta = \varphi \otimes \xi$ and $\psi = \varphi \otimes \eta$ are the elements of $L^2(G, \mathcal{H})$ defined as in the proof of Lemma 2.2.11 above, then

$$\begin{aligned}
\langle \tilde{\pi} \rtimes \lambda(f)\zeta, \psi \rangle = \int_G \langle \tilde{\pi}(f(s))\lambda_s \zeta, \psi \rangle \mathrm{d}s &= \int_G \int_G \langle \tilde{\pi}(f(s))\lambda_s \zeta(r), \psi(r) \rangle \mathrm{d}r\mathrm{d}s \\
&= \int_G \int_G \langle \pi(\alpha_{r^{-1}}(f(s)))\zeta\left(r^{-1}s\right)\psi(s) \rangle \mathrm{d}r\mathrm{d}s \\
&= \int_G \int_G \varphi(r^{-1}s)\varphi(r)\langle \pi(\alpha_{r^{-1}}(f(s)))\xi, \eta \rangle \mathrm{d}r\mathrm{d}s
\end{aligned}$$

and hence for all $r, s \in W$ we have

$$\begin{aligned}
|\langle \tilde{\pi} \rtimes \lambda(f)\zeta, \psi \rangle - \langle \pi(f(e))\xi, \eta \rangle| &= \left| \int_G \int_G \varphi(r^{-1}s)\varphi(s)(\langle \pi(\alpha_{r^{-1}} f(s))\xi, \eta \rangle - \langle \pi(f(e))\xi, \eta \rangle \mathrm{d}r\mathrm{d}s \right| \\
&\leq \int_G \int_G \varphi(r^{-1}s)\varphi(s)|\langle \pi(\alpha_{r^{-1}}(f(s)))\xi, \eta \rangle - \langle \pi(f(e))\xi, \eta \rangle|\mathrm{d}r\mathrm{d}s \\
&< \int_G \int_G \varphi(r^{-1}s)\varphi(s)\frac{|\langle \pi(f(e))\xi, \eta \rangle|}{3}\mathrm{d}r\mathrm{d}s \\
&= \frac{|\langle \pi(f(e))\xi, \eta \rangle|}{2},
\end{aligned}$$

and in particular, it follows that $\tilde{\pi} \rtimes \lambda(f) \neq 0$ as $\langle \tilde{\pi} \rtimes \lambda(f)\zeta, \psi \rangle$ is non-zero. □

With the above result in hand, we may define a universal norm for the general convolution algebra $C_c(G, A, \alpha)$. Like in the discrete case, we define the *universal norm* of $f \in C_c(G, A, \alpha)$ to be non-negative real number

$$\begin{aligned}
\|f\|_u &:= \sup \{\|\pi \rtimes u(f)\| \mid \pi \rtimes u \text{ is a covariant representation of } (A, G, \alpha)\} \\
&\leq \|f\|_1,
\end{aligned}$$

and it is easy to see that the assignment $C_c(G, A, \alpha) \ni f \mapsto \|f\|_u$ is a norm: It clearly satisfies $\|f + g\|_u \leq \|f\|_u + \|g\|_u$ and $\|cf\|_u = |c| \, \|f\|_u$ for $f, g \in C_c(G, A, \alpha)$ and $c \in \mathbb{C}$. If $\|f\|_u = 0$ then $\tilde{\pi} \rtimes \lambda(f) = 0$ for all faithful π, which implies by Lemma 2.2.15 that $f = 0$. Moreover, we see that $C_c(G, A, \alpha)$ is a pre C^*-algebra, since $\pi \rtimes u$ is a homomorphism for all covariant representations (π, u), and thus $\|\pi \rtimes u(f^* * f)\| = \|\pi \rtimes u(f)^* \circ \pi \rtimes u(f)\| = \|\pi \rtimes u(f)\|^2$, so that $\|f^* * f\|_u = \|f\|_u^2$.

Note 2.2.16
While it indeed is not possible to take the supremum over a class, it can be shown that it is enough to take the supremum over cyclic covariant representations, which is indeed a set - this is due to the fact that every non-degenerate representation of a $*$-algebra is a direct sum of cyclic representations, see e.g. [Mur14, Theorem 5.1.3]. For now, we stick with the following:

Lemma 2.2.17. Let $G \overset{\alpha}{\curvearrowright} A$ be a C^*-dynamical system and let (π, u) be a covariant representation on $\mathcal{H}$. Let

$$V_\pi = \overline{\operatorname{span}\{\pi(a)\xi \mid a \in A \text{ and } \xi \in \mathcal{H}\}} \subseteq \mathcal{H},$$

then V_π is invariant under both π and u by covariance, and if π^{V_π} and u^{V_π} denotes the corresponding subrepresentations on V_π, we have for all $f \in C_c(G, A, \alpha)$

$$\left\|\pi^{V_\pi} \rtimes u^{V_\pi}(f)\right\| = \|\pi \rtimes u(f)\|,$$

so that

$$\|f\|_u = \sup\left\{\|\pi \rtimes u(f)\| \mid (\pi, u) \text{ is a non-degenerate covariant representation of } G \overset{\alpha}{\curvearrowright} A\right\}.$$

We refer to [Wil07, p. 52] a proof.

This allows us to define the general crossed product as the completion of $C_c(G, A, \alpha)$:

Definition 2.2.18. For a C^*-dynamical system (A, G, α), we define the *full crossed product* (or simply the crossed product) of A and G to be norm closure of $C_c(G, A, \alpha)$ in the norm $\|\cdot\|_u$, and we will denote it by $A \rtimes_\alpha G$.

Remark 2.2.19
Another (equivalent) formulation of the crossed product is to consider the representation of $L^1(G, A)$ given by the direct sum of all non-degenerate covariant representations of $L^1(G, A)$. Again, one can reformulate the above to only consider cyclic covariant representations, and use the fact that every representation of a C^*-algebra is a sum of cyclic representations.

The perhaps most trivial example is in fact not a degenerate example, for it gives us a way to define the full group C^*-algebra of a locally compact group G:

Example 2.2.20
Recall the definition of the full group C^*-algebra $C^*(G)$, which is given as the closure of $C_c(G)$ (or $L^1(G)$) in the universal norm induced by the direct sum of all non-degenerate $*$-representations of the Banach *-algebra $L^1(G)$. This is equivalent to the construction above when we let $G \curvearrowright \mathbb{C}$ be the trivial action 1.

Another fundamental example is the following:

Example 2.2.21
Let $G = \mathbb{Z}$ and let $\theta \in \mathbb{R}$. Define an action of $\mathbb{Z}$ on $C(S^1)$ by $\alpha_\theta(n)f(w) = f(e^{-2\pi\theta n}w)$. The corresponding crossed product $C(S^1) \rtimes_{\alpha_\theta} \mathbb{Z}$ is called the *(Ir)rational Rotation Algebra* (depending on wether θ is irrational or rational).

In fact, for θ irrational, we obtain one of the prototypes of simple crossed product algebras, which we recall is the topic of this thesis. Nice!

Example 2.2.22
The Cuntz algebras $\mathcal{O}_n$ can be thought of as a sort of crossed product: One can show that there is an AF-algebra A and an action of Z on A such that the stabilized Cuntz algebra $\mathcal{O}_n \otimes \mathbb{K}$ is isomorphic to $A \rtimes \mathbb{Z}$. While we will not go through the details (see e.g., [Bla06, Example II.10.3.15] for a short exposition) of this construction, we mention it as it provide a good angle for showing that $\mathcal{O}_n \not\cong \mathcal{O}_m$ for $n \neq m$, by using K-theory and the Pimsner-Voiculescu theorem, see e.g. [Bla06, V.1.3.1 and V.1.3.3]. For a good introduction and on K-theory, AF-algebras, the Cuntz algebras and the Pimsner-Voiculescu result see the excellent book [Rør+00].

Inconvenient as it is, there is no guarantee that $A \rtimes_\alpha G$ contains a copy of A or G except under certain additional conditions on A and G. However, we can always embed A and G into the Multiplier Algebra $M(A \rtimes_\alpha G)$ of $A \rtimes_\alpha G$ in a nice way:

Lemma 2.2.23. Suppose that $G \overset{\alpha}{\curvearrowright} A$ is a C^*-dynamical system. For $a \in A$ define the map $\iota_A(a) \colon A \rtimes_\alpha G \to A \rtimes_\alpha G$ to be the extension of the map defined on $C_c(G, A, \alpha)$ by

$$\iota_A(a) f(s) = a f(s), \text{ for } f \in C_c(G, A, \alpha) \text{ and } s \in G.$$

Then $\iota_A \colon A \to M(A \rtimes_\alpha G)$, $a \mapsto \iota_A(a)$, is a faithful non-degenerate *-representation, satisfying

$$\overline{\pi \rtimes u}\ \iota_A(a) = \pi(a),$$

for all covariant representations (π, u) of $G \overset{\alpha}{\curvearrowright} A$.

Proof.
First, for covariant representations (π, u) of $G \overset{\alpha}{\curvearrowright} A$, we see that

$$\pi \rtimes u(\iota_A(a) f) = \int_G \pi(a)\pi(f(s)) u_s \mathrm{d}s = \pi(a) \pi \rtimes u(f),$$

for all $f \in C_c(G, A, \alpha)$ and $a \in A$, so $\|\iota_A(a) f\| \leq \|a\| \, \|f\|$. Hence $\iota_A(a)$ extends to an operator, which we also denote by $\iota_A(a)$. If $a \in A$ and $f \in C_c(G, A, \alpha)$, then

$$(\iota_A(a) f)^* (s) = \Delta(s^{-1}) \alpha_s(f(s^{-1})^*) \alpha_s(a^*) = f^*(s) \alpha_s(a^*), \text{ for } s \in G,$$

implying that

$$\begin{aligned}(\iota_A(a) f)^* *_\alpha g(s) = \int_G f^*(r) \alpha_r(a^*) \alpha_r(g(r^{-1}s)) \mathrm{d}r &= \int_G f^*(r) \alpha_r \left((\iota_A(a^*) g)(r^{-1}s) \right) \mathrm{d}r \\ &= f^* *_\alpha (\iota_A(a^*) g)(s).\end{aligned}$$

We conclude that $\iota_A(a^*)$ is an adjoint (with respect to the canonical $A \rtimes_\alpha G$)-valued inner product) on the dense $*$-subalgebra $C_c(G, A, \alpha) \subseteq A \rtimes_\alpha G$, hence everywhere by continuity, i.e., $\iota_A(a) \in M(A \rtimes_\alpha G)$.

For injectivity, let π be any faithful, non-degenerate $*$-representation of A, and let $\overline{\tilde{\pi} \rtimes \lambda}$ denote the extension of $\tilde{\pi} \rtimes \lambda$ to $M(A \rtimes_\alpha G)$. By Proposition 2.2.6 and the above, we see that $\tilde{\pi}(a) = \overline{\tilde{\pi} \rtimes \lambda} \circ \iota_A(a)$, hence $\iota_A(a) = 0$ implying that $tilde\pi(a) = 0$ so $a = 0$, since $\tilde{\pi}$ is faithful.

For non-degeneracy, recall that the set of functions $\varphi \otimes b$ for $\varphi \in C_c(G)$ and $b \in A$ span a dense subset of $C_c(G, A)$ in the inductive limit norm and hence in L^1-norm by Lemma 2.1.4. This means that this set also spans a dense subset of $A \rtimes_\alpha G$ in the universal norm. Also, for $a \in A$, we see that $\iota_A(a)\varphi \otimes b = \varphi \otimes ab$, from which we conclude that the set $\iota_A(A)A \rtimes_\alpha G$ is dense, finishing the proof. ☐

Lemma 2.2.24. There is an injective and strictly continuous group homomorphism

$$\iota_G \colon G \to \mathcal{U}M(A \rtimes_\alpha G),$$

such that for $f \in C_c(G, A, \alpha) \subseteq A \rtimes_\alpha G$ we have

$$\iota_G(s)f(t) = \alpha_s(f(s^{-1}t)),$$

for all $s, t \in G$, and if (π, u) is a covariant representation of $G \overset{\alpha}{\curvearrowright} A$, then

$$\overline{\pi \rtimes u}\iota_G(s) = u_s,$$

for $s \in G$.

Before we commence the proof of the above, we want to point out that the above two lemmas ensure that for each covariant representation (π, u) of $G \overset{\alpha}{\curvearrowright} A$ we obtain the following commutative diagram

$$\begin{tikzcd} G \arrow[r, hook, "\iota_G"] \arrow[dr, "u"'] & M(A \rtimes_\alpha G) \arrow[d, "\overline{\pi \rtimes u}"] & A \arrow[l, hook', "\iota_A"'] \arrow[dl, "\pi"] \\ & \mathbb{B}(\mathcal{H}) & \end{tikzcd} \tag{2.10}$$

And we shall see that the pair $\iota_G, \iota_A)$ is in fact a covariant pair. But first, the proof of the above lemma:

Proof.
Let $\iota_G(s)$ be defined on $C_c(G, A, \alpha)$ as earlier. By Lemma 2.2.14, we see that

$$\|\iota_G(r)f\|_u = \|f\|_u \,,$$

for all $f \in C_c(G, A, \alpha)$, and so it extends to $A \rtimes_\alpha G$. Let $s, r \in G$, then for $f \in C_c(G, A, \alpha)$, we see that

$$(\iota_G(r)f)^*(s) = \Delta(s^{-1})\alpha_s((\alpha_r(f(r^{-1}s^{-1}))^*) = \Delta(s^{-1})\alpha_{sr}(f(r^{-1}s^{-1})^*),$$

which, combined with the fact that $\Delta(s^{-1})\mathrm{d}s$ is the integrand of a right Haar measure, can be used to show that $\iota_G(r)^* = \iota_G(r^{-1})$, so that $\iota_G(r) \in M(A \rtimes_\alpha G)$. In fact, since clearly $\iota_G(rs) = \iota_G(r) \circ \iota_G(s)$ for $s, r \in G$, it is a unitary element.

Let (π, u) be any non-degenerate covariant representation of $G \overset{\alpha}{\curvearrowright} A$. By Lemma 2.2.14, we see that $u_s = \overline{\pi \rtimes u}\iota_G(s)$ for all $s \in G$. Moreover, for faithful representations π (so that $\tilde{\pi}$ is faithful) we have that the corresponding representation $\tilde{\pi} \rtimes \lambda$ is faithful on $C_c(G, A, \alpha)$. This implies that for $s \neq e$ we have $\mathrm{id} \neq \lambda_s = \overline{\tilde{\pi} \rtimes \lambda}\iota_G(s)$ for $s \neq e$, hence $\iota_G(s) \neq \mathrm{id}$ as well, so it is injective.

For continuity, let $f \in C_c(G, A, \alpha)$. We will see that ι_G is strongly continuous, which by Lemma 2.1.12 is equivalent to being strictly continuous: Let $(s_i) \subseteq G$, $s_i \to e$. Then we claim that $\iota_G(s_i)x \to x$ for $x \in A \rtimes_\alpha G$. To see this, note that we know that the set of functions $\varphi \otimes a$, $\varphi \in C_c(G)$ and $a \in A$, spans a dense subset of $C_c(G, A)$, hence it suffices to show that $\iota_G(s_i)\varphi \otimes a \xrightarrow{L^1} \varphi \otimes a$, but

$$\begin{aligned} \|\iota_G(r_i)\varphi \otimes a - \varphi \otimes a\|_1 &= \int_G \left\|\varphi(r_i^{-1}s)\alpha_{r_i}(a) - \varphi(r_i^{-1}s)a + \varphi(r_i^{-1}a)a - \varphi(s)a\right\|_A \mathrm{d}s \\ &\leq \|\alpha_{r_i}(a) - a\| \|\varphi\|_1 + \|\lambda_{r_i}\varphi - \varphi\|_1 \|a\|, \end{aligned}$$

which goes to 0, by uniform continuity of φ and the fact that $\alpha_{r_i}(a) \to a$ for all $a \in A$ as $r_i \to e$. ☐

And, we summarize the above in the following

Proposition 2.2.25. For a C^*-dynamical system $G \overset{\alpha}{\curvearrowright} A$, there exists a non-degenerate faithful $*$-homomorphsim

$$\iota_A \colon A \to M(A \rtimes_\alpha G),$$

and an injective strictly continuous unitary representation

$$\iota_G \colon G \to \mathcal{U}M(A \rtimes_\alpha G),$$

such that $\iota_G(r)f(s) = \alpha_r(f(r^{-1}s))$ and $\iota_A(a)f(s) = af(s)$ for $f \in C_c(G, A, \alpha)$ and $r, s \in G$ and $a \in A$. Moreover, the pair (ι_A, ι_G) is covariant with respect to α, i.e.,

$$\iota_A(\alpha_r(a)) = \iota_G(r)\iota_A(a)\iota_G(r)^*,$$

and for non-degenerate covariant representations (π, u) we have

$$\overline{\pi \rtimes u} \circ \iota_A(a) = \pi(a) \text{ and } \overline{\pi \rtimes u} \circ \iota_G(s) = u_s.$$

Proof.
The only thing left to be shown is the covariance relation, but this follows immediately from the following calculation:

$$\iota_G(r)\iota_A(a)\iota_G(r)^* f(s) = \alpha_r(\iota_A(a)\iota_G(r)^* f(r^{-1}s)) = \alpha_r(a)\alpha_r(\alpha_{r^{-1}} f(rr^{-1}s)) = \iota_A(\alpha_r(a))f(s),$$

for $f \in C_c(G, a, \alpha)$, $a \in A$ and $s, r \in G$. □

We now embark on our final task in our construction and picture painting of the general crossed product $A \rtimes_\alpha G$, namely we wish to properly characterize its representation theory. We are going to see that $A \rtimes_\alpha G$ is the C^*-algebra whose representation theory is in a one-to-one correspondance with the covariant representations of (A, G, α) via the map $(\pi, u) \mapsto \pi \rtimes u$, but before we finalize the description of the representation theory of $A \rtimes_\alpha G$, we need to extend the map ι_G to $C^*(G)$, the full group C^*-algebra of G, since this will provide us with a useful characterization of a dense subset of $A \rtimes_\alpha G$.

Lemma 2.2.26. There exists a $*$-homomorphsim $\tilde{\iota}_G \colon C^*(G) \to M(A \rtimes_\alpha G)$ such that

$$\tilde{\iota}_G(z) = \int_G z(s)\iota_G(s)\mathrm{d}s,$$

for all $z \in C_c(G)$

Proof.
For all $z \in C_c(G)$, the map $s \mapsto z(s)\iota_G(s)$ is strictly continuous, and we may define the integral by Theorem 2.1.13. Let $z, w \in C_c(G)$, then

$$\begin{aligned}
\tilde{\iota}_G(z * w) &= \int_G \int_G z(r)w(r^{-1}s)\mathrm{d}r\iota_G(s)\mathrm{d}s \\
&= \int_G \int_G z(r)w(r^{-1}s)\iota_G(s)\mathrm{d}s\mathrm{d}r \\
&= \int_G \int_G z(r)w(s)\iota_G(rs)\mathrm{d}s\mathrm{d}r \\
&= \int_G z(r)\iota_G(r)\mathrm{d}r \int_G w(s)\iota_G(s)\mathrm{d}s \\
&= \tilde{\iota}_G(z)\tilde{\iota}_G(w),
\end{aligned}$$

so if it is well-defined and continuous on $C^*(G)$, it will be a $*$-homomorphism, since the integral also respects taking adjoints.

Let π be any faithful and non-degenerate representation of $A \rtimes_\alpha G$, so that its extension $\overline{\pi}$ to $M(A \rtimes_\alpha G)$ is faithful aswell.

Consider the composition $u\colon s \mapsto \overline{\pi}(\iota_G(s))$, so that each u_s is a unitary. To see it is strongly continuous: Suppose that $f \in C_c(G, A)$ and $\xi \in \mathcal{H}_\pi$, then for $s \in G$ we see that $\overline{\pi}(\iota_G(s))\pi(f)\xi = \pi(\iota_G(s)f)\xi$, and it follows that $s \mapsto u_s\xi$ is continuous for all $\xi \in \mathcal{H}$ by continuity of $s \mapsto \iota_G(s)f$ and non-degeneracy of π. Hence:

$$\begin{aligned}\overline{\pi}(\tilde{\iota}_G(z)) = \overline{\pi}\left(\int_G z(s)\iota_G(s)\mathrm{d}s\right) &= \int_G z(s)\overline{\pi}(\iota_G(s))\mathrm{d}s \\ &= u(z),\end{aligned}$$

where the last line is the induced representation of $L^1(G)$ corresponding to the unitary representation $\overline{\pi} \circ \iota_G$ of G, which we shall discuss later on. In particular, this implies that

$$\|\tilde{\iota}_G(z)\| = \|u(z)\| \leq \|z\|, \tag{2.11}$$

for all $z \in C_c(G)$, so it extends to a $*$-homomorphism $C^*(G) \to M(A \rtimes_\alpha G)$, which we will also denote by ι_G. □

Corollary 2.2.27. Let $G \overset{\alpha}{\curvearrowright} A$ be a C^*-dynamical system and let $a \in A$, $f, h \in C_c(G, A)$ and $\varphi \in C_c(G)$. Then

$$\begin{gathered}\iota_A(a)\iota_G(\varphi) = \varphi \otimes a, \\ \int_G \iota_A(f(s))\iota_G(s)h\mathrm{d}s = f *_\alpha h \text{ and} \\ \int_G \iota_A(f(s))\iota_G(s)\mathrm{d}s = f,\end{gathered}$$

where we identified $C_c(G, A, \alpha) \subseteq A \rtimes_\alpha G \subseteq M(A \rtimes_\alpha G)$.

Proof.
Let (π, u) be a non-degenerate covariant representation of $G \overset{\alpha}{\curvearrowright} A$ and notice that

$$\overline{\pi \rtimes u}\left(\int_G \iota_A(f(s))\iota_G(s)\mathrm{d}s\right) = \pi \rtimes u(f),$$

which combined with Lemma 2.2.17 shows the first and last identity. The second equality follows from the last and Theorem 2.1.13. □

Remark 2.2.28
The above and Lemma 2.1.4 implies that the set of functions of the form $\iota_A(a)\iota_G(z)$ for $a \in A$ and $z \in C_c(G)$ spans a dense subset of $A \rtimes_\alpha G$. This will be useful in the following:

Theorem 2.2.29. Let $G \overset{\alpha}{\curvearrowright} A$ be a C^*-dynamical system and let $\mathcal{H}$ be a Hilbert space. If $L\colon A \rtimes_\alpha G \to \mathbb{B}(\mathcal{H})$ is a non-degenerate $*$-representation, then there exists a non-degenerate covariant representation (π, u) of $G \overset{\alpha}{\curvearrowright} A$ into $\mathbb{B}(\mathcal{H})$ such that $L = \pi \rtimes u$ which can be realised by

$$u_s = \overline{L}(\iota_G(s)) \text{ and } \pi(a) = \overline{L}(\iota_A(a)), \tag{2.12}$$

for all $s \in G$ and $a \in A$. In fact, there is a bijective correspondance between covariant representations of $G \overset{\alpha}{\curvearrowright} A$ and $*$-representations of $A \rtimes_\alpha G$.

Proof.
We already have seen the assignment $(\pi, u) \mapsto \pi \rtimes u$. For the converse, suppose that L is a non-degenerate representation of $A \rtimes_\alpha G$. Let π, u be defined by Equation (2.12). It is clear that π is a $*$-representation of A on $\mathbb{B}(\mathcal{H})$ and similarly that u is a strongly continuous unitary-valued group representation on $\mathbb{B}(\mathcal{H})$, since ι_G is strictly and hence strongly continuous.

To see that (π, u) is non-degenerate, let (e_i) be an approximate unit for A. Then $\iota_A(e_i) \to 1$ strictly in $M(A \rtimes_\alpha G)$, and so $\pi(e_i) = \overline{L}(\iota_A(e_i)) \to I$ strictly, so in particular $\pi(e_i)\xi \to \xi$ for all $\xi \in \mathcal{H}$ showing that π is non-degenerate.

For covariance, suppose that $s \in G$ and $a \in A$, and note that

$$u_s \pi(a) u_s^* = \overline{L}(\iota_G(s))\overline{L}(\iota_A(a))\overline{L}(\iota_G(s))^* = \overline{L}(\iota_A(\alpha_s(a))) = \pi(\alpha_s(a)).$$

Thus the pair is a proper non-degenerate covariant representation of $G \overset{\alpha}{\curvearrowright} A$. Finally, using Theorem 2.1.13, we see that

$$\begin{aligned}
\pi \rtimes u(\iota_A(a)\iota_G(z)) = \pi(a) \circ \overline{\pi \rtimes u}(\iota_G(z)) &= \overline{L}(\iota_A(a)) \int_G z(s)\overline{L}(\iota_G(s))\mathrm{d}s \\
&= \overline{L}(\iota_A(a))\overline{L}(\iota_G(z)) \\
&= \overline{L}(\iota_A(a)\iota_G(z)),
\end{aligned}$$

for all $z \in C_c(G)$ and $a \in A$, which by the above remark ensures that $L = \pi \rtimes u$ on all of $A \rtimes_\alpha G$. □

2.3 Properties of Crossed products

We want to examine some interesting properties of crossed products, such as what happens when G is amenable or how can we concretize the statespace $\mathcal{S}(A \rtimes_\alpha G)$. To do this, we will need to recall the usual definition of a positive definite function $G \to \mathbb{C}$ and extend it to the case of positive definite functions $G \to A^*$. The following is from [Ped79].

Definition 2.3.1. Suppose that $G \overset{\alpha}{\curvearrowright} A$ is a dynamical system, then a bounded continuous (with respect to the norm on A^*) function $\varphi\colon G \to A^*$ is *positive definite* if any of the following equivalent statements hold:

1. There is a positive functional $\psi \in (A \rtimes_\alpha G)^*$ such that $\varphi(t)(a) = \psi(\iota_A(a)\iota_G(t))$ for all $t \in G$ and $a \in A$,
2. $\sum_{i,j} \varphi(s_i^{-1}s_j)(\alpha_{s_i^{-1}}(x_i^* x_j)) \geq 0$ for all finite sets $s_1, \dots, s_n \in G$ and $x_1, \dots, x_n \in A$,
3. $\sum_{i,j} \int_G \varphi(t)(x_i^* \alpha_t(x_j))(f_i^* \times f_j)(t)\mathrm{d}t \geq 0$ for all finite sets $f_1, \dots, f_n \in C_c(G, A, \alpha)$ and $x_1, \dots, x_n \in A$, and
4. $\int \varphi(t)(f^* \rtimes f)(t)\mathrm{d}t \geq 0$ for all $f \in C_c(G, A, \alpha)$.

Remark 2.3.2
Note that in the above, a priori without explanation the first definition does not make sense, since $\iota_A(a)\iota_G(t)$ might not be an element of $A \rtimes_\alpha G$, however, since states of $A \rtimes_\alpha G$ correspond to non-degenerate representations which extend to $M(A \rtimes_\alpha G)$, we may in fact extend each positive functional ψ on $A \rtimes_\alpha G$ to all of $M(A \rtimes_\alpha G)$. We will also denote this extension by ψ.

If G acts trivially on $\mathbb{C}$, then the above becomes the regular result about equivalent definitions of positive definite functions on G, as described in [Fol16] or in [Ped79].

The first definition is the general motivation behind the above for us, since we will be dealing with a class of functions arising this way, which leads to the following definition:

Definition 2.3.3. Suppose that $G \overset{\alpha}{\curvearrowright} A$. For each $\varphi \in (A \rtimes_\alpha G)^*$, define the map $\Gamma_\varphi\colon G \to A^*$, given by

$$\Gamma_\varphi(t)(a) := \varphi(\iota_A(a)\iota_G(t)) \in \mathbb{C},$$

for $t \in G$ and $a \in A$. The set of functions $G \to A^*$ defined in this way is denoted by $B(A \rtimes_\alpha G)$, and the set of $\Gamma_\varphi \in B(A \rtimes_\alpha G)$ for $\varphi \geq 0$, is denoted by $B_+(A \rtimes_\alpha G)$. We denote by $B_+^1(A \rtimes_\alpha G)$ the functions such that $\|\Gamma_\varphi(e)\| = 1$ and $\varphi \geq 0$.

Lemma 2.3.4. Each $\Gamma_\varphi \in B(A \rtimes_\alpha G)_+$ is continuous with respect to the norm topology on A^*.

Proof.
Let $t \in G$ and $a \in A$, $\|a\| = 1$. Then

$$|\Gamma_\varphi(t)(a)| = |\varphi(\iota_A(A)\iota_G(t))| \leq |\varphi(\iota_A(a^*a))||\varphi(\iota_G(e))| \leq \|\varphi\| \, \|\Gamma_\varphi(e)\| \, ,$$

by positivity of φ and the functional Cauchy-Schwarz. This shows that it is bounded hence continuous. □

Remark 2.3.5
We may use the fact that every non-degenerate representation of $A \rtimes_\alpha G$ is of the form $\pi \rtimes u$ for some covariant, non-degenerate representation (π, u) of $G \overset{\alpha}{\curvearrowright} A$ to show that for all $\varphi \in \mathcal{S}(A \rtimes_\alpha G)$ we have

$$\varphi(y) = \int_G \Gamma_\varphi(t)(y(t))\mathrm{d}t \tag{2.13}$$

for all $y \in C_c(G, A, \alpha) \subseteq A \rtimes_\alpha G$: To such φ we associate a GNS triple $(\pi_\varphi, \xi, \mathcal{H})$ with π_φ non-degenerate. By Theorem 2.2.29, we see that π_φ is of the form $\pi \rtimes u$ for a suitable covariant representation (π, u), and so

$$\begin{aligned}
\varphi(y) = \langle \pi_\varphi(y)\xi, \xi\rangle = \langle \pi \rtimes u(y)\xi, \xi\rangle &= \int_G \langle \pi(y(t))u_t\xi, \xi\rangle \mathrm{d}t \\
&= \int_G \langle \overline{\pi \rtimes u}(\iota_A(y(t)))\overline{\pi \rtimes u}(\iota_G(t))\xi, \xi\rangle \mathrm{d}t \\
&= \int_G \langle \overline{\pi \rtimes u}(\iota_A(y(t))\iota_G(t))\xi, \xi\rangle \mathrm{d}t \\
&= \int_G \varphi(\iota_A(y(t))\iota_G(t))\mathrm{d}t \\
&= \int_G \Gamma_\varphi(t)(y(t))\mathrm{d}t.
\end{aligned}$$

In particular, density of $C_c(G, A, \alpha) \subseteq A \rtimes_\alpha G$ implies by the above that the assignment $\mathcal{S}(A \rtimes_\alpha G) \ni \varphi \mapsto \Gamma_\varphi \in B^1_+(A \rtimes_\alpha G)$ is a bijection. Combining this with linearity of the assignment, i.e., that $\Gamma_{\alpha\varphi+\beta\psi} = \alpha\Gamma_\varphi + \beta\Gamma_\psi$ for $\alpha, \beta \in \mathbb{C}$ and $\varphi, \psi \in (A \rtimes_\alpha G)^*$, this shows that the assignment also induces a bijection between $(A \rtimes_\alpha G)^*$ and $B(A \rtimes_\alpha G)$.

For $A = \mathbb{C}$ again and $\alpha = 1$, we may represent G on $M(C^*(G))$ via $t \mapsto \delta_t$, and then Γ_φ becomes the map $t \mapsto \varphi(\delta_t)$ for $\varphi \in C^*(G)^*$ which leads to the following result (see [Ped79, p. 7.1.11]):

Proposition 2.3.6. For $A = \mathbb{C}$ and $\alpha = 1$, the isomorphism above is a homeomorphism from $\mathcal{S}(C^*(G))$ with the weak*-topology onto the set of positive-definite functions on G with $\Gamma_\varphi(e) = 1$ equipped with the topology of uniform convergence on compacta.

Generalizing the above, we introduce a topology on $B^1_+(A \rtimes_\alpha G)$, which extends the topology of uniform convergence on compacta of G to the general case:

Definition 2.3.7. We define the *topology of weak* uniform convergence on compacta* on $B^1_+(A \rtimes_\alpha G)$ to be the topology satisfying that a net $\Gamma_{\varphi_i} \subseteq B^1_+(A \rtimes_\alpha G)$ converges to $\Gamma_\varphi \in B^1_+(A \rtimes_\alpha G)$ if and only if for each $a \in A$, the restriction of the map

$$t \mapsto \Gamma_{\varphi_i}(t)(a)$$

to any compact subset $K \subseteq G$ is uniformly convergent to the map $t \mapsto \Gamma_\varphi(t)(a)$ restricted to K. A basis for the topology consists of the sets

$$V(\varepsilon, f, K, a) := \left\{ f' \in B^1_+(A \rtimes_\alpha G) \mid \sup_{t \in K} |f'(t)(a) - f(t)(a)| < \varepsilon \right\},$$

where $f \in B^1_+(A \rtimes_\alpha G)$, $\varepsilon > 0$, $a \in A$ and $K \subseteq G$ is compact.

We then obtain:

Proposition 2.3.8. The map $\mathcal{S}(A \rtimes_\alpha G) \ni \varphi \mapsto \Gamma_\varphi \in B^1_+(A \rtimes_\alpha G)$ is a homeomorphism with respect to the weak*-topology on $\mathcal{S}(A \rtimes_\alpha G)$ and the topology of weak* convergence uniformly on compacta of G.

Proof.
We have already established the bijectivity. For the last part, suppose that we have a convergent net $(\varphi_i) \subseteq \mathcal{S}(A \rtimes_\alpha G)$ with limit $\varphi \in \mathcal{S}(A \rtimes_\alpha G)$. We will in the following also write Γ_φ to denote the extension from A to A^1. For $x, y \in A^1$, define the map $\Psi_\varphi(x, y) \colon G \to \mathbb{C}$ by $t \mapsto \Gamma_\varphi(x^* \alpha_t(y))$. The polarization identity and linearity ensures that if for all $x \in A^1$ we have $\Psi_{\varphi_i}(x, x) \to \Psi_\varphi(x, x)$ in the topology of convergence uniformly on compacta then $\Psi_{\varphi_i}(x, y) \to \Psi_\varphi(x, y)$ as well. By Proposition 2.3.6, we see that $\Psi_{\varphi_i}(x, x) \to \Psi_\varphi(x, x)$ in the topology of convergence uniformly on compacta of G, and setting $y = 1_{A^1}$, we see that $\Gamma_{\varphi_i} \to \Gamma_\varphi$.

Conversely, suppose that $\Gamma_{\varphi_i} \to \Gamma_\varphi$, then in particular $\varphi_i(f \otimes x) \to \varphi(f \otimes x)$ for all $x \in A$ and $f \in C_c(G)$ by Equation (2.13) and hence on all of $A \rtimes_\alpha G$, since the functions $f \otimes x$ span a dense subset of $A \rtimes_\alpha G$. □

For discrete groups, it is well-known that if G is amenable (or more generally acts amenably on A), i.e., has a left-invariant mean $m \in \ell^\infty(G)$, then there is only one crossed product of G and any C^*-algebra A (if G is not amenable, but admits an amenable action on A, then the statement is true for crossed products with A as well, see e.g., [BO08, p. 4.3.4] or the authors notes on the subject [Kar16, Part 2, pg. 26, 32]).

We will see that this is true for locally compact groups as well - in particular, it is true for abelian groups locally compact groups as they are amenable, c.f. [BLHV08, appendix G].

Lemma 2.3.9. Suppose that $G \overset{\alpha}{\curvearrowright} A$ and $(\pi, \mathcal{H})$ is a cyclic representation of A with respect to a set $(\xi_i) \subseteq \mathcal{H}$, and suppose that $(\eta_j) \subseteq L^2(G)$ is cyclic for the left-regular representation λ of G. Then the set of $\xi_i \otimes \eta_j$ is cyclic for the representation $(\tilde{\pi} \rtimes \lambda, L^2(G, \mathcal{H}))$.

Proof.
We will show that the orthogonal complement of $\sum_{i,j}(A \rtimes_\alpha G)(\xi_i \otimes \eta_j)$ is 0. Suppose that ξ is orthogonal to $\tilde{\pi} \rtimes \lambda(x)\xi_i \otimes \eta_j$ for all i, j. Then for all i, j and $a \otimes f \in C_c(G, A) \subseteq A \rtimes_\alpha G$ we have

$$\begin{aligned} 0 &= \langle \tilde{\pi} \rtimes \lambda(a \otimes f)(\xi_i \otimes \eta_j), \xi \rangle_{L^2} \\ &= \int_G \langle \tilde{\pi} \rtimes \lambda(a \otimes f)(\xi_i \otimes \eta_j)(s), \xi(s) \rangle_{\mathcal{H}} \mathrm{d}s \\ &= \int_G \int_G \langle \pi(\alpha_{t^{-1}}\left((a \otimes f)(s)\right) \lambda_t \eta_j(s) \xi_i, \xi \rangle_{\mathcal{H}} \mathrm{d}t \mathrm{d}s \\ &= \int_G \int_G \langle \pi\left(\alpha_{t^{-1}}(a)\right) f(s) \eta_j(t^{-1}s) \xi_i, \xi \rangle_{\mathcal{H}} \mathrm{d}t \mathrm{d}s \\ &= \int_G \int_G \langle \pi(\alpha_{t^{-1}}(a)) \xi_i, \xi \rangle_H f(s) \eta_j(t^{-1}s) \mathrm{d}t \mathrm{d}s \\ &= \int_G \langle \pi(\alpha_{t^{-1}}(a)) \xi_i, \xi \rangle_{\mathcal{H}} (f * \eta_j)(s) \mathrm{d}s. \end{aligned}$$

Now, let $E \subseteq G$ be any compact set. We will argue that $\xi = 0$ by showing that $\xi = 0$ on any arbitrarily large compact subset F of E: Let $\varepsilon > 0$, and use a Banach space-valued analogue of Lusin's theorem (e.g. as in [Wil07, Appendix B]) to pick $F \subseteq E$ compact with $\mu(E \backslash F) < \varepsilon$ and such that $\xi|_F \in C(F, H)$. Cyclicity of $\{\eta_j\}_{j \in J}$ implies that the set $\{f * \eta_j \mid j \in J \text{ and } f \in C_c(G)\}$ spans a dense subset of $C_c(G)$ (ee e.g. [Fol16, Proposition 3.33]), so

$$\int_F \langle \pi(\alpha_{t^{-1}}(x)) \xi_i, \xi(t) \rangle_H f(t) \mathrm{d}t = 0,$$

for all $f \in C(F)$, from which we conclude that $F \ni t \mapsto \langle \pi(\alpha_{t^{-1}}(x)) \xi_i, \xi(t) \rangle = 0$, and since ξ_i was cyclic for π, this means that $\xi|_F = 0$. In particular, since $\varepsilon > 0$ was arbitrary, this means that $\xi|_E = 0$ and since $E \subseteq G$ was arbitrary, this means that $\xi = 0$, finishing the proof. $\square$

We now will go through a result which will allow us to describe regular representations pointwise through approximations of states of a certain form:

Proposition 2.3.10. Let (A, G, α) be a C^*-dynamical system. Given $\varphi \in \mathcal{S}(A)$, for each $z \in C_c(G, A, \alpha)$ such that $\varphi(z^* *_\alpha z(e)) = 1$, there is a vector state $\tilde{\varphi}_z \in \mathcal{S}(A \rtimes_\alpha G)$ of the representation $(\tilde{\pi}_\varphi \rtimes \lambda, L^2(G, \mathcal{H}_\varphi))$ satisfying

$$\tilde{\varphi}_z(f) = \varphi\left((z^* *_\alpha f *_\alpha z)(e)\right), \quad \text{for all } f \in C_c(G, A, \alpha).$$

Moreover, the set of vector states contained in $(\tilde{\pi}_\varphi \rtimes \lambda, L^2(G, \mathcal{H}_\varphi))$ is a norm limit of vector states of this form.

Proof.
For $z \in C_c(G, A, \alpha)$ with $t \mapsto \|z(t)\| \in L^2(G)$, define

$$\xi_z(t) = \pi_\varphi\left(\alpha_{t^{-1}}(z(t)\right)\xi_\varphi,$$

so that $\xi_z \in L^2(G, \mathcal{H}_\varphi)$. A few calculations then show that

$$\begin{aligned}\langle \tilde{\pi}_\varphi \rtimes \lambda(y^* *_\alpha y)\xi_z, \xi_z\rangle = \|\tilde{\pi}_\varphi \rtimes \lambda(y)\xi_z\|^2 &= \int_G \|\pi_\varphi(\alpha_{t^{-1}}(y *_\alpha z)(t)\xi_\varphi\|^2 \,\mathrm{d}t \\ &= \int_G \varphi\left((y *_\alpha z)^*(t)\alpha_t((y *_\alpha z)(t^{-1}))\right) \mathrm{d}t \\ &= \varphi(\underbrace{(y *_\alpha z)^*}_{=z^* *_\alpha y^*} *_\alpha (y *_\alpha z)(e)) = \tilde{\varphi}_z(y^* *_\alpha y),\end{aligned}$$

for all $y \in C_c(G, A, \alpha)$. By linearity, it follows that $\tilde{\varphi}_z$ is a vector state on $A \rtimes_\alpha G$ of $(\tilde{\pi} \rtimes \lambda, L^2(G, \mathcal{H}_\varphi))$.

For the last claim, we show that the set $\{\xi_z \mid z \in C_c(G, A, \alpha)\}$ is dense in $L^2(G, \mathcal{H}_\varphi)$: Note that for $z \in C_c(G, A, \alpha)$, $f \in C_c(G)$ and $t \in G$ we have

$$\tilde{\pi}_\varphi(z(s))(\lambda_s(f \otimes \xi_\varphi))(t) = \pi_\varphi(\alpha_t^{-1}(z(s)))f(s^{-1}t)\xi_\varphi,$$

which implies that

$$\tilde{\pi}_\varphi \rtimes \lambda(z)(f \otimes \xi_\varphi)(t) = \xi_{z*f}(t),$$

where $z * f(s) = \int_G z(s)f(s^{-1}t)\mathrm{d}s \in A$. By cyclicity of ξ_φ and Lemma 2.3.9, since $C_c(G) \subseteq L^2(G)$ is dense, we see that the set $\{f \otimes \xi_\varphi \mid f \in C_c(G)\}$ is cyclic for $\tilde{\pi}_\varphi \rtimes \lambda$, which finishes the proof. $\square$

Recall the definition of the universal representation $(\pi_u, \mathcal{H}_u)$ of a C^*-algebra A. The induced representation $\mathrm{Ind}(\pi_u)$ gives rise to the reduced crossed product of a C^*-dynamical system (A, G, α):

Definition 2.3.11. For a C^*-dynamical system (A, G, α), we define the *reduced crossed product* of G and A to be the algebra

$$A \rtimes_{\alpha,r} G := \overline{\tilde{\pi}_u \rtimes \lambda(C_c(G, A, \alpha))},$$

where π_u is the universal representation of A. Equivalently, we may define $A \rtimes_{\alpha,r} G$ as the image of $A \rtimes_\alpha G$ of the extension of $\tilde{\pi}_u \rtimes \lambda$ to $A \rtimes_\alpha G$.

Remark 2.3.12
We will soon see that the above is actually independent of the choice of the representation π_u of A, and may just as well be replaced with any faithful representation π of A.

We need define yet another subset of functionals on $A \rtimes_\alpha G$, one which closely resembles vector states of $\tilde{\pi}_u \rtimes \lambda$:

Definition 2.3.13. The set of functionals $\psi \in (A \rtimes_\alpha G)^*$ of the form

$$\psi(x) = \sum_i \langle \tilde{\pi}_u \rtimes \lambda(x)\xi_i, \eta_i \rangle,$$

where $(\xi_i) \subseteq L^2(G, \mathcal{H}_u)$ and $(\eta_i) \subseteq L^2(G, \mathcal{H}_u)$ have finite ℓ^2-norm, i.e., $\sum_i \|\eta_i\|^2 < \infty$ and $\sum_i \|\xi_i\|^2 < \infty$, is easily seen to be a norm-closed subspace of $(A \rtimes_\alpha G)^*$. We define the set

$$A(A \rtimes_\alpha G) := \{\Gamma_\psi \in B(A \rtimes_\alpha G) \mid \psi(x) = \sum_i \langle \tilde{\pi}_u \rtimes \lambda(x)\xi_i, \eta_i \rangle,\ (\xi_i),\ (\eta_i) \text{ is } \ell^2-\text{summable}\}.$$

Remark 2.3.14
It is an easy consequence of the commutativity of Equation (2.10) that if $\varphi \in \mathcal{S}(A)$ satisfies the condition of Proposition 2.3.10 with respect to some $z \in C_c(G, A, \alpha)$, then

$$\begin{aligned}\Gamma_{\tilde{\varphi}_z}(t)(x) = \varphi_z(\iota_A(x)\iota_G(t)) = \langle \overline{\tilde{\pi} \rtimes \lambda(\iota_A(x)\iota_G(t))}\xi_z, \xi_z \rangle &= \int_G \varphi(z^*(s)\alpha_s(x)\alpha_{st}(z\,(st)^{-1}))\mathrm{d}s \\ &= \varphi\left(\int_G z^*(s)\alpha_s(x)\alpha_{st}(z(st)^{-1})\mathrm{d}s\right),\end{aligned}$$

and in particular, we have $\Gamma_{\tilde{\varphi}_z} \in A_+(A \rtimes_\alpha G)$. Another consequence of Proposition 2.3.10 is that every function in $A_+(A \rtimes_\alpha G)$ is a norm limit of linear combinations of functions $\Gamma_{\tilde{\varphi}_z}$, for states $\varphi \in \mathcal{S}(A)$. This is because π_u is the direct sum of cyclic representations of A.

For a representation $(\pi, \mathcal{H})$ of A, we will also use $\tilde{\pi} \rtimes \lambda$ to denote the extension from $C_c(G, A, \alpha) \subseteq A \rtimes_{\alpha,r} G$ to all of $A \rtimes_{\alpha,r} G$ (doable, since the norm π_u is the universal representation of A, hence the norm will be bounded by it).

Theorem 2.3.15. For a C^*-dynamical system (A, G, α), given any faithful representation $(\pi, \mathcal{H})$ of A, it holds that the associated integrated form $\tilde{\pi} \rtimes \lambda$ will be faithful on $A \rtimes_{\alpha,r} G$.

Proof.
For this particular proof, we will let $\tilde{\pi} \rtimes_r \lambda$ denote the corresponding representation of $A \rtimes_{\alpha,r} G$.

We will show that $\tilde{\pi}_u \rtimes_r \lambda$ is *weakly contained* in $\tilde{\pi} \rtimes_r \lambda$ for all faithful representations π of A, i.e., that $\ker \tilde{\pi} \rtimes_r \lambda \subseteq \ker \tilde{\pi}_u \rtimes_r \lambda = 0$. By [Dix69, p. 80], this is equivalent to showing that the set of vector states of $\tilde{\pi} \rtimes_r \lambda$ is weak* dense in the set of states in $\tilde{\pi}_u \rtimes_r \lambda$.

Let E be the set of states contained in $(\pi, \mathcal{H})$, which will be weak* dense in $\mathcal{S}(A)$ by faithfulness of π. For now, let σ denote the surjective $*$-representation $A \rtimes_\alpha G \to A \rtimes_{\alpha,r} G$. Let F denote set of functions

$$F = \left\{ \Gamma_\psi \in A_+^1(A \rtimes_\alpha G) \mid \Gamma_\psi = \Gamma_{\varphi\circ\sigma} \text{ for some state } \varphi \text{ contained in } \tilde{\pi} \rtimes_r \lambda \right\}$$

Note that the set of vector states in $\tilde{\pi}_u \rtimes_r \lambda$ is exactly the set of functionals which give rise to $A_+^1(A \rtimes_\alpha G)$, by definition and the fact that we may view $A \rtimes_{\alpha,r} G$ as the image of $A \rtimes_\alpha G$ under $\tilde{\pi}_u \rtimes \lambda$.

By the isomorphism between the statespace of $A \rtimes_\alpha G$ and $B(A \rtimes_\alpha G)$, it suffices to show that $F \subseteq A_+^1(A \rtimes_\alpha G)$ is dense in the topology of weak* convergence uniformly on compacta, and by the above remark and Proposition 2.3.10, it is enough to consider approximations of the functions of the form

$$\Gamma_{\tilde{\varphi}_z}(t)(x) = \varphi\left(\int_G z^*(s)\alpha_s(x)\alpha_{st}(z\left((st)^{-1}\right) \mathrm{d}s\right),$$

for suitable $\varphi \in \mathcal{S}(A)$ and $z \in C_c(G, A, \alpha)$, i.e., with $\varphi(z^* * z(e)) = 1$, since they span a dense subset of $A_+^1(A \rtimes_\alpha G)$.

Let such φ and z be given. By density of E, pick a convergent net $\varphi_i \to \varphi$ in the weak*-topology. This implies that $\Gamma_{\tilde{\varphi}_{i_z}}(t) \to \Gamma_{\tilde{\varphi}_z}(t)$ weak*, uniformly on compacta of G. Since $\varphi_i \in E$, the corresponding representation $(\pi_{\varphi_i}, H_{\varphi_i})$ will be a sub-representation of (π, H), from which we conclude that $\Gamma_{\tilde{\varphi}_{i_z}} \in F$, finishing the proof. ☐

In particular, this means that we may always consider $A \rtimes_{\alpha,r} G$ as the completion of $C_c(G, A, \alpha)$ under the regular representation induced by a faithful representation of A, which will be convenient, as we shall see, for in a large case of groups there is only one crossed product. The following useful lemma can be found in [Ped79, Lemma 7.7.6]:

Lemma 2.3.16. If $\Gamma_\varphi \in B_+^1(A \rtimes_\alpha G)$ has compact support, then $\Gamma_\varphi \in A_+^1(A \rtimes_\alpha G)$

We recall one of the several equivalent definition of amenability for locally compact groups G:

Definition 2.3.17. A group G is *amenable* if there is a net of unit vectors $(f_i) \subseteq L^2(G)$ such that $(f_i * \tilde{f}_i)$ converges uniformly on compacta of G to 1, where $\tilde{f}(t) = \overline{f}(t^{-1})$ for $f \in L^2(G)$ and $t \in G$.

See e.g., [BLHV08, Appendix G], [Ped79, Proposition 7.3.7 and 7.3.8] for the above and many more useful characterizations.

Definition 2.3.18. We say that a C^*-dynamical system (A, G, α) is a *regular C^*-dynamical system* whenever $A \rtimes_\alpha G \cong A \rtimes_{\alpha,r} G$ via a regular representation.

We now obtain the following

Theorem 2.3.19. If G is amenable, then any G C^*-dynamical system (A, G, α) is regular, i.e., whenever $G \overset{\alpha}{\curvearrowright} A$, we have $A \rtimes_\alpha G \cong A \rtimes_{\alpha,r} G$.

Proof.
By applying the above to $G \overset{1}{\curvearrowright} \mathbb{C}$ as the trivial action, we obtain a net $\Psi_i \in B_+(\mathbb{C} \rtimes_\alpha G)$ with compact support converging uniformly on compacta to 1, by amenability of G and [Ped79, Lemma 7.2.4]. It is easy to see that the pointwise product $\Psi_i \Gamma_\varphi \in B_+^1(A \rtimes_\alpha G)$ for all $\varphi \in \mathcal{S}(A \rtimes_\alpha G)$, and since Ψ_i has compact support, we see that $\Psi_i \Gamma_\varphi \in A_+(A \rtimes_\alpha G)$ and $\Psi_i \Gamma_\varphi \to \Gamma_\varphi$ in the topology of weak* uniformly convergence on compacta of G. Hence, $A_+(A \rtimes_\alpha G) \subseteq B_+(A \rtimes_\alpha G)$ is dense. In particular, this implies that if π denotes the universal faithful representation of $A \rtimes_\alpha G$, then $\ker \tilde{\pi}_u \rtimes \lambda \subseteq \ker \pi = 0$, by weak containment ([Dix69, p. 80]), so $A \rtimes_\alpha G \cong \tilde{\pi}_u \rtimes \lambda(A \rtimes_\alpha G) = A \rtimes_{\alpha,r} G$. □

Combining the above, we obtain

Corollary 2.3.20. If $G \overset{\alpha}{\curvearrowright} A$ with G amenable, then $A \rtimes_\alpha G = \overline{\tilde{\pi} \rtimes \lambda(C_c(G, A, \alpha))}$ for any faithful representation π of A.

The above allows us to concretely work with the crossed products of a large class of groups which include finite groups, abelian groups, compact groups and others. This will be very useful.

For a fixed group G, the crossed product constructions described above are in fact functors from the category of G-algebras to the category of C^*-algebras, where the objects in G-algebras are C^*-algebras A equipped with an action $G \overset{\alpha}{\curvearrowright} A$ and morphism given by G-equivariant $*$-morphism, i.e., $\varphi\colon A \to B$ for G-algebras is a morphism if the diagram

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ {\scriptstyle\alpha_g}\downarrow & & \downarrow{\scriptstyle\beta_g} \\ A & \xrightarrow[\varphi]{} & B \end{array} \tag{2.14}$$

commutes for every $g \in G$. This can bee seen by the following:

Proposition 2.3.21. Suppose that $G \overset{\alpha}{\curvearrowright} A$ and $G \overset{\beta}{\curvearrowright} B$ and $\varphi\colon A \to B$ is a G-equivariant morphism. Then the map $\tilde{\varphi}\colon C_c(G, A, \alpha) \to C_c(G, B, \beta)$, $\tilde{\varphi}(f)(s) := \varphi(f(s))$, extends to a $*$-morphism $A \rtimes_\alpha G \to B \rtimes_\beta G$.

Proof.
It is clear that $\tilde{\varphi}$ is a $*$-morphism. To see that it extends, we show that $\|\tilde{\varphi}(f)\| \leq \|f\|$, where the norms comes from the embeddings $C_c(G, A, \alpha) \subseteq A \rtimes_\alpha G$ and $C_c(G, B, \beta) \subseteq B \rtimes_\beta G$. For this, let (π, u) be a covariant representation of $G \overset{\beta}{\curvearrowright} B$. Then, clearly $(\pi \circ \varphi, u)$ is a covariant representation of $G \overset{\alpha}{\curvearrowright} A$, and moreover, if $f \in C_c(G, A, \alpha)$, then

$$(\pi \circ \varphi) \rtimes u(f) = \int_G \pi(\varphi(f(s)))u_s \mathrm{d}s = \int_G \pi(\tilde{\varphi}(f)(s))u_s \mathrm{d}s = \pi \rtimes u \circ \tilde{\varphi}(f),$$

so $\pi \circ \varphi \rtimes u = \pi \rtimes u \circ \tilde{\varphi}$, and hence

$$\|\pi \rtimes u \circ \tilde{\varphi}(f)\| = \|\pi \circ \varphi \rtimes u(f)\| \leq \|f\|,$$

since the integrated form of a covariant representation is norm decreasing. Hence, since the norm of $\tilde{\varphi}(f)$ is the supremum of integrated forms of covariant representations of $G \overset{\beta}{\curvearrowright} B$, we see that $\tilde{\varphi}$ extends to a morphism $A \rtimes_\alpha G \to B \rtimes_\beta G$. $\square$

The above result also applies to the case of the reduced crossed product. This is easily seen by applying the above proof to regular representations $(\tilde{\pi}, \lambda)$ rather than any covariant representation. The remaining properties for a functor are easily verified, i.e., preserves composition and sends the identity map to the identity map.

2.4 The case of Abelian groups and Takai Duality

In the 70's, Hiroshi Takai published a paper ([Tak75]) showing that the crossed product construction in a way captures the Pontrjagin duality of locally compact abelian groups, i.e., that $\widehat{\widehat{G}} \cong G$, where $\widehat{G}$ denotes the dual group of G. Takai proved that taking the crossed product twice with G (in the sense that $\widehat{G} \overset{\hat{\alpha}}{\curvearrowright} A \rtimes_\alpha G$), one obtains an isomorphism

$$A \otimes \mathbb{K}(L^2(G)) \cong (A \rtimes_\alpha G) \rtimes_{\hat{\alpha}} \widehat{G}, \tag{2.15}$$

and in fact that both of these are non-trivial G-C^*-dynamical systems and the isomorphism is covariant. We will follow both Raeburns and Pedersens take on this, see [Rae88] and [Ped79, Chapter 7.9], since it is more aligned with the results and notation we have gone through.

Note 2.4.1
In the above, and in the rest, unless otherwise stated, we will use $A \otimes B$ between two C^*-algebras to mean the minimal tensor product. We will generally be dealing with tensor products of so called *nuclear C^*-algebra*, which we recall are C^*-algebras A such that for all other C^*-algebras B there is only one C^*-tensorproduct. This class includes e.g., a commutative C^*-algebra or $\mathbb{K}(H)$. For more on this, see e.g., [BO08].

For the two C^*-algebras in Equation (2.15), we first show that the left-hand side is a C^*-dynamical system for any (also non-abelian) locally compact group G with respect to the action $\alpha \otimes \mathrm{Ad}\rho$ of G, but first we need a lemma:

Lemma 2.4.2. Suppose that $G \overset{\alpha}{\curvearrowright} A$. Then the set $A^c \subseteq A$ given by

$$A^c := \{x \in A \mid t \mapsto \alpha_t(x) \text{ is continuous with respect to the norm topology on } A\},$$

is a G-invariant C^*-subalgebra of A.

Proof.
Clearly A^c is a $*$-subalgebra by continuity of the binary operatios in A. Suppose that $(x_i) \subseteq A^c$ is a net with limit $x \in A$. Let $(t_j) \subseteq G$ be any net with limit t and let $\varepsilon > 0$. Let i_0 be such that for $i \geq i_0$,

$$\|x_i - x\| < \frac{\varepsilon}{3},$$

Since $\alpha_{t_j}(x_i) \to \alpha_t(x_i)$ for each i, we may for $i \geq i_0$ choose (relative to i_0) j_0 such that for $j \geq j_0$ we have

$$\left\|\alpha_{t_j}(x_i) - \alpha_t(x_i)\right\| < \frac{\varepsilon}{3},$$

Hence for all such (i, j) we have

$$\left\|\alpha_{t_j}(x) - \alpha_t(x)\right\| \leq \left\|\alpha_{t_j}(x) - \alpha_{t_j}(x_i)\right\| + \left\|\alpha_{t_j}(x_i) - \alpha_t(x_i)\right\| + \|\alpha_t(x_i) - \alpha_t(x)\| < \varepsilon,$$

so $x \in A^c$. □

Proposition 2.4.3. Let $G \overset{\alpha}{\curvearrowright} A$ and let ρ denote the right-regular representation of G on $L^2(G)$, i.e., given by $\rho_t f(s) = f(st)\Delta(t)^{\frac{1}{2}}$. Then $(A \otimes \mathbb{K}(L^2(G)), G, \alpha \otimes \mathrm{Ad}\rho)$ is a C^*-dynamical system.

Proof.
With lemma 2.4.2 in mind, let (π, u) be a covariant representation of (A, G, α) on a Hilbert space H with π faithful. Let σ denote the diagonal representtion of G on $H \otimes L^2(G)$ given by

$$\sigma = u \otimes \rho,$$

which is a strongly continuous unitary representation. For $x \in A \subseteq \mathbb{B}(H)$ and $y \in \mathbb{K}(L^2(G))$ we see that

$$\sigma_t(x \otimes y)\sigma_t^* = \alpha_t(x) \otimes \rho_t y \rho_t^* \in A \otimes \mathbb{K}(L^2(G)) \subseteq \mathbb{B}(H \otimes L^2(G)).$$

Hence $A \otimes \mathbb{K}(L^2(G))$ is invariant under $\mathrm{Ad}\sigma$. Moreover, for finite dimensional $y \in \mathbb{K}(L^2(G))$, the map $t \mapsto \mathrm{Ad}\sigma_t(y)$ is norm continuous. The closure of $F = \mathrm{span}\{a \otimes y \mid a \in A,\ y \text{ finite rank}\}$ equals $A \otimes \mathbb{K}(L^2(G))$. Let B be the C^*-subalgebbra of $A \otimes \mathbb{K}(L^2(G))$ defined by

$$B := C^*(\{a \otimes K \ \mid\ t \mapsto \mathrm{Ad}\sigma_t(a \otimes K) \text{ is continuous}\}),$$

which is a G-invariant C^*-algebra containing F, so that B equals all of $A \otimes \mathbb{K}(L^2(G))$. Clearly $\alpha \otimes \mathrm{Ad}\rho = \mathrm{Ad}\sigma$, hence $(A \otimes \mathbb{K}(L^2(G)), G, \alpha \otimes \mathrm{Ad}\rho)$ is a C^*-dynamical system. □

The way we show Takai duality is through a series of appropriate isomorphisms of C^*-algebras and C^*-dynamical systems, some of which requires G to be abelian. We show the first step in this in the following:

Theorem 2.4.4. Let $G \overset{\alpha}{\curvearrowright} A$. Then there is a C^*-dynamical system $(C_0(G, A) \rtimes_{\gamma,r} G, G, \sigma)$ where we define the actions $G \overset{\gamma}{\curvearrowright} C_0(G, A)$ and $G \overset{\sigma}{\curvearrowright} C_0(G, A)$ by

$$\sigma_s(f)(t) = f(ts) \text{ and } \gamma_s f(t) = \alpha_s(f(s^{-1}t)),$$

for $f \in C_0(G, A)$ and $s, t \in G$. This system is covariantly isomorphic to $(A \otimes \mathbb{K}(L^2(G)), G, \alpha \otimes \mathrm{Ad}\rho)$.

Proof.
If we consider the isomorphism $C_0(G,A) \cong C_0(G) \otimes A$, it is not hard to see that the representations σ and γ are diagonal actions in disguise:

$$\sigma = \mathrm{rt} \otimes \mathrm{id} \text{ and } \gamma = \mathrm{lt} \otimes \alpha,$$

where rt and lt denote the left- and right-translation actions on $C_0(G)$, which implies that they are strongly continuous actions of G. Now, represent A faithfully on a Hilbert space $\mathcal{H}$, and define a representation of $C_0(G,A)$ on $L^2(G,\mathcal{H})$ as multiplication operators, i.e.,

$$\pi(f)\xi(s) = f(s)\xi(s),$$

for $f \in C_0(G,A)$, $\xi \in L^2(G,\mathcal{H})$ and $s \in G$. This is easily seen to be a faithful representation, so by Theorem 2.3.15 we obtain a faithful representation $\tilde{\pi} \rtimes \lambda$ of $C_0(G,A) \rtimes_{\gamma,r} G$ on $L^2(G, L^2(G,\mathcal{H})) \cong L^2(G \times G, \mathcal{H})$. Consider $C_c(G \times G, A) \subseteq C_c(G, C_0(G,A))$, via the map $(f\colon (x,y) \mapsto f(x,y)) \mapsto (x \mapsto (y \mapsto f(x,y)))$, and note that there is not necessarily equality. For $z \in C_c(G \times G, A)$, we see that

$$\tilde{\pi} \rtimes \lambda(z)\xi(s,t) = \int_G \alpha_{s^{-1}}(z(r,st))\xi(r^{-1}s,t)\mathrm{d}r.$$

Define the operator w on $L^2(G \times G, \mathcal{H})$ by

$$w\xi(s,t) := \xi(st,t)\Delta(t)^{\frac{1}{2}},$$

then w is a unitary operator with adjoint $w^*\xi(s,t) = \xi(st^{-1},t)\Delta(t)^{-\frac{1}{2}}$. We then calculate for $z \in C_c(G \times G, A)$

$$\begin{aligned} w^*(\tilde{\pi} \rtimes \lambda(z))w(\xi)(s,t) &= \int_G \alpha_{ts^{-1}}(z(r,s))w\xi(r^{-1}st^{-1},t)\mathrm{d}r\Delta(t)^{-\frac{1}{2}} \\ &= \int_G \alpha_{ts^{-1}}(z(r,s))\xi(r^{-1}s,t)\mathrm{d}r, \end{aligned}$$

for all $\xi \in L^2(G \times G, \mathcal{H})$ and $s,t \in G$. It is not hard to see that any function of $C_c(G \times G, A)$ can be approximated by linear combinations of maps of the form $(s,t) \mapsto z(s)g(t)\alpha_t(x)$ for $z,g \in C_c(G)$ and $x \in A$, and hence also by functions of the form

$$f(s,t) = \alpha_t(x)z(s^{-1}t)g(t)\Delta(s^{-1}t), \tag{2.16}$$

for fixed $z,g \in C_c(G)$ and $x \in A$. Applying Adw to functions of the form Equation (2.16) this we get

$$\begin{aligned}
w^*(\tilde{\pi} \rtimes \lambda(f))w(\xi)(s,t) &= \int_G \alpha_{ts^{-1}}(f(r,s))\xi(r^{-1}s,t)\mathrm{d}r \\
&= \int_G \alpha_{ts^{-1}}(\alpha_s(x)z(r^{-1}s)g(s)\Delta(r^{-1}s))\xi(r^{-1}s,t)\mathrm{d}r \\
&= \int_G \alpha_t(x)z(r^{-1}s)g(s)\xi(r^{-1}s,t)\Delta(r^{-1}s)\mathrm{d}r \\
&= \alpha_t(x)g(s)\int_G z(r)\xi(r,t)\mathrm{d}r.
\end{aligned}$$

Consider the faithful representation $\breve{\pi}$ of A on $L^2(G,\mathcal{H})$ by $\breve{\pi}(x)\xi(s) = \alpha_s(x)\xi(s)$. Given $K \in \mathbb{K}(\mathcal{H})$ and $x \in A$, if U denotes the unitary operator identifying $L^2(G) \otimes L^2(G,\mathcal{H})$ with $L^2(G, L^2(G,\mathcal{H})$, then for $h \in L^2(G)$ and $\xi \in L^2(G,\mathcal{H})$ we see that

$$U(\breve{\pi}(x) \otimes K(h \otimes \xi))(s)(t) = \alpha_s(x)h(s)K\xi(t),$$

so after identifying $L^2(G, L^2(G,H))$ with $L^2(G \times G, H)$, we see that for f as in Equation (2.16):

$$w^*(\tilde{\pi} \rtimes \lambda(f))w = \breve{\pi} \otimes p_{zg} \in A \otimes \mathbb{K}(L^2(G)) \subseteq \mathbb{B}(L^2(G \times G, \mathcal{H})),$$

where is $p_{z,g}$ the rank-one operator $\eta \mapsto \langle \eta, \overline{z}\rangle g$, $\eta \in L^2(G)$. In particular, this implies that

$$A \otimes \mathbb{K}(L^2(G)) \subseteq w^*(\tilde{\pi} \rtimes \lambda(C_0(G,A) \rtimes_{\lambda,r} G))w,$$

since $\breve{\pi}$ is faithful and $\mathbb{K}(L^2(G))$ is the closure of the span of rank-one operators. Since we may approximate $C_c(G \times G, A)$ by linear combinations of forms $(s,t)\alpha_s(x)f(r^{-1}s)g(s)\Delta(r^{-1}s)$, and $C_c(G \times G, A)$ is dense in $C_0(G,A) \rtimes_{\lambda,r} G$, the above inclusion is an equality, so the two algebras are spatially isomorphic with respect to w.

Denote also by σ_t the representation of G on $C_c(G, C_0(G,A))$, $\sigma_t f(x)(y) = f(s)(yt)$. Then σ_t extends to all of $C_0(G,A) \rtimes_{\lambda,r} G$ by continuity and density. If f is as above, then

$$\begin{aligned}
w^*(\tilde{\pi} \rtimes \lambda\sigma_r(f))w\xi(s,t) &= \alpha_{tr}(x) \underbrace{g(sr)\Delta(r)}_{=\rho_r g(s)\Delta(r)^{\frac{1}{2}}} \int_G \underbrace{z(qr)}_{=\rho_r z(q)\Delta(r)^{-\frac{1}{2}}} \xi(q,t)\mathrm{d}q \\
&= \alpha_t(\alpha_r(x))\rho_r g(s)\int_G \rho_r z(q)\xi(q,t)\mathrm{d}q \\
&= (\breve{\pi}(\alpha_r(x)) \otimes \rho_r p_{z,g}\rho_r^*)\xi(s,t) \\
&= (\breve{\pi}(\alpha_r(x)) \otimes \mathrm{Ad}\rho_r p_{z,g})\xi(s,t).
\end{aligned}$$

In particular, after identifying $\breve{\pi}(A)$ with A, we have

$$w^*(\tilde{\pi} \rtimes \lambda(\sigma_r(f)))w = (\alpha \otimes \mathrm{Ad}\rho)_r(x \otimes p_{z,g}),$$

and hence by density, we see that $(C_0(G,A)\rtimes_{\lambda,r}, G, \sigma)$ is covariantly isomorphic to $(A \otimes \mathbb{K}(L^2(G)), G, \alpha \otimes \mathrm{Ad}\sigma)$ via w, in the sense that w implements a G-equivariant spatial isomorphism between the two systems. □

Applying the above to the case where $A = \mathbb{C}$ and $\alpha = 1$, we obtain the Stone-von Neumann theorem for reduced crossed products:

Corollary 2.4.5. Let lt denote the representation of G on $C_0(G)$ given by left-translation, i.e., $\mathrm{lt}_s f(t) = f(s^{-1}t)$. Then lt corresponds to γ in the above case with $G \overset{1}{\curvearrowright} \mathbb{C}$, so

$$C_0(G) \rtimes_{\mathrm{lt},r} G \cong \mathbb{K}(L^2(G)) \otimes \mathbb{C} \cong \mathbb{K}(L^2(G)).$$

It is worth noting that the result also holds for the full crossed product:

$$C_0(G) \rtimes_{\mathrm{lt}} G \cong \mathbb{K}(L^2(G)),$$

however, we mention it without proof. The inclined reader is refered to [Wil07, Theorem 4.24] or [RW98, §C.6]. In particular, this implies that

$$C_0(G) \rtimes_{\mathrm{lt},r} G \cong C_0(G) \rtimes_{\mathrm{lt}} G,$$

for all groups G.

In the following, we will assume that G in addition to being locally compact is abelian, and we will use $+$ to denote the binary operation. We briefly recall the definition of the dual group $\widehat{G}$ and associated notions. For a better exposition on these topics, we again refer the reader to the excellent work in [Fol16] or the author's written project on the subject [Kar18, Chapter 1-4]

Definition 2.4.6. For a locally compact abelian group G, the dual group $\widehat{G}$ is the group of continuous homomorphisms $\xi\colon G \to S^1$, where S^1 denotes the circle. For $\xi \in \widehat{G}$ and $t \in G$, we will write $(t,\xi) := \xi(t)$. An alternative and equivalent definition of $\widehat{G}$ is as the set of irreducible unitary representations of G, which by commutativity of G are all one-dimensional, or we may identify $\widehat{G}$ with the spectrum, $\sigma(L^1(G))$, of the group algebra via the pairing

$$\xi(f) := \int_G (t,\xi) f(t) \mathrm{d}t, \tag{2.17}$$

for $f \in L^1(G)$ and $\xi \in \widehat{G}$. For a subset $S \subseteq G$, we will denote by $S^\perp$ the annihilator of S in $\widehat{G}$, i.e., the set

$$S^\perp := \left\{ \xi \in \widehat{G} \;\middle|\; (s,\xi) = 1 \text{ for all } s \in S \right\}.$$

Usually, one will examine the Fourier transformation of $M(G)$, where $M(G)$ is the space of bounded Radon Measures on G. It is useful to recall also that $L^1(G) \subseteq M(G)$ by assigning f to $f \cdot \mathrm{d}x$, where $\mathrm{d}x$ is the integrand of the fixed Haar measure on G. However, we will instead consider the inverse Fourier transformaton:

Definition 2.4.7. The inverse Fourier transformation of a measure $\mu \in M(G)$ is the bounded continuous map on $\widehat{G}$ given by

$$\widehat{\mu}(\xi) := \int_G (t, \xi) \mathrm{d}\mu(t).$$

For $f \in L^1(G) \subseteq M(G)$, we have

$$\widehat{f}(\xi) := \int_G (t, \xi) f(t) \mathrm{d}t.$$

Note that $\widehat{\mu} \in C_b(\widehat{G})$ and $\widehat{f} \in C_0(\widehat{G})$ for $\mu \in M(G)$ and $f \in L^1(G)$.

And it will be worth noting that there is a topological duality between G and $\widehat{G}$, in the following sense:

Proposition 2.4.8. A group G is discrete if and only if $\widehat{G}$ is compact and vice versa.

It will also be important to note that there is a one-to-one correspondance between unitary representations of G and representations of $M(G)$ as a Banach $*$-algebra:

Proposition 2.4.9. There is a bijective correspondance between unitary representations of G and representations of $M(G)$ which restricts to a non-degenerate representation of $L^1(G)$. If u is a unitary representation of G on $\mathcal{H}$, then the associated representation of $L^1(G)$ is given by

$$u_f(\xi) = \int_G u_s \xi f(s) \mathrm{d}s,$$

in the weak sense, i.e., that

$$\langle u_f(\xi), \eta \rangle = \int_G \langle u_s \xi, \eta \rangle f(s) \mathrm{d}s,$$

for $\xi, \eta \in \mathcal{H}$.

For the rest of this section, we will assume that G is any abelian locally compact group, and we shall think of $A \rtimes_\alpha G$ as operators on $L^2(G, \mathcal{H}_u)$, where $\mathcal{H}_u$ is the universal Hilbert space for A.

Definition 2.4.10. Suppose that $G \overset{\alpha}{\curvearrowright} A$. We define the *dual action*, $\widehat{\alpha}$, of $\widehat{G}$ on $C_c(G, A, \alpha)$ by

$$\widehat{\alpha}_\xi y(t) = (t, \xi)y(t),$$

for $y \in C_c(G, A, \alpha)$, $\xi \in \widehat{G}$ and $t \in G$.

To see that this extends to a well-defined action on $A \rtimes_\alpha G$, we use the following:

Lemma 2.4.11. Let $G \overset{\alpha}{\curvearrowright} A$. Define a unitary representation u of $\widehat{G}$ on $L^2(G, H)$ by

$$u_\chi \xi(t) = (t, \chi)\xi(t).$$

Then, with $\widehat{\alpha}_\chi$ as above, it holds that

$$\widehat{\alpha}_\chi = \mathrm{Ad} u_\chi,$$

so that $(A \rtimes_\alpha G, \widehat{G}, \widehat{\alpha})$ becomes a C^*-dynamical system.

Proof.
Let $y \in C_c(G, A, \alpha)$, so that $y\xi(s) = \int_G \alpha_{-s}(y(t))\xi(t - s)\mathrm{d}s$. Then

$$\begin{aligned}\mathrm{Ad} u_\chi(y)\xi(t) &= (t, \chi) \int_G \alpha_{-t}(y(s))(t, -\chi)(-s, -\chi)\xi(t - s)\mathrm{d}s \\ &= \int_G \alpha_{-t}((s, \chi)y(s))\xi(t - s)\mathrm{d}s \\ &= \widehat{\alpha}_\chi y\xi(s).\end{aligned}$$

Hence, since $\widehat{\alpha}_\chi y \in C_c(G, A, \alpha)$, and conjugation by unitaries is an isometry, we may define

$$\widehat{\alpha}_\chi := (A \rtimes_\alpha G \ni x \mapsto \mathrm{Ad} u_\chi x \in A \rtimes_\alpha G) \in \mathrm{Aut}(A \rtimes_\alpha G),$$

and we will think of it as the automorphism satisfying $\widehat{\alpha}_\chi y(s) = (s, \chi)y(s)$ for $y \in C_c(G, A, \alpha)$. □

The above example is one of the motivations for the notion of a G-product:

Definition 2.4.12. Let G be a group with dual group $\widehat{G}$. We say that a C^*-algebra A is a *G-product* if:

1. There is a homomorphism $\lambda \colon G \to \mathcal{U}(M(A))$ such that the map $t \mapsto \lambda_t y$, $y \in A$ is continuous,

2. There is a homomorphism $\widehat{\alpha}\colon \widehat{G} \to \mathrm{Aut}(A)$ such that $(A, \widehat{G}, \widehat{\alpha})$ is a C^*-dynamical system satisfying
$$\widehat{\alpha_\chi}\lambda_t = (t,\chi)\lambda_t,$$
for all $t \in G$ and $\chi \in \widehat{G}$.

Moreover, we say that $x \in M(A)$ satisfies *Landstad's conditions* if:

1. $\widehat{\alpha}_\chi(x) = x$ for all $\chi \in \widehat{G}$,
2. $x\lambda_f \in A$ and $\lambda_f x \in A$ for all $f \in L^1(G)$,
3. the map $t \mapsto \lambda_t x \lambda_t^*$ is continuous.

Lemma 2.4.13. Suppose that $G \overset{\alpha}{\curvearrowright} A$. Then with the above action, it holds that $G \rtimes_\alpha A$ is a G-product, and each element of $A \subseteq M(A \rtimes_\alpha G)$ satisfies Landstad's condition.

Proof.
Let u and $\widehat{\alpha}$ be as above. Let λ_t be the operator on $L^2(G, H)$ given by
$$\lambda_t \xi(s) = \xi(s-t).$$
Then we may view λ as a map $G \to \mathcal{U}(M(A \rtimes_\alpha G))$, $\lambda_t y(s) = \alpha_t(y(s-t))$ for $y \in C_c(G, A) \subseteq A \rtimes_\alpha G$, by Proposition 2.2.25 such that $t \mapsto \lambda_t y$ is continuous for all $y \in A \rtimes_\alpha G$. Combining this with the calculations in Lemma 2.4.11, we see that $A \rtimes_\alpha G$ is a G-product, since
$$u_\chi \lambda_s u_{-\chi}\xi(t) = (t,\chi)(t-s,-\chi)\xi(t-s) = (s,\chi)\lambda_s\xi(t),$$
for all $\xi \in L^2(G,H)$, $s,t \in G$ and $\chi \in \widehat{G}$. For the last claim, we see that if $a \in A \subseteq M(A \rtimes_\alpha G)$, then $\widehat{\alpha}_\chi a = u_\chi a u_\chi^* = a$, since $ay(t) = \alpha_t(a)y(t)$ for all $y \in C_c(G, A, \alpha)$, so
$$\widehat{\alpha}_\chi(a)y(t) = (t,\chi)\alpha_{-t}(a)(t,-\chi)y(t) = \alpha_{-t}(a)y(t) = ay(t).$$

If $f \in C_c(G)$ and $x \in A$, then for all $\xi \in L^2(G,H)$ and $s \in G$ we see that
$$x\lambda_f\xi(s) = \int_G f(t)x\lambda_t\xi(s)\mathrm{d}t = \int_G f(t)\alpha_{-t}(x)\lambda_t\xi(s)\mathrm{d}t = \int_G (f\otimes x)(t)\lambda_t\xi(s)\mathrm{d}t = (f\otimes x)\xi(s),$$
where we have identified $f \otimes x \in C_c(G, A, \alpha) \subseteq A \rtimes_\alpha G$. In particular $x\lambda_f \in A \rtimes_\alpha G$ and likewise $\lambda_f x \in A \rtimes_\alpha G$ for $f \in C_c(G)$ and hence for $f \in L^1(G)$, by dominance of the norm on $A \rtimes_\alpha G$. $\square$

Now, we are going to show the theorem of Takai by combining the previous results with the following result, for which we recall that given any group G we denote by $G \overset{1}{\curvearrowright} A$ the trivial action of A, i.e., the map $t \mapsto \mathrm{id} \in \mathrm{Aut}(A)$. Then

Lemma 2.4.14. For all groups G and C^*-algebras A, there is an isomorphism $A \rtimes_1 \widehat{G} \cong C_0(G, A)$. Moreover, the isomorphism is spatially implemented by a unitary u.

Proof.
Let $C_0(G, A)$ be faithfully represented on $L^2(G, H)$ as multiplication operators $f\xi(s) = f(s)\xi(s)$, where A is faithfully represented on H. By the Plancheral Theorem, we have $L^2(G) \cong L^2(\widehat{G})$ via the (proper) Fourier transformation, and so we may define the isometry $u\colon L^2(G, H) \to L^2(\widehat{G}, H)$ by

$$u\xi(\chi) = \int_G \xi(t)\overline{(t,\chi)}\mathrm{d}t, \text{ for } \xi \in L^2(G, H) \text{ and } \chi \in \widehat{G},$$

so that

$$u^*\eta(t) = \int_{\widehat{G}} \eta(\chi)(t,\chi)\mathrm{d}\chi, \text{ for } \eta \in L^2(\widehat{G}, H) \text{ and } t \in G,$$

where the integrals are defined in the L^2-sense. Then, for all $f \in C_c(\widehat{G}, A, 1)$ we see that

$$\begin{aligned} fu\xi(\chi) = \int_{\widehat{G}} \mathrm{id}(f(\chi))\lambda_\sigma(u\xi)(\chi)\mathrm{d}\sigma &= \int_{\widehat{G}} f(\chi) \int_G \xi(t)\overline{(t,\chi-\sigma)}\mathrm{d}t\mathrm{d}\sigma \\ &= \int_{\widehat{G}}\int_G f(\chi)(t,\sigma)\xi(t)\overline{(t,\chi)}\mathrm{d}t\mathrm{d}\sigma \\ &= \int_{\widehat{G}} \widehat{f}(t)\xi(t)\overline{(t,\chi)}\mathrm{d}t \\ &= \int_{\widehat{G}} (\widehat{f}\xi)(t)\overline{(t,\chi)}\mathrm{d}t \\ &= u\widehat{f}\xi(\chi) \end{aligned}$$

implying that $u^*fu = \widehat{f} \in C_0(G, A)$. By density of $C_c(\widehat{G}, A, 1) \subseteq A \rtimes_1 \widehat{G}$, it follows that u implements the desired isomorphism. $\square$

Remark 2.4.15
Recall that $C_0(\widehat{G}, A) \cong C_0(\widehat{G}) \otimes A$ and that $C_0(\widehat{G}) \cong C^*(G)$ for G abelian, hence the above result shows that the crossed product associated to the trivial C^*-dynamical system $G \overset{1}{\curvearrowright} A$ is covariantly isomorphic to $C^*(G) \otimes A$. This generalizes the result for when $A = \mathbb{C}$. We will in the follow write $A \rtimes_1 \widehat{G}$ to emphasize that we consider the dense subset $C_c(\widehat{G}, A)$.

We are now ready to define the C^*-dynamical system which we will show is covariantly isomorphic to $(A \otimes \mathbb{K}(L^2(G)), G, \alpha \otimes \mathrm{Ad}\lambda)$ and $\Big((A \rtimes_\alpha G) \rtimes_{\widehat{\alpha}} \widehat{G}, G, \widehat{\widehat{\alpha}}\Big)$:

Proposition 2.4.16. Suppose that $G \overset{\alpha}{\curvearrowright} A$. Define for each $t \in G$ the map $\beta_t\colon C_c(\widehat{G}, A, 1) \to C_c(\widehat{G}, A, 1)$ by

$$\beta_t f(\chi) = \overline{(t,\chi)}\alpha_t(f(\chi)),$$

for $f \in C_c(\widehat{G}, A, 1)$ and $\chi \in \widehat{G}$. Then β_t extends to an automorphism of $A \rtimes_1 \widehat{G}$ such $t \mapsto \beta_t$ is an action of G on $A \rtimes_1 \widehat{G}$. With this β, we have a spatial isomorphism $(A \rtimes_1 \widehat{G}) \rtimes_\beta G \cong (A \rtimes_\alpha G) \rtimes_{\widehat{\alpha}} \widehat{G}$.

Proof.
Define $\gamma_t \colon C_0(G, A) \to C_0(G, A)$ to be the diagonal action

$$\gamma_t y(s) = \alpha_t(y(s-t)),$$

here diagonal comes from the fact that $\gamma_t = \alpha_t \otimes \mathrm{lt}_t$, when we identify $C_0(G, A) \cong C_0(G) \otimes A$. Let u be as in Lemma 2.4.14, i.e., so $\widehat{\beta_s(y)} = u^* \beta_s(y) u$. Then for $\xi \in L^2(G, H)$ it holds that

$$\widehat{\beta_s(y)}\xi(t) = \widehat{\beta_s(y)}(t)\xi(t) = \int_{\widehat{G}} (t, \sigma)\beta_s(y)(\sigma)\xi(t)\mathrm{d}\sigma = \int_{\widehat{G}} (t-s, \sigma)\alpha_s(y(\sigma))\xi(t)\mathrm{d}\sigma,$$

and

$$\gamma_s(\widehat{y})\xi(t) = \alpha_s(\widehat{y}(t-s))\xi(t) = \int_{\widehat{G}} (t-s, \sigma)\alpha_s(y(\sigma))\xi(t)\mathrm{d}\sigma = \widehat{\beta_s(y)}\xi(t).$$

In particular, this allows us to extend β_t to an automorphism of $A \rtimes_1 \widehat{G}$ such that the associated dynamical system is covariantly isomorphic to $(C_0(G, A), G, \gamma)$ via u.

We now show that the system $(\widehat{G} \rtimes_1 A, G, \beta)$ is spatially isomorphic to $(G \rtimes_\alpha A, \widehat{G}, \widehat{\alpha})$. For this, we note that an easy calculation shows for $x \in C_c(\widehat{G} \times G, A)$ and $\xi \in L^2(\widehat{G} \times G, H)$ that

$$\begin{aligned} x\xi(t, \chi) &= \int_G \int_{\widehat{G}} \beta_{-t}(x(s, \sigma))\xi(t-s, \chi-\sigma)\mathrm{d}\sigma\mathrm{d}s \\ &= \int_G \int_{\widehat{G}} \alpha_{-t}(x(s, \sigma))\xi(t-s, \chi-\sigma)(s, \sigma)\mathrm{d}\sigma\mathrm{d}s, \end{aligned}$$

and similarly for $y \in C_c(G \times \widehat{G}, A)$ and $\eta \in L^2(G \times \widehat{G}, H)$ we have

$$y\eta(\chi, t) = \int_{\widehat{G}} \int_G \alpha_{-t}(y(\sigma, s))\xi(\chi-\sigma, t-s)\overline{(t, \sigma)}\mathrm{d}s\mathrm{d}\sigma,$$

for all $\chi \in \widehat{G}$ and $t \in G$. Define an isomorphism $\Phi \colon C_c(G \times \widehat{G}, A) \to C_c(\widehat{G} \times G, A)$ and isometry $w \colon L^2(G \times \widehat{G}, H) \to L^2(\widehat{G} \times G, H)$ by

$$\Phi(y)(\chi, t) = (t, \chi)y(t, \chi) \text{ and } w\xi(\chi, t) = \overline{(t, \chi)}\xi(t, \chi)$$

so that

$$\Phi^{-1}(x)(t, \chi) = \overline{(t, \chi)}x(\chi, t) \text{ and } w^*\eta(t, \chi) = (t, \chi)\eta(\chi, t)$$

for $t \in G$, $\chi \in \widehat{G}$. Note now that for $\sigma, \chi \in \widehat{G}$ and $s, t \in G$ we have

$$(s,\sigma)\overline{(t-s,\chi-\sigma)(s,\chi)} = \overline{(t,\chi)}(t,\sigma),$$

hence for $x \in C_c(G \times \widehat{G}, A)$ and $\xi \in L^2(G \times \widehat{G}, H)$ we get using the above identities:

$$\begin{aligned}\Phi(x)w\xi(\chi,t) &= \int_{\widehat{G}}\int_G \alpha_{-t}(x(s,\sigma))\xi(t-s,\chi-\sigma)(s,\sigma)\overline{(t-s,\sigma-\chi)(s,\chi)}\mathrm{d}s\mathrm{d}\sigma \\ &= \overline{(t,\chi)}\int_{\widehat{G}}\int_G \alpha_{-t}(x(s,\sigma))\xi(t-s,\chi-\sigma)(t,\sigma)\mathrm{d}s\mathrm{d}\sigma \\ &= wx\xi(\chi,t),\end{aligned}$$

so that $w^*\Phi(x)w = x$ for all $x \in C_c(G \times \widehat{G}, A)$. By density, it follows that the two algebras $(G \rtimes_\alpha A) \rtimes_{\widehat{\alpha}} \widehat{G}$ and $(A \rtimes_1 \widehat{G}) \rtimes_\beta G$ are isomorphic. □

We are now ready to prove the main theorem by combining the above:

Theorem 2.4.17 (Takai Duality)**.** Suppose that $G \overset{\alpha}{\curvearrowright} A$. Then the double dual system $\left((A \rtimes_\alpha G) \rtimes_{\widehat{\alpha}} \widehat{G}, G, \widehat{\widehat{\alpha}}\right)$ is covariantly isomorphic to the C^*-dynamical system $(A \otimes \mathbb{K}(L^2(G)), G, \alpha \otimes \mathrm{Ad}\rho)$.

Proof.
We already have the following diagram of isomorphisms of C^*-algebras

$$A \otimes \mathbb{K}(L^2(G)) \xrightarrow{\cong} C_0(G,A) \rtimes_\gamma G \xrightarrow{\cong} (A \rtimes_1 \widehat{G}) \rtimes_\beta G \xrightarrow{\cong} (A \rtimes_\alpha G) \rtimes_{\widehat{\alpha}} \widehat{G}$$

Moreover, the dynamical systems $(A \otimes \mathbb{K}(L^2(G), G, \alpha \otimes \mathrm{Ad}\rho)$ and $(C_0(G,A) \rtimes_\gamma G, G, \sigma)$ are covariantly isomorphic, c.f. Theorem 2.4.4. We finish the proof by showing that $(C_0(G,A) \rtimes_\gamma G, G, \sigma)$ is covariantly isomorphic to $((A \rtimes_\alpha G) \rtimes_{\widehat{\alpha}} \widehat{G}, G, \widehat{\widehat{\alpha}})$, which is done by showing the relation on the generating set of functions $z \in C_c(G, C_0(G,A))$ of the form

$$z(s,t) = xf(s)\widehat{g}(t),$$

for some $x \in A$, $f \in C_c(G)$ and $g \in C_c(\widehat{G})$. Fix such z, and denote by $\breve{z} \in C_c(G \times \widehat{G}, A)$ the image of z under the middle isomorphism above, so that

$$\breve{z}(s,\chi) = xf(s)g(\chi),$$

for $(s,\chi) \in G \times \widehat{G}$, and note that the map $g_r \colon \chi \mapsto g(\chi)(r,\chi)$ is mapped to the function $\widehat{g}_r \colon t \mapsto \widehat{g}(r+t)$. Denote by Φ the last isomorphism in the above, so that $\Phi(\breve{z}) \in C_c(\widehat{G} \times G, A)$ is the map

$$\Phi(\breve{z})(\chi,s) = (s,\chi)\breve{z}(s,\chi) = xf(s)g(\chi)(s,\chi).$$

In particular, for the action $G \overset{\sigma}{\curvearrowright} C_0(G, A) \rtimes_\gamma G$ given by $\sigma_r y(s,t) = y(s, t+r)$ for $y \in C_c(G \times G, A)$, then we see that

$$\sigma_r z(s,t) = xf(s)\widehat{g}(t+r) = xf(s)\widehat{g}_r(t),$$

whence we see that

$$\widetilde{\sigma_r z}(s,\chi) = xf(s)g(\chi)(r,\chi),$$

so that

$$\Phi(\widetilde{\sigma_r z})(\chi, s) = xf(s)g(\chi)(s,\chi)(r,\chi) = \Phi(\breve{z})(\chi,s)(r,\chi).$$

Note also, that if $G \overset{\widehat{\widehat{\alpha}}}{\curvearrowright} (A \rtimes_\alpha G) \rtimes_{\widehat{\alpha}} \widehat{G}$ is the dual of the dual action, then for $y \in C_c(\widehat{G} \times G, A) \subseteq (A \rtimes_\alpha G) \rtimes_{\widehat{\alpha}} \widehat{G}$ we have

$$\widehat{\widehat{\alpha}}_r y(\chi, t) = (r,\chi)y(\chi,t),$$

whence we see that $\widehat{\widehat{\alpha}}_r(\Phi(\breve{z})) = \Phi(\widetilde{\sigma_r(z)})$. Since the set of functions of the form as z generates $C_0(G,A) \rtimes_\gamma G$, this shows the desired covariant isomorphism of the systems $(C_0(G,A) \rtimes_\gamma G, G, \sigma)$ and $((A \rtimes_\alpha G) \rtimes_{\widehat{\alpha}} \widehat{G}, G, \widehat{\widehat{\alpha}})$. $\square$

In particular, since $\widehat{\mathbb{R}} \cong \mathbb{R}$ (self-duality), we observe using Takai duality that the C^*-algebra $\mathbb{K}(L^2(\mathbb{R}))$ is isomorphic to $C_0(\mathbb{R}) \rtimes_{\mathrm{lt}} \mathbb{R}$, which according to the author is an especially beautiful application of the theory.

Before we begin our discussion of simplicity in crossed product in the next chapter, we will actually go back to the concept of a G-product and properties thereof, since it will give us a useful characterization of crossed products. We follow the guidelines of [OP78] in doing this, but the results are due to Landstad ([Lan79]).

Definition 2.4.18. For a G-product A, an element $y \in M(A)_+$ is said to be $\widehat{\alpha}$*-integrable* if there is an element $I(y) \in M(A)_+$ such that

$$\varphi(I(y)) = \int_G \varphi(\widehat{\alpha}_\chi(y))\mathrm{d}\chi,$$

for all $\varphi \in A^*$. If $x \in M(A)$ is such that x may be decomposed into $\widehat{\alpha}$-integrable positive parts, we then say that x is also $\widehat{\alpha}$-integrable.

Note 2.4.19
If $0 \leq x \leq y \in M(A)_+$ and y is $\widehat{\alpha}$-integrable, then both x is $\widehat{\alpha}$-integrable. For an argument, we refer the reader to [Ped79, Remark 7.8.4]. It is based on the fact that if y is $\widehat{\alpha}$ integrable

and $0 \leq x \leq y$, then they induce a lower semicontinuous and bounded map on the state space of A.

In particular, if z is $\widehat{\alpha}$-integrable, then for all positive $y \in A$ we have $yzy \leq \|y\| z$ is $\widehat{\alpha}$-integrable.

Lemma 2.4.20. Let A be a G-product. Assume that $x, y \in M(B)$ satisfy that x^*x and y^*y are $\widehat{\alpha}$-integrable. Then y^*x is $\widehat{\alpha}$-integrable with

$$\|I(y^*x)\|^2 \leq \|I(x^*x)\| \, \|I(y^*y)\| \, .$$

Proof.
It follows from the polarization identity that y^*x is $\widehat{\alpha}$-integrable, since $0 \leq (x + i^k y)^*(x + i^k y) \leq 2(x^*x + y^*y)$. Assume that $M(B)$ is faithfully represented in $\mathbb{B}(H)$ for some Hilbert space H. Then for all $\xi, \eta \in H$ we have

$$\begin{aligned} |\langle I(y^*x)\xi, \eta\rangle| &= \left| \int_{\widehat{G}} \langle \widehat{\alpha}_\sigma(x)\xi, \widehat{\alpha}_\sigma(y)\eta \rangle \mathrm{d}\sigma \right| \\ &\leq \left(\int_{\widehat{G}} \|\widehat{\alpha}_\sigma(x)\xi\|^2 \, \mathrm{d}\sigma \int_{\widehat{G}} \|\widehat{\alpha}_\sigma(y)\eta\|^2 \, \mathrm{d}\sigma \right)^{\frac{1}{2}} \\ &= \left(\langle I(x^*x)\xi, \xi \rangle \langle I(y^*y)\eta, \eta \rangle \right)^{\frac{1}{2}} , \end{aligned}$$

finishing the proof. $\square$

Remark 2.4.21
For the following, recall that the Fourier transformation embeds $L^1(G)$ into a dense $*$-subalgebra of $C_0(\widehat{G})$, and recall that we saw that $M(C_0(\widehat{G})) \cong C_b(\widehat{G})$ in chapter 1, and hence given a G-product A, the (induced) $*$-representation $\lambda\colon L^1(G) \to M(A)$ extends by continuity to a $*$-representation of $\widehat{\lambda}\colon C_0(\widehat{G}) \to M(A)$. Taking the multiplier on each side, we obtain a representation $\widehat{\lambda}\colon M(C_0(\widehat{G})) = C_b(\widehat{G}) \to M(M(A)) = M(A)$.

Lemma 2.4.22. Let A be a G-product. If $f \in L^1(G) \cap L^2(G)$, then $\lambda_f^* \lambda_f$ is $\widehat{\alpha}$-integrable and

$$I(\lambda_f^* \lambda_f) = \|f\|_2^2 \, 1,$$

where 1 is the identity operator on $M(A)$.

Proof.
Recall that for $f \in L^1(G)$, the extension $\widehat{\lambda}$ to $C_0(\widehat{G})$ satisfies $\widehat{\lambda}(\widehat{f}) = \lambda_f$. Hence, if $\sigma \in \widehat{G}$

and $f \in L^1(G)$, then $\widehat{\sigma \cdot f} = \delta_{-\sigma} * \widehat{f}$, and so

$$\widehat{\lambda}(\delta_\sigma * \widehat{f}) = \lambda_{\sigma f} = \int_{\widehat{G}} \widehat{\alpha}_\sigma(\lambda_t) f(t) \mathrm{d}t = \widehat{\alpha}_\sigma(\lambda_f) = \widehat{\alpha}_\sigma(\widehat{\lambda}(\widehat{f})),$$

where $\delta_\sigma * \widehat{f}$ denotes the measure convolution with the dirac measure at σ. Continuity of the extension $\widehat{\lambda}$ ensures that $\widehat{\lambda}(\delta_{-\sigma} * g) = \widehat{\alpha}_\sigma(\widehat{\lambda}(g))$ for all $g \in C_b(\widehat{G})$. Letting $f \in L^1(G) \cap L^2(G)$, we then obtain

$$\widehat{\alpha}_\sigma(\lambda_f^* \lambda_f) = \widehat{\alpha}_\sigma(\widehat{\lambda}(|\widehat{f}|^2)) = \widehat{\lambda}(\delta_{-\sigma} * |\widehat{f}|^2).$$

By the Plancherel theorem we have $\left\|\widehat{f}\right\|_2 = \|f\|_2$, and so

$$I(\lambda_f^* \lambda_f) = \int_{\widehat{G}} \widehat{\alpha}_\sigma(\lambda_f^* \lambda_f) \mathrm{d}\sigma = \widehat{\lambda}\left(\int_{\widehat{G}} \delta_{-\sigma} * |\widehat{f}|^2 \mathrm{d}\sigma\right) = \widehat{\lambda}(\left\|\widehat{f}\right\|_2^2 1) = \|f\|_2^2 1,$$

as for all $\chi \in \widehat{G}$ we have

$$\int_{\widehat{G}} \delta_{-\sigma} * |\widehat{f}|^2(\chi) \mathrm{d}\sigma = \int_{\widehat{G}} |\widehat{f}(\chi + \sigma)|^2 \mathrm{d}\sigma = \int_{\widehat{G}} |\widehat{f}(\sigma)|^2 \mathrm{d}\sigma = \left\|\widehat{f}\right\|_2^2 = \|f\|_2^2.$$

□

Lemma 2.4.23. Let A be a G-product and $f, g \in L^1(G) \cap L^2(G)$ and $x \in M(A)$. Then $\lambda_f^* x \lambda_g$ is $\widehat{\alpha}$-integrable. If we further assume that $x \in A$, then the function $t \mapsto I(\lambda_f^* x \lambda_g \lambda_t)$ is continuous with respect to the norm topology and $\lambda_f^* x \lambda_f - x \to 0$ whenever f ranges through an approximate identity of $L^1(G)$ which is also in $L^2(G)$.

Proof.
We first note that by Lemma 2.4.20 and Lemma 2.4.22, whenever $f \in L^1(G) \cap L^2(G)$ and $x \in B_+$, we have $0 \leq \lambda_f^* x \lambda_f \leq \|x\| \lambda_f^* \lambda_f$. Hence, replacing x with $x^{\frac{1}{2}}$ and letting $f, g \in L^1(G) \cap L^2(G)$ be arbitrary we see that $(x^{\frac{1}{2}} \lambda_f)^* x^{\frac{1}{2}} g = \lambda_f^* x \lambda_g$ will be $\widehat{\alpha}$-integrable. Hence it holds for every $x \in B$ as well.

For the last statement, we see using Lemma 2.4.20 that

$$\begin{aligned} \left\|I(\lambda_f^* x \lambda_g \lambda_t) - I(\lambda_f^* x \lambda_g \lambda_s)\right\|^2 &= \left\|I(\lambda_f^* x \lambda_g (\lambda_t - \lambda_s)\right\|^2 \\ &\leq \left\|I(\lambda_f^* x x^* \lambda_f)\right\| \left\|I((\lambda_t - \lambda_s)^* \lambda_g^* \lambda_g (\lambda_t - \lambda_s))\right\| \\ &\leq \|f\|_2^2 \|x\|^2 \|g * (\delta_t - \delta_s)\|_2^2, \end{aligned}$$

which tends to 0 as $s \to t$, showing that the claim of continuity is true. For the final statment, if $f \geq 0$ with $\|f\|_1 = 1$, then

$$\begin{aligned}
\|\lambda_f^* x \lambda_f - x\| &= \left\| \int \int (\lambda_t x \lambda_s - x) f(s) f^*(t) \mathrm{d}t \mathrm{d}s \right\| \\
&= \left\| \int \int (\lambda_t x - x\lambda_s^*) f^*(t) \mathrm{d}t \lambda_s f(s) \mathrm{d}s \right\| \\
&\leq \int \int \|(\lambda_t x - x + (x - x\lambda_s^*)\| \, |f^*(t)| \mathrm{d}t \, \|\lambda_s\| \, |f(s)| \mathrm{d}s \\
&\leq \int \int (\|\lambda_t x - x\| + \|x - x\lambda_s^*\|) f^*(t) f(s) \mathrm{d}t \mathrm{d}s \\
&= \int \|\lambda_t x - x\| \, f^*(t) \mathrm{d}t + \int \|x\lambda_s - x\lambda_s^* \lambda_s\| \, f(s) \mathrm{d}s,
\end{aligned}$$

which, whenever $x \in A$ will go to zero whenever if f ranges over an approximate identity of $L^1(G)$ of continuous functions with shrinking compact support approximating the dirach measure at 0 by continuity of the maps $t \mapsto \lambda_t x$ and $s \mapsto x\lambda_s$. □

Proposition 2.4.24. Let A be a G-product. If $x = \lambda_f^* y \lambda_f \in A$ for $f \in L^1(G) \cap L^2(G)$ and $y \in A$, then $I(x)$ satisfies Landstad's condition and x is the norm limit of elements given by

$$\int I(x\lambda_{-t})\lambda_t f_i(t) \mathrm{d}t,$$

for some net $(f_i) \subseteq L^1(G)$ satisfying $(\widehat{f_i}) \subseteq L^1(\widehat{G})$ is an approximate identity.

Proof.
Note first that given $x \in A$ of the form given above, then by Lemma 2.4.23, and x is $\widehat{\alpha}$-integrable and the map $t \mapsto I(x\lambda_t)$ is continuous in norm. We calculate

$$\begin{aligned}
I(x\lambda_{-t})\lambda_t = \int_{\widehat{G}} \widehat{\alpha}_\sigma(x\lambda_{-t})\lambda_t \mathrm{d}\sigma &= \int_{\widehat{G}} \widehat{\alpha}_\sigma(x\lambda_{-t})\overline{(t,\sigma)}\widehat{\alpha}_\sigma(\lambda_t) \mathrm{d}\sigma \qquad (2.18) \\
&= \int_{\widehat{G}} \widehat{\alpha}_\sigma(x)\overline{(t,\sigma)} \mathrm{d}\sigma, \qquad (2.19)
\end{aligned}$$

hence if $(f_i) \subseteq L^1(G)$ satisfies the conditions of the proposition, then

$$\int_G I(x\lambda_{-t})\lambda_t f_i(t) \mathrm{d}t = \int_{\widehat{G}} \int_G \widehat{\alpha}_\sigma(x)\overline{(t,\sigma)} f_i(t) \mathrm{d}t \mathrm{d}\sigma = \int_{\widehat{G}} \widehat{\alpha}_\sigma(x)\widehat{f_i}(-\sigma) \mathrm{d}\sigma \in A,$$

and in particular

$$\int_G I(x\lambda_{-t})\lambda_t f_i(t) \mathrm{d}t \to x.$$

in norm, since (f_i) can be chosen such that $(\widehat{f_i})$ converges uniformly to 1 on compact subsets of $\widehat{G}$, see e.g., [Kar18, Lemma 4.38]. To see that $I(x)$ satisfies Landstad's condition for each x of the above form, we first note that the first condition is immediately satisfied, since G is abelian and hence unimodular. For the second, let $g \in L^1(G)$ and $\varepsilon > 0$. Pick a symmetric neighborhood E of 0 satisfying

$$\|I(x\lambda_t) - I(x)\| < \varepsilon \text{ and } \|g - \delta_t \times g\|_1 < \varepsilon,$$

for all $t \in E$. Pick now positive $f \in L^1(G)$ satsifying $\|f\|_1 = 1$, $\operatorname{supp} f \subseteq E$ and $\widehat{f} \in L^1(\widehat{G})$, so that

$$\begin{aligned}
\|g - f \times g\|_1 = \int_G \left| g(s) - \int_G f(t) g(s-t) \mathrm{d}t \right| \mathrm{d}s &= \int_G \left| \int_G f(t)(g(s) - g(s-t)) \mathrm{d}t \right| \mathrm{d}s \\
&\leq \int_G f(t) \int_G |g(s) - g(s-t)| \mathrm{d}s \mathrm{d}t \\
&= \int_G f(t) \|g - \delta_t * g\| \,\mathrm{d}t \\
&< \varepsilon.
\end{aligned}$$

With this, we have

$$\begin{aligned}
\left\| I(x)\lambda_g - \int_G I(x\lambda_{-t})\lambda_t f(t)\mathrm{d}t\lambda_g \right\| &= \left\| I(x)\lambda_g - I(x)\lambda_f\lambda_g + I(x)\lambda_f\lambda_g - \int_G I(x\lambda_{-t})\lambda_t f(t)\mathrm{d}t\lambda_g \right\| \\
&\leq \|I(x)(\lambda_g - \lambda_f\lambda_g)\| + \left\| \int_G (I(x) - I(x\lambda_{-t}))\lambda_t f(t)\mathrm{d}t\lambda_g \right\| \\
&\leq \|I(x)\| \, \|g - f * g\|_1 + \int_G \|I(x) - I(x\lambda_{-t})\| \, f(t)\mathrm{d}t \, \|g\|_1 \\
&\leq (\|I(x)\| + \|g\|_1)\varepsilon,
\end{aligned}$$

from which we conclude, by arbitrariness of ε, that $I(x)\lambda_g \in A$. Similarly $\lambda_g I(x) \in A$. For the third condition, note that $\lambda_t I(x)\lambda_t^* = I(\lambda_t I(x)\lambda_t^*)$, and the map $t \mapsto I(\lambda_t x \lambda_t^*)$ is continuous by Lemma 2.4.23, finishing the proof. $\square$

The above proposition was the final ingredient in the theorem of Landstad, which shows that every G-product B is also a crossed product $A \rtimes_\alpha G$ for some C^*-algebra A with $G \overset{\alpha}{\curvearrowright} A$:

Theorem 2.4.25. *A C^*-algebra B is a G-product if and only if there exists a C^*-algebra A and an action $G \overset{\alpha}{\curvearrowright} A$ such that $B = A \rtimes_\alpha G$. The C^*-algebra A consists of exactly the elements of $M(B)$ satisfying Landstad's conditions, with $\alpha_t = \mathrm{Ad}\lambda_t|_A$, $t \in G$.*

Proof.
We have already seen one implication earlier, i.e., that a crossed product $A \rtimes_\alpha G$ is a G-product. For the converse, assume that B is a G-product and $A \subseteq M(B)$ be the set of

elements which satisfy Landstad's condition. The three conditions are all based on continuous operations in such a way that A is closed, this is immediate. It is easy to check that A is also a $*$-subalgebra, hence a C^*-subalgebra of $M(B)$.

The last of the Landstad conditions ensures that the map $t \mapsto \alpha_t := \mathrm{Ad}\lambda_t|_A$ is a strongly continuous action of G on A, so that (A, G, α) becomes a C^*-dynamical system.

Let $M(B)$ be faithfully represented on H for some Hilbert space. In particular, this means that $A \subseteq \mathbb{B}(H)$ so that we may identify $A \rtimes_\alpha G \subseteq \mathbb{B}(L^2(G, H))$. Note that by assumption that B is a G-product, the action $\widehat{G} \overset{\widehat{\alpha}}{\curvearrowright} B$ gives a faithful non-degenerate representation π of $B \rtimes_{\widehat{\alpha}} \widehat{G}$ on $\mathbb{B}(L^2(\widehat{G}, H))$ as well, given by

$$\pi(y)\eta(\sigma) = \widehat{\alpha}_{-\sigma}(y)\eta(\sigma),$$

for all $y \in B, \eta \in L^2(\widehat{G}, H)$ and $\sigma \in \widehat{G}$. This representation extends to a faithful representation $\overline{p}$ of $M(B)$ (by non-degeneracy) on $\mathbb{B}(L^2(\widehat{G}, H))$. We will also denote the extension by π. As in the proof of Lemma 2.4.14, let $u\colon L^2(G, H) \to L^2(\widehat{G}, H)$ denote the isometry

$$u\xi(\sigma) = \int_G \lambda_t \xi(t)\overline{(t, \sigma)}\, d\sigma \text{ and}$$
$$u^*\eta(t) = \lambda_{-t} \int_{\widehat{G}} \eta(\sigma)(t, \sigma)\mathrm{d}\sigma.$$

If $f \otimes x \in C_c(G, A)$, $f \in C_c(G)$ and $x \in A$, then an easy calculation shows that $(f \otimes x)u^*\eta(\sigma) = (u^*\pi(x\lambda_f)\eta)(t)$, so that

$$u(f \otimes x)u^* = \pi(x\lambda_f) \in \pi(B), \tag{2.20}$$

since $x\lambda_f \in B$. Density of the span of functions $f \otimes x$ in $A \rtimes_\alpha G$ shows that

$$u(A \rtimes_\alpha G)u^* \subseteq \pi(B).$$

For the converse, let $x \in B$ be $\widehat{\alpha}$-integrable of the form $\lambda_g^* y \lambda_g$, for some $g \in L^1(G) \cap L^2(G)$ and $y \in B$. By Proposition 2.4.24, we have $I(x) \in A$. For each $f \in C_c(G)$ and $t \in G$, we then calculate, using the identity of Equation (2.20):

$$u(I(x\lambda_{-t} \otimes \delta_t * f)u^* = \pi(I(x\lambda_{-t})\lambda_{\delta_t * f}) = \pi(I(x\lambda_{-t})\lambda_t\lambda_f).$$

Let $(f_i) \subseteq L^1(G)$ be a net satisfying the conditions of Lemma 2.4.23. Then,

$$x_i := \int_G I(x\lambda_{-t})\lambda_t f_i(t)\mathrm{d}t \to x,$$

in norm. By Proposition 2.4.24, we know that the map, $t \mapsto I(x\lambda_t) = I(\lambda_g^* y \lambda_g t)$, is a continuous map $G \to M(B)$. Continuity of $\mathrm{Ad}u$ and π plus the identity above gives that the map $t \mapsto \delta_t * f \otimes I(x\lambda_{-t})$ is continuous from $G \to C_c(G, A) \subseteq A \rtimes_\alpha G$. In particular, density of $C_c(G) \subseteq L^1(G)$ ensures that the integral of $t \mapsto (I(\delta_t * f) \otimes x\lambda_{-t}) \cdot f_i(t)$ makes sense for all i, so that

$$\begin{aligned} u^*\pi(x_i\lambda_f)u &= \int_G u^*(\pi(I(x_i\lambda_{-t}))\lambda_t\lambda_f)f_i(t))u\mathrm{d}t \\ &= \int_G (I(\delta_t * f) \otimes x_i\lambda_{-t})f_i(t)\mathrm{d}t \in A \rtimes_\alpha G. \end{aligned}$$

In the limit of i, we get that $u^*\pi(x\lambda_f)u \in A \rtimes_\alpha G$, implying by Proposition 2.4.24 that $u^*\pi(z\lambda_f)u \in A \rtimes_\alpha G$ for all $z \in B$. Letting f go through an approximate identity of $L^1(G)$ with shrinking compact support, we know that $z\lambda_f - z \to 0$ for all $z \in B$. Hence

$$u^*\pi(B)u \subseteq A \rtimes_\alpha G.$$

□

It is quite easy to show that the above C^*-dynamical system is unique up to covariant isomorphism, however we will not go into details and refer the reader to [OP78, Theorem 2.9]. The proof relies on the following result, which we also wont go into detail with, but include.

Lemma 2.4.26 ([OP78, Lemma 2.10])**.** Let (A, G, α) be a C^*-dynamical system and let $B \subseteq A$ be a G-invariant C^*-subalgebra of A. Then there is a natural injection $B \rtimes_\alpha G \subseteq A \rtimes_\alpha G$ which is surjective if and only if $B = A$.

We end this section with the promise that the results describes here, in particular Takai Duality, has concrete utilizations in describing the ideal structure of crossed products, even if they are not apparent just yet. In particular, in a series papers of Olesen and Pedersen (we have covered some of the theory of one of them), see e.g., [OP78], [OP80] and [OP82], Takai duality is used to describe a necessary and sufficient condition for simplicity of $A \rtimes_\alpha G$ in terms of a certain subgroup of $\widehat{G}$ and certain properties of the action $G \overset{\alpha}{\curvearrowright} A$. These particular articles established the foundation of much of the theory which we will examine in the next chapter.

2.5 An Important Example: Transformation Groups and actions on Spaces

Recall the example of $\mathbb{R} \overset{\theta}{\curvearrowright} C(S^1)$ via translation by $\theta \in \mathbb{R}$ from earlier. This is a more general construction of great importance which gives many great examples and ways to describe non-trivial examples of C^*-algebras and topological spaces. We will see that there is a correspondance between classic topological dynamic system and C^*-dynamical systems of abelian C^*-algebras and properties thereof.

Shortly put, we will describe C^*-dynamical systems associated with transformation groups, properties of actions on spaces and important examples of transformation groups including G-boundaries and the Furstenberg boundary.

Some of the theory discussed and covered in the following are commonly available in teaching books on the subject, for instance we use [Wil07] as an inspiration for the first part. Some theory is newer and not-so-common in teaching books, e.g., the theory of boundary actions. We will refer to the authors written projects (when possible) on these subjects as well as original source material on the subject.

2.5.1 Topological Dynamics

Throughout this chapter, we will let X denote an arbitrary locally compact Hausdorff space and G an arbitrary locally compact group.

Definition 2.5.1. We say that G acts on X, abbreviated $G \curvearrowright X$, if there is a jointly continuous map

$$\begin{aligned} G \times X &\to X \\ (t, x) &\mapsto t.x \in X \end{aligned}$$

such that for each fixed $t \in G$ the map $x \mapsto t.x$ is a homeomorphism of X and such that $s.(t.x) = (st).x$ and $e.x = x$ for all $s, t \in G$. Whenever $G \curvearrowright X$, the pair (X, G) is called a *Transformation Group* and X is called a *G-space*.

Example 2.5.2
If $h \in \text{Homeo}(X)$, then we obtain an action $\mathbb{Z} \curvearrowright X$, $n.x := h^n(x)$, $n \in \mathbb{Z}$, and we may always convert X into a $\mathbb{Z}$-space this way.

It is evident from the definition of transformation groups that we are interested in decribing continuous maps from G to $\text{Homeo}(X)$. To do this, we must describe the topology on it.

Definition 2.5.3. Given topological spaces X and Y, we endow $C(X, Y)$ with a topology called the *compact-open* topology. It is the topology generated by the subbasis of the form

$$V(K, U) := \{f \in C(X, Y) \ | \ f(K) \subseteq U\},$$

for $K \subseteq X$ compact and $U \subseteq Y$ open. The topology on $\text{Homeo}(X)$ is the topology such that $(h_i) \subseteq \text{Homeo}(X)$ converges to h if and only if $h_i \to h$ and $h_i^{-1} \to h^{-1}$ in the compact-open topology of $C(X, X)$.

Remark 2.5.4
A net $(f_i) \subseteq C(X,Y)$ converges in the compact-open topology if and only if $f_i(x_i) \to f(x)$ for whenever (x_i) is a net with the same index set converging to $x \in X$, see e.g. [Wil07, Lemma 1.30].

And, luckily for us, there is a nice correspondance between C^*-dynamical systems $(C_0(X), G, \alpha)$ and Transformation Groups (X, G):

Proposition 2.5.5. There is a bijective correspondance between C^*-dynamical systems $(C_0(X), G, \alpha)$ and Transformation groups (X, G) such that when $G \curvearrowright X$ the corresponding action $G \overset{\alpha}{\curvearrowright} C_0(X)$ is given by

$$\alpha_s(f)(x) := f(s^{-1}.x), \text{ for } f \in C_0(X), \ s \in G \text{ and } x \in X.$$

For a proof, see [Wil07, Lemma 1.33, 2.5 and Proposition 2.6].

Definition 2.5.6. If $G \curvearrowright X$, then for $x \in X$ the *stabilizer of* x, denoted by G_x, is the normal subgroup of G which fixes x, i.e., $G_x := \{t \in G \mid t.x = x\}$.

Remark 2.5.7
It is a normal subgroup, for if $t \in G_x$, then $t^{-1}.x = t^{-1}t.x = x$, and hence for $s \in G$ we have $sts^{-1}.x = st(st)^{-1}.x = x$.

We shall see that the stabilizer subgroups is an object of interest in the theory of crossed product later on. In the next section we will discuss a certain important example of G-spaces, called G-boundaries. In the last chapter we will show that these are important object, for instance we are going to connect the theory concerning G-boundaries to characterizations of simplicity of the reduced group C^*-algebra of discrete groups and various other results.

2.5.2 Boundary Actions and the Furstenberg boundary

The rest of this chapter is largely based on discussion and results covered authors written project (see [Kar17]), which covers and discusses (in a lesser generality) results of Furman and Glasner, see e.g., [Fur03] and [Gla76] respectively. We will not go through the proofs and techincal details, and instead offer a heuristic explanation with references as needed.

Definition 2.5.8. An action $G \curvearrowright X$ is said to be *minimal* if there are no non-trivial G-invariant closed subsets of X.

Remark 2.5.9
It is evident that the action $G \curvearrowright X$ being minimal is equivalent to every orbit $G.x := \{t.x \mid t \in G\}$ being dense in X. This will be used repeatedly.

Example 2.5.10
If $\theta \in \mathbb{R}\backslash\mathbb{Q}$, then the action $\mathbb{Z} \curvearrowright S^1$, $k.e^{2\pi it} := e^{2\pi i(t+k\theta)}$, is minimal.

See e.g., [Kar17, Example 4.6].

Definition 2.5.11. A subset $Y \subseteq X$ of a G-space is a minimal subset if it is non-empty, closed and G-invariant and contains no other non-empty closed G-invariant subsets.

Remark 2.5.12
It is evident that $Y \subseteq X$ is minimal if and only if the action restricts to a minimal action on Y. Moreover, an application of Zorn's lemma ensures that every compact G-space admits a minimal subset.

An easy application of abstract change-of-variable and dominated convergence shows that if we let $\mathrm{Prob}(X)$ denote the set of Radon probability measures on a compact G-space X, then we obtain a continuous action on $\mathrm{Prob}(X)$ of G as well (with respect to the weak*-topology of $C_0(X)^*$), which is given by $t.\mu(f) := \mu(\alpha_{t^{-1}}.f)$, where α is the induced action on $C_0(X)$ of G. This leads us the following:

Definition 2.5.13. Let X be a compact G-space. We say that $G \curvearrowright X$ is a *strongly proximal action* if for each $\mu \in \mathrm{Prob}(X)$ we have

$$\overline{G.\mu}^{w^*} \cap \{\delta_x \mid x \in X\} \neq \emptyset.$$

Note 2.5.14
Given a compact Hausdorff space X, the map $x \mapsto \delta_x$ is a homeomorphism between X and the point-mass measures on X as a subset of $\mathrm{Prob}(X)$. With this in mind, we can think of strongly proximal actions on compact spaces to correspond to having the ability to continuously deform subsets of spaces to points.

Definition 2.5.15. A compact G-space X is caled a *G-boundary* if the action is a *boundary action* which are minimal and strongly proximal actions.

The following result is arguably the most important result of the theory of boundary actions, and is due to Furstenberg and a proof can be found in [Kar17, Proposition 4.23].

Proposition 2.5.16. If X is a G-boundary and Y is a minimal compact G-space, then any continuous G-equivariant map $\varphi\colon Y \to \mathrm{Prob}(X)$ has $\delta_X \cong X$ as its range. If one such map exists, then it is unique.

Remark 2.5.17
The above proposition is equivalent to the statement that any G-equivariant to any injective u.c.p map $\psi\colon C(X) \to C(X)$ being an isomorphism of C^*-algebras. This is because for such ψ we may define a G-equivariant map $\psi_*\colon Y \to \mathrm{Prob}(X)$, $\psi_*(y)(f) = \psi(f)(y)$, $y \in Y$ and $f \in C(X)$, and this assignment is a one-to-one correspondance. See e.g., [Kar17, Lemma 4.16 and 4.17] for a proof.

For discrete countable groups, the theory of boundary actions and compact G-boundaries have strongly influenced the theory of classification of group C^*-algebras and other areas, for instance by E. Breuillard, M. Kalantar, M. Kennedy, N. Ozawa in [Bre+14]. If G is any (also non-discrete) group, then there is a particularly important G-boundary:

Definition 2.5.18. The *Furstenberg boundary* of a group G is the universal compact G-boundary, abbreviated $\partial_F G$, such that every other compact G-boundary X is the continuous image of a G-equivariant map $\partial_F G \to X$.

Both the existence and uniqueness is covered in [Kar17, Chapter 4]. For discrete countable groups, properties of $C(\partial_F G) \rtimes_\alpha G$ are intertwined with properties of G. We will discuss this further later on, and mention as an example that the free group on two generators gives rise to a boundary action on S^1, where (say $\mathbb{F}_2$ is generated by a, b)

$$a \mapsto (e^{2\pi i t} \mapsto e^{2\pi i(t+\theta)}) \text{ and } b \mapsto (e^{2\pi i t} \mapsto e^{2\pi i t^2}),$$

for $\theta \in \mathbb{R}\backslash\mathbb{Q}$, see e.g., [Kar17, Example 4.13].

As mentioned, the Furstenberg boundary has a lot of interesting and nice properties. It is an *extremally disconnected space* (meaning that the closure of open sets are open), see e.g. [Bre+14, Proposition 2.4] for a newer proof or [Ham79, p. 4.18] for the original proof.

We will get back to the connection of crossed product (in particular of the form $C(X) \rtimes_r G$ for G-boundaries) and boundary actions, where we will examine connections between the ideal structure of $C^*_r(G)$ and $A \rtimes_r G$ for certain C^*-algebras.

The Group Spectra And Applications

As promised, we will now begin describing the theory, both old and new, which is the main topic of this thesis: We will be describing the ideal structure of crossed products $A \rtimes_\alpha G$ for topological groups G. Since we will describe theory which only describes dynamical systems (A, G, α) where G is abelian and also theory which describes the general case, we introduce the notation that a C^*-dynamical system (A, G, α) is a *C^*-dynamic a-system* if G is abelian.

The theory we will now cover provided some of the initial ways of characterizing the ideal structure (and lack thereof) in crossed products of C^*-algebras and topological groups. It also provided a new angle to tackle some problems and led to the discovery of other results which paved the way for more modern results.

A main motivation of the following can be seen as the following: Given an abelian group G acting on a C^*-algebra A via an action $G \overset{\alpha}{\curvearrowright} A$, can we describe the ideal structure of $A \rtimes_\alpha G$ by means of properties of G and properties of A? We will develop a theory which allows us to answer this question for a large class of C^*-algebras and groups.

3.1 The Arveson and Connes spectra

For the time being, we will unless otherwise stated, assume that G is abelian with dual group $\widehat{G}$. Recall that whenever we have a C^*-dynamical a-system (A, G, α), we also obtain an association of $L^1(G)$ as operators on A, via to defining to each $f \in L^1(G)$ the operator α_f on A satisfying

$$\varphi(\alpha_f(a)) = \int_G \varphi(\alpha_t(x)) f(t) \mathrm{d}t, \text{ for all } a \in A \text{ and } \varphi \in A^*.$$

This association extends also to the set of finite Radon measures on G, $M(G)$. We say that α_f and α_μ are the *weak*-integrals* of f and μ, respectively. For a more detailed exposition of vector-valued integration, see e.g., [Ped79, Appendix 3, Lemma 7.4.4] or [Fol16, Appendix A.3].

For a group G we define the subset $C_c^1(G)$ of $C_c(G)$ by

$$C_c^1(G) := \{f \in C_c(G) \mid \widehat{f} \in C_c(\widehat{G})\},$$

which is a dense ideal of $L^1(G)$, see [Fol16, Theorem 4.60].

Definition 3.1.1. Let (A, G, α) be a C^*-dynamical a-system. For open sets $U \subseteq \widehat{G}$, we define the *spectral R-subspace* $R^\alpha(U)$ to be the subspace of A defined by

$$R^\alpha(U) := \overline{\operatorname{span}\left\{\alpha_f(a) \mid a \in A, f \in C_c^1(G) \text{ with } \operatorname{supp} \widehat{f} \subseteq U\right\}}^w,$$

and similarly, for each closed subset $K \subseteq \widehat{G}$ we define the *spectral M-subspace* $M^\alpha(K)$ to be the subspace of A defined by

$$M^\alpha(K) := \overline{\operatorname{span}\left\{x \in A \mid \alpha_f(x) = 0 \text{ for all } f \in C_c^1(G) \text{ with } \operatorname{supp} \widehat{f} \subseteq K^c\right\}},$$

By [Ped79, Theorem 8.1.4], it follows that the R- and M-spaces are G-invariant subsets, and

$$\bigcap_i M^\alpha(K_i) = M^\alpha\left(\bigcap_i K_i\right),$$

for all collections of subsets (K_i) of $\widehat{G}$. This allows us to give the following definition:

Definition 3.1.2. The *Arveson Spectrum* of an action $G \overset{\alpha}{\curvearrowright} A$ is defined to be the smallest closed subset K of $\widehat{G}$ such that $M^\alpha(K) = A$, and we denote it by $\operatorname{Sp}(\alpha)$.

In his dissertation (see [Con73]), Alain Connes proved a series of equivalent characterizations of $\operatorname{Sp}(\alpha)$, which we will need. We include it without proof and instead refer the reader to [Ped79, Proposition 8.1.9 and 8.1.8] for the proof.

Proposition 3.1.3. Let (A, G, α) be a C^*-dynamical a-system. Then for $\sigma \in \widehat{G}$ the following are equivalent:

1. $\sigma \in \operatorname{Sp}(\alpha)$,
2. For all neighborhoods U of σ: $R^\alpha(U) \neq 0$,
3. There is a net $(a_i) \subseteq A$, $\|a_i\| = 1$, such that $\|\alpha_t(x_i) - (t, \sigma)x_i\| \to 0$ uniformly on compacts of G,
4. For every finite Radon measure μ on G: $|\widehat{\mu}(\sigma)| \leq \|\alpha_\mu\|$,
5. For every $f \in L^1(G)$: $|\widehat{f}(\sigma)| \leq \|\alpha_f\|$,
6. If $f \in L^1(G)$, $\alpha_f = 0$, then $\widehat{f}(\sigma) = 0$ and
7. It holds that $M^\alpha(\{\sigma\}) := \{x \in A \mid \alpha_t(x) = (t, \sigma)x \text{ for all } t \in G\} \neq 0$.

One common use of the above proposition is to take the last condition as the definition:

$$\mathrm{Sp}(\alpha) := \left\{\chi \in \widehat{G} \mid \widehat{f}(\chi) = 0 \text{ whenever } f \in L^1(G) \text{ satisfies } \alpha_f(x) = 0 \text{ for all } x \in A\right\}.$$

We will frequently switch between the above equivalent conditions.

The main motivation (in our case) for introducing the Arveson spectrum is to define the Connes Spectrum. For this, we remind the reader of the concept of hereditary sub-C^*-algebras:

Definition 3.1.4. We say that a C^*-subalgebra $B \subseteq A$ is *hereditary* if for all positive $x \in A_+$ we have $x \in B_+$ whenever $y \in B_+$ with $0 \leq x \leq y$.

We define $\mathscr{H}(A)$ to be the set of non-zero hereditary C^*-subalgebras of A.

If furthermore we assume that (A, G, α) is a C^*-dynamical system, we define $\mathscr{H}^\alpha(A)$ to be the set

$$\mathscr{H}^\alpha(A) := \{D \in \mathscr{H}(A) \mid \alpha_g(D) = D \text{ for all } g \in G\}.$$

Remark 3.1.5
There is a bijection between the set of closed left-ideals of a C^*-algebra A and the set of hereditary C^*-subalgebras of A which maps a closed left-ideal L to $L^* \cap L$, see e.g. [Bla06, p. II.5.3.2].

This bijection will be useful soon, since we may define the Connes spectrum of an action $G \overset{\alpha}{\curvearrowright} A$ now:

Definition 3.1.6. The *Connes Spectrum* of an action $G \overset{\alpha}{\curvearrowright} A$ is the subset $\Gamma(\alpha)$ of $\widehat{G}$ given by

$$\widehat{G}(\alpha) := \bigcap_{D \in \mathscr{H}^\alpha(A)} \mathrm{Sp}(\alpha|_D).$$

3.2 The Connes Spectrum and Simplicity

As we shall see, the Connes spectrum serves as a useful tool in the theory of classification of crossed products. It both serves as an invariant, but also as a way to characterize the ideal structure for a large class of crossed products, and served as a bridge between different results and theorems. For instance it is an invariant under an equivalence relation on the class of G-C^*-dynamical systems called *exterior equivalence* (yet to be defined) and the Connes spectrum of the dual system detects ideals in A.

But first, a lemma which we will use several times throughout this chapter:

Lemma 3.2.1. Let A be a C^*-algebra and let I be a non-zero ideal of A. Then there are non-zeor positive elements $x, y \in I$ such that $yx = y$.

Proof.
Let $z \in I_+$, $z \neq 0$. Pick $0 < \varepsilon < \|z\|$, and note that I is hereditary in the unitization of A, so $z - \varepsilon \in I_+$. Consider the two continuous functions on $\mathbb{R}$, h_ε and f_ε, given by

$$f_\varepsilon(t) = \begin{cases} 0, & t \leq 0 \\ \frac{t}{\varepsilon}, & 0 \leq t \leq \varepsilon \\ 1 & \text{else} \end{cases}, \quad h_\varepsilon(t) = \begin{cases} 0, & t < \varepsilon \\ t - \varepsilon, & t \geq \varepsilon \end{cases}$$

Setting $y := h_\varepsilon(z)$ and $x = f_\varepsilon(z)$ gives the desired result by the continuous functional calculus. $\square$

Remark 3.2.2
The proof of the following result due to Olesen and Pedersen, relies on the theory of W^*-dynamical systems, which quite similar to that of C^*-dynamical systems. We will use some properties without proof, as it will take us too much off-course to develop in depth, and instead refer the reader to relevant material on the subject whenever needed.

Definition 3.2.3. A W^**-system* is a triple $(\mathscr{M}, G, \beta)$ where $\mathscr{M}$ is a von Neumann Algebra, G a locally compact Hausdorff group and $\beta \colon G \to \mathrm{Aut}(\mathscr{M})$ is group homomorphism which is continuous with respect to the pointwise weak topology on $\mathrm{Aut}(\mathscr{M})$.

We refer the curious reader to [Ped79, Chapter 7.4 and 7.10] for a thorough exposition of the subjects.

Proposition 3.2.4. Let (A, G, α) be a C^*-dynamical a-system. Then $t \in G(\widehat{\alpha})$ if and only if $I \cap \alpha_t(I) \neq 0$ for all non-zero ideals I of A.

Proof.
We will be considering $A \rtimes_\alpha G$ as a G-product, so that $A \subseteq M(A \rtimes_\alpha G)$ is the set of elements satisfying Landstad's conditions. Assume that $I \cap \alpha_t(I) = 0$ for non-zero ideals I of A. Using lemma 3.2.1 pick elements $0 \neq x, y \in I_+$ be such that $yx = y$. Let K be a compact neighborhood of $0 \in G$ such that $\|\alpha_s(x) - x\| < 1$ for $s \in K$. Then $1 - (\alpha_s(x) - x)$ is invertible in the unitization A^1 of A with inverse $\sum_{n=0}^\infty (\alpha_s(x) - x)^n$. We see that

$$\alpha_s(y) = \alpha_s(y)(1 - (\alpha_s(x) - x)) \sum_{n=0}^\infty (\alpha_s(x) - x)^n = \alpha_s(y) x \sum_{n=0}^\infty (\alpha_s(x) - x)^n \in I,$$

as $\alpha_s(y) - \alpha_s(yx) = 0$, whence

$$ya\alpha_{t-s}(y) = \alpha_{-s}(\alpha_s(y)\alpha_s(a)\alpha_t(y)) \in \alpha_{-s}(IA\alpha_t(I)) = 0,$$

for all $a \in A$ and $s \in K$.

Every y in A satisfies $\widehat{\alpha}_\sigma(y) = y$, $\sigma \in \widehat{G}$, by Landstad's condition. Hence $B := C^*(\{yzy \mid z \in A \rtimes_\alpha G\}) \subseteq A \rtimes_\alpha G$ is an $\widehat{\alpha}$-invariant hereditary C^*-subalgebra, i.e., $B \in \mathscr{H}^{\widehat{\alpha}}(A \rtimes_\alpha G)$. We claim that $t \notin \mathrm{Sp}(\widehat{\alpha}|_B)$: Pick any $g \in L^1(G)$ with $\mathrm{supp}\widehat{g} \subseteq t - K$. To see that $t \notin \mathrm{Sp}(\widehat{\alpha}|_B)$, it suffices to show that $\widehat{\alpha}_g|_B = 0$, as the Fourier transform densely embed $L^1(\widehat{G})$ in $C_0(G)$. To see this, let $b \in B$ be of the form

$$b = y\lambda_f^* a \lambda_f y,$$

for some $f \in L^1(G) \cap L^2(G)$ and $a \in A \rtimes_\alpha G$, so that b is $\widehat{\alpha}$-integrable by note 2.4.19. Then, by the Fourier Inversion formula, we have

$$\begin{aligned}
\widehat{\alpha}_g(b) &= \int_{\widehat{G}} y\widehat{\alpha}_\sigma(\lambda_f^* a\lambda_f) y g(\sigma)\mathrm{d}\sigma \\
&= \int_{\widehat{G}}\int_G y\widehat{\alpha}_\sigma(\lambda_f^* a\lambda_f)\overline{(s,\sigma)} y\widehat{g}(s)\mathrm{d}s\mathrm{d}\sigma \\
&= \int_{\widehat{G}}\int_G y\widehat{\alpha}_\sigma(\lambda_f^* a\lambda_f)\lambda_{-s}\lambda_s\overline{(s,\sigma)} y\widehat{g}(s)\mathrm{d}s\mathrm{d}\sigma \\
&= \int_{\widehat{G}}\int_G y\widehat{\alpha}_\sigma(\lambda_f^* a\lambda_f\lambda_{-s})\lambda_s y\widehat{g}(s)\mathrm{d}s\mathrm{d}\sigma \\
&= \int_G y\left(\int_{\widehat{G}} \widehat{\alpha}_\sigma(\lambda_f^* a\lambda_f\lambda_{-s}\right)\lambda_s \mathrm{d}\sigma y\widehat{g}(s)\mathrm{d}s \\
&= \int_G yI(\lambda_f^* a\lambda_f\lambda_{-s})\lambda_s y\widehat{g}(s)\mathrm{d}s \\
&= \int_G yI(\lambda_f^* a\lambda_f\lambda_{-s})\alpha_s(y)\lambda_s\widehat{g}(s)\mathrm{d}s = 0,
\end{aligned}$$

where we used the identity of eq. (2.19) and covariance of (α_s, λ_s). The last equality follows from $I(\lambda_f^* a\lambda_f\lambda_{-s}) \in A$ by proposition 2.4.24 and theorem 2.4.25 combined with the fact that $ya'\alpha_s(y) = 0$ for all $a' \in A$ and $s \in \mathrm{supp}\widehat{g}$ by the above.

The set of elements of the form $y\lambda_f^* a\lambda_f y$ spans a dense subset of B by lemma 2.4.23, as every $x \in A$ may be approximated by $\lambda_f^* x\lambda_f$ in norm. Thus $\widehat{\alpha}_g|B \neq 0$, and since g can be chosen such that $\widehat{g}(t) \neq 0$, we conclude that $t \notin \mathrm{Sp}(\widehat{\alpha}|_B)$, implying in particular that $t \notin G(\widehat{\alpha})$.

For the converse, assume $t \notin G(\widehat{\alpha})$, and let U be a neighborhood of t and $B \in \mathscr{H}^{\widehat{\alpha}}(A \rtimes_\alpha G)$ witness this:

$$\widehat{\alpha}_g|B = 0 \text{ whenever } g \in L^1(\widehat{G}) \text{ with } \mathrm{supp}\widehat{g} \subseteq U.$$

Choose a neighborhood $U' \subseteq U$ and a neighborhood U_0 of 0 such that $U' + U_0 - U_0 \subseteq U$. Suppose that $f_1, f_2 \in L^1(G)$, $\mathrm{supp} f_i \subseteq U_0$. If $g \in L^1(\widehat{G})$ with $\mathrm{supp}\widehat{g} \subseteq U'$, then for each $s \in U_0 - U_0$ we have

$$(\widehat{(s,\cdot)g})(t) = \int_{\widehat{G}} (t+s,\sigma)g(\sigma)\mathrm{d}\sigma = \widehat{g}(s+t) \neq 0 \iff t \in \mathrm{supp}\widehat{g} - s \subseteq U' + U_0 - U_0 \subseteq U. \tag{3.1}$$

For $s \in G$, let $sg := (s,\cdot)g$ and see that the above shows that $\widehat{\alpha}_{sg} = 0$ on B. Now, for $y \in B$ we see that

$$\widehat{\alpha}_g(\lambda_{f_1}^* y \lambda_{f_2}) = \int_{G\times G\times\widehat{G}} \lambda_{-r}\widehat{\alpha}_\sigma\lambda_s(s-r,\sigma)g(\sigma)\overline{f_1}(r)f_2(s)\mathrm{d}s\mathrm{d}r\mathrm{d}\sigma \tag{3.2}$$

$$= \int_{G\times G\times\widehat{G}} \alpha_{-r}(\widehat{\alpha}_\sigma(y))\lambda_{-r}\lambda_{s+r}(s,\sigma)g(\sigma)\overline{f_1}(r)f_2(s+r)\mathrm{d}s\mathrm{d}r\mathrm{d}\sigma \tag{3.3}$$

$$= \int_G\int_G \alpha_{-r}(\widehat{\alpha}_{sg}(y))\lambda_s\overline{f_1}(r)f_2(s+r)\mathrm{d}s\mathrm{d}r \tag{3.4}$$

$$= 0, \tag{3.5}$$

for if $r \in \mathrm{supp} f_1$ and $s + r \in \mathrm{supp} f_2$, then $s \in U_0 - U_0$ so $\widehat{\alpha}_{sg}(y) = 0$.

Since B is a hereditary C^*-subalgebra, there is a left ideal of $A \rtimes_\alpha G$, which necessarily must be $\widehat{G}$-invariant by invariance of B, such that $B = L^* \cap L$.

Pick an arbitrary non-zero $f \in C_c(G)$ with $\mathrm{supp} f \subseteq U_0$ and define

$$L_0 := \overline{\mathrm{span}\left\{\bigcup_{\sigma\in\widehat{G}} L\lambda_{\sigma f}\right\}},$$

so that L_0 is a closed $\widehat{G}$-invariant left ideal of $A \rtimes_\alpha G$. Define now the a hereditary C^*-subalgebra $B_0 := L_0^* \cap L_0 \in \mathscr{H}^{\widehat{\alpha}}(A \rtimes_\alpha G)$.

Since f was chosen such that $\mathrm{supp}\, f \subseteq U_0$, if $\sigma_1, \sigma_2 \in \widehat{G}$, then for all $g \in L^1(G)$ with $\mathrm{supp}\,\widehat{g} \subseteq U'$ and $y_1, y_2 \in L$ we have

$$\widehat{\alpha}_g(\lambda_{\sigma_1 f}^* y_1^* y_2 \lambda_{\sigma_2 f}) = 0,$$

by Equations (3.1) and (3.5). Since B_0 is densely spanned by elements of the form evaluated in the previous equation, this implies that $t \notin \mathrm{Sp}(\widehat{\alpha}|_{B_0})$.

Let $\mathscr{M}_0$ denote the strong operator closure of B_0 in $\mathbb{B}(L^2(G,H))$, and let β denote a σ-weakly extension of $\widehat{\alpha}|_{B_0}$ to $\mathscr{M}_0$ making $(\mathscr{M}_0, \widehat{G}, \beta)$ into a W^*-dynamical system. We define $R^\beta(\Omega)$ to be σ-weak closure of $R^{\widehat{\alpha}}(\Omega)$ for open sets $\Omega \subseteq \widehat{G}$. Then $\mathrm{Sp}(\widehat{\alpha}|_{B_0}) = \mathrm{Sp}(\beta|_{\mathscr{M}_0})$ cf. e.g. [Ped79, Proposition 8.8.9]. In particular, $t \notin \mathrm{Sp}(\beta|_{\mathscr{M}_0})$.

Since B_0 is the closure of a set of integrable positive elements (see lemma 2.4.23), let $0 \neq y_0 \in B_0$ be a positive and integrable element and let $x_0 := I(y_0) \in A$. Then $x_0 \in \mathscr{M}_0$: Let $g_i \in C_c(\widehat{G})$ be a net of positive functions such that $g_i(\sigma) \to 1$ for all $\sigma \in \widehat{G}$ and set $x_i := \widehat{\alpha}_{g_i}(y_0) = \int_{\widehat{G}} \widehat{\alpha}_\sigma(y_0) g_i(\sigma) \mathrm{d}\sigma$. It is easy to see that x_i increases in strongly to x_0.

If $x \in A$, then $x_0 x \lambda_t x_0 = \lim_{i-\text{strong}} x_i x \lambda_t x_i \in \mathscr{M}_0$, and hence

$$\beta_\sigma(x_0 x \lambda_t x_0) = (t, \sigma) x_0 x \lambda_t x_0 = x_0 x \widehat{\alpha}_\sigma(\lambda_t) x_0 = \widehat{\alpha}_\sigma(x_0 x \lambda_t x_0),$$

for all $\sigma \in \widehat{G}$, by Landstad's condition and the properties of a G-product. However, the only element which satisfies $\widehat{\alpha}_\sigma(a) = (t,\sigma)a$ is 0, as $t \notin \mathrm{Sp}(\widehat{\alpha}|_{B_0})$. In particular this implies that

$$x_0 x \alpha_t(x_0) = (x_0 x \lambda_t x_0)\lambda_{-t} = 0,$$

for all $x \in A$. If $I = \overline{(x_0)} \subseteq A$ is the smallest ideal containing x_0, then $0 \neq I \subseteq A$ with $I \cap \alpha_t(I) = I \cdot \alpha_t(I) = 0$. ☐

We now define an equivalence on the class of (A,G)-C^*-dynamical systems, i.e., the class of C^*-dynamical systems (A, G, α) for fixed (A, G).

Definition 3.2.5. We say that the C^*-dynamical systems (A, G, α) and (A, G, α') are *exterior equivalent* if there is a C^*-dynamical system $(M_2(A), G, \lambda)$ such that

$$\lambda_t(x \oplus y) = \alpha_t(x) \oplus \alpha'_t(y),$$

for all $x, y \in A$ and $t \in G$.

Lemma 3.2.6. A necessary and sufficient condition for (A, G, α) and (A, G, α') being exterior equivalent is that there exists a strongly continuous map $G \to \mathcal{U}M(A)$ such that

1. $u_{s+t} = u_s \alpha_s(u_t)$ for all $s, t \in G$ and
2. $\alpha'_t = (\mathrm{Ad} u_t)\alpha_t$.

Proof.
Suppose that (A, G, α) and (A, G, α') are exterior equivalent. Extend λ to $M(M_2(A))$, and define for each $t \in G$ the operator

$$m_t := \lambda_t \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

We then see that

$$m_t^* m_t = \lambda_t \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

since $\lambda_t \in \operatorname{Aut}(M(M_2(A)))$. Similarly $m_t m_t^* = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. It follows that $m_t = \begin{pmatrix} 0 & 0 \\ u_t & 0 \end{pmatrix}$ for some unitary operator $u_t \in M(A)$, $t \in G$. It is easy to verify that u_t satisfies the desired properties: let $s, t \in G$, then

$$\begin{pmatrix} 0 & 0 \\ u_{s+t} & 0 \end{pmatrix} = \lambda_s \left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_t & 0 \\ 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & 0 \\ u_t & 0 \end{pmatrix} \begin{pmatrix} \alpha_s(u_t) & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ u_s \alpha_s(u_t) & 0 \end{pmatrix}.$$

The other properties follows analogously. For the converse, define λ_t on $M_2(A)$ by

$$\lambda_t \begin{pmatrix} a & b \\ c & d \end{pmatrix} := \begin{pmatrix} \alpha_t(a) & \alpha_t(b) u_t^* \\ u_t \alpha_t(c) & \alpha_t'(d) \end{pmatrix}$$

It is straight forward to check that this is an action fulfilling the desired requirements. □

Traditionally, the second condition was called exterior equivalence while the first is called the *Cocycle equation*. In his thesis ([Con73, p. 2.2.5]), Connes showed that whenever $p \sim q$ are G-invariant equivalent projections in $M(A)$ (in the traditional sense, i.e., $p = v^* v$ and $q = vv^*$ for a partial isometry v), then

$$\widehat{G}(\alpha|_{pAp}) = \widehat{G}(\alpha|_{qAq}),$$

see also [OP78, lemma 4.3]. This is particularly useful for us, since we obtain:

Proposition 3.2.7. If (A, G, α) and (A, G, α') are exteriorly equivalent, then $\widehat{G}(\alpha) = \widehat{G}(\alpha')$.

Proof.
Let $p = 1 \oplus 0$ and $q = 0 \oplus 1$ in $M_n(M(A))$. Then $p \sim q$ and both are G-invariant, and clearly $\alpha = \gamma|_{p(M_2(A))p}$ and $\alpha' = \gamma|_{q(M_2(A))q}$, so

$$\widehat{G}(\alpha) = \widehat{G}(\gamma|_{p(M_2(A)p)}) = \widehat{G}(\gamma|_{q(M_2(A))q}) = \widehat{G}(\alpha').$$

□

Remark 3.2.8
If (A, G, α) and (B, G, β) are covariantly isomorphic C^*-dynamical systems, then (after identifying A with B), we see that (A, G, α) and (A, G, β) are trivially exteriorly equivalent.

Theorem 3.2.9. Let (A, G, α) be a C^*-dynamical a-system. Then $\widehat{G}(\alpha) = \widehat{G}(\widehat{\widehat{\alpha}})$ for (A, G, α) and $(A \rtimes_\alpha G \rtimes_{\widehat{\alpha}} \widehat{G}, G, \widehat{\widehat{\alpha}})$.

Proof.
First, let I be the trivial representation of G on $\mathbb{K}(L^2(G))$. Then $\widehat{G}(\alpha) = \widehat{G}(\alpha \otimes I)$: Let $\{\xi_i\}$

be a basis of $L^2(G)$, and let p_i be the minimal projection onto $\mathbb{C}\xi_i$. Let ι_i be the hereditary embedding of A into $A \otimes \mathbb{K}(L^2(G))$, $a \mapsto a \otimes p_i$. Since $A \subseteq A \otimes \mathbb{K}(L^2(G))$ as a hereditary C^*-subalgebra, $\widehat{G}(\alpha \otimes I) \subseteq \widehat{G}(\alpha)$, as given a projection in p a C^*-algebra A, the cut-off C^*-algebra $pAp \subseteq A$ is hereditary, see e.g., [Bla06, Example II.7.3.14].

For the other inclusion, we claim that given any non-zero G-invariant hereditary C^*-subalgebra $B \subseteq A \otimes \mathbb{K}(L^2(G))$ it must hold that $\iota_i(A) \cap B \neq 0$ for some i. To see this, given $0 < b \in B$, define $b_i := (1 \otimes p_i) b (1 \otimes p_i)$. Then $b_i \leq b$ as $b - b_i = (1 \otimes I - p_i) b (1 \otimes I - p_i) \geq 0$, so $b_i \in B$ hence $b_i \in \iota_i(A) \cap B = 0$ for all i. From this we infer that $b_i \eta \otimes \xi_i = 0$ for all i and all $\eta \in H$, where $A \subseteq \mathbb{B}(H)$. Since $\{\xi_i\}$ is a basis, this implies that $b = 0$, showing the claim. Then $\mathrm{Sp}(\alpha \otimes I|_B) \supseteq \mathrm{Sp}(\alpha|_{\iota_i^{-1}(B \cap \iota_i(A))})$, so that $\widehat{G}(\alpha \otimes I) \supseteq \widehat{G}(\alpha)$.

We already know that $\widehat{G}(\widehat{\widehat{\alpha}}) = \widehat{G}(\alpha \otimes \mathrm{Ad}\rho)$ by Takai duality. Hence it suffices to show that

$$(A \otimes \mathbb{K}(L^2(G)), G, \alpha \otimes I) \sim_{\text{ext}} (A \otimes \mathbb{K}(L^2(G)), G, \alpha \otimes \mathrm{Ad}\rho).$$

To see this, let $u_t := 1 \otimes \rho_t \in M(A \otimes \mathbb{K}(L^2(G)))$. Then for each $s, t \in G$

$$(\alpha_s \otimes I)(u_t) = \alpha_s(1) \otimes \rho_t = 1 \otimes \rho_t = u_t \text{ implying that } u_{s+t} = u_s (\alpha_s \otimes I)(u_t).$$

Also $\alpha \otimes \mathrm{Ad}\rho = \mathrm{Ad}u \circ \alpha \otimes I$, and the diagonal representation $t \mapsto u_t$ is strongly continuous, whence we conclude that

$$\widehat{G}(\widehat{\widehat{\alpha}}) = \widehat{G}(\alpha \otimes \mathrm{Ad}\rho) = \widehat{G}(\alpha \otimes I) = \widehat{G}(\alpha),$$

finishing the proof. □

This gives us the first ingredient in determining the ideal structure of $A \rtimes_\alpha G$ via means of the Connes spectrum:

Corollary 3.2.10. Let (A, G, α) be a C^*-dynamical a-system. Then $\sigma \in \widehat{G}(\alpha)$ if and only if $I \cap \widehat{\alpha}_\sigma(I) \neq 0$ for all non-zero ideals $I \subseteq A \rtimes_\alpha G$.

Proof.
Apply proposition 3.2.4 to $\widehat{\widehat{\alpha}}$ and combine it with theorem 3.2.9 to obtain the desired result. □

Definition 3.2.11. Let $G \overset{\alpha}{\curvearrowright} A$. We say that A is *G-prime* if whenever I, J are non-zero G-invariant ideals in A it holds that $I \cap J \neq 0$.

Remark 3.2.12
Analogously with the regular definition of a prime C^*-algebra, the definition of A being G-prime is equivalent to every non-zero G-invariant ideal of A being essential or if

$\alpha_t(x)A\alpha_s(y) = 0$ for all $s,t \in G$ implies $x = 0$ or $y = 0$. We will not go into details with this, the details are almost identical to the ones for regular primitivity of A.

Also worth noting is that if (A, G, α) and (B, Γ, β) are covariantly isomorphic, then A is G-prime if and only if B is Γ-prime, obviously.

With this, we obtain:

Lemma 3.2.13. Let (A, G, α) be a C^*-dynamical a-system. Then the following are equivalent:

1. A is prime;
2. (a) A is G-prime and (b) $G(\widehat{\alpha}) = G$.

Proof.
That A is prime implies A being G-prime is obvious, and (b) follows directly from proposition 3.2.4, as $\alpha_t(I)$ is a non-zero ideal of A whenever I is.

For the converse, assume towards a contradiction that (a) and (b) holds and A is not prime, and let I, J be non-zero ideals of A such that $I \cap J = 0$. If $t \in G$ is such that $\alpha_t(I) \cap J \neq 0$ then setting $L := \alpha_t(I)) \cap J$ we must have $L \cap \alpha_{-t}(L) \neq 0$ as $-t \in G(\widehat{\alpha})$. However, $L \cap \alpha_{-t}(L) \subseteq J \cap I = 0$, so $\alpha_t(I) \cap J = 0$. Let I_G be the smallest ideal containing $\bigcap_{t\in G} \alpha_t(I)$ and note that in particular I_G is a non-zero G-invariant ideal of A, hence essential by (a). Then $I_G \cap J = 0$, but since I_G is essential in A, this implies that $J = 0$, a contradiction, so A is prime. ☐

Lemma 3.2.14. Let (A, G, α) be a C^*-dynamical a-system. Then A is G-prime if and only if $A \rtimes_\alpha G$ is $\widehat{G}$-prime.

Proof.
It is quite easy to see that if I, J are non-zero $\widehat{G}$-invariant ideals of $A \rtimes_\alpha G$ with $I \cap J = 0$, then whenever $f \in L^1(G) \cap L^2(G)$, $\|f\|_1 \neq 0$ and $x \in I_+$ and $y \in J_+$ are non-zero, then by proposition 2.4.24 we have $a := I(\lambda_f^* y \lambda_f) \in A$ and $b := I(\lambda_f^* x \lambda_f)$ since A are the elements of $M(A \rtimes_\alpha G)$ satisfying Landstad's condition.

It is not hard to see that $azb = 0$ for all $z \in M(A \rtimes_\alpha G)$ (in particular for all $z \in A \subseteq M(A \rtimes_\alpha G)$) since:

1. For $z \in M(A \rtimes_\alpha G)$ we have $z\lambda_f^* y \lambda_f \in J$
2. The ideals are $\widehat{G}$-invariant

which together implies that

$$\widehat{\alpha}_\sigma(\lambda_f^* y \lambda_f) z \widehat{\alpha}_\chi(\lambda_f^* x \lambda_f) = 0$$

for all $z \in M(A \rtimes_\alpha G)$. In particular, by covariance of λ and α, we have

$$\alpha_s(a) z \alpha_t(b) = \lambda_s a(\lambda_{-s} z \lambda_t) b \lambda_{-t} = 0,$$

so if I' and J' are the smallest ideals generated by $\{\alpha_s(a) \mid s \in G\}$ and $\{\alpha_s(b) \mid s \in G\}$, respectively, which must be G-invariant, then $I' \cdot J' = 0$ implying that A is not G-prime.

For the converse, given non-zero G-invariant ideals I, J of A with $I \cap J = 0$. Setting $J' := J \otimes \mathbb{K}(L^2(G))$ and $I' := I \otimes \mathbb{K}(L^2(G))$, then I' and J' are G-invariant ideals of $A \otimes \mathbb{K}(L^2(G))$ with respect to the action $\alpha \otimes \mathrm{Ad}\rho$ and clearly satisfying $J' \cap I' = 0$. By Takai duality, this implies that $A \rtimes_\alpha G$ is not $\widehat{G}$-prime. □

Combining the previous round of results, we obtain a nice result:

Corollary 3.2.15. For a C^*-dynamical a-system (A, G, α), the following are equivalent:

1. $A \rtimes_\alpha G$ is prime;
2. (a) A is G-prime and (b) $\widehat{G}(\alpha) = \widehat{G}$.

Analogously to the case of primitivity and G-primitivity, we introduce the notion of G-simplicity of a dynamical system, which is closely related to minimality of actions on spaces:

Definition 3.2.16. Let (A, G, α) be a C^*-dynamical system. We say that A is *G-simple* if there are no non-trivial G-invariant ideals J of A.

It is clear that G-simple is a stronger requirement than being G-prime and is more closely related to simplicity of A. We will now see that the set of G-invariant ideals of A is in bijection with the set of $\widehat{G}$-invariant ideals of the crossed product $A \rtimes_\alpha G$, and list a necessary and sufficient condition for simplicity of $A \rtimes_\alpha G$ covering a large class of groups, but first, we have the following:

Proposition 3.2.17. Given a C^*-dynamical a-systems (A, G, α), it holds that A is G-simple if and only if $A \rtimes_\alpha G$ is $\widehat{G}$-simple.

Proof.
We begin by showing that if $A \rtimes_\alpha G$ is not $\widehat{G}$-simple then A is G-simple: Suppose towards a contradiction that $A \rtimes_\alpha G$ is not $\widehat{G}$-simple, but A is G-simple and choose a non-trivial $\widehat{G}$-invariant ideal J witnessing this. Then there is some state $\varphi \in \mathcal{S}(A \rtimes_\alpha G)$ such that $\varphi(J) = 0$, i.e., by letting φ be the state obtained by the representation with kernel J. Extend φ to all of $M(A \rtimes_\alpha G)$ and note that $\varphi(A) \neq 0$: Whenever u_i is an approximate identity of A we have $u_i x \to x$ for all $x \in A \rtimes_\alpha G$, which by Cauchy-Schwarz gives

$$|\varphi(u_i^* x)| \leq \varphi(u_i^* u_i) \varphi(x^* x),$$

implying that if $\varphi(A) = 0$ then $\varphi(x) = \lim \varphi(u_i^* x) \leq 0$ for all $x \in A \rtimes_\alpha G$ contradicting our assumptions.

Set now N to be the set

$$N := \{I(y) \mid y \in J \text{ is } \hat{\alpha} - \text{integrable}\},$$

which by theorem 2.4.25 and proposition 2.4.24 is a subset of A such that $\varphi(N) = 0$ by $\widehat{G}$-invariance of J. In particular, this implies that N can not be dense in A.

The $*$-linearity of I shows that N is a $*$-subspace, and in fact an algebraic $*$-ideal of A: To each $a, b \in A$, $t \in G$ and $I(y) \in N$ we have

- $aI(y)b = I(ayb)$,
- $\alpha_t(I(y)) = \lambda_t I(y) \lambda_{-t} = I(\lambda_t y \lambda_{-t})$,

since the action on A is given by $\alpha_t(a) = \lambda_t a \lambda_{-t}$ and A satisfying Landstad's condition. Also N is a non-zero ideal, as $I(\lambda_f^* y \lambda_f) \in N$ for all $y \in J$ and $f \in L^1(G) \cap L^2(G)$ by proposition 2.4.24. Thus the closure of N is a non-trivial G-invariant ideal of A.

For the converse, if J is a non-zero G-invariant ideal of A, then $J \otimes \mathbb{K}(L^2(G))$ is a non-zero $\widehat{G}$-invariant ideal of $A \otimes \mathbb{K}(L^2(G))$ with respect to the action $\alpha \otimes \mathrm{Ad}\rho$, and the result follows from Takai duality. □

Allowing us to, analogously with the case of primitivity, to obtain the following:

Proposition 3.2.18. For a C^*-dynamical a-system (A, G, α), if A is simple then

1. $A \rtimes_\alpha G$ is $\widehat{G}$-simple;
2. $G(\hat{\alpha}) = G$.

Proof.
If A is simple, then A is in particular G-simple implying that $A \rtimes_\alpha G$ is $\widehat{G}$-simple by proposition 3.2.17. Similarly, if A is simple then A is prime so $G(\hat{\alpha}) = G$ by lemma 3.2.13. □

Proposition 3.2.19. For a C^*-dynamical a-system (A, G, α), if $A \rtimes_\alpha G$ is simple then

1. A is G-simple;
2. $\widehat{G}(\alpha) = \widehat{G}$.

Proof.
The first condition follows again from proposition 3.2.17. The second condition follows from corollary 3.2.15 since simplicity implies primitivity. □

We proceed to show that whenever G is discrete, then the converse of the above holds. For this, we will apply the fact that G is discrete if and only if $\widehat{G}$ is compact with the following:

Lemma 3.2.20. Suppose that G is a compact group and $G \overset{\alpha}{\curvearrowright} A$. If A is prime and G-simple then A is simple.

Proof.
Assume that I is a non-trivial ideal of A, and choose non-zero elements $x, y \in I_+$ with $yx = y$. Let $0 \in U$ be a neighborhood such that

$$\|\alpha_s(x) - x\| < 1, \text{ for all } s \in U,$$

so that $\alpha_s(y) \in I$ for all $s \in U$, as per the proof of proposition 3.2.4. Let $t_1, \dots, t_n \in G$ be such that $\{U - t_i\}$ covers G. For all non-zero $a, b \in A$ there is $z \in A$ with $azb \neq 0$, as A is prime. With this, we recursively, find $a_1, \dots, a_{n-1} \in A\backslash\{0\}$ such that

$$0 \neq y_0 = \alpha_{t_1}(y)a_1\alpha_{t_2}(y)a_2 \cdots \alpha_{t_{n-1}}(y)a_{n-1}\alpha_{t_n}(y).$$

If $t \in G$, then there is $1 \leq k \leq n$ such that $t - t_k \in U$ and hence $\alpha_t(y_0) = \alpha(t)(\alpha_{t-1}(y)) \cdots \alpha_{t+t_k}(y) \cdots \in I$, implying that the orbit $\{\alpha_s(y_0) \mid s \in G\}$ is contained in I. Let I_0 be the smallest ideal containing this orbit, then $y_0 \in I_0 \subseteq I \neq A$, however this is contradicting the assumption that A is G-simple by invariance of I_0. This implies that A is simple. □

Theorem 3.2.21. Let (A, G, α) be a C^*-dynamical a-system with G discrete, then the following are equivalent:

1. $A \rtimes_\alpha G$ is simple;
2. (a) A is G-simple and (b) $\widehat{G}(\alpha) = \widehat{G}$.

Proof.
The implication $1 \implies 2$ is proposition 3.2.19.

For the converse, note that $A \rtimes_\alpha G$ is prime by corollary 3.2.15 and by proposition 3.2.17 it is also $\widehat{G}$-simple. As G is discrete, $\widehat{G}$ is compact so lemma 3.2.20 implies that $A \rtimes_\alpha G$ is simple. □

We will now see that the Connes spectrum, in a way, measures how well A sees ideals of $A \rtimes_\alpha G$ whenever G is discrete and abelian.

Definition 3.2.22. Let A be a C^*-subalgebra of a C^*-algebra B. We say that A *detects ideals* in B if $J \cap A = 0$ happens if and only if $J = 0$ for each ideal J of B. Similarly, we say that A *separates ideals* in B if $I \cap A = J \cap A$ implies $I = J$ whenever I, J are ideals of B.

Lemma 3.2.23. Let (A, G, α) be a C^*-dynamical a-system and assume that for all ideals $0 \neq I \subseteq A$ and each $t \in G$ we have $I \cap \alpha_t(I) \neq 0$. Then for each non-zero ideal $0 \neq J \subseteq A$ and each compact $K \subseteq G$ there is $z \in J \backslash \{0\}$ such that $\alpha_t(z) \in J$ for all $t \in K$.

Proof.
Without loss of generality, assume that $0 \in K$, for we may translate K to obtain the desired result else. Choose $0 \neq x, y \in J_+$ such that $yx = x$ and a neighborhood U of $0 \in G$ such that $\|\alpha_t(x) - x\| < 1$ for all $t \in U$ so that $\alpha_t(y) \in J$ whenever $t \in U$. Let now $t_1, \ldots, t_n \in G$ be such that $K \subseteq \bigcup_{1 \leq i \leq n}(U - t_n)$, by compactness of K. Assume that for some $2 \leq m < n$ we can find $a_2, \ldots, a_m \in A$ such that $z_1 := \alpha_{t_1}(y) \neq 0$ and $z_m = z_{m-1} a_m \alpha_{t_m}(y) \neq 0$.

We claim that then there is $a_{m+1} \in A$ such that $z_{m+1} = z_m a_{m+1} \alpha_{t_{m+1}}(y) \neq 0$: Suppose not, so that $z_m A \alpha_{t_{m+1}}(y) = 0$. Then $z_m A \alpha_{t_{m+1} - t_m}(z_m) \subseteq z_m A \alpha_{t_{m+1}}(y) = 0$, by assumption. Since $z_m \neq 0$, if I is the ideal generated by z_n, then $I \cap \alpha_{t_{m+1} - t_m}(I) = I \alpha_{t_{m+1} - t_m}(I) = 0$, contradicting our assumption. Hence we can pick $a_{m+1} \in A$ such that $z_{m+1} = z_m a_{m+1} \alpha_{t_{m+1}}(y) \neq 0$.

Let $z = z_n$, so that $\alpha_t(z) \in J$ for all $t \in K$ (as $t + t_m \in U$ for some m so $\alpha_{t+t_n}(y)$ is a factor of $\alpha_t(z)$). As $0 \in K$, we have $z \in J$, finishing the proof. $\square$

Proposition 3.2.24. Let (A, G, α) be a C^*-dynamical system and assume that for each non-zero ideal $0 \neq I \subseteq A$ it holds that $I \cap \alpha_t(I) \neq 0$ for all $t \in G$. If $0 \neq J \subseteq A$ is an ideal invariant under a subgroup G_0 of G with compact quotient G/G_0, then there is a non-zero G-invariant ideal $J_0 \subseteq J$.

Proof.
If q denotes the quotient map $G \to G/G_0$, then, by compactness of the quotient, there is a finite number of pre-compact sets of G whose image cover G/G_0, in particular, we may choose $E \subseteq G$ compact such that $G/G_0 = q(E)$. Choose nonzero $x \in J$ satisfying lemma 3.2.23. Each $s \in G$ is of the form $t + s_0$ for some $t \in E$ and $s_0 \in G_0$, as $q(E) = G/G_0$. In particular, this means that $\alpha_s(x) = \alpha_{s_0}(\alpha_t(x)) \in \alpha_{s_0}(J) \subseteq J$, by assumption and our choice of x. Hence the smallest ideal generated by the orbit $\{\alpha_t(x) \mid t \in G\}$ will be a non-zero G-invariant ideal contained in J. $\square$

Proposition 3.2.25. Let (A, G, α) be a C^*-dynamical system with G discrete. Then $\sigma \in \widehat{G}(\alpha)$ if and only if given any non-zero ideal J of $A \rtimes_\alpha G$ there is a non-zero $\hat{\alpha}_\sigma$-invariant ideal $J_0 \subseteq J$.

Proof.
We know that $\sigma \in \widehat{G}(\alpha)$ if and only if $\hat{\alpha}_\sigma(I) \cap I \neq 0$ for all non-zero ideals $I \subseteq A \rtimes_\alpha G$, cf. corollary 3.2.10. Note that since G is assumed discrete, $\widehat{G}$ is compact and since $\widehat{G}(\alpha)$ is

closed in $\widehat{G}$, this implies that $\widehat{G}(\alpha)$ is compact as well. By proposition 3.2.24, since $\widehat{G}(\alpha)/\{0\}$ is compact and the conditions are satisfied, there is a non-zero $\widehat{G}(\alpha)$-invariant ideal $I_0 \subseteq I$, hence in particular it will be $\widehat{\alpha}_\sigma$-invariant. □

We summarize, for discrete groups (though some of them are also true for non-discrete groups), our characterization of the Connes spectrum in the following theorem:

Theorem 3.2.26. Let (A, G, α) be a C^*-dynamical system with G discrete. Then the following are equivalent:

1. $\widehat{G}(\alpha) = \widehat{G}$.
2. For each $\sigma \in \widehat{G}$ and each non-zero ideal $I \subseteq A \rtimes_\alpha G$ we have $I \cap \widehat{\alpha}_\sigma(I) \neq 0$.
3. For each non-zero ideal $I \subseteq A \rtimes_\alpha G$ there is a non-zero $\widehat{G}$-invariant ideal $I_0 \subseteq I$.
4. A detects ideals in $A \rtimes_\alpha G$.

Proof.
The implication (1) $\iff$ (2) is corollary 3.2.10 and the implication (2) $\implies$ (3) is proposition 3.2.24. We show that (3) $\implies$ (4) $\implies$ (2) for discrete groups:

Assume (3). Let I be a non-zero ideal of $A \rtimes_\alpha G$, which without loss of generality may be assumed to be $\widehat{G}$-invariant by (3). If $0 \neq x \in I_+$, then $a = \int_{\widehat{G}} \widehat{\alpha}_\sigma(x) d\sigma$ is a non-zero element of I_+ which is fixed under $\widehat{G}$. By [LOP79, lemma 3.2], for discrete groups G, the elements of $A \rtimes_\alpha G$ which are fixed by $\widehat{G}$ correspond exactly to A, hence $a \in A \cap I$, showing (4).

Assume (4). Then given $\sigma \in \widehat{G}$ and a non-zero ideal $I \subseteq A \rtimes_\alpha G$, if $a \in A \cap I$, then again by [LOP79, lemma 3.2] we have $a = \widehat{\alpha}_\sigma(a)$, so $a \in I \cap \widehat{\alpha}_\sigma(I)$, showing (2). □

Theorem 3.2.21 of Olesen and Pedersen is from 1978, and was one of the first results characterizing simplicity by the means of Connes spectrum. In year after, following a paper of Gootman and Rosenberg (see [GR79]) on the Generalized Effros-Hahn conjecture, Olesen and Pedersen improved it for the case where A and G are separable to include a larger class of groups, including $\mathbb{R}$ and S^1, based on the previous results. We include a corollary of their main theorem for completeness, and refer the reader to [OP80] for the full discussion.

Corollary 3.2.27 ([OP80, Corollary 3.3])**.** Let (A, G, α) be a C^*-dynamical system with A separable and $G = \mathbb{R}$ or $G = S^1$. If A is not primitive, then $G \rtimes_\alpha A$ is simple if and only if A is G-simple and $\widehat{G}(\alpha) = \widehat{G}$.

To show that this theory is indeed usable (i.e., one can for instance actually determine the Connes spectrum in a non-trivial case), we will use it to give a simple proof of simplicity of the irrational rotation algebras. For this, we will use the following theorem of Pedersen and Gootman, characterizing the Connes spectrum in the commutative case in terms of stabilizers:

Theorem 3.2.28 ([GO81, Theorem 1.2 and Remark 1.3])**.** Suppose that $(C_0(X), G, \alpha)$ is a C^*-dynamical system with A and G separable. Then

$$\widehat{G}(\alpha) = \overline{\bigcup_{x \in X} G_x^{\perp}}.$$

In particular, if $G = \mathbb{Z}$ then $\widehat{\mathbb{Z}}(\alpha) = \widehat{\mathbb{Z}} \cong S^1$ whenever then $G_x = \{0\}$ for some $x \in X$.

Example 3.2.29
Let $\theta \in \mathbb{R} \backslash \mathbb{Q}$ and let $\mathbb{Z} \overset{\alpha}{\curvearrowright} C(S^1)$ denote rotation by θ. Then $C(S^1) \rtimes_\alpha \mathbb{Z}$ is simple.

Proof.
It is easy to establish that $C(S^1)$ is $\mathbb{Z}$-simple, since the system is minimal. It remains to be shown that $\widehat{\mathbb{Z}}(\alpha) = \widehat{\mathbb{Z}} = S^1$. By theorem 3.2.28 it suffices to show that some isotropy group is 0, but clearly $\alpha_k(z) = z$ if and only if $k = 0$ (since θ is irrational), hence $\mathbb{Z}_z = \{0\}$ so $\widehat{\mathbb{Z}}(\alpha) = S^1$, implying by theorem 3.2.21 that $C(S^1) \rtimes_\alpha \mathbb{Z}$ is simple. $\square$

While the above result might be an example of 'nuking the ant', as there are standard ways of achieving the same result without developing the general machinery we have covered thus far, we use it to show that the theory is relevant: We can, in not-degenerate examples, compute the Connes spectrum and use the theorem to answer questions about interesting C^*-algebras. It is to the authors knowledge not a standard proof seen in common literature.

Actions And Automorphisms

In this chapter, we will go through theory describing automorphisms on a C^*-algebra A and actions of groups G on C^*-algebras. The goal of this chapter is to characterize the ideal structure of crossed products with discrete groups through properties of the associated action. We will cover a theorem of Archbold and Spielberg (see [AS94]) stating that the reduced crossed product $A \rtimes_{\alpha,r} G$ is simple whenever the action of G is topologically free, which is something we will describe later on. After that we will cover some important consequences of this result, including an important result in the classification of group C^*-algebras.

4.1 Action Properties

We are going to discuss the concept of topologically free actions $G \overset{\alpha}{\curvearrowright} A$ and its implications on the crossed product. In order to do this, we introduce notation and discuss some results on the spectrum of a general C^*-algebra as well as the primitive ideal space of A:

Definition 4.1.1. For a C^*-algebra A, we define the *primitive ideal space* of a C^*-algebra A to be the set

$$\mathrm{Prim}(A) := \{\ker\pi \mid \pi \text{ is an irreducible representation of } A\}.$$

Furthermore, we define the *spectrum of a C^*-algebra A* to be the set

$$\widehat{A} := \{[\pi]_u \mid \pi \text{ is an irreducible representation of } A\}.$$

where $[\cdot]_u$ denotes the class of unitarily equivalent representations.

If $\pi_1 \sim_u \pi_2$, then $\ker\pi_1 = \ker\pi_2$ so the map $\kappa\colon \widehat{A} \to \mathrm{Prim}(A)$, $[\pi]_u \mapsto \ker\pi$, is surjective and well-defined. We endow $\mathrm{Prim}(A)$ with a topology called the *hull-kernel topology* defined such that for $S = \{s_i\}_{i\in I} \subseteq \mathrm{Prim}(A)$ we have

$$\overline{S} := \{p \in \mathrm{Prim}(A) \mid \cap_{i\in I} s_i \subseteq p\}.$$

To see that this is in fact a topology, we refer the reader to [RW98, Appendix A.2] for the details. We also endow $\widehat{A}$ with the corresponding initial topology stemming from the surjective map κ.

Remark 4.1.2
For a commutative C^*-algebra (i.e., $A = C_0(X)$), the spectrum of A is the Gelfand Space of A, i.e., non-zero $*$-homomorphisms $A \to \mathbb{C}$, since irreducible representations of commutative spaces are one-dimensional. Moreover, in this case it holds that $\widehat{A} \cong \mathrm{Prim}(A)$.

These objects interact nicely with dynamical systems (we already saw this for $A = C_0(X)$): Whenever $G \overset{\alpha}{\curvearrowright} A$ then we may define the *transposed action of* G on both $\widehat{A}$ and $\mathrm{Prim}(A)$ (as topological spaces) by

$$\alpha_r^*([\pi]_u) := [\pi \circ \alpha_{r^{-1}}] \text{ and } \alpha_r^*(\ker \pi) = \ker \pi \circ \alpha_{r^{-1}},$$

for all irreducible representations π of A. These are indeed actions on topological spaces, i.e., the corresponding maps out of $G \times \widehat{A} \to \widehat{A}$ and $G \times \mathrm{Prim}(A) \to \mathrm{Prim}(A)$ are jointly continuous, see e.g., [RW98, Lemma 7.1].

Definition 4.1.3. For an action $G \overset{\alpha}{\curvearrowright} A$ we say that the action is *topologically free* if for all $t_1, \dots, t_n \in G \backslash \{e\}$ the set $\bigcap_{i=1}^n \left\{[\pi]_u \in \widehat{A} | \alpha_{t_i}^*([\pi]_u) \neq [\pi]_u\right\}$ is dense in $\widehat{A}$.

Remark 4.1.4
If $A = C_0(X)$, then the action $G \overset{\alpha}{\curvearrowright} A$ if and only if for all $e \neq t \in G$, the set of fixed points of the induced action $G \curvearrowright X$ have empty interior or equivalently dense complement. Moreover, define

$$G_x^\circ := \{t \in G \mid t|_U = \mathrm{id} \text{ for some neighborhood } U \text{ of } x\},$$

which is clearly a normal subgroup of G, then the action is topologically free if and only $G_x^\circ = \{e\}$ for all x: If $t.y = y$ for all $y \in U$ of some neighborhood U of x, then the fixpoint set of t has non-empty interior.

Definition 4.1.5. We say that an action $G \curvearrowright X$ is free if every stabilizer is trivial, i.e., if for each $x \in X$ we have $G_x = \{e\}$.

Remark 4.1.6
In [OP82], Olesen and Pedersen characterized topological freeness of actions $\mathbb{Z} \overset{\alpha}{\curvearrowright} A$, $\alpha \in \mathrm{Aut}(A)$, in terms of the Connes spectrum (and many other characterizations). They showed the equivalence of the following:

1. For each $n \neq 0$, the set $\{[\pi]_u \in \widehat{A} \mid [\pi \circ \alpha^n]_u = [\pi]_u\}$ is has empty interior,
2. $\widehat{\mathbb{Z}}(\alpha) = \widehat{\mathbb{Z}} = S^1$.

The result is [OP82, Theorem 10.4], and the proof goes beyond the scope of this thesis. However, we include it to highlight that the theory developed in the previous chapter served as a stepping stone towards newer results which we will cover in this chapter.

Lemma 4.1.7. Suppose that $G \curvearrowright X$ is a minimal action. If $G_x^\circ = \{e\}$ for some $x \in X$, then $G_y^\circ = \{e\}$ for all $y \in X$, i.e., the action is topologically free.

Proof.
Let $x \in X$ be such that $G_x^\circ = \{e\}$. Let $y \in X$ and suppose towards a contradiction that $s \in G_y^\circ\{e\}$. Let $U \subseteq X$ witness this, i.e, be open with $y \in U$ and $s|_U = \mathrm{id}$. By minimality of the action, there is $t \in G$ such that $x \in tU$. Then $tst^{-1}.(t.v) = ts.v = t.v$ for all $v \in U$, so $tst^{-1}|_{tU} = \mathrm{id}$. By assumption, this implies that $tst^{-1} = e$, which forces $s = e$ (if $t = e$, then $s \in G_x^\circ = \{e\}$ and if $t \neq e$ then $s = e$), contradicting the assumption that $s \neq e$. □

Corollary 4.1.8. The action of G on $\partial_F G$ (the Furstenberg boundary) is topologically free if and only if it is free.

Proof.
This follows directly from the earlier remark that $\partial_F G$ is extremally disconnected, for $\partial_F G_s$ is clopen by a theorem of Frolik, see e.g., [Fro71, Theorem 3.1], which states that the set of fixed points of a homeomorphism $f\colon X \to X$, where X is a compact extremally disconnected space, is clopen in X. □

Definition 4.1.9. For a C^*-algebra A, we say that $\alpha \in \mathrm{Aut}(A)$ is *properly outer* if for all α-invariant non-zero ideals $I \subseteq A$ and all inner automorphisms β of I it holds that

$$\|\alpha|_I - \beta\| = 2.$$

Elliot and Olesen + Pedersen showed in [Ell80, Theorem 3.2] and [OP82, Theorem 7.2], respectively, that for discrete groups G with $G \overset{\alpha}{\curvearrowright} A$ minimal, a sufficient condition for C^*-simplicity of $A \rtimes_{\alpha,r} G$ for a large class of C^*-algebras was that α_t be properly outer for $t \neq e$. Archbold and Spielberg later improved on this result in [AS94]. We will go through their result and some recent applications.

Proposition 4.1.10 ([AS94, Proposition 1])**.** Let (A, G, α) be a C^*-dynamical system. If α is topologically free, then α_t is properly outer for all $t \neq e$.

Remark 4.1.11
The proof is based on theory of derivations of C^*-algebras, which we will not go into detail with. Shortly put, a derivation of a C^*-algebra A is a linear map on A which behaves similar

to derivation of differentiable functions. For an exhibition on this, see e.g., [Ped79, Chapter 8.6] (or the books they refer to, [Sak78] and [BR12], by Sakai and Bratteli et al., respectively.)

The proof is based on a result of Kadison-Ringrose, which characterizes proper outerness with not never locally being the composition of an inner automorphism composed with the exponential of a derivation, see [KR67] or [OP82, Theorem 6.6 (i) and (ii)].

Definition 4.1.12. Let A denote an arbitrary C^*-algebra with two representations π and τ on Hilbert spaces $\mathcal{H}_\pi$ and $\mathcal{H}_\tau$. We say that π and τ are *disjoint representations* if there is no subrepresentations of one which is unitarily equivalent to a non-zero subrepresentation of the other.

Remark 4.1.13
The above definition is the (to the author's knowledge) original definition. We will use the following fact: If π and τ are disjoint if and only if $1 \oplus 0$ is an element of the strong operator closure of $\pi \oplus \tau(A)$, i.e., there is some bounded net $(x_i) \subseteq A$ with $\pi(x_i) \to 1$ and $\tau(x_i) \to 0$ in their strong operator topology, and this happens if and only if $\pi \not\sim_u \tau$. See e.g. [Arv12, Proposition 2.1.4] for a proof.

4.2 Consequences of Topological Freeness

Recall that we may identify G as a subset of the unitary group of $M(A \rtimes_\alpha G)$. We denote this identification by u_t, $t \in G$.

Lemma 4.2.1. Let G be discrete and $G \overset{\alpha}{\curvearrowright} A$ and let π be any non-degenerate representation of A. If there is $t \in G$ such that $\pi \circ \alpha_t^{-1}$ and π are disjoint representations, then for all completely positive extension φ of π to $A \rtimes_\alpha G$ of norm 1 it holds that $\varphi(au_t) = 0$ for all $a \in A$ and $u_t \in G \subseteq M(A \rtimes_\alpha G)$.

Proof.
Let $(e_i) \subseteq A$ be a bounded net such that $\pi \circ \alpha_t^{-1}(e_i) \to 1$ in the strong operator topology and $\pi(e_i) \to 0$. Let φ be a completely positive extension of π to $A \rtimes_\alpha G$ of norm one. Then, for $a \in A$ and $t \in G$:

$$\varphi(au_t) = \lim_{\mathrm{SOT}} \varphi(x_i)\varphi(au_t) = \lim_{\mathrm{SOT}} \varphi(u_t\alpha_t^{-1}(x_i a)) = \lim_{\mathrm{SOT}} \varphi(u_t)\varphi(\alpha_t^{-1}(x_i))\varphi(\alpha_t^{-1}(a)) = 0,$$

Move to prelim

as $A \subseteq \mathrm{mult}(\varphi)$. see [BO08, Proposition 1.5.7 p.t (2)]. ☐

With this, we obtain the following:

Theorem 4.2.2. Let (A, G, α) be a C^*-dynamical system, G discrete. If α is topologically free, then A detects ideals in $A \rtimes_\alpha G$ modulo the kernel of the canonical surjection $A \rtimes_\alpha G \to A \rtimes_{\alpha,r} G$.

Proof.
Let I_r denote the kernel of the canonical surjection $\pi_r\colon A \rtimes_\alpha G \to A \rtimes_{\alpha,r} G$ and let E denote Include in prelim
the canonical faithful conditional expectation (see e.g., [BO08, Proposition 4.1.9]) given by include in Prelim

$$E_r\colon A \rtimes_{\alpha,r} G \to A,$$
$$c_c(G, A) \ni \sum_{g \in G} a_g u_g \mapsto a_e \in A,$$

and let $E = E_r \circ \pi_r$ denote the canonical conditional expectation $A \rtimes_\alpha G \to A$.

Suppose that I is an ideal of $A \rtimes_\alpha G$ with $A \cap I = 0$ and $I \not\subseteq I_r$. Since E_r is faithful. By density of $C_c(G, A)$ in $A \rtimes_\alpha G$, pick $b = \sum_{g \in F} b_g u_g$ for some finite set $F \subseteq G$ with

$$\|a - b\| < \frac{\|E(a)\|}{2}.$$

By assumption, the set $P := \bigcap_{e \neq t \in F} \left\{ p \in \widehat{A} \mid \alpha_t^*(p) \neq p \right\}$ is dense in $\widehat{A}$. Let $[\pi] \in P$. Let π' denote the composition

$$A + I \xrightarrow{q} (A + I)/I \xrightarrow{\cong} A \xrightarrow{\pi} \mathbb{B}(\mathcal{H}_\pi),$$

where q is the quotient map. We are going to extend π' to a completely positive map on $A \rtimes_\alpha G$ in the following way: For $x \in A \rtimes_\alpha G$, let $\tau(x) := P_e \tilde{\pi} \rtimes \lambda(x) P_e$, where P_e is the minimal projection $\mathcal{H}_\pi \otimes \ell^2(G) \to \mathcal{H}_\pi \otimes \mathbb{C}\delta_e$, then $\tau\colon A \rtimes_\alpha G$ is a completely positive map extending π' (after identifying $\mathcal{H}_\pi$ with $\mathcal{H}_\pi \otimes \mathbb{C}\delta_e$) to all of $A \rtimes_\alpha G$; if $a \in A$ and $s, t \in G$ then

$$\tau(a u_s)\xi \otimes \delta_t = \begin{cases} \pi(a)\xi \otimes \delta_e & \text{if } s = t = e, \\ 0 & \text{else} \end{cases}$$

for all $\xi \in \mathcal{H}_\pi$. If $x \in I$, then $P_e \tilde{\pi} \rtimes_r \lambda(x) P_e = 0$ as $A \cap I = 0$. In particular, τ is completely positive and $\|\tau\| = \lim \|\tau(e_i)\| = 1 \, \|\pi\| = 1$, where e_i is an approximate identity of A.

For $t \in F \backslash \{e\}$ the representations π and $\pi \circ \alpha_t^{-1}$ are disjoint, since $[\pi] \in P$. By lemma 4.2.1 we then see that

$$\tau(b) = \sum_{t \in F} \tau(b_t u_t) = \tau(b_e) = \tau(E(b)).$$

Since $\tau(I) = 0$, we have

$$\|\pi(E(b))\| = \|\tau(b)\| = \|\tau(b - a)\| \leq \|b - a\|.$$

As P is dense in $\widehat{A}$, we see that $\|E(b)\| \leq \|b - a\|$ yielding the contradiction

$$\|E(a)\| \leq \|E(a-b)\| + \|E(b)\| \leq 2\,\|a-b\| < \|E(a)\|\,,$$

so $I \subseteq I_r$. □

Corollary 4.2.3. If G is amenable, then A detects ideals in $A \rtimes_\alpha G$ whenever α is topologically free.

If we further assume that A is abelian, then the converse holds as well:

Theorem 4.2.4. Let X be a locally compact Hausdorff space. Given C^*-dynamical sytem $(C_0(X), G, \alpha)$ with G discrete, if $C_0(X)$ detect ideals in $C_0(X) \rtimes_\alpha G$ modulo I_r, then the action is topologically free.

Proof.
Let $x \in X$, and consider the Hilbert space $\ell^2(G.x) = \ell^2(\{t.x \mid t \in G\})$. We define the covariant pair (π_x, u) for $(C_0(X), G, \alpha)$ by

$$\pi_x(f)\delta_{t.x} = f(t.x)\delta_{t.x} \text{ and } u_s\delta_{t.x} = \delta_{st.x},$$

for $f \in C_0(X)$ and $t, s \in G$. That they are representations is easy to verify, we show covariance instead:

$$u_s\pi_x(f)u_{s^{-1}}\delta_{t.x} = u_s f(s^{-1}t.x)\delta_{s^{-1}t.x} = f(s^{-1}t.x)\delta_{t.x} = s.f(t.x)\delta_{t.x} = \pi_x(s.f)\delta_{t.x},$$

so that we obtain a representation $\pi_x \rtimes u_x$ of $C_0(X) \rtimes_\alpha G$ on $\ell^2(G.x)$. Let $I = \bigcap_{x\in X} \ker \pi_x$. If $f \in I$, then $f(t.x) = 0$ for all $t \in G$ (in particular for $t = e$) and $x \in X$, so $f = 0$. Hence $I \cap C_0(X) = 0$.

Suppose that $G \overset{\alpha}{\curvearrowright} C_0(X)$ is not topologically free. Then there is $e \neq s \in G$ such that $V_s = \{y \in X \mid s.y \neq y\}$ is not dense in X. Let $0 \neq f \in C_0(X)$ with $\operatorname{supp} f \subseteq \overline{V_s}^c$. Let $t \in G$ and note that

- If $t.x \in \operatorname{supp} f$ then $st.x = t.x$ and we see that

$$\pi_x \rtimes u_x(f - fu_s)\delta_{tx} = f(t.x)\delta_{t.x} - f(st.x)\delta_{st.x} = 0.$$

- If $st.x \in \operatorname{supp} f$, then $t.x = s^{-1}st.x = st.x$ as well.
- If $t.x \notin \operatorname{supp} f$ then $st.x \notin \operatorname{supp} f$ and we see that

$$\pi_x \rtimes u_x(f - fu_s)\delta_{tx} = f(t.x)\delta_{t.x} - f(st.x)\delta_{st.x} = 0.$$

Thus $f - fu_s \in I \subseteq I_r$ as $x \in X$ was arbitrary. Since $E(fu_s) = E_r(\pi_r(fu_s)) = 0$, we see that

$$0 \neq f = E(f - fu_s) = E_r(\pi(f - fu_s)) = E_r(0) = 0,$$

a contradiction. We conclude that the action of $C_0(X)$ is topologically free. $\square$

Corollary 4.2.5. Suppose that $G \overset{\alpha}{\curvearrowright} C_0(X)$ with G discrete. Then $C_0(X) \rtimes_\alpha G$ is simple if and only if the action is minimal, topologically free and $I_r = 0$.

Proof.
This follows directly from theorem 4.2.2, theorem 4.2.4 and lemma 4.2.1. $\square$

Once again, this gives an easy proof of simplicity of $C(S^1) \rtimes_\theta \mathbb{Z}$ for θ irrational, since the action is very clearly topologically free (there are no fixed points on S^1 for irrational rotation), and we already know that it is minimal and $I_r = 0$.

4.3 A Relation between Ideals of $C_r^*(G)$ and $A \rtimes_{\alpha,r} G$ and a theorem of four

From here on, let G denote a discrete countable group and X a compact G-space. In the following, we will discuss recent developments regarding the classification of C^*-simple groups. In particular results based on theory of boundary actions and topological freeness from earlier will be reviewd and their relations to simplicity of associated crossed products $A \rtimes_{\alpha,r} G$.

We say that a group G is *C^*-simple* if its reduced group is C^*-simple, and we say that G has the *unique trace property* if the canonical faithful tracial state $\tau_0 \colon f \mapsto \langle f\delta_e, \delta_e\rangle_{\ell^2(G)}$ is the only tracial state on $C_r^*(G)$. It was an open question for a long time wether having the unique tracial state was equivalent to being C^*-simple. In 2014, using theory of boundary actions, this was shown by to false by Le Boudec (see [LB17, Thoerem A]). However, C^*-simplicity does imply unique tracial state which is equivalent to having no non-trivial amenable normal subgroups (see e.g. the author's project on this subject: [Kar17, Chapter 5]). The breakthrough which started this development was due to Kennedy and Kalantar (see [KK14]).

It is quite a strong condition on a group G to be C^*-simple, for example discrete amenable (in particular abelian) groups are not C^*-simple. Groups which are C^*-simple include the class of Powers groups (an homage to R.T. Powers' proof on simplicity of the free group on two generators, see [Pow+75] or [Kar17, chapter 3]) which includes the free groups on finite number of generators.

In order to show our desired results, we need to cover a bit of theory on conditional expectations and crossed products. For instance whenever N is a normal subgroup of a discrete group G, then the inclusion $N \subseteq G$ gives rise to a conditional expecation $E_N^G \colon C_r^*(G) \to C_r^*(N)$ given by $\lambda_s \mapsto 1_N(s)\lambda_s$, see e.g. [BO08, Corollary 2.5.12]. Moreover, supposing that $A = C(X)$, then for all $x \in X$, the associated conditional expectation $E_{G_x}^G : C_r^*(G) \to C_r^*(G_x)$ extends to a conditional expecation

$$E_x \colon C(X) \rtimes_{\alpha,r} G \to C_r^*(G_x), \tag{4.1}$$

$$f\lambda_s \mapsto f(x)E_{G_x}^G(\lambda_s). \tag{4.2}$$

The following result shows the close relationship between topological freeness of actions and C^*-simplicty of groups with the ideal structure of crossed products of G-boundaries. It is (to the author's knowledge) due to [Bre+14], however we follow a proof of Ozawa (see e.g., [Oza14]). The result is heavily inspired by the proof of the Archbold-Spielberg result covered earlier.

Proposition 4.3.1. Let $G \curvearrowright X$ be a minimal action on a compact Hausdorff space X. If the stabilizer group G_x is C^*-simple for some $x \in X$, then the corresponding reduced crossed product $C(X) \rtimes_{\alpha,r} G$ is simple.

Proof.
Suppose that $I \subseteq C(X) \rtimes_r G$ is a non-trivial closed ideal and let $x \in X$. Let E denote the canoncial faithful conditional expectation $C(X) \rtimes_r G \to C(X)$. Then

- $E(I) \neq 0$ as E is faithful,
- $E(I)$ is dense in $C(X)$ by minimality of $G \curvearrowright X$, as $\overline{E(I)}$ is a non-empty G-invariant ideal of $C(X)$ so $\overline{E(I)} = C(X)$.

Let τ_0 denote the canoncial faithful tracial state on $C_r^*(G)$ let $f \in C(X)\{0\}$ with $f(x) \neq 0$. Then

$$\tau_0(E_x(f\lambda_e)) = f(x) = \delta_x(E(f\lambda_e)),$$

so $E_x(I) \neq \{0\}$ in $C_r^*(G_x)$.

We claim that $E_x(I)$ is not dense in $C_r^*(G_x)$, and similar to the proof of theorem 4.2.2, we consider state on $C(X) + I$ given by the composition:

$$C(X) + I \to (C(X) + I)/I \cong C(X) \xrightarrow{\delta_x} \mathbb{C},$$

where we used that the proper ideal $C(X) \cap I$ of $C(X)$ must be 0 by minimality of $G \curvearrowright X$. Let φ_x denote any state extending the above composition with $\varphi_x(I) = 0$.

The result will follow once we show that $\varphi_x = \varphi_x \circ E_x$, for then $E_x(I)$ can not be dense: If $\varphi_x = \varphi_x \circ E_x$ and $E_x(I)$ is dense in $C_r^*(G_x)$, then

$$\varphi_x(C(X) \rtimes_r G) = \varphi_x(E_x(C(X) \rtimes_r G)) = \varphi_x(C_r^*(G_x)) = \{0\},$$

a contradiction. To see that $\varphi_x = \varphi_x \circ E_x$ it suffices to show that $\varphi_x(\lambda_s) = 0$ for $s \notin G_x$: If $s \in G_x$, then as $C(X)$ is in the multiplicative domain of φ_x, it holds for each $f \in C(X)$ that

$$\varphi_x(f\lambda_s) = \varphi_x(f(x)\lambda_s) = \varphi_x(E_x(f\lambda_s)).$$

So assume that $s \notin G_x$, i.e., that $s.x \neq x$. Using Uryhson's lemma we pick a function $h \in C(X)$ with $h(x) = 1$ and $h(s^{-1}.x) = 0$. Then

$$\varphi_x(\lambda_s) = \varphi_x(h)\varphi_x(\lambda_s)\varphi_x(h) = \varphi_x(h\lambda_s h) = \varphi_x(h(s.h)\lambda_s) = f(x)f(s^{-1}.x)\varphi_x(\lambda_s) = 0,$$

finishing the proof. $\square$

Remark 4.3.2
Given a group G, if N is an amenable normal subgroup of G, then the quotient map $\mathbb{C}(G) \to \mathbb{C}(G/N)$ extends to a $*$-homomorphism $\pi_N : C_r^*(G) \to C_r^*(G/N)$ such that

$$\begin{tikzcd} \mathbb{C}G \arrow[r, "q"] \arrow[d, "\lambda_G"'] & \mathbb{C}(G/N) \arrow[d, "\lambda_{G/N}"] \\ C_r^*(G) \arrow[r, "\pi_N"'] & C_r^*(G/N) \end{tikzcd} \tag{4.3}$$

is commutative with the vertical maps being the left-regular representations, see e.g. [DLH07, Proposition 3] for a proof. In general, the associated representation of G as operators on $\ell^2(G/N)$ given by composition of the above is often refered to as the *quasi-regular representation*. We will use this result in a moment.

We have seen that topological freeness implies simplicity of $A \rtimes_{\alpha,r} G$ for G-simple C^*-algebras. We now prepare to cover a result from the same paper as above allows us to determine topological freeness from properties of G and $C(X) \rtimes_\alpha G$, but first we need a lemma which will allow us to describe a useful representation:

Lemma 4.3.3. Suppose that X is a minimal G-space such that for some $x \in X$ the normal subgroup G_x° is amenable. Then there is a representation π_x of $C(X) \rtimes_r G$ on $\ell^2(G/G_x^\circ)$ such that

$$\Pi_x(f)\delta_{[t]} = f(tx)\delta_{[t]} \text{ and } \Pi_x(\lambda_s)\delta_{[t]} = \delta_{[st]},$$

for $s, t \in G$ and $f \in C(X)$.

Proof.
For the remainder of this proof, we will let λ^x denote the representation of G in $C_r^*(G/G_x^\circ)$ from remark 4.3.2 and λ denote the left-regular representation of G. Observe that we may define a $*$-homomorphism $\pi_x \colon C(X) \to \ell^\infty(G/G_x^\circ)$, given by $\pi_x(f)([t]) = f(tx)$: It is well-defined, for if $[s] = [t] \in G/G_x^\circ$, then

$$\pi_x(f)([t]) = f(tx) = f(sx) = \pi_x(f)([s]),$$

for all $s, t \in G$ and $f \in C(X)$ as $s^{-1}tx = x$ if and only if $sx = tx$. It is easy to verify that it is a $*$-homomorphism. In fact, it is injactive: If $\pi_x(f) = \pi_x(g)$, then $f(tx) = g(tx)$ for all $t \in G$, which by minimality implies that $f = g$.

Redefine now π_x to denote the associated representation of $C(X)$ as multiplication operators on $\ell^2(G/G_x^\circ)$ and note that for all $t, s \in G$ and $f \in C(X)$:

$$\lambda_t^x \pi_x(f) \lambda_{t^{-1}}^x \delta_{[s]} = f(t^{-1}sx) \lambda_{t^{-1}}^x \delta_{[ts]} = f(t^{-1}sx)\delta_{[s]} = t.f(sx)\delta_{[t]} = \pi_x(t.f)\delta_{[s]},$$

so that the pair $(\lambda^x, \pi_x, \ell^2(G/G_x^\circ))$ is a covariant pair for the C^*-dynamical system $G \curvearrowright C(X)$. By Fell's absorbtion principle (see [BO08, Proposition 4.1.7] or [Kar16, p. 25]), we may identify

$$C(X) \rtimes_r G = C^*\left(\{\lambda_t^x \otimes \lambda_t\}_{t \in G}, \{\pi_x(f) \otimes 1\}_{f \in C(X)}\right) \subseteq \mathbb{B}(\ell^2(G/G_x^\circ)) \otimes C_r^*(G),$$

where f is mapped to $\pi_x(f) \otimes 1$ and λ_t is mapped to $\lambda_t^x \otimes \lambda_t$. Now, since G is discrete and countable, $C_r^*(G) \cong C^*(\{\lambda_t\}_{t\in G})$, and by remark 4.3.2 we obtain a $*$-homomorphism

$$C^*(\{\lambda_t\}_{t \in G}) \to C^*(\{\lambda_t^x\}_{t \in G}), \quad \lambda_t \mapsto \lambda_t^x,$$

for all $t \in G$. Combining everything, we obtain a representation σ of $C(X) \rtimes_r G$ on $\mathbb{B}(\ell^2(G/G_x^\circ)) \otimes \mathbb{B}(\ell^2(G/G_x^\circ))$ such that

$$\sigma_x(f) = \pi_x(f) \otimes 1 \text{ and } \sigma_x(f)(\lambda_t) = \lambda_t^x \otimes \lambda_t^x.$$

Let $\iota \colon \mathbb{B}(\ell^2(G/G_x^\circ)) \otimes \mathbb{B}(\ell^2(G/G_x^\circ)) \subseteq \mathbb{B}(\ell^2(G/G_x^\circ) \otimes \ell^2(G/G_x^\circ))$ be the canonical unique embedding of C^*-algebras and let u be the unitary operator embedding $\ell^2(G/G_x^\circ)$ in the diagonal space of $\ell^2(G/G_x^\circ) \otimes \ell^2(G/G_x^\circ)$, i.e., $u\delta_{[t]} = \delta_{[t]} \otimes \delta_{[t]}$.

Then the *-homomorphism $\Pi_x \colon C(X) \rtimes_r G \to \mathbb{B}(\ell^2(G/G_x^\circ))$ given by $\Pi_x(a) := u^*\iota(\sigma_x(a))u$ has the desired properties. $\square$

Proposition 4.3.4. Suppose that X is a minimal G-space and $C(X) \rtimes_\alpha G$ is simple. If there is some $x \in X$ such that the normal subgroup G_x° of G is amenable, then the action $G \curvearrowright X$ is topologically free.

Proof.
First, let Π_x denote the representation of lemma 4.3.3 and suppose that $s \in G_x^\circ$ and let U be a neighborhood of x witnessing this, i.e, be such that $s|_U = \mathrm{id}$, and let $f \in C(X)\backslash\{0\}$ with $\operatorname{supp} f \subseteq U$. By the same argument as in the proof of theorem 4.2.4, if $t.x \in \operatorname{supp} f$ then $st.x = t.x$ and if $t.x \notin \operatorname{supp} f$ then $st.x \notin \operatorname{supp} f$, so

$$\Pi_x\left(f(1-\lambda_s)\right)\right)\delta_{[t]} = f(t.x)\delta_{[t]} - f(st.x)\delta_{[st]} = 0.$$

Since we assumed simplicity of $C(X) \rtimes_r G$, this implies that $s = e$ and hence G_x° is trivial whence the minimality assumption finishes the proof. $\square$

In order to prove the main theorem of this section, we need two results which, arguably, is outside the scope of this thesis. We include them with reference to proofs.

Lemma 4.3.5 ([Ham85, Lemma 3.1])**.** Let A and B be unital C^*-algebras and G a discrete group such that $G \overset{\alpha}{\curvearrowright} A$ and $G \overset{\beta}{\curvearrowright} B$. If $\psi\colon A \to B$ is a G-equivariant unital completely positive map, then there a G-equivariant unital completely positive map

$$\tilde{\psi}\colon A \rtimes_{\alpha,r} G \to B \rtimes_{\alpha,r} G a\lambda_s \mapsto \psi(a)\lambda_s$$

which extends ψ. Moreover, ψ is an isometry or surjection if and only if $\tilde{\psi}$ is and similarly ψ is a $*$-homomorphism if and only if $\tilde{\psi}$ is.

Lemma 4.3.6 ([Oza14, Theorem 6 + Corollary 7])**.** whenever A is a unital G-C^*-algebra, then there is a G-equivariant unital completely positive map of A into $C(\partial_F G)$ extending the map sending 1 to 1.

With these results, we continue.

Proposition 4.3.7. Let $G \curvearrowright A$ with $1 \in A$ and let X be a G-boundary. If $A \rtimes_r G$ is non-simple then $(A \otimes C(X)) \rtimes_r G$ is non-simple in the following way: If $I \subseteq A \rtimes_r G$ is a non-trivial ideal, then the ideal generated by I in $(A \otimes C(X)) \rtimes_r G$ is proper.

Proof.
Consider the embedding of $A \rtimes_r G$ in $(A \otimes C(X)) \rtimes_r G$, $a\lambda_s \mapsto (a \otimes 1)\lambda_s$, induced by the embedding $A \subseteq A \otimes C(X)$, $a \mapsto a \otimes 1$. Let π be a non-faithful representation of $A \rtimes_r G$ on $\mathbb{B}(\mathcal{H})$ with kernel I, and let $\overline{\pi}$ denote a unital completely positive extension $(A \otimes C(X)) \rtimes_r G \to \mathbb{B}(\mathcal{H})$. As $A \rtimes_r G$ is contained in the multiplicative domain of $\overline{\pi}$, the bimodule property implies that $\overline{\pi}(axb) = \pi(a)\overline{\pi}(x)\pi(b)$ for all $a, b \in A \rtimes_r G$ and $x \in (A \otimes C(X)) \rtimes_r G$, so that the image of $\pi(A)$ and $\overline{\pi}(C(X))$ commute, since

$$(a \otimes 1)\lambda_s(1 \otimes f)\lambda_e = (1 \otimes f)\lambda_e(a \otimes 1)\lambda_s,$$

for all $a \in A$, $f \in C(X)$ and $s \in G$ and the bimodule property for $\overline{\pi}$ of $A \rtimes_r G$ mentioned earlier.

Define an action of G on $C^*(\overline{\pi}(C(X)))$ by inner automorphisms $s \mapsto \mathrm{Ad}\overline{\pi}(\lambda_s)$. By the virtue of lemma 4.3.6 let ψ denote any G-equivariant unital completely positive map from $C^*(\overline{\pi}(C(X))) \to C(\partial_F G)$. By a result of Furstenberg, see e.g. [Kar17, lemma 4.16], the composition $\psi \circ \overline{\pi}|_{C(X)}$ must be the unique G-equivariant injective $*$-homomorphism, i.e., the inclusion of $C(X)$ in $C(\partial_F G)$ induced by the quotient map $\partial_F G \to X$ stemming from the universal property of $\partial_F G$.

In particular ψ is then a $*$-homomorphism: The multiplicative domain of ψ contains the image of $\overline{\pi}(C(X))$ and by linearity and continuity it must be all of $C^*(\overline{\pi}(C(X)))$.

We now consider the kernel $K := \ker \psi$, which is G-invariant, and the C^*-algebra

$$B := C^*(\overline{\pi}((A \otimes C(X)) \rtimes_r G)) = \overline{C^*(\overline{\pi}(C(X))) \cdot \pi(A \rtimes_r G)},$$

where the second equality comes from the earlier remark that $A \rtimes_r G$ is contained in the multiplicative domain of $\overline{\pi}$. In particular, the ideal generated by K in B is

$$K' := \overline{K \cdot \pi(A \rtimes_r G)}.$$

Let $x \in B$ and let (e_i) denote an approximate identity of K', and see that

- if $e_i x \to x$, then $x \in K'$,
- if $x \in K'$, then $e_i x \to x$,

so $K' \cap C^*(\overline{\pi}(C(X))) = K$. Let q denote the quotient map $B \to B/K'$, and note that the composition $q \circ \overline{\pi}$ will be multiplicative when restricted to $A \otimes C(X)$ as $A \rtimes_r G$ is contained in the multiplicative domain of $\overline{\pi}$. Moreover, q will be G-equivariant with respect to $\mathrm{Ad}\lambda_s$, $s \in G$.

We conclude that the ideal $\ker(q \circ \overline{\pi})$ contains I and is proper since the non-zero ideal K of $C^*(\overline{\pi}(C(X)))$ is proper, hence the ideal generated by I will be non-trivial and proper. $\square$

Definition 4.3.8. Let B be a C^*-algebra equipped with a G-action and let $K \subseteq B$ be a G-invariant ideal of B. We then define an ideal $K \bar{\rtimes}_r G$ of $B \rtimes_r G$ by

$$K \bar{\rtimes}_r G := \{ b \in B \rtimes_r G \ \mid \ E(b\lambda_s^*) \in K \text{ for all } s \in G \},$$

where E is the canonical conditional expectation onto B. It clearly contains $K \rtimes_r G$ as an ideal.

Remark 4.3.9
That $K \bar{\rtimes}_r G$ is a closed ideal comes by the virtue of lemma 4.3.5: If E (respectively E_K)

denotes the canonical conditional expectation on $B \rtimes_r G$ (respectively $B/K \rtimes_r G$), then the quotient map $\pi\colon B \to B/K$ extends to a surjective $*$-homomorphism $\Pi\colon B \rtimes_r G \to B/K \rtimes_r G$ such that $\Pi(a\lambda_s) = [a]\lambda_s$. The kernel of Π is $K \bar{\rtimes}_r G$, and Π satisfies

$$E_K \circ \Pi = \pi \circ E.$$

Proposition 4.3.10. Let A be a unital C^*-algebra, $G \curvearrowright A$, X be a free G-boundary. Supppose that J is a closed ideal in $(A \otimes C(X)) \rtimes_r G$ and define $J_A := J \cap (A \otimes C(X))$. Then we have inclusions of ideals

$$J_A \rtimes_r G \subseteq J \subseteq J_A \bar{\rtimes}_r G.$$

Proof.
We begin by noting that J_A will is a G-invariant ideal: If $s \in G$ and $y \in J_A$ then $s.y = \lambda_s y \lambda_s^* \in J$ and as $A \otimes C(X)$ is a G-C^*-algebra also $s.y \in A \otimes C(X)$ whenever $y \in A \otimes C(X)$.

Now, let $x \in X$ be arbitrary and define $J_A^x := \mathrm{id}_A \otimes \delta_x(J_A) \subseteq A \otimes \mathbb{C} \cong A$, where $\delta_x\colon f \mapsto f(x)$ for $f \in C(X)$. Then J_A^x is an ideal of A, however it could be improper. We then obtain a surjective $*$-homomorphism $\pi_x\colon (A \otimes C(X))/J_A \to A/J_A^x$ such that the diagram

$$\begin{tikzcd}
0 \arrow[r] & J_A \arrow[r] \arrow[d, "\mathrm{id}\otimes\delta_x"] & A \otimes C(X) \arrow[r, "q"] \arrow[d, "\mathrm{id}\otimes\delta_x"] & (A \otimes C(X))/J_A \arrow[r] \arrow[d, dashed, "\pi_x"] & 0 \\
0 \arrow[r] & J_A^x \arrow[r] & A \arrow[r, "q_x"] & A/J_A^x \arrow[r] & 0
\end{tikzcd}$$

commutes with exact rows. We claim that if $y \in \cap_{x \in X} \ker \pi_x$ then $y = 0$. To see this, let $\varepsilon > 0$ and pick some irreducible representation π of $(A \otimes C(X))/J_A$ on a Hilbert space $\mathcal{H}$ such that $\|\pi(a)\| \geq \|a\| - \varepsilon$.

Every element of $(A \otimes C(X))/J_A$ commutes with elements of the form $[1 \otimes f]_{J_A}$, $f \in C(X)$, hence $\pi\left((1 \otimes C(X))/J_A\right) \subseteq \pi(A \otimes C(X))/J_A)' \cong \mathbb{C}1_{\mathcal{H}}$, as π is irreducible so the commutant is trivial. This implies that the composition

$$C(X) \longrightarrow 1 \otimes C(X) \longrightarrow (1 \otimes C(X))/J_A \xrightarrow{\pi} \mathbb{C}1_{\mathcal{H}}$$

defines a character on $C(X)$ and thus is of the form $\pi([1 \otimes f]) = \delta_x(f)1_{\mathcal{H}}$, $f \in C(X)$, for some $x \in X$. Define yet another representation π_A^x by the composition

$$A \longrightarrow A \otimes 1 \longrightarrow (A \otimes 1)/J_A \xrightarrow{\pi} \mathbb{B}(\mathcal{H}),$$

i.e., $\pi_A^x \colon a \mapsto \pi([a \otimes 1])$. This gives us the following commutative diagram

$$\begin{tikzcd}
A \otimes C(X) \arrow[r, "q"] \arrow[d, "\mathrm{id}\otimes\delta_x"] \arrow[dr, "\pi_A\otimes\delta_x"] & (A \otimes C(X))/J_A \arrow[d, "\pi"] \\
A \arrow[r, "\pi_A^x"] & \mathbb{B}(\mathcal{H})
\end{tikzcd}$$

Finishing the jigsaw puzzle, if $a \in A \otimes C(X)$ is such that $\pi_x([a]_{J_A}) = 0$, then $\mathrm{id} \otimes \delta_x(a) \in J_a^x$. Pick a representative $b \in J_A$ such that $\mathrm{id} \otimes \delta_x(a) = \mathrm{id} \otimes \delta_x(b)$. Then

$$\pi([a]) = \pi(q(a)) = \pi(q(b)) = \pi([b]) = 0.$$

Letting $\varepsilon \to 0$, we obtain that $[a] = 0$ whenever all $\pi_x([a]) = 0$, as the norm of A is given by the supremum over norms under irreducible representations.

The rest of the proof follows analogously to the proof of the Archbold-Spielberg result: Let $x \in X$ such that $J_A^x \neq A$, and assume $A/J_A^x \subseteq \mathbb{B}(\mathcal{H})$ via some faithful representation π. Consider the composition

$$J + A \otimes C(X) \xrightarrow{Q} (J + A \otimes C(X))/J \cong (A \otimes C(X))/J_A \xrightarrow{\pi_x} A/J_A^x \xrightarrow{\pi} \mathbb{B}(\mathcal{H}), \tag{4.4}$$

and use Arveson's theorem to extend the above composition to a unital completely positive map $\Phi_x \colon (A \otimes C(X)) \rtimes_r G \to \mathbb{B}(\mathcal{H})$. We claim that $\Phi_x \circ E = \Phi_x$, where E is the canonical conditional expectation onto $A \otimes C(X)$: This is true, for $A \otimes C(X)$ is in the multiplicative domain of Φ_x and

$$\Phi_x(1 \otimes f) = \pi_x(q(1 \otimes f)) = f(x).$$

Let $s \in G \backslash \{e\}$. By assumption that the action of G on X being free, we have $s.x \neq x$ so there is some $h \in C(X)$, $0 \leq h \leq 1$, such that $h(x) = 1$ and $h(s^{-1}.x) = 0$. Then, we see that

$$\Phi_x(\lambda_s) = \Phi_x(h)\Phi_x(\lambda_s)\Phi_x(h) = \Phi_x(h\lambda_s h) = \Phi_x(h(s.h)\lambda_s) = h(x)h(s^{-1}.x)\Phi_x(\lambda_s) = 0,$$

finishing the claim, as s was arbitrary different from e. It follows that $\Phi_x \circ E = \Phi_x$ and in particular

$$\pi_x(Q(E(J))) = \Phi_x(E(J)) = \Phi_x(J) = 0$$

for all $x \in X$ so $E(J) \subseteq J_A$. By construction, $E(J) \subseteq J_A$ implies that $J \subseteq J_A \bar{\rtimes}_r G$, as wanted. That $J_A \rtimes_r G \subseteq J$ is obvious. $\square$

We will need the following small lemma for the big theorem of this section, and we note that the lemma holds in a more general setting if we weaken the requirement a bit, however, this is out of our scope.

Lemma 4.3.11. For G discrete, each stabilizer subgroup G_x for $x \in \partial_F G$ is amenable.

For a proof, see [Oza14, lemma 11]. With these results in hand, we may prove a (super substantial) result, again due to [Bre+14] and [KK14]:

Theorem 4.3.12. For a discrete group G, the following are equivalent:

1. G is C^*-simple.
2. $C(X) \rtimes_r G$ is simplie for all G-boundaries X.
3. $C(\partial_F G) \rtimes_r G$ is simple.
4. The action $G \curvearrowright \partial_F G$ is free.
5. The action $G \curvearrowright \partial_F G$ is topologically free.
6. There is some G-boundary X such that the action is topologically free.
7. Whenever X is a compact minimal G-space, amenability of G_x for some $x \in X$ implies topological freeness of the action on X.
8. If $G \overset{\alpha}{\curvearrowright} A$ with A unital, then $A \rtimes_{\alpha,r} G$ is simple whenever A is G-simple.

Proof.
$1 \implies 2$: Let G be a C^*-simple group and X a G-boundary. Let I be an ideal of $C(X) \rtimes_r G$, and let π denote the quotient map onto $A := C(X) \rtimes_r G/I$, so that $I = \ker \pi$. We then have the following observations:

- By simplicity of simplicity of $C_r^*(G)$, we may consider the canoncial tracial state as a map $\tau_0 \colon \pi(C_r^*(G)) \to \mathbb{C}$, $\pi(\lambda_s) \mapsto \tau_0(\lambda_s)$. which will be G-invariant with respect to $\mathrm{Ad}\lambda_s$.
- Identifying $\mathbb{C}$ with $\mathbb{C}1 \subseteq C(\partial_F G)$, we may use lemma 4.3.6 to extend τ_0 to a G-equivariant completely positive map $\Phi \colon \pi(C(X) \rtimes_r G) \to C(\partial_F G)$.
- As previously, $\Phi \circ \pi$ restricts to the unique G-equivariant map $C(X) \to C(\partial_F G)$, so it must be the inclusion map. In particular, this implies that $C(X)$ is in the multiplicative domain of $\Phi \circ \pi$.

With these observations, we see that

$$\Phi(\pi(f\lambda_s)) = \delta_e(s)\Phi(\pi(f)) = E(f)$$

after identifying $C(X) \subseteq C(\partial_F G)$. In particular, this implies that π must be faithfull, since E is, and since π is a $*$-homomorphism it is injective. In particular, since each G_x are amenable for $X = \partial_F G$ by lemma 4.3.11, simplicity of $C(\partial_F G) \rtimes_r G$ implies topological freeness (and freeness, since $\partial_F G$ is extremally disconnected as mentioned earlier) of the action.

$2 \implies 5$: Consider a G-C^*-algebra A which is G-simple and unital. Let I be a proper ideal of $A \rtimes_r G$. Let J be the ideal of $(A \otimes C(\partial_F G)) \rtimes_r G$ generated by I after identifying $A \rtimes_r G \subseteq (A \otimes C(\partial_F G)) \rtimes_r G$, and let i denote this identification of $A \rtimes_r G$. By proposition 4.3.7, it is a proper ideal, so by proposition 4.3.10, the G-invariant ideal $J_A = J \cap (A \otimes C(\partial_F G))$ satisfies

$$J_A \rtimes_r G \subseteq J \subseteq J_A \bar{\rtimes}_r G.$$

We then obtain an ideal $I_A := J \cap i(A) \cong i^{-1}(i(A))$ of A which is G-invariant: $i(s.a) = i(\lambda_s a \lambda_s^*) = \lambda_s i(a) \lambda_s^* = s.i(a)$, for all $s \in G$ and $a \in A$, which by assumption forces $I_A = \{0\}$. This gives rise to the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & I_A \bar{\rtimes}_r G \arrow[r] & A \rtimes_r G \arrow[r, "\pi_A"] \arrow[d, hook, "i"] & (A/I_A) \rtimes_r G \arrow[r] \arrow[d, hook, "\psi"] & 0 \\
0 \arrow[r] & J_A \bar{\rtimes}_r G \arrow[r] & (A \otimes C(X)) \rtimes_r G \arrow[r, "\pi"] & ((A \otimes C(X))/J_A) \rtimes_r G \arrow[r] & 0
\end{tikzcd}$$

It is easy to see that $I \subseteq I_A \bar{\rtimes}_r G$: if $y \in I$, then $i(y) \in J \subseteq J_A \bar{\rtimes}_r G$. By the above diagram, this implies that $y \in \ker \pi = I_A \bar{\rtimes}_r G$.

We finish the claim by showing that $I_A \bar{\rtimes}_r G = 0$: For all $y \in I_A \rtimes_r G$ we have $E_A(y^* y) \in I_A = 0$, so $y = 0$ by faithfulness. This shows that $I = 0$, since $I \subseteq I_A \bar{\rtimes}_r G = 0$.

$5 \implies 1$: Trivial when applied to the trivial G-boundary $\{\cdot\}$.

$5 \implies 4 \implies 3 \implies 2$: follows from proposition 4.3.1 and proposition 4.3.4. ☐

Corollary 4.3.13. If G is C^*-simple, then $C(X) \rtimes_r G$ is simple for all compact minimal G-spaces X.

Note 4.3.14
As mentioned in the original article, the above result (and the rest of the paper) provided a much needed take on the theory of C^*-simplicity of discrete groups. Prior to this, the most common way of checking C^*-simplicity was the technique initially found by Powers (see [Pow+75] or the author's take on the subject [Kar17, chapter 3]) in his proof for C^*-simplicity of $\mathbb{F}_2$. As mentioned, the new results of [Bre+14] provided a route to an answer of the long lasting question of equivalence of C^*-simplicity and the unique trace property.

We include also a result of the late Uffe Haagerup, which was based on the above article as well, and characterized C^*-simplicity and uniqueness of trace in terms of convex hulls and states and an averaging property similar to that of Powers' original paper. For a proof, see the original article [Haa15] or the author's discussion and covering of the result [Kar17, Chapter 5]:

Theorem 4.3.15. Let G be a discrete countable group with canonical tracial state τ_0 on $C_r^*(G)$. Then the following are equivalent:

1. G is C^*-simple,
2. G admits a free boundary action,
3. $\tau_0 \in \overline{\{s.\varphi \mid s \in G\}}^{w^*}$ for all states φ on $C_r^*(G)$, where $s.\varphi(a) = \varphi(\lambda_s a \lambda_s^*)$,
4. $\tau_0 \in \overline{\mathrm{conv}}^{w^*} \{s.\varphi \mid s \in G\}$, for all states φ on $C_r^*(G)$,
5. for all $t_1, \ldots, t_m \in G \backslash \{e\}$ and $\varepsilon > 0$ there exists $s_1, \ldots, s_n \in G$ such that
$$\left\| \frac{1}{n} \sum_{j=1}^{n} \lambda_{s_j t_k s_j^{-1}} \right\| < \varepsilon,$$
for $k = 1, \ldots, m$,
6. $C_r^*(G)$ has the Dixmier property.

Bibliography

[Arv12] William Arveson. *An invitation to C*-algebras*. Vol. 39. Springer Science & Business Media, 2012.

[AS94] Robert J Archbold and Jack S Spielberg. "Topologically free actions and ideals in discrete C*-dynamical systems". In: *Proceedings of the Edinburgh Mathematical Society* 37.1 (1994), pp. 119–124.

[BCR84] Christian Berg, Jens Peter Reus Christensen, and Paul Ressel. "Harmonic analysis on semigroups". In: (1984).

[Bla06] Bruce Blackadar. *Operator algebras: theory of C*-algebras and von Neumann algebras*. Vol. 122. Springer Science & Business Media, 2006.

[Bla96] Étienne Blanchard. "Déformations de C*-algèbres de Hopf". In: *Bulletin de la Société mathématique de France* 124.1 (1996), p. 141.

[BLHV08] Bachir Bekka, Pierre de La Harpe, and Alain Valette. *Kazhdan's property (T)*. Vol. 11. Cambridge university press, 2008.

[BO08] Nathanial Patrick Brown and Narutaka Ozawa. *C*-algebras and finite-dimensional approximations*. Vol. 88. American Mathematical Soc., 2008.

[BR12] Ola Bratteli and Derek William Robinson. *Operator Algebras and Quantum Statistical Mechanics: Volume 1: C*-and W*-Algebras. Symmetry Groups. Decomposition of States*. Springer Science & Business Media, 2012.

[Bre+14] Emmanuel Breuillard et al. "C*-simplicity and the unique trace property for discrete groups". In: *Publications mathématiques de l'IHÉS* 126.1 (2014), pp. 35–71.

[Con73] Alain Connes. *Une classification des facteurs de type III*. 1973.

[Dix69] Jacques Dixmier. "Les C* algLbres et leurs reprØsentations". In: (1969).

[DLH07] Pierre De La Harpe. "On simplicity of reduced C*-algebras of groups". In: *Bulletin of the London Mathematical Society* 39.1 (2007), pp. 1–26.

[Ell80] George A Elliott. "Some simple C*-algebras constructed as crossed products with discrete outer automorphism groups". In: *Publications of the Research Institute for Mathematical Sciences* 16.1 (1980), pp. 299–311.

[Fol13] Gerald B Folland. *Real analysis: modern techniques and their applications*. John Wiley & Sons, 2013.

[Fol16] Gerald B Folland. *A course in abstract harmonic analysis*. Chapman and Hall/CRC, 2016.

[Fro71] Zdeněk Frolík. "Maps of extremally disconnected spaces, theory of types, and applications". In: *General topology and its relations to modern analysis and algebra* (1971), pp. 131–142.

[Fur03] Alex Furman. "On minimal, strongly proximal actions of locally compact groups". In: *Israel Journal of Mathematics* 136.1 (2003), pp. 173–187.

[Gla76] Shmuel Glasner. "Proximal flows". In: *Proximal Flows*. Springer, 1976, pp. 17–29.

[GO81] Elliot C Gootman and Dorte Olesen. "Spectra of actions on type I C*-algebras". In: *Mathematica Scandinavica* (1981), pp. 329–349.

[GR79] Elliot C Gootman and Jonathan Rosenberg. "The structure of crossed productC*-algebras: a proof of the generalized Effros-Hahn conjecture". In: *Inventiones mathematicae* 52.3 (1979), pp. 283–298.

[Haa15] Uffe Haagerup. "A new look at C*-simplicity and the unique trace property of a group". In: *arXiv preprint arXiv:1509.05880* (2015).

[Ham79] Masamichi Hamana. "Injective envelopes of C*-algebras". In: *Journal of the Mathematical Society of Japan* 31.1 (1979), pp. 181–197.

[Ham85] Masamichi Hamana. "INJECTIVE ENVELOPES OF {C^*}-DYNAMICAL SYSTEMS". In: *Tohoku Mathematical Journal, Second Series* 37.4 (1985), pp. 463–487.

[Kar16] Malthe Karbo. *Course Presentation on Amenable Actions and Inner Amenability*. Lecture Notes for Course Presentation, 2016+2018. URL: `https://github.com/mkarbo/uni/blob/master/Talks/Approx(2017%262018)/all_approx.pdf`.

[Kar17] Malthe Karbo. *Simplicity and uniqueness of trace of group C^*-algebras*. Bachelor's thesis, 2017. URL: `https://github.com/mkarbo/uni/blob/master/Projects/Bachelorproject/master.pdf`.

[Kar18] Malthe Karbo. *Fourier Analysis and the theory of representation of topological groups*. Written Project, 2018. URL: `https://github.com/mkarbo/uni/blob/master/Projects/Fourier%20Analysis%20And%20The%20Theory%20Of%20Representations%20In%20Topological%20Groups/projekt.pdf`.

[KK14] Mehrdad Kalantar and Matthew Kennedy. "Boundaries of reduced C*-algebras of discrete groups". In: *Journal für die reine und angewandte Mathematik (Crelles Journal)* 2014.727 (2014), pp. 247–267.

[KR67] Richard V Kadison and John R Ringrose. "Derivations and automorphisms of operator algebras". In: *Communications in Mathematical Physics* 4.1 (1967), pp. 32–63.

[Lan79] Magnus B Landstad. "Duality theory for covariant systems". In: *Transactions of the American Mathematical Society* 248.2 (1979), pp. 223–267.

[Lan95] E Christopher Lance. *Hilbert C*-modules: a toolkit for operator algebraists*. Vol. 210. Cambridge University Press, 1995.

[LB17] Adrien Le Boudec. "C*-simplicity and the amenable radical". In: *Inventiones mathematicae* 209.1 (2017), pp. 159–174.

[LOP79] Magnus B Landstad, Dorte Olesen, and Gert Kjærgård Pedersen. "Towards a Galois theory for crossed products of C*-algebras". In: *Mathematica Scandinavica* 43.2 (1979), pp. 311–321.

[Mur14] Gerald J Murphy. *C*-algebras and operator theory*. Academic press, 2014.

[OP78] Dorte Olesen and Gert K Pedersen. "Applications of the Connes spectrum to C^*-dynamical systems". In: *Journal of Functional Analysis* 30.2 (1978), pp. 179–197.

[OP80] Dorte Olesen and Gert K Pedersen. "Applications of the Connes spectrum to C^*-dynamical systems, II". In: *Journal of Functional Analysis* 36.1 (1980), pp. 18–32.

[OP82] Dorte Olesen and Gert K Pedersen. "Applications of the Connes spectrum to C^*-dynamical systems, III". In: *Journal of Functional Analysis* 45.3 (1982), pp. 357–390.

[Oza14] Narutaka Ozawa. "Lecture on the Furstenberg boundary and C^*-simplicity". In: *Annual Meeting of Operator Theory and Operator Algebras (available online), Tokyo*. 2014.

[Ped79] Gert K Pedersen. "C^*-algebras and their automorphism groups". In: *London Mathematical Society Monographs* (1979).

[Ped89] Gert K Pedersen. "Analysis now, volume 118 of Graduate Texts in Mathematics". In: *Springer* 7 (1989), pp. 23–29.

[Pow+75] Robert T Powers et al. "Simplicity of the C*-algebra associated with the free group on two generators". In: *Duke Math. J* 42.1 (1975), pp. 151–156.

[Rae88] Iain Raeburn. "On crossed products and Takai duality". In: *Proceedings of the Edinburgh Mathematical Society* 31.2 (1988), pp. 321–330.

[Rud91] Walter Rudin. *Functional analysis. International series in pure and applied mathematics.* 1991.

[RW98] Iain Raeburn and Dana P Williams. *Morita equivalence and continuous-trace C*-algebras.* 60. American Mathematical Soc., 1998.

[Rør+00] Mikael Rørdam et al. *An Introduction to K-theory for C*-algebras.* Vol. 49. Cambridge University Press, 2000.

[Sak78] Shôichirô Sakai. "Recent developments in the theory of unbounded derivations in C*-algebras". In: *C*-algebras and applications to physics.* Springer, 1978, pp. 85–122.

[Sch17] René L Schilling. *Measures, Integrals and Martingales.* Cambridge University Press, 2017.

[Tak75] Hiroshi Takai. "On a duality for crossed products of C^*-algebras". In: *Journal of Functional Analysis* 19.1 (1975), pp. 25–39.

[Wil07] Dana P Williams. *Crossed products of C*-algebras.* 134. American Mathematical Soc., 2007.

[Zhu93] Kehe Zhu. *An introduction to operator algebras.* Vol. 9. CRC press, 1993.

Index