Simplicty of Crossed Products of C^*-algebras

Masters thesis defense

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Advisor: Søren Eilers

February 15, 2019

Table of Contents

1. Introduction, history and motivation
2. Some notation and preliminaries
3. The general crossed product construction
4. Simplicity
5. The case of G abelian
6. The case of A abelian and G discrete
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And of course, we have overlaps between the above cases.

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All of these C^*-algebras play a crucial role in the theory of crossed products.

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The Gelfand-Neimark-Segal result

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The left-regular representation of G is the tuple $(\lambda, L^2(G))$, where

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A covariant representation of (A, G, α) is a triple $(\pi, u, \mathcal{H})$ consisting of representations of A and G as bounded operators on a common Hilbert space $\mathcal{H}$ satisfying the relation

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When G is discrete and A is unital, it is easy to associate to such a triple a **nice** new C^*-algebra:

Discrete crossed product cookbook recipe

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1. Given (A, G, α) with G discrete and A unital, consider the $*$-algebra

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3. Define a representation $\pi \rtimes u$ of $C_c(G, A)$ as operators on $\mathcal{H}$ by

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This is the building stone for constructing crossed products.

Discrete crossed product cookbook recipe

1. Given (A, G, α) with G discrete and A unital, consider the $*$-algebra
$$C_c(G, A) = \left\{ \sum_{t \in F} a_t \delta_t \mid F \subseteq G \text{ finite} \right\}.$$
2. Pick a covariant representation $(\pi, u, \mathcal{H})$ of (A, G, α).
3. Define a representation $\pi \rtimes u$ of $C_c(G, A)$ as operators on $\mathcal{H}$ by
$$\pi \rtimes u \left(\sum a_t \delta_t \right) := \sum \pi(a_t) u_t.$$
4. Take the closure of $\pi \rtimes u(C_c(G, A))$ to obtain a C^*-algebra encoding some information about the dynamical system (A, G, α).

This is the building stone for constructing crossed products. Fails when G is non-discrete.

Table of Contents

1. Introduction, history and motivation
2. Some notation and preliminaries
3. The general crossed product construction
4. Simplicity
5. The case of G abelian
6. The case of A abelian and G discrete
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Theorem

Let A be a C^*-algebra. Then there is a unique linear map $\int\colon C_c(G, M_s(A)) \to M(A)$ such that for all non-degenerate representations $\pi\colon A \to \mathbb{B}(H)$, $\xi, \eta \in \mathcal{H}$ and $f \in C_c(G, M_s(A))$ we have

$$\langle \overline{\pi}\left(\int (f)\right)\xi, \eta\rangle = \int_G \langle \overline{\pi}(f(s))\xi, \eta\rangle \mathrm{d}s.$$

And we write $\int(f) := \int_G f(s)\mathrm{d}s$.

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$$\pi \rtimes u(f) := \int (\pi(f)u) = \int_G \pi(f(s))u_s \mathrm{d}s.$$

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The *full crossed product* of (A, G, α) is the C^*-algebra $A \rtimes_\alpha G$ obtained by taking the closure of $C_c(G, A)$ with respect to the norm

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There is a faithful homomorphism $A \hookrightarrow M(A \rtimes_\alpha G)$ and an injective strictly continuous group homomorphism $U\colon G \to \mathcal{U}(M(A \rtimes_\alpha G))$ such that for $a \in A$, $s, r \in G$ and $f \in C_c(G, A)$

$$af(s) = af(s), \quad U_r f(s) = \alpha_r(f(r^{-1}s)) \text{ and } \alpha_r(a) = U_r a U_r^*.$$

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Makes up for lack of copies inside $A \rtimes_\alpha G$.

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The triple $(\tilde{\pi}, \lambda, L^2(G, \mathcal{H}))$ is a covariant representation.

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The reduced crossed product is the C^*-algebra $A \rtimes_{\alpha,r} G$ obtained by taking the closure of $\tilde{\pi} \rtimes \lambda(C_c(G, A))$ in $\mathbb{B}(L^2(G, \mathcal{H}))$ where π is any faithful non-degenerate representation of A on $\mathcal{H}$.

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Theorem

If G is amenable (=admits left-invariant mean), then $A \rtimes_\alpha G \cong A \rtimes_{\alpha,r} G$

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Need a less general approach to answer Question 1.

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1. $\widehat{G}(\alpha) = \widehat{G}$.
2. For each $\sigma \in \widehat{G}$ and each non-zero ideal $I \subseteq A \rtimes_\alpha G$ we have $I \cap \widehat{\alpha}_\sigma(I) \neq 0$.
3. For each non-zero ideal $I \subseteq A \rtimes_\alpha G$ there is a non-zero $\widehat{G}$-invariant ideal $I_0 \subseteq I$.
4. A detects ideals in $A \rtimes_\alpha G$.

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Definition

The action α is topologically free if the set

$$G_x := \{t \in G \mid t.x = x\}$$

has dense complement for each $x \in X$

Archbold-Spielberg for commutative C^*-algebra

The algebra $C_0(X) \rtimes_\alpha G$ is simple if and only if α is topologically free, minimal and $A \rtimes_\alpha G \cong A \rtimes_{\alpha,r} G$ via a regular representation $\tilde{\pi} \rtimes \lambda$.

Table of Contents

1. Introduction, history and motivation
2. Some notation and preliminaries
3. The general crossed product construction
4. Simplicity
5. The case of G abelian
6. The case of A abelian and G discrete
7. An application of the theory

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Kennedy, Kalantar, Breulliard and Ozawa

The following are equivalent:

1. G is C^*-simple.
2. $C(X) \rtimes_r G$ is simple for all G-boundaries X.
3. $C(\partial_F G) \rtimes_r G$ is simple.
4. The action $G \curvearrowright \partial_F G$ is free.
5. The action $G \curvearrowright \partial_F G$ is topologically free.
6. There is some G-boundary X such that the action is topologically free.
7. Whenever X is a compact minimal G-space, amenability of G_x for some $x \in X$ implies topological freeness of the action on X.
8. If $G \overset{\alpha}{\curvearrowright} A$ with A unital, then $A \rtimes_{\alpha,r} G$ is simple whenever A is G-simple.

Thanks for listening.