

# Simplicity and uniqueness of trace of group $C^*$ -algebras

Bachelor thesis defense

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February 3, 2017

- 1 Introduction, history and motivation
- 2 An introduction to various topics
  - Group  $C^*$ -algebras
  - Amenability
  - Group actions and the Reduced crossed product
  - Dixmier property
- 3 Boundary actions
- 4  $C^*$ -simplicity using boundary actions

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- In 1975, Robert T. Powers proved that  $\mathbb{F}_2$  was  $C^*$ -simple and that this implied the unique trace property of  $\mathbb{F}_2$ .

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Today we see for discrete countable group  $G$  :

$$C_r^*(G) \text{ simple} \implies C_r^*(G) \text{ unique trace} \iff \text{trivial amenable radical of } G.$$

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## A three minute crash-course on group $C^*$ -algebras

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### Unitary representations

A *unitary representation* of a discrete group  $G$  is a pair  $(u, \mathcal{H})$ , where  $\mathcal{H}$  is a Hilbert space and  $u: G \rightarrow \mathcal{U}(\mathcal{H})$  is a group homomorphism into the group of unitary operators on  $\mathcal{H}$ . We use the following notation for unitary representations:

$$u(g) := u_g, \quad g \in G.$$

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## Group $C^*$ -algebra

Given a unitary representation  $(u, \mathcal{H})$  of a discrete group  $G$ , we define the *group  $C^*$ -algebra* associated to  $u$  by  $C_u^*(G) := C^*(\{u_g: g \in G\}) \subseteq B(\mathcal{H})$ , the  $C^*$ -subalgebra of  $B(\mathcal{H})$  generated by the unitaries  $\{u_g\}_{g \in G}$ .

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The pair  $(\lambda, \ell^2(G))$ , where  $\lambda: G \rightarrow \mathcal{U}(\ell^2(G))$  is defined by

$$(\lambda_g \xi)(s) = \xi(g^{-1}s), \quad s, g \in G, \quad \xi \in \ell^2(G),$$

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is called the *left-regular representation* of  $G$ , and the associated group  $C^*$ -algebra,

$$C_r^*(G) := C_\lambda^*(G) \subseteq B(\ell^2(G)),$$

called the *reduced group  $C^*$ -algebra* of  $G$ .



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The set  $\{\delta_g\}_{g \in G} \subseteq \ell^2(G)$ , where for  $s, g \in G$  we define

$$\delta_s(g) = \begin{cases} 1 & \text{if } s = g \\ 0 & \text{else} \end{cases},$$

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### Lemma

For discrete group  $G$ , the map  $\tau_0: C_r^*(G) \rightarrow \mathbb{C}$  defined by  $\tau_0(x) = \langle x\delta_e, \delta_e \rangle$  is a faithful tracial state on  $C_r^*(G)$ .

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For a discrete group  $G$ , we say that  $G$  is  $C^*$ -*simple* if the reduced group  $C^*$ -algebra,  $C_r^*(G)$ , is simple. We say that  $G$  has the *unique trace property* if  $\tau_0$ , defined as above, is the only tracial state on  $C_r^*(G)$ .

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Amenability is a strong property to satisfy, and has many consequences.



## Group actions and the Reduced crossed product

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An *action* of  $G$  on  $\mathcal{A}$  is a group homomorphism  $\alpha: G \rightarrow \text{Aut}(\mathcal{A})$  from  $G$  to the group of  $*$ -automorphism on  $\mathcal{A}$ . Abbreviated  $G \overset{\alpha}{\curvearrowright} \mathcal{A}$ .

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Whenever  $G \overset{\alpha}{\curvearrowright} \mathcal{A} \curvearrowright$  we make a  $C^*$ -algebra  $\mathcal{A} \rtimes_{\alpha,r} G$  which contains  $\mathcal{A}$  and  $C_r^*(G)$  in a nice way. This  $C^*$ -algebra is called the reduced crossed product.

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④  $G \curvearrowright X \rightsquigarrow G \curvearrowright C(X)$  by equipping  $G$  with discrete topology and defining

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A subset  $Y \subseteq X$  of a  $G$ -space  $X$  is *minimal* if it is non-empty, closed and  $G$ -invariant and a minimal element in the set of non-empty, closed and  $G$ -invariant subsets of  $X$ .

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An application of Zorn's lemma yields that every compact  $G$ -space  $X$  has a minimal subspace. Moreover a subset  $Y$  is minimal if and only if the action  $G \curvearrowright Y$  is minimal

## Definition

For a  $G$ -space  $X$ , the action of  $G$  is *strongly proximal* if for each  $\mu \in \text{Prob}(X)$  we have

$$\overline{G \cdot \mu}^{w^*} \cap \{\delta_x : x \in X\} \neq \emptyset.$$

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## Definition

If  $X$  is a compact  $G$ -space, we say that the action  $G \curvearrowright X$  is a *boundary action* if it is minimal and strongly proximal. A compact  $G$ -space  $X$  for which the action is a boundary action is called a  *$G$ -boundary*.

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### Proposition

For a compact  $G$ -space  $X$ , TFAE:

- ❶  $X$  is a  $G$ -boundary,
- ❷ For every  $\nu \in \text{Prob}(X)$  we have  $\{\delta_x : x \in X\} \subseteq \overline{G \cdot \nu}^{w^*}$  and
- ❸  $\text{Prob}(X)$  admits no non-trivial closed convex subsets which are  $G$ -invariant under the induced action  $G \curvearrowright \text{Prob}(X)$ .

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In laymans terms, a  $G$ -boundary is a space such that every  $G$ -orbit in  $\text{Prob}(X)$  becomes a black hole for all the point masses when going to weak\* closure.

## Properties of different actions

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- The  $G$ -boundary property is preserved under quotients,
- Strongly proximality is preserved under taking products,
- If  $G \curvearrowright Y$  is minimal and  $X$  is a  $G$ -boundary, then any continuous  $G$ -map  $\varphi: Y \rightarrow X$  will be unique and surjective.

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# Large $G$ -boundaries

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- 3 Use Zorn's lemma to pick a minimal subset.

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## Proposition

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And more generally we have:

## Proposition (Furman)

For any countable discrete group  $G$  with  $t \in G$ , it holds that  $t \in \text{AR}_G$  if and only if  $t \in \ker(G \curvearrowright X)$  for all compact  $G$ -boundaries  $X$ .

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- ❺  $G \curvearrowright X \rightsquigarrow G \curvearrowright \mathcal{S}(C(X) \rtimes_r G)$ : for  $t \in G$  and any state  $\varphi$  define  $t.\varphi$  by

$$t.\varphi(a) = \varphi(\lambda_t a \lambda_t^*), \quad a \in C(X) \rtimes_r G.$$

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### Theorem

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The proof of this relies on the fact that given a normal amenable subgroup  $N \trianglelefteq G$ , there is a  $*$ -homomorphism  $\pi: C_r^*(G) \rightarrow C_r^*(G/N)$  such that

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Using this, one creates a trace  $\tau$  on  $C_r^*(G)$  which is 0 on  $\lambda_t$  for all  $t \notin \text{AR}_G$ .

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### Theorem

Let  $G$  be a group and  $t \in G$ . Then  $t \notin \text{AR}_G$  if and only if

$$0 \in \overline{\text{conv}} \{ \lambda_{sts^{-1}} : s \in G \}.$$

Proposition [Matthew Kennedy, Emmanuel Breuillard, Merhdad Kalatar and Narutaka Ozawa, 2014]

A discrete group  $G$  is  $C^*$ -simple if and only if there is a compact  $G$ -boundary  $X$  such that the action is free.

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What is this? Why is this interesting?

## $C^*$ -simplicity and uniqueness of trace using boundary actions

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# $C^*$ -simplicity and uniqueness of trace using boundary actions

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## Theorem

For a group  $G$  TFAE:

- ❶  $C_r^*(G)$  is simple,
- ❷  $\tau_0 \in \overline{\{s.\varphi : s \in G\}}^{w^*}$  for all states  $\varphi$  on  $C_r^*(G)$ ,
- ❸  $\tau_0 \in \overline{\text{conv}}^{w^*} \{s.\varphi : s \in G\}$  for all states  $\varphi$  on  $C_r^*(G)$ ,
- ❹  $\omega(1)\tau_0 \in \overline{\text{conv}} \{s.\omega : s \in G\}$  for all  $\omega \in (C_r^*(G))^*$ ,
- ❺ For all  $t_1, t_2, \dots, t_m \in G \setminus \{e\}$ , we have

$$0 \in \overline{\text{conv}}^{w^*} \{ \lambda_s (\lambda_{t_1} + \lambda_{t_2} + \dots + \lambda_{t_m}) \lambda_s^* : s \in G \},$$

- ❻ For all  $t_1, t_2, \dots, t_m \in G \setminus \{e\}$  and  $\varepsilon > 0$ , there exists  $s_1, s_2, \dots, s_n \in G$  such that

$$\left\| \sum_{k=1}^n \frac{1}{n} \lambda_{s_k t_j s_k^{-1}} \right\| < \varepsilon,$$

for all  $1 \leq j \leq m$ .

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## Theorem

Let  $G$  be a discrete group and  $t \in G$ . TFAE:

- ①  $t \notin \text{AR}_G$ ;
- ② there is a boundary action  $G \curvearrowright X$  such that  $t$  acts non-trivially on  $X$ ;
- ③  $\lambda_t \in \ker(\tau)$  for all tracial states  $\tau$  on  $C_r^*(G)$ ;
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## Theorem

Let  $G$  be a discrete group. TFAE:

- ❶  $C_r^*(G)$  has unique tracial state;
- ❷  $G$  admits a faithful boundary action;
- ❸  $\text{AR}_G = \{e\}$ ;
- ❹ for all  $t \in G \setminus \{e\}$  and all  $\varepsilon > 0$ , there is  $s_1, s_2, \dots, s_n \in G$  such that

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## Theorem

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- ❶  $C_r^*(G)$  is simple;
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- ❻  $C_r^*(G)$  has the Dixmier property.

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$\text{AR}_G = \{e_G\} \iff G \text{ has unique trace property} \iff \text{for all } t \in G \setminus \{e\} \text{ and all } \varepsilon > 0, \text{ there is } s_1, s_2, \dots, s_n \in G \text{ such that}$

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Thus

$C_r^*(G)$  simple  $\implies C_r^*(G)$  unique trace  $\iff$  trivial amenable radical of  $G$ .



The end.  
thanks for listening