I. BOUND MUON DECAY PROCESS

A. General notation

This program is purposed to evaluate the total transition rate of the tree-level bound muon decay into a free electron

$$\mu^- \to e^- + \overline{\nu}_e + \nu_\mu. \tag{1}$$

The muon is assumed to be bound by a central-field nuclear potential and it's state is defined by three quantum numbers $n_\mu \varkappa_\mu \mu_\mu$, where n is the principal quantum number, $\varkappa$ is the relativistic quantum number, and μ is the projection of the total angular momentum j. The total transition rate averaged over the initial states of the muon and summed over the final states of the electron can be written as

$$\frac{W_{\rm tot}\left(n_\mu \varkappa_\mu\right)}{W_0} = \frac{16}{m_\mu^5}\frac{1}{\Pi_{j_\mu}^2}\sum_{\varkappa_e}\Pi_{j_e}^2\int_{m_e}^{E_\mu} dE_e \int_0^{k^0} dk\, k^2 \sum_L \left(C_{j_e\frac{1}{2}L0}^{j_\mu\frac{1}{2}}\right)^2 \left\{k^2\left(I_L^1 + I_L^4\right) - k^0 k 2{\rm Re} I_L^2 + \left[\left(k^0\right)^2 - k^2\right] I_L^3\right\}, \tag{2}$$

where E_μ is the energy of the muon, m_μ and m_e are the muon and electron masses, respectively, $k^0 = E_\mu - E_e$,

$$W_0 = \frac{G_{\rm F}^2 m_\mu^5}{192\pi^3} \tag{3}$$

is the free-muon decay rate, and $G_{\rm F}$ is the Fermi constant. The sum over $\varkappa_e$ is infinite whereas the sum over L is governed by the selection rules of Clebsch-Gordan coefficient.

Other notations used in Eq. (2) are: the symbol $\Pi_{ab\ldots c} = \sqrt{2a+1}\cdot\sqrt{2b+1}\cdot\cdots\cdot\sqrt{2c+1}$, $C_{j_1\mu_1 j_2\mu_2}^{j_3\mu_3}$ is the Clebsch-Gordan coefficient. The quantities I_L^{1-4} are given by

$$I_L^1\left(k\right) = \Pi_L^2\left|\sum_{\beta=\pm1}\left[E\left(l_e,l_\mu,L\right)R_{1,L}^\beta\left(e,\mu\right) - iO\left(l_e,l_\mu,L\right)\beta R_{2,L}^\beta\left(e,\mu\right)\right]\right|^2, \tag{4}$$

$$\begin{aligned} I_L^2\left(k\right) = \sum_l i^{L-l}\Pi_{lL}C_{L010}^{l0}\sum_{\beta=\pm1}\left[E\left(l_e,l_\mu,L\right)R_{1,L}^\beta\left(e,\mu\right) + iO\left(l_e,l_\mu,L\right)\beta R_{2,L}^\beta\left(e,\mu\right)\right]\times \\ \times\left[-iO\left(l_e,l_\mu,l\right)\sum_{\beta=\pm1}U_l^\beta\left(\varkappa_e,\varkappa_\mu|L\right)R_{2,l}^\beta\left(e,\mu\right) + E\left(l_e,l_\mu,l\right)\sum_\beta \beta S_l^\beta\left(\varkappa_e,\varkappa_\mu|L\right)R_{1,l}^\beta\left(e,\mu\right)\right], \end{aligned} \tag{5}$$

$$I_L^3\left(k\right) = \sum_l \Pi_l^2\left|\left[iO\left(l_e,l_\mu,l\right)\sum_{\beta=\pm1}U_l^\beta\left(\varkappa_e,\varkappa_\mu|L\right)R_{2,l}^\beta\left(e,\mu\right) - E\left(l_e,l_\mu,l\right)\sum_\beta\beta S_l^\beta\left(\varkappa_e,\varkappa_\mu|L\right)R_{1,l}^\beta\left(e,\mu\right)\right]\right|^2, \tag{6}$$

$$\begin{aligned} I_L^4\left(k\right) = &\sum_{l_1 l_2} i^{-l_1+l_2}\Pi_{l_1 l_2}C_{L010}^{l_2 0}C_{L010}^{l_1 0}\times \\ &\times\left[iO\left(l_e,l_\mu,l_1\right)\sum_{\beta=\pm1}U_{l_1}^\beta\left(\varkappa_e,\varkappa_\mu|L\right)R_{2,l_1}^\beta\left(\mu,e\right) - E\left(l_e,l_\mu,l_1\right)\sum_\beta\beta S_{l_1}^\beta\left(\varkappa_e,\varkappa_\mu|L\right)R_{1,l_1}^\beta\left(e,\mu\right)\right] \\ &\times\left[-iO\left(l_e,l_\mu,l_2\right)\sum_{\beta=\pm1}U_{l_2}^\beta\left(\varkappa_e,\varkappa_\mu|L\right)R_{2,l_2}^\beta\left(\mu,e\right) - E\left(l_e,l_\mu,l_2\right)\sum_\beta\beta S_{l_2}^\beta\left(\varkappa_e,\varkappa_\mu|L\right)R_{1,l_2}^\beta\left(e,\mu\right)\right]. \end{aligned} \tag{7}$$

The radial matrix elements of the first and second type are given by

$$\begin{aligned} R^{\beta}_{1,l}\left(e,\mu\right) &= \int F^{\beta}_{e}\left(r\right) F^{\beta}_{\mu}\left(r\right) j_{l}\left(kr\right)\, dr, \\ R^{\beta}_{2,l}\left(e,\mu\right) &= \int F^{\beta}_{e}\left(r\right) F^{-\beta}_{\mu}\left(r\right) j_{l}\left(kr\right)\, dr, \end{aligned} \tag{8}$$

where $\beta = \pm 1$ enumerates the large and small component of the relativistic central-field radial wave function, respectively, and $j_l\left(z\right)$ is the spherical Bessel function of order l. The symbols $E\left(a,b,c\right)$ and $O\left(a,b,c\right)$ are equal to 1 if $a+b+c$ is an even or odd number and 0 otherwise. The functions $U^{\beta}_{l}\left(\varkappa_a,\varkappa_b|L\right)$ and $S^{\beta}_{l}\left(\varkappa_a,\varkappa_b|L\right)$ are defined [1–3] by

$$\begin{aligned} U^{\beta}_{l}\left(\varkappa_a,\varkappa_b|L\right) &= \frac{c\left(l,L\right)}{b\left(l,L\right)}\left(\varkappa_a + \left(-1\right)^{l-L}\varkappa_b + \beta a\left(l,L\right)\right), \\ S^{\beta}_{l}\left(\varkappa_a,\varkappa_b|L\right) &= \frac{c\left(l,L\right)}{b\left(l,L\right)}\left(\varkappa_a + \left(-1\right)^{l-L+1}\varkappa_b + \beta a\left(l,L\right)\right), \end{aligned} \tag{9}$$

with

$$a\left(l,L\right) = \frac{l\left(l+1\right) - L\left(L+1\right)}{2}, \qquad b\left(l,L\right) = \sqrt{1-\frac{1}{2}\delta_{lL}} \qquad c\left(l,L\right) = \sqrt{\frac{2\left(2L+1\right)}{\left(l+L\right)\left(l+L+1\right)\left(l+L+2\right)}}. \tag{10}$$

B. Program structure

1. *Overview*

The program might be full of flaw and profanity. It consists of one function which implements Eq. (2). The program depends on the several libraries:

- The radial wave functions $F^{\beta}_{\mu}\left(r\right)$ and $F^{\beta}_{e}\left(r\right)$ are calculated using the package RADIAL [4].
- The spherical Bessel functions $j_l\left(z\right)$ are evaluated using Algorithm 644 of ACM digital library [5].
- The routine for evaluation of the Clebsch-Gordan coefficients is borrowed from the program hfd [6].

For convenience of the user, the source code of the above-mentioned routines is shipped with the program. The internal data structure engine as well as many other solutions are creatively adopted (stolen) from EXP-T [7].

2. *Settings*

The program settings are specified in an input file which is passed to the program as the command-line argument. The input file has the following list of keys

Table I. List of settings.

`muon` ⟨`integer`⟩ ⟨`integer`⟩

The principal and relativistic quantum numbers n_μ and $\varkappa_\mu$ of the muon

`akappa_e` ⟨`integer`⟩ ⟨`integer`⟩

The minimum and maximum values of $|\varkappa_e|$ used to evaluate the sum $\Sigma_{\varkappa_e}$ in Eq. (2)

`energy_e` ⟨`float`⟩ ⟨`float`⟩

The minimum and maximum energy of the emitted electron (MeV)
used to evaluate the integral $\int_{E_{\min}}^{E_{\max}} dE_e$ in Eq. (2)
The electron rest mass in included in the energy.

`znucl` ⟨`integer`⟩

The nuclear charge number Z

`nuclmodel` ⟨`integer`⟩

The nuclear model

- 1 corresponds to the point-like nucleus model (Coulomb potential)
- 2 corresponds to the Fermi model

`rms` ⟨`float`⟩

Root-mean-squared radius of the nucleus (fm).
It is not referenced if the point-like nucleus model has been specified.

`rbox` ⟨`float`⟩ ⟨`float`⟩

The radial box parameters (relativistic units), in other words,
the minimum and maximum limits of the radial integration $\int_{r_1}^{r_2} dr$ in Eq. (8).

`nquad_r` ⟨`integer`⟩ ⟨`integer`⟩

Number of the radial subintervals in which the interval $[r_1; r_2]$ is split
and number of quadrature nodes in each subinterval.
If the number of the radial subintervals equals to 0,
then the very interval $[r_1; r_2]$ is used to spawn the quadrature nodes.
The maximum number of the quadrature nodes on each subinterval is 64

`nquad_k` ⟨`integer`⟩ ⟨`integer`⟩

The same as parameter nquad_r but for the k quadrature grid
which is used to evaluate the integral $\int_0^{k^0} dk$ in Eq. (2)

`nquad_e` ⟨`integer`⟩ ⟨`integer`⟩

The same as parameter nquad_r but for the E_e quadrature grid
which is used to evaluate the integral $\int_{E_{\min}}^{E_{\max}} dE_e$ in Eq. (2)

`print` ⟨`low` ∧ `high`⟩

The printing level

- `low` prints $W_{\text{tot}}\left(n_\mu \varkappa_\mu\right)/W_0$ for each digested values of $|\varkappa_e|$
  and the current partial sum
- `high` in addition prints the differential transition rate
  for each value of the electron energy E_e
  and some additional diagnostics information.

3. Build

The build process is implemented using `make` software. To compile the dependencies which are put in a static libraries, use `make libs` which will call the nested Makefiles. To compile and link the program, call `make` or `make all`. To remove the C-object files and the program binary, call `make clear`. To purge everything except the source files, use `make clear_all`.

4. Test runs

The program lacks of the testing suite. The examples of the input and output files are collected in the directory `examples`. The results of the small-scale test runs are compared with Refs. [8, 9] in the following Table

Table III. Normalized total bound-muon decay rates $W_{\rm tot}/W_0$ for the muon in $1s_{1/2}$ state for various nuclear charge numbers Z evaluated in the test runs employing the Fermi model of the nuclear charge distribution with the specified root-mean-squared radius, rms, and comparison with Refs. [8, 9].

Z	rms	test run	Ref. [8]	Ref. [9]
8	2.6991	0.997	0.994	
13	3.0610	0.993	0.992	0.9934
14	3.1224	0.992	0.991	
20	3.4776	0.983	0.981	
26	3.7377	0.971	0.971	
40	4.2694	0.938	0.936	
42	4.3847	0.933	0.932	
50	4.6393	0.914	0.914	
82	5.5012	0.847	0.847	
83	5.5211	0.845	0.845	

5. Known issues

Because of the large dependence on the external routines, the program might be sensitive to the input parameters. Most of the known issues come from inconsistent specification of the intervals of the radial integration. Too large or too small values of `rbox` might cause the RADIAL routine to break since it will not be able to construct the solution. If the user suspects the issue comes from RADIAL, try to change `rbox`, advice the documentation or even introduce necessary corrections to the source code of the library.

Another issue might come from the fact that the end-point of the electron spectrum is not known in advance. Wrong specification of `energy_e` might force the program to obtain nonphysical transition rate. To deal with it, we advice the user to run the program with some input, find out the energy of the muon, and use this value in the input.

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