Simplicty of Crossed Products of C^*-algebras

Masters thesis defense

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Advisor: Søren Eilers

February 15, 2019

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And of course, we have overlaps between the above cases.

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All of these C^*-algebras play a crucial role in the theory of crossed products.

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The Gelfand-Neimark-Segal result

The Gelfand-Neimark-Segal result is an important result stating that all C^*-algebras A are faithfully represented on a Hilbert space.

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An important example is the *left-regular representation* of G:

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The left-regular representation of G is the tuple $(\lambda, L^2(G))$, where

$$\lambda_t \xi(s) = \xi(t^{-1}s),$$

for $\xi \in L^2(G)$ and $s, t \in G$.

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A covariant representation of (A, G, α) is a triple $(\pi, u, \mathcal{H})$ consisting of representations of A and G as bounded operators on a common Hilbert space $\mathcal{H}$ satisfying the relation

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When G is discrete and A is unital, it is easy to associate to such a triple a **nice** new C^*-algebra:

Cooking with C^*-algebras

Discrete crossed product cookbook recipe

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1. Given (A, G, α) with G discrete and A unital, consider the $*$-algebra

$$C_c(G, A) = \left\{ \sum_{t \in F} a_t \delta_t \mid F \subseteq G \text{ finite} \right\}.$$

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This is the building stone for constructing crossed products.

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3. Define a representation $\pi \rtimes u$ of $C_c(G, A)$ as operators on $\mathcal{H}$ by

$$\pi \rtimes u \left(\sum a_t \delta_t \right) := \sum \pi(a_t) u_t.$$

4. Take the closure of $\pi \rtimes u(C_c(G, A))$ to obtain a C^*-algebra encoding some information about the dynamical system (A, G, α).

This is the building stone for constructing crossed products. Fails when G is non-discrete.

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Theorem

Let A be a C^*-algebra. Then there is a unique linear map $\int : C_c(G, M_s(A)) \to M(A)$ such that for all non-degenerate representations $\pi : A \to \mathbb{B}(H)$, $\xi, \eta \in \mathcal{H}$ and $f \in C_c(G, M_s(A))$ we have

$$\langle \overline{\pi} \left(\int (f) \right) \xi, \eta \rangle = \int_G \langle \overline{\pi}(f(s))\xi, \eta \rangle \mathrm{d}s.$$

And we write $\int (f) := \int_G f(s) \mathrm{d}s$.

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$$\pi \rtimes u(f) := \int (\pi(f)u) = \int_G \pi(f(s))u_s \mathrm{d}s.$$

The crossed product construction

Definition

The *full crossed product* of (A, G, α) is the C^*-algebra $A \rtimes_\alpha G$ obtained by taking the closure of $C_c(G, A)$ with respect to the norm

$$\|f\|_u = \sup \{\|\pi \rtimes u(f)\| \mid (\pi, u, \mathcal{H}) \text{ is a (non-degenerate) covariant representation}\}$$

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There is a faithful homomorphism $A \hookrightarrow M(A \rtimes_\alpha G)$ and an injective strictly continuous group homomorphism $U \colon G \to \mathcal{U}(M(A \rtimes_\alpha G))$ such that for $a \in A$, $s, r \in G$ and $f \in C_c(G, A)$

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Makes up for lack of copies inside $A \rtimes_\alpha G$.

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Theorem

If G is amenable (=admits left-invariant mean), then $A \rtimes_{\alpha} G \cong A \rtimes_{\alpha,r} G$

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