Image Analysis with Shapelets

M. den Brok
denbrok@astro.rug.nl

Kapteyn Institute
University of Groningen

Guest lecture for the Applied Signal Processing course, 2008

Outline

Introduction

- Post-reduction data analysis important for images
- Examples: Fourier transform, wavelets, &c.
- Use *Shapelets* to analyse shape of object
- Applications of this technique include gravitational lensing and image simulation.
- Technique is fairly young (first article appeared in 2001)
- This lecture is based on the articles by Refregier (2003), Refregier & Bacon (2003)
- A lot of information can be found at `http://www.astro.caltech.edu/~rjm/shapelets/`, including IDL code.

One-dimensional basis functions

The goal is to describe a 1-d image (I(x)) as a series of basis functions. (cf. Fourier series)
The Shapelets dimensionless basis functions are defined as distorted Gaussians:

$$\phi_n = \left(2^n \pi^{\frac{1}{2}} n!\right)^{\frac{1}{2}} H_n(x) e^{-\frac{x^2}{2}} \tag{1}$$

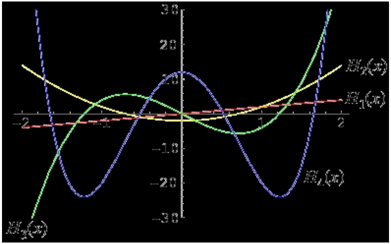
In practice, however, these are *rescaled* to the dimensional basis functions:

$$B_n(x, \beta) = \beta^{\frac{1}{2}} \phi_n(x/\beta) \tag{2}$$

So,

$$I(x) = \sum_n I_n B_n(x, \beta) \tag{3}$$

Hermite polynomials



First few Hermite polynomials:

$$
\begin{aligned}
H_0(x) &= 1 \\
H_1(x) &= 2x \\
H_2(x) &= 4x^2 - 2 \\
H_3(x) &= 8x^3 - 12x \\
H_4(x) &= 16x^4 - 48x^2 + 12
\end{aligned}
$$

Some useful properties:

$$
\begin{aligned}
H_{n+1}(x) &= 2xH_n(x) - 2nH_{n-1}(x) \\
H_n'(x) &= 2nH_{n-1}(x) \\
H_n(x) &= (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} \\
&= e^{x^2/2} \left(x - \frac{d}{dx} \right) e^{-x^2/2}
\end{aligned}
$$

We can define an inner product between continuous functions:

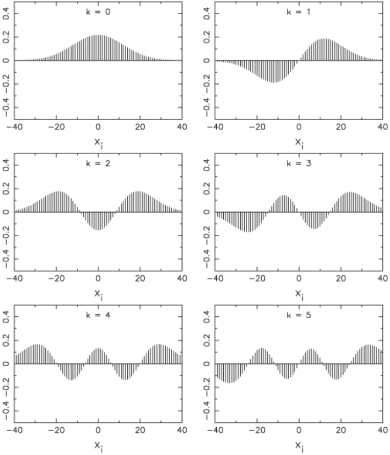
$$< f, g >= \int_{-\infty}^{\infty} f(x)g(x)dx$$

The Shapelet basis functions are orthonormal in this inner product space:

$$< B_m, B_n >= \delta_{m,n}$$

Since $I(x) = \sum_n I_n B_n(x, \beta)$, we can find the components I_n simply by taking the inner product with the right basis function:

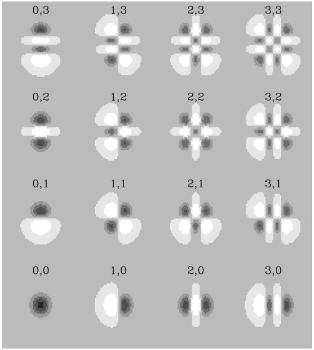
$$I_m =< I, B_m >$$



Berry, Hobson & Withington (2004)

However, in practice, the shapelet basisfunctions are discretized: in this case orthogonality between basisfunctions is lost. For this reason, decomposition into shapelets is usually done by a least squares fit, or a Singular Value Decomposition.

2-Dimensional Basisfunctions



Refregier (2003)

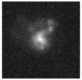
The one-dimensional formalism is easily extended to 2-D. Each two-dimensional basisfunction is the product of two one-dimensional basisfunction:

$$B_{n_x n_y}(x, y, \beta) = B_{n_x}(x, \beta) \cdot B_{n_y}(y, \beta)$$

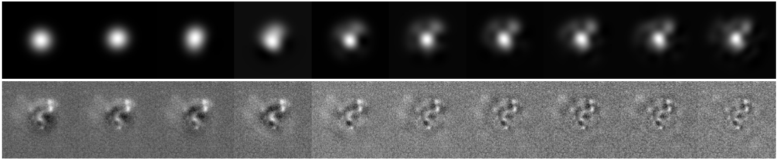
This set of 2-D basisfunctions is complete (you may prove that yourself.)

Example decomposition

In practice, only a finite number of components is used. High frequency components start sampling noise, and this is usually not what you want. Therefore, a maximum shapelet order is defined as $n_x + n_y <= n_{\text{max}}$.



Example decomposition of a galaxy into Shapelets. Only 56 numbers are necessary (order 9) to describe the shape of the galaxy quite well.



Application: Image compression (although I am not aware of anyone actually using this)

Application: weak gravitational lensing

Light of distant sources is bent around object of high mass: gravitational lensing. Three different kinds of lensing

- Strong lensing: Great distortions in the light distribution. Multiple images and Einstein rings occur.
- Microlensing: High mass object moves before background source, beaming the light of the background source and increasing the intensity. Transient effect (used, among other things, to detect exoplanets)
- Weak lensing: the shape distortion can only be detected by statistically analyzing the shear of a large number of galaxies.

Application: weak gravitational lensing

For weak lensing it is important to have a formalism that describes the shape of the galaxy in a quantitative way. One of the most widely used methods is the KSB method (Kaiser, Squires & Broadhurst 1995) This method determines the effect of weak gravitational lensing on the gaussian-weighted quadrupole moment of galaxies. In the shapelets formalism, this corresponds to the $n_1 + n_2 = 2$ coefficients.

- Remember: $\phi_n = \left(2^n \pi^{\frac{1}{2}} n!\right)^{\frac{1}{2}} H_n(x) e^{-\frac{x^2}{2}}$
- Question: Do the Shapelet basisfunctions somehow look familiar to you?

- Remember: $\phi_n = \left(2^n \pi^{\frac{1}{2}} n!\right)^{\frac{1}{2}} H_n(x) e^{-\frac{x^2}{2}}$
- Question: Do the Shapelet basisfunctions somehow look familiar to you?
- Answer: they are also the eigenfunctions of the `Quantum Harmonic Oscillator (QHO)`
- This is very useful, because now we can use the formalism of Quantum Mechanics.

Application: Fourier transform

Because we know that the Basisfunctions are the Eigenfunctions of the QHO, the Fourier transform of a Basisfunction is straightforward:

- The dimensionless 1-D equation of the Q.H.O looks like $\hat{H}\Psi = (\hat{x}^2 + \hat{p}^2)\Psi = E\Psi$

Position space

In position space, the base kets used are the position kets:

$$\hat{x}|x'>= x'|x'>$$

which are orthonormal:

$$<x''|x'>= \delta(x''-x')$$

Any physical state can be expanded in these base kets:

$$|\alpha>= \int dx'|x'><x'|\alpha>$$

Here we recogize $<x'|\alpha>$ as $\psi_\alpha(x')$ and hence $<\beta|\alpha>= \int dx' {\psi_\beta}^*(x')\psi_\alpha(x')$

Momentum space

For momentum space, we can do something similar:

$$\hat{p}|p' >= p'|p' >$$

The momentumspace base kets are also orthonormal:

$$< p''|p' >= \delta(p'' - p')$$
$$|\alpha >= \int dp'|p' >< p'|\alpha >$$

Using momentum as generator of translations, one can derive:

$$\begin{aligned} \hat{p}|\alpha > &= \int dx'|x' > \left(-i\hbar\frac{\partial}{\partial x'} < x'|\alpha >\right) \\ < x'|\hat{p}|\alpha > &= -i\hbar\frac{\partial}{\partial x'} < x'|\alpha > \end{aligned}$$

Relation between position/momentum space

From the above, we know:

$$< x'|\hat{p}|p' >= p' < x'|p' >= -i\hbar\frac{\partial}{\partial x'} < x'|p' >$$

From this, the relation between position and momentum space follows:

$$< x'|p' >= Ae^{\frac{ip'x'}{\hbar}}$$

Normalization requires that $A = \frac{1}{\sqrt{2\pi}}$ and hence:

$$|\Phi >= \int dp|p >< p|\Phi >= \int dp|p > \phi(p)$$

$$\phi(x) =< x|\Phi >= \int dp < x|p > \phi(p) = \int dp\frac{1}{\sqrt{2\pi}}e^{\frac{ip'x'}{\hbar}}\phi(p)$$

Application: Fourier transform

Because we know that the Basisfunctions are the Eigenfunctions of the QHO, the Fourier transform of a Basisfunction is straightforward:

- The dimensionless 1-D equation of the Q.H.O looks like $\hat{H}\Psi = (\hat{x}^2 + \hat{p}^2)\Psi = E\Psi$
- In x-space we have $(x^2 - \frac{\partial^2}{\partial x^2})\Psi(x) = E\Psi(x)$
- Transforming this equation to momentum space: $(p^2 - \frac{\partial^2}{\partial p^2})\Psi(p) = E\Psi(p)$
- From the symmetry between x and p, we deduce that the Fourier transform of the basisfunction should be, up to a multiplicative constant, equal to the same basisfunction.
- It turns out that $\tilde{\phi}(k) = i^n \phi_n(x)$

Fourier transform of an image

- The decomposition of a 2d image yields a vector with shapelets coefficients and a scale parameter (β.)
- The fourier transform of this image consists of the components of this vecor, multiplied with flying factors of i, and with scale $1/\beta$

Convolution

It is however more interesting to look at convolution of an image:

$$h(\mathbf{x}) = \int d^2x \; f(\mathbf{x}-\mathbf{x}')g(\mathbf{x}')$$

Each image can be decomposed into shapelets with scales α, β, γ and coefficients $f_\mathbf{n} =< \mathbf{n}, \alpha | f >$, $g_\mathbf{n} =< \mathbf{n}, \beta | g >, h_\mathbf{n} =< \mathbf{n}, \gamma | h >$.
Convolution is bilinear (remember, in Fourier space you have a simple product of two functions). Therefore, the shapelet coefficients of a convolved image should look like:

$$h_\mathbf{n} = \sum_{\mathbf{m},\mathbf{l}} C_{\mathbf{n},\mathbf{m},\mathbf{l}} f_\mathbf{m} g_\mathbf{l}$$

$C_{\mathbf{n},\mathbf{m},\mathbf{l}}(\alpha, \beta, \gamma)$ is a 2d convolution tensor. The coefficients of this tensor can be calculated analytically (a rather lengthy expression though.)

Deconvolution

Smoothing an image in Shapelet formalism is just matrix multiplication. Therefore, if the smoothing kernel is known, an image can be deconvolved by multiplying with the inverse matrix.

Lots of applications require PSF deconvolution

- weak lensing
- micro lensing
- photometry

Application: simple astro/photometric expressions

Using the basisfunctions, we can compute certain photo- and astrometric quantities fairly easy.

1. Total flux: $F = \pi^{\frac{1}{2}}\beta \sum_{n1,n2}^{\text{even}} 2^{\frac{1}{2}(2-n_1-n_2)} \binom{n_1}{n_1/2}^{\frac{1}{2}} f_{n_1,n_2}$
2. First moment: $x_f = \pi\beta^2 F^{-1} \sum_{n_1}^{odd} \sum_{n_2}^{even} (n_1 + 1)^{\frac{1}{2}} 2^{\frac{1}{2}(2-n_1-n_2)} \binom{n_1+1}{(n_1+1)/2}^{\frac{1}{2}} \binom{n_2}{n_2/2}^{\frac{1}{2}} f_{n_1,n_2}$
3. Rms radius: $r_f^2 = \pi\beta^2 F^{-1} \sum_{n_1}^{odd} \sum_{n_2}^{even} (n_1 + 1)^{\frac{1}{2}} (1 + n_1 + n_2) 2^{\frac{1}{2}(4-n_1-n_2)} \binom{n_1}{n_1/2}^{\frac{1}{2}} \binom{n_2}{n_2/2}^{\frac{1}{2}} f_{n_1,n_2}$

Polar shapelets

In 1d, the shapelet basisfunctions are the eigenfunctions of

$$\hat{H} = \hat{a}^{\dagger}\hat{a} + \frac{1}{2}$$

where $\hat{a}^{\dagger}$ and $\hat{a}$ are the creation and annihilation operators, with:

$$\hat{a}^{\dagger} = \hat{x} - i\hat{p}$$
$$\hat{a} = \hat{x} + i\hat{p}$$
$$\left[\hat{a}^{\dagger}, \hat{a}\right] = 1$$

We can define 2d angular momentum operators as follows:

$$\hat{L}_l = \hat{a}_l{}^\dagger \hat{a}_l$$
$$\hat{L}_r = \hat{a}_l{}^\dagger \hat{a}_r$$
$$\hat{a}_l = \frac{1}{\sqrt{2}}(\hat{a}_1 + i\hat{a}_2)$$
$$\hat{a}_r = \frac{1}{\sqrt{2}}(\hat{a}_1 - i\hat{a}_2)$$

The Hamiltonian may thus be written as:

$$\hat{H} = a_l^\dagger a_l + a_r^\dagger a_r + 1 = \hat{N}_l + \hat{N}_r + 1$$

Polar shapelets

The operators $\hat{N}_r$ and $\hat{N}_l$ form a complete set of observables (you can work out the commutation relations yourself). The eigenfunctions,

$$\hat{N}_l|n_l, n_r >= n_l|n_l, n_r >,$$

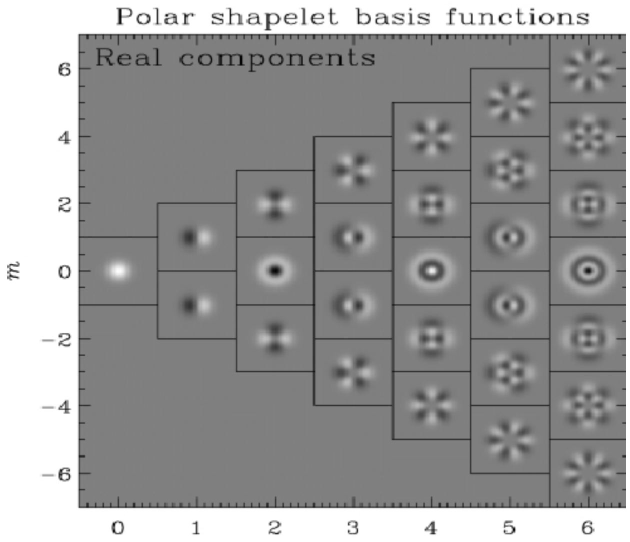
are the basisfunctions for polar shapelets. The eigenfunctions can be defined in terms of r and θ:

$$\chi_{n,m}(r,\theta,\beta) = \frac{-1^{(n-|m|)/2}}{\beta^{|m|+1}}\left[\frac{((n-|m|)/2)!}{\pi((n+|m|)/2)!}\right]^{\frac{1}{2}} \times \\ r^{|m|} L^{|m|}_{(n-|m|)/2}\frac{r^2}{\beta^2} e^{-\frac{r^2}{2\beta^2}} e^{-im\theta}$$

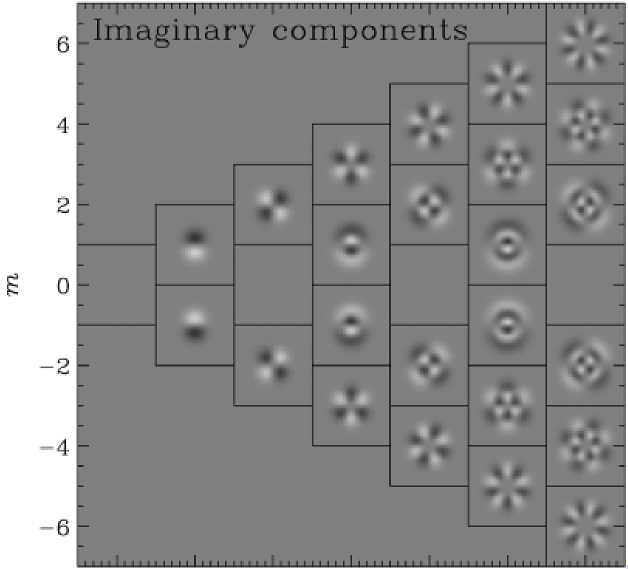
L^q_p is the associated Laguerre polynomial.

Polar Shapelets

Shapelets
Mark den Brok
Introduction
Cartesian Shapelets
Polar Shapelets
Summary
Appendix
For Further Reading



Polar Shapelets



Application: morphological classification

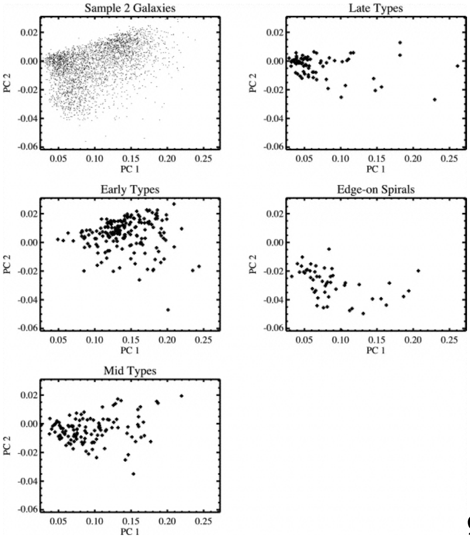
Galaxies come in different types, p.e. spirals, ellipticals. Classification is always a problem, because it is somewhat subjective. The authors below try to classify galaxies using Shapelets.

Recipe:

1. Select suitable sample of galaxies from SDSS
2. Put galaxies at same redshift (by rebinning)
3. Convolve galaxies to common PSF
4. Perform Principal Components Analysis of shapelet coefficients.

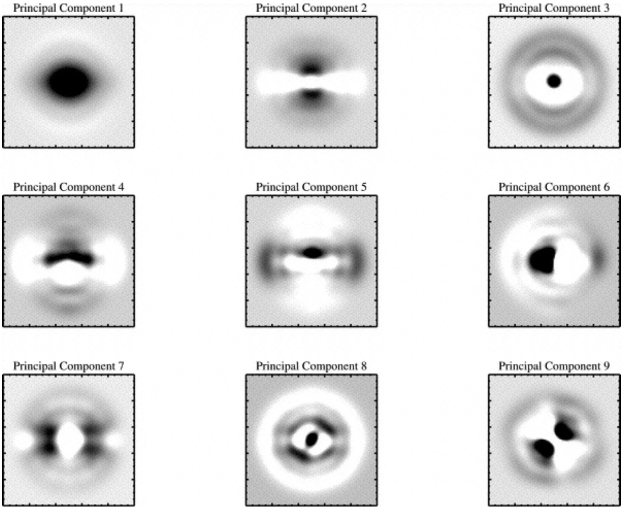
(Kelly & McKay, astro-ph:/307395)

Application: morphological classification



9 dimensions needed

Application: morphological classification



Application: image simulation

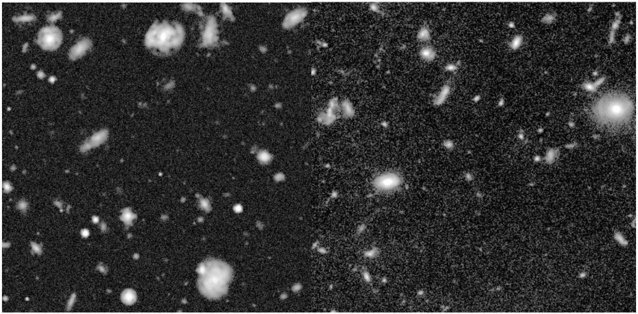
There exists some packages for image simulation (p.e. IRAF's *artdata), but most packages only provide symmetric profiles. The creation of bars, spiral arms &c is more difficult.*

Recipe:

1. *Draw large number of galaxies from sample (in this case, HDF).*
2. *Determine the distribution of shapelet coefficients*
3. *Pick shapelet coefficients at random from this distribution*

(Massey, Refregier, Conselice & Bacon, astro-ph/0301449)

Application: image simulation



Application: image simulation

Image simulation with Shapelets has another nicety: Gravitationally shearing an image is can be done analytically. First, let's look at a general image transformation, using infinitesimal displacement:

$$\mathbf{x} -> \mathbf{x}' = (1 + \mathbf{\Psi})\mathbf{x} + \epsilon$$
$$\mathbf{\Psi} = \begin{pmatrix} \kappa + \gamma_1 & \gamma_2 - \rho \\ \gamma_2 + \rho & \kappa - \gamma_1 \end{pmatrix}$$

Assuming all quantities are small, we can describe the transformed image as:

$$f' \approx (1 + \rho\hat{R} + \epsilon\hat{T} + \kappa\hat{K} + \gamma_i\hat{S}_i)f$$

Application: image simulation

One of the beautiful things of the Shapelet formalism is, that these transformation matrices are expressible in creation and annihilation operators.

$$
\begin{aligned}
\hat{R} &= -i\left(\hat{x}_1\hat{p}_2 - \hat{x}_2\hat{p}_1\right) = \hat{a}_1\hat{a}_2^\dagger - \hat{a}_1^\dagger\hat{a}_2 \\
\hat{K} &= -i\left(\hat{x}_1\hat{p}_1 + \hat{x}_2\hat{p}_2\right) = 1 + \frac{1}{2}\left(\hat{a}_1^{\dagger 2} + \hat{a}_2^{\dagger 2} - \hat{a}_1^2 - \hat{a}_2^2\right) \\
\hat{S}_1 &= -i\left(\hat{x}_1\hat{p}_1 - \hat{x}_2\hat{p}_2\right) = \frac{1}{2}\left(\hat{a}_1^{\dagger 2} - \hat{a}_2^{\dagger 2} - \hat{a}_1^2 + \hat{a}_2^2\right) \\
\hat{S}_2 &= -i\left(\hat{x}_1\hat{p}_2 + \hat{x}_2\hat{p}_1\right) = \hat{a}_1^\dagger\hat{a}_2^\dagger - \hat{a}_1\hat{a}_2 \\
\hat{T}_j &= -i\hat{p}_j = \frac{1}{\sqrt{2}}(\hat{a}_j^\dagger - \hat{a}_j), \quad j = 1, 2.
\end{aligned}
$$

The infinitesimal transformations are easily converted to finite displacements, by taking the exponent of the matrix.
This method has for example been used for STEP 2 (Shear Testing Programme), where a set of simulated sheared galaxies was created. The purpose of this was recovering the shear of these galaxies, to test how well weak lensing methods perform.

Summary

- Shapelets are a way to decompose (astronomical) images in a set of orthogonal basisfunctions.
- The basisfunctions come in two tastes: cartesian and polar shapelets
- The basisfunctions have nice properties:
  1. Analytical expressions for flux, first, second moment
  2. Analytical convolution/deconvolution (calculationally easy)
- Applications include:
  1. Weak gravitational lensing
  2. Accurate photometry
  3. Image compression
  4. Image simulation
  5. Morphological classification

For Further Reading I

- Refregier A. 2003, MNRAS 338 35
- Massey R. & Refregier A. 2005, MNRAS 363, 197
- Refregier A. & Bacon D. 2003, MNRAS 338 48
- Massey R., Refregier A., Conselice C. & Bacon D. 2004, MNRAS 348 214
- Kelly B. & McKay T. 2004, AJ, 127, 625
- Massey R. et al. 2006, MNRAS submitted