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Reflexivity in Banach spaces

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CONTENTS

INTRODUCTION

The purpose of this project is to illuminate a classic result due to Robert C. James, assuming only a basic understanding of functional analysis. The result is that a Banach space X is reflexive if and only if every bounded linear functional on X attains its supremum on the unit sphere of X. We follow a proof due to Pryce – one that is essentially the same as the James original proof – but it is generalised to the case of locally convex topological vector spaces, rather than Banach spaces.

In the first chapter we state basic definitions from topology and functional analysis, so that the reader, regardless of familiarity with functional analysis or not, may be able to follow the constructions of the project. In the second chapter we recall various results of functional analysis; some proofs are left out and referred to instead. In the third chapter, a classic result of Eberlein and Smulian is proven. This result allows us to view compactness in the weak topology as the same as sequential and limit point compactness in the weak topology. In chapter 4 we finally prove the James theorem, allowing us to classify weakly compact sets in Banach spaces, yielding a strong result for classifying reflexive Banach spaces. In chapter 5, we show some properties of uniformly convex Banach spaces, one of which is that every uniformly convex Banach space is reflexive, where we use the James theorem of chapter 4.

CHAPTER 1

BASIC DEFINITIONS

Throughout the following let X be a vector space over $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$. We will be working over $\mathbb{C}$, unless otherwise stated.

Definition 1.1
*A **Banach space** is a normed space X which is complete with respect to the metric induced by the norm.*

Definition 1.2
*A **linear functional** f on X is a linear map $f : X \to \mathbb{K}$.*

Definition 1.3
A directed set $(A, \leq)$ *is a set A equipped with a reflexive and transitive binary relation* $' \leq '$ *such that every pair of elements has an upper bound, i.e., that*

1. *$a \leq a$, for all $a \in A$.*
2. *$a \leq b$ and $b \leq c$ implies that $a \leq c$ for all $a, b, c \in A$.*
3. *If $a, b \in A$ then there is some $y \in A$ such that $a \leq y$ and $b \leq y$.*

Note: It may be the case that neither $a \leq b$ nor $b \leq a$ for $a, b \in A$, i.e., the ordering need not be total.

Definition 1.4
A net $(x_\alpha)_{\alpha \in A}$ *in X is a map $\alpha \mapsto x_\alpha$ from a directed set $(A, \leq)$ into X.*

If we let X be a topological space, and $E \subset X$, we use the following terminology for a net $(x_\alpha)_{\alpha \in A}$ having certain properties

1. **Eventually in:** A net $(x_\alpha)_{\alpha \in A}$ is said to be **eventually in** E if there exists a $\alpha_0 \in A$ such that $x_\alpha \in E$ for all $\alpha \geq \alpha_0$.
2. **Frequently in:** A net $(x_\alpha)_{\alpha \in A}$ is said to be **frequently in** E if for every $\alpha \in A$, there exists $\beta \geq \alpha$ such that $x_\beta \in E$.
3. **Converges to:** A net $(x_\alpha)_{\alpha \in A}$ is said to be **converging** to a point $x \in E$ if for all open neighborhoods $U_x \in E$ around x, $(x_\alpha)_{\alpha \in A}$ is **eventually** in U_x.
4. **Cluster point:** We say that x is a **cluster point** of $(x_\alpha)_{\alpha \in A}$ if for every neighborhood U_x of x, $(x_\alpha)_{\alpha \in A}$ is **frequently in** U.

Definition 1.5
*For a linear topological space X, we denote the space of all continuous linear functionals on X by X^* (the **dual space** of X). Whenever X is a normed space, then X^* is a normed linear space with respect to the operator norm defined by*

$$\|f\| = \sup\{\|f(x)\| \;\big|\; \|x\| = 1\}, \quad f \in X^*.$$

Remark 1.6
The dual space of a vector space X is a Banach space if X is a normed space. This is due to a result which states that the space of continuous linear maps $f\colon X \to Y$ is complete whenever Y is a complete normed space.

In this paper, we will be working with a very special and important topology on linear spaces, the **weak** topology, defined in the following way:

Definition 1.7
*The **weak-topology** is the Hausdorff topology on X induced by X^*, and it has a neighborhood basis at 0 given by*

$$V(0, f_1, \ldots, f_n, \varepsilon) = \{x \in X \big| \; |f_i(x)| < \varepsilon, 1 \le i \le n\}$$

where $n \in \mathbb{N}$ and $f_1, \ldots f_n \in X^*$ and $\varepsilon > 0$. This topology is the coarsest topology on X for which all the semi-norms $p_f\colon X \to \mathbb{R}$, defined by $p_f(x) = |f(x)|$, are continuous for $f \in X^*$. An equivalent notion to a net $(x_\alpha)_{\alpha\in A} \subset X$ converging weakly $x \in X$ if and only if $x_\alpha \to x$ with respect to the basis is the following: A net $(x_\alpha)_{\alpha\in A}$ is converging weakly to X if and only if $f(x_\alpha) \xrightarrow{|\cdot|} f(x)$ in $\mathbb{K}$ for all $f \in X^*$

Since X^* is also a normed vector space, it has a dual space, denoted by $(X^*)^*$ or simply X^{**}. This space is called the *bi-dual* of X. This space induces the weak-topology on X^*.

For each $x \in X$ we may define a semi-norm $p_x : X^* \to \mathbb{R}$ by $p_x(f) = |f(x)|$. This gives rise to the weak*-topology on X^*.

Definition 1.8
*The **weak*-topology** on X^* is the Hausdorff topology on X^* induced by the family of semi-norms $(p_x)_{x\in X}$ and it has a neighborhood base at 0, given by*

$$V^*(0, x_1, \ldots, x_n, \varepsilon) = \{f \in X^* \;\big|\; |f(x_i)| < \varepsilon, 1 \le i \le n\}$$

where $n \in \mathbb{N}$ and $x_1, \ldots x_n \in X$ and $\varepsilon > 0$. This topology is the coarsest topology on X for which all the semi-norms p_x are continuous.

Equivalently, a net $(f_\alpha)_{\alpha\in A} \subset X^*$ is said to be weak convergent to $f \in X^*$ if it converges pointwise for all $x \in X$, i.e., that we have $f_\alpha(x) \to f(x)$ for all $x \in X$.*

Now, a Banach space X can be embedded into it's bi-dual by a linear isometric map, $\Lambda : X \to X^{**}$ defined by $\Lambda(x)(f) = \hat{x}(f) = f(x)$ for $f \in X^*$. We denote the embedding of X in X^{**} by $\hat{X}$. If this map is an isomorphism, we say that $X = X^{**}$, and we say that X is a *reflexive* Banach space.

Definition 1.9
Let X, Y be topological vector spaces, and let Γ be a family of functions $\gamma : X \to Y$. We say that Γ is equicontinuous *at $x \in X$ if for every neighborhood W of $0 \in Y$ there is a neighborhood V of x such that for all $y \in V$ and all $\gamma \in \Gamma$ we have that*

$$\gamma(y) - \gamma(x) \in W.$$

Γ is said to be equicontinuous if it is equicontinuous at all $x \in X$.

Remark 1.10
If it happens that X is a Banach space and $Y = \mathbb{K}$ then a family of linear functionals $\Gamma \subset X^*$ is equicontinuous if for all $\varepsilon > 0$ there is some $\delta > 0$ such that for all $x, y \in X$ and all $f \in \Gamma$, if $\|x - y\| < \delta$ implies $|f(x) - f(y)| < \varepsilon$.

Definition 1.11
A topological space X is called separable*, if there is some countable dense subset $X_0 \subset X$.*

In a general topological space X, there are different notions of compactness; we will be dealing with three different forms; limit point compactness, sequential compactness and regular compactness. For general topological spaces, these forms of compactness are not equivalent, however in some cases they are - a prime example is in metric spaces. we recall the definitions of limit point compactness and sequential compactness:

Definition 1.12 (Limit point compactness)
Let X be a topological space. A subset $A \subset X$ is said to be ***limit point compact*** *if every countably infinite subset of A has a limit point in X, that is if every sequence $(x_n)_{n\geq 1}$ in A has a limit point x_0 in A. This means that there is a point $x_0 \in A$ such that for all neighborhoods V_{x_0} of x_0 there is some $j \in \mathbb{N}$ such that $x_j \in V_{x_0}$ and $x_j \neq x_0$.*

Definition 1.13 (Sequential compactness)
Let X be a topological space. A subset $A \subset X$ is said to be ***sequentially compact*** *if every sequence $(x_n)_{n\geq 1} \subset A$ has a convergent subsequence.*

CHAPTER 2

VARIOUS RESULTS FROM FUNCTIONAL ANALYSIS

Lemma 2.1
Let X be a normed linear space over $\mathbb{K}$. Then the closed span of a countable set in X is separable.

Proof.
Let $(x_n)_{n\geq 1} \subset X$ be any countable set. Now pick some countable dense subset $\mathbb{W} \subset \mathbb{K}$, and let $X_0 := \operatorname{span}(x_n)$. Concretely we can choose $\mathbb{W} = \mathbb{Q} \subset \mathbb{R}$ if X is real and $\mathbb{W} = \mathbb{Q} + i\mathbb{Q} \subset \mathbb{C}$ if it is complex. Now, the set

$$Y_0 := \left\{ \sum_{j=1}^{\infty} q_j x_j : q_j \in \mathbb{W}, x_j \in (x_n)_{n\geq 1} \right\},$$

is countable, as the elements can be identified with sequences in $\mathbb{W}$. For some $x \in X_0$ we can write $x = \sum_{j=1}^{\infty} a_j x_j$ with $a_j \in \mathbb{K}$, and hence we can pick q_j for each j such that $\sum_{j=1}^{\infty} (a_j - q_j)\|x_j\| < \varepsilon$ as $\mathbb{W}$ was dense. Since this is possible Y_0 is dense in X_0, and X_0 is dense in $\overline{X_0}$ and hence Y_0 is dense in $\overline{X_0}$ and so $\overline{X_0}$ is separable.

□

Lemma 2.2
Let X be a compact metric space. Then X is separable.

Proof.
For each $n \in \mathbb{N}$ consider the set of all open balls of radius $\frac{1}{n}$, i.e., of the form $\{B(x, \frac{1}{n}) | x \in X\}$. This is an open covering of the space, and hence we can choose a finite subcover, C_n, for each n. To each such finite subcover we associate the set of centers of these finitely many balls, L_n, and the union of these, $L := \bigcup_{i=1}^{\infty} L_n$. The union of all such sets is a countable union of finite sets, and hence countable itself, and we claim that it is furthermore dense in X. Each point x of the space is contained in a ball in each C_n, i.e., for each $x \in X$ and each $n \in \mathbb{N}$ there exists an $B(x_0, \frac{1}{n})$ such that $x \in B(x_0, \frac{1}{n})$. We now have that for each ε there exists an $x_0 \in L$ at most ε away from x, and so L is dense in X. □

We now recall some classic results, which are usually covered in any introductory course in functional analysis. We will omit some proofs, as most are readily available in the literature.

Theorem 2.3
If X is a topological vector space over $\mathbb{K}$ then

1. *every neighborhood of* 0 *contains a balanced neighborhood of* 0

2. *every convex neighborhood of* 0 *contains a balanced convex neighborhood of* 0

For a proof of this see 1.14 Rudin 1991, p. 12.

Theorem 2.4 (Alaoglu's theorem)
Let X be a normed vector space over $\mathbb{K}$. Then the closed unit ball of X^* is compact in the w^*-topology.

Proof found in Folland 2013, p. 169.

Theorem 2.5 (The Uniformed Boundedness Principle.)
Let X and Y be normed vector spaces over $\mathbb{K}$ and let A be a set of continuous linear maps $T\colon X \to Y$.

1. *If $\sup_{T\in A}\|T(x)\| < \infty$ for all x in some nonmeager subset of X, then*
$$\sup_{T\in A}\|T\| < \infty$$
.

2. *If X is a Banach space and $\sup_{T\in A}\|T(x)\| < \infty$ for all $x \in X$ then*
$$\sup_{T\in A}\|T\| < \infty$$
.

For a proof of this see Folland 2013, p. 163.

Theorem 2.6 (The Open Mapping Theorem)
If X and Y are Banach spaces over $\mathbb{K}$ and $T\colon X \to Y$ is continuous linear and surjective, then T is open (i.e., T maps open sets to open sets).

For a proof of this see 5.10 Folland 2013, p. 162.

Corollary 2.7
If X and Y are Banach spaces over $\mathbb{K}$ and $T\colon X \to Y$ is continuous, linear and bijective, then T is an isomorphism, that is, T^{-1} is a continuous linear map $T^{-1}\colon Y \to X$.

For proof see 5.11 Folland 2013, p. 162.

Theorem 2.8
If X is a Banach space over $\mathbb{K}$, then the following are equivalent:

1. *X is reflexive.*
2. *X^* is reflexive.*
3. *The weak- and weak*-topologies coincide on X^*.*
4. *The unit ball S_X of X is weakly compact.*

For proof see Theorem 4.2 Conway 2013, p. 136.

Definition 2.9
If X is a Banach space over $\mathbb{K}$, then a subset $A \subset X$ is said to be weakly bounded if $f(A)$ is bounded in $\mathbb{K}$ for all $f \in X^*$.

Theorem 2.10
Let X be a Banach space, a subset $A \subset X$ is weakly bounded if and only if it is norm bounded.

For proof see Theorem 1.10 Conway 2013, p. 130.

Lemma 2.11
If X is a Banach space over $\mathbb{K}$ and X^* its dual space, and $(x_n)_{n\geq 1}$ a sequence in X. Assume that x_0 is a weak limit point of $(x_n)_{n\geq 1}$ and that $\lim_{n\to\infty} g(x_n)$ exists for some $g \in X^*$. Let $y_0 := \lim_{n\to\infty} g(x_n)$. Then $y_0 = g(x_0)$.

Proof.
By the definition of a limit point we can assume that x_0 does not occur in the sequence. Furthermore we can assume that no element occurs twice. This assumption is due to the definition of limit points in combination with the topology being Hausdorff. So we assume without loss of generality that $x_n \neq x_m$ whenever $n \neq m$, and that $x_n \neq x_0$ for all $n \geq 1$. Let $\varepsilon > 0$ be given and choose $N_\varepsilon \in \mathbb{N}$ such that for all $n \geq N_\varepsilon$ we have that $x_n \in g^{-1}(\{y \in \mathbb{C} \big| \ |y - y_0| < \frac{\varepsilon}{3}\})$. For each $i \in \mathbb{N}$, let V_i be a weak neighborhood of x_0 with $x_i \notin V_i$. Then $V := V\left(x_0, g, \frac{\varepsilon}{3}\right) \cap \left(\bigcap_{i=1}^{N_\varepsilon} V_i\right)$ is a neighborhood of x_0. Since x_0 is a weak limit point of $(x_n)_{n\geq 1}$, the intersection $(x_n)_{n\geq 1} \cap V$ is non-empty, but by construction, for $k \leq N_\varepsilon$ we have

$$x_k \notin V\left(x_0, g, \frac{\varepsilon}{3}\right) \cap \left(\bigcap_{i=1}^{N_\varepsilon} V_i\right)$$

Thus there is some $r > N_\varepsilon$ such that x_r is an element of it. Now, if we look at

$$V\left(x_0, g, \frac{\varepsilon}{3}\right) \cap \left(\bigcap_{i=1}^{N_\varepsilon} V_i\right) \cap g^{-1}\left(\left\{y \in \mathbb{C} \big| \ |y - y_0| < \frac{\varepsilon}{3}\right\}\right),$$

since $r \geq N_\varepsilon$ x_r is an element of each of the sets we are intersecting, it is in the intersection. Thus we get for all $n \geq N_\varepsilon$ that

$$|g(x_0 - x_n)| \leq |g(x_0 - x_r)| + |g(x_r) - y_0| + |y_0 - g(y_n)| < \varepsilon,$$

as wanted. □

Theorem 2.12
If X is a Banach space over $\mathbb{K}$, then the weak**-topology of the closed unit sphere of X^* is a metric topology if and only if X is separable.*

For proof see Theorem 5.1 Conway 2013, p. 138.

Corollary 2.13
If X is a Banach space, a subset $A \subseteq X^*$ is weak**-compact if and only if it is closed in the weak***-topology and bounded in the metric topology.*

See Corollary 3 Dunford et al. 1971, p. 424.

Theorem 2.14
Let X be any complex vector space. If f is a complex linear functional on X, and $u := Re(f)$, then u is a real linear functional on X, and $f(x) = u(x) - iu(ix)$ for all $x \in X$. Conversely, if u is a real linear functional on X and $f\colon X \to \mathbb{C}$ is defined by $f(x) = u(x) - iu(ix)$ for all $x \in X$, then f is a complex linear functional on X. If X is normed, we have $\|u\| = \|f\|$.

See Proposition 5.5 Folland 2013, p. 157.

Theorem 2.15 (Hahn-Banach extension theorem)
Let X be a real vector space, p be a sublinear functional on X, and M some subspace of X. If f is a linear functional on M such that for all $x \in M$ we have

$$f(x) \leq p(x).$$

Then there exists a functional F on X such that for all $x \in X$,

$$F(x) \leq p(x)$$

And $F_{|M} = f$.

For proof see Theorem 5.6 Folland 2013, p. 157.

Theorem 2.16 (Complex Hahn-Banach)
Let X be a complex vector space, p a sublinear functional on X and M a subspace of X. Given a complex linear functional f on M such that for all $x \in M$,

$$|f(x)| \leq p(x).$$

Then there exists a complex linear functional F on X such that for all $x \in X$

$$|F(x)| \leq p(x).$$

and $F|_M = f$.

For proof see Theorem 5.7 Folland 2013, p. 158.

Theorem 2.17
Let Y be a topological vector space over $\mathbb{C}$, and let φ be any non-zero linear functional on Y. Then the following are equivalent.

(a) φ is continuous with respect to the topology on Y.

(b) $\ker(\varphi)$ is closed is closed in the topology on Y.

(c) $\ker(\varphi)$ is not dense in Y.

(d) φ is bounded in some neighborhood $V \subset Y$ of 0.

Proof.
(a) implies (b) If φ is continuous, then, since $\{0\}$ is a closed set, $\varphi^{-1}(\{0\}) = \ker(\varphi)$ is also closed.

(b) implies (c) if $\ker(\varphi)$ is closed, we have that $\overline{\ker(\varphi)} = \ker(\varphi)$. Since $\varphi \neq 0$ we have $\ker(\varphi) \neq X$, so $\ker(\varphi)$ is not dense in Y.

(c) implies (d) if $\ker(\varphi)$ is not dense in Y, then its complement has non-empty interior, and therefore contains some interior point y. Being an interior point, there is by Theorem 2.3 some balanced neighborhood V of y such that $\ker(\varphi) \cap V = \emptyset$. Now we have that $V - y$ is a balanced neighborhood of 0. If $\varphi(V)$ is bounded on this, the result follows. Assume it is not, then by linearity of φ, the image of V is a balanced subset of $\mathbb{C}$, since the only balanced subsets of $\mathbb{C}$ are the empty set, the closed and open balls and $\mathbb{C}$ itself, thus $\varphi(V) = \mathbb{C}$. Then there is some $x \in V$ such that $\varphi(y) = -\varphi(x)$, and we see that $y + x \in V$ but $\varphi(y) + \varphi(x) = 0$, contradicting our assumption that $\ker(\varphi) \cap V = \emptyset$.

(d) implies (a) if φ is bounded in some neighborhood V of 0, then there is some $N > 0$ such that for all y in V we have

$$|\varphi(y)| < N.$$

Now, let $\varepsilon > 0$ be arbitrary, and set $W_\varepsilon = \frac{\varepsilon}{N} V$. Then W_ε is a neighborhood of 0, and all $x \in W_\varepsilon$ are of the form $x = \frac{\varepsilon}{N} y$ for some $y \in V$, and thus we have for all $x \in W_\varepsilon$,

$$|\varphi(x)| = \frac{\varepsilon}{N} |\varphi(y)| < \varepsilon.$$

So φ is continuous at 0, and thus everywhere by linearity. □

Theorem 2.18
Let X be a Banach space over $\mathbb{K}$. A linear functional $\varphi\colon X^* \to \mathbb{K}$ is weak*-continuous if and only if there is $x \in X$ such that $\varphi(f) = f(x)$ for all $f \in X^*$.

For proof, see Proposition 11 Musat 2015, p. 29.

Corollary 2.19
Let X be a Banach space over $\mathbb{K}$, and let $\varphi \in X^{**}$. Let $\hat{X}$ denote the image of X under the canonical embedding in X^{**}. Then the following statements are equivalent.

1. *φ belongs to $\hat{X}$.*
2. *φ is weak*-continuous.*
3. *$\ker(\varphi)$ is closed in the weak*-topology.*

Proof.
In Theorem 2.17 let $Y := X^*$ equipped with the weak*-topology. Then we have that φ is weak*-continuous if and only if $\ker(\varphi)$ is closed in the weak*-topology. By Theorem 2.18 we have that φ is weak*-continuous if and only if φ is given by evaluation, i.e., that $\varphi \in \hat{X}$. □

Corollary 2.20
If X is a Banach space over $\mathbb{K}$, a subset of X^* is compact in the weak*-topology if and only if it is closed in the weak*-topology and bounded in the norm topology.

See Corollary V.4.3 Dunford et al. 1971, p. 424 for a proof.

Definition 2.21
*Let X be a vector space over $\mathbb{K}$, and $A \subset X$ non-empty. The **convex hull** of A, denoted by $co(A)$ is the smallest convex set in X containing A. It is given by*

$$co(A) := \left\{ \sum_{i=1}^{n} \alpha_i x_i \ : n \geq 1, \quad x_i \in A, \ \alpha_i > 0, \ \sum_{i=1}^{n} \alpha_i = 1 \right\}.$$

Corollary 2.22
If X is a Banach space, and $(x_n)_{n\geq 1}$ is a sequence in X converging weakly to some x in X, there is a sequence $(u_n)_{n\geq 1}$ of convex combinations of $(x_n)_{n\geq 1}$ such that $u_n \overset{\|\cdot\|}{\to} x$, i.e., for all $j \geq 1$ we have that $u_j = \sum_{i=1} a_i x_i$ with $\sum_{i=1}^{k} a_i = 1$.

For proof see Corollary v.3.14 Dunford et al. 1971, p. 422.

Theorem 2.23 (Krein-Smulian)
A convex set in X^* is weak*-closed if and only if its intersection with every positive multiple of the closed unit sphere of X^* is weak*-closed.

For a proof of this see Dunford et al. 1971, p. 429.

CHAPTER 3

THE EBERLEIN-SMULIAN THEOREM

We are now almost ready to prove the Eberlein-Smulian Theorem, which ensures equivalence of three forms of compactness in the weak topology of a Banach space. We first prove some lemmas:

Lemma 3.1
Let X be a Banach space over $\mathbb{K}$. If $A \subset X$ is either

1. *weakly sequentially compact,*
2. *weakly limit point compact,*
3. *weakly compact,*

then A is bounded.

Proof.
Recall that by Theorem 2.10, A is bounded if and only if $f(A)$ is bounded in $\mathbb{K}$ for all $f \in X^*$. We will show that under the conditions *(1),(2),(3)*, the conclusion is true.

(1) Let $f \in X^*$ and let $A \subset X$ be weakly sequentially compact. Let $(x_n)_{n\geq 1} \subset f(A)$ be any sequence such that $x_n \neq x_m$ for $n \neq m$. Then, since $f : A \to f(A)$ is surjective, there is a sequence $(a_n)_{n\geq 1} \subset A$ such that $f(a_n) = x_n,\ n \geq 1$. By assumption, there is a weakly convergent subsequence $(a_{n_k})_{k\geq 1}$ of $(a_n)_{n\geq 1}$, thus $(f(a_{n_k}))_{k\geq 1} = (x_{n_k})_{k\geq 1}$ converges to some point. So $\overline{f(A)}$ is sequentially compact, hence compact since $\mathbb{K}$ is a metric space, and so $f(A)$ is bounded and thus A is bounded.

(2) Let $f \in X^*$ and let $A \subset X$ be weakly limit point compact. Now, let $(z_n)_{n\geq 1} \subset f(A)$ be any sequence such that $z_j \neq z_k$ for $j \neq k$. Now, as above, let $(a_n)_{n\geq 1} \subset A$ be a sequence such that $f(a_n) = z_n$ for $n \geq 1$. By hypothesis the sequence has a weak limit point, i.e., there exists $x_0 \in A$ such that whenever $U \subset X$ is a weak neighborhood of x_0, then U contains some $a_n \neq x_0$. Now we claim that there is a subsequence of $(a_n)_{n\geq 1}$ such that $(f(a_{n_k})) \to f(x_0)$. To see this, let $n_1 \geq 1$ be such that $a_{n_1} \neq x_0$ and $|f(a_{n_1}) - f(x_0)| < 1$.

Suppose that $1 < n_1 < n_2 < \cdots < n_k$ are chosen such that

$$|f(a_{n_i}) - f(x_0)| < \frac{1}{i} \quad \text{for } 1 \leq i \leq k$$

Let $F := \{n \geq 1 | a_n \neq x_0\}$ and pick U a weak neighbourhood of x_0 such that

$$a_n \notin U, \ \forall n \leq n_k, n \in F$$

with a construction similar to the one used in the proof of Lemma 2.11. Now there is some $p \geq 1$ such that $a_p \in V\left(x_0, f, \frac{1}{k+1}\right) \cap U$ with $a_p \neq x_0$. Furthermore, we must have that $p > n_k$, for if $p \leq n_k$ with $p \in F$ then $a_p \notin U$ and if $p \notin F$ then $a_p = x_0$. So $p > n_k$, and setting $n_{k+1} = p$ we get

$$|f(a_{n_{k+1}}) - f(x_0)| < \frac{1}{k+1}$$

thus $f(a_{n_k}) = z_{n_k} \to f(x_0)$, giving us that $\overline{f(A)}$ is compact and thus bounded, and so $f(A)$ is bounded, giving us that A is bounded.

(3) If $\bar{A}$ is weakly compact, then the image $f(\bar{A})$ is compact, and hence bounded, for each $f \in X^*$. This gives us that $f(A)$ is bounded since $f(A) \subset f(\bar{A})$, resulting in A being bounded. $\square$

Lemma 3.2
If X is a Banach space, any equicontinuous family Γ of linear functionals is contained in a weak*-compact set in X^*.

Proof.
If Γ is equicontinuous, then by Remark 1.10 it is also pointwise bounded, and thus Theorem 2.5 implies that $\sup_{f\in\Gamma}\|f\| < \infty$, and hence $\Gamma \subset \overline{B_{X^*}(0, \sup_{f\in\Gamma}(\|f\|))}$ which by Alaoglu is a weak*-compact set. $\square$

Theorem 3.3 **(Eberlein-Smulian)**
Let X be a Banach space, and let $A \subset X$. The following statements are equivalent:

(a) A is weakly sequentially compact,

(b) A is limit point compact in the weak topology,

(c) the closure of A in the weak topology is compact in the weak topology.

Proof.
First of all, note that due to Lemma 3.1, A is a bounded subset of X. We split the proof of the theorem up into three bits.

(c) implies (b). Let $B = (b_n)_{n\geq 1}$ be any sequence in A. Since B is a sequence it has an accumulation point b_0 in the closure of A by compactness. Since some subsequence of B is eventually in every neighbourhood of b_0, the intersection $U \cap B$

is nonempty for every open U containing b_0. This means that b_0 is also a limit point for B and since B was arbitrary, we are done.

(b) implies (a). Let $(x_n)_{n\geq 1}$ be any sequence in A, and define $X_0 = \overline{\text{span}}(x_n)$. Being a closed subspace of a Banach space, it is also a Banach space, and by Lemma Lemma 2.1, we have that X_0 is a separable Banach space. Being a Banach space, Theorem 2.4**(Alaoglu)** tells us that the closed unit ball of the dual space, $\overline{B_{X_0^*}(0,1)}$ is compact in the weak*-topology, and since X_0 is separable, we have by Theorem 2.12 that the weak*-topology on the closed unit ball of X^* is metrizable. This means that the closed unit ball is a metric space with the metric of Theorem 2.12 which is also compact in the weak*-topology, combining this with Lemma 2.2, we get that $\overline{B_{X_0^*}(0,1)}$ is separable. For each $n \geq 1$ we have that $n \cdot \overline{B(0,1)} = \overline{B(0,n)}$ is separable.

Now since $X_0^* = \bigcup_{n\in\mathbb{N}} \overline{B_{X_0^*}(0,n)}$ we have that the entire dual space of X_0 is separable. Now pick a countable dense set $H_0 \subset X_0^*$. H_0 is separating on X_0, for let $x, y \in X_0,\ x \neq y$ and let $f \in X_0^*$ be such that

$$f(x) \neq f(y) \quad \text{and} \quad |f(x) - f(y)| = 1$$

Since H_0 is countably dense in X_0^*, there is a sequence $(g_n)_{n\geq 1} \subset H_0$ such that $f = \lim\limits_{n\to\infty} g_n$. And thus we see that

$$f(x) = \lim_{n\to\infty} g_n(x) \neq \lim_{n\to\infty} g_n(y) = f(y).$$

Let $0 < \varepsilon < \frac{1}{2}$ be given and pick $N \geq 1$ such that $|g_N(x) - f(x)| < \varepsilon$ and $|g_N(y) - f(y)| < \varepsilon$, then by reverse triangle inequality we see that

$$\begin{aligned} |g_N(x) - f(y)| &\geq \Big| |g_N(x) - f(x)| - |f(x) - f(y)| \Big| \\ &> 1 - \varepsilon \end{aligned}$$

And thus

$$\begin{aligned} |g_N(x) - g_N(y)| &\geq \Big| g_N(x) - f(y)| - |f(y) - g_N(y)| \Big| \\ &> 1 - 2\varepsilon > 0. \end{aligned}$$

Using Hahn-Banach we can uniquely extend every element of H_0 to a linear functional on all of X. Let H denote the countable set of all the extension of H_0 to all of X, ie. $H \subset X^*$, clearly H is separating on X_0. First we write $H = (g_k)_{k\geq 1}$. By Lemma 3.1 we have that A is bounded, so the sequence $(g_1(x_n))_{n\geq 1}$ is bounded in $\mathbb{C}$, and thus has a convergent subsequence. Denote by $(1_n)_{n\geq 1}$ any subsequence of $(n)_{n\geq 1}$ for which $(g_1(x_{1_n})_{n\geq 1}$ is convergent for $n \to \infty$ and with the property that $n > k \geq 1$ implies $1_n > 1_k$. Now $(g_2(x_{1_n}))_{n\geq 1}$ is also bounded, and thus has a convergent subsequence. Define again a subsequence $(2_n)_{n\geq 1} \subset (1_n)_{n\geq 1}$ such that $(g_2(x_{2_n}))_{n\geq 1}$ converges in $\mathbb{C}$, and with the same property as above, i.e., $n > k$ implies $2_n > 2_k$. Inductively we get subsequences (m_n) of (m_{n-1}) such that $g_m(x_{m_n})$ converges in $\mathbb{C}$ as $n \to \infty$ for all $m \geq 1$. Moreover we also have that for any $n \geq 1$ and $m \geq 2$ that m_n must be an element of $((m-1)_n)_{n\geq 1}$, i.e., that it has the property ($\star$): $m_n = (m-1)_{n'}$ for some unique $n' \geq n$, and that $m_n \geq (m-1)_n$.

Define a function $q : \mathbb{N} \to \mathbb{N}$ by $q(n) = n_n$. Then $(x_{q(n)})_{n\geq 1}$ is a subsequence of (x_n). Note that for $n \geq 1$ we have

$$q(n+1) = (n+1)_{n+1} > n_{n+1} \geq n_n = q(n)$$

And for fixed $k \geq 1$ we have by construction that

$$q(n+k) = (n+k)_{n+k} = \cdots = k_{a_k(n)}$$

for some function $a_k \colon \mathbb{N}_0 \to \mathbb{N}$. doing the property ($\star$) k times. Since q is strictly increasing, a_k is increasing as well, and we see that the sequence with elements

$$\begin{aligned} g_k(x_{q(k)}) &= g_k(x_{k_{a_k(0)}}), \\ g_k(x_{q(k+1)}) &= g_k(x_{k_{a_k(1)}}), \\ g_k(x_{q(k+2)}) &= g_k(x_{k_{a_k(2)}}), \\ &\vdots \end{aligned}$$

is a subsequence of $(g_k(x_{k_n}))_{n\geq 1}$, i.e, $(g_k(x_{q(k+n)}))_{n\geq 1}$ converges as n tends to infinity. Let $y_j := x_{q(j)}$ for $j \geq 1$, then $(y_j)_{j\geq 1}$ has the desired property, that

$$\lim_{m\to\infty} g_k(y_m)$$

is well-defined for each k.

Since A is limit point compact in the weak topology, by assumption there is a point $y_0 \in X$ such that every weak neighborhood of y_0 has nonempty intersection with $(y_m)_{m\geq 1}$. Since this sequence is in X_0, which is weakly closed, we have that $y_0 \in X_0$. By Lemma 2.11, $\lim_{m\to\infty} g(y_m) = g(y_0)$ for all $g \in H$, and it remains to be shown that $\lim_{m\to\infty} f(y_m) = f(y_0)$ for all $f \in X^*$, for then A is weakly sequentially compact.

Assume for contradiction that this is not the case. Then there exists $f_0 \in X^*$ and $\varepsilon > 0$ and some subsequence $(y_{m_k})_{k\geq 1} \subset (y_m)_{m\geq 1}$ such that

$$|f_0(y_{m_k}) - f_0(y_0)| > \varepsilon,\ k \geq 1. \tag{$\star$}$$

Since this new subsequence is also a countably infinite subset of A, it has a weak limit point y_0'. As before this limit point lies in X_0, and $\lim_{k\to\infty} g(y_{m_k}) = g(y_0')$, and since H is separating on X_0, we have that $y_0 = y_0'$. Since y_0' is a weak limit point of $(y_{m_k})_{k\geq 1}$, the intersection $V(y_0', f_0, \varepsilon) \cap (y_{m_k})_{k\geq 1}$ is non-empty. This means that there exists $y \in (y_{m_k})_{k\geq 1}$ such that

$$|f_0(y - y_0)| = |f_0(y - y_0')| < \varepsilon$$

which is a contradiction to ($\star$), and thus any sequence of A contains a weakly convergent subsequence, and thus A is weakly sequentially compact.

(a) implies (c) Let $\bar{A}$ denote that weak closure of A, and let Λ denote the natural embedding map $\Lambda : X \to X^{**}$. Since $\Lambda : X \to \Lambda(X) := \hat{X}$ is both continuous, isometric and one-to-one, by Corollary 2.7, it is an isometric homeomorphism between (X, τ_w) and $(\Lambda(X), \tau_{w^*})$, and so we have that $\bar{A}$ is weakly compact if and only if $\hat{\bar{A}}$ is weak* compact. By Lemma 3.1 we have that $\bar{A}$ is bounded, and thus by Corollary 2.13, we have that $\Lambda(\bar{A})$ is weak* compact if and only if it's a weak* closed subset of X^{**}. We will show this by showing that $\Lambda(\bar{A}) = \overline{\Lambda(A)}^{w^*}$.

We will get that $\Lambda(\bar{A}) = \overline{\Lambda(A)}^{w^*}$ by proving a useful lemma and that $\overline{\Lambda(A)}^{w^*} \subset \Lambda(X)$. The lemma in question states that $\Lambda(\bar{A}) = \overline{\Lambda(A)}^{w^*} \cap \Lambda(X)$.

We get $\Lambda(\bar{A}) \subset \overline{\Lambda(A)}^{w^*} \cap \Lambda(X)$ by the continuity and isometry of Λ. To get the reverse inclusion let $\varphi \in \overline{\Lambda(A)}^{w^*} \cap \Lambda(X)$. By definition this means that $\varphi = \Lambda(x)$ for some $x \in X$ and that there exists a net $(x_\alpha)_{\alpha \in I} \subset A$ such that $\Lambda(x_\alpha)(f) \to \varphi(f)$ for all $f \in X^*$. But since $\Lambda(y)(f) = f(y)$ by definition this is equivalent to $f(x_\alpha) \to f(x)$ for all $f \in X^*$, which is just weak convergence of $(x_\alpha)_{\alpha \in I}$ to x. This means that $x \in \bar{A}$, and therefore that $\Lambda(x) \in \Lambda(\bar{A})$.

So let $x^{**} \in \overline{\Lambda(A)}^{w^*}$ be arbitrary, we wish to show that there is $x \in X$ such that

$$x^{**}(x^*) = x^*(x), \; x^* \in X^*$$

By Corollary 2.19, we get that $x^{**} \in \Lambda(X)$ if and only if $\ker(x^{**})$ is a weak* closed subset of X^*, and by Theorem 2.23 we get that this happens if its intersection with the closed unit sphere of the dual is closed in the weak* topology. So our claim is that if $\psi_0^* \in \overline{D \cap S^*}^{w^*}$ then $\psi_0^* \in D \cap S^*$. In the following, let $D := \ker(x^{**}) \subset X^*$ and $S^* := \overline{B_{X^*}(0,1)}$.

Now first we **claim** (†) the following:
Let $\{\varphi_1^*, \ldots, \varphi_n^*\} \subset X^*$ be a finite subset of X^*, then there exists $z \in \bar{A}$ such that

$$x^{**}\varphi_i^* = \varphi_i^* z, \; i = 1, \ldots, n$$

proof of claim Let $\{\varphi_1, \ldots, \varphi_n\} \subset X^*$ and let $m \geq 1$ be given. Since $x^{**} \in \overline{\Lambda(A)}^{w^*}$, we have, for each basis element for the weak* topology on X^{**}, $V^*\left(x^{**}, \varphi_1, \ldots, \varphi_n, \frac{1}{m}\right)$, that the intersection

$$V_m^* := V^*\left(x^{**}, \varphi_1, \ldots, \varphi_n, \frac{1}{m}\right) \cap \Lambda(A)$$

is nonempty. For each m, pick $\widehat{z_m} \in V_m^*$, and let $z_m \in A$ be the corresponding evaluation element. Then we have that $(z_m)_{m \geq 1} \subset A$. Since A is weakly sequentially compact, there is a weakly convergent subsequence $(z_{m_j})_{j \geq 1}$ converging to some $z \in \overline{A}$. This z has the property that for all $\varepsilon > 0$ we have

$$|x^{**}(\varphi_i) - \varphi_i(z)| < \varepsilon, \quad i = 1, \ldots, n$$

Hence $x^{**}(\varphi_i) = \varphi_i(z)$ for $i = 1, \ldots, n$, as required.

Now we consider $\psi_0 \in \overline{D \cap S^*}^{w^*}$. Recall that the goal is to show that $\psi_0 \in D \cap S^*$.

Let $\varepsilon > 0$ be arbitrary. The set $\{\psi_0\}$ is finite, and thus has the property required for (†) to apply, so there is $z_1 \in \overline{A}$ such that

$$x^{**}(\psi_0) = \psi_0(z_1)$$

Now, since $z_1 \in \overline{A}$, the intersection

$$V_{(x,1)} := V\left(z_1, \psi_0, \frac{\varepsilon}{4}\right) \cap A$$

is non-empty, so pick $x_1 \in V_{(x,1)}$. Then $x_1 \in A$ and it has the property

$$|\psi_0(x_1) - \psi_0(z_1)| < \frac{\varepsilon}{4}$$

If we let $V^*\left(\psi_0, x_1, \frac{\varepsilon}{4}\right)$ be a weak* neighborhood of ψ_0 in X^*, then, since $\psi_0 \in \overline{D \cap S^*}^{w^*}$, we have that the intersection

$$V_{(\psi_0,1)} := V^*\left(\psi_0, x_1, \frac{\varepsilon}{4}\right) \cap D \cap S^* \subset X^*$$

is non-empty. So pick some $\psi_1 \in V_{(\psi,1)}$. Then $\psi_1 \in D \cap S^*$, so $x^{**}(\psi_1) = 0$, $\|\psi_1\| \leq 1$. Furthermore $\psi_1 \in V^*\left(\psi_0, x_1, \frac{\varepsilon}{4}\right)$ we have

$$|\psi_0(x_1) - \psi_1(x_1)| < \frac{\varepsilon}{4}$$

Now, let $n \in \mathbb{N}$ and assume that we have three sets

$$\{z_1, \ldots, z_n\} \subset \overline{A}^w, \quad \{x_1, \ldots, x_n\} \subset A, \quad \text{and } \{\psi_1, \ldots, \psi_n\} \subset D \cap S^*$$

with the following properties:

1. $\psi_i(z_n) = x^{**}(\psi_i) = 0$ for $1 \leq i \leq n-1$,
2. $|\psi_m(z_n) - \psi_m(x_n)| < \frac{\varepsilon}{4}$ for $0 \leq m \leq n-1$,
3. $|\psi_n(x_i) - \psi_0(x_i)| < \frac{\varepsilon}{4}$ for $1 \leq i \leq n$,
4. $\psi_0(z_i) = x^{**}(\psi_0)$ for $1 \leq i \leq n$,
5. $\|\psi_m\| \leq 1$.

Now, since the set $\{\psi_0, \ldots, \psi_n\} \subset X^*$ is finite, (†) produces $z_{n+1} \in \overline{A}$ such that

$$x^{**}(\psi_i) = \psi_i(z_{n+1}), \quad i = 0, \ldots, n.$$

Since $z_{n+1} \in \overline{A}$, the intersection $V_{x,n+1} := V\left(z_{n+1}, \psi_0, \ldots, \psi_n, \frac{\varepsilon}{4}\right) \cap A$ is non-empty. Pick x_{n+1} in this intersection. Then $x_{n+1} \in A$ and satisfies that $|\psi_m(z_{n+1}) - \psi_m(x_{n+1})| < \frac{\varepsilon}{4}$ for $0 \leq m \leq n$.

Now, since the intersection $V^*\left(\psi_0, x_1, \ldots, x_{n+1}, \frac{\varepsilon}{4}\right) \cap (D \cap S^*) = V_{\psi,n+1} \cap (D \cap S^*)$ is nonempty, we can pick some $\psi_{n+1} \in V_{\psi,n+1}$. This ψ_{n+1} is an element of $(D \cap S^*)$, so $x^{**}(\psi_{n+1}) = 0$ and $\|\psi_{n+1}\| \leq 1$. Furthermore we have $|\psi_n(x_i) - \psi_0(x_i)| < \frac{\varepsilon}{4}$ for $1 \leq i \leq n+1$. We have now shown that the sets

$$\{z_1, \ldots, z_{n+1}\} \subset \overline{A}, \quad \{x_1, \ldots, x_{n+1}\} \subset A, \quad \{\psi_1, \ldots, \psi_{n+1}\} \subset D \cap S^*$$

have properties 1-5 above. By this recursive process we obtain sequences

$$(z_n)_{n\geq 1} \subset \bar{A} \quad (x_n)_{n\geq 1} \subset A, \quad (\psi_m)_{m\geq 1} \subset D \cap S^*$$

with the following properties:

1. $\psi_i(z_n) = x^{**}(\psi_i) = 0$ for $1 \leq i \leq n-1$;
2. $|\psi_m(z_n) - \psi_m(x_n)| < \frac{\varepsilon}{4}$ for $0 \leq m \leq n-1$;
3. $|\psi_n(x_i) - \psi_0(x_i)| < \frac{\varepsilon}{4}$ for $1 \leq i \leq n$;
4. $\psi_0(z_i) = x^{**}(\psi_0)$ for $1 \leq i \leq n$;
5. $\|\psi_m\| \leq 1$.

If we set $m = 0$ and combine 2,3 and 4, we get for $1 \leq i \leq n$

$$\begin{aligned} |x^{**}(\psi_0) - \psi_n(x_i)| &\overset{4}{=} |\psi_0(z_i) - \psi_0(x_i) + \psi_0(x_i) - \psi_n(x_i)| \\ &\leq |\psi_0(z_i) - \psi_0(x_i)| + |\psi_0(x_i) - \psi_n(x_i)| \\ &\overset{2,3}{\leq} \frac{\varepsilon}{2} \end{aligned} \tag{†}$$

Since $(x_n)_{n\geq 1}$ is a sequence in A, we get by hypothesis that there is a subsequence converging weakly to some $x \in \bar{A}$. Assume without loss of generality that the entire sequence converges weakly to this limit. Combining 1 and 2 we get

$$|\psi_i(x_n)| < \frac{\varepsilon}{4}, \ i = 1, \ldots, n-1$$

So in the limit this gives us the following:

$$|\psi_m(x)| \leq \frac{\varepsilon}{4}$$

By Corollary 2.22 there is some sequence $(u_j)_{j\geq 1}$ of convex combinations of $(x_n)_{n\geq 1}$ with $u_j \overset{\|\cdot\|}{\to} x$ for $j \to \infty$. Thus, for the fixed $\varepsilon > 0$ from earlier, there is $u \in (u_n)_{n\geq 1}$,

$u = \sum_{j=1}^{n} a_j x_j$ with $\sum_{i=1}^{n} a_i = 1$ and $x_j \in (x_n)_{n\geq 1}$ for all $j \geq 1$ such that $\|x-u\| < \frac{\varepsilon}{4}$. Hence we have that

$$\begin{aligned}
|x^{**}(\psi_0) - \psi_n(u)| &= |\underbrace{\sum_{j=1}^{n} a_j}_{=1} x^{**}(\psi_0) - \sum_{j=1}^{n} a_j \psi_n(x_j)| \\
&\leq \sum_{j=1}^{n} a_j |x^{**}(\psi_0) - \psi_n(x_j)| && \text{(by †)} \\
&\leq \sum_{j=1}^{n} a_j \frac{\varepsilon}{2} = \frac{\varepsilon}{2}
\end{aligned}$$

And thus we see that

$$\begin{aligned}
|x^{**}(\psi_0)| &\leq |x^{**}(\psi_0) - \psi_n(u)| + |\psi_n(u) - \psi_n(x)| + |\psi_n(x)| \\
&< \frac{\varepsilon}{2} + \|\psi_n\| \|x-u\| + \frac{\varepsilon}{4} \\
&< \varepsilon
\end{aligned}$$

Since ε was arbitrary, we have that $x^{**}(\psi_0) = 0$, hence $\psi_0 \in D$. Since S^* is weak*-closed, $\psi_0 \in D \cap S^*$. ☐

Corollary 3.4
Let X be a Banach space and $A \subset X$ a (weakly) bounded subset, then the following are equivalent

1. *A is weakly sequentially compact*
2. *Let $(f_n)_{n\geq 1}$ be an equicontinuous sequence of linear functionals on X and $(x_m)_{m\geq 1}$ a sequence in A. Then if $\lim_m \lim_n f_m(x_n)$ and $\lim_n \lim_m f_m(x_n)$ both exist they are equal.*

Proof.
"1 $\implies$ 2" Since A is weakly sequentially compact, $(x_n)_{n\geq 1}$ has a weak clusterpoint x_0, and since $(f_m)_{m\geq 1}$ is contained in a weak*-compact set by Lemma 3.2, it has a weak*-clusterpoint f_0, so if $\lim_m \lim_n f_m(x_n)$ and $\lim_n \lim_m f_m(x_n)$ both exists, it must be true that

$$\lim_{m\to\infty} \lim_{n\to\infty} f_m(x_n) = \lim_{n\to\infty} \lim_{m\to\infty} f_m(x_n) = f_0(x_0),$$

as wanted.

"2 $\implies$ 1" Let $(x_n)_{n\geq 1} \subset A$. Since A is weakly bounded it is also norm bounded, so $\Lambda(A) \subset X^{**}$ is bounded, and thus the sequence $(\Lambda(x_n))_{n\geq 1}$ has a weak*-clusterpoint $x^{**} \in X^{**}$. The **claim** is that $x^{**} \in \Lambda(X)$. Assume not. Then, by Krein-Smulian, the restriction of x^{**} to the closed unit ball of X^*, S_* is not weak*-continuous at 0. For each $n \geq 1$ define $V_n := \left\{ f \in X^* \middle| |f(x_i)| < \frac{1}{n},\ 1 \leq i \leq n \right\}$, then $S_* \cap V_n$ is a weak*-neighborhood of 0, and by assumption, there is $\varepsilon > 0$ such that for all $n \geq 1$ and $f_n \in V_n \cap S_*$ we have

$$|x^{**}(f_n)| > \varepsilon$$

Now, for $m \geq 1$ define $U_m := \left\{ \varphi \in X^{**} \middle| \ |\varphi(f_k)| < \frac{1}{m},\ 1 \leq k \leq m \right\}$, for each $m \geq 1$ we have that U_m is a weak*-neighborhood of $0 \in X^{**}$. Since $(\Lambda(x_n))_{n \geq 1}$ has x^{**} as a weak*-clusterpoint, there is an increasing function $\phi : \mathbb{N} \to \mathbb{N}$ such that $\Lambda(x_{\phi(n)}) \in x^{**} + U_n$ for $n \geq 1$, and similarly by the boundedness of $(x^{**}(f_m))_{m \geq 1}$ there is an increasing function $\psi : \mathbb{N} \to \mathbb{N}$ such that $x^{**}(f_{\psi(m)})$ converges in $\mathbb{C}$.

Now, fix $n \geq 1$, for $m \geq \psi(n)$ we have that

$$\Lambda(x_{\phi(m)}) - x^{**} \in U_m$$

resulting in $|f_{\psi(n)}(x_{\phi(m)}) - x^{**}(f_{\psi(n)})| < \frac{1}{m}$, and hence

$$\lim_m f_{\psi(n)}(x_{\phi(m)}) = \lim_m \Lambda(x_{\phi(m)})(f_{\psi(n)}) = x^{**}(f_{\psi(n)}).$$

And we see that

$$\lim_n \left| \lim_m f_{\psi(n)}(x_{\phi(m)}) \right| = \lim_n \left| \lim_m \Lambda(x_{\phi(m)})(f_{\psi(n)}) \right| = \lim_n \left| x^{**}(f_{\psi(n)}) \right| \geq \varepsilon$$

since $|x^{**}(f_{\psi(j)})| > \varepsilon$ for all $j \geq 1$ by assumption. But, if we fix $m \geq 1$ with $\psi(n) \geq \phi(m)$ we have $f_{\psi(n)} \in V_{\psi(n)} \subset V_{\phi(m)}$, giving us that $|f_{\psi(n)}(x_{\phi(m)})| < \frac{1}{n}$, so $\lim_n f_{\psi(n)}(x_{\phi(m)}) = 0$, and we have

$$\lim_m \lim_n f_{\psi(n)}(x_{\phi(m)}) = 0 \neq \varepsilon \leq \lim_n \left| \lim_m f_{\psi(n)}(x_{\phi(m)}) \right|$$

Hence we have a contradiction, and thus $x \in \Lambda(X)$ $\square$

CHAPTER 4

THE JAMES THEOREM

We are now ready to tackle The James theorem, published in 1966, which states that if X is a complete locally convex linear topological space and $A \subset X$, then A is weakly compact if and only if all elements $f \in X^*$ attains their supremum on A. We will not be dealing with the theorem in the general setting of X being any complete locally convex linear topological space. Instead, we will assume that X is a Banach space and prove his theorem in this setting using our variant of the Eberlein-Smulian Theorem of the preceding chapter. Doing this will give us a criterion to determine when a Banach space X is reflexive.

To prove the James theorem, we need a series of lemmas which prove that if X is a real Banach space, and $C \subset X$ is a weakly closed bounded subset which is not weakly compact, then there is a bounded linear functional not attaining its supremum on C.

So for the rest of this chapter, assume that X is a real Banach space and $C \subset X$ is a weakly closed and bounded subset of X, which is **not** weakly compact unless otherwise is stated.

Lemma 4.1
There is some sequence $(x_n)_{n\geq 1} \subset C$ and some equicontinuous sequence $(f_m)_{m\geq 1} \subset X^*$ such that the limits $\lim_m \lim_n f_m(x_n)$ and $\lim_n \lim_m f_m(x_n)$ both exist, but differ.

Proof.
This is a direct consequence of Corollary 3.4. □

In the following, let $\mathcal{F}$ be the set of all continuous real-valued positive-homogenous functions, that is the set of continuous $f\colon X \to \mathbb{R}$ and for which

$$f(ax) = af(x)$$

for $a \geq 0$.

All functions $f \in \mathcal{F}$ are bounded on bounded sets, for if $f \in \mathcal{F}$, then by the continuity of f we have that if $I \subset \mathbb{R}$ is a neighborhood of 0 there is some neighborhood U of 0 in X such that $f(U) \subset I$. Setting $I = [-1, 1]$, we get that there is a neighborhood U' of 0 in X such that

$$|f(x)| = |f(x) - f(0)| \leq 1, \quad \text{for all } x \in U'$$

And all bounded sets $V \subset X$ are absorbed by the neighborhood U'. Thus for all $v \in V$ we have

$$|f(v)| = |f(\alpha x)| = |\alpha f(x)| \leq \alpha < \infty$$

for some $x \in U'$ with $v = \alpha x$ for some $\alpha \geq 0$ for which $V \subset \alpha U'$.

We equip $\mathcal{F}$ with the topology of pointwise convergence on X. Note that $X^* \subset \mathcal{F}$ and it is furthermore a closed subspace in this topology. The subspace topology on X^* coincides with the weak$-^*$ topology on X^*.

Now, define a functional $p \colon \mathcal{F} \to \mathbb{R}$ by

$$p(f) = \sup\{f(x) : x \in C\}, \; f \in \mathcal{F}.$$

This functional has the following properties:

1. p is finite valued for all $f \in \mathcal{F}$, since C is a weakly bounded set, it is also bounded in norm by Theorem 2.10.
2. The functional p is sublinear, that is, $p(\lambda f) = \lambda p(f)$ for $\lambda \geq 0$ and $p(f+g) \leq p(f) + p(g)$. This is due to the properties of the supremum and the positive homogenity of $f, g \in \mathcal{F}$.
3. We also have, by subadditivity of p, that

$$-p(g-f) \leq p(f) - p(g) \leq p(f-g)$$

   for all $f, g \in \mathcal{F}$.
4. If $A \subset \mathcal{F}$ is an equicontinuous family of functions, then p is bounded on A, for there is a neighborhood basis element U_A of $0 \in X$ such that $|f(x)| \leq 1$ for all $x \in U_A$ and $f \in A$, and U_A being a neighborhood basis element of 0 in a locally convex space, C is absorbed by U_A.

We will also consider the topology on $\mathcal{F}$ induced by the functional $P : \mathcal{F} \to \mathbb{R}$ defined by $P(f) = \sup\{|f(x)| : x \in C\}$. This topology is the topology of *uniform convergence* on C. We will write $f_n \overset{uc}{\to} f$ if we mean convergence in this topology.

Now, let $(f_i)_{i \geq 1} \subset \mathcal{F}$ be any equicontinuous sequence. We then define functions $G_- = \liminf_i f_i$ and $G_+ = \limsup_i f_i$ by

$$G_-(x) = \liminf_i f_i(x) \quad \text{and} \quad G_+(x) = \limsup_i f_i(x)$$

for $x \in X$.

G_- and G_+ are everywhere finite and continuous. To prove this, let $x_0 \in X$ and $\varepsilon > 0$ be given. There is, by the equicontinuity of $(f_i)_{i\geq 1}$, a neighborhood U_{x_0} of x_0 such that

$$\sup_{x\in U_{x_0}} \{|f_i(x) - f_i(x_0)| \ : \ i = 1, 2, \dots\} \leq \varepsilon.$$

Note, for fixed $i \geq 1$ we have

$$|\inf_{k\geq i} f_k(x) - \inf_{k\geq i} f_k(x_0)| \leq \sup_{k\geq i} |f_k(x) - f_k(x_0)| \tag{†}$$

To see this, let $(a_k)_{k\geq 1}, (b_k)_{k\geq 1}$ be defined by $a_k := -f_k(x), b_k := -f_k(x_0)$. Thux

$$\begin{aligned}
\sup_{k\geq i} b_k &= \sup_{k\geq i} (b_k - a_k + a_k) \\
&\leq \sup_{k\geq i} (b_k - a_k) + \sup_{k\geq i} a_k \\
&\leq \sup_{k\geq i} |b_k - a_k| + \sup_{k\geq i} a_k,
\end{aligned}$$

implying that

$$\sup_{k\geq i} b_k - \sup_{k\geq i} a_k \leq \sup_{k\geq i} |b_k - a_k|.$$

Replacing b_k with a_k in the above we obtain that

$$\left|\sup_{k\geq i} b_k - \sup_{k\geq i} a_k\right| \leq \sup_{k\geq i} |b_k - a_k|$$

But we know that

$$\begin{aligned}
\left|\sup_{k\geq i} b_k - \sup_{k\geq i} a_k\right| &= \left|\inf_{k\geq i} -a_k - \inf_{k\geq i} -b_k\right| \\
&= \left|\inf_{k\geq i} f_k(x) - \inf_{k\geq i} f_k(x_0)\right| \\
&\leq \sup_{k\geq i} |f_k(x) - f_k(x_0)|,
\end{aligned}$$

Thus achieving the desired result. Now, let $U_{x_0}, \varepsilon > 0$ be as above, then for fixed $x \in U_{x_0}$ we have

$$\begin{aligned}
|G_-(x) - G_-(x_0)| &= |\liminf_i f_i(x) - \liminf_i f_i(x_0)| \\
&= |\lim_i (\inf_{k\geq i} f_k(x)) - \lim_i (\inf_{k\geq i} f_k(x_0))| \\
&= \lim_i |\inf_{k\geq i} f_k(x) - \inf_{k\geq i} f_k(x_0)| \\
&\leq \lim_i \sup_{k\geq i} |f_k(x) - f_k(x_0)| && \text{(by †)} \\
&\leq \sup_{k\geq 1} |f_k(x) - f_k(x_0)| \leq \varepsilon,
\end{aligned}$$

and analogously for G_+. By the properties of $\liminf, \limsup$ they are also positive-homogeneous, so G_-, G_+ are elements of $\mathcal{F}$.
We are now ready to prove another result

Lemma 4.2
Let $(f_i)_{i\geq 1}$ be any equicontinous sequence of $\mathcal{F}$ and let the topology on $\mathcal{F}$ be that of the seminorm P, that is the topology of uniform convergence on C. Let H be a subset of $\mathcal{F}$, which is separable in the subspace-topology. Then there is a subsequence $(g_j)_{j\geq 1}$ of $(f_i)_{i\geq 1}$ such that if $G_- = \liminf_j g_j$ and $G_+ = \limsup_j g_j$ then

$$p(f - G_-) = p(f - G_+)$$

for all $f \in H$, with $p(f) = \sup\limits_{x\in C}\{f(x)\}$ as earlier.

Proof.
By assumption, H has a countable dense subset $(\omega_k)_{k\geq 1}$. Let $(\varphi_j)_{j\geq 1} \subset H$ be defined by

$$(\varphi_j)_{j\geq 1} := (\omega_1,\ \omega_1, \omega_2,\ \omega_1, \omega_2, \omega_3,\ \dots)$$

This new sequence has every point of H as a clusterpoint. Now, we build a sequence $(x_n)_{n\geq 1} \subset C$ and sequences $((f_i^n)_{i\geq 1})_{n\geq 1}$ recursively. First note that since

$$p(f) = \sup_{x\in C}\{f(x)\},$$

we have that there is $y \in C$ such that

$$|\varphi_1(y) - \liminf_i f_i(y) - p(\varphi_1 - \liminf_i f_i)| < \frac{1}{2}.$$

For $n = 1$ choose $x_1 \in C$ such that

$$\begin{aligned}\varphi_1(x_1) - \liminf_i f_i(x_1) &> p(\varphi_1 - \liminf_i f_i) - \frac{1}{2} \\ &= \sup_{x\in C}\{\varphi_1(x) - \liminf_i f_i(x)\} - \frac{1}{2}.\end{aligned}$$

For x_1 in particular we can pick a subsequence $(f_i^1)_{i\geq 1} \subset (f_i)_{i\geq 2}$ such that $(f_i^1(x_1))_{i\geq 1}$ converges to $\liminf_i f_i(x_1)$. Similarly for $n > 1$ choose $x_n \in C$ and a subsequence $(f_i^n)_{i\geq 1}$, with

$$(f_i^n)_{i\geq 1} \subset (f_i^{n-1})_{i\geq 2} \subset \cdots \subset (f_i^1)_{i\geq n},$$

in such a way that

$$\varphi_n(x_n) - \liminf_i f_i^{n-1}(x_n) > p(\varphi_n - \liminf_i f_i^{n-1}) - \frac{1}{2^n}$$

and

$$f_i^n(x_n) \overset{i\to\infty}{\to} \liminf_i f_i^{n-1}(x_n).$$

Define a new sequence $(g_i)_{i\geq 1}$ by $g_k := f_1^k$. Note that for every $n \geq 1$ we have $(g_k)_{k\geq n}$ is a subsequence of $(f_i^n)_{i\geq 1}$ and thus if we define G_-, G_+ by $G_- = \liminf_j g_j$ and $G_+ = \limsup_j g_j$, we have

(i) $\lim_k g_k(x_n)$ is well defined and equals $\lim_i f_i^n(x_n)$ for each $n \geq 1$

(ii) Since $(g_k)_{k\geq 1} \subset (f_i^n)_{i\geq 1}$ for all n, we have

$$\begin{aligned}
\varphi_n(x_n) - G_-(x_n) &= \varphi_n(x_n) - \liminf_i g_i(x_n) \\
&= \varphi_n(x_n) - \lim_k g_k(x_n) \\
&= \varphi_n(x_n) - \liminf_i f_i^{n-1}(x_n) \\
&> p(\varphi_n - \liminf_i f_i^{n-1}) - \frac{1}{2^n} \\
&\geq p(\varphi_n - G_-) - \frac{1}{2^n}.
\end{aligned}$$

The last step comes from $G_- = \liminf_i g_i \geq \liminf_i f_i^n$ for all n.

Now, let $f \in H$ be arbitrary. Since $(\varphi_k)_{k\geq 1}$ has every point as a clusterpoint, given any $\varepsilon > 0$ there is an $n \in \mathbb{N}$ such that (a): $\frac{1}{2^n} < \varepsilon$ and (b): For all $x \in C$ we have $|f(x) - \varphi_n(x)| < \varepsilon$. Combining everything we get that

$$\begin{aligned}
p(f - G_-) &= \sup_{x\in C}\{f(x) - \liminf_i g_i(x)\} \\
&\leq \varepsilon + \sup_{x\in C}\{\varphi_n(x) - \liminf_i g_i(x)\} && \text{(by (b))} \\
&< 2\varepsilon + \varphi_n(x_n) - \liminf_i g_i(x_n) && \text{(by (a) and (ii))} \\
&= 2\varepsilon + \varphi_n(x_n) - \limsup_i g_i(x_n).
\end{aligned}$$

Now, since $\limsup_i g_i(x_n) = G_+$, we have that

$$\begin{aligned}
&2\varepsilon + \varphi_n(x_n) - \limsup_i g_i(x_n) \\
&< 3\varepsilon + f(x_n) - \limsup_i g_i(x_n) && \text{(by (b))} \\
&\leq 3\varepsilon + \sup_{x\in C}\{f(x) - G_+(x)\} \\
&= 3\varepsilon + p(f - G_+).
\end{aligned}$$

Since ε was arbitrary, we have that $p(f - G_-) \leq p(f - G_+)$. The other inequality follows from $G_- \leq G_+$, proving that $p(f - G_-) = p(f - G_+)$ for $f \in H$. □

Now, if we let $(f_i)_{i\geq 1} \subset X^* \subset \mathcal{F}$ be the equicontinuous sequence of Lemma 4.1, and let $H = \text{span}(f_i)$, then H is separable in the P-topology of $\mathcal{F}$. To see this, note that if $f \in H$, then f is on the form $f = \sum_{i\geq 1} \lambda_i f_i$ with $\lambda_i \in \mathbb{R}$, for each $i \geq 1$. Also note that, by the equicontinuity of $(f_i)_{i\geq 1}$ we have

$$\sup_{k\geq 1} P(f_k) = \sup_{k} \sup_{x\in C} \{|f_k(x)|\} < \infty.$$

Let $f \in H$, $\varepsilon > 0$ be given and pick $(q_i)_{i\geq 1} \in \mathbb{Q}$ with $q_i = 0$ whenever $\lambda_i = 0$ and such that

$$0 < \sum_{i\geq 1}^{\infty} (\lambda_i - q_i) < \varepsilon$$

and $g = \sum_{i=1}^{\infty} q_i f_i \in H$. Then we see that

$$\begin{aligned}
P(f-g) &= \sup_{x\in C} \{f(x) - g(x)\} \\
&= \sup_{x\in C} \left\{ \left| \sum_{i=1}^{\infty} \lambda_i f_i(x) - \sum_{i=1}^{\infty} q_i f_i(x) \right| \right\} \\
&= \sup_{x\in C} \left\{ \left| \sum_{i=1}^{\infty} (\lambda_i - q_i) f_i(x) \right| \right\} \\
&\leq \sum_{i=1}^{\infty} (\lambda_i - q_i) \sup_{x\in C} \{|f_i(x)|\} \\
&\leq \sum_{i=1}^{\infty} (\lambda_i - q_i) \sup_{k} \sup_{x\in C} \{|f_k(x)|\} \\
&< \varepsilon \sup_{k} P(f_k)
\end{aligned}$$

And since $\varepsilon > 0$ was arbitrary, the set of finite linear combinations of f_i with rational coefficients are P-dense in H. So Lemma 4.2 ensures a subsequence $(f_{i_k})_{k\geq 1}$ of $(f_i)_{i\geq 1}$ such that

$$p(f - G_-) = p(f - G_+)$$

for $f \in H$, and such the hypothesis of *Lemma* 4.1 holds, i.e., that

$$\lim_{k} \lim_{i} f_k(x_i) \neq \lim_{i} \lim_{k} f_k(x_i).$$

Assume without loss of generality that

$$\lim_{k} \lim_{i} f_k(x_i) > \lim_{i} \lim_{k} f_k(x_i).$$

This means that there exist $r > 0$ such that for some fixed $J \geq 1$ we have for all $j \geq J$

$$\lim_{i} f_j(x_i) - \lim_{i} \lim_{k} f_k(x_i) > 2r.$$

And equivalently we get $\lim_i \left(f_j(x_i) - \lim_k f_k(x_i) \right) > 2r$. Hence, for some $I \geq 1$ we have $f_j(x_i) - \lim_k f_k(x_i) > r$ for all $i \geq I$. Using this, we can, without loss of generality, assume that for every $j \geq 1$ and sufficiently large k we have

$$f_j(x_k) - \lim_i f_i(x_k) \geq r$$

for some $r > 0$. Now, let $m \in \mathbb{N}$ and define $K_m := \operatorname{co}\big((f_i)_{i \geq m}\big)$. Note that then we have that $\mathcal{F} \supseteq X^* \supseteq H \supseteq K_1 \supseteq K_2 \supseteq \dots$

Lemma 4.3
For all $f \in K_1$ we have that $p(f - G_-) \geq r$

Proof.
Let $f \in K_1$ be given and let $(x_j)_{j \geq 1} \subset C$ be that of Lemma 4.1, then we have that $f = \sum_{i=1}^k \lambda_i f_{n_i}$ with $\lambda_i \geq 0$ and $\sum_{i=1}^k \lambda_i = 1$. And we see for sufficiently large $j \geq 1$ that

$$\begin{aligned} p(f - G_-) = \sup_{x \in C}\{f(x) - G_-(x)\} &\geq f(x_j) - G_-(x_j) \\ &= \sum_{i=1}^k \lambda_i (f_{n_i}(x_j) - G_-(x_j)) \\ &= \sum_{i=1}^k \lambda_i (f_{n_i}(x_j) - \liminf_m f_m(x_j)) \\ &= \sum_{i=1}^k \lambda_i \underbrace{(f_{n_i}(x_j) - \lim_m f_m(x_j))}_{\geq r} \\ &\geq \sum_{i=1}^k \lambda_i r \\ &= r, \end{aligned}$$

as wanted. □

Lemma 4.4
Let X be a Banach space, and $\alpha, \beta, \xi > 0$. Let $A \subset X$ be any convex subset, and let $u \in A$. Let $p\colon X \to \mathbb{K}$ be any sublinear functional. Suppose that

$$\inf_{a \in A} p(u + \beta a) > \beta\alpha + p(u).$$

Then there is some point $a_0 \in A$ such that

$$\inf_{b \in A} p(u + \beta a_0 + \xi b) > \xi\alpha + p(u + \beta a_0).$$

Proof.
Let $x, y \in A$ and define $c(x,y) := \frac{\beta x+\xi y}{\beta+\xi} = \frac{\beta}{\beta+\xi}x + \frac{\xi}{\beta+\xi}y \in A$ since $\frac{\beta}{\beta+\xi} + \frac{\xi}{\beta+\xi} = 1$. Moreover, we also have that for $u \in A$,

$$\begin{aligned} u + \beta x + \xi y &= u + c(x,y) \\ &= (1+\frac{\xi}{\beta})(u+\beta c(x,y)) - \frac{\xi}{\beta}u. \end{aligned}$$

Now, let $u \in A$, and assume that we have $\inf_{a\in A} p(u+a\beta) > p(u) + \beta\alpha$, then there is some $\delta > 0$ such that

$$-p(u) = \beta\alpha - \inf_{a\in A} p(u+\beta a) + \delta, \tag{$\star$}$$

yielding the following

$$\begin{aligned} p(u+\beta x+\xi y) &= p(u + c(x,y)(\beta+\xi)) \\ &= p\left(\left(1+\frac{\xi}{\beta}\right)(u+\beta c(x,y)) - \frac{\xi}{\beta}u\right) \\ &\geq \left(1+\frac{\xi}{\beta}\right)p(u+\beta c(x,y)) - \frac{\xi}{\beta}p(u). \end{aligned} \tag{$\dagger$}$$

Now, for fixed $a_0 \in A$ we combine the above to achieve, where $c(a_0, b) = \frac{\beta a_0+\xi b}{\beta+\xi} \in A$

$$\begin{aligned} \inf_{b\in A} p(u+\beta a_0+\xi b) &\overset{\text{by}(\dagger)}{\geq} \left(1+\frac{\xi}{\beta}\right)\inf_{b\in A} p(u+\beta c(a_0,b)) - \frac{\xi}{\beta}p(u) \\ &\geq \left(1+\frac{\xi}{\beta}\right)\inf_{a\in A} p(u+\beta a) - \frac{\xi}{\beta}p(u) \\ &\overset{\text{by}\star}{=} \left(1+\frac{\xi}{\beta}\right)\inf_{a\in A} p(u+\beta a) + \frac{\xi}{\beta}(\beta\alpha - \inf_{a\in A} p(u+a\beta) + \delta) \\ &= \inf_{a\in A} p(u+\beta a) + \frac{\xi}{\beta}\inf_{a\in A} p(u+\beta a) + \xi\alpha + \frac{\xi\delta}{\beta} - \frac{\xi}{\beta}\inf_{a\in A} p(u+\beta a) \\ &= \inf_{a\in A} p(u+\beta a) + \xi\alpha + \frac{\xi\delta}{\beta}. \end{aligned}$$

Now, it is possible to pick $a_0 \in A$ such that

$$p(u+a_0\beta) < \inf_{a\in A} p(u+\beta a) + \frac{\xi\delta}{\beta}, \tag{$\heartsuit$}$$

since $\frac{\xi\delta}{\beta}$ is just some fixed positive constant at this point. And thus, picking $a_0 \in A$ such that ($\heartsuit$) holds, we get that

$$\begin{aligned} \inf_{b\in A} p(u+\beta a_0+\xi b) &\geq \inf_{a\in A} p(u+\beta a) + \xi\alpha + \frac{\xi\delta}{\beta} \\ &\overset{(\text{by}\heartsuit)}{>} p(u+a_0\beta) + \xi\alpha, \end{aligned}$$

as wanted $\square$

Recall that we previously defined $p\colon \mathcal{F} \to \mathbb{R}$ as $p(f) = \sup\limits_{x \in C} f(x),\ \ f \in \mathcal{F}$. Another definition we will need is that of K_n. We defined $K_n = \operatorname{co}(f_n, f_{n+1}, \dots)$, with $(f_n)_{n\geq 1} \subset \mathcal{F}$ being the sequence of Lemma 4.1, with the additional properties obtained after Lemma 4.2.

Lemma 4.5
If $(\beta_n)_{n\geq 1} \subset \mathbb{R}_+\backslash\{0\}$ is an arbitrary sequence of positive real numbers, then there is a sequence $(g_n)_{n\geq 1} \subset \mathcal{F}$ such that for all $n \geq 1$ one has $g_n \in K_n$ and

$$p\left(\sum_{i=1}^{n} \beta_i(g_i - G_-)\right) > \frac{1}{2}\beta_n r + p\left(\sum_{i=1}^{n-1} \beta_i(g_i - G_-)\right)$$

with $r > 0$ as in Lemma 4.3 and G_- as before.

Proof.
This will be proven using induction. As in Lemma 4.4, let now $u = 0 \in \mathcal{F}$, and set $\beta_1 = \beta$ and $\beta_2 = \xi$ and let $A := K_1 - G_- = \{f - G_- : f \in K_1\}$. We have that

$$\begin{aligned}\inf_{f\in A} p(u + \beta_1 f) &= \inf_{f\in K_1} p(\beta_1(f - G_-))\\ &\geq \beta_1 r > \frac{1}{2}\beta_1 r + \underbrace{p(u)}_{=0},\end{aligned}$$

by Lemma 4.3. Thus by Lemma 4.4 we have that there is $g_1 \in K_1$ such that

$$\inf_{g\in K_1} p(\beta_1(g_1 - G_-) + \beta_2(g - G_-)) > \frac{1}{2}\beta_2 r + p(\beta_1(g_1 - G_-))$$

For $n \geq 2$, in order to use Lemma 4.4, let $u := \sum_{i=1}^{n-1} \beta_i(g_i - G_-)$, $\beta = \beta_n$, $\xi = \beta_{n+1}$ and let $A := K_n - G_-$. Recall that $\mathcal{F} \supseteq X^* \supseteq H \supseteq K_1 \supseteq K_2 \supseteq \dots$, so in particular we get, by our inductive hypothesis and the fact that $K_{n-1} \supseteq K_n$ that

$$\inf_{f\in A} p(u + \beta_n f) \geq \inf_{f\in K_{n-1}-G_-} p(u + \beta_n f) > \frac{1}{2}\beta_n r + p(u).$$

Thus Lemma 4.4 gives the existence of $g_n \in K_n$ such that if we define

$$v := u + \beta_n(g_n - G_-) = \sum_{j=1}^{n-1} \beta_j(g_j - G_-) + \beta_n(g_n - G_-) = \sum_{j=1}^{n} \beta_j(g_j - G_-),$$

we get that

$$\inf_{f\in A} p(v + \beta_{n+1} f) > \frac{1}{2}\beta_{n+1} r + p(v).$$

Which means that there is $g_{n+1} \in K_{n+1} \subset K_n$ such that $p(v + \beta_{n+1}(g_{n+1} - G_-)) > \inf_{f \in A} p(v + \beta_{n+1} f)$. This gives us that

$$p\left(v + \beta_{n+1}(g_{n+1} - G_-)\right) = p\left(\sum_{j=1}^{n+1} \beta_j(g_j - G_-)\right) > \frac{1}{2}\beta_{n+1} r + p\left(\sum_{j=1}^{n} \beta_j(g_j - G_-)\right),$$

as wanted. □

Lemma 4.6
There is a linear functional on X, $G_0 \in X^*$, for which

1. $\liminf g_n(x) \leq G_0(x) \qquad (x \in X)$.
2. $p(h - G_0) = p(h - G_-) \quad (h \in H)$.

Proof.
Since K_1 is the convex hull of the equicontinuous sequence $(f_i)_{i\geq 1}$, it is by Lemma 3.2 contained in a weak* compact set. So the weak*-closure of K_1 in X^* is weak*-compact. Hence the sequence $(g_n)_{n\geq 1}$ is contained in a weak*-compact set and has a weak* clusterpoint. The **claim** is that this clusterpoint has the properties listed in the lemma, and we call it G_0 from here onward.

Proof of claim: For $x \in X$, we have that $G_0(x)$ is a clusterpoint of the sequence $(g_n(x))_{n\geq 1}$; in particular $\liminf g_n(x) \leq G_0(x) \leq \limsup g_n(x)$, which is exactly the first property G_0 should have.

As $g_n \in K_n = \mathrm{co}(\{f_n, f_{n+1} ...\})$ we can write $g_n(x) = \sum \lambda_i f_{m_i}(x)$, with all $m_i \geq n$. Since this is a convex combination it must be true that for some i we have $f_{m_i}(x) \leq g_n(x)$ for each $x \in C$. For fixed $x \in C$ this process can be repeated for any given n, i.e., for all n we can choose $m \geq n$ such that $f_m(x) \leq g_n(x)$, so we conclude that $G_- = \liminf f_n(x) \leq \liminf g_n(x)$. Similarly we can, for all n, choose m such that $f_m(x) \geq g_n(x)$, yielding the conclusion that $G_+ = \limsup f_n(x) \geq \limsup g_n(x)$. Collecting these inequalities we get that $G_- \leq G_0 \leq G_+$. Consequently we have $p(h - G_-) \geq p(h - G_0) \geq p(h - G_+)$ for all $h \in H$. Recall now that $p(h - G_-) = p(h - G_+)$, and so $p(h - G_-) = p(h - G_0) = p(h - G_+)$, which was the second property we wished for.

□

Corollary 4.7
If we let $(\beta_n)_{n\geq 1} \subset \mathbb{R}_+ \backslash \{0\}$ be any arbitrary sequence of positive real numbers, then there is a sequence $(g_n)_{n\geq 1} \subset \mathcal{F}$ such that for all $n \geq 1$ one has $g_n \in K_n$ and

$$p\left(\sum_{i=1}^{n} \beta_i(g_i - G_0)\right) > \frac{1}{2}\beta_n r + p\left(\sum_{i=1}^{n-1} \beta_i(g_i - G_0)\right)$$

with the $r > 0$ as in Lemma 4.3, i.e., the conclusion of Lemma 4.5 holds with G_- replaced by G_0

Proof.
Fix $n \geq 1$, then we see that

$$\begin{aligned} p\left(\sum_{j=1}^{n} \beta_j (g_j - G_0)\right) &= \sum_{j=1}^{n} \beta_j p\left(\frac{1}{\sum_{j=1}^{n} \beta_j}\left(\sum_{j=1}^{n} \beta_j g_j - \sum_{j=1}^{n} \beta_j G_0\right)\right) \\ &= \sum_{j=1}^{n} \beta_j p\left(\left(\frac{1}{\sum_{j=1}^{n} \beta_j} \sum_{j=1}^{n} \beta_j g_j\right) - G_0\right) \\ \text{(By Lemma 4.6 part (ii))} &= \sum_{j=1}^{n} \beta_j p\left(\left(\frac{1}{\sum_{j=1}^{n} \beta_j} \sum_{j=1}^{n} \beta_j g_j\right) - G_-\right) \\ &= p\left(\sum_{j=1}^{n} \beta_j (g_j - G_-)\right), \end{aligned}$$

as wanted. □

Lemma 4.8
If the sequence $(\beta_n)_{n \geq 1}$ of strictly positive numbers decreases sufficiently fast, i.e., if

$$\lim_{n\to\infty} \frac{\sum_{j=n+1}^{\infty} \beta_j}{\beta_n} = 0,$$

then the series

$$\sum_{i=1}^{\infty} \beta_i (g_i - G_0)$$

with $(g_n)_{n \geq 1}$ as in Corollary 4.7 defines a functional $g \in X^*$ which does not attain its supremum on C.

Proof.
To start with, assume that $\sum_{j=1}^{\infty} \beta_j < \infty$. By construction, we have that K_1 is equicontinuous, and thus $K_1 - G_0$ is equicontinuous, as it is just a translation of K_1. This gives a neighborhood U of $0 \in X$ such that for $x \in U$ and for $f \in K_1 - G_0$ we have

$$|f(x)| \leq 1.$$

For all $j \geq 1$ we have that $g_j \in K_j$ by construction, and thus $g_j - G_0 \in K_j - G_0$, so for $x \in U$ we have

$$|g_j(x) - G_0(x)| \leq 1.$$

Thus, for $x \in U$ we have

$$\sum_{j=1}^{\infty} \beta_j \left(g_j(x) - G_0(x)\right) \leq \sum_{j=1}^{\infty} \beta_j.$$

So $g := \sum_{i=1}^{\infty} \beta_i \left(g_i - G_0\right)$ is well defined and continuous on X, since X^* is weak*-closed, i.e., we have that $g \in X^*$. Now, since $K_1 - G_0$ is equicontinuous, there is $M \geq 0$ such that if $x \in C$ and $f \in K_1 - G_0$ then

$$|f(x)| \leq M. \tag{4.1}$$

When we defined p we saw that $-p(f-g) \leq p(g) - p(f)$ and as a consequence $p(f) - p(f-g) \leq p(g)$, which yields, with g defined as above and $f := \sum_{i=1}^{n} \beta_i(g_i - G_0)$, that

$$p(g) \geq p\left(\sum_{i=1}^{n} \beta_i(g_i - G_0)\right) - p\left(\sum_{i=1}^{n} \beta_i(g_i - G_0) - \sum_{i=1}^{\infty} \beta_i(g_i - G_0)\right). \tag{$\star$}$$

Now, suppose towards a contradiction that g attains its supremum on C at some point $u \in C$. Then, for all $n \geq 1$ we have that

$$\begin{aligned}
\sum_{i=1}^{n} \beta_i(g_i - G_0)(u) &= \sum_{i=1}^{\infty} \beta_i(g_i - G_0)(u) - \sum_{i=n+1}^{\infty} \beta_i(\underbrace{g_i - G_0}_{\in K_1 - G_0})(u) \\
\text{(By (4.1))} \quad &\geqslant g(u) - M \sum_{i=n+1}^{\infty} \beta_i \\
&= p(g) - M \sum_{i=n+1}^{\infty} \beta_i \\
\text{(By } \star\text{)} \quad &\geq p\left(\sum_{i=1}^{n} \beta_i(g_i - G_0)\right) - p\left(\sum_{i=1}^{n} \beta_i(g_i - G_0) - g\right) - M \sum_{i=n+1}^{\infty} \beta_i \\
&\geq p\left(\sum_{i=1}^{n} \beta_i(g_i - G_0)\right) - p\left(-\sum_{i=n+1}^{\infty} \beta_i(g_i - G_0)\right) - M \sum_{i=n+1}^{\infty} \beta_i \\
\text{(By (4.1))} \quad &\geq p\left(\sum_{i=1}^{n} \beta_i(g_i - G_0)\right) - 2M \sum_{i=n+1}^{\infty} \beta_i \\
\text{(By Corollary 4.7)} \quad &> \frac{1}{2}\beta_n r + p\left(\sum_{i=1}^{n-1} \beta_i(g_i - G_0)\right) - 2M \sum_{i=n+1}^{\infty} \beta_i \\
&\geq \frac{1}{2}\beta_n r + \sum_{i=1}^{n-1} \beta_i(g_i - G_0)(u) - 2M \sum_{i=n+1}^{\infty} \beta_i.
\end{aligned}$$

And thus, we have that

$$\beta_n(g_n - G_0)(u) = \sum_{i=1}^{n} \beta_i(g_i - G_0)(u) - \sum_{i=1}^{n-1} \beta_i(g_i - G_0)(u) > \frac{1}{2}\beta_n r - 2M \sum_{i=n+1}^{\infty} \beta_i.$$

Dividing both sides by β_n yields

$$(g_n - G_0)(u) > \frac{1}{2}r - \frac{2M\sum_{i=n+1}^{\infty}\beta_i}{\beta_n}$$

If we choose $(\beta_n)_{n\geq 1}$ such that $\frac{\sum_{i=n+1}^{\infty}\beta_i}{\beta_n} \to 0,\ n \to \infty$, for an example $\beta_n = \frac{1}{n!}$, we see that

$$\liminf_n (g_n - G_0)(u) \geq \frac{1}{2}r$$

But this is a contradiction, since $\liminf_n g_n(u) \leq G_0(u)$ by Lemma 4.6. Thus g cannot attain its supremum on C. □

Theorem 4.9
Let X be Banach space over $\mathbb{R}$ and $C \subset X$ a weakly closed bounded subset. Then C is ***weakly compact*** *if and only if given any element f of the dual X^*, there is $x \in C$ such that $f(x) = \sup\{f(y) : y \in C\}$. (Or that all elements of the dual attain their supremum on C)*

Proof.
"$\Rightarrow$" This implication is trivial, for if $f \in X^*$ then f is weakly continuous, and if C is weakly compact, f will attain its supremum on C.

"$\Leftarrow$" Assuming that C is not weakly compact, Lemma 4.8 gives us a linear functional $g \in X^*$ which does not attain its supremum on the closed unit ball of X. This gives us the desired implication, and the theorem is proved. □

The following proposition allows us to generalize the above theorem to Banach spaces over $\mathbb{C}$

Proposition 4.10
Let X be a complex locally convex topological vector space over $\mathbb{C}$. Let $X_{\mathbb{R}}$ be the space viewed as a vector space over $\mathbb{R}$ rather than $\mathbb{C}$. Then the weak topology on X induced by X^* and the weak topology on $X_{\mathbb{R}}$ induced by $X_{\mathbb{R}}^*$ coincide.

Proof.
We show that given any weak basis element for X, there is a weak basis element of $X_{\mathbb{R}}$ which is a subset of it and vice versa. So let $n \in \mathbb{N}$ be given, and $f_1, \ldots, f_n \in X^*$,

and $\varepsilon > 0$ be given. Then $V(0, f_1, \ldots, f_n, \varepsilon) = \{x \in X \big| |f_i(x)| < \varepsilon,\ 1 \le i \le n\}$ is a basis element for the weak topology on X, and we see that

$$\left\{x \in X \Big| |\mathrm{Re} f_i(x)| < \sqrt{\frac{\varepsilon}{2}} \text{ and } |\mathrm{Re} f_i(ix)| < \sqrt{\frac{\varepsilon}{2}},\ 1 \le i \le n\right\} \subset V(0, f_1, \ldots, f_n, \varepsilon).$$

Now, if $g_1, \ldots, g_n \in X^*_{\mathbb{R}}$, $\varepsilon > 0$ are given, then $U(0, g_1, \ldots, g_n, \varepsilon)$ is a weak basis element for the weak topology on $X_{\mathbb{R}}$. Let f_i be defined by $\mathrm{Re} f_i = u_i$ as in Theorem 2.14, then

$$\{x \in X \big| |f_i(x)| < \varepsilon,\ 1 \le i \le n\} \subset U(0, g_1, \ldots, g_n, \varepsilon),$$

finishing the proof. $\square$

Theorem 4.11 (The James theorem)
Let X be Banach space and $C \subset X$ a weakly closed bounded subset. Then C is ***weakly compact*** *if and only if given any element f of the dual X^*, there is $x \in C$ such that $|f(x)| = \sup\{|f(y)| : y \in C\}$. (Or that all elements of the dual attain their absolute supremum on C)*

Proof.
In the following we will by $X_{\mathbb{R}}$ denote X viewed as a real Banach space. If C is weakly compact, then every bounded linear functional $f \in X^*$ attain their absolute supremum on C. Conversely, assume that $f \in X^*$ and that f attain its absolute supremum on C. Since C is bounded by Theorem 2.10, it may be assumed that C is contained in the closed unit ball of X. Now, let

$$B := \bigcap_{f \in X^*} \{x \ :\ x \in X,\ |f(x)| \le \sup\{|f(y)|,\ y \in C\}\}$$

then B is balanced, for given $\alpha \in \mathbb{C}$ with $|\alpha| \le 1$ we have that $|f(\alpha x)| = |\alpha||f(x)| \le |f(x)|$ for $f \in X^*,\ x \in B$. Furthermore, we also have that B is weakly closed, for let $(x_\lambda)_{\lambda \in \Lambda} \subset B$ be a net with weak limit x, we then have that for every $f \in X^*$

$$\begin{aligned} |f(x)| &= |\lim_{\lambda \in \Lambda} f(x_\lambda)| \\ &= \lim_{\lambda \in \Lambda} |f(x_\lambda)| \\ &\le \sup\{|f(y)|,\ y \in C\}, \end{aligned}$$

so $x \in B$. We also have that $C \subset B$, and that B is a subset of the closed unit ball of X. Now, if $x \in B$, then $|f(x)| \le \sup_{y \in C}\{|f(y)|\}$, so $\sup_{x \in B}\{|f(x)|\} \le \sup_{y \in C}\{|f(y)|\}$. But since $C \in B$, we have that whenever $f \in X^*$, $|f|$ attains its supremum on B, infact on C. Now, if X is a Banach space over $\mathbb{C}$, then we have that $f(x) = u(x) - iu(ix)$, with $u := \mathrm{Re}(f)$, by Theorem 2.14 . We obviously have $|u(x)| \le |f(x)|$. So

$$\sup_{x \in B}\{|u(x)|\} \le \sup_{x \in B}\{|f(x)|\}.$$

Now, there is some $\alpha \in \mathbb{C}$ with $|\alpha| = 1$ such that $|f(x)| = \alpha f(x) = u(\alpha x)$. Since B is balanced, we have for $x \in B$ that $\alpha x \in B$. This show that

$$\begin{aligned} |f(x)| &= u(\alpha x) \\ &\leq |u(\alpha x)| \\ &\leq \sup\{|u(y)|,\ y \in B\}, \end{aligned}$$

resulting in

$$\sup\{|f(x)|,\ x \in B\} = \sup\{|\mathrm{Re} f(x)|,\ x \in B\}.$$

Thus we have that whenever $u \in X_{\mathbb{R}}^*$, then $|u|$ attains its supremum on B. So by Theorem 4.9 B is a weakly compact subset of $X_{\mathbb{R}}$, and thus by Theorem 4.9 a weakly compact subset of X. Since C is a weakly closed subset of B, it is also weakly compact. □

We now have a useful tool for telling when a Banach space X is reflexive:

Theorem 4.12
Let X be a Banach space, then X is reflexive if and only if all elements of the dual f attain their supremum on the closed unit ball of X.

Proof.
From Theorem 4.11, we have that the closed unit ball of X is weakly compact if and only if every linear functional $f \in X^*$ attain their supremum on the closed unit ball of X. Theorem 2.8 tells us that a Banach space X is reflexive if and only if the closed unit ball is weakly compact. Hence the theorem is proved. □

CHAPTER 5

VARIOUS APPLICATIONS OF THE JAMES THEOREM

In this chapter we'll prove, using Theorem 4.12, that certain spaces are reflexive.

Definition 5.1
A Banach space X is said to be strictly convex *if to every $x, y \in X$ with $x \neq y$ and $\|x\| = \|y\| = 1$ it holds that $\left\|\frac{x+y}{2}\right\| < 1$*

Definition 5.2
A Banach space X is said to be uniformly convex *if to every $\varepsilon > 0$ there is some $\delta > 0$ such that for $x, y \in X$ with $\|x\| = \|y\| = 1$ and $\|x - y\| \geq \varepsilon$ we have that*

$$\left\|\frac{x+y}{2}\right\| \leq 1 - \delta$$

Note that every uniformly convex Banach space X is also strictly convex.

The above definition of Uniform Convexity in Banach spaces is equivalent to the following

Remark 5.3
A Banach space X is uniformly convex if for every $\varepsilon > 0$ there is some $2 \geq \delta > 0$ such that for every $x, y \in X$ with $\|x\| \leq 1,\ \|y\| \leq 1$ with $\left\|\frac{x+y}{2}\right\| \geq 1 - \delta$ it holds that $\|x - y\| \leq \varepsilon$

Which means that if $\|x + y\| \to 2$ then $\|x - y\| \to 0$

Lemma 5.4
Let X be a uniformly convex Banach space, and $B \subset X$ a closed convex subset. Then there is a unique $y \in B$ such that

$$\|y\| = \inf_{x \in B}\{\|x\|\}$$

Proof.
We first show existence. Let $\xi := \inf_{x\in B}\{\|x\|\}$, and let $(x_n)_{n\geq 1} \subset B$ be a sequence such that $\|x_n\| \to \xi$. We wish to show that $(x_n)_{n\geq 1}$ is Cauchy. For this, note that

$$x_n - x_m = \|x_n\|\left(\frac{x_n}{\|x_n\|} - \frac{x_m}{\|x_m\|}\right) + \frac{x_m}{\|x_m\|}(\|x_n\| - \|x_m\|)$$

So it will be enough to show that $\frac{x_n}{\|x_n\|}$ is Cauchy. Since X is uniformly convex and $\left\|\frac{x_n}{\|x_n\|}\right\| = 1$, it will be enough to show that

$$\left\|\frac{x_n}{\|x_n\|} + \frac{x_m}{\|x_m\|}\right\| \to 2$$

By convexity of B, we have $\frac{1}{2}(x_n + x_m) \in B$, and the definition of ξ and $(x_n)_{n\geq 1}$, we have

$$\xi \leq \left\|\frac{1}{2}(x_n + x_m)\right\| \leq \frac{1}{2}(\|x_n\| + \|x_m\|) \to \xi$$

so $\|\frac{1}{2}(x_n + x_m)\| \to \xi$. We also have that

$$\begin{aligned}\frac{x_n}{\|x_n\|} + \frac{x_m}{\|x_m\|} &= \frac{\|x_m\|x_n + x_m\|x_n\|}{\|x_n\|\|x_m\|} \\ &= \frac{1}{\|x_n\|\|x_m\|}((x_n + x_m)\|x_m\| + x_m(\|x_n\| - \|x_m\|)),\end{aligned}$$

so

$$\begin{aligned}2 &\geq \left\|\frac{x_n}{\|x_n\|} + \frac{x_m}{\|x_m\|}\right\| \\ &= \frac{1}{\|x_n\|\|x_m\|}\,\|(x_n + x_m)\|x_m\| + x_m(\|x_n\| - \|x_m\|)\| \\ &\geq \frac{1}{\|x_n\|\|x_m\|}\,\big(\|x_n + x_m\|\|x_m\| - \|x_m\|(\big|\|x_n\| - \|x_m\|\big|)\big) \\ &= \frac{1}{\|x_n\|}(\|x_n + x_m\|) - \big|\|x_n\| - \|x_m\|\big|) \\ &\overset{n,m\to\infty}{\to} \frac{1}{\xi}(2\xi - |\xi - \xi|) = 2.\end{aligned}$$

Since we are in a uniformely convex space, this means that

$$\left\|\frac{x_n}{\|x_n\|} - \frac{x_m}{\|x_m\|}\right\| \to 0$$

This proves the existence of some element $y \in B$ with $\|y\| = \xi$. The uniqueness comes from the following: assume there are $x, y \in B$ with $x \neq y$ and $\|x\| = \|y\| = \xi$. Since X is strictly convex and B is convex, we have that

$$\xi \leq \left\|\frac{x+y}{2}\right\| \leq \frac{1}{2}(\|x\| + \|y\|) = \xi$$

But we must have that $\frac{1}{2}(\|x + y\|) < \xi$ by the definition of strict convexity of X, a contradiction. $\square$

Lemma 5.5
Let X be a uniformly convex Banach space and assume $(x_n)_{n\geq 1} \subset X$ with $\|x_n\| = 1$ for all $n \geq 1$, $\varphi \in X^*$ with $\|\varphi\| = 1$ and $\varphi(x_n) \to 1$ for $n \to \infty$. Then $(x_n)_{n\geq 1}$ is a Cauchy sequence in the norm topology.

Proof.
To see this, note that

$$\varphi(x_n + x_m) = \varphi(x_n) + \varphi(x_m) \to 2$$

And so we have

$$|\varphi(x_n) + \varphi(x_m)| \leq \underbrace{\|\varphi\|}_{=1}\|x_n + x_m\| \leq \|x_n\| + \|x_m\| = 2$$

and by the above we have that $\|x_n + x_m\| \to 2$ for $n, m \to \infty$. By the definition of uniform convexity, $\|x_n - x_m\| \to 0$ and thus $(x_n)_{n\geq 1}$ is Cauchy. □

Lemma 5.6
If X is a uniformly convex Banach space, then every $\varphi \in X^*$ attain its maximum on the closed unit ball, $\overline{B_x}$, of X.

Proof.
Let X be a uniformly convex Banach space, So, just as in Lemma 5.5, let $\varphi \in X^*$, and assume that $\|\varphi\| = 1$. For each $n \geq 1$ pick x_n with $\|x_n\| = 1$ such that

$$|\varphi(x_n) - \underbrace{1}_{=\|\varphi\|}| < \frac{1}{n}$$

Then $|\varphi(x_n)| \to 1$ as $n \to \infty$. By Lemma 5.5 we have that $(x_n)_{n\geq 1}$ is a Cauchy sequence. Let $x := \lim_n x_n$. Then we have that

$$\begin{aligned} 0 &\leq \big||\varphi(x_n)| - |\varphi(x)|\big| \\ &\leq |\varphi(x_n) - \varphi(x)| \\ &= |\varphi(x_n - x)| \\ &\leq \|\varphi\|\|x_n - x\| \to 0, \ n \to \infty \end{aligned}$$

Hence $\big||\varphi(x_n)| - |\varphi(x)|\big| \to 0$ and therefore $|\varphi(x)| = \lim_n |\varphi(x_n)| = 1$ and $\|x\| = 1$. All of this results in

$$\|\varphi\| = 1 \quad \text{and } |\varphi(x)| = 1$$

It was assumed that $\|\varphi\| = 1$, if this is not the case, one can observe $\frac{\varphi}{\|\varphi\|}$ and use the same argument. □

Theorem 5.7
Let X be a uniformly convex Banach space. Then X is reflexive.

Proof.
By Lemma 5.6, if X is a uniformly convex Banach space, then all functionals $f \in X^*$ attain their supremum on the closed unit ball of X and by Theorem 4.12, X is reflexive. $\square$

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