Simplicty and uniqueness of trace of group C^*-algebras

Bachelor thesis defense

Malthe Munk Karbo

Advisor: Mikael Rørdam

February 3, 2017

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And of course, we have overlaps between the above cases.

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All of these C^*-algebras play a crucial role in the theory of crossed products.

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The Gelfand-Neimark-Segal result

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A short primer on topological group theory

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The left-regular representation of G is the tuple $(\lambda, L^2(G))$, where

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for $\xi \in L^2(G)$ and $s, t \in G$.

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4. Take the closure of $\pi \rtimes u(C_c(G, A))$

This is the building stone for constructing crossed products.

The perhaps shortest primer on C^*-dynamical systems and crossed products

When G is discrete and A is unital, it is easy to associate to such a triple a **nice** new C^*-algebra:

Discrete crossed product cookbook recipe

1. Given (A, G, α) with G discrete and A unital, consider the $*$-algebra
$$C_c(G, A) = \left\{ \sum_{t \in F} a_t \delta_t \mid F \subseteq G \text{ finite} \right\}.$$
2. Pick a covariant representation $(\pi, u, \mathcal{H})$ of (A, G, α) (they exist!)
3. Define a representation $\pi \rtimes u$ of $C_c(G, A)$ by
$$\pi \rtimes u \left(\sum a_t \delta_t \right) := \sum \pi(a_t) u_t$$
4. Take the closure of $\pi \rtimes u(C_c(G, A))$

This is the building stone for constructing crossed products. Fails when G is non-discrete.

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