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REMARKS ON GENERATORS AND DIMENSIONS OF TRIANGULATED CATEGORIES

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To Pierre Deligne with admiration

ABSTRACT. In this paper we prove that the dimension of the bounded derived category of coherent sheaves on a smooth quasi-projective curve is equal to one. We also discuss dimension spectrums of these categories.

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Let $\mathcal{T}$ be a triangulated category. We say that an object $E \in \mathcal{T}$ is a *classical generator* for $\mathcal{T}$ if the category $\mathcal{T}$ coincides with the smallest triangulated subcategory of $\mathcal{T}$ which contains E and is closed under direct summands.

If a classical generator generates the whole category for a finite number of steps then it called a *strong generator*. More precisely, let $\mathcal{I}_1$ and $\mathcal{I}_2$ be two full subcategories of $\mathcal{T}$. We denote by $\mathcal{I}_1 * \mathcal{I}_2$ the full subcategory of $\mathcal{T}$ consisting of all objects such that there is a distinguished triangle $M_1 \to M \to M_2$ with $M_i \in \mathcal{I}_i$. For any subcategory $\mathcal{I} \subset \mathcal{T}$ we denote by $\langle \mathcal{I} \rangle$ the smallest full subcategory of $\mathcal{T}$ containing $\mathcal{I}$ and closed under finite direct sums, direct summands and shifts. We put $\mathcal{I}_1 \diamond \mathcal{I}_2 = \langle \mathcal{I}_1 * \mathcal{I}_2 \rangle$ and we define by induction $\langle \mathcal{I} \rangle_k = \langle \mathcal{I} \rangle_{k-1} \diamond \langle \mathcal{I} \rangle$. If $\mathcal{I}$ consists of an object E we denote $\langle \mathcal{I} \rangle$ as $\langle E \rangle_1$ and put by induction $\langle E \rangle_k = \langle E \rangle_{k-1} \diamond \langle E \rangle_1$.

Definition 1. Now we say that E is a *strong generator* if $\langle E \rangle_n = \mathcal{T}$ for some $n \in \mathbb{N}$.

Note that E is classical generator if and only if $\bigcup_{k \in \mathbb{Z}} \langle E \rangle_k = \mathcal{T}$. It is also easy to see that if a triangulated category $\mathcal{T}$ has a strong generator then any classical generator of $\mathcal{T}$ is strong as well.

Following to [2] we define the dimension of a triangulated category.

Definition 2. The *dimension* of a triangulated category $\mathcal{T}$, denoted by $\dim \mathcal{T}$, is the minimal integer $d \geqslant 0$ such that there is $E \in \mathcal{T}$ with $\langle E \rangle_{d+1} = \mathcal{T}$.

We also can define the dimension spectrum of a triangulated category as follows.

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Definition 3. The *dimension spectrum* of a triangulated category $\mathcal{T}$, denoted by $\sigma(\mathcal{T})$, is a subset of $\mathbb{Z}$, which consists of all integer $d \geqslant 0$ such that there is $E \in \mathcal{T}$ with $\langle E\rangle_{d+1} = \mathcal{T}$ and $\langle E\rangle_d \neq \mathcal{T}$.

A. Bondal and M. Van den Bergh showed in [1] that the triangulated category of perfect complexes $\mathfrak{Perf}(X)$ on a quasi-compact quasi-separated scheme X has a classical generator. (Recall that a complex of $\mathcal{O}_X$-modules is called perfect if it is locally quasi-isomorphic to a bounded complex of vector bundles.)

For the triangulated category of perfect complexes on a quasi-projective scheme we can present a classical generator directly.

Theorem 4. *Let X be a quasi-projective scheme of dimension d over an arbitrary field k and let $\mathcal{L}$ be a very ample line bundle on X. Then the object $\mathcal{E} = \bigoplus_{i=k-d}^{k} \mathcal{L}^i$ is a classical generator for the triangulated category of perfect complexes $\mathfrak{Perf}(X)$.*

Proof. The scheme X is an open subscheme of a projective scheme $X' \subset \mathbb{P}^N$ and $\mathcal{L}$ is the restriction of $\mathcal{O}_{\mathbb{P}^N}(1)$ on X. Let us take $N+1$ linear independent hyperplanes $H_i \subset \mathbb{P}^N$, $i = 0, \dots, N$. In this case the intersection $H_0 \cap \dots \cap H_N$ is empty. The hyperplanes H_i give a section s of the vector bundle $U = \mathcal{O}(1)^{\oplus(N+1)}$ which does not have zeros. This implies that the Koszul complex induced by s

$$0 \to \Lambda^{N+1}(U^*) \to \Lambda^N(U^*) \to \cdots \to \Lambda^2(U^*) \to U^* \to \mathcal{O}_{\mathbb{P}^N} \to 0$$

is exact on $\mathbb{P}^N$. Consider the restriction of the truncated complex on X

$$\Lambda^{d+1}(U_X^*) \to \cdots \to \Lambda^2(U_X^*) \to U_X^*.$$

It has two nontrivial cohomologies, one of which is $\mathcal{O}_X$. And, moreover, since the dimension of X is equal to d the sheaf $\mathcal{O}_X$ is a direct summand of this complex. Tensoring this complex with $\mathcal{L}^{k+1}$ we obtain that the triangulated subcategory which contains $\mathcal{L}^i$ for $i = k-d, \dots, k$ also contains $\mathcal{L}^{k+1}$. Thus, it contains $\mathcal{L}^i$ for all $i \geqslant k-d$. By duality this category contains also all $\mathcal{L}^i$ for all $i \leqslant k$. Thus we have all powers $\mathcal{L}^i$, where $i \in \mathbb{Z}$.

Finally, it easy to see that $\{\mathcal{L}^i\}_{i\in\mathbb{Z}}$ classically generate the triangulated category of perfect complexes $\mathfrak{Perf}(X)$. Indeed, for any perfect complex E we can construct a bounded above complex $P^{\boldsymbol{\cdot}}$, where all P^k are direct sums of line bundles $\mathcal{L}^i$, together with a quasi-isomorphism $P^{\boldsymbol{\cdot}} \xrightarrow{\sim} E$. Consider the brutal truncation $\sigma^{\geqslant -m}P^{\boldsymbol{\cdot}}$ for sufficiently large m and the map $\sigma^{\geqslant -m}P^{\boldsymbol{\cdot}} \to E$. The cone of this map is isomorphic to $\mathcal{F}[m+1]$, where $\mathcal{F}$ is a vector bundle. And since the $\mathrm{Hom}(E, \mathcal{F}[m+1]) = 0$ for sufficiently large m we get that E is a direct summand of $\sigma^{\geqslant -m}P^{\boldsymbol{\cdot}}$. $\square$

A. Bondal and M. Van den Bergh also proved that for any smooth separated scheme X the triangulated category of perfect complexes $\mathfrak{Perf}(X)$ has a strong generator [1, Th. 3.1.4]. Furthermore, R. Rouquier showed that for quasi-projective scheme X the property to be regular is equivalent to the property that the triangulated category of perfect complexes $\mathfrak{Perf}(X)$ has a strong generator (see [2, Prop. 7.35]). On the other hand, there is a remarkable result of R. Rouquier which says that under some general conditions the bounded derived category of coherent sheaves $\mathrm{D}^b(\mathrm{coh}(X))$ has a strong generator. More precisely it says

Theorem 5 (R. Rouquier, [2, Th. 7.39])**.** *Let X be a separated scheme of finite type. Then there are an object $E \in \mathrm{D}^b(\mathrm{coh}(X))$ and an integer $d \in \mathbb{Z}$ such that $\mathrm{D}^b(\mathrm{coh}(X)) \cong \langle E\rangle_{d+1}$. In particular, $\dim \mathrm{D}^b(\mathrm{coh}(X)) < \infty$.*

Keeping in mind this theorem we can ask about the dimension of the derived category of coherent sheaves on a separated scheme of finite type. It is proved in [2] that

- for a reduced separated scheme X of finite type $\dim \mathrm{D}^b(\mathrm{coh}(X)) \geqslant \dim X$;
- for a smooth affine scheme $\dim \mathrm{D}^b(\mathrm{coh}(X)) = \dim X$;
- for a smooth quasi-projective scheme $\dim \mathrm{D}^b(\mathrm{coh}(X)) \leqslant 2\dim X$.

In this paper we show that the dimension of the derived category of coherent sheaves on a smooth quasi-projective curve C is equal to 1. For affine curves it is known and for $\mathbb{P}^1$ it is evident. Thus, it is sufficient to consider a smooth projective curve of genus $g \geqslant 1$.

Theorem 6. *Let C be a smooth projective curve of genus $g \geqslant 1$. Then the dimension of the triangulated category $\mathrm{D}^b(\mathrm{coh}(C))$ equals 1.*

At first, we should bring an object which generates $\mathrm{D}^b(\mathrm{coh}(C))$ for one step. Let $\mathcal{L}$ be a line bundle on C such that $\deg \mathcal{L} \geqslant 8g$. Let us consider $\mathcal{E} = \mathcal{L}^{-1} \oplus \mathcal{O}_C \oplus \mathcal{L} \oplus \mathcal{L}^2$. We are going to show that $\mathcal{E}$ generates the bounded derived category of coherent sheaves on C for one step, i.e. $\langle \mathcal{E}\rangle_2 = \mathrm{D}^b(\mathrm{coh}\, X)$.

Since any object of $\mathrm{D}^b(\mathrm{coh}(X))$ is a direct sum of its cohomologies it is sufficient to prove that any coherent sheaf $\mathcal{G}$ belongs to $\langle \mathcal{E}\rangle_2$. Further, each coherent sheaf $\mathcal{G}$ on a curve is a direct sum of a torsion sheaf T and a vector bundle $\mathcal{F}$.

Lemma 7. *Let C be a smooth projective curve of genus $g \geqslant 1$ and let $\mathcal{L}$ be a line bundle on C as above. Then there is an exact sequence of the form*

$$(\mathcal{L}^{-1})^{\oplus r_1} \to \mathcal{O}_C^{\oplus r_0} \to T \to 0$$

for any torsion coherent sheaf T on C.

Let $\mathcal{F}$ be a vector bundle on the curve C. Consider the Harder–Narasimhan filtration $0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_n = \mathcal{F}$. It is such filtration that every quotient $\mathcal{F}_i/\mathcal{F}_{i-1}$ is semi-stable and $\mu(\mathcal{F}_i/\mathcal{F}_{i-1}) > \mu(\mathcal{F}_{i+1}/\mathcal{F}_i)$ for all $0 < i < n$, where $\mu(\mathcal{G})$ is the slope of a vector bundle $\mathcal{G}$ and is equal to $c_1(\mathcal{G})/r(\mathcal{G})$.

Main Lemma 8. *Let $\mathcal{L}$ be a line bundle with $\deg \mathcal{L} \geqslant 8g$. Let $\mathcal{F}$ be a vector bundle on C and let $0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_n = \mathcal{F}$ be its Harder–Narasimhan filtration. Choose $0 \leqslant i \leqslant n$ such that $\mu(\mathcal{F}_i/\mathcal{F}_{i-1}) \geqslant 4g > \mu(\mathcal{F}_{i+1}/\mathcal{F}_i)$. Then there are exact sequences of the form*

$$\text{(a)}\quad (\mathcal{L}^{-1})^{\oplus r_1} \xrightarrow{\alpha} \mathcal{O}_C^{\oplus r_0} \to \mathcal{F}_i \to 0, \qquad \text{(b)}\quad 0 \to \mathcal{F}/\mathcal{F}_i \to \mathcal{L}^{\oplus s_0} \xrightarrow{\beta} (\mathcal{L}^2)^{\oplus s_1}.$$

To prove Lemma 7 and Main Lemma 8 we need the following lemma which is well-known.

Lemma 9. *Let $\mathcal{G}$ be a vector bundle on a smooth projective curve C over a field k. Denote by $\overline{\mathcal{G}}$ its pullback on $\overline{C} = C \otimes_k \overline{k}$. Assume that for any line bundle $\mathcal{M}$ on $\overline{C}$ with $\deg \mathcal{M} = d$ we have $H^1(\overline{C}, \overline{\mathcal{G}} \otimes \mathcal{M}) = 0$. Then*

(i) $H^1(C, \mathcal{G}\otimes\mathcal{N}) = 0$ *for any* $\mathcal{N}$ *on* C *with* $\deg\mathcal{N} \geqslant d$;
(ii) *any sheaf* $\mathcal{G}\otimes\mathcal{N}$ *is generated by the global sections for all* $\mathcal{N}$ *with* $\deg\mathcal{N} > d$.

Proof. (i) Since any field extension is strictly flat it is sufficient to check that $H^1(\overline{C}, \overline{\mathcal{G}}\otimes\overline{\mathcal{N}}) = 0$. From an exact sequence

$$0 \to \overline{\mathcal{G}}\otimes\overline{\mathcal{N}}(-x) \to \overline{\mathcal{G}}\otimes\overline{\mathcal{N}} \to (\overline{\mathcal{G}}\otimes\overline{\mathcal{N}})_x \to 0 \tag{1}$$

on $\overline{C}$ we deduce that if $H^1(\overline{C}, \overline{\mathcal{G}}\otimes\overline{\mathcal{N}}(-x)) = 0$ then $H^1(\overline{C}, \overline{\mathcal{G}}\otimes\overline{\mathcal{N}}) = 0$. This implies (i).

(ii) By the same reason as above it is enough to show that the sheaf $\overline{\mathcal{G}}\otimes\overline{\mathcal{N}}$ is generated by the global sections. Since by $H^1(\overline{C}, \overline{\mathcal{G}}\otimes\overline{\mathcal{N}}(-x)) = 0$ the map

$$H^0(\overline{C}, \overline{\mathcal{G}}\otimes\overline{\mathcal{N}}) \to H^0(\overline{C}, (\overline{\mathcal{G}}\otimes\overline{\mathcal{N}})_x)$$

is surjective for any $x \in \overline{C}$. Hence, $\overline{\mathcal{G}}\otimes\overline{\mathcal{N}}$ and $\mathcal{G}\otimes\mathcal{N}$ are generated by the global sections for all $\mathcal{N}$ of degree greater than d. □

Proof of Lemma 7. Any torsion sheaf T is generated by the global sections. Consider the surjective map $\mathcal{O}_C^{\oplus r_0} \to T$, where $r_0 = \dim H^0(T)$. Denote by U the kernel of this map. Now it is evident that $H^1(\overline{U}\otimes\mathcal{M}) = 0$ for any line bundle $\mathcal{M}$ on $\overline{C}$ with $\deg\mathcal{M} \geqslant 2g-1$, because $H^1(\mathcal{M}) = 0$. Applying Lemma 9 we get that $U\otimes\mathcal{L}$ is generated by the global sections. Hence, there is an exact sequence of the form

$$(\mathcal{L}^{-1})^{\oplus r_1} \to \mathcal{O}_C^{\oplus r_0} \to T \to 0.$$

for any torsion sheaf T. □

Proof of the Main Lemma. If $\mathcal{G}$ is a semi-stable vector bundle on C with $\mu(\mathcal{G}) \geqslant 2g$ then by Serre duality we have $H^1(\overline{C}, \overline{\mathcal{G}}\otimes\mathcal{M}) = 0$ for all $\mathcal{M}$ with $\deg\mathcal{M} \geqslant -1$. Therefore, by Lemma 9 the bundle $\mathcal{G}$ is generated by the global sections.

Now $\mathcal{F}_i \subseteq \mathcal{F}$ as an extension of semi-stable sheaves with $\mu \geqslant 4g$ is generated by the global sections as well. Consider the short exact sequence

$$0 \to U \to \mathcal{O}_C^{\oplus r_0} \to \mathcal{F}_i \to 0,$$

where r_0 is the dimension of $H^0(\mathcal{F}_i)$. Take a line bundle $\mathcal{M}$ on $\overline{C}$ of degree $2g$ and consider the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \overline{U}\otimes\mathcal{M}^{-1} \arrow[r] \arrow[d] & (\mathcal{M}^{-1})^{\oplus r_0} \arrow[r] \arrow[d] & \overline{\mathcal{F}}_i\otimes\mathcal{M}^{-1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \overline{U}^{\oplus 2} \arrow[r] \arrow[d] & \mathcal{O}_{\overline{C}}^{\oplus 2r_0} \arrow[r] \arrow[d] & \overline{\mathcal{F}}_i^{\oplus 2} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \overline{U}\otimes\mathcal{M} \arrow[r] \arrow[d] & \mathcal{M}^{\oplus r_0} \arrow[r] \arrow[d] & \overline{\mathcal{F}}_i\otimes\mathcal{M} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Since the sheaf $\overline{\mathcal{F}}_i \otimes \mathcal{M}^{-1}$ is the extension of semi-stable sheaves with $\mu \geqslant 2g$ we have $H^1(\overline{\mathcal{F}}_i \otimes \mathcal{M}^{-1}) = 0$. Hence, the map $H^0(\overline{\mathcal{F}}_i^{\oplus 2}) \to H^0(\overline{\mathcal{F}}_i \otimes \mathcal{M})$ is surjective. Further, we know that the map $H^0(\mathcal{O}_{\overline{C}}^{\oplus 2r_0}) \to H^0(\overline{\mathcal{F}}_i^{\oplus 2})$ is surjective and the map $H^0(\mathcal{O}_{\overline{C}}^{\oplus 2r_0}) \to H^0(\mathcal{M}^{\oplus r_0})$ is injective. This implies that the map $H^0(\mathcal{M}^{\oplus r_0}) \to H^0(\overline{\mathcal{F}}_i \otimes \mathcal{M})$ is surjective as well. Hence $H^1(\overline{U} \otimes \mathcal{M}) = 0$. Therefore, by Lemma 9 the bundle $\overline{U} \otimes \mathcal{M}'$ is generated by the global sections for all $\mathcal{M}'$ with $\deg \mathcal{M}' \geqslant 2g+1$. In particular, $U \otimes \mathcal{L}$ is generated by the global sections. Thus, we get an exact sequence

$$(\mathcal{L}^{-1})^{\oplus r_1} \xrightarrow{\alpha} \mathcal{O}_C^{\oplus r_0} \to \mathcal{F}_i \to 0.$$

Sequence (b) can be obtained by dualizing of sequence a) applied for the sheaf $\mathcal{F}^* \otimes \mathcal{L}$. □

Proof of Theorem 6. At first, since the category of coherent sheaves on C has homological dimension one we see that any torsion sheaf T is a direct summand of the complex of the form $(\mathcal{L}^{-1})^{\oplus r_1} \to \mathcal{O}_C^{\oplus r_0}$. Hence, it belongs to $\langle \mathcal{E} \rangle_2$.

Now consider a vector bundle $\mathcal{F}$ on C with the Harder–Narasimhan filtration $0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_n = \mathcal{F}$. As above let us fix $0 \leqslant i \leqslant n$ such that $\mu(\mathcal{F}_i/\mathcal{F}_{i-1}) \geqslant 4g > \mu(\mathcal{F}_{i+1}/\mathcal{F}_i)$. Applying the Main Lemma we obtain the following long exact sequence

$$0 \to \operatorname{Ker}\alpha \to (\mathcal{L}^{-1})^{\oplus r_1} \xrightarrow{\alpha} \mathcal{O}_C^{\oplus r_0} \to \mathcal{F} \to \mathcal{L}^{\oplus s_0} \xrightarrow{\beta} (\mathcal{L}^2)^{\oplus s_1} \to \operatorname{Coker}\beta \to 0.$$

Furthermore, it is easy to see that the canonical map

$$\operatorname{Ext}^1(\mathcal{L}^{\oplus s_0}, \mathcal{O}_C^{\oplus r_0}) \to \operatorname{Ext}^1(\mathcal{F}/\mathcal{F}_i, \mathcal{F}_i)$$

is surjective. Let us fix $e \in \operatorname{Ext}^1(\mathcal{F}/\mathcal{F}_i, \mathcal{F}_i)$ which defines $\mathcal{F}$ as the extension and choose some its pull back $e' \in \operatorname{Ext}^1(\mathcal{L}^{\oplus s_0}, \mathcal{O}_C^{\oplus r_0})$.

Now let us consider the map

$$\phi\colon (\mathcal{L}^{-1})^{\oplus r_0} \oplus \mathcal{L}^{\oplus s_0}[-1] \to \mathcal{O}_C^{\oplus r_0} \oplus (\mathcal{L}^2)^{\oplus s_1}[-1], \qquad \text{where } \phi = \begin{pmatrix} \alpha & e' \\ 0 & \beta \end{pmatrix} \tag{2}$$

and take a cone $C(\phi)$ of ϕ. The cone $C(\phi)$ is isomorphic to a complex that has three nontrivial cohomologies $H^{-1}(C(\phi)) \cong \operatorname{Ker}\alpha$, $H^1(C(\phi)) \cong \operatorname{Coker}\beta$ and, finally, $H^0(C(\phi)) \cong \mathcal{F}$. Thus, $\mathcal{F}$ is a direct summand of $C(\phi)$ and, consequently, it belongs to $\langle \mathcal{E} \rangle_2$. This implies that the whole bounded derived category of coherent sheaves on C coincides with $\langle \mathcal{E} \rangle_2$ and the dimension of $\mathrm{D}^b(\mathrm{coh}(C))$ is equal to 1. □

Having in view of the given theorem we may assume, that the following conjecture can be true.

Conjecture 10. *Let X be a smooth quasi-projective scheme of dimension n. Then* $\dim \mathrm{D}^b(\mathrm{coh}(X)) = n$.

Remark 11. For a non regular scheme it is evidently not true. For example, the dimension of the bounded derived category of coherent sheaves on the zero-dimension scheme $\operatorname{Spec}(k[x]/x^2)$ equals to 1.

It is also very interesting to understand what the spectrum $\sigma(\mathrm{D}^b(\mathrm{coh}(X)))$ forms. In particular we can ask the following questions

Question 12. *Is the spectrum of the bounded derived category of coherent sheaves on a smooth quasi-projective scheme bounded? Is it bounded for a non smooth scheme?*

Question 13. *Does the spectrum of the bounded derived category of coherent sheaves on a (smooth) quasi-projective scheme form an integer interval?*

Let us try to calculate the dimension spectra of the derived categories of coherent sheaves on some smooth curves.

Proposition 14. *Let C be a smooth affine curve. Then the dimension spectrum $\sigma(\mathrm{D}^b(\mathrm{coh}\, C))$ coincides with $\{1\}$.*

Proof. If $\mathcal{E}$ is a strong generator then it has a some locally free sheaf $\mathcal{F}$ as a direct summand. Now since C is affine then there is an exact sequence of the form

$$\mathcal{F}^{\oplus r_1} \to \mathcal{F}^{\oplus r_0} \to \mathcal{G} \to 0$$

for any coherent sheaf $\mathcal{G}$ on C. Hence, any coherent sheaf $\mathcal{G}$ belongs to $\langle \mathcal{E} \rangle_2$. Since the global dimension of $\mathrm{coh}\, C$ is equal to 1 we obtain that $\langle \mathcal{E} \rangle_2 = \mathrm{D}^b(\mathrm{coh}\, C)$. □

We can also find the dimension spectrum of the projective line.

Proposition 15. *The dimension spectrum $\sigma(\mathrm{D}^b(\mathrm{coh}\, \mathbb{P}^1))$ coincides with the set $\{1, 2\}$.*

Proof. Indeed, 1 is the dimension. And, for example, the object $\mathcal{E} = \mathcal{O}(-1) \oplus \mathcal{O}$ generate the whole category $\mathrm{D}^b(\mathrm{coh}\, \mathbb{P}^1)$ for one step. Now, the object $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_p$, where p is a point, is a generator, because $\mathcal{O}(-1)$ belongs to $\langle \mathcal{E} \rangle_2$. This also implies that $\langle \mathcal{E} \rangle_3 \cong \mathrm{D}^b(\mathrm{coh}\, \mathbb{P}^1)$. On the other hand, $\langle \mathcal{E} \rangle_2 \ncong \mathrm{D}^b(\mathrm{coh}\, \mathbb{P}^1)$. To see it we can check that an object $\mathcal{O}_q$, where $q \neq p$, doesn't belong to $\langle \mathcal{E} \rangle_2$. Indeed, $\mathcal{O}_q$ is completely orthogonal to $\mathcal{O}_p$ (i.e. all Ext groups between these objects are trivial) and doesn't belong to subcategory generated by $\mathcal{O}$. Finally, it easy to see that any object $\mathcal{E}$, which generates the whole category, generates it at least for two steps, i.e. $\langle \mathcal{E} \rangle_3 \cong \mathrm{D}^b(\mathrm{coh}\, \mathbb{P}^1)$. If $\mathcal{E}$ contains as direct summands two different line bundles than it generates the whole category for one step. If $\mathcal{E}$ has only one line bundle as a direct summand then it also has a torsion sheaf as a direct summand. This implies that $\langle \mathcal{E} \rangle_2$ has another line bundle. Therefore, $\langle \mathcal{E} \rangle_3$ is the whole category. □

Another simple result says

Proposition 16. *Let C be a smooth projective curve of genus $g > 0$ over a field k. Assume that C has at least two different points over k. Then the dimension spectrum $\sigma(\mathrm{D}^b(\mathrm{coh}\, C))$ contains $\{1, 2\}$ as a proper subset, i.e. $\{1, 2\}$ is strictly contained in the dimension spectrum.*

Proof. The spectrum contains 1 as the dimension of the category. Let us now take a line bundle $\mathcal{L}$ on C which satisfies the condition as in Theorem 8, i.e. $\deg \mathcal{L} \geqslant 8g$ and consider the object $\mathcal{E} = \mathcal{O}_C \oplus \mathcal{L}^2$. It is easy to see that the line bundles $\mathcal{L}^{-1}$ and $\mathcal{L}$ belong to $\langle \mathcal{E} \rangle_2$, because there are exact sequences

$$0 \to \mathcal{O}_C \to \mathcal{L}^{\oplus 2} \to \mathcal{L}^2 \to 0 \qquad \text{and} \qquad 0 \to (\mathcal{L}^{-1})^{\oplus 2} \to \mathcal{O}_C^{\oplus 3} \to \mathcal{L}^2 \to 0.$$

The proof of Theorem 6 (see the map (2)) implies that $\langle\mathcal{E}\rangle_3 \cong \mathrm{D}^b(\mathrm{coh}\, C)$. On the other hand, the subcategory $\langle\mathcal{E}\rangle_2$ doesn't coincides with the whole $\mathrm{D}^b(\mathrm{coh}\, C)$. For example, a nontrivial line bundle $\mathcal{M}$ from $\mathrm{Pic}^0\, C$ doesn't belong to $\langle\mathcal{E}\rangle_2$, because it is completely orthogonal to the structure sheaf $\mathcal{O}_C$ and, evidently, could not be obtained from the line bundle $\mathcal{L}^2$.

Let us take a point $p \in C$ and consider the object $\mathcal{E} = \mathcal{O}_C \oplus \mathcal{O}_p$, where $\mathcal{O}_p$ is the skyscraper in p. This object is a strong generator and we can show that $\langle\mathcal{E}\rangle_3 \neq \mathrm{D}^b(\mathrm{coh}\, C)$. Take another point $q \neq p$ and consider the skyscraper sheaf $\mathcal{O}_q$. It is completely orthogonal to $\mathcal{O}_p$ and has only one-dimensional Ext^1 to $\mathcal{O}_C$. Hence, if $\mathcal{O}_q$ belongs to $\langle\mathcal{E}\rangle_3$ then it should be a direct summand of an object M which is included in an exact triangle of the form

$$\mathcal{O}_C^{\oplus k} \to N \to M \to \mathcal{O}_C^{\oplus k}[1],$$

where $N \in \langle\mathcal{E}\rangle_2$. Since Ext^1 from $\mathcal{O}_q$ to $\mathcal{O}_C$ is one-dimensional we can take $k = 1$. The composition of the map $\mathcal{O}_q \to M$ with $M \to \mathcal{O}_C$ should be the nontrivial Ext^1 from $\mathcal{O}_q$ to $\mathcal{O}_C$. Now object N is a direct sum of indecomposable objects from $\langle\mathcal{E}\rangle_2$. It is easy to see that we can consider only objects for which there are nontrivial homomorphisms from $\mathcal{O}_C$ and nontrivial homomorphisms to $\mathcal{O}_q$. All other can be removed from N. Thus N is a direct sum of $\mathcal{O}(p)$ and objects U that are extensions

$$0 \to \mathcal{O}_C^{\oplus r_1} \to U \to \mathcal{O}_C^{\oplus r_2} \to 0. \tag{3}$$

Finally, split embedding $\mathcal{O}_q \to M$ gives us a nontrivial map from $\mathcal{O}(q)$ to N. But there are no nontrivial maps from $\mathcal{O}(q)$ to $\mathcal{O}(p)$ and to U of the form (3). Therefore, $\mathcal{O}_q$ can not belong to $\langle\mathcal{E}\rangle_3$. □

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