CSCD396

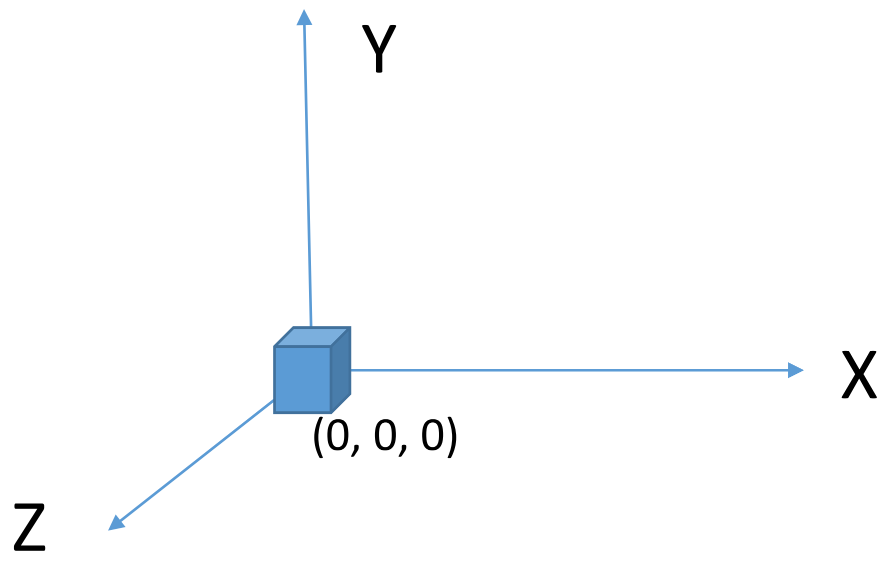
Beginning Graphics

Today’s topic

- Rotation about an arbitrary axis;
- Shear transformation;
- Reflection.

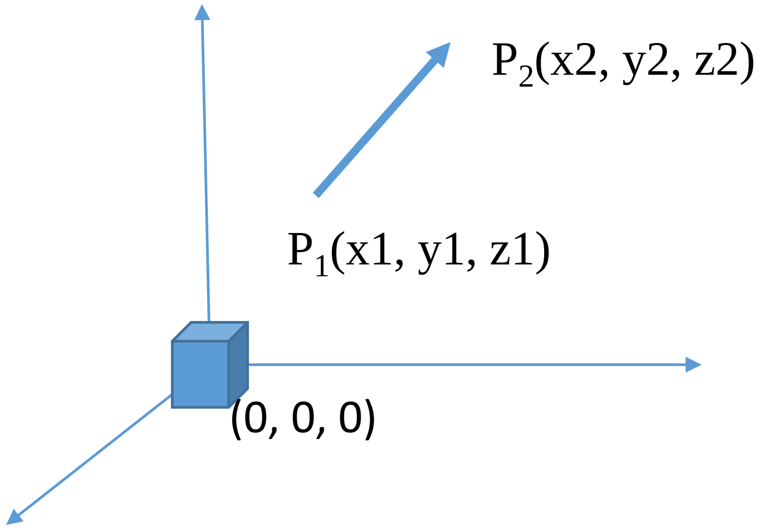
Rotation about Arbitrary Axis

- Suppose, we'd like to rotate an object centered at origin by rotation angle 'ɵ' about an arbitrary axis.



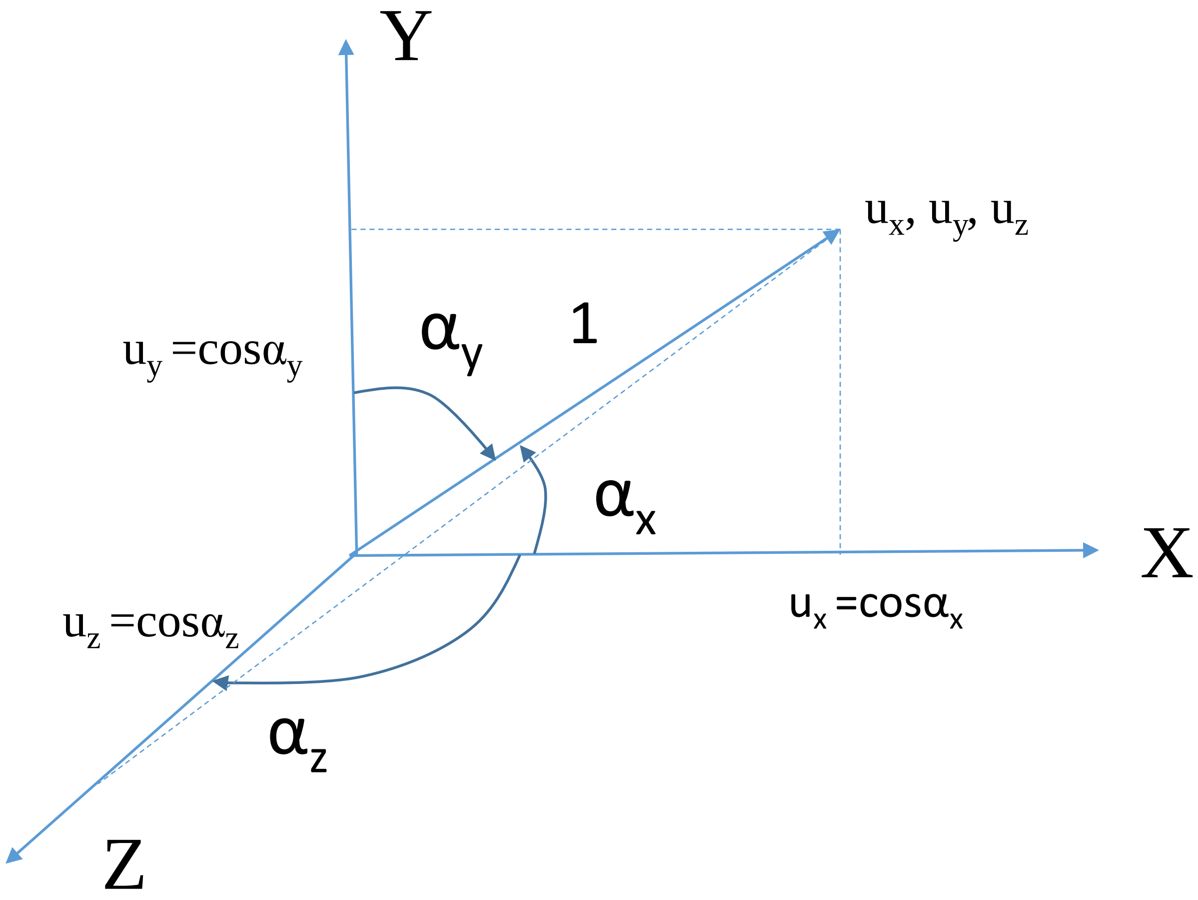
Rotation about Arbitrary Axis

- An arbitrary axis can be formed from two arbitrary points: $P_1(x, y, z)$ and $P_2(x, y, z)$



- U = P2 – P1 ← defines a vector with direction from P1 to P2
- A unit vector $u = U/|U|$ where $|U| = |\text{sqrt}(U_x^2 + U_y^2 + U_z^2)|$

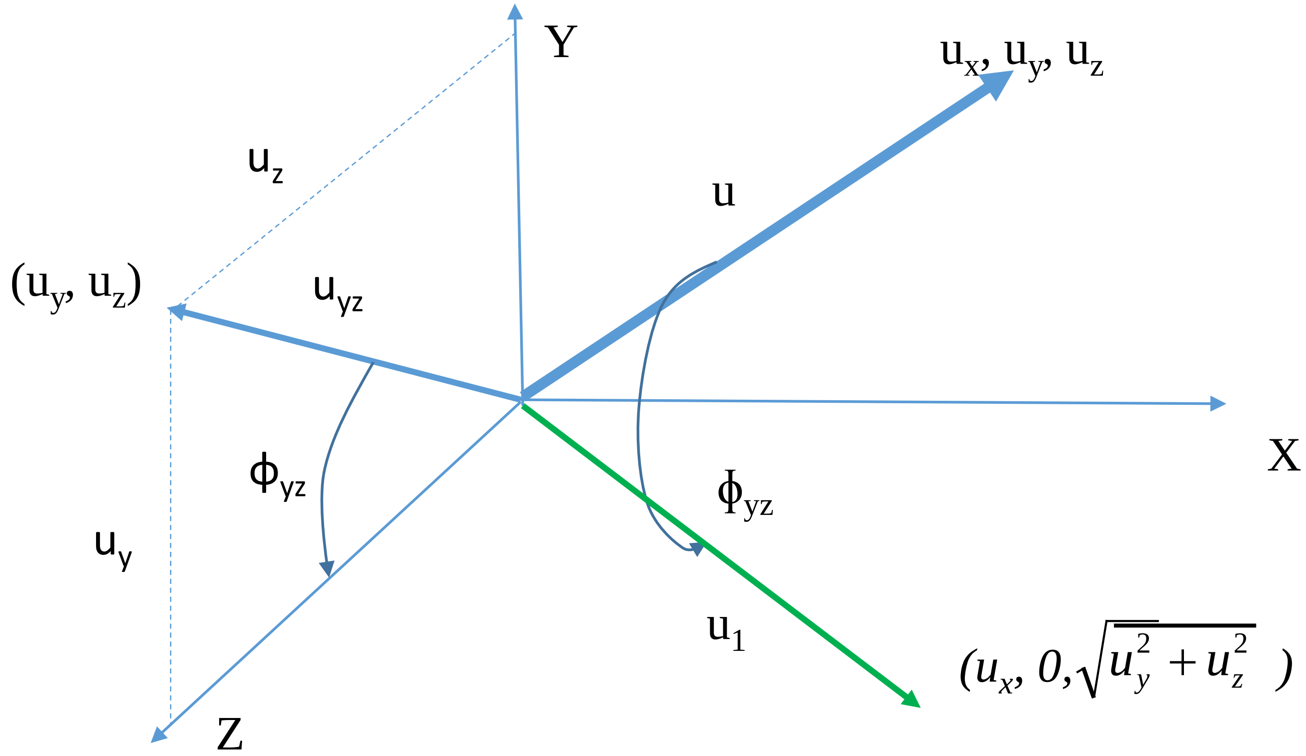
Rotation about Arbitrary Axis



$$u_x^2 + u_y^2 + u_z^2 = 1$$
$$\cos^2\alpha_x + \cos^2\alpha_y + \cos^2\alpha_z = 1$$

Rotation about Arbitrary Axis

- ***Step 1***: Rotate 'u' about x-axis by angle 'ϕ_{yz}' so that '***u***' rotates into the upper half of xz plane (similar to rotating u_{yz} about x-axis by angle 'ϕ_{yz}')



$$u_{yz} = \sqrt{u_y^2 + u_z^2}$$

Rotation about Arbitrary Axis

- Hence in step 1, the required rotation is as follows
- $R_x(\phi_{yz}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\phi_{yz}) & -\sin(\phi_{yz}) & 0 \\ 0 & \sin(\phi_{yz}) & \cos(\phi_{yz}) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

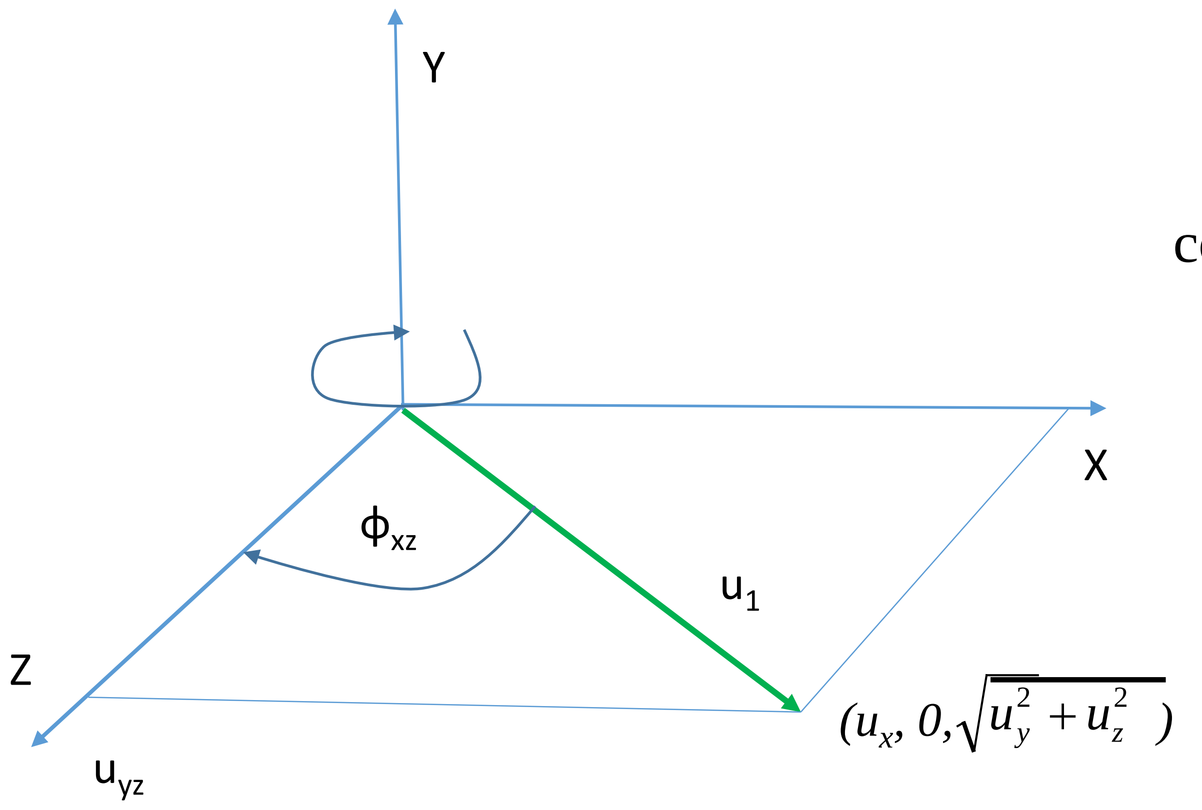
$$\cos\phi_{yz} = {u_z}/{u_{yz}}$$

$$\sin\phi_{yz} = {u_y}/{u_{yz}}$$

$$u_{yz} = \sqrt{u_y^2 + u_z^2}$$

Rotation about Arbitrary Axis

- **Step 2:** Rotate ‘u_1’ about y-axis by ‘ϕ_{xz}’ so that ‘u_1’ aligns with the z-axis



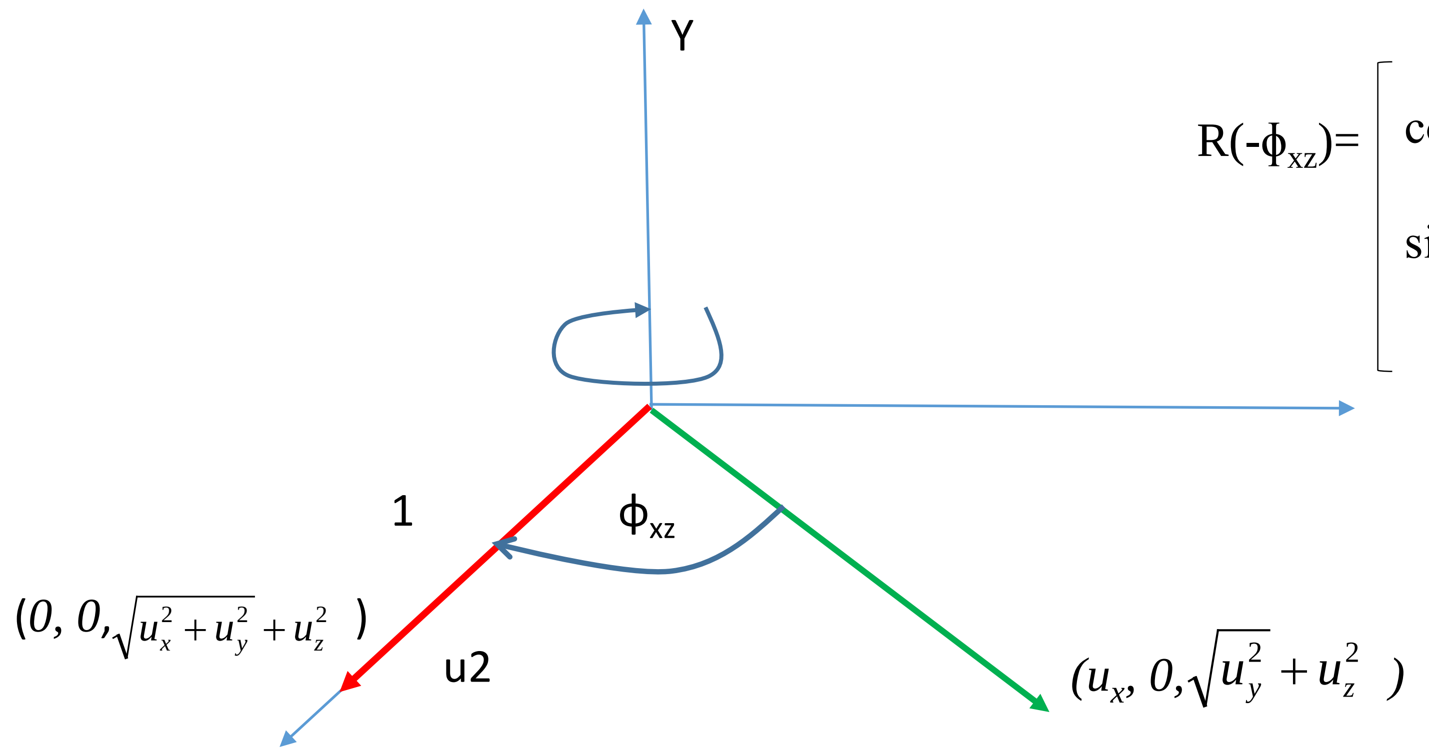
$$\cos\phi_{xz} = u_{yz} \Big/ \sqrt{u_x^2 + u_y^2 + u_z^2}$$

$$\cos\phi_{xz} = u_{yz} \quad , u_{yz} = \sqrt{u_y^2 + u_z^2}$$

$$\sqrt{u_x^2 + u_y^2 + u_z^2} = 1$$

Rotation about Arbitrary Axis

- Hence in step 2, the required rotation is as follows: $R_y(-\phi_{xz})$



$$R(-\phi_{xz}) = \begin{bmatrix} \cos(\phi_{xz}) & 0 & -\sin(\phi_{xz}) & 0 \\ 0 & 1 & 0 & 0 \\ \sin(\phi_{xz}) & 0 & \cos(\phi_{xz}) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Note: rotation is clockwise here!
'ϕ_{xz}' is negative

Rotation about Arbitrary Axis

- Step3: perform simple rotation ‘ɵ’ around z axis;

$$\text{Now, the Rotation } R_z(\theta)= \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotation about Arbitrary Axis

- Step 4: Undo rotation in Step 2

$$R_y(\phi_{xz}) = \begin{bmatrix} \cos(\phi_{xz}) & 0 & \sin(\phi_{xz}) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(\phi_{xz}) & 0 & \cos(\phi_{xz}) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotation about Arbitrary Axis

- Step 5: Undo the rotation in Step 1

- $R_x(-\phi_{yz}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\phi_{yz}) & \sin(\phi_{yz}) & 0 \\ 0 & -\sin(\phi_{yz}) & \cos(\phi_{yz}) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

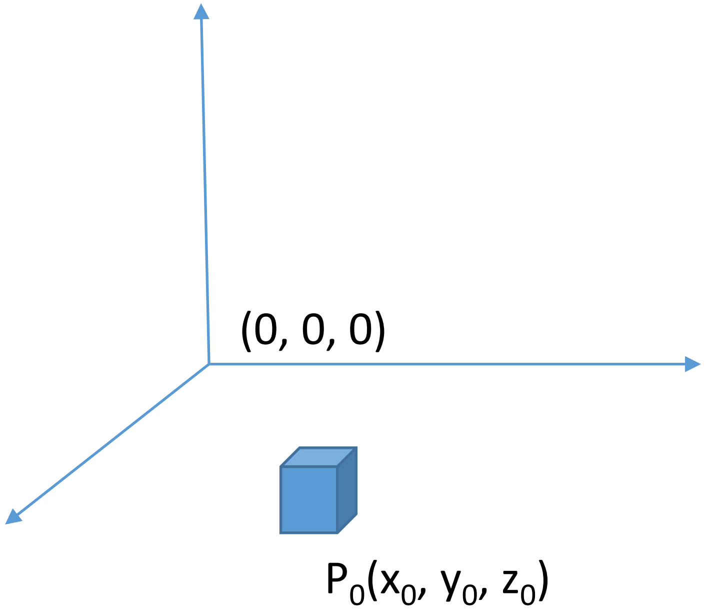
Rotation about Arbitrary Axis

- Now the composite matrix for rotation of an object by angle ‘ɵ’ around any arbitrary axis is as follows:

$$R_{arbitrary}(\theta)= R_x(-\phi_{yz})\, R_y(\phi_{xz}) R_z(\theta)\, R_y(-\phi_{xz})\, R_x(\phi_{yz})$$

Rotation about an arbitrary axis

- Object may not be centered at origin;
- Suppose, object is placed at point $P_0(x_0, y_0, z_0)$

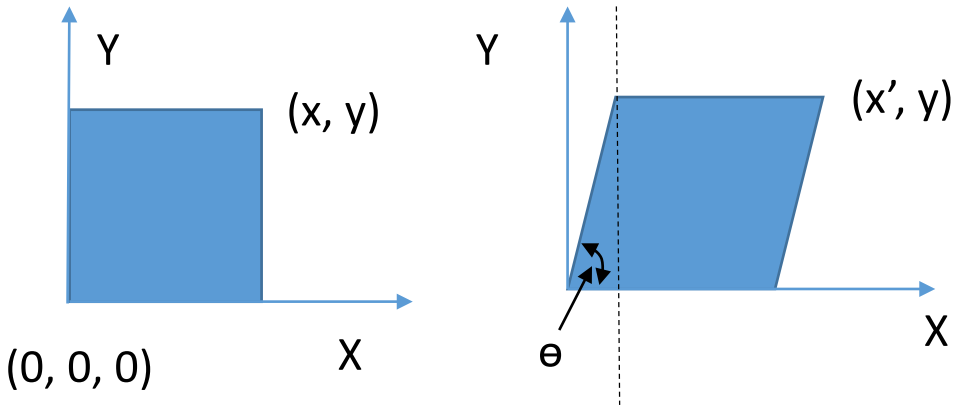


Rotation about an arbitrary axis

- In this case, at first object needs to centered at origin, so we need to apply translation $T(-x_0, -y_0, -z_0)$;
- Now perform Step 1 to Step 5
- Lastly, bring the object back to its original position, $T(x_0, y_0, z_0)$
- So the composite matrix will look as follows:
- $R_{arbitrary\_translated}(\theta, x_0, y_0, z_0) =$
  $T(x_0, y_0, z_0)\, R_x(-\phi_{yz})\, R_y(\phi_{xz}) R_z(\theta)\, R_y(-\phi_{xz})\, R_x(\phi_{yz})\, T(-x_0, -y_0, -z_0)$

Shear

- Non-rigid affine transformation;
- Suppose, the top of the object ‘A’ is pulled to the right and bottom to the left



$x' = x + y\cot(\theta) = x + ay$
$y' = y;$
$z' = z;$

- Shear(a,0, 0) = $\begin{bmatrix} 1 & a & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Shear

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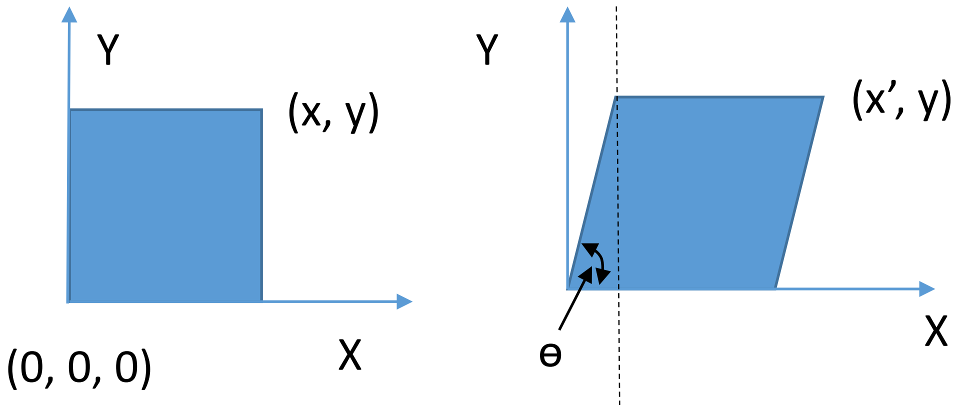
- Suppose, the top of the object ‘A’ is pulled to the right and bottom to the left;



x’ = x + ycot(ɵ) = x + ay
y’ = y;
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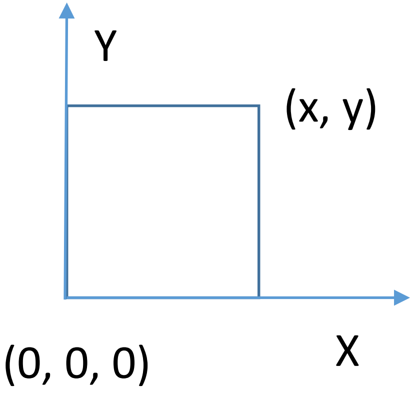
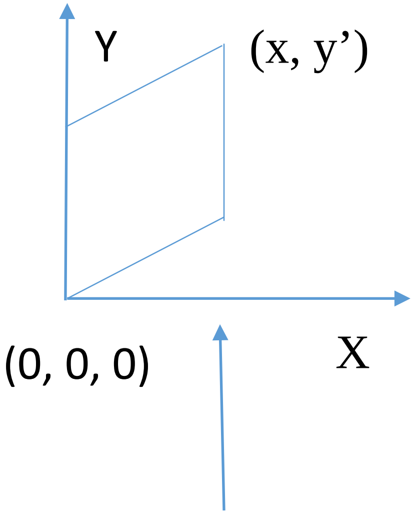
Shear

- Suppose, the left of the object ‘A’ is pushed downward and right pushed upward





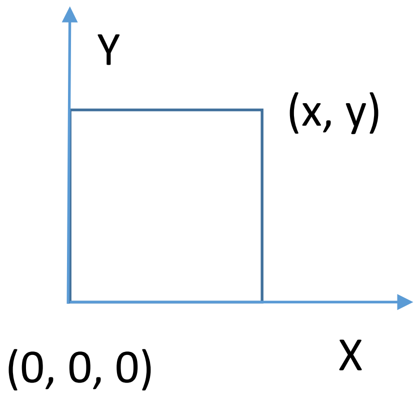
$x' = x$
$y' = bx + y;$
$z' = z;$

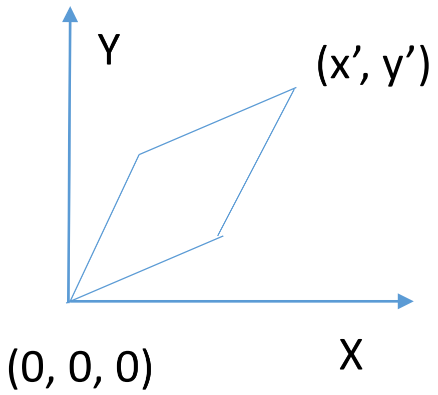
- Shear (0, b,0) = $\begin{bmatrix} 1 & 0 & 0 & 0 \\ b & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Shear

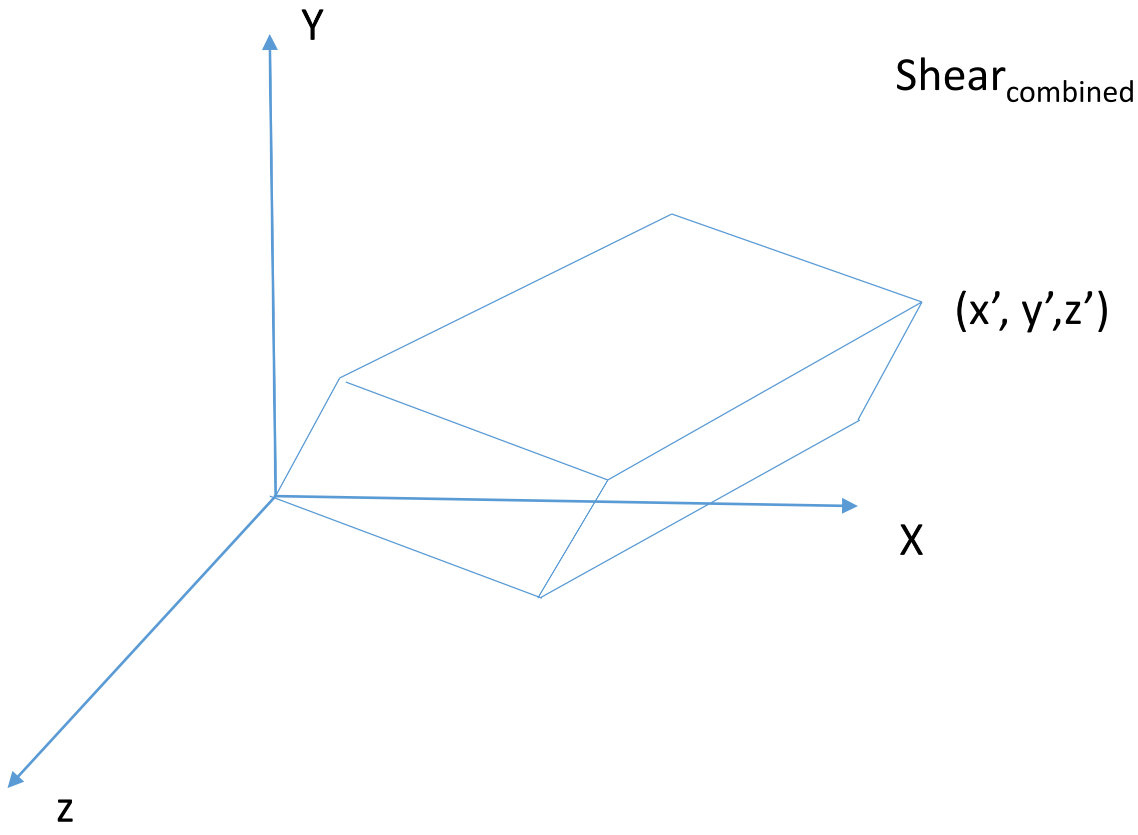
- Simultaneous shear along x, y:
- Shear(a, b, 0) = $\begin{bmatrix} 1 & a & 0 & 0 \\ b & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$x' = x + ay$
$y' = b x + y$





Shear

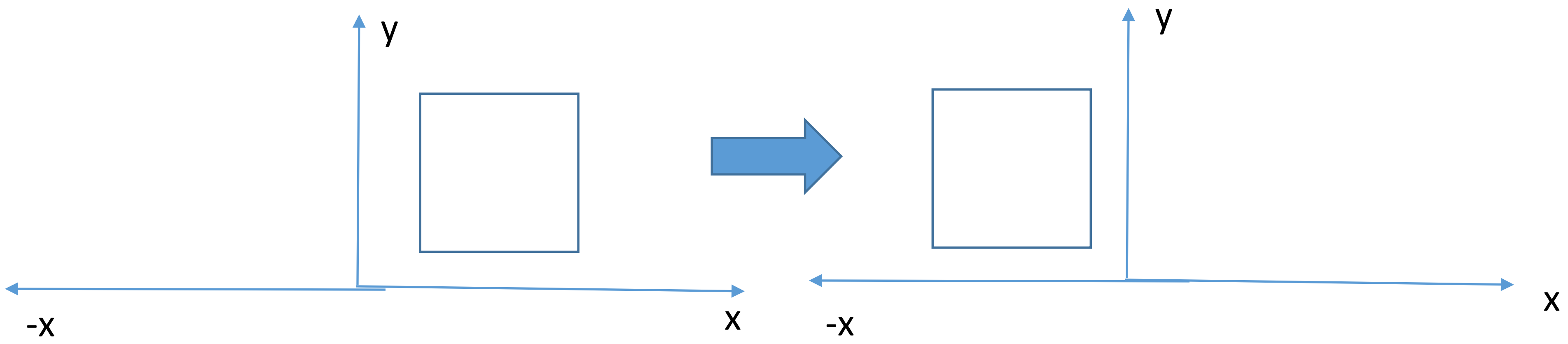


$$\text{Shear}_{\text{combined}} = \begin{bmatrix} 1 & a & e & 0 \\ b & 1 & f & 0 \\ c & d & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x' = x + ay + ez; \\ y' = bx + y + fz; \\ z' = cx + dy + z; \end{bmatrix}$$

Reflection

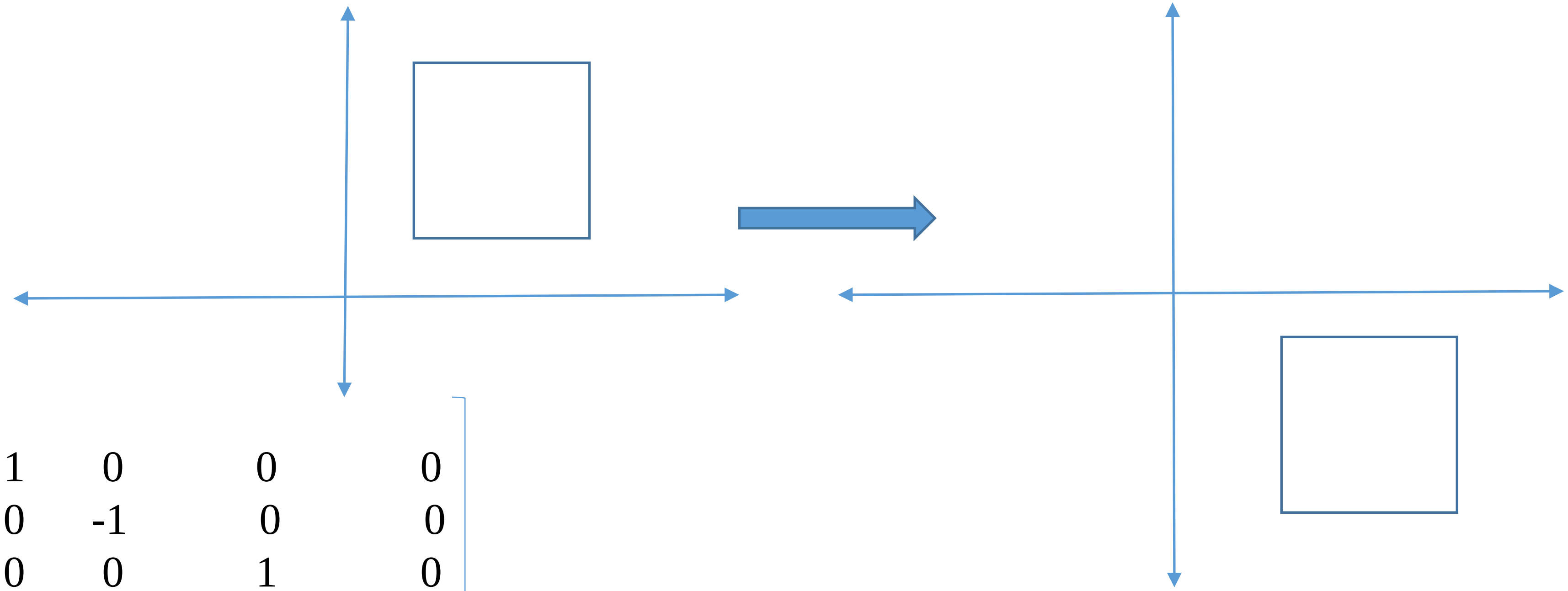
- A mirror reflection around y axis



- $M_{ref\_y} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Reflection

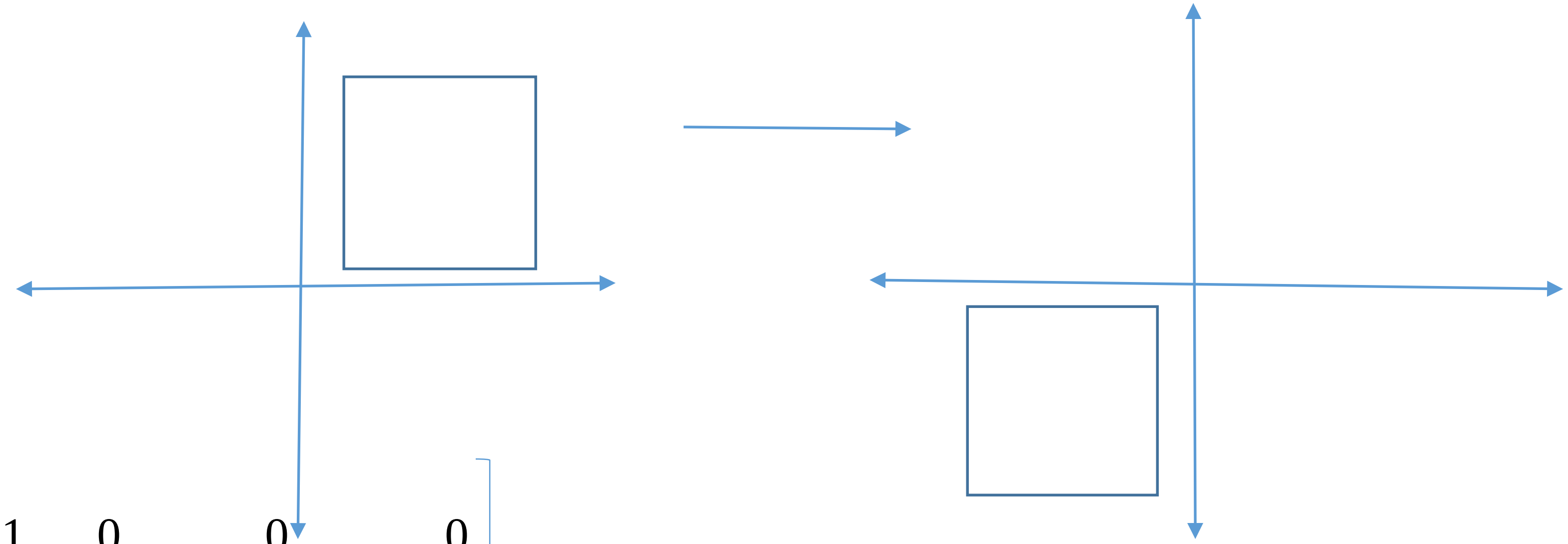
- Around x axis:



$$M_{ref\_x} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Reflection

xy plane:



$$M_{ref\_xy} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$