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RESEARCH ARTICLE

Rouquier dimension is Krull dimension for normal toric varieties

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Abstract
We prove that for any normal toric variety, the Rouquier dimension of its bounded derived category of coherent sheaves is equal to its Krull dimension. Our proof uses the coherent-constructible correspondence to translate the problem into the study of Rouquier dimension for certain categories of constructible sheaves.

Keywords Rouquier dimension · Derived categories · Toric varieties

Mathematics Subject Classification 14M25 · 32S60 · 14F08 · 14J33

1 Introduction

The Rouquier dimension of a triangulated category $\mathcal{T}$ is a measure of its homological complexity. More precisely, given a generator $G \in \mathcal{T}$, the *generation time* is the minimal number of exact triangles needed to form every object of $\mathcal{T}$ from G. The Rouquier dimension is simply the infimum of these generation times.

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Rouquier, who introduced this invariant in [29], primarily studied it for $\mathcal{T} = \mathrm{Coh}(X)$, the derived category of coherent sheaves of a scheme X. Notably, he showed that the Rouquier dimension of $\mathrm{Coh}(X)$ is bounded below by the Krull dimension of X and remarked that he knew of no cases where the two were not equal. Orlov [26] went on to prove that the two invariants agree in dimension 1 and proposed that Rouquier dimension and Krull dimension should agree for all smooth algebraic varieties.

Conjecture 1.1 (Orlov) *For any smooth algebraic variety, the Rouquier dimension of its derived category of bounded coherent sheaves agrees with the Krull dimension of the variety.*

While this conjecture is still largely open, is has been established in various special cases [1, 2, 4–7, 9, 12, 25, 26, 28, 29, 33].[1] In this note, we settle the toric case:

Theorem 1.2 (Corollary 3.7) *For any normal toric variety Conjecture 1.1 holds.*

This result has also been obtained independently and will appear in forthcoming work of Hanlon–Hicks–Lazarev [20].

Our method of proof passes through a form of homological mirror symmetry for toric varieties (and certain toric stacks) called the coherent-constructible correspondence. This correspondence goes back to the pioneering works of Fang–Liu–Treumann–Zaslow [13–15], however, we use the most general form due to Kuwagaki [23]. This provides an equivalence

$$\kappa\colon \mathrm{Coh}(X_\Sigma) \cong \mathrm{Sh}^{\mathrm{w}}(\Lambda_\Sigma)$$

between the derived categories of coherent sheaves on the toric variety X_Σ and a certain class of constructible sheaves on a torus.

Hence, to obtain the theorem, it is enough to study Rouquier dimension for certain categories of constructible sheaves. It turns out that such categories have a natural generator $\mathcal{P}^\Lambda$ which we call the *probe generator*. Therefore, we study the generation time of this object, providing upper and lower bounds which in the case of a torus are sharp (yielding our main result as a consequence).

Theorem 1.3 (Example 2.16) *Let $\mathbb{T} = (S^1)^n$ be a real torus and Λ be a Lagrangian skeleton. Then the generation time of the probe generator is n.*

For completeness, we also track the probe generator through the equivalence κ^{-1}. As it turns out, the summands of the probe generator correspond to a natural stratification of the torus first considered by Bondal–Ruan [8] who established a bijection between strata and summands of the toric Frobenius pushforward of $\mathcal{O}$. Therefore, unsurprisingly, we find that the probe generator is taken to the toric Frobenius pushforward of $\mathcal{O}$ (the same generator studied in [2]).

[1] We are not sure if this list is exhaustive.

Notation and conventions

- All categories are dg derived.
- We use the notation in e.g. [24] for various sheaf categories equipped with either a stratification or with singular support conditions. $\mathrm{Sh}^{\diamond}$ means the dg derived category of constructible sheaves of $\mathbb{C}$-modules (of possibly infinite rank), Sh^{w} means compact objects in $\mathrm{Sh}^{\diamond}$, and Sh^{c} means the dg derived category of constructible sheaves of $\mathbb{C}$-modules of finite rank.
- The generation time of a generator G in a triangulated category is denoted by $t(G)$. The Loewy length of an algebra A is denoted by $\mathrm{LL}(A)$.

2 Generation time for probe generators

Definition 2.1 (*Poset stratified topological space*) Let S be a poset. Equip S with the Alexandroff topology. An S-*stratification* of a topological space X is the level sets of a surjective continuous map $f: X \to S$.

Let X be a finite regular CW complex satisfying the *axiom of the frontier*

$$\overline{e_\alpha} \cap e_\beta \neq \varnothing \quad \Longleftrightarrow \quad e_\beta \subset \overline{e_\alpha} \quad \text{for cells } \ e_\beta \text{ and } e_\alpha.$$

The axiom of the frontier guarantees that the collection of cells S_{CW} form a poset. Hence X obtains an S_{CW}-stratification. The incidence algebra, A, of this poset is an HPA [16, Definition 3.1] whose underlying quiver Q has:

$$Q_0 := \{e_i \in S_{\mathrm{CW}}\}$$
$$\# Q_1(e_j \to e_i) = \begin{cases} 1 & \text{if } \ \overline{e_i} \cap e_j \neq \varnothing \text{ and } j = i-1, \\ 0 & \text{otherwise} \end{cases}$$

with ideal generated by path homotopy relation among exit paths. One can write $A = \mathrm{End}(G_A)$, with $G_A = \bigoplus_{S_\alpha \in S_{\mathrm{CW}}} \mathcal{P}_\alpha$ and $\mathcal{P}_\alpha$ are the exit sheaves from strata S_α [16, Proposition 4.16].

Lemma 2.2 *The algebra A is Koszul. The direct sum of the left dual exceptional collection is*

$$G_A^! := \bigoplus_{\alpha \in S_{\mathrm{CW}}} i_{\alpha !} \mathbb{C}_{S_\alpha}[k_\alpha],$$

where k_α is the position $\mathcal{P}_\alpha$ appears in the exceptional collection. The endomorphism algebra of $G_A^!$ in $\mathrm{Sh}^{\mathrm{c}}_{S_{\mathrm{CW}}}(X)$ is the Koszul dual $A^!$.

Proof The order complex of any open interval (e_α, e_β) in the cell poset, which is the face link of the minimal cell e_α inside the maximal cell e_β bounding the interval, is always homeomorphic to a sphere of dim $\dim e_\beta - \dim e_\alpha - 2$, hence the cell poset is locally Cohen–Macaulay. By [27, Proposition, Section 1.6], A is Koszul. (See also [31, 32].)

Now observe that

$$\mathrm{Hom}(\mathcal{P}_\beta, i_{\alpha!}\mathbb{C}_{S_\alpha}) = \begin{cases} \mathbb{C} & \alpha = \beta, \\ 0 & \text{else.} \end{cases}$$

Hence $i_{\alpha!}\mathbb{C}_{S_\alpha}[k]$ is the left dual of $\mathcal{P}_\alpha$. □

Remark 2.3 Explicitly, we can define an ideal $I \subseteq kQ$ generated by

$$I := \langle a(e_i, e_{i-1})a(e_{i-1}, e_{i-2}) - a(e_i, e'_{i-1})a(e'_{i-1}, e_{i-2}) \rangle.$$

Then $A = kQ/I$. Our assumptions on X imply, by [10, Theorem 4.2], that for any codimension 2 indicence relation $\sigma <_2 \sigma'$, there are exactly two intermediate τ, such that $\sigma <_1 \tau <_1 \sigma'$. Hence the perpendicular of I in kQ^{op} is given by

$$I^\perp := \langle a(e_i, e_{i-1})a(e_{i-1}, e_{i-2}) + a(e_i, e'_{i-1})a(e'_{i-1}, e_{i-2}) \rangle,$$

which gives the quadratic dual algebra

$$A^! := kQ^{\mathrm{op}}/I^\perp.$$

Remark 2.4 In this simple case, $\mathrm{Sh}^{\mathrm{w}}_{S_{\mathrm{CW}}}(X) = \mathrm{Sh}^{\mathrm{c}}_{S_{\mathrm{CW}}}(X)$.

Remark 2.5 The A_∞-structure on both dg endomorphism algebras can be seen to be formal. Namely, an exit sheaf $\mathcal{P}_\alpha \in \mathrm{Sh}^{\mathrm{c}}_{S_{\mathrm{CW}}}(X)$ is a complex of sheaves concentrated in degree 0. Since $\mathcal{P}_\alpha$ represents the stalk functor at $p \in S_\alpha$ [16], we have $\mathrm{Hom}_{\mathrm{Sh}^{\mathrm{c}}_{S_{\mathrm{CW}}}(X)}(\mathcal{P}_\alpha, \mathcal{P}_{\alpha'}) = \mathcal{P}_{\alpha'}|_p$, which is in degree 0. Hence all morphisms among the exit sheaves are in degree 0, which implies $\mathrm{End}_{\mathrm{Sh}^{\mathrm{c}}_{S_{\mathrm{CW}}}(X)}(G_A)$ is formal. The Koszul property also forces the A_∞-structure of $\mathrm{End}_{\mathrm{Sh}^{\mathrm{c}}_{S_{\mathrm{CW}}}(X)}(G^!_A)$ to be formal for degree reasons.

Proposition 2.6 *Given* (X, S_{CW})*, we have*

$$t(G_A) = t(G^!_A) = \dim X.$$

Proof By Lemma 2.2, A is Koszul, and the quadratic dual $\mathrm{End}(G^!_A) = A^!$. Hence by [5, Proposition 3.26], we have $t(G_A) = \mathrm{LL}(A^!) - 1 = \dim X$. Symmetrically, $\mathrm{End}(G_A) = A$ and $t(G^!_A) = \mathrm{LL}(A) - 1 = \dim X$. □

We now give a (somewhat restrictive) definition of Lagrangian skeleta for convenience.

Definition 2.7 A *Lagrangian skeleton* on a smooth manifold X is a conical Lagrangian in T^*X containing the zero section embedded as a closed subset in the union of conormals of all strata in a Whitney stratification S' of X.

If S is a coarsening S' such that the quotient map $S' \to S$ induces a poset structure on S, we call the Lagrangian skeleton

$$\Lambda_S := \bigcup_{S_\alpha \in S} \mathrm{ss}(i_{\alpha !}\mathbb{C}_{S_\alpha})$$

the *stratification skeleton* of S.

Now suppose X is a smooth manifold and $\Lambda' \subset T^*X$ is a Lagrangian skeleton. Recall from [24] that if $\Lambda \subset \Lambda'$ is a closed subskeleton, there is a localization functor

$$\rho^{\mathrm{L}} \colon \mathrm{Sh}^{\diamond}_{\Lambda'}(X) \to \mathrm{Sh}^{\diamond}_{\Lambda}(X),$$

defined as the left adjoint to the inclusion $\rho \colon \mathrm{Sh}^{\diamond}_{\Lambda}(X) \to \mathrm{Sh}^{\diamond}_{\Lambda'}(X)$. Moreover, ρ^{L} preserves compact objects. Restriction to compact objects gives the stop removal functor

$$\rho^{\mathrm{L,w}} \colon \mathrm{Sh}^{\mathrm{w}}_{\Lambda'}(X) \to \mathrm{Sh}^{\mathrm{w}}_{\Lambda}(X).$$

Definition 2.8 Let $\Lambda \subseteq T^*X$ be a Lagrangian skeleton. We call the representative of the stalk functor at v, that is, $\mathcal{P}^{\Lambda}_v \in \mathrm{Sh}^{\mathrm{w}}_{\Lambda}(X)$ satisfying

$$\mathrm{Hom}_{\mathrm{Sh}^{\diamond}_{\Lambda}(X)}(\mathcal{P}_v, F) = F_v, \quad \text{for all} \quad F \in \mathrm{Sh}^{\diamond}_{\Lambda}(X),$$

the *stalk probe* at v.

Remark 2.9 Here, we do not require v to be inside the smooth locus of Λ. This is allowed since taking the stalk at any point $v \in X$ is a cocontinuous functor $\mathrm{Sh}^{\diamond}_{\Lambda} \to \mathbb{C}$-Mod.

Lemma 2.10 *Let $\Lambda \subseteq \Lambda'$ be a closed subskeleton, then*

$$\rho^{\mathrm{L,w}}\big(\mathcal{P}^{\Lambda'}_v\big) = \mathcal{P}^{\Lambda}_v.$$

Proof. Let $\mathcal{F} \in \mathrm{Sh}^{\diamond}_{\Lambda}(X)$. We have

$$\begin{aligned} \mathrm{Hom}_{\Lambda}\big(\rho^{\mathrm{L,w}}\big(\mathcal{P}^{\Lambda'}_v\big), \mathcal{F}\big) &= \mathrm{Hom}_{\Lambda'}\big(\mathcal{P}^{\Lambda'}_v, \mathcal{F}\big) && \text{by adjunction} \\ &= \mathcal{F}_v && \text{by definition.} \end{aligned}$$

□

Definition 2.11 Suppose S_{CW} is a regular CW complex stratification on a smooth manifold X with stratification skeleton Λ_{CW} (see Definition 2.7), and let $\Lambda \subseteq \Lambda_{\mathrm{CW}}$ be a closed subskeleton. For each cell e_α, choose a base point $e_{\alpha,v}$. The *probe generator* is the direct sum of stalk probes

$$\mathcal{P}^{\Lambda} := \bigoplus_{S_\alpha \in S_{\mathrm{CW}}} \mathcal{P}^{\Lambda}_{e_{\alpha,v}}$$

considered as an object of $\mathrm{Sh}^{\mathrm{w}}_{\Lambda}$.

Lemma 2.12 *Let $\Lambda \subseteq \Lambda'$ be a closed subskeleton of a Lagrangian skeleton $\Lambda \subset T^*X$, then*

$$t\big(\mathcal{P}^{\Lambda}\big) \leqslant t\big(\mathcal{P}^{\Lambda'}\big).$$

Proof. The functor $\rho^{\mathrm{L,w}}$ has dense image by [18, Corollary 4.22]. Hence

$$\begin{aligned} t\big(\mathcal{P}^{\Lambda'}\big) &\geqslant t\big(\rho^{\mathrm{L,w}}\big(\mathcal{P}^{\Lambda'}\big)\big) && \text{since } \rho^{\mathrm{L,w}} \text{ has dense image} \\ &= t\big(\mathcal{P}^{\Lambda}\big) && \text{by Lemma 2.10.} \end{aligned}$$

□

The proof of the next theorem requires the use of ghost maps. We recall the following definition:

Definition 2.13 ([5, Definition 2.10]) Let $\mathcal{T}$ be a triangulated category, f be a morphism, and $\mathcal{I}$ be a full subcategory. We say that f is $\mathcal{I}$ *ghost* if, for all $I \in \mathcal{I}$, the induced map, $\mathrm{Hom}_{\mathcal{T}}(I, X) \to \mathrm{Hom}_{\mathcal{T}}(I, Y)$, is zero.

We now recall the Ghost Lemma, which will be used to prove the theorem below.

Lemma 2.14 ([5, Lemma 2.12] or [29, Lemma 4.11]) *Let $\mathcal{T}$ be a triangulated category and let G be an object of $\mathcal{T}$. If there exists a sequence of morphisms, $f_i : X_{i-1} \to X_i$, $1 \leqslant i \leqslant t$, in $\mathcal{T}$ where each f_i is G-ghost and $f_t \circ \cdots \circ f_1 \neq 0$, then G generates X_0 in time at least t.*

Theorem 2.15 *Let X, Λ_{CW}, and $\Lambda \subset \Lambda_{\mathrm{CW}}$ be as in Definition 2.11. Then,*

$$\mathrm{LL}(H^*(X, k)) - 1 \leqslant t\big(\mathcal{P}^{\Lambda}\big) \leqslant \dim X.$$

In particular, the Rouquier dimension of $\mathrm{Sh}^{\mathrm{w}}_{\Lambda}(X)$ *is at most* $\dim X$.

Proof. Let $c \in H^i(X, \mathbb{C})$ be a cohomology class. This induces a map of constructible sheaves

$$c\colon \mathbb{C}_X \to \mathbb{C}_X[i].$$

For $i > 0$, the map c is $\mathcal{P}^{\Lambda}$-ghost as

$$\mathrm{Hom}\big(\mathcal{P}^{\Lambda}, \mathbb{C}_X[i]\big) = \bigoplus_{e_v} \mathbb{C}[i]$$

is concentrated in degree i. Hence any nontrivial cup product $c_1 \cup \cdots \cup c_n$ gives a nontrivial sequence of $\mathcal{P}^{\Lambda}$-ghost maps. The Ghost Lemma 2.14 then gives the lower bound.

For the upper bound, we have

$$\begin{aligned} t\big(\mathcal{P}^{\Lambda}\big) &\leqslant t\big(\mathcal{P}^{\Lambda_{\mathrm{CW}}}\big) && \text{by Lemma 2.12} \\ &= \dim X && \text{by Proposition 2.6.} \end{aligned}$$

□

Example 2.16 If X is a torus then $H^*(X,k)$ is isomorphic to the exterior algebra $\Lambda^\bullet k^{\dim X}$. Hence $\mathrm{LL}(H^*(X,k)) = \dim X + 1$ and $t(\mathcal{P}) = \dim X$.

3 Application to toric varieties

We now consider the Langragian skeleta appearing as mirrors of toric varieties. For this, let $U \subseteq \mathbb{A}^n$ be an open toric subset of $\mathbb{A}^n$ defined by a subfan $\widetilde{\Sigma} \subseteq \Sigma_{\mathbb{A}^n}$. Let G be any subgroup of $\mathbb{G}_m^n$, the maximal torus of $\mathbb{A}^n$. We define $\widetilde{M} := \mathrm{Hom}(\mathbb{G}_m^n, \mathbb{G}_m)$ and $\widehat{G} := \mathrm{Hom}(G, \mathbb{G}_m)$. Since $\mathrm{Hom}(-, \mathbb{G}_m)$ is exact, applying it to the inclusion we get an exact sequence

$$0 \to M \xrightarrow{\beta^\vee} \widetilde{M} \xrightarrow{\mu} \widehat{G} \to 0$$

where M is defined as the kernel of μ.

Taking $\mathrm{Hom}(-, \mathbb{Z})$ gives a four-term exact sequence

$$0 \to \widehat{G}^\vee \xrightarrow{\mu^\vee} \widetilde{N} \xrightarrow{\beta} N \to \operatorname{coker}\beta \to 0.$$

We adopt the setup in [23], asking that $\beta_{\mathbb{R}}$ induces a combinatorial isomorphism of fans. That is, we assume

(i) The restricted map $\beta_{\mathbb{R}}|_{\widetilde{\sigma}} : \widetilde{\sigma} \xrightarrow{\sim} \sigma$ is an isomorphism of cones for all $\widetilde{\sigma} \in \widetilde{\Sigma}$.
(ii) There is a poset isomorphism

$$\widetilde{\Sigma} \simeq \Sigma$$

where Σ is a fan consisting of all cones $\sigma = \beta_{\mathbb{R}}(\widetilde{\sigma})$.

The above assumption implies Σ is simplicial. The quotient stack $[U/G]$ is a smooth Deligne–Mumford toric stack (see [17]) whose coarse moduli space is the toric variety X_Σ. We note that if X_Σ is smooth, then X_Σ is isomorphic to $[U/G]$.

Furthermore, since $\widetilde{\Sigma}$ is a subfan of the standard fan for $\mathbb{A}^n$, all cones $\widetilde{\sigma} \in \widetilde{\Sigma}$ are of the form $\widetilde{\sigma}_I = \mathbb{R}^I_{\geqslant 0} \subset \widetilde{N}_{\mathbb{R}}$, with dual cone $\widetilde{\sigma}_I^\vee = \mathbb{R}^I_{\geqslant 0} \times \mathbb{R}^{I^c}$. Let

$$\widetilde{\Lambda} := \bigcup_{\substack{\widetilde{\sigma}_I \in \widetilde{\Sigma} \\ \widetilde{D} \in \widetilde{M}}} (\widetilde{\sigma}_I^\perp + \widetilde{D}) \times (-\widetilde{\sigma}_I) = \bigcup_{\substack{\widetilde{\sigma}_I \in \widetilde{\Sigma} \\ \widetilde{D} \in \widetilde{M}}} \left(\mathbb{R}^{I^c} + \widetilde{D}\right) \times \mathbb{R}^I_{\leqslant 0}$$

be the $\widetilde{M}$-periodic FLTZ skeleton associated to the fan $\widetilde{\Sigma}$. Let $\pi : \widetilde{M}_{\mathbb{R}} \to \widetilde{M}_{\mathbb{R}}/M$ be the projection, and define

$$\Lambda_{\widetilde{\Sigma}} := \widetilde{\Lambda}/M.$$

Then, the fiber skeleton

$$\Lambda_\Sigma := \Lambda_{\widetilde{\Sigma}}|_{M_{\mathbb{R}}/M}$$

is the (non-equivariant) mirror skeleton to $[U/G]$.

Lemma 3.1

$$\mathcal{P}^{\Lambda_{\widetilde{\Sigma}}}_{\pi(p)} = \pi_! \mathcal{P}^{\widetilde{\Lambda}}_{p}.$$

Proof.

$$\begin{aligned} \operatorname{Hom}\left(\pi_! \mathcal{P}^{\widetilde{\Lambda}}_{p}, F\right) &= \operatorname{Hom}\left(\mathcal{P}^{\widetilde{\Lambda}}_{p}, \pi^! F\right) && \text{by adjunction} \\ &= \operatorname{Hom}\left(\mathcal{P}^{\widetilde{\Lambda}}_{p}, \pi^* F\right) && \text{since } \pi \text{ is a covering map} \\ &= (\pi^* F)_p && \text{by definition of stalk probe} \\ &= F_{\pi(p)}. \end{aligned}$$

□

Since $\pi_! \mathcal{P}^{\widetilde{\Lambda}}_{\widetilde{D}}$ does not depend on the choice of lift $\widetilde{D}$, we write $\mathcal{P}^{\Lambda_{\widetilde{\Sigma}}}_{D} := \pi_! \mathcal{P}^{\widetilde{\Lambda}}_{\widetilde{D}}$.

We will use the following noncharacteristic deformation lemma.

Lemma 3.2 ([30, Theorem 14]) *Let $i_b \colon \mu_{\mathbb{R}}^{-1}(b)/M \hookrightarrow \widetilde{M}_{\mathbb{R}}/M$. The pullback*

$$i_b^* \colon \mathrm{Sh}^{\diamond}_{\Lambda_{\widetilde{\Sigma}}} \to \mathrm{Sh}^{\diamond}_{\Lambda_{\widetilde{\Sigma}}|_{\mu_{\mathbb{R}}^{-1}(b)}}$$

is an equivalence of categories.

In particular, taking $b = 0$ gives an equivalence of categories

$$i^* \colon \mathrm{Sh}^{\diamond}_{\Lambda_{\widetilde{\Sigma}}} \xrightarrow{\simeq} \mathrm{Sh}^{\diamond}_{\Lambda_{\Sigma}}.$$

We now recall the stratification refining Λ_Σ defined by Bondal–Ruan [8]. Let D_i for $1 \leqslant i \leqslant n$ be the standard basis of $\mathbb{Z}^n$. The *Bondal–Ruan map* is defined by

$$\begin{aligned} \Phi \colon \mu_{\mathbb{R}}^{-1}(0)/M &\to \widehat{G} \\ \sum_i a_i D_i + M &\mapsto \mu\Big(-\sum_i \lfloor a_i \rfloor D_i\Big) \end{aligned}$$

where $\lfloor a_i \rfloor$ is the floor of a_i. We take the associated stratification S_{BR} on $\mathbb{T}^n$ whose strata are given by the level sets of $S_D := \Phi^{-1}(D)$.

Lemma 3.3 *Let $[p] \in \widetilde{M}_{\mathbb{R}}/M$ denote the coset $p + M$, where $p \in \widetilde{M}_{\mathbb{R}}$. The probe sheaf $\mathcal{P}^{\Lambda_\Sigma}_{[p]}$ at the point $[p] \in S_D$ is given by*

$$\mathcal{P}^{\Lambda_\Sigma}_{[p]} = i^* \mathcal{P}^{\Lambda_{\widetilde{\Sigma}}}_{D}.$$

Proof. First choose $\widetilde{D}$ such that $\mu(\widetilde{D}) = D$. For $p \in S_{\widetilde{D}} \subset \widetilde{D} + (-1, 0]^n$ for some $\widetilde{D} \in \widetilde{M}$, write $p = \widetilde{D} + x$, where $x \in (-1, 0]^n$, i.e., p, $\widetilde{D}$ lie in the same stratum of the cube stratification S_{cube}. Hence, $\mathcal{P}_p^{S_{\text{cube}}} = \mathcal{P}_{\widetilde{D}}^{S_{\text{cube}}}$ which implies

$$\begin{aligned}
\mathcal{P}_{\tilde{p}}^{\widetilde{\Lambda}} &= \rho^{\mathrm{L,w}}\left(\mathcal{P}_p^{S_{\text{cube}}}\right) && \text{by Lemma 2.10} \\
&= \rho^{\mathrm{L,w}}\left(\mathcal{P}_{\widetilde{D}}^{S_{\text{cube}}}\right) && \text{from the observation above} \\
&= \mathcal{P}_{\widetilde{D}}^{\widetilde{\Lambda}} && \text{by Lemma 2.10.}
\end{aligned} \tag{1}$$

Now we have

$$\begin{aligned}
\mathcal{P}_{[p]}^{\Lambda_{\widetilde{\Sigma}}} &= \pi_! \mathcal{P}_p^{\widetilde{\Lambda}} && \text{by Lemma 3.1} \\
&= \pi_! \mathcal{P}_{\widetilde{D}}^{\widetilde{\Lambda}} && \text{by (1)} \\
&= \mathcal{P}_D^{\Lambda_{\widetilde{\Sigma}}} && \text{by Lemma 3.1.}
\end{aligned} \tag{2}$$

$$\begin{aligned}
\operatorname{Hom}\left(i^*\mathcal{P}_D^{\Lambda_{\widetilde{\Sigma}}}, G\right) &= \operatorname{Hom}\left(i^*\mathcal{P}_D^{\Lambda_{\widetilde{\Sigma}}}, i^*F\right) \text{ for some } F && \text{since } i^* \text{ is essentially surjective} \\
&= \operatorname{Hom}\left(\mathcal{P}_D^{\Lambda_{\widetilde{\Sigma}}}, F\right) && \text{since } i^* \text{ is an equivalence} \\
&= \operatorname{Hom}\left(\mathcal{P}_{[p]}^{\Lambda_{\widetilde{\Sigma}}}, F\right) && \text{by (2)} \\
&= F_{[p]} && \text{by the definition of stalk probe} \\
&= (i^*F)_{[p]} && \text{since } [p] \in S_D \subset \mu_{\mathbb{R}}^{-1}(0)/M \\
&= G_{[p]} && \text{by definition of } F. \qquad \square
\end{aligned}$$

Lemma 3.4 *Let*

$$\kappa\colon \operatorname{Coh}([U/G]) \to (\mathrm{Sh}^{\mathrm{w}}_{\Lambda_\Sigma})^{\mathrm{op}}$$

be the covariant *equivalence (mirror functor) from* [23]. *Then,*

$$\kappa\left(\bigoplus_{D \in \operatorname{Im}\Phi} \mathcal{O}(D)\right) = \mathcal{P}^{\Lambda_\Sigma}.$$

Proof. We observe that each κ_I in the mirror functor

$$\operatorname{Ind}\kappa = \varprojlim_{C(\Sigma)} \operatorname{Ind}\kappa_{\sigma_I}$$

defined by Kuwagaki [23] takes $\mathcal{O}_{[\mathbb{C}^I \times (\mathbb{C}^*)^{I^c}/G]}(D)$ to $\mathcal{P}_D^{\Lambda_{\Sigma_I}|_{M_{\mathbb{R}}/M}} = i^*\mathcal{P}_D^{\Lambda_{\widetilde{\Sigma}_I}}$. Then the Čech resolution of $\mathcal{O}_{[U/G]}(D)$ realizes

$$\mathcal{O}_{[U/G]}(D) = \varprojlim_{C(\Sigma)} \mathcal{O}_{[\mathbb{C}^I \times (\mathbb{C}^*)^{I^c}/G]}(D). \tag{3}$$

On the other hand for $[p] \in S_D$,

$$
\begin{aligned}
\mathcal{P}^{\Lambda_\Sigma}_{[p]} &= i^*\mathcal{P}^{\Lambda_{\tilde{\Sigma}}}_{D} && \text{by Lemma 3.3}\\
&= i^*\pi_!\mathcal{P}^{\widetilde{\Lambda}_I}_{\widetilde{D}} && \text{by Lemma 3.1}\\
&= i^*\pi_! \varinjlim_{C(\Sigma)} \mathcal{P}^{\widetilde{\Lambda}_I}_{\widetilde{D}} && \text{by [21, Proposition 4.5]}\\
& && \text{Note: [21] used opposite indexing, i.e., } \sigma_I \text{ here is } \sigma_{I^c} \text{ in [21]}\\
&= i^* \varinjlim_{C(\Sigma)} \pi_!\mathcal{P}^{\widetilde{\Lambda}_I}_{\widetilde{D}} && \text{since } \pi_! \text{ is a left adjoint hence preserves colimit}\\
&= i^* \varinjlim_{C(\Sigma)} \mathcal{P}^{\Lambda_{\tilde{\Sigma}_I}}_{D} && \text{by Lemma 3.1}\\
&= \varinjlim_{C(\Sigma)} i^*\mathcal{P}^{\Lambda_{\tilde{\Sigma}_I}}_{D} && \text{since } i^* \text{ is an equivalence.}
\end{aligned}
\tag{4}
$$

In the opposite category, the colimit becomes the limit.

Hence

$$
\begin{aligned}
\kappa(\mathcal{O}_{[U/G]}(D)) &= \kappa\big(\varprojlim_{C(\Sigma)} \mathcal{O}_{[\mathbb{C}^I\times(\mathbb{C}^*)^{I^c}/G]}(D)\big) && \text{by (3)}\\
&= \varinjlim_{C(\Sigma)} \kappa\big(\mathcal{O}_{[\mathbb{C}^I\times(\mathbb{C}^*)^{I^c}/G]}(D)\big) && \text{since } \kappa \text{ is an equivalence}\\
&= \varinjlim_{C(\Sigma)} i^*\mathcal{P}^{\Lambda_{\tilde{\Sigma}_I}}_{D} && \text{by definition of } \kappa\\
&= \mathcal{P}^{\Lambda_\Sigma}_{[p]} && \text{by (4).} \quad \square
\end{aligned}
$$

The following corollary will appear in forthcoming works of Ballard–Duncan–McFaddin [3] and Hanlon–Hicks–Lazarev [20]; both results were publicly claimed before us and use algebro-geometric methods. See also [1, 19] for more general discussions on the Rouquier dimension of Fukaya categories and applications via mirror symmetry.

Corollary 3.5 *The object* $\bigoplus_{D\in \operatorname{Im}\Phi} \mathcal{O}(D)$ *generates* $\operatorname{Coh}(X_\Sigma)$.

Proof Since κ is an equivalence [23], this follows from Lemmas 3.1 and 3.4. $\square$

As a consequence, we have

Corollary 3.6 *For any toric stack of the form* $[U/G]$ *in the setup of* [23], *Conjecture* 1.1 *holds, i.e.,*

$$\operatorname{rdim}\operatorname{Coh}([U/G]) = \dim [U/G].$$

Furthermore, this generation time is achieved by the collection of line bundles $\bigoplus_{D\in \operatorname{Im}\Phi} \mathcal{O}(D)$.

Proof By [4, Lemma 2.17][2],

$$\operatorname{rdim} \operatorname{Coh}([U/G]) \geqslant \dim [U/G].$$

Recall that the Bondal–Ruan stratification S_{BR} is a block stratification by [16, Corollary 5.7]. By definition, this means it is the coarsening of a CW stratification S_{CW}. Let S^1_{CW} be the first barycentric subdivision of S_{CW}. Then S^1_{CW} is a regular CW stratification.

We have a chain of embeddings of closed subskeletons

$$\Lambda_\Sigma \subset \Lambda_{S_{\mathrm{BR}}} \subset \Lambda_{S_{\mathrm{CW}}} \subset \Lambda_{S^1_{\mathrm{CW}}}.$$

The upper bound then follows from Theorem 2.15 and the non-equivariant coherent-constructible correspondence [23, Theorem 1.2] which provides an equivalence

$$\operatorname{Coh}([U/G]) \simeq \operatorname{Sh}^{\mathrm{w}}_{\Lambda_\Sigma}.$$

The fact that this is achieved by $\bigoplus_{D \in \operatorname{Im} \Phi} \mathcal{O}(D)$ is Lemma 3.4. □

We now deduce the result for any normal toric variety $X_{\Sigma'}$, using the result for stacks. The following corollary will be translated entirely into toric methods in forthcoming work of Hanlon–Hicks–Lazarev [20] inspired by a similar approach seen here; their result was claimed publicly before our paper appeared.

Corollary 3.7 *For any normal toric variety $X_{\Sigma'}$, Conjecture 1.1 holds, i.e.,*

$$\operatorname{rdim} \operatorname{Coh}(X_{\Sigma'}) = \dim X_{\Sigma'}.$$

Proof First, observe that we can choose a smooth resolution X_Σ (see e.g. [11, Theorem 11.1.9].) Since X_Σ is smooth, it is isomorphic to the quotient stack $X_\Sigma = [U/G]$ where U is obtained from the Cox construction. Therefore by Corollary 3.6,

$$\operatorname{rdim} \operatorname{Coh}(X_\Sigma) = \dim X_\Sigma.$$

Furthermore, since any normal toric variety has rational singularities (see e.g. [11, Theorem 11.4.2]), the functor

$$g_* \colon \operatorname{Coh}(X_\Sigma) \to \operatorname{Coh}(X_{\Sigma'})$$

has dense image (see e.g. [22, Lemma 7.4]). It follows that

$$\operatorname{rdim} \operatorname{Coh}(X_\Sigma) \leqslant \operatorname{rdim} \operatorname{Coh}(X_{\Sigma'}) = \dim X'_\Sigma = \dim X_\Sigma.$$

Once again, the lower bound for is [29, Proposition 7.17]. □

[2] For completeness, we give the reference to the stacky case. The original proof (for schemes) is due to Rouquier [29, Proposition 7.17].

Remark 3.8 To deduce the general case from the smooth case, one may also observe that $\Lambda_{\Sigma'} \subseteq \Lambda_{\Sigma}$ and then appeal directly to Theorem 2.15.

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