Simplicty and uniqueness of trace of group C^*-algebras

Bachelor thesis defense

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Today we see for discrete countable group G :

$$C_r^*(G) \text{ simple } \implies C_r^*(G) \text{ unique trace } \iff \text{ trivial amenable radical of } G.$$

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A three minute crash-course on group C^*-algebras

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Unitary representations

A *unitary representation* of a discrete group G is a pair $(u, \mathcal{H})$, where $\mathcal{H}$ is a Hilbert space and $u\colon G \to \mathcal{U}(\mathcal{H})$ is a group homomorphism into the group of unitary operators on $\mathcal{H}$. We use the following notation for unitary representations: $u(g) := u_g,\ g \in G$.

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Group C^*-algebra

Given a unitary representation $(u, \mathcal{H})$ of a discrete group G, we define the *group C^*-algebra* associated to u by $C^*_u(G) := C^*(\{u_g : g \in G\}) \subseteq B(\mathcal{H})$, the C^*-subalgebra of $B(\mathcal{H})$ generated by the unitaries $\{u_g\}_{g \in G}$.

An important example

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The pair $(\lambda, \ell^2(G))$, where $\lambda\colon G \to \mathcal{U}(\ell^2(G)$ is defined by

$$\left(\lambda_g \xi\right)(s) = \xi(g^{-1}s), \quad s, g \in G,\ \xi \in \ell^2(G),$$

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is called the *left-regular representation* of G, and the associated group C^*-algebra,

$$C_r^*(G) := C_\lambda^*(G) \subseteq B(\ell^2(G)),$$

called the *reduced group C^*-algebra* of G.

A three minute crash-course on group C^*-algebras

The set $\{\delta_g\}_{g \in G} \subseteq \ell^2(G)$, where for $s, g \in G$ we define

$$\delta_s(g) = \begin{cases} 1 & \text{if } s = g \\ 0 & \text{else} \end{cases},$$

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Lemma

For discrete group G, the map $\tau_0 \colon C_r^*(G) \to \mathbb{C}$ defined by $\tau_0(x) = \langle x\delta_e, \delta_e\rangle$ is a faithful tracial state on $C_r^*(G)$.

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Definition

For a discrete group G, we say that G is ***C^*-simple*** if the reduced group C^*-algebra, $C_r^*(G)$, is simple. We say that G has the ***unique trace property*** if τ_0, defined as above, is the only tracial state on $C_r^*(G)$.

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Amenability is a strong property to satisfy, and has many consequences.

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Group actions and the Reduced crossed product

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Definition

An *action* of G on $\mathcal{A}$ is a group homomorphism $\alpha\colon G \to \mathrm{Aut}(\mathcal{A})$ from G to the group of $*$-automorphism on $\mathcal{A}$. Abbreviated $G \overset{\alpha}{\curvearrowright} \mathcal{A}$.

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Whenever $G \overset{\alpha}{\curvearrowright} \mathcal{A} \rightsquigarrow$ we make a C^*-algebra $\mathcal{A} \rtimes_{\alpha,r} G$ which contains $\mathcal{A}$ and $C_r^*(G)$ in a nice way. This C^*-algebra is called the reduced crossed product.

Dixmier property

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Proposition

Let $\mathcal{A}$ be a unital C^*-algebra with a faithful tracial state τ. If $\mathcal{A}$ has the Dixmier property, then $\mathcal{A}$ is simple.

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Boundary actions: G-spaces

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An application of Zorn's lemma yields that every compact G-space X has a minimal subspace. Moreover, it is easy to see that a subset Y is minimal if and only if the action $G \curvearrowright Y$ is minimal

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For a G-space X, the action of G is *strongly proximal* if for each $\mu \in \mathrm{Prob}(X)$ we have

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Definition

If X is a compact G-space, we say that the action $G \curvearrowright X$ is a *boundary action* if it is minimal and strongly proximal. A compact G-space X for which the action is a boundary action is called a *G-boundary*

Boundary actions

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Proposition

For a compact G-space X, TFAE:

1. X is a G-boundary,
2. For every $\nu \in \mathrm{Prob}(X)$ we have $\{\delta_x : x \in X\} \subseteq \overline{G.\nu}^{w^*}$ and
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In laymans terms, a G-boundary is a space such that every G-orbit in $\mathrm{Prob}(X)$ becomes a black hole for all the point masses when going to weak* closure.

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- The G-boundary property is preserved under quotients,
- Strongly proximality is preserved under taking products,
- If $G \curvearrowright Y$ is minimal and X is a G-boundary, then any continuous G-map $\varphi\colon Y \to X$ will be unique and surjective.

Large G-boundaries

We now proceed to construct a largest G-boundary, in the sense that it has the universal property that every other G-boundary is a quotient of it.

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1. There is a bound on the cardinality of every compact G-boundary.
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3. Use Zorn's lemma to pick a minimal subset.

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Proposition (Furman)

For any countable discrete group G with $t \in G$, it holds that $t \in \mathrm{AR}_G$ if and only if $t \in \ker(G \curvearrowright X)$ for all compact G-boundaries X.

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5. $G \curvearrowright X \rightsquigarrow G \curvearrowright \mathcal{S}(C(X) \rtimes_r G)$: for $t \in G$ and any state φ define $t.\varphi$ by

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C^*-simplicity using boundary actions

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The proof of this relies on the fact that given a normal amenable subgroup $N \trianglelefteq G$, there is a $*$-homomorphism $\pi\colon C_r^*(G) \to C_r^*(G/N)$ such that

$$\pi(\lambda_G(t)) = \lambda_{G/N}([t]), \quad \text{for all } t \in G.$$

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Using this, one creates a trace τ on $C_r^*(G)$ which is 0 on λ_t for all $t \notin \mathrm{AR}_G$.

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C^*-simplicity and uniqueness of trace using boundary actions

Proposition [Matthew Kennedy, Emmanuel Breuillard, Merhdad Kalatar and Narutaka Ozawa, 2014]

A discrete group G is C^*-simple if and only if there is a compact G-boundary X such that the action is free.

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What is this? Why is this interesting?

Using this, we may prove the following:

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For a group G TFAE:

1. $C_r^*(G)$ is simple,
2. $\tau_0 \in \overline{\{s.\varphi\colon s \in G\}}^{w^*}$ for all states φ on $C_r^*(G)$,
3. $\tau_0 \in \overline{\text{conv}}^{w^*} \{s.\varphi\colon s \in G\}$ for all states φ on $C_r^*(G)$,
4. $\omega(1)\tau_0 \in \overline{\text{conv}} \{s.\omega\colon s \in G\}$ for all $\omega \in (C_r^*(G))^*$,
5. For all $t_1, t_2, \dots, t_m \in G \setminus \{e\}$, we have
$$0 \in \overline{\text{conv}}^{w^*} \{\lambda_s(\lambda_{t_1} + \lambda_{t_2} + \cdots + \lambda_{t_m})\lambda_s^*\colon s \in G\},$$
6. For all $t_1, t_2, \dots, t_m \in G \setminus \{e\}$ and $\varepsilon > 0$, there exists $s_1, s_2, \dots, s_n \in G$ such that
$$\left\| \sum_{k=1}^{n} \frac{1}{n} \lambda_{s_k t_j s_k^{-1}} \right\| < \varepsilon,$$
for all $1 \leq j \leq m$.

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Theorem

Let G be a discrete group and $t \in G$. TFAE:

1. $t \notin \mathrm{AR}_G$;
2. there is a boundary action $G \curvearrowright X$ such that t acts non-trivially on X;
3. $\lambda_t \in \ker(\tau)$ for all tracial states τ on $C_r^*(G)$;
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Theorem

Let G be a discrete group. TFAE:

1. $C_r^*(G)$ has unique tracial state;
2. G admits a faithful boundary action;
3. $\mathrm{AR}_G = \{e\}$;
4. for all $t \in G \backslash \{e\}$ and all $\varepsilon > 0$, there is $s_1, s_2, \ldots, s_n \in G$ such that

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Theorem

Let G be a discrete group. TFAE:

1. $C_r^*(G)$ is simple;
2. G admits a free boundary action;
3. $\tau_0 \in \overline{\{s.\varphi \colon g \in G\}}^{w^*}$, for all states φ on $C_r^*(G)$;
4. $\tau_0 \in \overline{\mathrm{conv}}^{w^*} \{s.\varphi \colon s \in G\}$, for all states φ on $C_r^*(G)$;
5. for all $t_1, t_2, \ldots, t_m \in G \setminus \{e\}$ and all $\varepsilon > 0$, there exist $s_1, s_2, \ldots, s_n \in G$ such that
$$\left\| \frac{1}{n} \sum_{j=1}^{n} \lambda_{s_j t_k s_j^{-1}} \right\| < \varepsilon,$$
for all $k = 1, 2, \ldots, m$;
6. $C_r^*(G)$ has the Dixmier property.

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$\mathrm{AR}_G = \{e_G\} \iff G$ has unique trace property $\iff$ for all $t \in G \setminus \{e\}$ and all $\varepsilon > 0$, there is $s_1, s_2, \ldots, s_n \in G$ such that

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G is C^*-simple $\iff$ for all $t_1, t_2, \ldots, t_m \in G \setminus \{e\}$ and all $\varepsilon > 0$, there exist $s_1, s_2, \ldots, s_n \in G$ such that

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Thus

$$C_r^*(G) \text{ simple} \implies C_r^*(G) \text{ unique trace} \iff \text{trivial amenable radical of } G.$$

The end.

thanks for listening