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Simplicity and uniqueness of trace of group C^*-algebras

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Abstract

A discrete group is C^**-simple* if the reduced group C^*-algebra $C_r^*(G)$ has no non-trivial proper two-sided closed ideals, and it has the *unique trace property* if the canonical faithful tracial state on $C_r^*(G)$ is the only tracial state. In 1975 Powers published an article proving that the free group on two generators is C^*-simple and has the unique trace property. His result and calculations have been generalized since, and some of these new results will be dealt with in this thesis. In 2014 [Bre+14] was released, using theory of boundary actions of discrete groups G on compact Hausdorff spaces to classify when G is C^*-simple, respectively, has the unique trace property, and to show that C^*-simplicity implies the unique trace property. Following up on this an article, due to the late Uffe Haagerup ([Haa15]), was released in 2015. This article provided new proofs of precisely when a discrete group G is C^*-simple, respectively, has the unique trace property, using *boundary actions*. We will in this thesis describe C^*-simplicity and the unique trace property in terms of both Dixmier properties and properties of boundary actions of G, found in [Haa15].

Contents

Introduction

The topic of this thesis is C^*-simplicty and the unique trace property of discrete groups. A discrete group G is *C^*-simple* if the reduced group C^*-algebra $C_r^*(G)$ is simple, and G has the *unique trace property* if $C_r^*(G)$ has unique faithful tracial state. This thesis aims to describe these properties both in terms of classical Dixmier properties and in terms of boundary actions of G.

In the first chapter we will discuss amenability of discrete groups G and amenability of normal subgroups of G. We prove that every discrete group G has a maximal amenable normal subgroup, called the *amenable radical* of G.

The second chapter is about the topic of representing a group G unitarily in a C^*-algebra, specifically the *left-regular representation* of G on $B(\ell^2(G))$, and the *reduced group C^*-algebra* $C_r^*(G)$. We also show that this C^*-algebra has a faithful trace. We then show that for a discrete group G acting on a C^*-algebra $\mathcal{A}$, there is a way of constructing a C^*-algebra, called the *reduced crossed product*, which contains a copy of $C_r^*(G)$ and $\mathcal{A}$.

In the third chapter we discuss simplicity and uniqueness of trace of unital C^*-algebras in terms of Dixmier properties. We end the chapter by proving that the free group on two generators is C^*-simple and has unique faithful tracial state.

In the fourth chapter, we discuss *boundary actions*, which are *minimal* and *strongly proximal* actions of second countable locally compact groups on compact Hausdorff spaces. We prove that for such a group G, there is a universal G-boundary, i.e. a compact Hausdorff space X such that the action of G on X is a boundary action, satisfying a certain universal property. This universal G-boundary is called the *Furstenberg boundary*, and we show that every closed amenable normal subgroup of G acts trivially on every G-boundary. We end the chapter by noting that if G is discrete and non-amenable, then there is a boundary action of G which is non-trivial.

The last chapter of this thesis will combine the topics discussed in the preceeding chapters, we will prove sufficient and necessary conditions for discrete groups G to have the unique trace property and to be C^*-simple by combining results about the amenable radical of G, boundary actions and dixmier properties. We also show that C^*-simplicity implies the unique trace property. The results of this chapter are due to the late Uffe Haagerup, and we follow [Haa15].

Preliminaries

This thesis relies heavily on results and notions from topics such as functional analysis, group theory, C^*-algebras, point-set topology and such. We will for the sake of completeness provide a proofless list of definitions and results, which will be used throughout this thesis.

For any set Ω, we denote the powerset of Ω by $\mathcal{P}(\Omega)$. For two disjoint subsets $A, B \in \mathcal{P}(\Omega)$, we use $A \sqcup B$ to denote their disjoint union.

Groups

- We use G to denote a group, and we will frequently use H to denote a subgroup of G.
- The neutral element of G will be denoted by e_G.
- A topological group G is a group G with a topology such that multiplication is jointly continuous and inversion is continuous.

Hilbert spaces

We use $\mathcal{H}$ to denote an arbitary Hilbert space, and the inner product of two elements $\xi, \eta \in \mathcal{H}$ is denoted by $\langle \xi, \eta \rangle$, and if needed we add a subscript to emphasize that it is the inner product in $\mathcal{H}$. For discrete groups G, we will often look at $\ell^2(G)$, the Hilbert space of square summable functions with inner product

$$\langle \xi, \eta \rangle = \sum_{g \in G} \xi(g)\overline{\eta(g)}, \ \xi, \eta \in \ell^2(G).$$

For $s \in G$, the point mass at s is a function $\delta_s \colon G \to \mathbb{C}$ defined by

$$\delta_s(t) = \begin{cases} 1 & \text{if } s = t \\ 0 & \text{else} \end{cases}, \quad t \in G.$$

One may verify that the family of all point masses $\{\delta_s\}_{s \in G}$ is an orthonormal basis for $\ell^2(G)$. All bounded linear functionals on a Hilbert space $\mathcal{H}$ are of the form $f_y(x) = \langle x, y \rangle$ for some unique $y \in \mathcal{H}$ cf. [5.25 Fol13, p. 174]. We denote by $B(\mathcal{H})$ the C^*-algebra of bounded linear operators $T \colon \mathcal{H} \to \mathcal{H}$.

C^*-algebras

A C^*-algebra $\mathcal{A}$ is a complex Banach algebra which is also a $*$-algebra satisfying the C^*-identity;

$$\|x^*x\| = \|x\|^2, \quad \text{for all } a \in \mathcal{A}$$

We use $\mathcal{A}$ to denote an arbitary C^*-algebra. We always assume a C^*-algebra to be unital, and denote the unit of $\mathcal{A}$ by $1_{\mathcal{A}}$.

- A C^*-algebra $\mathcal{A}$ is simple if it contains no non-trivial proper closed two-sides ideals.
- For a C^*-algebra $\mathcal{A}$ and $a \in \mathcal{A}$ we say that
  - a is *self-adjoint* if $a^* = a$. The set of all self-adjoint elements of $\mathcal{A}$ is denoted by $(\mathcal{A})_{\text{s-a}}$,
  - a is *positive* if $a = c^*c$ for some $c \in \mathcal{A}$. We write $a \geq 0$ if and only if a is positve. The *positive cone* of all positive elements in $\mathcal{A}$ is denoted by $(\mathcal{A})_+$,
  - a is a *projection* if $a = a^2 = a^*$, i.e. idempotent and self-adjoint,
  - a is *unitary* if $a^*a = aa^* = 1_{\mathcal{A}}$. The group of all unitary elements of $\mathcal{A}$ is denoted by $\mathcal{U}(\mathcal{A})$.
- A $*$-*homomorphism* $\Phi\colon \mathcal{A} \to \mathcal{B}$ between unital C^*-algebras $\mathcal{A}, \mathcal{B}$ is a unital, multiplicative and linear map which carries involution. We say that two C^*-algebras $\mathcal{A}, \mathcal{B}$ are isomorphic if there is a bijective $*$-homomorphism between them.
- Every commutative C^*-algebra is isomorphic to the C^*-algebra of continuous functions $C(K)$ on some compact Hausdorff space K cf. [Theorem 9.4 Zhu93, p. 56].
- A linear map $\varphi\colon \mathcal{A} \to \mathcal{B}$ between C^*-algebras is said to be positive if it carries positive elements to positive elements, i.e., if $\varphi\left((\mathcal{A})_+\right) \subseteq (\mathcal{B})_+$. Hence a positive continuous linear functional $\varphi \in \mathcal{A}^*$ is a functional such that $\varphi\left((\mathcal{A})_+\right) \subseteq \mathbb{R}_+$. By [Theorem 13.5 Zhu93, p. 79], a functional φ on $\mathcal{A}$ is positive if and only if it is bounded and $\|\varphi\| = \varphi(1)$. For a positive linear functional φ on $\mathcal{A}$ we say that
  - φ is a *state* if $\varphi(1) = 1 = \|\varphi\|$. The space of all states on $\mathcal{A}$ is called the state space of $\mathcal{A}$, and is denoted by $\mathcal{S}(\mathcal{A})$.
  - φ is said to be *faithful* if $\varphi(x) > 0$ for all non-zero positive elements $x \in (\mathcal{A})_+$.
  - φ is said to be a *trace* is it is a state and it is tracial, i.e., if $\varphi(ab) = \varphi(ba)$ for all $a, b \in A$ and $\varphi(1) = 1$.

- A representation of a C^*-algebra on a Hilbert space $\mathcal{H}$ is a $*$-homomorphism $\pi\colon \mathcal{A} \to B(\mathcal{H})$. For any C^*-algebra $\mathcal{A}$, there is a Hilbert space $\mathcal{H}$ such that $\mathcal{A}$ is faithfully represented on it by the GNS-construction [Thoerem 14.4 Zhu93, p. 87]

- A linear map $\varphi\colon \mathcal{A} \to \mathcal{B}$ between C^*-algebras is *c.c.p* (*contractive completely positive*) if for all $n \geq 1$, the map $\varphi_n\colon M_n(A) \to M_n(B)$ defined by

$$\varphi_n([a_{ij}]) := [\varphi(a_{ij})],\ [a_{ij}] \in M_n(A),$$

  is positive and contractive.

- We will by X and Y denote compact topological spaces, and we denote by $C(X)$ the unital C^*-algebra of continuous functions $f\colon X \to \mathbb{C}$ equipped the supremum norm and involution $f \mapsto \overline{f}$.

- Given a bounded linear functional ω on a unital C^*-algebra $\mathcal{A}$, we define its adjoint ω^* by $\omega^*(a) = \overline{\omega(a^*)}$ for $a \in \mathcal{A}$.

- Every bounded linear functional ω on a unital C^*-algebra $\mathcal{A}$ can be decomposed into $\mathrm{Re}\omega + i\mathrm{Im}\omega$, where $\mathrm{Re}\omega, \mathrm{Im}\omega$ are the self-adjoint linear functionals defined by

$$\mathrm{Re}\omega = \frac{\omega + \omega^*}{2},\ \mathrm{Im}\omega = \frac{\omega - \omega^*}{2i}.$$

- Every self-adjoint linear functional γ on $C_r^*(G)$ has a unique *Jordan decomposition*, there exists two positive linear functionals $\gamma_\pm$ on $C_r^*(G)$ such that $\gamma = \gamma_+ - \gamma_-$ which satisfy $\|\gamma\| = \|\gamma_+\| + \|\gamma_-\|$, and they are unique.

- As a consequence of the above, we see that every bounded linear functional can be decomposed into a linear combination of four states.

1. Amenability and The Amenable Radical

For a countable discrete group G, we denote by $\mathrm{Prob}(G)$ the space of all probability measures on G, i.e.,

$$\mathrm{Prob}(G) = \{\mu \in \ell^1(G) \colon \mu \geq 0 \text{ and } \sum_{g \in G} \mu(g) = 1\}.$$

For a set Ω, a map $\mu \colon \mathcal{P}(\Omega) \to [0,1]$ is a finitely additive probability measure on Ω if $\mu(\Omega) = 1$ and for disjoint sets $A, B \in \mathcal{P}(\Omega)$ we have $\mu(A \cup B) = \mu(A) + \mu(B)$. We denote by $PM(\Omega)$ the set of finitely additive probability measures on Ω.

Definition 1.1. *The left-translation action of G on $\mathrm{Prob}(G)$ is defined by*

$$s.\mu(E) := \mu(s^{-1}E), s \in G,\ \mu \in \mathrm{Prob}(G),\ E \subseteq G.$$

If $s.\mu = \mu$, then μ is said to be a left-invariant probability measure.

Definition 1.2. *A group G is* amenable *if there exists a finitely additive left-invariant (or right-invariant) measure $\mu \colon \mathcal{P}(G) \to [0, \infty]$ such that $\mu(G) = 1$*

Remark 1.3
This is equivalent to saying that G is amenable if and only if there exists a left-invariant state on $\mu \in \ell^\infty(G)$, where left-invariance if with respect to the left translation action on $\ell^\infty(G)$. Such a state is called an invariant mean.

Definition 1.4. *We say that a group G has an* approximate invariant mean *if for any finite $E \subseteq G$ and $\varepsilon > 0$ there is $\mu \in \mathrm{Prob}(G)$ such that*

$$\max_{s \in E} \|s.\mu - \mu\|_1 < \varepsilon$$

The following proposition is just a small snippet of equivalent conditions for a discrete group G to be amenable.

Proposition 1.5 ([Theorem 2.6.8 BO08, p. 50])**.** *For a discrete group G, the following are equivalent:*

1. *G is amenable,*
2. *G has an approximate invariant mean.*

Proposition 1.6. *If G is a discrete group, then the following statements hold:*

1. *If G is amenable, then any subgroup $H \leq G$ is amenable.*
2. *If $N \trianglelefteq G$ is amenable and G/N is amenable, then G is amenable.*
3. *If $\{H_i\}_{i\in I}$ is a family of upward directed normal amenable subgroups of G, where upward directed means that for every $i, j \in I$ there is $k \in I$ such that $H_i, H_j \subseteq H_k$, then $H = \bigcup\limits_{i\in I} H_i$ is an amenable subgroup of G.*

Proof.
(1): Let $H \leq G$. Using the Axiom of Choice, choose a right transversal $R \subseteq G$ of H, i.e., a set R containing a representant for each coset of H. Let $\varepsilon > 0$ be arbitrary and $E \subseteq H$ be any finite subset. By Proposition 1.5, there is an approximate invariant mean $\mu \in \mathrm{Prob}(G)$ such that $\|s.\mu - \mu\|_1 \leq \varepsilon$, for all $s \in E$. Define $\tilde{\mu} \in \mathrm{Prob}(H)$ by

$$\tilde{\mu}(g) = \sum_{r\in R} \mu(gr).$$

Then, for every $s \in E$, we have

$$\begin{aligned}
\|s.\tilde{\mu} - \tilde{\mu}\|_1 &= \sum_{h\in H} |\sum_{r\in R} s.\mu(hr) - \mu(hr)| \\
&\leq \sum_{h\in H} \sum_{r\in R} |s.\mu(hr) - \mu(hr)| \\
&\leq \|s.\mu - \mu\|_1 < \varepsilon,
\end{aligned}$$

since $\bigcup_{r\in R} \bigcup_{h\in H} \{hr\} \subseteq G$. Hence $\tilde{\mu}$ is an approximate invariant mean for H, and again by Proposition 1.5 we see that H is amenable.

(2): Assume that $N \trianglelefteq G$ is amenable and G/N is amenable. Then there are left-invariant finitely additive probability measures μ, ν on N and respectively G/N. For each $A \subset G$ define $f_A\colon G \to [0,1]$ by $f_A(g) = \mu(N \cap g^{-1}A)$ for $g \in G$. Let $A \subset G$ and $a \in G$. Let $g, h \in [a] \in G/N$ be any two elements belonging to the coset of a, i.e., $g = an_1$ and $h = an_2$ for some $n_1, n_2 \in N$. We then see that

$$\begin{aligned} f_A(g) &= \mu(N \cap g^{-1}A) \\ &= \mu(N \cap n_1^{-1}a^{-1}A) \\ &= \mu(n_1^{-1}a^{-1}an_1N \cap n_1^{-1}a^{-1}A) \\ &= \mu(n_1^{-1}a^{-1}(an_1N \cap A)) \\ &= \mu(an_1N \cap A) \\ &= \mu(n_2^{-1}n_1N \cap n_2^{-1}a^{-1}A) \\ &= \mu(N \cap n_2^{-1}a^{-1}A) \\ &= f_A(h). \end{aligned}$$

So f_A is well defined on left cosets of N, so we define $\tilde{f}_A : G/N \to [0,1]$ by

$$\tilde{f}_A([g]) = f_A(g), [g] \in G/N.$$

Clearly $\tilde{f}_G([g]) = 1$, and if $A, B \subset G$ are disjoint sets, then, for all $g \in G$,

$$\tilde{f}_{A\cup B}([g]) = \mu(N \cap g^{-1}(A \cup B) = \mu(N \cap g^{-1}A) + \mu(N \cap g^{-1}B) = \tilde{f}_A([g]) + \tilde{f}_B([g]).$$

Now, define a measure $\tau : \mathcal{P}(G) \to [0, \infty]$ by

$$\tau(A) := \int_{G/N} \tilde{f}_A \, \mathrm{d}\nu, \quad A \subseteq G,$$

and since G is discrete this becomes

$$\tau(A) = \sum_{[g]\in G/N} \nu(\{[g]\})\tilde{f}_A([g]), \quad A \subseteq G.$$

By the above arguments we have that τ is finitely additive and $\tau(G) = 1$. Moreover, we see that τ is also left-invariant, since for $h \in G$ we have, by abstract change of variable, that

$$\begin{aligned} \tau(hA) &= \int_{G/N} \tilde{f}_{hA}([g]) \, \mathrm{d}\nu([g]) \\ &= \int_{G/N} \tilde{f}_A([h^{-1}g]) \, \mathrm{d}\nu([g]) \\ &= \int_{G/N} \tilde{f}_A([g]) \, \mathrm{d}\nu([hg]) \\ &= \int_{G/N} \tilde{f}_A([g]) \, \mathrm{d}\nu([g]) \\ & \tau(A). \end{aligned}$$

Therefore G is amenable.

(3) Suppose now that $\{H_i\}_{i\in I}$ is a family of upward directed amenable subgroups of G. Then $H := \bigcup_{i\in I} H_i$ is a group, and for each $i \in I$ define

$$\mathcal{M}_i := \{\mu\colon \mathcal{P}(H) \to [0,1]\colon \mu \text{ is finitely additive and left } H_i \text{ invariant}\}.$$

We identify $[0,1]^{\mathcal{P}(H)}$ by the space of functions $f\colon \mathcal{P}(H) \to [0,1]$ equipped with the product topology. Then each $\mathcal{M}_i$ is a closed subset of $[0,1]^{\mathcal{P}(H)}$. Denote by μ_i a finitely additive left-invariant probability measure ensuring the amenability of H_i. Let $A \subseteq H$, define $\mu_i'(A) := \mu_i(A \cap H_i)$, then $\mu_i' \in \mathcal{M}_i$. So they are non-empty as well. Now, by assumption of $\{H_i\}_{i\in I}$ being upward directed we have for $i, j \in I$ that there is $k \in I$ such that $H_i, H_j \subseteq H_k$. But if $\mu \in \mathcal{M}_k$, then $\mu(gA) = \mu(A)$ for every $g \in H_i, H_j$ aswell, so $\mathcal{M}_k \subset \mathcal{M}_i \cap \mathcal{M}_j$. So the family $\{\mathcal{M}_i\}_{i\in I}$ of non-empty closed subsets has the finite intersection property. Since $[0,1]^{\mathcal{P}(H)}$ is compact by Tychonoff, we conclude that there is a measure $\mu \in \bigcap_{i\in I} \mathcal{M}_i$, which is a left H-invariant probability measure, so H is amenable. □

Lemma 1.7. *If H_1, H_2 are normal amenable subgroups of a discrete group G, then H_1H_2 is a normal amenable subgroup of G.*

Proof.
If H_1, H_2 are normal amenable subgroups of G, then by normality and the second isomorphism theorem for groups we have that H_1H_2 is a group and that $h_1h_2h_1^{-1} \in H_2$ for every $h_1 \in H_1, h_2 \in H_2$. So H_1 acts on H_2 by conjugation, and we may define the semi-direct product $H_1 \rtimes H_2$, with respect to the conjugation action $H_1 \curvearrowright H_2$, i.e., the group with $H_1 \times H_2$ as the underlying set and the following binary operation

$$(h_1, h_2)(h_1', h_2') = (h_1h_1', h_2(h_1h_2'h_1^{-1})), \quad h_1, h_1' \in H_1,\ h_2, h_2' \in H_2.$$

Then we have an isomorphism of H_1 into $H_1 \rtimes H_2$, $h_1 \mapsto (h_1, e_{H_2})$. We write $\overline{H_1}$ for this identification. We then have that

$$(H_1 \rtimes H_2)\big/\overline{H}_1 \simeq H_2,$$

so by Proposition 1.6 we have that $H_1 \rtimes H_2$ is amenable. The map $H_1 \rtimes H_2 \to H_1H_2$, $(h_1, h_2) \mapsto h_1h_2$ is a surjective group homomorphism which shows that H_1H_2 is also amenable. It is also clearly normal. □

Proposition 1.8. *Every countable discrete group G has a largest normal amenable subgroup H, in the sense that it contains all other normal amenable subgroups.*

Proof.
Let $\{H_i\}_{i\in I}$ be the family of every normal amenable subgroups of G. Then for

all $i, j \in I$ we have that $H_i, H_j \subset H_j H_i$, so setting $H_k = H_i H_j$ we get that this family is upward directed. So by Lemma 1.7 and Proposition 1.6 We get that the subgroup $H = \bigcup_{i \in I} H_i$ is a normal amenable subgroup containing every other normal amenable subgroup of G. □

Definition 1.9. *For a discrete group G we denote the largest amenable subgroup of G by AR_G, called the* amenable radical *of G.*

We say that the amenable radical of a group G is trivial if and only if $\mathrm{AR}_G = \{e\}$, and G is amenable if and only if $\mathrm{AR}_G = G$.

2. Representations of groups

2.1 The reduced group C^*-algebra

For a Hilbert space $\mathcal{H}$ we denote by $\mathcal{U}(\mathcal{H}) \subseteq B(\mathcal{H})$ the set of unitary operators on $\mathcal{H}$. The set $\mathcal{U}(\mathcal{H})$ forms a group under composition. If $u_1, u_2 \in \mathcal{U}(\mathcal{H})$, then $u_1u_2 \in \mathcal{U}(\mathcal{H})$, $u^{-1} = u^* \in \mathcal{U}(\mathcal{H})$ for all $u \in \mathcal{U}(\mathcal{H})$ and $I \in \mathcal{U}(\mathcal{H})$.

Definition 2.2. *Let G be a topological group. A unitary representation of G on a Hilbert space $\mathcal{H}$ is a strongly continuous group homomorphism $\pi\colon G \to \mathcal{U}(\mathcal{H})$. We define $\pi_s := \pi(s) \in \mathcal{U}(\mathcal{H})$ for all $s \in G$ and we note that $\pi_e = I$. By the defined notation we see that*

$$\pi_s^* = \pi_s^{-1} = \pi_{s^{-1}},$$

for every $s \in G$.

For a discrete group G consider the Hilbert space

$$\ell^2(G) = \{f\colon G \to \mathbb{C} \ : \ \sum_{g\in G} |f(g)|^2 < \infty\},$$

with inner-product $\langle \xi, \eta\rangle = \sum_{g\in G} \xi(g)\overline{\eta(g)}$, $\xi, \eta \in \ell^2(G)$. It has a canonical orthonormal basis $\{\delta_g\}_{g\in G}$ defined by

$$\delta_g(s) = \begin{cases} 1 & g = s \\ 0 & g \neq s \end{cases}$$

Definition 2.3. *For a discrete group G define $\lambda\colon G \to B(\ell^2(G))$ by*

$$\lambda(s)\xi(t) := \xi(s^{-1}t), \quad s, t \in G,\ \xi \in \ell^2(G).$$

This is the left-regular *representation of G, and we use the notation $\lambda(s) := \lambda_s$. This is a unitary representation of G, since for any $\xi, \eta \in \ell^2(G)$ and $s \in G$ we have*

$$\begin{aligned}
\langle \lambda_s\xi, \eta\rangle &= \sum_{g\in G} \xi(s^{-1}g)\overline{\eta(g)} \\
&= \sum_{g\in G} \xi(g)\overline{\eta(sg)} \\
&= \sum_{g\in G} \xi(g)\overline{\lambda_{s^{-1}}\eta(g)} \\
&= \langle \xi, \lambda_{s^{-1}}\eta\rangle,
\end{aligned}$$

so $(\lambda_s)^* = \lambda_{s^{-1}}$. *For* $s, t \in G$, *we see that*

$$\lambda_s \delta_t(g) = \delta_t(s^{-1}g) = \delta_{st}(g), \quad g \in G.$$

For a discrete group G, we consider the group ring $\mathbb{C}G$ of G, which is the the set of finite linear combinations of elements of G with coefficients in $\mathbb{C}$, i.e.,

$$\mathbb{C}G = \left\{ \sum_{g \in G} \alpha_g g : \alpha_g \in \mathbb{C} \text{ with only finitely many } \alpha_g \text{ are non-zero} \right\}.$$

One can check that defining multiplication and involution by

$$\left(\sum_{s \in G} a_s s \right) \left(\sum_{t \in G} b_t t \right) = \sum_{s,t \in G} a_s b_t st$$
$$\left(\sum_{g \in G} a_g g \right)^* = \sum_{g \in G} \overline{a_g} g^{-1},$$

turns $\mathbb{C}G$ into a $*$-algebra. If $u \colon G \to B(\mathcal{H})$, $g \mapsto u_g$ is a unitary representation of G, it induces a unital $*$-homomorphism $\pi \colon \mathbb{C}G \to B(\mathcal{H})$ by

$$\pi(g) = u_g, \quad g \in G.$$

There is a one-to-one correspondance between unitary representations of G and $*$-representations of $\mathbb{C}G$. Moreover, if we let $u = \lambda$ be the left regular representation, we denote also by λ the induced $*$-representation $\lambda \colon \mathbb{C}G \to B(\ell^2(G))$. This $*$-representation is injective, indeed, if $t \in G$ and $x = \sum_{g \in G} a_g g \in \mathbb{C}G$ we then see that

$$\langle \lambda(x)\delta_e, \delta_t \rangle = \sum_{g \in G} a_g \langle \delta_g, \delta_t \rangle = a_t.$$

So if $\lambda(x) = 0$, then $a_t = 0$ for all $t \in G$, so $x = 0$.

Definition 2.4. *For a discrete group* G, *we define the reduced group* C^*-*algebra, abbreviated* $C_r^*(G)$, *to be the completion of the image of* $\mathbb{C}G$ *under* λ, *i.e.,*

$$C_r^*(G) = \overline{\lambda(\mathbb{C}G)}^{\|\cdot\|} \subset B(\ell^2(G)),$$

or equivalently, by the completion under the norm $\|a\|_r = \|\lambda(a)\|_{B(\ell^2(G))}$ *for* $a \in \mathbb{C}G$.

Besides the left-regular representation, there is also a right-regular representation $\rho \colon G \to B(\ell^2(G))$, acting on the orthonormal basis by

$$\rho_s(\delta_t) = \delta_{ts^{-1}}, \quad s, t \in G.$$

The right-regular representation evidently commutes with $C_r^*(G)$, since it commutes with the generators $\{\lambda_s\}_{s\in G}$, indeed if $s,t,h \in G$ then

$$\rho_t \lambda_s \delta_h = \delta_{sht^{-1}} = \lambda_s \rho_t \delta_h.$$

With this in mind we may prove the following lemma:

Lemma 2.5. *Let G be a discrete group and $x, y \in C_r^*(G)$. If $x\delta_e = y\delta_e$, then $x = y$.*

Proof.
Let $s \in G$, and assume that $x\delta_e = y\delta_e$. Since the right-regular representation of G, ρ, commutes with $C_r^*(G)$ we see that

$$x\delta_s = x\delta_{e(s^{-1})^{-1}} = x\rho_{s^{-1}}\delta_e = \rho_{s^{-1}}x\delta_e = \rho_{s^{-1}}y\delta_e = y\delta_s.$$

Since $\ell^2(G)$ is a Hilbert space with $\{\delta_s\}_{s\in G}$ as a complete orthonormal basis, every $\xi \in \ell^2(G)$ can be written as

$$\xi = \sum_{g\in G} \langle \xi, \delta_g \rangle \delta_g.$$

Hence, for every $\xi \in \ell^2(G)$ we have

$$x\xi = \sum_{g\in G} \langle x\xi, \delta_g\rangle \delta_g = \sum_{g\in G} \langle \xi, x^*\delta_g\rangle \delta_g = \sum_{g\in G} \langle \xi, y^*\delta_g\rangle \delta_g = \sum_{g\in G} \langle y\xi, \delta_g\rangle \delta_g = y\xi$$

So $x = y$. $\square$

Proposition 2.6. *For a discrete group G, the function $\tau_0 \colon C_r^*(G) \to \mathbb{C}$ defined by $\tau_0(x) = \langle x\delta_e, \delta_e\rangle$ is a faithful tracial state on $C_r^*(G)$.*

Proof.
τ_0 is positive, for if $x \in C_r^*(G)$, then $\tau_0(x^*x) = \langle x\delta_e, x\delta_e\rangle = \|x\delta_e\|^2 \geq 0$, and by [Corollary 13.6 Zhu93, p. 80] it also a state, since $\tau_0(\lambda_e) = 1 = \|\tau_0\|$. We also see that τ_0 is tracial on the dense subset $\lambda(\mathbb{C}G) \subseteq C_r^*(G)$, for if $s, t \in G$ then

$$\tau_0(\lambda_s\lambda_t) = \begin{cases} 1 & \text{if } st = e \iff s = t^{-1} \\ 0 & \text{else} \end{cases}$$
$$\tau_0(\lambda_t\lambda_s) = \begin{cases} 1 & \text{if } ts = e \iff s = t^{-1} \\ 0 & \text{else} \end{cases},$$

Linearity and continuity of τ_0 allows us to conclude that $\tau_0(xy) = \tau_0(yx)$ for all $x, y \in C_r^*(G)$. We now show that it is also faithful. Let $0 \leq x \in C_r^*(G)$ with $\tau_0(x) = 0$. The positivity of x allows us to define the positive element

$x^{\frac{1}{2}} \in C_r^*(G)$ using the continuous functional calculus cf. [Theorem 10.3 Zhu93, p. 62]. We then see that

$$0 = \langle x\delta_e, \delta_e\rangle = \langle x^{\frac{1}{2}}\delta_e, x^{\frac{1}{2}}\delta_e\rangle = \|x^{\frac{1}{2}}\delta_e\|,$$

so $x^{\frac{1}{2}}\delta_e = 0$. By Lemma 2.5, this implies that $x^{\frac{1}{2}} = 0$, which in turn implies that $x = 0$. So τ_0 is faithful. $\square$

Proposition 2.7 ([Proposition 2.5.9 BO08, p. 46])**.** *If $H \leq G$ is a subgroup of a discrete group G, then $C_r^*(H) \subseteq C_r^*(G)$ canonically.*

Proposition 2.8 ([Corollary 2.5.12 BO08, p. 47])**.** *If $H \leq G$ is a subgroup of a discrete group G, and $C_r^*(H) \subseteq C_r^*(G)$ is the inclusion above, then there is a conditional expectation $E_H\colon C_r^*(G) \to C_r^*(H)$ satsfying (and is uniquely determined by) the following*

$$E_H(\lambda_s) = \begin{cases} \lambda_s & s \in H \\ 0 & s \notin H \end{cases}.$$

2.9 Crossed products

Definition 2.10. *For a discrete group G and a C^*-algebra $\mathcal{A}$, an action of G on $\mathcal{A}$ is a group homomorphism $\alpha\colon G \to \mathrm{Aut}(\mathcal{A})$, the group of *-automorphism on $\mathcal{A}$ with composition as product. We write $G \overset{\alpha}{\curvearrowright} \mathcal{A}$ to say that G acts on $\mathcal{A}$ by an action α. If the action α is understood we simply write $G \curvearrowright \mathcal{A}$.*

The triple $(\mathcal{A}, \alpha, G)$ is called a C^*-dynamical system. For $g \in G$ define the function $\delta_g\colon G \to \mathbb{C}$ by $\delta_g(g) = 1$ and $\delta_g(s) = 0$ for $g \neq s \in G$. For a triple $(\mathcal{A}, \alpha, G)$ one may check that we turn

$$C_c(G, \mathcal{A}) = \left\{ \sum_{g \in G} a_g\delta_g\colon a_g \in \mathcal{A} \text{ only finitely many } a_g \text{ are non-zero} \right\}$$

into a $*$-algebra by defining a product by the twisted convolution and an involution by a twisted involution defined by

$$\left(\sum_{g\in G} a_g\delta_g\right) \cdot \left(\sum_{s\in G} b_s\delta_s\right) = \sum_{s,g\in G} a_g\alpha_g(b_s)\delta_{gs},$$
$$\left(\sum_{g\in G} a_g\delta_g\right)^* = \sum_{g\in G} \alpha_{g^{-1}}(a_g^*)\delta_{g^{-1}}.$$

Then $C_c(G, \mathcal{A})$ is the $*$-algebra generated by $\mathcal{A}$ and unitaries $\{u_g\}_{g\in G}$ given by $u_g = 1_{\mathcal{A}}\delta_g \in C_c(G, \mathcal{A})$ such that $\alpha_s(\cdot) = u_s(\cdot)u_s^*$. This way $C_c(G, \mathcal{A})$ contains a copy of $\mathcal{A}$ and a unitary copy of G by the maps $a \mapsto au_e$, respectively, $g \mapsto u_g$.

Definition 2.11. *A* ***covariant representation*** *of a C^*-dynamical system $(\mathcal{A}, \alpha, G)$ is a triple $(\pi, \rho, \mathcal{H})$ such that $\mathcal{H}$ is a Hilbert space, $\pi: \mathcal{A} \to B(\mathcal{H})$ is a representation and $\rho: G \to \mathcal{U}(\mathcal{H})$ is a unitary representation such that $\pi(\alpha_g(a)) = \rho_g(\pi(a))\rho_g^*$ for every $a \in \mathcal{A}$.*

Remark 2.12
Whenever $(\pi, \rho, \mathcal{H})$ is a covariant representation for $(\mathcal{A}, \alpha, G)$, then the map $\pi \times \rho: C_c(G, \mathcal{A}) \to B(\mathcal{H})$ defined by

$$(\pi \times \rho)\left(\sum_{g \in G} a_g \delta_g\right) = \sum_{g \in G} \pi(a_g)\rho_g$$

is a $*$-representation of $C_c(G, \mathcal{A})$ on $B(\mathcal{H})$. Furthermore, it is faithful resp. non-degenerate whenever π is faithful resp. non-degenerate.

Given a faithful represntation $\pi: \mathcal{A} \to B(\mathcal{H})$, define a new representation $\pi_\alpha: A \to B(\mathcal{H} \otimes \ell^2(G))$ by

$$\pi_\alpha(a)(\xi \otimes \delta_g) = \pi(\alpha_{g^{-1}}(a))\xi \otimes \delta_g,$$

for $g \in G, a \in \mathcal{A}$ and $\xi \in \mathcal{H}$. If $\lambda: G \to \mathcal{U}(\ell^2(G))$ denotes the left-regular representation, we get that the triple $(\pi_\alpha, I \otimes \lambda, \mathcal{H})$ is a covariant representation, and thus $\pi_\alpha \times (I \otimes \lambda)$ is a faithful $*$-representation of $C_c(G, \mathcal{A})$ on $B(\mathcal{H} \otimes \ell^2(G))$. Such a representation is called a regular representation, and it gives rise to a norm on $C_c(G, \mathcal{A})$ called the reduced norm, given by

$$\|x\|_r = \|(\pi_\alpha \times (I \otimes \lambda))(x)\|_{\mathcal{H} \otimes \ell^2(G)}.$$

When it is clear from context, we will denote $(I \otimes \lambda)$ by λ, i.e., $\lambda_s(\xi \otimes \delta_t) = \xi \otimes \delta_{st}$.

Definition 2.13. *The* ***reduced crossed product*** *$A \rtimes_{\alpha,r} G$ is the norm closure of the image under a faithful regular representation $\pi_\alpha \times (I \otimes \lambda)$. That is*

$$A \rtimes_{\alpha,r} G = \overline{(\pi_\alpha \times (I \otimes \lambda))(C_c(G, \mathcal{A}))}^{\|\cdot\|} \subseteq B(\mathcal{H} \otimes \ell^2(G)).$$

Then $A \rtimes_{\alpha,r} G = C^*(\{(\pi_\alpha \times (I \otimes \lambda))(au_g): a \in \mathcal{A}, g \in G\})$. Moreover, it contains an isometrically isomorphic copy of $C_r^*(G)$, indeed, for all $s, t \in G$, $\xi \in H$ and $a \in A$ we have

$$(\pi_\alpha \times (I \otimes \lambda))(au_s)(\xi \otimes \delta_t) = (\pi(\alpha_s(a)) \otimes \lambda_s)(\xi \otimes \delta_t),$$

where $\pi(\alpha_s(a)) \otimes \lambda_s \in \pi(A) \otimes C_r^*(G)$. The representation $I \otimes \lambda: \mathbb{C}G \to B(\mathcal{H} \otimes \ell^2(G))$ is a faithful representation, and since the norm on $B(\mathcal{H} \otimes \ell^2(G))$ is a cross-norm, $\mathcal{A} \rtimes_{\alpha}, G$ contains an isometrically isomorphic copy of $C_r^*(G)$.

Throughout this thesis, G will often be a group acting on a compact Hausdorff space X by an action $\alpha: G \to \mathrm{Aut}(X)$, where $\alpha_s(x) := s.x$. This action induces an action on $C(X)$, defined by

$$\alpha_s(f)(x) = f(s^{-1}.x), \quad f \in C(X),\ s \in G,\ x \in X.$$

We write $s.f$ instead of $\alpha_s(f)$. Since $C(X)$ is a C^*-algebra, we may now form the reduced crossed product $C(X) \rtimes_{\alpha,r} G$ which contains both a copy of $C_r^*(G)$ and $C(X)$ which are related by $\lambda_s f \lambda_s^* = \alpha_s(f) := s.f$. The reader may note that we omitted the representation $\pi\colon C(X) \to B(\mathcal{H} \otimes \ell^2(G))$ and we wrote λ_s instead of $(I \otimes \lambda)(s)$. We will continue this notation for sake of readability, if needed we will note when to distinguish between notations.

3. C^*-simplicity and uniqueness of trace

3.1 Dixmier Property

Definition 3.2. *A discrete group G is* C^*-simple *if the reduced group C^*-algebra $C_r^*(G)$ is simple, i.e., contains no non-trivial closed two-sided ideals. We say that G has the unique trace property if the canonical faithful tracial state τ_0 on $C_r^*(G)$, $x \mapsto \langle x\delta_e, \delta_e\rangle$, is the only tracial state on $C_r^*(G)$.*

For a C^*-algebra $\mathcal{A}$, we define for each $a \in \mathcal{A}$ the set $D(a) \subseteq \mathcal{A}$ by

$$D(a) = \operatorname{conv}\{uau^* : u \in \mathcal{U}(\mathcal{A})\}.$$

If we need to make clear what C^*-algebra we work in, we will denote the above set by $D_{\mathcal{A}}(a)$

Definition 3.3. *Let $\mathcal{A}$ be a unital C^*-algebra. Let $a \in \mathcal{A}$, we say that $\mathcal{A}$ has the* Dixmier property *at a if*

$$\overline{D(a)} \cap \mathbb{C}1_{\mathcal{A}} \neq \emptyset.$$

We say that $\mathcal{A}$ has the Dixmier property *if it has the Dixmier property at every point.*

Lemma 3.4. *Let $\mathcal{A}$ be a unital C^*-algebra with a tracial state τ. Then $\tau\left(\overline{D(a)}\right) = \{\tau(a)\}$ for every $a \in \mathcal{A}$.*

Proof.
For $a \in \mathcal{A}$, it holds that $\tau(x) = \tau(a)$ for every $x \in D(a)$. If $(b_i)_{i\in I} \subseteq D(a)$ is any net converging to some $b \in \mathcal{A}$, then $\tau(a) = \tau(b_i)$ for every $i \in I$ and hence $\tau(a) = \tau(b)$, by continuity of τ. Thus $\tau(b) = \tau(a)$ for every $b \in \overline{D(a)}$. □

Lemma 3.5. *Let $\mathcal{A}$ be a unital C^*-algebra with a tracial state τ. Assume that $\mathcal{A}$ has the Dixmier property. Then τ is unique.*

Proof.
Let $a \in \mathcal{A}$. Let τ' be another tracial state on $\mathcal{A}$, the assumption that $\mathcal{A}$ has the Dixmier property yields $\lambda \in \mathbb{C}$ such that $\lambda 1_{\mathcal{A}} \in \overline{D(a)}$

$$\tau'(a) = \tau'(\lambda 1_{\mathcal{A}}) = \lambda = \tau(\lambda 1_{\mathcal{A}}) = \tau(a).$$

Since a was arbitrary, $\tau = \tau'$. □

Proposition 3.6. *Let $\mathcal{A}$ be a unital C^*-algebra with a faithful tracial state τ. If $\mathcal{A}$ has the Dixmier property, then $\mathcal{A}$ is simple.*

Proof.
Assume that $\mathcal{I} \subseteq A$ is a non-trivial two-sided closed ideal, i.e. $\mathcal{I} \neq \{0\}$. Then there is $x \in \mathcal{I}$ such that $a := x^*x > 0$, so $\tau(a) > 0$. Since $\mathcal{I}$ is a two sided closed ideal, $a \in \mathcal{I}$. And so $D(a) \subseteq \mathcal{I}$, and since $\mathcal{I}$ is closed, $\overline{D(a)} \subseteq \mathcal{I}$. We have $0 \notin \overline{D(a)}$, since else $0 = \tau(0) \in \tau\left(\overline{D(a)}\right) = \{\tau(a)\}$, a contradiction. Since $\mathcal{A}$ has the Dixmier property, there is $0 \neq \beta \in \mathbb{C}$ such that $\beta 1_{\mathcal{A}} \in \overline{D(a)} \subseteq \mathcal{I}$. Therefore $\mathcal{I} = \mathcal{A}$. □

Definition 3.7. *For a unital C^*-algebra $\mathcal{A}$ we denote by $\mathcal{F}(\mathcal{A})$ the set of functions $f\colon \mathcal{A} \to \mathcal{A}$, where*

$$f(a) = \sum_{i=1}^{n} \alpha_i u_i a u_i^*,$$

where $\alpha_i \geq 0$ with $\sum_{i=1}^n \alpha_i = 1$ and $u_i \in \mathcal{U}(\mathcal{A})$. Such a function f is called an averaging process.

Clearly, for a unital C^*-algebra $\mathcal{A}$ and $a \in A$, we see that

$$D(a) = \{f(a)\colon f \in \mathcal{F}(\mathcal{A})\}.$$

Moreover, each f is u.c.p (unital c.c.p), since they are convex combinations of positive unital $*$-homomorphisms. If $f, g \in \mathcal{F}(\mathcal{A})$, then $(f \circ g)(a) = \sum_{i=1}^k \sum_{j=1}^n \beta_i \alpha_j u_i' u_j a u_j{}^* u_i'{}^*$. Defining

$$\lambda_s := \sum_{i=1}^{n} \beta_s \alpha_i \in [0, 1]$$

for $1 \leq s \leq k$ and $\tilde{u}_s := \sum_{i=1}^n u_s' u_i$. Then $(f \circ g)(a) = \sum_{s=1}^k \lambda_s \tilde{u}_s a \tilde{u}_s^*$, and $\sum_{s=1}^k \lambda_s = 1$, $\tilde{u}_s \in \mathcal{U}(\mathcal{A})$, for $1 \leq s \leq k$, so $f \circ g \in \mathcal{F}(\mathcal{A})$.

Lemma 3.8. *Let $\mathcal{A}$ be a unital C^*-algebra. If $\mathcal{A}$ has the Dixmier property at every self-adjoint $a \in \mathcal{A}_{sa}$, then $\mathcal{A}$ has the Dixmier property.*

Proof.
Let $a \in \mathcal{A}$. And let $a_1, a_2 \in \mathcal{A}_{sa}$ be the self-adjoint elements such that $a = a_1 + ia_2$, i.e. a_1 is the real part of a and a_2 is the imaginary part. Let $\varepsilon > 0$ be given. Since a_1 is self-adjoint, $\overline{D(a_1)} \cap \mathbb{C}1_{\mathcal{A}} \neq \emptyset$, so there is $f \in \mathcal{F}(\mathcal{A})$ and $\lambda_1 \in \mathbb{C}$ such that

$$\|f(a_1) - \lambda_1 1_{\mathcal{A}}\| < \frac{\varepsilon}{2}.$$

And since f is a $*$-homomorphism, $f(a_2) \in \mathcal{A}_{sa}$. So there is $g \in \mathcal{F}(\mathcal{A})$ and $\lambda_2 \in \mathbb{C}$ such that

$$\|g(f(a_2)) - \lambda_2 1_{\mathcal{A}}\| < \frac{\varepsilon}{2}.$$

Note that $g(a+b) = g(a) + g(b)$, and $g(\beta 1_{\mathcal{A}}) = \beta 1_{\mathcal{A}}$ for every $\beta \in \mathbb{C}$. Now, set $\lambda := \lambda_1 + i\lambda_2$. Then

$$\begin{aligned}
\|g(f(a)) - \lambda 1_{\mathcal{A}}\| &= \|g(f(a_1 + ia_2)) - \lambda_1 1_{\mathcal{A}} - i\lambda_2 1_{\mathcal{A}}\| \\
&= \|g(f(a_1)) - \lambda_1 1_{\mathcal{A}} + i(g(f(a_2)) - \lambda_2 1_{\mathcal{A}})\| \\
&\leq \|g(f(a_1) - \lambda_1 1_{\mathcal{A}})\| + \|g(f(a_2)) - \lambda_2 1_{\mathcal{A}}\| \\
&\leq \|f(a_2) - \lambda_1 1_{\mathcal{A}}\| + \|g(f(a_2)) - \lambda_2 1_{\mathcal{A}}\| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Thus we conclude that $\mathcal{A}$ has the Dixmier property at $a \in \mathcal{A}$, and since $a \in \mathcal{A}$ was arbitrary, $\mathcal{A}$ has the Dixmier property. □

Using this lemma, we now show that it is enough to check for Dixmier property of a unital C^*-algebra $\mathcal{A}$ on a unital dense $*$-subalgebra $\mathcal{A}_0 \subseteq \mathcal{A}$ satisfying certain properties, i.e., we have the following lemma

Lemma 3.9. *Let $\mathcal{A}$ be a unital C^*-algebra with tracial state τ. If $\mathcal{A}_0 \subseteq \mathcal{A}$ is a unital dense $*$-subalgebra and $\overline{D(a)} \cap \mathbb{C}1_{\mathcal{A}} \neq \emptyset$ for all $a \in (\mathcal{A}_0)_{sa}$ with $\tau(a) = 0$, then $\mathcal{A}$ has the Dixmier property.*

Proof.
Let $a \in \mathcal{A}$ be self-adjoint and $\varepsilon > 0$ be arbitrary. Since $\mathcal{A}_0$ is dense in $\mathcal{A}$, pick $a_0 \in \mathcal{A}_0$ such that $\|a - a_0\| < \frac{\varepsilon}{3}$. The element $b_0 := \frac{a_0 + a_0^*}{2}$ is a self-adjoint element of $\mathcal{A}_0$ such that

$$\|a - b_0\| = \|\frac{a - a_0}{2} + \frac{a^* - a_0^*}{2}\| \leq \frac{1}{2}(\|a - a_0\| + \|a - a_0^*\|) < \frac{\varepsilon}{3}.$$

Now, if we define $x_0 := \tau(b_0)1_{\mathcal{A}} - b_0$, then $x_0 \in \mathcal{A}_0$ is self-adjoint. Moreover, $\tau(x_0) = 0$, so by assumption there is $\beta \in \mathbb{C}$ such that $\beta 1_{\mathcal{A}} \in \overline{D(x_0)}$. By Lemma 3.4, we have that $\beta = \tau(\beta 1_{\mathcal{A}}) \in \tau(\overline{D(x_0)} = \{\tau(x_0)\}$. Therefore $0 = \tau(x_0)1_{\mathcal{A}} = \beta 1_{\mathcal{A}} \in \overline{D(x_0)}$. Thus, there is $f \in \mathcal{F}(\mathcal{A})$ such that

$$\|\tau(b_0)1_{\mathcal{A}} - f(b_0)\| = \|f(x_0) - 0\| < \frac{\varepsilon}{3}.$$

Now, $|\tau(x-y)| \leq \|x-y\|$, since τ is a state, and $f \in \mathcal{F}(\mathcal{A})$ means that it is unital and contractive, thus we have

$$\begin{aligned}
\|(f(a) - \tau(a)1_{\mathcal{A}}\| &= \|f(a) - f(b_0) + f(b_0) - \tau(b_0)1_{\mathcal{A}} + \tau(b_0)1_{\mathcal{A}} - \tau(a)1_{\mathcal{A}}\| \\
&\leq \|f(a) - f(b_0)\| + \|f(b_0) - \tau(b_0)1_{\mathcal{A}}\| + \|\tau(b_0)1_{\mathcal{A}} - \tau(a)1_{\mathcal{A}}\| \\
&= \|f(a - b_0)\| + \|f(b_0) - \tau(b_0)1_{\mathcal{A}}\| + \|(\tau(b_0 - a))1_{\mathcal{A}}\| \\
&\leq \|a - b_0\| + \|f(b_0) - \tau(b_0)1_{\mathcal{A}}\| + \|b_0 - a\| \\
&< \varepsilon,
\end{aligned}$$

hence $\tau(a)1_{\mathcal{A}} \in \overline{D(a)}$. Since $a \in \mathcal{A}$ was any arbitrary self-adjoint element, Lemma 3.8 tells us that $\mathcal{A}$ has the Dixmier property. □

3.10 Powers groups

Definition 3.11. *A discrete group G is a* Powers group *if to every non-empty finite subset $F \subseteq G \backslash \{e\}$ and any $n \in \mathbb{N}$, there are disjoint subsets $C, D \subseteq G$ satisfying $G = C \cup D$ and a finite set of elements $s_1, \ldots, s_n \in G$ such that*

1. *$fC \cap C = \emptyset$ for all $f \in F$ and*
2. *$s_i D \cap s_j D = \emptyset$ for all $1 \leq j \neq i \leq n$.*

In the following, let $\mathbb{F}_2$ be the free group on two generators a, b.

Lemma 3.12. *Suppose $X \subseteq \mathbb{F}_2$ is any finite set, then there is $k \in \mathbb{Z}$ such that for all $g \in X$ we have that $b^k g b^{-k}$ in reduced form begins and ends with a non-zero power of b.*

Proof.
Let $X = \{g_1, \ldots, g_n\}$ be any finite subset of $\mathbb{F}_2$. Define $n_b : \mathbb{F}_2 \to \mathbb{Z}$ by $n_b(g)$ as the power of b at the beginning of the reduced form of g, e.g., $n_b(b^2ab) = 2$, $n_b(ab^2) = 0$ and $n_b(b^{-3}a) = -3$. Define $m_b : \mathbb{F}_2 \to \mathbb{Z}$ by $m_b(g)$ is the power of b at the end of g. If $g \in \mathbb{F}_2$ and $j \in \mathbb{Z}$ then $n_b(b^j g) = j + n_b(g)$ and $m_b(gb^{-j}) = m_b(g) - j$. Since the sets $\{n_b(g_1), \ldots, n_b(g_n\}$ and $\{m_b(g_1), \ldots, m_b(g_n)\}$ are finite, we can choose $k_1, k_2 \in \mathbb{Z}$ such that

$$k_1 > -n_b(g_i) \quad \text{and} \quad k_2 > m_b(g_i)$$

for $1 \leq i \leq n$. Set $k = max\{k_1, k_2\}$, then

$$b^k g_i b^{-k}$$

in reduced form begins and ends with a non-zero power of b for $1 \leq i \leq n$. □

Example 3.13
The free group on two generators $\mathbb{F}_2$ is a Powers group

Proof.
Let $F \subseteq \mathbb{F}_2 \backslash \{e\}$ and $n \in \mathbb{N}$. Let $k \in \mathbb{Z}$ satisfy the conditions of Lemma 3.12 and let n_b, m_b be functions as in the same lemma. Setting $C := \{s \in \mathbb{F}_2 \colon n_b(s) = -k\}$ results in $n_b(b^k s) = 0$ for all $s \in C$, by construction. Let $f \in F$, then $k \in \mathbb{Z}$ was chosen such that $n_b(b^k f) \neq 0$ and $m_b(fb^{-k}) \neq 0$, and we see that for all $s \in C$

$$n_b(b^k fs) = n_b(b^k fb^{-k} b^k s) \neq 0.$$

Thus $fs \notin C$, i.e., so $fC \cap C = \emptyset$. Let $D = \mathbb{F}_2 \backslash C$. For each $1 \leq j \leq n$, set $s_j := a^j b^k$, then $n_b(s_j s) = 0$ and $n_a(s_j s) = j$ for all $s \in D$. And we see that for all $1 \leq i \neq j \leq n$

$$n_a(s_j s) = j \neq i = n_a(s_i s),$$

so $s_j D \cap s_i D = \emptyset$. And from this we conclude that $\mathbb{F}_2$ is a Powers group. $\square$

In the following, we let $\lambda \colon \mathbb{C}G \to B(\ell^2(G))$ be the left-regular representation and $C_r^*(G)$ denote the reduced group C^*-algebra of G as in chapter 2. We denote by τ_0 the canonical faithful tracial state on $C_r^*(G)$, $\tau_0(x) = \langle x\delta_e, \delta_e \rangle$. For $D \subset G$, we denote by $\ell^2(D)$ the closed subspace

$$\ell^2(D) := \{\xi \in \ell^2(G) : \xi(s) = 0 \text{ for } s \notin D\} \subseteq \ell^2(G)$$

Then $\ell^2(D)^\perp = \ell^2(G \backslash D)$, indeed, if $\xi \in \ell^2(G \backslash D)$ and $\eta \in \ell^2(D)$ then their support is disjoint, hence

$$\begin{aligned} \langle \xi, \eta \rangle &= \sum_{g \in G} \xi(g)\overline{\eta(g)} \\ &= \sum_{g \in D} \xi(g)\overline{\eta(g)} + \sum_{g \in G \backslash D} \xi(g)\overline{\eta(g)} \\ &= 0, \end{aligned}$$

so $\xi \perp \eta$.

Proposition 3.14. *Let G be a Powers group, if $a \in \mathbb{C}G$ is a self-adjoint element with $\tau_0(a) = 0$, then for $n \geq 1$ there are $s_1, \dots, s_n \in G$ such that*

$$\left\| \frac{1}{n} \sum_{j=1}^{n} \lambda_{s_j} a \lambda_{s_j}^* \right\| \leq \frac{2}{\sqrt{n}} \|a\|.$$

Proof.
Identify $\mathbb{C}G$ with as a dense subset of $C_r^*(G)$, i.e., as $\lambda(\mathbb{C}G)$. Let $a \in \mathbb{C}G$ be self-adjoint, and let $z_1, \dots, z_n \in \mathbb{C}$ and $f_1, \dots, f_n \in G$ be such that $a = \sum_{i=1}^n z_i \lambda_{f_i}$. Then

$$\begin{aligned}\tau_0(a) &= \langle \sum_{j=1}^n z_j \lambda_{f_j} \delta_e, \delta_e \rangle \\ &= \sum_{j=1}^n z_j \langle \delta_{f_j e}, \delta_e \rangle \\ &= \begin{cases} z_k & \text{if } f_k = e \text{ for some } 1 \leq k \leq n \\ 0 & \text{else} \end{cases}\end{aligned}$$

Assuming that $\tau_0(a) = 0$, we may assume that $f_1, \dots, f_k \in G \backslash \{e\}$. Since G is a Powers group and the set $F := \{f_1, \dots, f_n\}$ is finite, there is a partition $G = C \cup D$ and elements $s_1, \dots, s_n \in G$ such that for $1 \leq j \leq n$ and $s_j D \cap s_i D = \emptyset$ for $1 \leq i \neq j \leq n$ and $C \cap f_j C = \emptyset$. Then, for $f_j \in \mathcal{F}$, we have

$$f_j^{-1}(C \cap f_j C) = f_j^{-1} C \cap C = \emptyset,$$

which implies that $f_j^{-1} C \subseteq D$. Let $1 \leq j \leq n$ and $g \in s_j C$, then $g = s_j c$ for some $c \in C$

$$s_j f_i^{-1} s_j^{-1} g = s_j f_i^{-1} c \in s_j f_i^{-1} C \subseteq s_j D.$$

Thus, if $\xi \in \ell^2(s_j C)$, defined in the passage above, then for $g \in s_j C$ we have $s_j f_i^{-1} s_j^{-1} g \in s_j D$. And since $s_j D \cap s_j C = \emptyset$, we have

$$0 = \xi(s_j f_i^{-1} s_j^{-1} g) = \lambda_{s_j f_i s_j^{-1}} \xi(g).$$

So $\lambda_{s_j f_i s_j^{-1}}$ maps $\ell^2(s_j C)$ into $\ell^2(s_j D)$ for all $1 \leq j \leq n$. Define

$$b_j := \lambda_{s_j} a \lambda_{s_j}^* = \sum_{k=1}^n z_k \lambda_{s_j f_k s_j^{-1}},$$

then b_j maps $\ell^2(s_j C)$ into $\ell^2(s_j D)$ for all j. For each $1 \leq j \leq n$, let p_j denote the projection of $\ell^2(G)$ onto $\ell^2(s_j D)$. By assumption $G = C \cup D$, so $G \backslash s_j D = s_j C$. Since p_j is a projection, we have that

$$\begin{aligned}\operatorname{Ran}(p_j) &= \ker(1 - p_j) = \ell^2(s_j D) \\ \operatorname{Ran}(1 - p_j) &= \ker(p_j) = \ell^2(s_j C).\end{aligned}$$

Thus, since b_j maps $\ell^2(s_j C)$ into $\ell^2(s_j D) = \ker(1 - p_j)$, we have

$$(1 - p_j) b_j (1 - p_j) = 0.$$

Using this fact, if $\xi \in \ell^2(G)$ with $\|\xi\| = 1$, we get for $1 \leq j \leq n$:

$$\begin{aligned}
|\langle b_j\xi, \xi\rangle| &= |\langle b_j p_j \xi + b_j(1-p_j)\xi, p_j\xi + (1-p_j)\xi\rangle| \\
&= |\langle b_j p_j \xi, p_j\xi\rangle + \langle b_j p_j\xi, (1-p_j)\xi\rangle + \langle b_j(1-p_j)\xi, p_j\xi\rangle + \langle b_j(1-p_j)\xi, (1-p_j)\xi\rangle| \\
&= |\langle b_j \xi p_j \xi\rangle + \langle b_j p_j\xi, (1-p_j)\xi\rangle + \langle b_j(1-p_j)\xi, (1-p_j)\xi\rangle| \\
&\leq |\langle b_j\xi, p_j\xi\rangle| + |\langle b_j p_j \xi, (1-p_j)\xi\rangle + \langle (1-p_j)b_j(1-p_j)\xi, \xi\rangle| \\
&= |\langle b_j\xi, p_j\xi\rangle| + |\langle b_j p_j \xi, (1-p_j)\xi\rangle| \\
&\leq 2\|b_j\|\|p_j\xi\| \leq 2\|a\|\|p_j\xi\|,
\end{aligned}$$

since $\|b_j\| = \|\lambda_{s_j} a \lambda_{s_j}^*\|$ and λ_{s_j} is a unitary operator and that both $p_j, (1-p_j)$ are projections and such have norm one. If we define

$$b := \frac{1}{n}\sum_{k=1}^{n} b_j = \sum_{k=1}^{n} \lambda_{s_j} a \lambda_{s_j}^*,$$

then we get

$$\begin{aligned}
|\langle b\xi, \xi\rangle| &= \left| \frac{1}{n}\sum_{k=1}^{n} \langle b_j\xi, \xi\rangle \right| \\
&\leq \frac{1}{n}\sum_{k=1}^{n} |\langle b_j\xi, \xi\rangle| \\
&\leq \frac{2}{n}\|a\| \sum_{k=1}^{n} \|p_j\xi\|.
\end{aligned}$$

Viewing the vectors $x := (\|p_1\xi\|, \|p_2\xi\|, \ldots, \|p_n\xi\|)$ and $y = (1, 1, \ldots, 1)$ in $\mathbb{R}^n$ we get

$$\sum_{k=1}^{n} \|p_j\xi\| = |\langle x, y\rangle_{\mathbb{R}^n}| \leq \left(\sum_{k=1}^{n} \|p_j\xi\|^2\right)^{\frac{1}{2}} \sqrt{n}.$$

Since the projections p_j are orthogonal, the Pythagorean theorem for inner product spaces and the above gives us

$$\begin{aligned}
\frac{2}{n}\|a\| \sum_{k=1}^{n} \|p_j\xi\| &\leq \frac{2}{\sqrt{n}}\|a\| \left(\sum_{k=1}^{n} \|p_j\xi\|^2\right)^{\frac{1}{2}} \\
&\leq \frac{2}{\sqrt{n}}\|a\| \left\| \left(\sum_{k=1}^{n} p_j\right)\xi \right\| \leq \frac{2}{\sqrt{n}}\|a\|.
\end{aligned}$$

Since b_j is self-adjoint for $1 \leq j \leq n$, so is b. For self-adjoint elements in $B(\mathcal{H})$, the norm of b is given by $\|b\| = \sup\{|\langle b\xi, \xi\rangle| : \|\xi\| = 1\}$, and since ξ was chosen with $\|\xi\| = 1$, this concludes the proof. $\square$

Theorem 3.15. *Let G be a discrete Powers group, then $C_r^*(G)$ has the Dixmier property and hence is simple with unique tracial state.*

Proof.
Since τ_0 is a faithful tracial state on $C_r^*(G)$, it is by Proposition 3.6 enough to check if it has the Dixmier property. Identifying $\mathbb{C}G$ with $\lambda(\mathbb{C}G) \subseteq C_r^*G$ as a dense $*$-subalgebra, it is by Lemma 3.9 enough to check if it has the Dixmier property at all $a \in \mathbb{C}G$ self-adjoint element with $\tau_0(a) = 0$. So let $a \in \mathbb{C}G$ satisfy these conditions. For each $n \geq 1$, apply Proposition 3.14 to achieve elements $s_1, \ldots, s_n \in G$ such that if $a_n := \frac{1}{n}\sum_{j=1}^n \lambda_{s_j} a \lambda_{s_j}^*$ we have

$$\|a_n\| \leq \frac{2}{\sqrt{n}}\|a\|.$$

Then $a_n \in D_{C_r^*(G)}(a)$ for all $n \geq 1$. and $a_n \to 0$ as n tends to infinity, so $0 \in \overline{D_{C_r^*(G)}(a)}$. So $C_r^*(G)$ has the Dixmier property at a, implying that $C_r^*(G)$ is simple with unique tracial state. □

4. Boundary actions

4.1 G-spaces

In the following, we let X be a compact Hausdorff space and G be a second countable locally compact group.

Definition 4.2. *A G-*action *on a compact Hausdorff space X is a continuous group action*

$$\alpha\colon G \times X \to X$$

We abbreviate G-actions by $G \curvearrowright X$ and write

$$\alpha(g)(x) := g.x, \quad x \in X,\ g \in G.$$

*A space X equipped with a G-action is called a G-*space*.*

Remark 4.3
If X is a G-space, then one may turn $C(X)$ into a G-space by defining $G \curvearrowright C(X)$ by

$$g.f(x) = f(g^{-1}.x),\ g \in G,\ x \in X, f \in C(X),$$

where G is viewed as a discrete group.

For a compact Hausdorff space X, given any state φ on $C(X)$, the Riesz Representation Theorem, cf. [Theorem 7.2 Fol13, p. 212] ensures that there is a unique Radon probability measure $\mu \in \mathrm{Prob}(X)$, where $\mathrm{Prob}(X)$ is the set of all Radon probability measures on X, such that

$$\varphi(f) = \int_X f \,\mathrm{d}\mu,\ f \in C(X).$$

The correspondance between the state space of $C(X)$ and $\mathrm{Prob}(X)$ is a bijective and affine correspondance, and so we write μ for both the state on $C(X)$ and the measure. We give $\mathrm{Prob}(X)$ the weak* topology inherited from the state space of $C(X)$ through this isomorphism. The state space of $C(X)$ is convex and compact in the weak*-topology, cf. [Proposotition 13.8 Zhu93, p. 81].

Again, if X is a compact G-space, we put a G-space structure on $\mathrm{Prob}(X)$, where G is viewed as a discrete space, by defining

$$(g.\mu)(E) := \mu(g^{-1}.E), \quad g \in G, \mu \in \mathrm{Prob}(X),\ E \subseteq X \text{ Borel}.$$

The map $\mu \mapsto g.\mu$ is indeed continuous: If $g \in G$ and $(\mu_\alpha)_{\alpha\in A} \subseteq \text{Prob}(X)$ is a net converging in the weak*-topology to $\mu \in \text{Prob}(X)$, then for all $f \in C(X)$, using the abstract change-of-variable formula, we have

$$(g.\mu_\alpha)(f) = \int_X f \,\mathrm{d}(g.\mu_\alpha) = \int_X g^{-1}.f \,\mathrm{d}\mu_\alpha \to \int_X g^{-1}.f \,\mathrm{d}\mu = \int_X f \,\mathrm{d}g.\mu = (g.\mu)(f).$$

Moreover, it is also bijective, since the map $\mu \mapsto g.\mu$ has inverse $\mu \mapsto g^{-1}.\mu$. Since $\text{Prob}(X)$ is compact in the weak*-topology, it follows that for $g \in G$, the map $\mu \mapsto g.\mu$, $\mu \in \text{Prob}(X)$, is a homeomorphism of $\text{Prob}(X)$ into $\text{Prob}(X)$. The action $G \curvearrowright \text{Prob}(X)$ is affine, for if $0 \le \alpha \le 1$ and $\mu_1, \mu_2 \in \text{Prob}(X)$ we get for all $g \in G$ and $E \subseteq X$ that

$$\begin{aligned} g.(\alpha\mu_1 + (1-\alpha)\mu_2)(E) &= \alpha\mu_2(g^{-1}.E) + (1-\alpha)\mu_2(g^{-1}.E) \\ &= \alpha(g.\mu_1)(E) + (1-\alpha)(g.\mu_2)(E). \end{aligned}$$

4.4 Boundary actions

Definition 4.5. *A G-space X is said to be* minimal *if it contains no non-trivial closed G-invariant subsets, or equivalently, if every G-orbit is dense in X, i.e., for each $x \in X$ the set*

$$G.x = \{g.x : g \in G\}$$

is dense in X.

Example 4.6
Let $\theta \in \mathbb{R}\backslash\mathbb{Q}$ be any irrational number and define the group

$$G_\theta := \{\left(e^{2i\pi\theta}\right)^n : n \in \mathbb{Z}\} \cong \mathbb{Z}.$$

Then $Z \curvearrowright S^1$ by rotation via

$$e^{2\pi i\theta n}.e^{2\pi it} := e^{2\pi i(\theta n + t)}, \quad \text{for all } t \in [0,1),\ n \in \mathbb{Z}.$$

Since θ is irrational, the orbit $G_\theta.x \subseteq S^1$ is dense in S^1 for every $x \in S^1$. Indeed, since $\theta \in \mathbb{R}\backslash\mathbb{Q}$, we have for $n \in \mathbb{Z}\backslash\{0\}$ that $e^{2\pi in\theta} \neq 1$, for else there would be $k \in \mathbb{Z}\backslash\{0\}$ so that $\theta = \frac{n}{k}$, from this it follows that all points of $(e^{2\pi in\theta})_{n\in\mathbb{N}}$ are distinct. Given $\varepsilon > 0$, it follows that there are positive integers $m < n$ so that $\left|e^{2\pi in\theta} - e^{2\pi im\theta}\right| < \varepsilon$. Setting $N := m - n$, see that

$$\left|e^{2\pi iN\theta} - 1\right| = \left|e^{2\pi in\theta} - e^{2\pi im\theta}\right| < \varepsilon.$$

For all $k \ge 1$ set $N_k := kN = kn - km$, then

$$\left|e^{2\pi iN_k\theta} - e^{2\pi iN_{k+1}\theta}\right| = \left|e^{2\pi ikN\theta}\left(1 - e^{2\pi iN\theta}\right)\right| < \varepsilon,\ k \ge 0.$$

So the sequence $\left(e^{2\pi i N_k \theta}\right)_{k\geq 1}$ partitions S^1 into arcs of length ε. Thus, given any $t \in [0,1)$, there is $k \geq 1$ so that $\left|e^{2\pi i k N\theta} - e^{2\pi i t}\right| < \varepsilon$, so S^1 is a minimal G_θ-space.

Definition 4.7. *For a G-space X, the action of G is said to be* strongly proximal *if for each $\mu \in \mathrm{Prob}(X)$ it holds that*

$$\overline{G.\mu}^{w^*} \cap \{\delta_x : x \in X\} \neq \emptyset.$$

If X is a compact Hausdorff space, then the map $X \to \{\delta_x : x \in X\} \subseteq \mathrm{Prob}(X)$, $x \mapsto \delta_x$, is bijective continuous map, in fact a homeomorphism. Indeed, by Uryhson's lemma a net $(x_\alpha)_{\alpha\in A} \subseteq X$ converges to some x if and only if $f(x_\alpha) \to f(x)$ for all $f \in C(X)$. But then

$$\delta_{x_\alpha}(f) = f(x_\alpha) \to f(x) = \delta_x(f), \quad \text{for all } f \in C(X),$$

so δ_{x_α} converges to δ_x in the weak* topology. Likewise, if δ_{x_α} converges to δ_x in the weak*-topology, then $f(x_\alpha) \to f(x)$ for all $f \in C(X)$, and by Uryhson's lemma $x_\alpha \to x$. We abbreviate the set of point masses by $\delta_X := \{\delta_x : x \in X\}$. Also, given $g \in G$ and $x \in X$, then for all $f \in C(X)$ we have

$$g.\delta_x(f) = \delta_x(g^{-1}.f) = g^{-1}.f(x) = f(g.x) = \delta_{g.x}(f),$$

so $g.\delta_x = \delta_{g.x}$.

Definition 4.8. *A compact G-space X for which the action $G \curvearrowright X$ is both minimal and strongly proximal will be called a G-*boundary*.*

Definition 4.9. *A subset $Y \subseteq X$ of a G-space X is* minimal *if it is non-empty, closed and G-invariant and minimal with respect to these properties in the set of non-empty, closed and G-invariant subsets of X partially ordered by inclusion.*

Remark 4.10
It is worth noting that a subset $Y \subseteq X$ of a compact G-space X is minimal if and only if $G \curvearrowright Y$ is minimal. Moreover, for every compact G-space X, Zorn's lemma yields a minimal subset.

Proof.
Let $\mathcal{F}$ denote the set of all non-empty compact G-invariant subsets of X, partially ordered by inclusion. Given a descending chain $(F_i)_{i\in I} \subseteq \mathcal{F}$, compactness of X yields that the intersection $\bigcap_{i\in I} F_i$ is non-empty. It is also G-invariant and compact, hence belongs to $\mathcal{F}$. Thus, by Zorn's lemma, there is a minimal G-space $Y \subseteq X$. ☐

Lemma 4.11. *For a compact Hausdorff space X, δ_X is the set of extreme points of the weak* compact and convex set $\mathrm{Prob}(X)$.*

Proof.
By [Theorem 6.2 Zhu93, p. 35] X is homeomorphic to δ_X, and δ_X is the space of multiplicative linear functionals on $C(X)$. Since $C(X)$ is a commutative C^*-algebra, the set of multiplicative linear functionals are precisely the pure states on $C(X)$ cf. [Exercise 13.3 and 13.4 Zhu93, p. 83]. $\square$

Proposition 4.12. *For a compact G-space X, the following are equivalent:*

1. *X is a G-boundary,*
2. *For every $\nu \in \mathrm{Prob}(X)$ we have $\delta_X \subseteq \overline{G.\nu}^{w^*}$,*
3. *There are no proper non-empty closed convex subsets of $\mathrm{Prob}(X)$ which are invariant under the induced affine action $G \curvearrowright \mathrm{Prob}(X)$.*

Proof.
(1)⇒(2): Let $\nu \in \mathrm{Prob}(X)$ and $\delta_x \in \overline{G.\nu}^{w^*}$. Then there is some net $(g_\alpha)_{\alpha\in A} \subset G$ such that $g_\alpha.\nu(f) \to \delta_x(f)$ for every $f \in C(X)$, hence $g.(g_\alpha.\nu)(f) \to g.\delta_x(f)$, for every $g \in G$. So $G.\delta_x \subseteq \overline{G.\nu}^{w^*}$, but then $\overline{G.\delta_x}^{w^*} \subseteq \overline{G.\nu}^{w^*}$. Since the map $X \to \delta_X$, $x \mapsto \delta_x$, is a homeomorphism between X and δ_X, minimality of X give us

$$\delta_X = \overline{\delta_{G.x}}^{w^*} \subseteq \overline{G.\nu}^{w^*}.$$

(2)⇒(1): Let $x \in X$. Since $\delta_x \in \mathrm{Prob}(X)$, we have $\delta_y \in \overline{G.\delta_x}^{w^*}$ for all $y \in X$, so there is a net $(g_i)_{i\in I} \subset G$ such that $g_i.\delta_x \to \delta_y$ in the weak* topology. Since the map $x \mapsto \delta_x$ is a homeomorphism, it follows that $g_i.x \to y$, so X is minimal, and our original assumption ensures that X is also strongly proximal. So X is a G-boundary.

(2)⇒(3): Let $V \subseteq \mathrm{Prob}(X)$ be a non-empty closed convex G-invariant subset, and let $\mu \in V$. Since V is closed, the closure of the orbit $G.\mu$ belongs to V, and by assumption we have $\delta_X \subseteq \overline{G.\mu}^{w^*} \subseteq V$. Since V is closed and convex, we have $\overline{\mathrm{conv}}(\delta_X) \subseteq V$. But by the Krein-Milman theorem, $\overline{\mathrm{conv}}(\delta_X) = \mathrm{Prob}(X)$, so $V = \mathrm{Prob}(X)$.

(3)⇒(2): Assume that the action $G \curvearrowright \mathrm{Prob}(X)$ admits no G-invariant proper non-empty closed convex subsets. Let $\mu \in \mathrm{Prob}(X)$. Since the action of G on $\mathrm{Prob}(X)$ is affine, the convex set $\mathrm{conv}(G.\mu) \subseteq \mathrm{Prob}(X)$ is G-invariant, and by continuity of the action, the closure $\overline{\mathrm{conv}}^{w^*}(G.\mu)$ is also G-invariant. By assumption, we now have a subset which is non-empty, for it contains μ, closed, convex and G-invariant, so it must equal all of $\mathrm{Prob}(X)$. By a Theorem of Milman cf. [Theorem 3.25 Rud91, p. 76], we then have $\mathrm{Ext}(\mathrm{Prob}(X)) \subseteq \overline{G.\mu}^{w^*}$, but $\mathrm{Ext}(\mathrm{Prob}(X)) = \delta_X$. $\square$

Example 4.13
Let $\gamma\colon [0,1) \to [0,1)$ be the function $\gamma(t) = t^2$ with inverse $\gamma^{-1}(t) = t^{\frac{1}{2}}$, and let θ be irrational. Consider the compact space $S^1 := \{e^{2i\pi t}\colon t \in [0,1)\}$, and let $\alpha\colon S^1 \to S^1$ be the homeomorphism $\alpha(e^{2i\pi t}) = e^{2i\pi\gamma(t)},\ t \in [0,1)$, and $\beta\colon S^1 \to S^1$ be the rotation by θ, i.e., $\beta\left(e^{2i\pi t}\right) = e^{2i\pi(t+\theta)}$. The orbit $\{\beta^n x\colon n \in \mathbb{Z}\}$, $x \in S^1$, is dense and given any $t \in [0,1)$, we have

$$\alpha^n(e^{2i\pi t}) = e^{2i\pi t^{2n}} \to 1, \tag{†}$$

as n tends to $\pm\infty$. Let $\mathbb{F}_2 = \langle \alpha, \beta \rangle$, then $\mathbb{F}_2 \curvearrowright S^1$ by $\beta.x = \beta(x),\ \alpha.x = \alpha(x)$. Given any $\mu \in \mathrm{Prob}(S^1)$, using abstract change-of-variable, we then see that for all $f \in C(S^1)$

$$\alpha^n.\mu(f) = \int_{S^1} f(x)\,\mathrm{d}(\alpha^n.\mu)(x) = \int_{S^1} f \circ \alpha^n(x)\,\mathrm{d}\mu(x) \overset{(†)}{\to} \int_{S^1} f(1)\,\mathrm{d}\mu(x) = f(1),$$

so $\alpha^n.\mu$ converges to δ_1 in the weak* topology. So the action is strongly proximal and by Example 4.6 also minimal, hence a boundary action.

Definition 4.14. *If X, X' are compact G-spaces, we say that a map $p\colon X \to X'$ is a G-map or G-equivariant if the diagram*

$$\begin{tikzcd} X \arrow[r, "x \mapsto g.x"] \arrow[d, "p"'] & X \arrow[d, "p"] \\ X' \arrow[r, "x' \mapsto g.x'"'] & X' \end{tikzcd}$$

commutes, i.e, $p(g.x) = g.p(x)$ for every $g \in G$ and $x \in X$.

Definition 4.15. *If X, Y are G-spaces and $p\colon X \to Y$ is a continuous surjective G-map, we say that Y is a quotient of X.*

Lemma 4.16. *If Y is a quotient of a compact G-space X with respect to a G-equivariant quotient map q, then the induced map $q_*\colon C(Y) \to C(X)$ given by*

$$q_*(f) = f \circ q,$$

is an injective G-equivariant *-homomorphism.

Proof.
Let $f \in C(Y)$. Then

$$q_*(f^*)(x) = \overline{f}(q(x)) = \overline{f(q(x))} = \overline{q_*(f)(x)} = (q_*(f))^*(x),$$

where $\overline{f}$ denotes the complex conjugate of f. Let $f, h \in C(Y)$ and assume that $q_*(f) = q_*(h)$. Then

$$f(q(x)) = h(q(x)), \quad x \in X,$$

and since q is surjective this implies $f = h$. Now, let $g \in G$ and $f \in C(Y)$ and $x \in X$. Then

$$q_*(g.f)(x) = f(g^{-1}.q(x)) = f(q(g^{-1}.x)) = g.q_*(f)(x).$$

Clearly, q_* is linear and mulitplicative, so the lemma is shown. □

Lemma 4.17. *Let X be a compact G-space and Y a quotient of X with respect to a G-map $q\colon X \to Y$. Then the push-forward map $\varphi_q\colon \mathrm{Prob}(X) \to \mathrm{Prob}(Y)$ defined by*

$$\varphi_q(\mu)(E) := \mu(q^{-1}(E)), \ E \subseteq Y \ \textit{Borel},$$

is affine and G-equivariant. Moreover, by identifying $\mathrm{Prob}(X)$ and $\mathrm{Prob}(Y)$ with the state space of $C(X)$, respectively, $C(Y)$, it is also weak* continuous. When we identify $\mathrm{Prob}(X)$ with the state space of $C(X)$, then for $\mu \in \mathrm{Prob}(X)$ and $f \in C(X)$, we have

$$\varphi_q(\mu)(f) = \int_Y f \,\mathrm{d}\varphi_q\mu = \int_X q_*f \,\mathrm{d}\mu = \int_X f \circ q \,\mathrm{d}\mu.$$

Moreover, $\delta_Y \subset \varphi_q(\mathrm{Prob}(X))$.

Proof.
Let $(\mu_\alpha) \subseteq \mathrm{Prob}(X)$ be a net converging in the weak* topology to some $\mu \in \mathrm{Prob}(X)$. Then for all $f \in C(Y)$ we have

$$\varphi_q(\mu_\alpha)(f) = \int_X \underbrace{f \circ q}_{\in C(X)} \,\mathrm{d}\mu_\alpha \to \int_X f \circ q \,\mathrm{d}\mu = \varphi_q(\mu)(f).$$

For $f \in C(Y)$ and $x \in X$ we have

$$\begin{aligned}\varphi_q(\delta_x)(f) &= \int_X f \circ q \,\mathrm{d}\delta_x \\ &= (f \circ q)(x) \\ &= f(q(x)) \\ &= \delta_{q(x)}(f),\end{aligned}$$

and by surjectivity of q, we have $\delta_Y = \varphi_q(\delta_X) = \delta_{q(X)}$. The affinity of φ_q is clear. □

Corollary 4.18. *Let X be a compact G-space and Y a quotient of X with respect to a map $q\colon X \to Y$. Then the map $\varphi_q\colon \mathrm{Prob}(X) \to \mathrm{Prob}(Y)$ defined above is surjective.*

Proof.
To see this, note that since $\operatorname{Ext}(\operatorname{Prob}(X)) = \delta_X$ and that $\operatorname{Prob}(X)$ is a weak* compact and convex space, the Krein-Milman theorem states that

$$\operatorname{Prob}(X) = \overline{\operatorname{conv}(\operatorname{Ext}(\operatorname{Prob}(X)))}.$$

By Lemma 4.17, the push-forward map φ_q is affine and weak* continuous with $\varphi_q(\delta_X) = \delta_Y$. Hence

$$\begin{aligned}\operatorname{Prob}(Y) &= \overline{\operatorname{conv}(\delta_Y)} \\ &= \overline{\operatorname{conv}(\varphi_q(\delta_X))} \\ &= \overline{\varphi_q \operatorname{conv}(\delta_X)} \\ &\subseteq \overline{\varphi_q(\operatorname{Prob}(X))}.\end{aligned}$$

Now, since the state space is weak* closed and $\varphi_q(\operatorname{Prob}(X)) \subseteq \operatorname{Prob}(Y)$, it follows that

$$\varphi_q(\operatorname{Prob}(X)) = \operatorname{Prob}(Y).$$

□

Proposition 4.19. *Every quotient of a G-boundary is a G-boundary.*

Proof.
Let $q\colon X \to Y$ be a continuous surjective G-map. If $y \in Y$, then by surjectivity of q there is $x \in X$ such that $q(x) = y$. If $y' \in Y$ with $y' \neq y$, then again by surjectivity there is $x' \in X$ such that $q(x') = y'$. Since every G orbit is dense in X, there is a net $(g_i)_{i\in I} \subseteq G$ such that $g_i.x \to x'$. But then

$$g_i.y = g_i.q(x) = q(g_i.x) \to q(x') = y'.$$

Thus we conclude that Y is a minimal G-space. Now, let $\varphi_q\colon \operatorname{Prob}(X) \to \operatorname{Prob}(Y)$, be the induced weak* continuous, affine and G-equivariant map defined in Lemma 4.17. Let $\mu \in \operatorname{Prob}(Y)$. By surjectivity of the map $\varphi_q\colon \operatorname{Prob}(X) \to \operatorname{Prob}(Y)$ there is $\nu \in \operatorname{Prob}(X)$ such that $\varphi_q(\nu) = \mu$. Since X is a G-boundary, there is a net $(g_i)_{i\in I} \subset G$ such that $(g_i.\mu)_{i\in I}$ converges in the weak*-topology to δ_x for some $x \in X$. By the continuity and G-equivariance of φ_q we see that

$$g_i.\mu = g_i.\varphi_q(\nu) = \varphi_q(g_i.\nu) \to \varphi_q(\delta_x) = \delta_{q(x)} \in \delta_Y.$$

So Y is both minimal and strongly proximal, i.e., a G-boundary. □

If $G \curvearrowright X$ and $G \curvearrowright Y$, then $G \times G \curvearrowright X \times Y$ via the map

$$(g,h).(x,y) := (g.x, h.y).$$

Definition 4.20. *If $G \curvearrowright X$ and $G \curvearrowright Y$, then identifying G with the diagonal of $G \times G$, i.e., by*

$$G \cong \{(g,g)\colon g \in G\} \subseteq G \times G,$$

defines an action $G \curvearrowright X \times Y$ by $g.(x,y) := (g.x, g.y)$ called the diagonal action.

The above action is defined for an arbitrary family X_i of topological spaces such that $G \curvearrowright X_i$ for every $i \in I$. We now embark on the task of defining and proving the uniqueneuss and existence of a G-boundary satisfying a universal property of G-boundaries, the property that all other G-boundaries are quotients of it. For this, we need to show that arbitrary products of G-boundaries are again G-boundaries with respect to the diagonal action. First, however, we restrict us to the case of finite products.

Lemma 4.21. *If X and Y are compact and strongly proximal G-spaces, then $X \times Y$ is strongly proximal G-space with respect to the diagonal action of G on $X \times Y$.*

Proof.
Let π_X denote the projection of $X \times Y$ onto X. Let $\pi_{X,*}$ denote the push forward map $\mathrm{Prob}(X \times Y) \to \mathrm{Prob}(X)$ and let $\nu \in \mathrm{Prob}(X, Y)$, then the push forward measure $\pi_{X,*}(\nu)$ is defined by

$$\pi_{X,*}(\nu)(f) = \nu(f \circ \pi_X), \;\; f \in C(X).$$

By Lemma 4.17 this map is G-equivariant. Since X and Y are compact and X is strongly proximal we have

$$\delta_x \in \overline{G.\pi_{X,*}(\nu)}^{w^*} = \overline{\pi_{X,*}(G.\nu)}^{w^*} = \pi_{X,*}\left(\overline{G.\nu}^{w^*}\right)$$

for some $x \in X$. By Lemma A.4, there is some $\nu_0 \in \mathrm{Prob}(Y)$ such that $\delta_x \otimes \nu_0 \in \overline{G.\nu}^{w^*}$. Since Y is strongly proximal, there is a net $(g_i)_{i \in I} \subseteq G$ such that $g_i.\nu_0 \to \delta_y$ for some $y \in Y$. Again by compactness of X, the net $(g_i.x)_{i \in I} \subseteq X$ has a subnet $(g_j.x)_{j \in J}$ converging to some $x' \in X$. Then, by the Fubini-Tonelli theorem for Radon measures, we have

$$\delta_{(x',y)} = \delta_{x'} \otimes \delta_y = \lim_{j \in J} g_j.(\delta_x \otimes \nu_0) = \lim_{j \in J}(\delta_{g_j.x} \otimes g_j.\nu_0) \in \overline{G.\nu}^{w^*}.$$

□

.

Proposition 4.22. *If $(X_i)_{i \in I}$ is a family of compact and strongly proximal G-spaces, then the product space $X = \prod_{i \in I} X_i$ is strongly proximal with respect to the diagonal action.*

Proof.
Let $\nu \in \mathrm{Prob}(X)$, and let $\mathcal{F}$ be the directed set of finite subsets of I ordered by inclusion. For $F \in \mathcal{F}$, let $\pi_{F,*}\colon \mathrm{Prob}(X) \to \mathrm{Prob}(\prod_{i\in F} X_i)$ denote the pushforward map with respect to the G-equivariant projection from X onto the the finite product $\prod_{i\in F} X_i$, and let $\nu_F := \pi_{F,*}(\nu)$ denote the push forward measure with respect to this map. Define subsets of $\overline{G.\nu}^{w^*}$ by

$$P(F) := \{\mu \in \overline{G.\nu}^{w^*} \colon \pi_{F,*}(\mu) \text{ is a point mass}\},\ \ F \in \mathcal{F}.$$

By Lemma 4.21, $P(F)$ is nonempty for every $F \in \mathcal{F}$. Moreover, if $F, F' \in \mathcal{F}$ and $\mu \in P(F \cup F')$, then $\pi_{F,*}(\mu)$ is a point-mass and $\pi_{F',*}(\mu)$ is a point mass, so $P(F \cup F') \subseteq P(F) \cap P(F')$. Thus the family $\{P(F)\}_{F\in\mathcal{F}}$ has the finite intersection property. Let $F \in \mathcal{F}$ and let $(\mu_\alpha)_{\alpha\in A} \subseteq P(F)$ be a net converging to some μ in the weak* topology. By Lemma 4.17, the map $\pi_{F,*}$ is weak* continuous, so $\pi_{F,*}(\mu_\alpha) \to \pi_{F,*}(\mu)$. But $\pi_{F,*}(\mu_\alpha) \subseteq \delta_{\prod_{i\in F} X_i}$, which is weak$*$ closed, so $\pi_{F,*}(\mu)$ is a point-mass, and thus in $P(F)$. The family $\{P(F)\}_{F\in\mathcal{F}}$ is a family of closed subsets of a compact space with the finite intersection property, so there is $\mu \in \bigcap_{F\in\mathcal{F}} P(F)$. By Lemma A.5, μ is a point mass, concluding the proof. □

Proposition 4.23. *Let Y be a minimal compact G-space and X a G-boundary. Any continuous G-equivariant map $\varphi\colon Y \to \mathrm{Prob}(X)$, has image δ_X. Moreover, if there is such a map, then there is at most one continuous G-equivariant map $Y \to X$, and it is surjective.*

Proof.
Since Y is compact, the image under φ is compact. This together with the G-equivariance of φ yields

$$\varphi(Y) = \overline{\varphi(Y)}^{w^*} = \overline{\varphi(G.Y)}^{w^*} = \varphi(G.Y) = G.\varphi(Y).$$

Since X is strongly proximal, there is some $x \in X$ such that $\delta_x \in \varphi(Y)$, i.e., $\varphi^{-1}(\delta_X) \neq \emptyset$. If $y \in Y$ is such that $\varphi(y)y\delta_x$ we get that $\varphi(G.y) = G.\delta_x \subseteq \delta_X$. So $G.y \subseteq \varphi^{-1}(\delta_X)$. But δ_X is weak*-closed, thus $\varphi^{-1}(\delta_X) = Y$.

If $\psi_1, \psi_2\colon Y \to X$ are continuous G-equivariant maps, then the map $\alpha\colon Y \to \mathrm{Prob}(X)$, $y \mapsto \frac{1}{2}(\delta_{\psi_1(y)} + \delta_{\psi_2(y)})$, is a continuous equivariant map, and by the above argument, it has image δ_X. So if $y \in Y$ then there exists $x \in X$ such that $\alpha(y) = \delta_x$. Then

$$\delta_x = \alpha(y) = \frac{1}{2}\delta_{\psi_1(y)} + \frac{1}{2}\delta_{\psi_2(y)},$$

and since δ_X is the set of pure states on $C(X)$, we have that $\psi_2(y) = \psi_1(y) = x$, implying that $\psi_1 = \psi_2$. Identifying X with δ_X by the homeomorphism $\iota\colon \delta_X \to X$, $\iota(\delta_x) = x$, then $\iota \circ \varphi\colon Y \to X$ must be the only continuous G-equivariant map from X to Y, and it is surjective. □

Definition 4.24. *A compact G-space Z is a* universal boundary *for G if Z itself is a G-boundary satisfying that to every G-boundary X there is a surjective continuous G-equivariant map $\rho\colon Z \to X$.*

Remark 4.25
We can easen the requirement that ρ must be surjective, for Proposition 4.23 ensures surjectivity of ρ.

Recall that the Stone-Céch compactification of G, βG, has the universal property that any continuous function from G with compact Hausdorff codomain extends to a continuous function from βG.

Before proving existence and uniqueness of the universal boundary of G, note that for a compact G-boundary X, there is a bound on the cardinality of X. Let $x \in X$, define a function $\alpha\colon G \to X$ by $\alpha(g) = g.x$. This is a continuous function with compact Hausdorff codomain, so it extends to a continuous function $\tilde{\alpha}\colon \beta G \to X$. By compactness of βG we have $\tilde{\alpha}(\beta G) = \overline{G.x} = X$, so $\tilde{\alpha}$ is surjective, and we get the following bound

$$|X| \leq |\beta G|$$

Theorem 4.26. *There exists a universal boundary of G. We denote it by $\partial_F G$ and it is called the Furstenburg boundary.*

Proof.
By the above note, we can index the family of all the G-boundaries up to isomorphism, i.e., G-equivariant homeomorphism. Let $\{X_i\}_{i \in I}$ be this family, by 4.22, the product

$$Z = \prod_{i \in I} X_i,$$

is strongly proximal. By Remark 4.10 there is a minimal compact G-space $\partial_F G \subseteq Z$. The inclusion map $i_{\partial_F G}\colon \partial_F G \to Z$ is continuous and G-equivariant. For a G-boundary X, let π_X denote the projection from Z to X. Then the map $\pi_{\partial_F G, X} := \pi_X \circ i_{\partial_F G}\colon \partial_F G \to X$ is continuous and G-equivariant, hence by Remark 4.25 it has image X, so it is surjective. Thus $\partial_F G$ is a universal boundary for G. $\square$

Proposition 4.27. *The Furstenberg boundary is unique up to isomorphism.*

Proof.
Assume that X, Y are universal boundaries for G. By universitality, there are maps $\varphi_1\colon X \to Y$ and $\varphi_2\colon Y \to X$ which are both continuous surjective and G-equivariant, and thus by Proposition 4.23 we see that $\varphi_2 \circ \varphi_1\colon X \to X$ is the identity on X and likewise $\varphi_1 \circ \varphi_2$ is the identity on Y. Thus $X \cong Y$. Therefore the Furstenberg boundary $\partial_F G$ is unique up to isomorphism. $\square$

Proposition 4.28. *An amenable subgroup N of a discrete group G acts trivially on every compact G-boundary X.*

Proof.
Let N be an amenable subgroup of G. Let X be a compact G-boundary. Since N is amenable, the induced action $N \curvearrowright C(X)$ is amenable, hence the action $N \curvearrowright X$ is amenable and there is an N-invariant probability measure $\mu_0 \in \mathrm{Prob}(X)$. Let $\mathcal{P}_N \subseteq \mathrm{Prob}(X)$ denote the set of N-invariant measures on X. This set is closed, by continuity of the action $N \curvearrowright \mathrm{Prob}(X)$, and it is convex, since $N \curvearrowright \mathrm{Prob}(X)$ is affine. We now **claim** that the set $\mathcal{P}_N$ is G-invariant. Indeed, let $\mu \in \mathcal{P}_N$ and $g \in G$. Since N is normal, if $n \in N$, there is $n' \in N$ such that $ng = gn'$. We then have

$$(ng).\mu = (gn').\mu = g.(n'.\mu) = g.\mu.$$

So $\mathcal{P}_N$ is a non-empty closed convex G-invariant subset of $\mathrm{Prob}(X)$, and since X is a G-boundary, Proposition 4.12 tells us $\mathcal{P}_N = \mathrm{Prob}(X)$. Identifying X with $\delta_X \subseteq \mathrm{Prob}(X)$, we get that N acts trivially on X. $\square$

Proposition 4.29. *Let G be a discrete group and $t \in G$. Then $t \in \mathrm{AR}_G$ if and only if $t \in \ker(G \curvearrowright X)$ for all compact G-boundaries X.*

Proof.
By the above proposition, if $t \in \mathrm{AR}_G$, then $t \in \ker(G \curvearrowright X)$ for all compact G-boundaries. For the converse, see [Proposition 7 Fur03, p. 179]. $\square$

5. C^*-simplicity and boundary actions

5.1 Uniqueness of trace, a new way

The reader may have asked themselves, what does C^*-simplicity and boundary actions of a discrete group G have to do with each other? It turns out that they relate very much. In this chapter, we will show results which define exactly when a discrete group is C^*-simple using boundary actions, and exactly when it has the unique trace property using boundary actions. The result is thus a more complete toolkit for characterizing C^*-simplicity and uniqueness of trace of discrete groups.

Recall that when a discrete group G admits an action α on a compact Hausdorff space X, we can form the reduced crossed product $C(X) \rtimes_{\alpha,r} G$ defined as in Chapter 2 which contains both $C_r^*(G)$ and $C(X)$, and that they are related in the following way: $\lambda_t f \lambda_t^* = \alpha_t(f) := t.f$ for $t \in G$ and $f \in C(X)$, where $t.f(x) = f(t^{-1}.x)$ for $x \in X$. If the action α is understood, we will omit it from the notation $C(X) \rtimes_r G$. Throughout the rest of this chapter, G will denote a countable discrete group and X will denote a compact Hausdorff space.

Lemma 5.2. *Let G be a discrete group admitting an action on a compact Hausdorff space X and let $x \in X$. If φ is a state on $C(X) \rtimes_r G$ such that $\varphi_{|C(X)} = \delta_x$, then $\varphi(\lambda(t)) = 0$ for all $t \in G$ such that t does not act trivially on x, i.e., $t.x \neq x$.*

Proof.
Let $x \in X$ and φ be a state on $C(X) \rtimes_r G$ satisfying the conditions in the statement. Since $\mathbb{C}$ and $C(X)$ are commutative, we see that for $f \in C(X)$,

$$\varphi(ff^*) = \varphi(f^*f) = \delta_x(f^*f) = \delta_x(f)^*\delta_x(f) = \varphi(f)^*\varphi(f).$$

Since states are unital completely positive maps *(u.c.p)* on $C(X)$, the above shows that φ satisfies the bimodule propery of [Proposition 1.5.7 BO08, p. 12] on every $f \in C(X)$. Thus $\varphi(af) = \varphi(a)\varphi(f)$ for all $a \in C(X) \rtimes_r G$. So for $t \in G$ and $f \in C(X)$ we get

$$\varphi(\lambda_t)f(x) = \varphi(\lambda_t f) = \varphi((t.f)\lambda_t) = \varphi(t.f)\varphi(\lambda_t) = t.f(x)\varphi(\lambda_t) = f(t^{-1}.x)\varphi(\lambda_t)$$

By Urysohn's lemma, there is an $f \in C(X)$ such that $f(t^{-1}.x) \neq f(x)$, and since the above holds for all $f \in C(X)$, this implies $\varphi(\lambda_t) = 0$ for $t^{-1}.x \neq x$, or equivalently when $t.x \neq x$. $\square$

Given a state on either $C_r^*(G)$ or $C(X) \rtimes_r G$ and $t \in G$, we define a new linear functional $t.\varphi$ by

$$(t.\varphi)(a) = \varphi(\lambda_t a \lambda_t^*), \quad a \in C_r^*(G) \text{ or } C(X) \rtimes_r G.$$

Then $t.\varphi$ is again a state, since $t.\varphi(1) = \varphi(\lambda_{tt^{-1}}) = \varphi(1) = 1$ and the fact that the positive elements are stable under conjugation.

Lemma 5.3. *Let τ be a tracial state on $C_r^*(G)$, and assume that X is a G-boundary, i.e, $G \curvearrowright X$ is a boundary action. Then for all $x \in X$ there is a state φ on $C(X) \rtimes_r G$ extending τ such that the restriction of φ to $C(X)$ is δ_x.*

Proof.
$C_r^*(G)$ sits naturally inside $C(X) \rtimes_r G$, and τ is a state and thus a u.c.p map. Using Hahn-Banach we may extend τ to a state ψ on $C(X) \rtimes_r G$. Let ρ denote the restriction of ψ to $C(X)$. Then ρ is a state on $C(X)$. Since X is a G-boundary, for all $x \in X$ we have $\delta_x \in \overline{G.\rho}^{w^*}$, i.e., there is a net $(g_i)_{i \in I} \subseteq G$ such that $g_i.\rho$ converges to δ_x in the weak*-topology. Since the state space of $C(X) \rtimes_r G$ is weak*-compact, the net $(g_i.\psi)_{i \in I}$ has a subnet $(g_j.\psi)_{j \in J}$ converging to some state φ in the weak*-topology. Then, for $f \in C(X)$, we see that

$$\varphi(f) = \lim_{j \in J} g_j.\rho(f) = \delta_x(f) = f(x),$$

so the restriction of φ to $C(X)$ is $\delta_x \in \mathcal{S}(C(X))$. Since τ is tracial, we get for $s, t \in G$

$$(s.\psi)(\lambda_t) = \psi(\lambda_{sts^{-1}}) = \tau(\lambda_{sts^{-1}}) = \tau(\lambda_t).$$

Thus $\varphi(\lambda_t) = \lim_{j \in J} g_j.\psi(\lambda_t) = \tau(\lambda_t)$ for all $t \in G$. So φ is an extension of τ satisfying the desired properties. $\square$

Given a normal amenable subgroup N of a discrete group G, denote by λ_G and $\lambda_{G/N}$ the left-regular representations of G, respectively, G/N on $B(\ell^2(G))$, respectively, $B(\ell^2(G/N))$. By [Proposition 3 DLH07, p. 3] there is $*$-homomorphism $\pi\colon C_r^*(G) \to C_r^*(G/N)$ extending the canonical quotient map $q\colon \mathbb{C}G \to \mathbb{C}(G/N)$ such that the following diagram commutes

$$\begin{tikzcd}
\mathbb{C}G \arrow[r, "q"] \arrow[d, "\lambda_G"'] & \mathbb{C}(G/N) \arrow[d, "\lambda_{G/N}"] & (\star) \\
C_r^*(G) \arrow[r, "\pi"'] & C_r^*(G/N) &
\end{tikzcd}$$

With this in mind, we now are now ready to prove the following theorem

Theorem 5.4. *Let G be a discrete group, and $t \in G$. Then $\tau(\lambda_t) = 0$ for all tracial states τ on $C_r^*(G)$ if and only if $t \notin \mathrm{AR}_G$.*

Proof.
"$\Leftarrow$": Suppose that $t \notin \mathrm{AR}_G$. By Proposition 4.29, there is a compact Hausdorff space X such that $G \curvearrowright X$ is a boundary action with $t \notin \ker(G \curvearrowright X)$, so let $x \in X$ witness this, i.e., such that $t.x \neq x$. Let τ denote any tracial state on $C_r^*(G)$. By Lemma 5.3, there is a state ψ extending τ to $C(X) \rtimes_r G$ such that the restriction of ψ to $C(X)$ is δ_x. By Lemma 5.2, since $t.x \neq x$, we have $\tau(\lambda_t) = \psi(\lambda_t) = 0$.

"$\Rightarrow$": Assume $t \in G$ and that $\tau(\lambda_G(t)) = 0$ for every tracial state τ on $C_r^*(G)$. Let π denote the induced $*$-homomorphism $C_r^*(G) \to C_r^*(G/\mathrm{AR}_G)$, which exists since AR_G is an amenable normal subgroup of G, and let τ_0' denote the canonical faithful tracial state on $C_r^*(G/\mathrm{AR}_G)$. Define a new tracial state $\tau := \tau_0' \circ \pi \colon C_r^*(G) \to \mathbb{C}$. Then for $s \in G$ we have

$$\tau(\lambda_G(s)) = \tau_0'(\pi(\lambda_G(s)) = \tau_0'(\lambda_{G/AR_G}([s])) = \begin{cases} 1 & \text{if } s \in AR_G \\ 0 & \text{else} \end{cases},$$

by $(\star)$. And, since we assumed that $\tau(\lambda_G(t)) = 0$, we conclude that $t \notin AR_G$. □

Corollary 5.5. *The reduced group C^*-algebra of a discrete group G has the unique trace property if and only if the amenable radical is trivial.*

Proof.
By the above theorem, if AR_G is non-trivial, and if $t \in \mathrm{AR}_G$ with $t \neq e$, the trace τ obtained by composition of the canonical trace on $C_r^*(G/\mathrm{AR}_G)$ and the $*$-homomorphism $\pi \colon C_r^*(G) \to C_r^*(G/\mathrm{AR}_G)$ satisfies $\tau(\lambda_t) = 1$ where $\tau_0(\lambda_t) = 0$, so $\tau \neq \tau_0$.

If AR_G is trivial, then every tracial state τ on $C_r^*(G)$ satisfies $\tau(\lambda_t) = 0$ for every $t \in G \backslash \{e\}$ by the above theoreom. Since τ_0 is the unique trace such that $\tau_0(\lambda_t) = 0$ for all $t \neq e$, it follows that $\tau = \tau_0$. □

5.6 C^*-simplicity and the unique trace property

In the following, we let $\ell^2(G)_+$ denote the subset of $\ell^2(G)$ of positively valued functions, i.e., the set of $\xi \in \ell^2(G)$ such that $\xi(G) \subseteq [0, \infty)$.

Lemma 5.7. *If $x, y \in C_r^*(G)$ are finite positive linear combinations of elements of $\{\lambda_t \colon t \in G\}$, then $\|x + y\| \geq \|x\|$.*

Proof.
If $n \in \mathbb{N}$, $\alpha_1, \ldots, \alpha_n \geq 0$, $g_1, \ldots, g_n \in G$ and $z = \sum_{i=1}^n \alpha_i \lambda_{g_i} \in C_r^*(G)$, then z maps $\ell^2(G)_+$ to $\ell^2(G)_+$, since $z\xi(t) = \sum_{i=1}^n \alpha_i \xi(g_i^{-1}t) \geq 0$ for all $\xi \in \ell^2(G)_+$ and $t \in G$. Moreover, for all $\xi, \eta \in \ell^2(G)$ we have

$$\begin{aligned}
|\langle z\xi, \eta \rangle| &= \left| \left\langle \sum_{i=1}^n \alpha_i (\lambda_{s_i} \xi), \eta \right\rangle \right| \\
&\leq \sum_{i=1}^n \alpha_i \left| \sum_{g \in G} \xi(s_i^{-1} g) \overline{\eta(g)} \right| \\
&\leq \sum_{i=1}^n \alpha_i \sum_{g \in G} |\xi(s_i^{-1} g)| |\overline{\eta(g)}| \\
&= \sum_{g \in G} \sum_{i=1}^n \alpha_i |(\lambda_{s_i} \xi)(g)| |\eta(g)| \\
&= \sum_{g \in G} z |\xi(g)| |\eta(g)| \\
&= \langle z|\xi|, |\eta| \rangle.
\end{aligned}$$

If $\xi \in \ell^2(G)$ with $\|\xi\| = 1$ then $\|z\xi\| = \sup_{\|\eta\|=1} |\langle \eta, z\xi \rangle|$, by the Riesz' representation theorem for Hilbert spaces. Thus, we see that

$$\|z\| = \sup\{|\langle z\xi, \eta \rangle| \colon \|\xi\| \leq 1,\ \|\eta\| \leq 1\}.$$

Combining the above we get that

$$\|z\| = \sup\{\langle z\xi, \eta \rangle \colon \xi, \eta \in \ell^2(G)_+,\ \|\xi\| \leq 1,\ \|\eta\| \leq 1\}.$$

If x, y are finite positive linear combination of elements of $\{\lambda_t \colon t \in G\}$, then

$$\begin{aligned}
\|x\| &= \sup\{\langle x\xi, \eta \rangle \colon \xi, \eta \in \ell^2(G)_+,\ \|\xi\| \leq 1,\ \|\eta\| \leq 1\} \\
&\leq \sup\{\langle x\xi, \eta \rangle + \langle y\xi, \eta \rangle \colon \xi, \eta \in \ell^2(G)_+,\ \|\xi\| \leq 1,\ \|\eta\| \leq 1\} \\
&= \|x + y\|.
\end{aligned}$$

□

Lemma 5.8. *Let G be a group and $t \in G$. Then the following are equivalent:*

1. $0 \notin \overline{\mathrm{conv}}\{\lambda_{sts^{-1}} : s \in G\}$,
2. $0 \notin \overline{\mathrm{conv}}\{\lambda_{sts^{-1}} + \lambda_{sts^{-1}}^* : s \in G\}$,
3. *There exists a self-adjoint linear functional ω on $C_r^*(G)$ of norm 1 and a constant $c > 0$ such that $\mathrm{Re}\omega(\lambda_{sts^{-1}}) \geq c$ for all $s \in G$.*

Proof.
(1)⇒(2): Assume $t \in G$ such that $0 \notin \overline{\mathrm{conv}}\{\lambda_{sts^{-1}} : s \in G\}$. Given $c_1, \dots, c_n \geq 0$ such that $\sum_{j=1}^n c_j = 1$ and $s_1, \dots, s_n \in G$ then the element $z = \sum_{j=1}^n c_j \lambda_{s_j t}$ is non-zero and satisfies the conditions of Lemma 5.7, so

$$\|z\| \leq \|z + z^*\|$$

Since every element of $\overline{\mathrm{conv}}\{\lambda_{sts^{-1}} + \lambda^*_{sts^{-1}}\}$ is a limit of elements of the form $z + z^*$, continuity yields that 1 implies 2.

(2)⇒(1): This is trivial.

(2)⇒(3): The space $(C^*_r(G))_{\mathrm{sa}} = \{a \in C^*_r(G) \colon a = a^*\}$ is a real vector space, with $\{0\}$ as a compact subset disjoint from the closed subset $\overline{\mathrm{conv}}(\lambda_{sts^{-1}} + \lambda^*_{sts^{-1}} : s \in G\}$. By The Hahn-Banach separation theorem ([Theorem 3.4 Rud91, p. 59]), there is a real linear functional ω on $(C^*_r(G))_{\mathrm{sa}}$ and $c > 0$ such that

$$\omega\left(\lambda_{sts^{-1}} + (\lambda_{sts^{-1}})^*\right) \geq c, \tag{$\star$}$$

for all $s \in G$. Setting $\omega' := \frac{\omega}{\|\omega\|}$, and $c' := c\|\omega\|$, we see that ω' is a real linear functional of norm 1 satisfying $(\star)$.

Defining a linear functional ω_0 on $C^*_r(G)$ by

$$\omega_0(a) = \omega'\left(\frac{a + a^*}{2}\right) + i\omega'\left(\frac{a - a^*}{2i}\right), \quad a \in C^*_r(G),$$

we see that ω_0 is a self-adjoint linear functional of norm 1 on $C^*_r(G)$ satisfying

$$\begin{aligned}\mathrm{Re}\omega_0\left((\lambda_{sts^{-1}}\right) &= \frac{1}{2}\omega'\left(\lambda_{sts^{-1}} + (\lambda_{sts^{-1}})^*\right) \\ &\geq \frac{1}{2}c' > 0\end{aligned}$$

for all $s \in G$.

(3)⇒(1): By the linearity of linear functionals, $0 \in \ker(\omega)$ for all linear functionals ω on $C^*_r(G)$. Thus 0 cannot be in $\overline{\mathrm{conv}}\{\lambda_{sts^{-1}} : s \in G\}$ assuming (3). □

Theorem 5.9. *Let G be a group and $t \in G$. Then $t \notin \mathrm{AR}_G$ if and only if*

$$0 \in \overline{\mathrm{conv}}\{\lambda_{sts^{-1}} : s \in G\}.$$

Proof.
"⇒": Suppose $0 \notin \overline{\mathrm{conv}}(\lambda_{sts^{-1}} : s \in G)$, and assume towards contradiction that $t \notin \mathrm{AR}_G$. By Proposition 4.29, there is a compact G-boundary X and

$x \in X$ such that $t.x \neq x$. By Lemma 5.8, there is a self-adjoint functional f on $C_r^*(G)$ of norm 1 and a $c > 0$ such that

$$\mathrm{Re} f(\lambda_{sts^{-1}}) \geq c, \quad \text{for all } s \in G.$$

Since f is a self-adjoint linear functional, it has a Jordan decomposition (c.f. [Theorem II.6.3.4 Bla06, p. 106]), i.e., there are positive linear functionals $f_\pm$ such that $f = f_+ - f_-$ and $\|f\| = \|f_+\| + \|f_-\|$. The functional $f_+ + f_-$ is a positive functional, so its norm is given by

$$\begin{aligned}\|f_+ + f_-\| = (f_+ + f_-)(1_{C_r^*(G)}) &= f_+(1_{C_r^*(G)}) + f_-(1_{C_r^*(G)}) \\ &= \|f_+\| + \|f_-\| \\ &= 1.\end{aligned}$$

Thus, $f_+ + f_-$ is a state on $C_r^*(G)$. The crossed product, $C(X) \rtimes_r G$, contains $C_r^*(G)$ as a C^*-subalgebra. Using Hahn-Banach and [Lemma 1.5.15 BO08, p. 16], the positive linear functionals $f_\pm$ extend to positive linear functionals $\psi_\pm$ on $C(X) \rtimes_r G$ with $\|\psi_\pm\| = \|f_\pm\|$. Then $\psi := \psi_+ + \psi_-$ is a state which extends the state $f_+ + f_-$ to $C(X) \rtimes_r G$. Let $p_\pm$ be the restrictions of $\psi_\pm$ to $C(X)$, and set $p := p_+ + p_-$. Since X is a G-boundary, there is a net $(g_i)_{i \in I} \subseteq G$ such that $g_i.p$ converges to δ_x in the weak*-topology. Since $\|\psi_\pm\| \leq 1$, Alaoglu's theorem ensures the existence of a subnet $(g_j)_{j \in J}$ such that $g_j.\psi_\pm$ converges to positive functionals $\varphi_\pm$ in the weak$*$-topology. Then $\varphi_\pm$ has the same norm as $\psi_\pm$, indeed, since they are positive functionals their norm are given by

$$\begin{aligned}\|\varphi_\pm\| = \varphi_\pm(1_{C(X) \rtimes_r G}) &= \lim_{j \in J} g_j \psi_\pm(1_{C(X) \rtimes_r G}) \\ &= \lim_{j \in J} \psi_\pm(\lambda_j 1_{C(X) \rtimes_r G} \lambda_j^*) \\ &= \lim_{j \in J} \|\psi_\pm\| \\ &= \|\psi_\pm\|.\end{aligned}$$

Recall that $(g_j)_{j \in J} \subseteq G$ is a subnet of a net $(g_i)_{i \in I} \subseteq G$ such that $g_j.p$ converges in the weak* topology to δ_x. Thus if $\rho_\pm$ denotes the restrictions of $\varphi_\pm$ to $C(X)$, then $\rho := \rho_+ + \rho_- = \delta_x$. Setting $\mu_\pm := \frac{\rho_\pm}{\|\rho_\pm\|}$, we get that

$$\delta_x = \mu_+ \|\rho_+\| + \mu_- \|\rho_-\|.$$

And since δ_x is a pure state, and $1 = \|\rho_+\| + \|\rho_-\|$, we have that $\mu_\pm = \delta_x$. So $\rho_\pm = \delta_x \|\rho_\pm\|$. By assumption, $t.x \neq x$, so $\varphi_\pm(\lambda_t) = 0$: By Lemma 5.2, we see that $\left.\frac{\varphi_\pm}{\|\varphi_\pm\|}\right|_{C_r^*(G)} (\lambda_t) = 0$. Hence we get the following

$$\begin{aligned}0 &= \left.\varphi_+\right|_{C_r^*(G)} (\lambda_t) - \left.\varphi_-\right|_{C_r^*(G)} (\lambda_t) \\ &= \left.\varphi\right|_{C_r^*(G)} (\lambda_t) \\ &= \lim_{j \in J} g_j.f(\lambda_t) \\ &= \lim_{j \in J} f\left(\lambda_{g_j t g_j^{-1}}\right),\end{aligned}$$

contradicting our assumption that $f(\lambda_{sts^{-1}}) > 0$ for all $s \in G$, so we conclude that $t \in \mathrm{AR}_G$. □

Corollary 5.10. *A discrete group G has the unique trace property if and only if for all $t \in G \backslash \{e\}$ we have*

$$0 \in \overline{\mathrm{conv}}\{\lambda_{sts^{-1}} : s \in G\}.$$

Proof.
Assume that for all $t \neq e$ we have $0 \in \overline{\mathrm{conv}}\{\lambda_{sts^{-1}} : s \in G\}$. By Theorem 5.9, this happens if and only if $t \notin \mathrm{AR}_G$ for all $t \neq e$. So $\mathrm{AR}_G = \{e\}$. By Corollary 5.5 this happens if and only if $C_r^*(G)$ has unique trace τ_0. □

Definition 5.11. *Let G be a discrete group, and X a compact G-space. We say that the action $G \curvearrowright X$ is* topologically free *if for all $s \in G \backslash \{e\}$ the set of s-fixed points $X^s := \{x \in X : s.x = x\}$ has empty interior. We say that the action $G \curvearrowright X$ is* free *if $X^s = \emptyset$ for all $s \neq e$.*

The following result is [Proposition 3.1 Bre+14, p. 11], a result in a recent article by Emmanuel Breuillard, Merhdad Kalantar, Matthew Kennedy and Narutaka Ozawa, expanding a result of Kennedy and Kalantar. It will be useful later on

Theorem 5.12. *Let G be a discrete group. The G is C^*-simple if and only if there is a compact Hausdorff space X such that $G \curvearrowright X$ is a free boundary action.*

Recall that for $s \in G$ and $\varphi \in \mathcal{S}(C_r^*(G))$ we define a new state $s.\varphi \colon C_r^*(G) \to \mathbb{C}$ by

$$(s.\varphi)(a) = \varphi(\lambda_s a \lambda_s^*), \quad a \in C_r^*(G).$$

Using the above proposition, we get the following.

Theorem 5.13. *Let G be a discrete group. If τ_0 denotes the canonical tracial state on $C_r^*(G)$, then the following are equivalent:*

1. *$C_r^*(G)$ is simple,*
2. *$\tau_0 \in \overline{\{s.\varphi : s \in G\}}^{w^*}$ for all states φ on $C_r^*(G)$,*
3. *$\tau_0 \in \overline{\mathrm{conv}}^{w^*}\{s.\varphi : s \in G\}$ for all states φ on $C_r^*(G)$,*
4. *$\omega(1)\tau_0 \in \overline{\mathrm{conv}}\{s.\omega : s \in G\}$ for all $\omega \in (C_r^*(G))^*$,*
5. *For all $t_1, t_2, \dots, t_m \in G \backslash \{e\}$, we have*

$$0 \in \overline{\mathrm{conv}}\{\lambda_s(\lambda_{t_1} + \lambda_{t_2} + \cdots + \lambda_{t_m})\lambda_s^* : s \in G\},$$

6. *For all* $t_1, t_2, \ldots, t_m \in G\backslash\{e\}$ *and* $\varepsilon > 0$*, there exists* $s_1, s_2, \ldots, s_n \in G$ *such that*

$$\left\| \sum_{k=1}^{n} \frac{1}{n} \lambda_{s_k t_j s_k^{-1}} \right\| < \varepsilon,$$

for all $1 \leq j \leq m$.

Proof.
(1)⇒(2): By Theorem 5.12, there is a compact G-space X such that the action $G \curvearrowright X$ is a free boundary action. Let φ be any state on $C_r^*(G)$. Extend φ to a state ψ on $C(X) \rtimes_r G$, and let ρ denote its restriction to $C(X)$. Since X is a boundary action, there is an $x \in X$ and a net $(g_i.\rho)_{i \in I} \subseteq \mathrm{Prob}(X)$ such that $g_i.\rho$ converges to δ_x in the weak* topology. By Alaoglu's theorem, the net $(g_i.\psi)_{i \in I}$ in the state space of $C(X) \rtimes_r G$ has a weak* convergent subnet $(g_j.\psi)_{j \in J}$ with limit Ψ, and the restriction of Ψ to $C(X)$ is δ_x. By assumption the action $G \curvearrowright X$ is free, so for all $t \in G\backslash\{e\}$ the set of t-fixed points $\{x \in X : t.x = x\}$ is empty, so by Lemma 5.2, $\Psi_{|C_r^*(G)}(\lambda_t) = 0$. And since the restriction of Ψ to $C_r^*(G)$ is a state, we see that $\Psi_{|C_r^*(G)}(\lambda_e) = 1$. Since τ_0 is the unique tracial state which vanishes on λ_t for $t \neq e$, we conclude that

$$\lim_{j \in J} g_j.\varphi = \Psi_{|C_r^*(G)} = \tau_0.$$

(2)⇒(3): Is trivially true.

(3)⇒(4): Suppose $\varphi_1, \ldots, \varphi_m$ are states on $C_r^*(G)$. Then the set

$$\overline{\mathrm{conv}}^{w^*}\{(s.\varphi_1, \ldots, s.\varphi_m) : s \in G\} \subseteq \mathcal{S}(C_r^*(G)))^m$$

is a weak* closed, convex and G-invariant subset with respect to the diagonal action. Define the set $\mathcal{M}$ to be the set of functions $\Phi \colon \mathcal{S}(C_r^*(G)) \to \mathcal{S}(C_r^*(G))$ of the form $\Phi(\gamma) = \sum_{i=1}^n \alpha_i s_i.\gamma$ for $\gamma \in \mathcal{S}(C_r^*(G))$, for some $n \geq 1$, $\alpha_1, \ldots, \alpha_n \geq 0$, $\sum_{i=1}^n \alpha_i = 1$ and $s_1, \ldots, s_n \in G$. If $\rho \in \mathcal{S}(C_r^*(G))$, we see that

$$\mathrm{conv}(G.\rho) = \{\Phi(\rho) : \Phi \in \mathcal{M}\}.$$

If $\Phi \in \mathcal{M}$, then for all $a \in C_r^*(G)$ we have

$$\Phi(\tau_0)(a) = \sum_{i=1}^{n} \alpha_i (s_i.\tau_0)(a) = \sum_{i=1}^{n} \alpha_i \tau_0(\lambda_{s_i} a \lambda_{s_i}^*) = \tau_0(a),$$

by traciality of τ_0, hence $\Phi(\tau_0) = \tau_0$ for all $\Phi \in \mathcal{M}$. By assumption, there is a net $(\Phi_i)_{i \in I} \subseteq \mathcal{M}$ such that $\Phi_i(\varphi_1)$ converges to τ_0 in the weak* topology. Since $(\Phi_i(\varphi_j))_{i \in I}$ is a net in $\mathcal{S}(C_r^*(G))$, which is weak* compact, taking convergent subnets $m-1$ times yields a subnet $(\Phi_k)_{k\in K}$ such that $\Phi_k(\varphi_j)$ converges to $\psi_j \in \mathcal{S}(C_r^*(G))$ in the weak* topology for $2 \le j \le m$. So we have that

$$(\tau_0, \psi_2, \psi_3, \ldots, \psi_m) \in \overline{\text{conv}}^{w^*}\{(s.\varphi_1, s.\varphi_2, \ldots, s.\varphi_m) : s \in G\} \subseteq \mathcal{S}(C_r^*(G))^m.$$

Since the set $\overline{\text{conv}}^{w^*}\{(s.\varphi_1, s.\varphi_2, \ldots, s.\varphi_m) : s \in G\}$ is weak* closed, convex and G-invariant, minimality of the convex hull gives that

$$\overline{\text{conv}}^{w^*}\{(s.\tau_0, s.\psi_2, \ldots, s.\psi_m) : s \in G\} \subseteq \overline{\text{conv}}^{w^*}\{(s.\varphi_1, \ldots, s.\varphi_m) : s \in G\}.$$

Once again, our intial assumption ensures the existence of a net $(\Phi_i)_{i\in I} \subseteq \mathcal{M}$ such that $\Phi_i(\psi_2)$ converges to τ_0 in the weak* topology. Again, taking subnets $m-2$ times we obtain a subnet $(\Phi_k)_{k\in K} \subseteq \mathcal{M}$ such that for $3 \le j \le m$ the net $(\Phi_k(\psi_j))_{k\in K}$ converges to some $\psi'_j \in \mathcal{S}(C_r^*(G))$ in the weak* topology. So we now have

$$(\tau_0, \tau_0, \psi'_3, \ldots, \psi'_m) \in \overline{\text{conv}}^{w^*}\{(\tau_0, s.\psi_2, s.\psi_3, \ldots, s.\psi_m) : s \in G\} \subseteq \mathcal{S}(C_r^*(G))^m.$$

Continuing this process recursively, we see that

$$(\tau_0, \tau_0, \ldots, \tau_0) \in \overline{\text{conv}}^{w^*}\{(s.\varphi_1, s.\varphi_2, \ldots, s.\varphi_m) : s \in G\}.$$

If ω is a bounded linear functional on $C_r^*(G)$, it can be decomposed into a linear combination of four states $\varphi_1, \varphi_2, \varphi_3$ and φ_4 such that

$$\omega = \alpha_1\varphi_1 + \alpha_2\varphi_2 + \alpha_3\varphi_3 + \alpha_4\varphi_4,$$

for some $\alpha_i \in \mathbb{C}$. We may assume without loss of generality that $\omega \neq 0$, so that $\alpha_i \neq 0$ for some $1 \le i \le 4$. Since $\varphi_1, \ldots, \varphi_4$ are states, we see that

$$\omega(1_{C_r^*(G)}) = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4.$$

By the above argument, we have

$$(\tau_0, \tau_0, \tau_0, \tau_0) \in \overline{\text{conv}}^{w^*}\{(s.\varphi_1, \ldots, s.\varphi_4) : s \in G\}.$$

So given a finite subset $F \subseteq C_r^*(G)$ and $\varepsilon > 0$, there are $s_1, \ldots s_n \in G$, $\beta_1, \ldots, \beta_n \ge 0$ with $\sum_{i=1}^n \beta_i = 1$ such that

$$\left|\sum_{i=1} \beta_i s_i.\varphi_j(a) - \tau_0(a)\right| < \frac{\varepsilon}{\sum_{i=1}^4 |\alpha_i|},$$

for all $a \in F$ and $j = 1, 2, 3, 4$. Therefore,

$$\begin{aligned}\left|\sum_{j=1}^{n} \beta_j s_j.\omega(a) - \omega(1)\tau_0(a)\right| &= \left|\sum_{j=1}^{n} \beta_j s_j. \left(\sum_{i=1}^{4} \alpha_i \varphi_i(a)\right) - \sum_{i=1}^{4} \alpha_i \tau_0(a)\right| \\ &= \left|\sum_{i=1}^{4} \alpha_i \left(\sum_{j=1}^{n} \beta_j s_j.\varphi_i(a) - \tau_0(a)\right)\right| \\ &\leq \sum_{i=1}^{4} |\alpha_i| \left|\sum_{j=1}^{n} \beta_j s_j.\varphi_i(a) - \tau_0(a)\right| \\ &< \sum_{i=1}^{4} |\alpha_i| \frac{\varepsilon}{\sum_{i=1}^{4} |\alpha_i|} = \varepsilon,\end{aligned}$$

so that $\omega(1)\tau_0 \in \overline{\text{conv}}^{w^*}\{s.\omega : s \in G\}$.

(4)⇒(5): Assume that (5) does not hold. Let $t_1, \dots, t_m \in G$ be such that

$$0 \notin \overline{\text{conv}} \left\{ \lambda_s \left(\sum_{j=1}^{m} \lambda_{t_j} \right) \lambda_{s^{-1}} : s \in G \right\}.$$

Let $z \in \text{conv} \left\{ \lambda_s \left(\sum_{j=1}^{m} \lambda_{t_j} \right) \lambda_{s^{-1}} : s \in G \right\}$. Then z, z^* satisfies the conditions of Lemma 5.7, so $0 < \|z\| \leq \|z+z^*\|$. If $y \in \text{conv} \left\{ \lambda_s \left(\sum_{j=1}^{m} \lambda_{t_j} \right) \lambda_{s^{-1}} : s \in G \right\}$, then $y = z + z^*$ for some $z \in \text{conv} \left\{ \lambda_s \left(\sum_{j=1}^{m} \lambda_{t_j} \right) \lambda_{s^{-1}} : s \in G \right\}$, so $0 < \|y\|$. By continuity, Lemma 5.7 and the assumption that $0 \notin \overline{\text{conv}} \left\{ \lambda_s \left(\sum_{j=1}^{m} \lambda_{t_j} \right) \lambda_{s^{-1}} : s \in G \right\}$ we conclude that

$$0 \notin \overline{\text{conv}} \left\{ \lambda_s \left(\sum_{j=1}^{m} \lambda_{t_j} + \lambda_{t_j^{-1}} \right) \lambda_{s^{-1}} : s \in G \right\}.$$

By the Hahn-Banach separation theorem, there is a self-adjoint linear functional ω on $C_r^*(G)$ of norm 1 and a $c > 0$ such that for all $s \in G$ we have

$$\omega \left(\sum_{j=1}^{m} \lambda_{st_js^{-1}} + \left(\lambda_{st_js^{-1}}\right)^* \right) \geq c.$$

And so we see that, since ω is self-adjoint,

$$\begin{aligned}\omega\left(\sum_{j=1}^{m}\lambda_{st_js^{-1}}+{\lambda_{st_js^{-1}}}^*\right) &= \omega\left(\sum_{j=1}^{m}\lambda_{st_js^{-1}}\right)+\omega\left(\sum_{j=1}^{m}\left(\lambda_{st_js^{-1}}\right)^*\right)\\ &= \omega\left(\sum_{j=1}^{m}\lambda_{st_js^{-1}}\right)+\overline{\omega\left(\sum_{j=1}^{m}\lambda_{st_js^{-1}}\right)}\\ &= 2\mathrm{Re}\omega\left(\sum_{j=1}^{m}\lambda_{st_js^{-1}}\right).\end{aligned}$$

So we have $2\mathrm{Re}\omega\left(\sum_{j=1}^{m}\lambda_{st_js^{-1}}\right)\geq c$. Let $\rho\in\mathrm{conv}\{s.\omega : s\in G\}$ be defined by $\rho=\sum_{k=1}^{n}\alpha_i s_i.\omega$ for some $\alpha_k\geq 0$, $s_i\in G$ and $\sum_{k=1}^{n}\alpha_k=1$. We then see that

$$\begin{aligned}2\mathrm{Re}\rho\left(\sum_{j=1}^{n}\lambda_{t_j}\right) &= 2\sum_{k=1}^{n}\alpha_i s_i.\mathrm{Re}\omega\left(\sum_{j=1}^{m}\lambda_{t_j}\right)\\ &= 2\sum_{k=1}^{n}\alpha_i\mathrm{Re}\omega\left(\sum_{j=1}^{m}\lambda_{s_it_js_i^{-1}}\right)\\ &\geq 2\sum_{k=1}^{n}\alpha_i c = c.\end{aligned}$$

By continuity this also holds for all $\rho\in\overline{\mathrm{conv}}^{w^*}\{s.\omega : s\in G\}$. But, since $t_1,\dots,t_m\in G\backslash\{e\}$, we have $\tau_0\left(\sum_{j=1}^{m}\lambda_{t_j}\right)=0$. Thus τ_0 cannot be in $\overline{\mathrm{conv}}^{w^*}\{s.\omega : s\in G\}$.

(5)⇒(6): Assuming (5), given $t_1,\dots,t_m\in G\backslash\{e\}$ and $\varepsilon>0$, by Lemma B.1 there are $s_1,\dots,s_n\in G$ (where $s_j=s_i$ is allowed for $1\leq i\neq j\leq n$) such that

$$\left\|\sum_{k=1}^{n}\frac{1}{n}\lambda_{s_k}\left(\sum_{j=1}^{m}\lambda_{t_j}\right)\lambda_{s_k^{-1}}\right\|<\varepsilon.$$

By Lemma 5.7, we see that

$$\left\|\sum_{k=1}^{n}\frac{1}{n}\lambda_{s_kt_js_k^{-1}}\right\|<\varepsilon,$$

for all $j=1,2,\dots,m$.

(6)⇒(1): Identifying $\mathbb{C}G$ with its image under the left-regular representation as a dense unital $*$-subalgebra of $C_r^*(G)$, we may use Lemma 3.9 to check for Dixmier property on self-adjoint elements $a\in\mathbb{C}G$ such that $\tau_0(a)=0$. So let $a = \sum_{j=1}^n \alpha_j \lambda_{t_j} \in \mathbb{C}G$ be such an element. Assuming $\tau_0(a) = 0$, we may assume that $t_1, \dots, t_n \neq e$. To each $m \geq 1$, there is, by assumption, a function $f_m \colon C_r^*(G) \to C_r^*(G)$ of the form

$$f_m(x) = \sum_{k=1}^{M_m} \frac{1}{M_m} \lambda_{s_k} x \lambda_{s_k^{-1}}, \quad x \in C_r^*(G),$$

for some $s_1, s_2, \dots s_{M_m} \in G$, such that

$$\|f_m(\lambda_{t_j})\| < \frac{1}{m}, \quad \text{for } j = 1, 2, \dots n.$$

Letting $D(a) \subseteq C_r^*(G)$ being defined as in chapter 3, we see that $f_m(a) \in D(a)$ for each $m \geq 1$, and by the triangle inequality we have

$$\begin{aligned} \|f_m(a)\| &= \left\| \sum_{k=1}^{M_m} \frac{1}{M_m} \lambda_{s_k} \left(\sum_{j=1}^n \alpha_j \lambda_{t_j} \right) \lambda_{s_k^{-1}} \right\| \\ &\leq \sum_{j=1}^n |\alpha_j| \|f_m(\lambda_{t_j})\| \\ &< \sum_{j=1}^n |\alpha_j| \frac{1}{m}, \end{aligned}$$

So $f_m(a) \to 0$ as $m \to \infty$. So $0 \in \overline{D(a)}$. By Lemma 3.9, $C_r^*(G)$ has the Dixmier property, and since τ_0 is a faithful trace, $C_r^*(G)$ is simple. $\square$

5.14 Summary

We have throughout this thesis shown quite a few different conditions for discrete groups having the unique trace property and the C^*-simplicity property. We will in this section give a summary of the shown results.

Theorem 5.15. *Let G be a discrete group and $t \in G$. Then the following are equivalent:*

1. $t \notin \mathrm{AR}_G$;
2. *there is a boundary action $G \curvearrowright X$ such that t acts non-trivially on X;*
3. $\lambda_t \in \ker(\tau)$ *for all tracial states τ on $C_r^*(G)$;*
4. $0 \in \overline{\mathrm{conv}}\{\lambda_{sts^{-1}} : s \in G\}$.

Proof.
The implication (1)⇔(2) follows from Proposition 4.29, (1)⇔(3) follows from Theorem 5.4 and (1)⇔(4) follows from Theorem 5.9. $\square$

Theorem 5.16. *Let G be a discrete group. Then the following are equivalent:*

1. *$C_r^*(G)$ has unique tracial state;*
2. *G admits a faithful boundary action;*
3. *$\mathrm{AR}_G = \{e\}$;*
4. *for all $t \in G \backslash \{e\}$ and all $\varepsilon > 0$, there is $s_1, s_2, \ldots, s_n \in G$ such that*

$$\left\| \sum_{j=1}^{n} \frac{1}{n} \lambda_{s_k t s_k^{-1}} \right\| < \varepsilon.$$

Proof.
We first show (2)⇔(3): Assume that AR_G is trivial. By Proposition 4.29, given $t \in G\backslash\{e\}$ there is is G-boundary X such that $t.x \neq x$ for some $x \in X$. By the universitality of the Furstenberg boundary $\partial_F G$, there is $x' \in \partial_F G$ so that if $\pi_X : \partial_F G \to X$ is the G-equivariant projection onto X, then $\pi_X(x') = x$. By the G-equivariance of π_X, we see that

$$\pi_X(t.x') = t.\pi_X(x') = t.x \neq x = \pi_X(x'),$$

so $t.x' \neq x'$. Since there is such x' for each $t \in G\backslash\{e\}$, we see that the action $G \curvearrowright \partial_F G$ is a faithful boundary action.

(1)⇔(3) follows from Corollary 5.5, while (3)⇔(4) follows from Corollary 5.10. □

Theorem 5.17. *Let G be a discrete group, and let τ_0 be the canonical tracial state on $C_r^*(G)$. Then the following are equivalent:*

1. *$C_r^*(G)$ is simple;*
2. *G admits a free boundary action;*
3. *$\tau_0 \in \overline{\{s.\varphi : g \in G\}}^{w^*}$, for all states φ on $C_r^*(G)$;*
4. *$\tau_0 \in \overline{\mathrm{conv}}^{w^*}\{s.\varphi : s \in G\}$, for all states φ on $C_r^*(G)$;*
5. *for all $t_1, t_2, \ldots, t_m \in G\backslash\{e\}$ and all $\varepsilon > 0$, there exist $s_1, s_2, \ldots, s_n \in G$ such that*

$$\left\| \frac{1}{n} \sum_{j=1}^{n} \lambda_{s_j t_k s_j^{-1}} \right\| < \varepsilon,$$

for all $k = 1, 2, \ldots, m$;

6. *$C_r^*(G)$ has the Dixmier property.*

Proof.
(1)$\Leftrightarrow$(2) follows from Theorem 5.12 and the implications (1)$\Leftrightarrow$(3)$\Leftrightarrow$(4)$\Leftrightarrow$(5) follows from Theorem 5.13. $\square$

Corollary 5.18. *If G is a C^*-simple discrete group, then G has the unique trace property.*

Proof.
Assuming G is C^*-simple, then G has property (5) of Theorem 5.17, which implies property (4) of Theorem 5.16, so $C_r^*(G)$ has unique tracial state τ_0. In $\square$

It is worth noting that the converse does not hold, i.e., the unique trace property does not imply C^*-simplicty of discrete groups, as seen in [Theorem A LB15, p. 2].

A. Measure theory

Throughout this appendix we denote by X a locally compact Hausdorff space.

Definition A.1. *For a Borel measure μ on X and a Borel subset $E \subset X$, we say that μ is* outer regular *on E if*

$$\mu(E) = \inf\{\mu(U) \colon E \subseteq U,\ U \text{ open}\}$$

and inner regular *on E if*

$$\mu(E) = \sup\{\mu(K) \colon K \subseteq E, K \text{ compact}\}.$$

If μ is both inner and outer regular on all Borel subsets of X, we say that μ is a regular measure*.*

Definition A.2. *A measure μ on X is a* Radon measure *if it is finite on compact sets, outer regular on Borel sets and inner regular on open sets.*

By the Riesz Representation theorem [Theorem 7.2 Fol13, p. 212], which ensures that to every positive linear functional φ on $C(X)$, there is a unique Radon measure μ on X such that $\varphi(f) = \int_X f \,\mathrm{d}\mu$, we have a bijective affine correspondance between the set of Radon probability mesures on X and the state space of $C(X)$. We endow $\mathrm{Prob}(X)$ with the weak* topology inherited from the state space of $C(X)$, so that a net $(\mu_\alpha)_{\alpha \in A} \subseteq \mathrm{Prob}(X)$ converges to μ if and only if $\int_X f \,\mathrm{d}\mu_\alpha \to \int_X f \,\mathrm{d}\mu$ for every $f \in C(X)$.

If X, Y are locally compact Hausdorff spaces, we define for $f \in C_c(X)$ and $g \in C_c(Y)$ the function $f \otimes g \in C_c(X \times Y)$ by $(x, y) \mapsto f(x)g(y)$. Similarly for Radon measures μ and ν og X and Y, we define the Radon product $\mu \otimes \nu$ by $\mu \otimes \nu(E \times B) = \mu(E)\nu(B)$ for measurable $E \subseteq X$ and $B \subseteq Y$.

The space $C_c(X \times Y)$ is the closure of the linear span of functions $f \otimes g$ for $f \in C_c(X)$ and $C_c(Y)$ cf. [Proposition 7.21 Fol13, p. 226].

Lemma A.3. *Let X, Y be compact Hausdorff spaces and $\mu \in \mathrm{Prob}(X \times Y)$ and π_1, π_2 be the projection on to the first and respectively second coordinate. If $\pi_1(\mu) = \delta_x$ for some $x \in X$, then $\mu = \delta_x \otimes \mu_2$. for some $\mu_2 \in \mathrm{Prob}(Y)$.*

Proof.
Let $\mu_2 = \pi_Y(\mu)$. For each Borel subset $E \subseteq X \times Y$, if $E \cap \pi_X^{-1}(\{x\}) = \emptyset$ then

$$\begin{aligned} 0 \leq \mu(E) &\leq \mu(E) + \mu(\pi_X^{-1}(\{x\})) \\ &= \mu(E) + \delta_x(x) \\ &= \mu(E) + 1. \end{aligned}$$

But by assumption $E \cap \pi_X^{-1}(\{x\})$ is empty, so

$$\begin{aligned} 0 \leq \mu(E) \leq \mu(E) + \mu(\pi_X^{-1}(\{x\})) = \mu(E \cup \pi_X^{-1}(\{x\})) \leq \mu(X \times Y) = 1 \\ \implies \mu(E) = 0 \end{aligned}$$

If $E \subseteq X$ and $B \subseteq Y$ are Borel sets such that $x \in E$. Then

$$\begin{aligned} \mu(E \times B) = \mu(E \times B) + \underbrace{\mu(E^c \times B)}_{=0} = \mu(X \times B) = \mu(\pi_2^{-1}(B)) = \mu_2(B) \\ = \mu_1(E)\mu_2(B). \end{aligned}$$

The paving $\{A \times B \colon A \subseteq X \text{ Borel} : B \subseteq Y \text{ Borel }\}$ is stable under intersections and generates the Borel sigma-algebra on $X \times Y$ cf. [Theorem 4.22 Han06, p. 85]. And since μ and $\delta_x \otimes \mu_2$ agrees on this paving, [Theorem 3.8 Han06, p. 63] ensures that $\mu = \delta_x \otimes \mu_2$, since Radon measures by definition are finite on compact sets. □

For a family of compact Hausdorff spaces $\{X_i\}_{i \in I}$, the product space X is a compact Hausdorff space, by Tychonoff's theorem. Then the unital $*$-subalgebra

$$B = \{f \in C(X) \mid \exists F \subseteq I \text{ finite, such that } f = \prod_{i \in F} f_i \circ \pi_i\}$$

of $C(X)$ is dense by the Stone-Weistrass theorem.

Lemma A.4. *Let $\{X_i\}_{i \in I}$ is a family of compact Hausdorff spaces and $X = \prod_{i \in I} X_i$ and let $\nu \in \mathrm{Prob}(X)$. If $\nu_F = \pi_F(\nu)$ is a point-mass every finite $F \subseteq I$ with $\pi_F \colon X \to \prod_{i \in F} X_i$ being the projection, then ν is a point mass.*

Proof.
Let $\mathcal{F} = \{F \subseteq I \colon F \text{ finite}\}$. For every $F \in I$ let $x_F \in X$ be such that $\nu_F = \delta_{\pi_F(x_F)}$. Since X is compact, there is a subnet $(x_\alpha)_{\alpha \in A} \subseteq X$ such that $x_\alpha \to x$ for some $x \in X$. Let $h \colon A \to \mathcal{F}$ be the monotone and final function defining this subnet. Let $f \in B$, where B is the $*$-subalgebra of $C(X)$ as above. Then

$$f = \prod_{j \in F} f_j \circ \pi_j$$

for some $F \in \mathcal{F}$. Since h is a final function, there is $\alpha_0 \in A$ such that $F \subseteq h(\alpha_0)$. For any $\alpha \geq \alpha_0$, we have $F \in h(\alpha)$. Define for every $j \in h(\alpha)$ function $g_j \in C(X_j)$ by

$$g_j = \begin{cases} f_j & j \in F \\ 1 & j \notin F \end{cases}$$

Define the function $f_\alpha \colon \prod_{j \in h(\alpha)} X_j \to \mathbb{C}$ by $f_\alpha = \prod_{j \in h(\alpha)} g_j \circ \pi_{\alpha,j}$, where $\pi_{\alpha,j}$ for $j \in h(\alpha)$ is the projection from $\prod_{i \in h(\alpha)} \to X_j$. Then for $x \in \prod_{j \in h(\alpha)} X_j$ we have

$$f_\alpha(x) = \prod_{j \in h(\alpha)} (g_j \circ \pi_{\alpha,j})(x) = \prod_{i \in F} f_i(x_i),$$

But then $f_\alpha \circ \pi_{h(\alpha)} = f$. And we see that

$$\begin{aligned} \int f \, \mathrm{d}\nu = \int f_\alpha \circ \pi_{h(\alpha)} \, \mathrm{d}\nu &= \int f_\alpha \, \mathrm{d}\pi_{h(\alpha)}(\nu) \\ &= \int f_\alpha \, \mathrm{d}\delta_{\pi_{h(\alpha)}(x_\alpha)} \\ &= f_\alpha(\pi_{h(\alpha)}(x_\alpha)) \\ &= f(x_\alpha). \end{aligned}$$

And since this holds for every $\alpha \geq \alpha_0$, we see that

$$\int f \, \mathrm{d}\nu = f(x).$$

And since B was dense in $C(X)$, it holds that $\int g \, \mathrm{d}\nu = g(x)$ for every $g \in C(X)$. So $\nu = \delta_x$. $\square$

B. A result on convex hulls

Lemma B.1. *Let X be a topological vector space over $\mathbb{C}$, $C \subseteq X$ a convex subset and $M \subseteq C$ any subset. Define subsets $A(M), \mathrm{conv}_{\mathbb{Q}}(M) \subseteq C$ by*

$$A(M) := \left\{ \frac{1}{n} \sum_{j=1}^{n} c_j \colon n \in \mathbb{N},\ c_j \in M \right\}$$

$$\mathrm{conv}_{\mathbb{Q}}(M) := \left\{ \sum_{j=1}^{n} \alpha_j c_j \colon n \in \mathbb{N},\ \forall 1 \leq j \leq n\ \alpha_j \in [0,1] \cap \mathbb{Q},\ \sum_{j=1}^{n} \alpha_j = 1,\ c_j \in M \right\}.$$

Then $A(M) = \mathrm{conv}_{\mathbb{Q}}(M)$.

Proof.
Let $\sum_{j=1}^{n} \alpha_i c_i \in \mathrm{conv}_{\mathbb{Q}}(M)$. Then every $\alpha_i = \frac{p_i}{q_i}$ for some positive integers p_i, q_i, with $q_i \neq 0$. Define for $1 \leq i \leq n$

$$b_i := p_i \prod_{j=1, j \neq i}^{n} q_j, N := \prod_{i=1}^{n} q_i.$$

Then we see that

$$\sum_{j=1}^{n} \alpha_i c_i = \frac{1}{N} \sum_{i=1}^{n} b_i c_i = \frac{1}{N} \sum_{i=1}^{n} \sum_{j=1}^{b_i} c_i.$$

Define $\tilde{c}_1, \ldots, \tilde{c}_N \in M$ by

$$\tilde{c}_j := c_i, \quad j = b_{i-1}, (b_{i-1}+1), \ldots, (b_{i-1}+b_i)$$

where $1 \leq i \leq n$ and $b_0 = 1$. Then

$$\frac{1}{N} \sum_{i=1}^{n} \sum_{j=1}^{b_i} c_i = \frac{1}{N} \sum_{i=1}^{N} \tilde{c}_i.$$

$\square$

Since the set $\mathrm{conv}_Q(M)$ is dense in $\mathrm{conv}(M)$, every point in $\mathrm{conv}(M)$ can be approximated by averages from M, by the above result.

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