

Simplicity and uniqueness of trace of group C^* -algebras

Bachelor thesis defense

Malthe Munk Karbo

Advisor: Mikael Rørdam

February 3, 2017

- 1 Introduction, history and motivation
- 2 An introduction to various topics
 - Group C^* -algebras
 - Amenability
 - Group actions and the Reduced crossed product
 - Dixmier property
- 3 Boundary actions
- 4 C^* -simplicity using boundary actions

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Today we see for discrete countable group G :

$$C_r^*(G) \text{ simple} \implies C_r^*(G) \text{ unique trace} \iff \text{trivial amenable radical of } G.$$

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A three minute crash-course on group C^* -algebras

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Unitary representations

A *unitary representation* of a discrete group G is a pair (u, \mathcal{H}) , where \mathcal{H} is a Hilbert space and $u: G \rightarrow \mathcal{U}(\mathcal{H})$ is a group homomorphism into the group of unitary operators on \mathcal{H} . We use the following notation for unitary representations:

$$u(g) := u_g, \quad g \in G.$$

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Group C^* -algebra

Given a unitary representation (u, \mathcal{H}) of a discrete group G , we define the *group C^* -algebra* associated to u by $C_u^*(G) := C^*(\{u_g: g \in G\}) \subseteq B(\mathcal{H})$, the C^* -subalgebra of $B(\mathcal{H})$ generated by the unitaries $\{u_g\}_{g \in G}$.

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$$(\lambda_g \xi)(s) = \xi(g^{-1}s), \quad s, g \in G, \quad \xi \in \ell^2(G),$$

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is called the *left-regular representation* of G , and the associated group C^* -algebra,

$$C_r^*(G) := C_\lambda^*(G) \subseteq B(\ell^2(G)),$$

called the *reduced group C^* -algebra* of G .

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The set $\{\delta_g\}_{g \in G} \subseteq \ell^2(G)$, where for $s, g \in G$ we define

$$\delta_s(g) = \begin{cases} 1 & \text{if } s = g \\ 0 & \text{else} \end{cases},$$

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Lemma

For discrete group G , the map $\tau_0: C_r^*(G) \rightarrow \mathbb{C}$ defined by $\tau_0(x) = \langle x\delta_e, \delta_e \rangle$ is a faithful tracial state on $C_r^*(G)$.

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Definition

For a discrete group G , we say that G is C^* -*simple* if the reduced group C^* -algebra, $C_r^*(G)$, is simple. We say that G has the *unique trace property* if τ_0 , defined as above, is the only tracial state on $C_r^*(G)$.

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Amenability is a strong property to satisfy, and has many consequences.

Group actions and the Reduced crossed product

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Whenever $G \overset{\alpha}{\curvearrowright} \mathcal{A} \curvearrowright$ we make a C^* -algebra $\mathcal{A} \rtimes_{\alpha,r} G$ which contains \mathcal{A} and $C_r^*(G)$ in a nice way. This C^* -algebra is called the reduced crossed product.

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A G -action of G on X is a continuous group homomorphism $\alpha: G \rightarrow \text{Homeo}(X)$. We write $G \curvearrowright X$ to mean that G acts on X , and as before, we write $\alpha(g)(x) := g.x$ for $x \in X$ and $g \in G$.

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④ $G \curvearrowright X \rightsquigarrow G \curvearrowright C(X)$ by equipping G with discrete topology and defining

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- ② $G \curvearrowright X \rightsquigarrow G \curvearrowright \text{Prob}(X)$ by equipping G with discrete topology and defining

$$g.\mu(E) := \mu(g^{-1}.E), \quad \mu \in \text{Prob}(X), \quad g \in G \text{ and } E \subseteq X \text{ Borel},$$

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An application of Zorn's lemma yields that every compact G -space X has a minimal subspace. Moreover a subset Y is minimal if and only if the action $G \curvearrowright Y$ is minimal

Definition

For a G -space X , the action of G is *strongly proximal* if for each $\mu \in \text{Prob}(X)$ we have

$$\overline{G \cdot \mu}^{w^*} \cap \{\delta_x : x \in X\} \neq \emptyset.$$

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Definition

If X is a compact G -space, we say that the action $G \curvearrowright X$ is a *boundary action* if it is minimal and strongly proximal. A compact G -space X for which the action is a boundary action is called a *G -boundary*.

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Proposition

For a compact G -space X , TFAE:

- ❶ X is a G -boundary,
- ❷ For every $\nu \in \text{Prob}(X)$ we have $\{\delta_x : x \in X\} \subseteq \overline{G \cdot \nu}^{w^*}$ and
- ❸ $\text{Prob}(X)$ admits no non-trivial closed convex subsets which are G -invariant under the induced action $G \curvearrowright \text{Prob}(X)$.

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In laymans terms, a G -boundary is a space such that every G -orbit in $\text{Prob}(X)$ becomes a black hole for all the point masses when going to weak* closure.

Properties of different actions

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- Strongly proximality is preserved under taking products,
- If $G \curvearrowright Y$ is minimal and X is a G -boundary, then any continuous G -map $\varphi: Y \rightarrow X$ will be unique and surjective.

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Proposition (Furman)

For any countable discrete group G with $t \in G$, it holds that $t \in \text{AR}_G$ if and only if $t \in \ker(G \curvearrowright X)$ for all compact G -boundaries X .

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- ❺ $G \curvearrowright X \rightsquigarrow G \curvearrowright \mathcal{S}(C(X) \rtimes_r G)$: for $t \in G$ and any state φ define $t.\varphi$ by

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The proof of this relies on the fact that given a normal amenable subgroup $N \trianglelefteq G$, there is a $*$ -homomorphism $\pi: C_r^*(G) \rightarrow C_r^*(G/N)$ such that

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The proof of this relies on the fact that given a normal amenable subgroup $N \trianglelefteq G$, there is a $*$ -homomorphism $\pi: C_r^*(G) \rightarrow C_r^*(G/N)$ such that

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Using this, one creates a trace τ on $C_r^*(G)$ which is 0 on λ_t for all $t \notin \text{AR}_G$.

We now start to show some of results mentioned in the beginning.

Theorem

Let G be a group and $t \in G$. Then $\tau(\lambda_t) = 0$ for all tracial states τ on $C_r^*(G)$ if and only if $t \notin \text{AR}_G$.

Corollary

A group G has the unique trace property if and only if it has trivial amenable radical.

C^* -simplicity using boundary actions

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Theorem

Let G be a group and $t \in G$. Then $t \notin \text{AR}_G$ if and only if

$$0 \in \overline{\text{conv}} \{ \lambda_{sts^{-1}} : s \in G \}.$$

Proposition [Matthew Kennedy, Emmanuel Breuillard, Merhdad Kalatar and Narutaka Ozawa, 2014]

A discrete group G is C^* -simple if and only if there is a compact G -boundary X such that the action is free.

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C^* -simplicity and uniqueness of trace using boundary actions

Using this, we may prove the following:

Theorem

For a group G TFAE:

- ❶ $C_r^*(G)$ is simple,
- ❷ $\tau_0 \in \overline{\{s.\varphi : s \in G\}}^{w^*}$ for all states φ on $C_r^*(G)$,
- ❸ $\tau_0 \in \overline{\text{conv}}^{w^*} \{s.\varphi : s \in G\}$ for all states φ on $C_r^*(G)$,
- ❹ $\omega(1)\tau_0 \in \overline{\text{conv}} \{s.\omega : s \in G\}$ for all $\omega \in (C_r^*(G))^*$,
- ❺ For all $t_1, t_2, \dots, t_m \in G \setminus \{e\}$, we have

$$0 \in \overline{\text{conv}}^{w^*} \{ \lambda_s (\lambda_{t_1} + \lambda_{t_2} + \dots + \lambda_{t_m}) \lambda_s^* : s \in G \},$$

- ❻ For all $t_1, t_2, \dots, t_m \in G \setminus \{e\}$ and $\varepsilon > 0$, there exists $s_1, s_2, \dots, s_n \in G$ such that

$$\left\| \sum_{k=1}^n \frac{1}{n} \lambda_{s_k t_j s_k^{-1}} \right\| < \varepsilon,$$

for all $1 \leq j \leq m$.

To summarize everything, we have

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Theorem

Let G be a discrete group and $t \in G$. TFAE:

- ❶ $t \notin \text{AR}_G$;
- ❷ there is a boundary action $G \curvearrowright X$ such that t acts non-trivially on X ;
- ❸ $\lambda_t \in \ker(\tau)$ for all tracial states τ on $C_r^*(G)$;
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Theorem

Let G be a discrete group. TFAE:

- ❶ $C_r^*(G)$ has unique tracial state;
- ❷ G admits a faithful boundary action;
- ❸ $\text{AR}_G = \{e\}$;
- ❹ for all $t \in G \setminus \{e\}$ and all $\varepsilon > 0$, there is $s_1, s_2, \dots, s_n \in G$ such that

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Theorem

Let G be a discrete group. TFAE:

- ❶ $C_r^*(G)$ is simple;
- ❷ G admits a free boundary action;
- ❸ $\tau_0 \in \overline{\{s.\varphi : g \in G\}}^{w^*}$, for all states φ on $C_r^*(G)$;
- ❹ $\tau_0 \in \overline{\text{conv}}^{w^*} \{s.\varphi : s \in G\}$, for all states φ on $C_r^*(G)$;
- ❺ for all $t_1, t_2, \dots, t_m \in G \setminus \{e\}$ and all $\varepsilon > 0$, there exist $s_1, s_2, \dots, s_n \in G$ such that

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for all $k = 1, 2, \dots, m$;

- ❻ $C_r^*(G)$ has the Dixmier property.

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$\text{AR}_G = \{e_G\} \iff G \text{ has unique trace property} \iff \text{for all } t \in G \setminus \{e\} \text{ and all } \varepsilon > 0, \text{ there is } s_1, s_2, \dots, s_n \in G \text{ such that}$

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Theorem

G is C^* -simple \iff for all $t_1, t_2, \dots, t_m \in G \setminus \{e\}$ and all $\varepsilon > 0$, there exist $s_1, s_2, \dots, s_n \in G$ such that

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for all $k = 1, 2, \dots, m$

Thus

$C_r^*(G)$ simple $\implies C_r^*(G)$ unique trace \iff trivial amenable radical of G .

The end.
thanks for listening