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Simplicity of Crossed Products of C^*-algebras

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Abstract

Write abstract

CONTENTS

1 Setting The Stage - Constructions **1**
1.1 Hilbert C^*-modules and Multiplier Algebras 1
1.2 Crossed Products 11

Bibliography **13**

Appendix A **14**
A.1 Integration Theory In Groups 14

Setting The Stage - Constructions

In this chapter we will consider the constructions of certain important objects, e.g., *the Crossed Product* of general locally compact groups acting on a C^*-algebra or the *Dual Action* of an Abelian locally compact group, both of which are at the center of the topic of this thesis. Since the inception of the theory of some of the following constructions, there has been several modifications to the proofs and the generality of the theory covered, and we will cherry pick the most suitable results and variations thereof from both the old and new school of the subjects.

1.1 Hilbert C^*-modules and Multiplier Algebras

Given a non-unital C^*-algebra A, we can extend a lot of the theory of unital C^*-algebras to A by considering the unitalization $A^\dagger = A + 1\mathbb{C}$ with the canonical operations. This is the 'smallest' unitalization in the sense that A is an essential ideal of $A^\dagger$ and $A^\dagger / A \cong \mathbb{C}$. We can add to prelim
also consider a maximal unitalization, called the *Multiplier Algebra* of A, denoted by $M(A)$. We will now construct it and some theory revolving it, in particular that of Hilbert-modules:

Definition 1.1. Let A be any C^*-algebra. An *inner-product A-module* is a complex vector space E which is also a right-A-module which is equipped with a map $\langle \cdot, \cdot \rangle_A \colon E \times A \to A$ such that for $x, y, z \in E,\ \alpha, \beta \in \mathbb{C}$ and $a \in A$ we have

$$\langle a, \alpha y + \beta z \rangle_A = \alpha \langle x, y \rangle_A + \beta x, z \rangle_A \tag{1.1}$$

$$\langle x, y \cdot a \rangle - A = \langle x, y \rangle_A \cdot a \tag{1.2}$$

$$\langle y, x \rangle_A = \langle x, y \rangle_A^* \tag{1.3}$$

$$\langle x, x \rangle_A \geq 0 \tag{1.4}$$

$$\langle x, x \rangle_A = 0 \implies x = 0. \tag{1.5}$$

If $\langle \cdot, \cdot \rangle$ satisfy (1.1)-(1.4) but not (1.5) above, we say that E is a *semi-inner-product A-module*. If there is no confusion, we simply write $\langle \cdot, \cdot \rangle$ to mean the A-valued inner-product on E. Similarly, we will often denote $x \cdot a$ by xa.

It is easy to see that this generalizes the notion of Hilbert spaces, for if $A = \mathbb{C}$, then any Hilbert space (over $\mathbb{C}$) is a inner-product $\mathbb{C}$-module. Even when E is only a semi-inner-product A-module, we still have a lot of the nice properties of regular inner-producte, e.g.:

Proposition 1.2 (Cauchy-Schwarz Inequality)**.** Let E be a semi-inner-product A-module. For $x, y \in E$, it holds that

$$\langle x, y\rangle^* \langle x, y\rangle \leq \|\langle x, x\rangle\|_A \langle y, y\rangle.$$

Proof.
Let x, yE and let $a = \langle x, y\rangle \in A$ and $t > 0$. Then, since $(\langle x, y\rangle a)^* = a^* \langle y, x\rangle$, we have

$$0 \leq \langle xa - ty, xa - ty\rangle = t^2 \langle y, y\rangle + a^* \langle x, x\rangle a - 2ta^* a.$$

For $c \geq 0$ in A, it holds that $a^* ca \leq \|c\|_A a^* a$: Let $A \subseteq \mathbb{B}(\mathcal{H})$ faithfully. The regular Cauchy-Schwarz implies that $\langle a^* ca\xi, \xi\rangle_{\mathcal{H}} \leq \|c\| \langle a^* a\xi, \xi\rangle_{\mathcal{H}}$ for all $\xi \in \mathcal{H}$. With this in mind, we then have

$$0 \leq a^* a \|\langle x, x\rangle\|_A - 2ta^* a + t^2 \langle y, y\rangle.$$

If $\langle x, x\rangle = 0$, then $0 \leq 2a^* a \leq t\langle y, y\rangle \to 0$, as $t \to 0$ so that $a^* a = 0$ as wanted. If $\langle x, x\rangle \neq 0$, then $t = \|\langle x, x\rangle\|_A$ gives the wanted inequality. □

Recall that for $0 \leq a \leq b \in A$, it holds that $\|a\|_A \leq \|b\|_A$. This and the above implies that $\|\langle x, y\rangle\|_A^2 \leq \|\langle x, x\rangle\|_A \|\langle y, y\rangle\|_A$.

Definition 1.3. If E is an (semi-)inner-product A-module, we define a (semi-)norm on E by

$$\|x\| := \|\langle x, x\rangle\|_A^{\frac{1}{2}}, \text{ for } x \in E.$$

If E is an inner-product A-module such that this norm is complete, we say that E is a *Hilbert C^*-module over A* or just a *Hilbert A-module.*

If the above defined norm is but a semi-norm, then we may form the quotient space of E with respect to $N = \{x \in E \mid \langle x, x\rangle = 0\}$ to obtain an inner-product A-module with the obvious operations. Moreover, the usual ways of passing from pre-Hilbert spaces extend naturally to pre-Hilbert A-modules. For $x \in E$ and $a \in A$, we see that

$$\|xa\|^2 = \left\| a^* \underbrace{\langle x, x\rangle}_{\geq 0} a \right\| \leq \|a\|^2 \|x\|^2,$$

so that $\|xa\| \leq \|x\| \|a\|$, which shows that E is a normed A-module. For $a \in A$, let $R_a \colon x \mapsto xa \in E$. Then R_a is a linear map on E satisfying $\|R_a\| = \sup_{\|x\|=1} \|xa\| \leq \|a\|$, and the map $R \colon a \mapsto R_a$ is a contractive anti-homomorphism of A into the space of bounded linear maps on E (anti in the sense that $R_{ab} = R_b R_a$ for $a, b \in A$).

Example 1.4
The classic example of a Hilbert A-module is A itself with $\langle a, b\rangle = a^* b$, for $a, b \in A$. We will write A_A to specify that we consider A as a Hilbert A-module.

In the following, we assume that E is a Hilbert A-module.

Remark
Let $I_0(E) = \{\langle x, y\rangle \mid x, y \in E\}$, and let $\langle E, E\rangle := \overline{I_0(E)}$. Then $\langle E, E\rangle$ is a closed ideal in A.

And this gives us the following useful lemma:

Lemma 1.5. The set $E\langle E, E\rangle$ is dense in E.

Proof.
Let (u_i) denote an approximate unit for $\langle E, E\rangle$. Then clearly $\langle x - xu_i, x - xu_i\rangle \to 0$ for $x \in E$ (seen by expanding the expression), which implies that $xu_i \to x$ in E □

Definition 1.6. If $\langle E, E\rangle$ is dense in A we say that the Hilbert A-module is *full*.

closedness under direct sum and $\mathcal{H}_A = \mathcal{H} \otimes A$

Note
There are similarities and differences between Hilbert A-modules and regular Hilbert spaces. One major difference is that for closed submodules $F \subseteq E$, it does not hold that $F \oplus F^\perp = E$, where $F^\perp = \{y \in E \mid \langle x, y\rangle = 0,\ x \in F\}$: Let $A = C([0,1])$ and consider A_A. Let $F \subseteq A$ be the set $F = \{f \in A_A \mid f(0) = f(1) = 0\}$. Then F is clearly a closed sub A-module of A_A. Consider the element $f \in F$ given by

$$f(t) = \begin{cases} t & t \in [0, \frac{1}{2}] \\ 1 - t & t \in (\frac{1}{2}, 1] \end{cases}$$

For all $g \in F^\perp$ it holds that $\langle f, g\rangle = fg = 0$, so $g(t) = 0$ for $t \in (0,1)$ and by uniform continuity on all of $[0,1]$, from which we deduce that $F^\perp = \{0\}$ showing that $A \neq F \oplus F^\perp$. The number of important results in the theory of Hilbert spaces relying on the fact that $\mathcal{H} = F^\perp \oplus F$ for closed subspaces F of $\mathcal{H}$ is enormous, and hence we do not have the entirety of our usual toolbox available.

However, returning to the general case, we are pleased to see that it does hold for Hilbert A-modules E that for $x \in E$ we have

$$\|x\| = \sup\{\|\langle x, y\rangle\| \mid y \in E,\ \|y\| \leq 1\}.$$

This is an obvious consequence of the Cauchy-Schwarz theorem for Hilbert A-modules.

Another huge difference is that adjoints don't automatically exist: Let $A = C([0,1])$ again and F as before. Let $\iota\colon F \to A$ be the inclusion map. Clearly ι is a linear map from F to A, however, if it had an adjoint $\iota^*\colon A \to F$, then for any self-adjoint $f \in F$, we have

$$f = \langle \iota(f), 1\rangle_A = \langle f, \iota^*(1)\rangle = f\iota^*(1),$$

so $\iota^*(1) = 1$, but $1 \notin F$. Hence ι^* is not adjointable.

The set of adjointable maps between Hilbert A-modules is of great importance and has many nice properties.

Definition 1.7. For Hilbert A-modules E_i with A-valued inner-products $\langle \cdot, \cdot \rangle_i$, $i = 1, 2$, we define $\mathcal{L}_A(E_1, E_2)$ to be the set of all maps $t\colon E_1 \to E_2$ for which there exists a map $t^*\colon E_2 \to E_1$ such that

$$\langle tx, y \rangle_2 = \langle x, t^* y \rangle_1,$$

for all $x \in E_1$ and $y \in E_2$, i.e., *adjointable maps*. When no confusion may occur, we will write $\mathcal{L}(E_1, E_2)$ and ommit the subscript.

Proposition 1.8. if $t \in \mathcal{L}(E_1, E_2)$, then

1. t is A-linear and

2. t is a bounded map.

Proof.
Linearity is clear. To see A-linearity, note that for $x_1, x_2 \in E_i$ and $a \in A$ we have $a^* \langle x_1, x_2 \rangle_i = \langle x_1 a, x_2 \rangle_i$, and hence for $x \in E_1$, $y \in E_2$ and $a \in A$ we have $\langle t(xa), y \rangle_2 = a^* \langle x, t^*(y) \rangle_1 = \langle t(x)a, y \rangle_1$.

To see that t is bounded, we note that T has closed graph: Let $x_n \to x$ in E_1 and assume that $tx_n \to z \in E_2$. Then

$$\langle tx_n, y \rangle_2 = \langle x_n, t^* y \rangle_1 \to \langle x, t^* y \rangle_1 = \langle tx, y \rangle_2$$

so $tx = z$, and hence t is bounded, by the Closed Graph Theorem (CGT hereonforth). □

In particular, similar to the way that the algebra $\mathbb{B}(\mathcal{H}) = \mathcal{L}_{\mathbb{C}}(\mathcal{H}_{\mathbb{C}}, \mathcal{H}_{\mathbb{C}})$ is of great importance, we examine the for a fixed Hilbert A-module E, the $*$-subalgebra $\mathcal{L}(E) := \mathcal{L}(E, E)$ of bounded operators on E:

Proposition 1.9. The algebra $\mathcal{L}(E)$ is a closed $*$-subalgebra of $\mathbb{B}(E)$ such that for $t \in \mathcal{L}(E)$ we have

$$\|t^* t\| = \|t\|^2,$$

hence is it a C^*-algebra.

Proof.
Since $\mathbb{B}(E)$ is a Banach algebra, we have $\|t^* t\| \leq \|t^*\| \|t\|$ for all $t \in \mathcal{L}(E)$. Moreover, by Cauchy-Schwarz, we have

$$\|t^* t\| = \sup_{\|x\| \leq 1} \{\|t^* tx\|\} = \sup_{\|x\|, \|y\| \leq 1} \{\langle tx, ty \rangle\} \geq \sup_{\|x\| \leq 1} \{\|\langle tx, tx \rangle\|\} = \|t\|^2,$$

so $\|t^* t\| = \|t\|^2$. Assume that $t \neq 0$, then we see that $\|t\| = \|t^* t\| \|t\|^{-1} \leq \|t^*\|$ and similarly $\|t^*\| \leq \|t^{**}\| = \|t\|$, so $\|t\| = \|t^*\|$, in particular $t \mapsto t^*$ is continuous. It is clear that $\mathcal{L}(E)$ is a $*$-subalgebra. To see that is is closed, suppose that (t_α) is a Cauchy-net, and denote its limit by $t \in \mathbb{B}(E)$. By the above, we see that (t_α^*) is Cauchy with limit $f \in \mathbb{B}(E)$, since Then, we see that for all $x, y \in E$

$$\langle tx, y\rangle = \lim_\alpha \langle t_\alpha x, y\rangle = \lim_\alpha x, t_\alpha^* y\rangle = \langle x, fy\rangle,$$

so t is adjointable, hence an element of $\mathcal{L}(E)$. □

If we for $x \in E$ denote the element $\langle x, x\rangle^{\frac{1}{2}} \geq 0$ by $|x|_A$, then we obtain a variant of a classic and important inequality:

Lemma 1.10. Let $t \in \mathcal{L}(E)$, then $|tx|^2 \leq \|t\|^2 |x|^2$ for all $x \in E$.

Proof.
The operator $\|t\|^2 - t^*t \geq 0$, since $\mathcal{L}(E)$ is a C^*-algebra, and thus equal to s^*s for some $s \in \mathcal{L}(E)$. Hence

$$\|t\|^2 \langle x, x\rangle - \langle t^*tx, x\rangle = \langle (\|t\|^2 - t^*t)x, x\rangle = \langle sx, sx\rangle \geq 0,$$

for all $x \in E$, as wanted. □

A quite useful result about positive operators on a Hilbert space carries over to the case of Hilbert C^*-modules, for we have:

Lemma 1.11. Let E be a Hilbert A-module and let T be a linear operator on E. Then T is a positive operator in $\mathcal{L}_A(E)$ if and only if for all $x \in E$ it holds that $\langle Tx, x\rangle \geq 0$ in A.

Proof.
If T is adjointable and positive, then $T = S^*S$ for some $S \in \mathcal{L}(E)$ and $\langle Tx, x\rangle = \langle Sx, Sx\rangle \geq 0$. For the converse, we note first that the usual polarization identity applies to Hilbert C^*-modules as well:

$$4\langle x, y\rangle = \sum_{k=0}^{3} i^k \langle u + i^k v, u + i^k v\rangle,$$

for all $u, v \in E$, since $\langle u, v\rangle = \langle v, u\rangle^*$. Hence, if $\langle Tx, x\rangle \geq 0$ for all $x \in E$, we see that $\langle Tx, x\rangle = \langle x, Tx\rangle$, so

$$4\langle Tx, y\rangle = \sum_{k=0}^{3} i^k \underbrace{\langle T(x + i^k y), x + i^k y\rangle}_{=\langle x + i^k y, T(x + i^k y)\rangle} = 4\langle x, Ty\rangle, \text{ for } x, y \in E.$$

Thus $T = T^*$ is an adjoint of T and $T \in \mathcal{L}_A(E)$. An application of the continuous functional calculus allows us to write $T = T^+ - T^-$ for positive operators $T^+, T^- \in \mathcal{L}_A(E)$ with $T^+T^- = T^-T^+ = 0$, see e.g., Zhu 1993, Theorem 11.2. Then, for all $x \in E$ we have

$$0 \leq \langle TT^- x, x\rangle = \langle -(T^-)^2 x, T^- x\rangle = -\underbrace{\langle (T^-)^3 x, x\rangle}_{\geq 0} \leq 0,$$

so $\langle (T^-)^3 x, x\rangle = 0$ for all x. The polarization identity shows that $(T^-)^3 = 0$ and hence $(T^-) = 0$, so $T = T^+ \geq 0$. □

And, as in the Hilbert Space situation, we also have

Lemma 1.12. Let E be a Hilbert A-module and $T \geq 0$ in $\mathcal{L}_A(E)$. Then

$$\|T\| = \sup_{\|x\|\leq 1} \{\|\langle Tx, x\rangle\|\}$$

Proof.
Write $T = S^*S$ for $S \in \mathcal{L}_A(E)$ to obtain the identity. □

It is well-known that the only ideal of $\mathbb{B}(\mathcal{H})$ is the set of compact operators, $\mathbb{K}(\mathcal{H})$, which is the norm closure of the finite-rank operators. We now examine the Hilbert C^*-module analogue of these two subsets:

Definition 1.13. Let E, F be Hilbert A-modules. Given $x \in E$ and $y \in F$, we let $\Theta_{x,y} \colon F \to E$ be the map

$$\Theta_{x,y}(z) := x\langle y, z\rangle,$$

for $z \in F$. It is easy to see that $\Theta_{x,y} \in \mathbb{B}(F, E)$.

Proposition 1.14. Let E, F be Hilbert A-modules. Then for every $x \in E$ and $y \in F$, it holds that $\Theta_{x,y} \in \mathcal{L}(F, E)$. Moreover, if H is another Hilbert A-module, then

1. $\Theta_{x,y}\Theta_{u,v} = \Theta_{x\langle y,u\rangle_F, v} = \Theta_{x, v\langle u,v\rangle_F}$, for $u \in F$ and $v \in G$,
2. $T\Theta_{x,y} = \Theta_{tx,y}$, for $T \in \mathcal{L}(E, G)$,
3. $\Theta_{x,y}S = \Theta_{x,S^*y}$ for $S \in \mathcal{L}(G, F)$.

Proof.
Let $x, w \in E$ and $y, u \in F$ for Hilbert A-modules E, F as above. Then, we see that

$$\langle \theta_{x,y}(u), w\rangle_E = \langle x\langle y, u\rangle_F, w\rangle_E = \langle u, y\rangle_F\langle x, w\rangle_E = \langle u, y\langle x, w\rangle_E\rangle_F,$$

so $\Theta^*_{x,y} = \Theta_{y,x} \in \mathcal{L}(E, F)$. Suppose that G is another Hilbert A-module and $v \in G$ is arbitrary, then

$$\Theta_{x,y}\Theta_{u,v}(g) = x\langle y, u\langle v, g\rangle_G\rangle_F = x\langle y, u\rangle_F\langle v, g\rangle = \Theta_{x\langle y,u\rangle_F, v}(g),$$

for $g \in G$. The other identity in (1) follows analogously. Let $T \in \mathcal{L}(E, G)$, then

$$T\Theta_{x,y}(u) = Tx\langle y, u\rangle_F = \Theta_{Tx,y}(u),$$

so (2) holds and (3) follows analogously. □

Definition 1.15. Let E, F be Hilbert A-modules. We define the set of compact operators $F \to E$ to be the set

$$\mathbb{K}_A(F, E) := \overline{\operatorname{Span}\{\Theta_{x,y} \mid x \in E,\ y \in F\}} \subseteq \mathcal{L}_A(F, E).$$

It follows that for $E = F$, the $*$-subalgebra of $\mathcal{L}_A(E)$ obtained by taking the span of $\{\Theta_{x,y} \mid x, y \in E\}$ forms a two-sided algebraic ideal of $\mathcal{L}_A(E)$, and hence its closure is a C^*-algebra:

Definition 1.16. For a Hilbert A-module E, we define the compact operators on E to be the C^*-algebra

$$\mathbb{K}_A(E) := \overline{\operatorname{Span}}\,\{\Theta_{x,y} \mid x, y \in E\} \subseteq \mathcal{L}_A(E).$$

Note
An operator $t \in \mathbb{K}_A(F, E)$ need not be a compact operator when E, F are considered as Banach spaces, i.e., a limit of finite-rank operators. This is easy to see by assuming that A is unital: The operator $\mathrm{id}_A = \Theta_{1,1}$ is in $\mathbb{K}_A(A_A)$, but the identity operator is not a compact operator on an infinite-dimensional Banach space.

Lemma 1.17. There is an isomorphism $A \cong \mathbb{K}_A(A_A)$. If A is unital, then $\mathbb{K}_A(A_A) = \mathcal{L}_A(A_A)$.

Proof.
Fix $a \in A$ and define $L_a \colon A \to A$ by $L_a(b) = ab$. Then $\langle L_a x, y\rangle = x^* a^* y = \langle x, L_{a^*} y\rangle$ for all $x, y \in A$, hence L_a is adjointable with adjoint L_{a^*}. Note that $L_{ab^*}(x) = ab^* x = a\langle b, x\rangle = \Theta_{a,b}(x)$ for $a, b, x \in A$, hence if $L \colon a \mapsto L_a$, we have $\mathbb{K}_A(A_A) \subseteq \operatorname{Im}(L)$. It is clear that L is a $*$-homomorphism of norm one, i.e., an isometric homomorphism, so it's image is a C^*-algebra. To see that it has image equal to $\mathbb{K}_A(A_A)$, let (u_i) be an approximate identity of A. Then for any $a \in A$ $L_{u_i a} \to L_a$, and since $L_{u_i a} \in \mathbb{K}_A(A_A)$ for all i, we see that $L_a \in \mathbb{K}_A(A_A)$.

If $1 \in A$, then for $t \in \mathcal{L}_A(A_A)$ we see that $t(a) = t(1 \cdot a) = t(1) \cdot a = L_{t(1)} a$, so $t \in \mathcal{K}_A(A_A)$, and $\mathcal{L}_A(A_A) \cong A$. ☐

We note that it does not hold in general that $A \cong \mathcal{L}_A(A_A)$.

We empose quite an assortment of topologies on $\mathbb{B}(\mathcal{H}, \mathcal{H}')$ for Hilbert spaces $\mathcal{H}$ and $\mathcal{H}'$ – this is not the case when we study Hilbert C^*-modules: Given Hilbert A-module E and F, we only consider two topologies – the norm topology and the *strict topology*:

Definition 1.18. The *strict topology* on $\mathcal{L}_A(E, F)$ is the locally convex topology on $\mathcal{L}_A(E, F)$ generated by the family of semi-norms

$$t \mapsto \|tx\|\,, \text{ for } x \in E \text{ and } t \mapsto \|t^* y\|\,, \text{ for } y \in F.$$

This topology turns out to be very important, and there is many results of the classic theory which apply in this setting as well:

Proposition 1.19. Let E be a Hilbert A-module. Then the unit ball of $\mathbb{K}_A(E, F)$, $(\mathbb{K}_A(E, F))_1$, is strictly dense in the unit ball, $(\mathcal{L}_A(E, F))_1$, of $\mathcal{L}_A(E, F)$.

Proof.
Let (e_i) be an approximate identity of the C^*-algebra $\mathbb{K}_A(E)$. Then, for all $w\langle x, y\rangle \in E\langle E, E\rangle$ we see that

$$e_i w\langle x, y\rangle = e_i \Theta_{w,x}(y) \to \Theta_{w,x}(y) = w\langle x, y\rangle,$$

and hence $e_i x \to x$ for all $x \in E$, since $E\langle E, E\rangle \subseteq E$ is dense by Lemma 1.5. Then we clearly have, for $t \in (\mathcal{L}_A(E, F))_1$ that

$$\|t(e_i x - x)\| \to 0 \text{ and } \|e_i t^* y - t^* y\| \to 0,$$

for $x \in E$ and $y \in F$, so $te_i \to t$ strictly, and by 1.14 we see that $te_i \in (\mathbb{K}_A(E, F))_1$, finishing the proof. $\square$

We are almost ready to introduce the Multiplier Algebra of a C^*-algebra A, however, we must introduce some new terminology first

Definition 1.20. An ideal I of a C^*-algebra A is essential if it is non-zero and satisfies $I \cap J \neq 0$ for all non-zero ideals J of A.

And there is another commonly used equivalent formulation of the definition of essential ideals:

Lemma 1.21. Let I, J be non-zero ideals of a C^*-algebra A. Then the following holds:

1. $I^2 = I$,
2. $I \cap J = IJ$,
3. I is essential if and only if $aI = 0$ implies $a = 0$ for all $a \in A$.

Proof.
(1): It holds that $I^2 \subseteq I$, since I is an ideal. For the other inclusion, note that if $a \geq 0$ is an element of I, then $a = (a^{\frac{1}{2}})^2 \in I^2$, so $I_+ \subseteq I^2$ hence $I = I^2$.

(2): Note that $I \cap J = (I \cap J)^2$ by (1), and $(I \cap J)(I \cap J) \subseteq IJ \subseteq I \cap J$.

(3): Assume that I is essential and $aI = 0$. Then $(a)I = (a) \cap I = 0$, where (a) is the ideal generated by a in A, and hence $(a) = 0$ so $a = 0$. For the converse, let $J \neq 0$ be any ideal of A and let $0 \neq a \in J$. By assumption this implies that $aI \neq 0$, so $(a)I = (a) \cap I \neq 0$, which shows that $J \cap I \neq 0$. $\square$

As mentioned earlier, it is a useful tool to be able to adjoin units to a non-unital C^*-algebra A. We formalize this process in the following definition:

Definition 1.22. A *unitalization* of a C^*-algebra A is a pair (B, ι) consisting of a unital C^*-algebra B and an injective homomorphism $\iota \colon A \hookrightarrow B$ such that $\iota(A)$ is an essential ideal of B.

The most common way of adjoining a unit to a non-unital C^*-algebra A is to consider the vector space $A \oplus \mathbb{C}$ and endow it with a proper $*$-algebra structure and give it a certain C^*-norm, see e.g., Zhu 1993, Theorem 15.1. This can also be done via the use of compact operators on A:

Example 1.23
Let $A^1 := A \oplus \mathbb{C}$ as a $*$-algebra with $(x,\alpha)(y,\beta) := xy + \alpha y + \beta x + \alpha\beta$ and entrywise addition and involution. Identify A with $\mathbb{K}_A(A_A)$ inside $\mathcal{L}_A(A_A)$ via the left-multipication operator, L, on A, and consider the algebra generated by 1 and $\mathbb{K}_A(A_A)$ in $\mathcal{L}_A(A_A)$. It is easy to see that it is a C^*-algebra and that we may extend L to an isomorphism of $*$-algebras between $\mathbb{K}_A(A_A) + \mathbb{C}$ and A^1, which we turn into a C^*-algebra by stealing the norm:

$$\|(a,\alpha)\| = \sup\left\{\|ax + \alpha x\| \mid \|x\| \leq 1,\ x \in A\right\},\ \text{for } (a,\alpha) \in A^1.$$

The pair (A^1, ι), where $\iota\colon A \to A^1$, $a \mapsto (1,0)$ is a unitalization of A: It is clear that it is injective, since $\|\iota(a)\| = \|L_a\| = \|a\|$ for all $a \in A$ and $\iota(A)$ is an ideal since $\mathbb{K}_A(A_A)$ is an ideal. It remains to be shown that it is essential:

Suppose that $(x,\beta)\iota(A) = 0$ so for all $a \in A$ we have $xa + \beta a = 0$. If $\beta = 0$ then $\|(x,0)^*(x,0)\| = \|x\|^2 = 0$ so $x = 0$ and $(x,\beta) = (0,0)$. Assume that $\beta \neq 0$, then $-\frac{x}{\beta}a = a$ for all $a \in A$, so $a^* = a^*\left(-\frac{x}{\beta}\right)^*$, and hence $a = a\left(-\frac{x}{\beta}\right)^*$, since $a \in A$ is arbitrary. Letting $a = -\frac{x}{\beta}$, we see that $\frac{x^*}{\beta}$ is self-adjoint and hence a unit of A, a contradiction. Therefore $(x,\beta) = (0,0)$ and $\iota(A)$ is an essential ideal of A^1.

Example 1.24
The pair $(\mathcal{L}_A(A_A), L)$, $L\colon a \mapsto L_a$, is a unitalization of A - to see this we need only show that $\mathbb{K}_A(A_A)$ is essential in $\mathcal{L}_A(A_A)$: Suppose that $T\mathbb{K}_A(A_A) = 0$, so $TK = 0$ for all compact K on A, then with $\Theta_{x,y}\colon z \mapsto x\langle y,z\rangle = xy^*z$, we see by 1.14 that $T\Theta_{x,y} = \Theta_{Tx,y} = 0$ for all $x,y \in A$, which shows that for $x,y \in A$ we have $\Theta_{Tx,y}(y) = (Tx)(y^*y) = L_{Tx}(y^*y) = 0$, implying that $L_{Tx} = 0 \iff Tx = 0$ so $T = 0$.

And, it makes sense to talk about a 'largest' unitalization, in the sense that it has a certain universal property:

Definition 1.25. A unitalization (B,ι) of a C^*-algebra A is said to be *maximal* if to every other unitalization (C,ι'), there is a $*$-homomorphism $\varphi\colon C \to B$ such that diagram:

$$\begin{tikzcd} A \arrow[r, hook, "\iota"] \arrow[dr, hook, "\iota'"'] & B \\ & C \arrow[u, "\exists\varphi"'] \end{tikzcd} \tag{1.6}$$

commutes.

Definition 1.26. For a C^*-algebra A, we define the *multiplier algebra* of A to be $M(A) := \mathcal{L}_A(A_A)$.

It turns out that this is a maximal unitalization of A, as we shall see in a moment – but first we need a bit more work, for instance we need the notion of non-degenerate representations of a C^*-algebra on a Hilbert space to be extended to the case of Hilbert C^*-modules:

Definition 1.27. Let B be a C^*-algebra and E a Hilbert A-module. A homomorphism $\varphi\colon B \to \mathcal{L}_A(E)$ is *non-degenerate* if the set

$$\varphi(B)X := \operatorname{Span}\left\{\varphi(b)x \mid b \in B \text{ and } x \in E\right\},$$

is dense in E.

Proposition 1.28. Let A and C be C^*-algebras, and let E denote a Hilbert A-module and let B be an ideal of C. If $\alpha\colon B \to \mathcal{L}_A(E)$ is a non-degenerate *-homomorphism, then it extends uniquely to a *homomorphism $\overline{\alpha}\colon C \to \mathcal{L}_A(E)$. If further $\iota(B)$ is an essential ideal of C, then the extension is injective.

Proof.
The last statement is clear, since $\ker\overline{\alpha} \cap B = \ker\alpha \cap B = 0$ which implies that $\ker\overline{\alpha} = 0$, since B is essential. Let (e_j) be an approximate identity of B. Define for $c \in C$ the operator $\overline{\alpha}(c)$ on $\alpha(B)X$ by

$$\overline{\alpha}(c)\left(\sum_{i=1}^{n} \alpha(b_i)(x_i)\right) := \sum_{i=1}^{n} \alpha(cb_i)(x_i).$$

This is well-defined and bounded, for we see that

$$\left\|\sum_{i=1}^{n} \alpha(cb_i)(x_i)\right\| = \lim_j \left\|\sum_{i=1}^{n} \alpha(\underbrace{ce_j}_{\in B})\alpha(b_i)(x_i)\right\| \leq \|c\| \left\|\sum_{i=1}^{n} \alpha(b_i)(x_i)\right\|.$$

This extends to a bounded operator on all of E, since $\alpha(B)E$ is dense which also implies uniqueness of the extension. It is easy to see that $c \mapsto \overline{\alpha}(c)$ is a homomorphism. It remains to be shown that is is adjointable: Let $c \in C$ and $b, b' \in B$ and $x, y \in E$, then

$$\langle\overline{\alpha}(c)\alpha(b)x, \alpha(b')y\rangle = \langle\alpha(cb)x, \alpha(b')y\rangle = \langle x, \alpha(cb)^*\alpha(b')y\rangle = \langle x, \alpha(b^*c^*b')y\rangle = \langle\alpha bx, \overline{\alpha}(c)\alpha(b')y\rangle,$$

by linearity this shows that $\overline{\alpha}(c^*)$ is an adjoint of $\overline{\alpha}(c)$, hence it is a *-homomorphism. $\square$

1.2 Crossed Products

Throughout this chapter, we let G be a locally compact topological group and A a C^*-algebra, with no added assumptions unless otherwise specified. We will write $G \curvearrowright_\alpha A$ to specify that $\alpha\colon G \to \mathrm{Aut}(A)$ is an action of G on A, i.e., a strongly continuous group homomorphism. We will always fix a base left Haar measure μ on G, whose integrand we will denote by $\mathrm{d}t$, and with modular function $\Delta\colon G \to (0,\infty)$.

In analysis, an object of interest is the group algebra $L^1(G)$ for locally compact groups G. One can turn it into a $*$-algebra by showing that the convolution gives rise to a well-defined multiplication, and one can use the modular function to define an involution. Analogously to this, we do the same for A valued functions:

Definition 1.29. Suppose that $G \curvearrowright_\alpha A$. We let

$$C_c(G,A) := \{f\colon G \to A \mid f \text{ continuous and compactly supported}\}.$$

For $f, g \in C_c(G,A)$, we define the α-twisted convolution of f and g at $t \in G$ by

$$(f *_\alpha g)(t) := \int_G f(s)\alpha_s(g(s^{-1}t))\mathrm{d}s,$$

and we define an α-twisted convolution of $f \in C_c(G,A)$ at $t \in G$ by

$$f^*(t) := \Delta(t^{-1})\alpha_t(f(t^{-1})^*).$$

Straight forward calculations similar to the one classic ones show that $C_c(G,A)$ becomes a $*$-algebra with multiplication given by the α-twisted convolution and involution given by the α-twisted involution. We denote the associated $*$-algebra by $C_c(G,A,\alpha)$.

To $f \in C_c(G,A,\alpha)$, we associate the number $\|f\|_1 \in [0,\infty)$ by

$$\|f\|_1 := \int_G \|f(t)\|\,\mathrm{d}t.$$

Straightforward calculations show that $f \mapsto \|f\|_1$ is a norm, and we denote the completion of $C_c(G,A,\alpha)$ in $\|\cdot\|_1$ by $L^1(G,A,\alpha)$.

For $G \overset{\alpha}{\curvearrowright} A$ and a covariant representation $(\pi,\mathcal{H})$ on some Hilbert space $\mathcal{H}$, we define for $f \in C_c(G,A,\alpha)$ the operator I_f on $\mathcal{H}$ point-wise, i.e., for $\xi,\eta \in \mathcal{H}$

$$\langle I_f\xi,\eta\rangle = \int_G \langle \pi(f(s))U_s\xi,\eta\rangle\mathrm{d}s,$$

and we will write $\int_G \pi(f(s))U_s\mathrm{d}s := I_f$. We denote the map $f \mapsto I_f$ by $\pi \rtimes U$.

Lemma 1.30. The map $\pi \rtimes U$ is a $*$-representation of $C_c(G,A,\alpha)$ called *the integrated form of π and U*. It is L^1-norm decreasing, i.e., for $f \in C_c(G,A,\alpha)$ we have

$$\|\pi \rtimes U(f)\| \leq \|f\|_1.$$

Proof.
Let $f \in C_c(G, A, \alpha)$ and $\xi, \eta \in \mathcal{H}$. Then

$$|\langle \pi \rtimes U(f)\xi, \eta\rangle| \leq \int_G |\langle \pi(f(s))U_s\xi, \eta\rangle| \mathrm{d}s \leq \int_G \|f(s)\| \|\xi\| \|\eta\| \, \mathrm{d}s,$$

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so that $\|\pi \rtimes U(f)\| \leq \|f\|_1$.Let $f, g \in C_c(G, A, \alpha)$ $\square$

Bibliography

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Appendix A

A.1 Integration Theory In Groups

Integration over groups plays a vital role in the theory developed and processed throughout this thesis. Therefore, we will build some of the cornerstone results in this theory, which will be used frequently. We will by G denote an arbitrary locally compact group, and by X an arbitrary Banach space. We will fix a left-Haar measure μ on G, whose integrand we denote by $\mathrm{d}t$.

For $f \in C_c(G)$ and $x \in X$, we denote by $f \otimes x$ the map $G \ni t \mapsto f(t)x \in X$. It is clear that then $f \otimes x \in C_c(G, X)$. [titlecase]title

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