The Homogeneous Coordinate Ring of a Toric Variety
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This paper will introduce the *homogeneous coordinate ring S* of a toric variety X. The ring S is a polynomial ring with one variable for each one-dimensional cone in the fan Δ determining X, and S has a natural grading determined by the monoid of effective divisor classes in the Chow group $A_{n-1}(X)$ of X (where $n = \dim X$). Using this graded ring, we will show that X behaves like projective space in many ways. The paper is organized into four sections as follows.

In §1, we define the homogeneous coordinate ring S of X and compute its graded pieces in terms of global sections of certain coherent sheaves on X. We also define a monomial ideal $B \subset S$ that describes the combinatorial structure of the fan Δ. In the case of projective space, the ring S is just the usual homogeneous coordinate ring $\mathbb{C}[x_0, \ldots, x_n]$, and the ideal B is the "irrelevant" ideal $\langle x_0, \ldots, x_n \rangle$.

Projective space $\mathbb{P}^n$ can be constructed as the quotient $(\mathbb{C}^{n+1} - \{0\})/\mathbb{C}^*$. In §2, we will see that there is a similar construction for any toric variety X. In this case, the algebraic group $G = \mathrm{Hom}_{\mathbb{Z}}(A_{n-1}(X), \mathbb{C}^*)$ acts on an affine space $\mathbb{C}^{\Delta(1)}$ such that the categorical quotient $(\mathbb{C}^{\Delta(1)} - Z)/G$ exists and is isomorphic to X. The exceptional set Z is the zero set of the ideal B defined in §1. If X is simplicial (meaning that the fan Δ is simplicial), then $X \simeq (\mathbb{C}^{\Delta(1)} - Z)/G$ is a geometric quotient, so that elements of $\mathbb{C}^{\Delta(1)} - Z$ can be regarded as "homogeneous coordinates" for points of X.

On $\mathbb{P}^n$, there is a correspondence between sheaves and graded $\mathbb{C}[x_0, \ldots, x_n]$-modules. For any toric variety, we will see in §3 that finitely generated graded S modules give rise to a coherent sheaves on X, and when X is simplicial, every coherent sheaf arises in this way. In particular, every closed subscheme of X is determined by a graded ideal of S. We will also study the extent to which this correspondence fails to be is one-to-one.

Another feature of $\mathbb{P}^n$ is that the action of $PGL(n+1, \mathbb{C})$ on $\mathbb{P}^n$ lifts to an action of $GL(n+1, \mathbb{C})$ on $\mathbb{C}^{n+1} - \{0\}$. In §4, we will see that for any complete simplicial toric variety X, the action of $\mathrm{Aut}(X)$ on X lifts to an action of an algebraic group $\widetilde{\mathrm{Aut}}(X)$ on $\mathbb{C}^{\Delta(1)} - Z$. This group is a extension of $\mathrm{Aut}(X)$ by the group G defined above. We will give an explicit description of $\widetilde{\mathrm{Aut}}(X)$, and in particular, we will see that the roots of $\mathrm{Aut}(X)$ (as defined in [Demazure]) have an especially nice interpretation.

For simplicity, we will work over the complex numbers $\mathbb{C}$. Our notation will be similar to that used by [Fulton] and [Oda]. I would like to thank Bernd Sturmfels for stimulating my interest in toric varieties.

§1. The Homogeneous Coordinate Ring

Let X be the toric variety determined by a fan Δ in $N \simeq \mathbb{Z}^n$. As usual, M will denote the $\mathbb{Z}$-dual of N, and cones in Δ will be denoted by σ. The one-dimensional cones of Δ form the set $\Delta(1)$, and given $\rho \in \Delta(1)$, we let n_ρ denote the unique generator of $\rho \cap N$. If σ is any cone in Δ, then $\sigma(1) = \{\rho \in \Delta(1) : \rho \subset \sigma\}$ is the set of one-dimensional faces of σ. We will assume that $\Delta(1)$ spans $N_\mathbb{R} = N \otimes_\mathbb{Z} \mathbb{R}$.

Each $\rho \in \Delta(1)$ corresponds to an irreducible T-invariant Weil divisor D_ρ in X, where $T = N \otimes_\mathbb{Z} \mathbb{C}^*$ is the torus acting on X. The free abelian group of T-invariant Weil divisors on X will be denoted $\mathbb{Z}^{\Delta(1)}$. Thus an element $D \in \mathbb{Z}^{\Delta(1)}$ is a sum $D = \sum_\rho a_\rho D_\rho$. The T-invariant Cartier divisors form a subgroup $\mathrm{Div}_T(X) \subset \mathbb{Z}^{\Delta(1)}$.

Each $m \in M$ gives a character $\chi^m : T \to \mathbb{C}^*$, and hence χ^m is a rational function on X. As is well-known, χ^m gives the Cartier divisor $\mathrm{div}(\chi^m) = -\sum_\rho \langle m, n_\rho \rangle D_\rho$. We find it convenient to ignore the minus sign, so that we will consider the map

$$M \longrightarrow \mathbb{Z}^{\Delta(1)} \text{ defined by } m \mapsto D_m = \sum_\rho \langle m, n_\rho \rangle D_\rho \ .$$

This map is injective since $\Delta(1)$ spans $N_\mathbb{R}$. By [Fulton, §3.4], we have a commutative diagram

(1)
$$
\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & \mathrm{Div}_T(X) & \longrightarrow & \mathrm{Pic}(X) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M & \longrightarrow & \mathbb{Z}^{\Delta(1)} & \longrightarrow & A_{n-1}(X) & \longrightarrow & 0
\end{array}
$$

where the rows are exact and the vertical arrows are inclusions. Thus a divisor $D \in \mathbb{Z}^{\Delta(1)}$ determines an element $\alpha = [D] \in A_{n-1}(X)$. Note that n is the dimension of X.

For each $\rho \in \Delta(1)$, introduce a variable x_ρ, and consider the polynomial ring

$$S = \mathbb{C}[x_\rho : \rho \in \Delta(1)] \ .$$

We will usually write this as $S = \mathbb{C}[x_\rho]$. Note that a monomial $\prod_\rho x_\rho^{a_\rho}$ determines a divisor $D = \sum_\rho a_\rho D_\rho$, and to emphasize this relationship, we will write the monomial as x^D. We will grade S as follows:

the *degree* of a monomial $x^D \in S$ is $\deg(x^D) = [D] \in A_{n-1}(X)$.

Using the exact sequence (1), it follows that two monomials $\prod_\rho x_\rho^{a_\rho}$ and $\prod_\rho x_\rho^{b_\rho}$ in S have the degree if and only if there is some $m \in M$ such that $a_\rho = \langle m, n_\rho \rangle + b_\rho$ for all ρ. Then let

$$S_\alpha = \bigoplus_{\deg(x^D)=\alpha} \mathbb{C} \cdot x^D \ ,$$

so that the ring S can be written as the direct sum

$$S = \bigoplus_{\alpha \in A_{n-1}(X)} S_\alpha \ .$$

Note also that $S_\alpha \cdot S_\beta \subset S_{\alpha+\beta}$. We call S the *homogeneous coordinate ring* of the toric variety X. Of course, "homogeneous" means with respect to the above grading. We should mention that the ring S, without the grading, appears in the Danilov-Jurkiewicz description of the cohomology of a simplicial toric variety (see [Danilov, §10], [Fulton, §5.2] or [Oda, §3.3]).

Here are some examples of what the ring S looks like:

Projective Space: When $X = \mathbb{P}^n$, it is easy to check that S is usual homogeneous coordinate ring $\mathbb{C}[x_0, \dots, x_n]$ with the standard grading.

Weighted Projective Space: When $X = \mathbb{P}(q_0, \dots, q_n)$, then S is the ring $\mathbb{C}[x_0, \dots, x_n]$, where the grading is determined by giving the variable x_i weight q_i.

Product of Projective Spaces: When $X = \mathbb{P}^n \times \mathbb{P}^m$, then $S = \mathbb{C}[x_0, \dots, x_n; y_0, \dots, y_m]$. Here, the grading is the usual bigrading, where a polynomial has bidegree (a, b) if it is homogeneous of degree a (resp. b) in the x_i (resp. y_j).

Our first result about the ring S shows that the graded pieces of S are isomorphic to the global sections of certain sheaves on X:

Proposition 1.1.
(i) *If $\alpha = [D] \in A_{n-1}(X)$, then there is an isomorphism*

$$\phi_D : S_\alpha \simeq H^0(X, \mathcal{O}_X(D)) \ ,$$

where $\mathcal{O}_X(D)$ is the coherent sheaf on X determined by the Weil divisor D (see [Fulton, §3.4]).
(ii) *If $\alpha = [D]$ and $\beta = [E]$, then there is a commutative diagram*

$$\begin{array}{ccc} S_\alpha \otimes S_\beta & \longrightarrow & S_{\alpha+\beta} \\ \downarrow & & \downarrow \\ H^0(X, \mathcal{O}_X(D)) \otimes H^0(X, \mathcal{O}_X(E)) & \longrightarrow & H^0(X, \mathcal{O}_X(D+E)) \end{array}$$

where the top arrows is multiplication, the bottom arrow is tensor product, and the vertical arrows are the isomorphisms $\phi_D \otimes \phi_E$ and ϕ_{D+E}.

Proof. Suppose that $D = \sum_\rho a_\rho D_\rho$. By [Fulton, §3.4], we know that $H^0(X, \mathcal{O}_X(D)) = \bigoplus_{\alpha \in P_D \cap M} \mathbb{C} \cdot \chi^m$, where

$$P_D = \{m \in M_\mathbb{R} : \langle m, n_\rho \rangle \geq -a_\rho \text{ for all } \rho\} \ .$$

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Given $m \in P_D \cap M$, let $D_m = \sum_\rho \langle m, n_\rho \rangle D_\rho$, so that $x^{D_m+D} = \prod_\rho x_\rho^{\langle m, n_\rho \rangle + a_\rho}$. Then $m \mapsto x^{D_m+D}$ defines a map from $P_D \cap M$ to the monomials in S_α (the monomial is in S since $m \in P_D$, and it has degree α since $m \in M$). This map is one-to-one since $M \to \mathbb{Z}^{\Delta(1)}$ is injective, and it is easy to see that every monomial in S_α arises in this way. Then $\phi_D(x^{D_m+D}) = \chi^m$ gives the desired isomorphism. The proof of the second part of the proposition is straightforward and is omitted. $\qquad \square$

We get the following corollary (see [Fulton, §3.4]):

Corollary 1.2. *Let X be a complete toric variety. Then:*
(i) *S_α is finite dimensional for every α, and in particular, $S_0 = \mathbb{C}$.*
(ii) *If $\alpha = [D]$ for an effective divisor $D = \sum_\rho a_\rho D_\rho$, then $\dim_{\mathbb{C}} S_\alpha = |P_D \cap M|$.*

In the complete case, the graded ring S has some additional structure. Namely, there is a natural order relation on the monomials of S defined as follows: if $x^D, x^E \in S$, then set

$$x^D < x^E \iff \text{there is } x^F \in S \text{ with } x^D | x^F, \ x^D \neq x^F \text{ and } \deg(x^F) = \deg(x^E) \ .$$

This relation has the following properties:

Lemma 1.3. *When X is a complete toric variety, the order relation defined above is transitive, antisymmetric and multiplicative (meaning that $x^D < x^E \Rightarrow x^{D+F} < x^{E+F}$) on the monomials of S.*

Proof. The transitive and multiplicative properties are trivial to verify. To prove antisymmety, note that $x^D | x^F$ and $\deg(x^F) = \deg(x^E)$ implies that $[E] - [D] = [F] - [D] = [F - D]$ is the class of an effective divisor. Thus, if $x^D < x^E$ and $x^E < x^D$, then $[E] - [D]$ and its negative are effective classes. Since X is complete, irreducible and reduced, we must have $[D] = [E]$. Then, for x^F as above, we would have $x^{F-D} \in S_0 \simeq \mathbb{C}$, which would imply $x^F = x^D$. This contradicts the definition of $x^D < x^E$, and antisymmetry is proved. $\qquad \square$

In the case of $\mathbb{P}^n$, this gives the usual ordering by total degree. For a complete simplicial toric variety X, we will use the order relation of Lemma 1.3 in §4 when we study the automorphism group of X.

An important observation is that the theory developed so far depends only on the one-dimensional cones $\Delta(1)$ of the fan Δ. More precisely, if Δ and Δ' are fans in N with $\Delta(1) = \Delta'(1)$, then $A_{n-1}(X) = A_{n-1}(X')$, and the corresponding rings S and S' are also equal as graded rings. It follows that we cannot reconstruct the fan from S and its grading—more information is needed. The ring tells us about $\Delta(1)$, but we need something else to tell us which cones belong in Δ.

The crucial object is the following ideal of S. For a cone $\sigma \in \Delta$, let $\hat{\sigma}$ be the divisor $\sum_{\rho \notin \sigma(1)} D_\rho$, and let the corresponding monomial in S be $x^{\hat{\sigma}} = \prod_{\rho \notin \sigma(1)} x_\rho$. Then define

$B = B_\Delta \subset S$ to be the ideal generated by the $x^{\hat{\sigma}}$, i.e.,

$$B = \langle x^{\hat{\sigma}} : \sigma \in \Delta \rangle \subset S .$$

Note the B is in fact generated by the $x^{\hat{\sigma}}$ as σ ranges over the *maximal* cones of Δ. Also, if Δ and Δ' are fans in N with $\Delta(1) = \Delta'(1)$, then $\Delta = \Delta'$ if and only if the corresponding ideals $B, B' \subset S$ satisfy $B = B'$. This follows because $\{x^{\hat{\sigma}} : \sigma$ is a maximal cone of $\Delta\}$ is the unique minimal basis of the monomial ideal B.

When X is a projective space or weighted projective space, the ideal B is just the "irrelevant" ideal $\langle x_0, \ldots, x_n \rangle \subset \mathbb{C}[x_0, \ldots, x_n]$. A basic theme of this paper is that the pair $B \subset S$ plays the role for an arbitrary toric variety that $\langle x_0, \ldots, x_n \rangle \subset \mathbb{C}[x_0, \ldots, x_n]$ plays for projective space. In particular, we can think of the variety

$$Z = \mathbf{V}(B) = \{x \in \mathbb{C}^{\Delta(1)} : x^{\hat{\sigma}} = 0 \text{ for all } \sigma \in \Delta\} \subset \mathbb{C}^{\Delta(1)}$$

as the "exceptional" subset of $\mathbb{C}^{\Delta(1)}$. In §2, we will show how to construct the toric variety X from $\mathbb{C}^{\Delta(1)} - Z$.

Lemma 1.4. $Z \subset \mathbb{C}^{\Delta(1)}$ *has codimension at least two.*

Proof. Since $B' = \langle x^{\hat{\rho}} : \rho \in \Delta(1) \rangle \subset B$, we have $Z = \mathbf{V}(B) \subset \mathbf{V}(B')$. But $\mathbf{V}(B')$ is the union of all codimension two coordinate subspaces, and the lemma follows. $\square$

§2. Homogeneous Coordinates

Given a toric variety X, the Chow group $A_{n-1}(X)$ defined in §1 is a finitely generated abelian group of rank $d - n$, where $d = |\Delta(1)|$. It follows that $G = \mathrm{Hom}_{\mathbb{Z}}(A_{n-1}(X), \mathbb{C}^*)$ is isomorphic to a product of the torus $(\mathbb{C}^*)^{d-n}$ and the finite group $\mathrm{Hom}_{\mathbb{Z}}(A_{n-1}(X)_{tor}, \mathbb{Q}/\mathbb{Z})$. Furthermore, if we apply $\mathrm{Hom}(-, \mathbb{C}^*)$ to the bottom exact sequence of (1), then we get the exact sequence

$$1 \longrightarrow G \longrightarrow (\mathbb{C}^*)^{\Delta(1)} \longrightarrow T \longrightarrow 1 ,$$

where $T = N \otimes \mathbb{C}^* = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{C}^*)$ is the torus acting on X. Note also that $A_{n-1}(X)$ is naturally isomorphic to the character group of G.

Since $(\mathbb{C}^*)^{\Delta(1)}$ acts naturally on $\mathbb{C}^{\Delta(1)}$, the subgroup $G \subset (\mathbb{C}^*)^{\Delta(1)}$ acts on $\mathbb{C}^{\Delta(1)}$ by

$$g \cdot t = (g([D_\rho]) \, t_\rho)$$

for $g : A_{n-1}(X) \to \mathbb{C}^*$ in G and $t = (t_\rho)$ in $\mathbb{C}^{\Delta(1)}$. We get a corresponding representation of G on $S = \mathbb{C}[x_\rho]$, and given a character $\alpha \in A_{n-1}(X)$ of G, the graded piece S_α is the α-eigenspace of this representation.

Using the action of G on $\mathbb{C}^{\Delta(1)}$, we can describe the toric variety X as follows:

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Theorem 2.1. *Let X be the toric variety determined by the fan Δ, and let $Z = \mathbf{V}(B) \subset \mathbb{C}^{\Delta(1)}$ be as in §1. Then:*

(i) *The set $\mathbb{C}^{\Delta(1)} - Z$ is invariant under the action of the group G.*

(ii) *X is naturally isomorphic to the categorical quotient of $\mathbb{C}^{\Delta(1)} - Z$ by G.*

(iii) *X is the geometric quotient of $\mathbb{C}^{\Delta(1)} - Z$ by G if and only if X is simplicial (i.e., the fan Δ is simplicial).*

In the analytic category, part (iii) of Theorem 2.1 was known to M. Audin (see Chapter VI of [Audin], which mentions earlier work by Delzant and Kirwan), and versions of the theorem were known to V. Batyrev (unpublished) and J. Fine (just now appearing in [Fine]). As this paper was being written, I. Musson discovered a result closely related to part (ii) of the theorem (see [Musson]), and more recently he proved an analog of part (iii).

In the first version of this paper, part (iii) only stated that the geometric quotient exists when X is simplicial. The referee asked if the converse were true, and simultaneously Musson proved this in a slightly different content. His argument is used below.

Proof of Theorem 2.1. Given a cone $\sigma \in \Delta$, let

$$U_\sigma = \mathbb{C}^{\Delta(1)} - \mathbf{V}(x^{\hat{\sigma}}) = \{x \in \mathbb{C}^{\Delta(1)} : x^{\hat{\sigma}} \neq 0\} \ .$$

Note that U_σ is a G-invariant affine open subset of $\mathbb{C}^{\Delta(1)}$ and that

$$\mathbb{C}^{\Delta(1)} - Z = \bigcup_{\sigma \in \Delta} U_\sigma \ .$$

This proves part (i) of the theorem.

For parts (ii) and (iii) of the theorem, we first need to study the quotient of G acting on the affine variety U_σ. The coordinate ring of U_σ is the localization of S at $x^{\hat{\sigma}}$, which we will denote S_σ. Note that S_σ has a natural grading by $A_{n-1}(X)$.

The following lemma describes the degree zero part of S_σ:

Lemma 2.2. *Let $\sigma \in \Delta$ be a cone and let $\check{\sigma} \subset M_{\mathbb{R}}$ be its dual cone. Then there is a natural isomorphism of rings $\mathbb{C}[\check{\sigma} \cap M] \simeq (S_\sigma)_0$.*

Proof. Given $m \in M$, let $x^{D_m} = \prod_\rho x_\rho^{\langle m, n_\rho \rangle}$, and note that

$$x^{D_m} \in S_\sigma \iff \langle m, n_\rho \rangle \geq 0 \text{ for all } \rho \in \sigma(1) \iff m \in \check{\sigma} \ .$$

Since a monomial has degree zero if and only if it is of the form x^{D_m} for some $m \in M$, we get a one-to-one correspondence between $\check{\sigma} \cap M$ and monomials in $(S_\sigma)_0$. Then the map sending $m \in \check{\sigma} \cap M$ to $x^{D_m} \in (S_\sigma)_0$ gives the desired ring isomorphism. $\square$

The action of G on U_σ induces an action on S_σ, and as we observed earlier about S, the graded pieces are the eigenspaces for the characters of G. In particular, the invariants

of G acting on S_σ are precisely the polynomials of degree zero. Thus Lemma 2.2 implies

$$(S_\sigma)^G = (S_\sigma)_0 \simeq \mathbb{C}[\check{\sigma} \cap M] \ .$$

Since G is a reductive group acting on the affine scheme U_σ, standard results from invariant theory (see [Fogarty and Mumford, Theorem 1.1]) imply that the categorical quotient is given by

$$U_\sigma/G = \mathrm{Spec}((S_\sigma)^G) = \mathrm{Spec}(\mathbb{C}[\check{\sigma} \cap M]) \ .$$

Thus U_σ/G is the affine toric variety X_σ determined by the cone σ. Note that X_σ is an affine piece of the toric variety X.

The next step is to see how the quotients U_σ/G fit together as we vary σ. So let τ be a face of a cone $\sigma \in \Delta$. It is well-known that $\tau = \sigma \cap \{m\}^\perp$ for some $m \in \check{\sigma} \cap M$. Then we can relate $(S_\tau)_0$ and $(S_\sigma)_0$ as follows:

Lemma 2.3. *Let τ be a face of σ, and let $m \in \check{\sigma} \cap M$ be as above. Under the isomorphism of Lemma 2.2, m maps to the monomial $x^{D_m} = \prod_\rho x_\rho^{\langle m, n_\rho \rangle} \in (S_\sigma)_0$. Then:*

(i) There is a natural identification

$$(S_\sigma)_0 = \left((S_\tau)_0\right)_{x^{D_m}} \ .$$

(ii) There is a commutative diagram

$$
\begin{array}{ccc}
(S_\tau)_0 & \longrightarrow & (S_\sigma)_0 = \left((S_\tau)_0\right)_{x^{D_m}} \\
\downarrow & & \downarrow \\
\mathbb{C}[\check{\tau} \cap M] & \longrightarrow & \mathbb{C}[\check{\sigma} \cap M] = \mathbb{C}[\check{\tau} \cap M]_m
\end{array}
$$

where the horizontal maps are localization maps and the vertical arrows are the isomorphisms of Lemma 2.2.

Proof. Since $\tau \in \sigma \cap \{m\}^\perp$, x^{D_m} has positive exponent for x_ρ when $\rho \in \sigma(1) - \tau(1)$ and zero exponent when $\rho \in \tau(1)$. It follows immediately that $S_\tau = (S_\sigma)_{x^{D_m}}$. Part (i) of the lemma follows since localization commutes with taking elements of degree zero (because x^{D_m} already has degree zero). It is trivial to check that the diagram of part (ii) commutes, and the lemma is proved. $\square$

We can now prove part (ii) of Theorem 2.1. Namely, Lemma 2.2 gives us categorical quotients $U_\sigma/G \cong X_\sigma$, and by the compatibility of Lemma 2.3, these patch to give a categorical quotient $(\mathbb{C}^{\Delta(1)} - Z)/G \cong X$.

For part (iii) of the theorem, first assume that X is simplicial. To show that X is a geometric quotient, it suffices to show that each X_σ is the geometric quotient of G acting on U_σ. By [Fogarty and Mumford, Amplification 1.3], we need only show that the orbits of G acting on U_σ are closed.

Fix a point $t = (t_\rho)$ of U_σ. In Lemma 2.2 we saw that $m \in \check{\sigma} \cap M$ gives the monomial $x^{D_m} = \prod_\rho x_\rho^{\langle m, n_\rho \rangle} \in (S_\sigma)_0$. Then consider the following subvariety $V \subset U_\sigma$:

$$V = \{x = (x_\rho) \in U_\sigma : x^{D_m} = t^{D_m} \text{ for all } m \in \check{\sigma} \cap M\} .$$

We will prove that $V = G \cdot t$. First observe that $t \in V$, and since x^{D_m} is G-invariant (it has degree zero), it follows that $G \cdot t \subset V$.

To prove the other inclusion, let $u = (u_\rho) \in V$. We must find $g : A_{n-1}(X) \to \mathbb{C}^*$ such that $u_\rho = g([D_\rho]) \, t_\rho$ for all $\rho \in \Delta(1)$. The first step is to show that

(2) $$\text{for all } \rho \in \Delta(1), \ u_\rho \neq 0 \iff t_\rho \neq 0 .$$

Note that (2) is trivially true when $\rho \notin \sigma(1)$. So suppose that $\rho \in \sigma(1)$. Since X is simplicial, the set $\{n_{\rho'} : \rho' \in \sigma(1)\}$ is linearly independent. Hence we can find $m \in M$ such that

$$\langle m, n_\rho \rangle > 0 \text{ and } \langle m, n_{\rho'} \rangle = 0 \text{ for all } \rho' \in \sigma(1) - \{\rho\} .$$

Setting $a = \langle m, n_\rho \rangle > 0$, we have

$$u^{D_m} = \prod_{\rho' \notin \sigma(1)} u_{\rho'}^{\langle m, n_{\rho'} \rangle} \cdot u_\rho^a$$

$$t^{D_m} = \prod_{\rho' \notin \sigma(1)} t_{\rho'}^{\langle m, n_{\rho'} \rangle} \cdot t_\rho^a .$$

Since these are equal, and since $u_{\rho'}$ and $t_{\rho'}$ are nonzero for $\rho' \notin \sigma(1)$, it follows that $u_\rho \neq 0$ if and only if $t_\rho \neq 0$. This completes the proof of (2).

Now let $A = \{\rho : t_\rho = 0\} = \{\rho : u_\rho = 0\}$ and note that $A \subset \sigma(1)$. For $\rho \in \Delta(1) - A$, set

$$g_A(D_\rho) = \frac{u_\rho}{t_\rho} .$$

This defines a map $g_A : \mathbb{Z}^{\Delta(1)-A} \to \mathbb{C}^*$. To relate this to $G = \operatorname{Hom}_\mathbb{Z}(A_{n-1}(X), \mathbb{C}^*)$, let $i_A : M \to \mathbb{Z}^A$ be the composition $M \to \mathbb{Z}^{\Delta(1)} \to \mathbb{Z}^A$, where the last map is projection. Then we have the commutative diagram

$$
\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
0 & \longrightarrow & 0 & \longrightarrow & M & = & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow i_A & & \\
0 & \longrightarrow & \mathbb{Z}^{\Delta(1)-A} & \longrightarrow & \mathbb{Z}^{\Delta(1)} & \longrightarrow & \mathbb{Z}^A & \longrightarrow & 0 \\
 & & \| & & \downarrow & & & & \\
 & & \mathbb{Z}^{\Delta(1)-A} & \longrightarrow & A_{n-1}(X) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

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Since the columns and two middle rows are exact, the snake lemma gives us an exact sequence

$$\text{(3)} \qquad\qquad \ker(i_A) \longrightarrow \mathbb{Z}^{\Delta(1)-A} \longrightarrow A_{n-1}(X) \ .$$

We next claim that $g_A(\ker(i_A)) = \{1\}$. An easy diagram chase shows that for $m \in \ker(i_A)$, we have

$$g_A(m) = \frac{u^{D_m}}{t^{D_m}} \ .$$

When $m \in \check{\sigma} \cap M$, we know that $u^{D_m} = t^{D_m}$, so that $g_A(m) = 1$ whenever $m \in \check{\sigma} \cap \ker(i_A)$. If we can show that this semigroup generates the group $\ker(i_A)$, then our claim will follow. But $\{n_\rho : \rho \in \sigma(1)\}$ is linearly independent since σ is simplicial. Hence we can find $\tilde{m} \in M$ such that

$$\langle \tilde{m}, n_\rho \rangle > 0 \text{ when } \rho \in \Sigma(1) - A, \text{ and } \langle \tilde{m}, n_\rho \rangle = 0 \text{ when } \rho \in A \ .$$

Note that $\tilde{m} \in \check{\sigma} \cap \ker(i_A)$. Now, given any $m \in \ker(i_A)$, we can pick an integer N such that $m + N\tilde{m} \in \check{\sigma} \cap \ker(i_A)$. Then $m = (m + N\tilde{m}) - N\tilde{m}$ shows that $\check{\sigma} \cap \ker(i_A)$ generates $\ker(i_A)$ as a group. This proves that $g_A(\ker(i_A)) = \{1\}$.

From the exact sequence (3), it follows that $g_A : \mathbb{Z}^{\Delta(1)-A} \to \mathbb{C}^*$ induces a map $\tilde{g}_A : \tilde{A} \to \mathbb{C}^*$, where $\tilde{A} \subset A_{n-1}(X)$ is the image of $\mathbb{Z}^{\Delta(1)-A} \to A_{n-1}(X)$. Since $\mathbb{C}^*$ is injective as an abelian group, $\tilde{g}_A$ extends to $g : A_{n-1}(X) \to \mathbb{C}^*$. Thus $g \in G$, and it is straightforward to check that $u_\rho = g([D_\rho]) \, t_\rho$ for all ρ. Then $u \in G \cdot t$, and we conclude that $G \cdot t = V$. Thus the orbits are closed and we have a geometric quotient.

To prove the converse, assume that X is not simplicial. The following argument of I. Musson will show that G has a non-closed orbit in $\mathbb{C}^{\Delta(1)} - Z$. Let $\sigma \in \Delta$ be a non-simplicial cone, so that we have a relation $\sum_{\rho \in \sigma(1)} b_\rho n_\rho = 0$. We can assume that $b_\rho \in \mathbb{Z}$ and that at least one $b_\rho > 0$. Then set $b_\rho = 0$ for $\rho \notin \sigma(1)$. Applying $\text{Hom}_{\mathbb{Z}}(-, \mathbb{Z})$ to the exact sequence at the bottom of (1), we see that $\phi(t)([D_\rho]) = t^{b_\rho}$ defines an element of G for $t \in \mathbb{C}^*$ (in fact, $\phi(t)$ is a 1-parameter subgroup of G). Now consider $u = (u_\rho) \in \mathbb{C}^{\Delta(1)}$, where

$$u_\rho = \begin{cases} 1 & \text{if } \rho \notin \sigma(1) \text{ or } m_\rho > 0 \\ 0 & \text{otherwise .} \end{cases}$$

It is easy to check that u and $\lim_{t \to 0} \phi(t)(u)$ are in $\mathbb{C}^{\Delta(1)} - Z$ but lie in different G-orbits. Thus $G \cdot u$ isn't closed in $\mathbb{C}^{\Delta(1)} - Z$, and Theorem 2.1 is proved. $\qquad\square$

The description of X as a quotient of $\mathbb{C}^{\Delta(1)} - Z$ gives a new way of looking at the torus action on X. Namely, an element of the torus $(\mathbb{C}^*)^{\Delta(1)}$ gives an automorphism of $\mathbb{C}^{\Delta(1)}$ which preserves Z. It also commutes with the action of G, so that we get an automorphism of X by the universal property of a categorical quotient. This gives a map

$(\mathbb{C}^*)^{\Delta(1)} \to \operatorname{Aut}(X)$, and the kernel is easily seen to be G (see §4 for a proof). Since we have the exact sequence

$$1 \longrightarrow G \longrightarrow (\mathbb{C}^*)^{\Delta(1)} \longrightarrow T \longrightarrow 1 \ ,$$

it follows that the torus T is naturally a subgroup of $\operatorname{Aut}(X)$. In §4, we will discuss $\operatorname{Aut}(X)$ in more detail. Note also that $(\mathbb{C}^*)^{\Delta(1)} \subset \mathbb{C}^{\Delta(1)} - Z$, and thus, taking the quotient with respect to G, we get the inclusion $T \subset X$.

For the final result of this section, we will use Theorem 2.1 to study closed subsets (= reduced closed subschemes) of a simplicial toric variety X. Given a graded ideal $I \subset S$, note that the set

$$\mathbf{V}(I) - Z = \{t \in \mathbb{C}^{\Delta(1)} - Z : f(t) = 0 \text{ for all } f \in I\} \subset \mathbb{C}^{\Delta(1)} - Z$$

is G-invariant and closed in the Zariski topology. Since we have a geometric quotient, the quotient map $\mathbb{C}^{\Delta(1)} - Z \to X$ is submersive (see [Fogarty and Mumford, Definition 0.6]). Thus there is a one-to-one correspondence between G-invariant closed sets in $\mathbb{C}^{\Delta(1)} - Z$ and closed sets in X. Hence $\mathbf{V}(I) - Z$ determines a closed subset of X which we will denote $\mathbf{V}_X(I)$. By abuse of notation, we can write

$$\mathbf{V}_X(I) = \{t \in X : f(t) = 0 \text{ for all } f \in I\} \ ,$$

provided we think of t as the homogeneous coordinates of a point of X.

The map sending $I \subset S$ to $\mathbf{V}_X(I) \subset X$ has the following properties:

Proposition 2.4. *Let X be a simplicial toric variety, and let $B = \langle x^{\hat{\sigma}} : \sigma \in \Delta \rangle \subset S$ be the ideal defined in §1.*
  (i) *(The Toric Nullstellensatz) For any graded ideal $I \subset S$, we have $\mathbf{V}_X(I) = \emptyset$ if and only if $B^m \subset I$ for some integer m.*
  (ii) *(The Toric Ideal–Variety Correspondence) The map $I \mapsto \mathbf{V}_X(I)$ induces a one-to-one correspondence between radical graded ideals of S contained in B and closed subsets of X.*

Proof. (i) Since $\mathbf{V}_X(I) = \emptyset$ in X if and only if $\mathbf{V}(I) \subset Z = \mathbf{V}(B)$ in $\mathbb{C}^{\Delta(1)}$, the first part of the proposition follows from the usual Nullstellensatz.

(ii) Given a closed subset $W \subset \mathbb{C}^{\Delta(1)}$, we get the ideal $\mathbf{I}(W) = \{f \in S : f_{|W} = 0\}$. It is straightforward to show that W is G-invariant if and only if $\mathbf{I}(W)$ is a graded ideal of S. Note also that B is radical since it is generated by squarefree monomials. Thus $B = \mathbf{I}(Z)$,

and we get one-to-one correspondences

$$\text{radical graded ideals of } S \text{ contained in } B$$
$$\longleftrightarrow G\text{-invariant closed subsets of } \mathbb{C}^{\Delta(1)} \text{ containing } Z$$
$$\longleftrightarrow G\text{-invariant closed subsets of } \mathbb{C}^{\Delta(1)} - Z \ ,$$

where the last correspondence is induced by Zariski closure since Z is G-invariant. The proposition now follows since $\mathbb{C}^{\Delta(1)} - Z \to X$ is submersive. $\qquad\square$

In §3, we will generalize this proposition by showing that there is a correspondence between graded S-modules and quasi-coherent sheaves on X.

§3. Graded Modules and Quasi-coherent Sheaves

In this section we will explore the relation between sheaves on a toric variety X and graded modules over its homogeneous coordinate ring S. The theory we will sketch is similar to what happens for sheaves on $\mathbb{P}^n$ (as given in [Hartshorne] or [Serre]).

An S-module F is *graded* if there is a direct sum decomposition

$$F = \bigoplus_{\alpha \in A_{n-1}(X)} F_\alpha$$

such that $S_\alpha \cdot F_\beta \subset F_{\alpha+\beta}$ for all $\alpha, \beta \in A_{n-1}(X)$. Given such a module F, we will construct a quasi-coherent sheaf $\widetilde{F}$ on X as follows. First, for $\sigma \in \Delta$, let S_σ be the localization of S at $x^{\hat{\sigma}} = \prod_{\rho \notin \sigma(1)} x_\rho$. Then $F_\sigma = F \otimes_S S_\sigma$ is a graded S_σ-module, so that taking elements of degree 0 gives a $(S_\sigma)_0$-module $(F_\sigma)_0$. From the proof of Theorem 2.1, we know that $X_\sigma = \mathrm{Spec}((S_\sigma)_0)$ is an affine open subset of X, and then standard theory tells us that $(F_\sigma)_0$ determines a quasi-coherent sheaf $\widetilde{(F_\sigma)}_0$ on X_σ.

To see how these sheaves patch together, let τ be a face of σ. Then $\tau = \sigma \cap \{m^\perp\}$ for some $m \in \check{\sigma} \cap M$, and as in Lemma 2.3, one can prove

$$(F_\sigma)_0 = \big((F_\tau)_0\big)_{x^{D_m}}$$

where $x^{D_m} = \prod_\rho x_\rho^{\langle m, n_\rho \rangle}$. It follows immediately that the sheaves $\widetilde{(F_\sigma)}_0$ on X_σ patch to give a quasi-coherent sheaf $\widetilde{F}$ on X.

The map $F \mapsto \widetilde{F}$ has the following basic properties of (we omit the simple proof):

Proposition 3.1. *The map sending F to $\widetilde{F}$ is an exact functor from graded S-modules to quasi-coherent $\mathcal{O}_X$-modules.* $\qquad\square$

As an example of how this works, consider the S-module $S(\alpha)$, where $S(\alpha)_\beta = S_{\alpha+\beta}$ for all β. We will denote the sheaf $\widetilde{S(\alpha)}$ by $\mathcal{O}_X(\alpha)$. If $\alpha = [D]$ for some T-invariant divisor

D, then there is an isomorphism

$$(5) \qquad \mathcal{O}_X(\alpha) \simeq \mathcal{O}_X(D) \ .$$

To prove this, it suffices to find compatible isomorphisms

$$(S(\alpha)_\sigma)_0 \simeq H^0(X_\sigma, \mathcal{O}_X(D))$$

for each affine open X_σ of X. If $D = \sum_\rho a_\rho D_\rho$, then [Fulton, §3.4] shows that this is equivalent to finding compatible isomorphisms

$$(S_\sigma)_\alpha \simeq \bigoplus_{\substack{m \in M \\ \langle m, n_\rho \rangle \geq -a_\rho \\ \text{for all } \rho \in \sigma(1)}} \mathbb{C} \cdot \chi^m \ .$$

As in Proposition 1.1, we can map χ^m to the monomial $x^{D_m + D} = \prod_\rho x^{\langle m, n_\rho \rangle + a_\rho} \in (S_\sigma)_\alpha$ The compatibility of these maps is easily checked, giving us the desired isomorphism (5).

From (5), we see that $\widetilde{S} = \mathcal{O}_X$. It follows easily that $H^0(X, \mathcal{O}_X(\alpha))$ is canonically isomorphic to S_α, so that

$$S \simeq \bigoplus_\alpha H^0(X, \mathcal{O}_X(\alpha)) \ .$$

From (5), we also conclude that $\mathcal{O}_X(\alpha)$ is a coherent sheaf (see [Fulton, §3.4]). However, it need not be a line bundle—in fact, $\mathcal{O}_X(\alpha)$ is invertible if and only if α is the class of a Cartier divisor. See [Dolgachev, §1.5] for an example of how $\mathcal{O}_X(\alpha)$ can fail to be invertible when X is a weighted projective space. Also note that the natural map

$$(6) \qquad \mathcal{O}_X(\alpha) \otimes_{\mathcal{O}_X} \mathcal{O}_X(\beta) \to \mathcal{O}_X(\alpha + \beta)$$

need not be an isomorphism—see [Dolgachev, §1.5] for an example. However, when $\mathcal{O}_X(\alpha)$ and $\mathcal{O}_X(\beta)$ are invertible, then (6) is an isomorphism.

Our next task is to see which sheaves on X come from graded S-modules. When X is simplicial, the answer is especially nice:

Theorem 3.2. *If X is a simplicial toric variety, then every quasi-coherent sheaf on X is of the form $\widetilde{F}$ for some graded S-module F.*

Proof. The proof is similar to the proof of the corresponding result in $\mathbb{P}^n$, as given in [Hartshorne] or [Serre]. The argument requires standard facts about the behavior of sections of line bundles (see [Hartshorne, Lemma II.5.14]). As we observed above, $\mathcal{O}_X(\alpha)$ need not be a line bundle. However, when X is simplicial, $\text{Pic}(X)$ has finite index in $A_{n-1}(X)$ (this follows from an exercise in [Fulton, §3.3]). Thus, given any α, there is a positive integer k such that $\mathcal{O}_X(k\alpha)$ is a line bundle.

Given a quasi-coherent sheaf $\mathcal{F}$ on X, let

$$F = \bigoplus_\alpha H^0(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(\alpha)) \ .$$

Using (6) and the natural isomorphism $S_\alpha \simeq H^0(X, \mathcal{O}_X(\alpha))$, we see that F has the natural structure of a graded S-module. Given $\sigma \in \Delta$, let

$$F'_\sigma = \bigoplus_\alpha H^0(X_\sigma, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(\alpha))$$

which is a graded module over S_σ. There is an obvious map

$$\phi_\sigma : (F_\sigma)_0 \longrightarrow (F'_\sigma)_0 = H^0(X_\sigma, \mathcal{F}) \ .$$

which patch to give $\phi : \widetilde{F} \to \mathcal{F}$. To show that ϕ is an isomorphism, it suffices to show that ϕ_σ is an isomorphism for all $\sigma \in \Delta$.

An element of $(F_\sigma)_0$ is written f/x^D, where $f \in H^0(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(\alpha))$ and $\deg(x^D) = \alpha$. Since $f/x^D = (x^{(k-1)D} f)/x^{kD}$ when $k > 0$, we can assume that $\mathcal{O}_X(\alpha)$ is a line bundle. We can also assume that $x^{\hat{\sigma}}$ divides x^D, which implies that $X_\sigma = \{x \in X : x^D \neq 0\}$. This guarantees that [Hartshorne, Lemma II.5.14] applies, and from here, the proof follows the arguments in [Hartshorne, Proposition II.5.15] without change. $\qquad\square$

We next study coherent sheaves on X:

Proposition 3.3. *Let X be a toric variety with coordinate ring S.*
 (i) *If F is a finitely generated graded S-module, the $\widetilde{F}$ is a coherent sheaf on X.*
 (ii) *If X is simplicial, then every coherent sheaf on X is of the form $\widetilde{F}$ for some finitely generated graded S-module F.*

Proof. (i) A set of homogeneous generators of F gives a surjective map $\oplus_i S(-\alpha_i) \to F$. By Proposition 3.1, we get a surjection $\oplus_i \mathcal{O}_X(-\alpha_i) \to \widetilde{F}$, and it follows that $\widetilde{F}$ is coherent.

(ii) Suppose that $\mathcal{F}$ is coherent. On the affine scheme X_σ, we can find finitely many sections $f_{i,\sigma} \in H^0(X_\sigma, \mathcal{F})$ which generate $\mathcal{F}$ over X_σ. Since X is simplicial, arguing as in the proof of Theorem 3.2 shows that we can find some x^{D_σ} invertible in S_σ such that $x^{D_\sigma} f_{i,\sigma}$ comes from a section $g_{i,\sigma}$ of $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(\alpha_\sigma)$, where $\alpha_\sigma = \deg(x^{D_\sigma})$. Now consider the graded S-module $F' \subset F = \oplus_\alpha H^0(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(\alpha))$ generated by the $g_{i,\sigma}$. Since $\widetilde{F} \simeq \mathcal{F}$, we see that $\widetilde{F}' \to \mathcal{F}$ is injective. Over X_σ, we have $f_{i,\sigma} = g_{i,\sigma}/x^{D_\sigma} \in (F'_\sigma)_0$, and since these sections generate $\mathcal{F}$ over X_σ, it follows that $\widetilde{F}' \simeq \mathcal{F}$. $\qquad\square$

The map sending graded S-module F to the sheaf $\widetilde{F}$ is not injective, and in fact, there are many nonzero modules which give the zero sheaf. To analyze this phenomenon, we will use the subgroup $\mathrm{Pic}(X) \subset A_{n-1}(X)$. The following lemma will be helpful:

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Lemma 3.4. *If $\alpha \in \mathrm{Pic}(X) \subset A_{n-1}(X)$, then for every $\sigma \in \Delta$, we can write α in the form $\alpha = \left[\sum_{\rho \notin \sigma(1)} a_\rho D_\rho\right]$.*

Proof. It is well-known (see, for example, the exercises in [Fulton, §3.3]) that if $\sum_\rho b_\rho D_\rho$ is a Cartier divisor, then for each $\sigma \in \Delta$, there is $m \in M$ such that $\langle m, n_\rho \rangle = -b_\rho$ for all $\rho \in \sigma(1)$. It follows that D is linearly equivalent to the divisor $\sum_{\rho \notin \sigma(1)} (b_\rho + \langle m, n_\rho \rangle) D_\rho$, and the lemma is proved. ∎

We can now describe which finitely generated graded S-modules F correspond to the zero sheaf on X. In the projective case, where $S = \mathbb{C}[x_0, \ldots, x_n]$, this happens if and only if $F_k = \{0\}$ for all $k \gg 0$. This can be rephrased in terms of the irrelevant ideal $\langle x_0, \ldots, x_n \rangle \subset S$: namely, $F_k = \{0\}$ for $k \gg 0$ exactly when $\langle x_0, \ldots, x_n \rangle^k F = \{0\}$ for some k. In the case of a simplicial toric variety, we can generalize this as follows:

Proposition 3.5. *Let X be a simplicial toric variety, and let $B \subset S$ be the ideal defined in §1. Then, for a finitely generated graded S-module F, we have $\widetilde{F} = 0$ on X if and only if there is some $k > 0$ such that $B^k F_\alpha = \{0\}$ for all $\alpha \in \mathrm{Pic}(X)$.*

Proof. First, suppose that $B^k F_\alpha = \{0\}$ for all $\alpha \in \mathrm{Pic}(X)$. Then fix $\sigma \in \Delta$ and take $f/x^D \in (F_\sigma)_0$. Because X is simplicial, we know that $l[D] \in \mathrm{Pic}(X)$ for some $l > 0$. If we let $\alpha = l[D]$, then $x^{(l-1)D} f \in F_\alpha$, and hence $(x^{\hat\sigma})^k x^{(l-1)D} f = 0$ for some $k > 0$. This easily implies $f/x^D = 0$, and it follows that $(F_\sigma)_0 = \{0\}$. Thus $\widetilde{F}$ is the zero sheaf.

Conversely, let F be a finitely generated graded S-module such that $\widetilde{F} = 0$. Fix $\alpha \in \mathrm{Pic}(X)$ and let $f \in F_\alpha$. By Lemma 3.4, we can write $\alpha = [D]$ where $D = \sum_{\rho \notin \sigma(1)} a_\rho D_\rho$. Then x^D is an invertible element in S_σ, and hence $f/x^D \in (F_\sigma)_0 = \{0\}$. This implies that $(x^{\hat\sigma})^k f = 0$ for some $k > 0$. To show that we can get a k that works for all $f \in F_\alpha$ and all $\alpha \in \mathrm{Pic}(X)$, note that $\sum_{\alpha \in \mathrm{Pic}(X)} SF_\alpha \subset F$ is a submodule and hence is finitely generated. The proposition now follows easily. ∎

When X is a smooth toric variety, we know that $\mathrm{Pic}(X) = A_{n-1}(X)$. Hence we get the following corollary of Proposition 3.5:

Corollary 3.6. *If X is a smooth toric variety and F is a finitely generated graded S-module, then $\widetilde{F} = 0$ on X if and only if there is some $k > 0$ with $B^k F = \{0\}$.* ∎

To see how Corollary 3.6 can fail when X not smooth, consider $X = \mathbb{P}(1,1,2)$. Here, $S = \mathbb{C}[x, y, z]$, where x, y have degree 1 and z has degree 2, and $B = \langle x, y, z \rangle$. The graded $F = S(1)/(xS(1) + yS(1))$ has only elements of odd degree. Then $(F_z)_0 = \{0\}$ since z has degree 2, and it is clear that $(F_x)_0 = (F_y)_0 = \{0\}$. It follows that $\widetilde{F} = 0$, yet one easily checks that $B^k F = z^k F \neq \{0\}$ for all k. Thus Corollary 3.6 does not hold.

We will end this section with a study of the ideal-variety correspondence. Our goal is to refine what we did in Proposition 2.4 by taking the scheme structure into account.

First note that if $I \subset S$ is a graded ideal, then by exactness we get an ideal sheaf $\widetilde{I} \subset \mathcal{O}_X$. Hence $I \subset S$ determines the closed subscheme $Y \subset X$ with $\mathcal{O}_Y = \mathcal{O}_X/\widetilde{I}$. The following theorem tells us which subschemes of X arise in this way:

Theorem 3.7. *Let X be a simplicial toric variety, and let $B \subset S$ be as usual.*

(i) *Every closed subscheme of X comes from a graded ideal of S (as described above).*

(ii) *Two graded ideals I and J of S correspond to the same closed subscheme of X if and only $(I{:}B^\infty)_\alpha = (J{:}B^\infty)_\alpha$ for all $\alpha \in \mathrm{Pic}(X)$.*

Proof. (i) Given an ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$, let $I' = \oplus_\alpha H^0(X, \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{O}_X(\alpha))$. Since X is simplicial, we know that $\widetilde{I'} = \mathcal{I}$. The map $I' \to S$ factors $I' \twoheadrightarrow I \hookrightarrow S$, which gives maps $\mathcal{I} \twoheadrightarrow \widetilde{I} \hookrightarrow \mathcal{O}_X$ by exactness. Since the composition is $\mathcal{I} \subset \mathcal{O}_X$, it follows that $\mathcal{I} = \widetilde{I}$.

(ii) First, note that if I is graded, then so is $I{:}B^\infty$. Now, given graded ideals $I, J \subset S$, assume that $\widetilde{I} = \widetilde{J}$. Then for every $\sigma \in \Delta$, we have $(I_\sigma)_0 = (J_\sigma)_0$ in $(S_\sigma)_0$. Let $\alpha \in \mathrm{Pic}(X)$ and pick $u \in (I{:}B^\infty)_\alpha$. For k sufficiently large, we have $(x^{\hat\sigma})^k u \in I$. Increasing k if necessary, we can assume that $\beta = k \deg(x^{\hat\sigma}) \in \mathrm{Pic}(X)$. Thus $\alpha + \beta \in \mathrm{Pic}(X)$, so Lemma 3.4 implies $\alpha + \beta = [D]$, where $D = \sum_{\rho \notin \sigma(1)} a_\rho D_\rho$. Then $((x^{\hat\sigma})^k u)/x^D \in (I_\sigma)_0 = (J_\sigma)_0$, which implies that $(x^{\hat\sigma})^{k+l} u \in J$ for some $l > 0$. This is true for all σ, so that $u \in (J{:}B^\infty)_\alpha$. The other inclusion $(J{:}B^\infty)_\alpha \subset (I{:}B^\infty)_\alpha$ follows by symmetry, and equality is proved.

Conversely, suppose $(I{:}B^\infty)_\alpha = (J{:}B^\infty)_\alpha$ for all $\alpha \in \mathrm{Pic}(X)$. To prove $\widetilde{I} = \widetilde{J}$, it suffices to show that $(I_\sigma)_0 = (J_\sigma)_0$ for all σ. But given $u \in (I_\sigma)_0$, we know that $u = f/(x^{\hat\sigma})^k$, and we can assume that $\alpha = \deg(f) \in \mathrm{Pic}(X)$. Then $f \in I_\alpha \subset (I{:}B^\infty)_\alpha = (J{:}B^\infty)_\alpha$. This easily implies $u = f/(x^{\hat\sigma})^k$ is in $(J_\sigma)_0$. Thus $(I_\sigma)_0 \subset (J_\sigma)_0$, and the opposite inclusion follows by symmetry. This completes the proof of the theorem. $\square$

As an example, consider the ideal $I = \langle x_\rho \rangle$ for some fixed $\rho \in \Delta(1)$. We will show that the corresponding subscheme of X is the divisor D_ρ. Given $\sigma \in \Delta$, we need to study $(I_\sigma)_0$. If $\rho \in \sigma(1)$, then it is easy to check that under the isomorphism $(S_\sigma)_0 \simeq \mathbb{C}[\check\sigma \cap M]$, we have

$$(I_\sigma)_0 \simeq \bigoplus_{\substack{m \in M \\ \langle m, \sigma \rangle \geq 0 \\ \langle m, n_\rho \rangle > 0}} \mathbb{C} \cdot \chi^m \ .$$

By an exercise in [Fulton, §3.1], this is exactly the ideal of D_ρ in the affine piece $X_\sigma \subset X$. On the other hand, if $\rho \notin \sigma(1)$, then $(I_\sigma)_0 = \langle 1 \rangle$, which is consistent with $D_\rho \cap X_\sigma = \emptyset$. Thus D_ρ is defined scheme-theoretically by the ideal $\langle x_\rho \rangle$. In particular, D_ρ has the global equation $x_\rho = 0$ even though it may not be a Cartier divisor.

In the smooth case, Theorem 3.7 has some nice consequences. We say that an ideal $I \subset S$ is *B-saturated* if $I{:}B = I$. Then we have:

Corollary 3.8. *Let X be a smooth toric variety.*

(i) *Two graded ideals I and J of S give the same closed subscheme of X if and only if $I{:}B^\infty = J{:}B^\infty$.*

(ii) *There is a one-to-one correspondence between closed subschemes of X and graded B-saturated ideals of S.*

Proof. Part (i) follows from Theorem 3.7 since $\mathrm{Pic}(X) = A_{n-1}(X)$, and part (ii) follows immediately since an ideal is B-saturated if and only if it is of the form $J{:}B^\infty$. □

When X is smooth, Corollary 3.8 gives a one-to-one correspondence between radical graded B-saturated ideals of S and closed reduced subschemes of X. This is slightly different from the correspondence given in Proposition 2.4, where we used radical ideals contained in B. However, the maps

$$I \text{ radical, graded, } B\text{-saturated} \;\mapsto\; I \cap B$$

$$J \text{ radical, graded, contained in } B \;\mapsto\; J{:}B$$

give bijections between these two classes of radical graded ideals, so the two descriptions of reduced subschemes are compatible.

Finally, one could ask if there is an analog of Corollary 3.8 in the simplicial case. There are two ways to approach this problem. One way is to work with the following special class of graded ideals. We say that a graded ideal $I \subset S$ is *Pic-generated* if I is generated by homogeneous elements whose degrees lie in $\mathrm{Pic}(X)$, and I is *Pic-saturated* if $I_\alpha = (I{:}B)_\alpha$ for all $\alpha \in \mathrm{Pic}(X)$. For example, if I is any graded ideal, it is easy to check that $\sum_{\alpha \in \mathrm{Pic}(X)} S(I{:}B^\infty)_\alpha$ is both Pic-generated and Pic-saturated. Then one can prove the following result (we will omit the proof):

Corollary 3.9. *If X is a simplicial toric variety, then there is a one-to-one correspondence between closed subschemes of X and graded ideals of S which are Pic-generated and Pic-saturated.* □

Another way to study sheaves on X is to replace the ring S with the subring

$$(7) \qquad\qquad R = \bigoplus_{\alpha \in \mathrm{Pic}(X)} S_\alpha \;.$$

When X is simplicial, we know that $\mathrm{Pic}(X) \subset A_{n-1}(X)$ has finite index. Thus

$$H = \mathrm{Hom}_{\mathbb{Z}}(A_{n-1}(X)/\mathrm{Pic}(X), \mathbb{C}^*) \subset G = \mathrm{Hom}_{\mathbb{Z}}(A_{n-1}(X), \mathbb{C}^*)$$

is a finite subgroup of G, and $R = S^H$ under the action of H on S. It follows that $Y = \mathrm{Spec}(R)$ is the quotient of $\mathbb{C}^{\Delta(1)}$ under the action of H. Furthermore, if we set $B' = B \cap R$ and let $Z' = \mathbf{V}(B') \subset Y$, then it is easy to see that $Y - Z' = (\mathbb{C}^{\Delta(1)} - Z)/H$. There is also a natural action of $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{Pic}(X), \mathbb{C}^*)$ on $Y - Z'$, and Theorem 2.1 implies the following result:

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Theorem 3.10. *If X is a simplicial toric variety and $Y = \mathrm{Spec}(R)$, where R is the ring (7), then X is the geometric quotient of the action of $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{Pic}(X), \mathbb{C}^*)$ on $Y - Z'$.* $\square$

Although the ring R may not be a polynomial ring, the relation between sheaves and R-modules is nicer than the relation between sheaves and S-modules. It is easy to show that a graded R-module F (where the grading is now given by $\mathrm{Pic}(X)$) gives a sheaf $\widetilde{F}$ on X. Then one can prove the following result (we omit the proof since the arguments are similar to what we did above):

Theorem 3.11. *If X is a simplicial toric variety and R is the ring defined in (7), then:*
(i) *The map sending a graded R-module F to $\widetilde{F}$ is an exact functor.*
(ii) *All quasi-coherent sheaves on X arise in this way.*
(iii) *$\widetilde{F}$ is coherent when F is finitely generated, and this gives all coherent sheaves on X.*
(iv) *If F is finitely generated, then $\widetilde{F} = 0$ if and only if $B'^k F = \{0\}$ for some $k > 0$ (where $B' = B \cap R$).* $\square$

Finally, a graded ideal $I \subset R$ gives an ideal sheaf $\widetilde{I} \subset \mathcal{O}_X$, so that I determines a closed subscheme of X. We will omit the proof of the following result:

Theorem 3.12. *If X is a simplicial toric variety and R is the ring defined in (7), then:*
(i) *Every closed subscheme of X comes from some graded ideal of R.*
(ii) *Two graded ideals I, J give the same closed subscheme if and only if $I{:}B'^\infty = J{:}B'^\infty$ (where $B' = B \cap R$).*
(iii) *There is a one-to-one correspondence between graded B'-saturated ideals of R and closed subschemes of X.* $\square$

§4. The Automorphism Group

This section will use the graded ring S to give an explicit description of the automorphism group $\mathrm{Aut}(X)$ of a complete simplicial toric variety X. To see what sort of results to expect, consider $\mathbb{P}^n \simeq (\mathbb{C}^{n+1} - \{0\})/\mathbb{C}^*$. Notice that $\mathrm{Aut}(\mathbb{P}^n) = PGL(n+1, \mathbb{C})$ doesn't act on $\mathbb{C}^{n+1} - \{0\}$; rather, we must use $GL(n+1, \mathbb{C})$, which is related to $PGL(n+1, \mathbb{C})$ by the exact sequence

$$1 \longrightarrow \mathbb{C}^* \longrightarrow GL(n+1, \mathbb{C}) \longrightarrow PGL(n+1, \mathbb{C}) \longrightarrow 1 \ .$$

For a complete simplicial toric variety X, we know $X \simeq (\mathbb{C}^{\Delta(1)} - Z)/G$ by Theorem 2.1. Then the analog of $GL(n+1, \mathbb{C})$ will be the following pair of groups:

Definition 4.1.
(i) *$\widetilde{\mathrm{Aut}}(X)$ is the normalizer of G in the automorphism group of $\mathbb{C}^{\Delta(1)} - Z$.*
(ii) *$\widetilde{\mathrm{Aut}}^0(X)$ is the centralizer of G in the automorphism group of $\mathbb{C}^{\Delta(1)} - Z$.*

It is clear that $\widetilde{\mathrm{Aut}}^0(X) \subset \widetilde{\mathrm{Aut}}(X)$ is a normal subgroup. Note also that every element of $\widetilde{\mathrm{Aut}}(X)$ preserves G-orbits and hence descends to an automorphism of X. Thus we get a natural homomorphism $\widetilde{\mathrm{Aut}}(X) \to \mathrm{Aut}(X)$.

We can now state the main result of this section:

Theorem 4.2. *Let X be a complete simplicial toric variety, and let $S = \mathbb{C}[x_\rho]$ be its homogeneous coordinate ring. Then:*

(i) *$\widetilde{\mathrm{Aut}}(X)$ is an affine algebraic group of dimension $\sum_\rho \dim_\mathbb{C} S_{\deg(x_\rho)}$, and $\widetilde{\mathrm{Aut}}^0(X)$ is the connected component of the identity.*

(ii) *The natural map $\widetilde{\mathrm{Aut}}(X) \to \mathrm{Aut}(X)$ induces an exact sequence*

$$1 \longrightarrow G \longrightarrow \widetilde{\mathrm{Aut}}(X) \longrightarrow \mathrm{Aut}(X) \longrightarrow 1 \ .$$

(iii) *$\widetilde{\mathrm{Aut}}^0(X)$ is naturally isomorphic to the group $\mathrm{Aut}_g(S)$ of graded $\mathbb{C}$-algebra automorphisms of S.*

In the smooth case, this theorem follows from results of [Demazure]. Before we can begin the proof of Theorem 4.2, we need to study the group $\mathrm{Aut}_g(S)$. Consider the set $\mathrm{End}_g(S)$ of graded $\mathbb{C}$-algebra homomorphisms $\phi : S \to S$ which satisfy $\phi(1) = 1$. Since X is complete, we know that $S_0 = \mathbb{C}$, and it follows that $S^+ = \sum_{\alpha \neq 0} S_\alpha$ is closed under multiplication. Thus there is a natural bijection $\mathrm{End}_g(S) \simeq \mathrm{End}_g(S^+)$. Since S^+ has no identity to worry about, $\mathrm{End}_g(S^+)$ has a natural structure as a $\mathbb{C}$-algebra, and in this way we can regard $\mathrm{End}_g(S)$ as a $\mathbb{C}$-algebra.

Note that an element $\phi \in \mathrm{End}_g(S)$ is uniquely determined by $\phi(x_\rho) \in S_{\deg(x_\rho)}$ for $\rho \in \Delta(1)$. Furthermore, each $S_{\deg(x_\rho)}$ is finite dimensional since X is complete, so that $\mathrm{End}_g(S)$ is a finite dimensional $\mathbb{C}$-algebra. Since $\mathrm{Aut}_g(S)$ is the set of invertible elements of $\mathrm{End}_g(S)$, standard arguments show that $\mathrm{Aut}_g(S)$ is an affine algebraic group.

To describe the structure of this group in more detail, we will use the equivalence relation on $\Delta(1)$ defined by

$$\rho \sim \rho' \iff \deg(x_\rho) = \deg(x_{\rho'}) \text{ in } A_{n-1}(X) \ .$$

This partitions $\Delta(1)$ into disjoint subsets $\Delta_1 \cup \cdots \cup \Delta_s$, where each Δ_i corresponds to a set of variables of the same degree α_i. Then each S_{α_i} can be written as

$$(7) \qquad\qquad\qquad S_{\alpha_i} = S'_{\alpha_i} \oplus S''_{\alpha_i}$$

where S'_{α_i} is spanned by the x_ρ for $\rho \in \Delta_i$, and S''_{α_i} is spanned by the remaining monomials in S_{α_i} (all of which are the product of at least two variables).

Before we state our next result, note that the torus $(\mathbb{C}^*)^{\Delta(1)}$ can be regarded as a subgroup of $\mathrm{Aut}_g(S)$ since $t = (t_\rho) \in (\mathbb{C}^*)^{\Delta(1)}$ gives the graded automorphism $\phi_t : S \to S$ defined by $\phi_t(x_\rho) = t_\rho x_\rho$. Then the group structure of $\mathrm{Aut}_g(S)$ can be described as follows:

Proposition 4.3. *Let X be a complete toric variety, and let S be its homogeneous coordinate ring. Then:*

(i) *$\mathrm{Aut}_g(S)$ is a connected affine algebraic group of dimension $\sum_{i=1}^{s} |\Delta_i| \dim_{\mathbb{C}} S_{\alpha_i}$, and $(\mathbb{C}^*)^{\Delta(1)} \subset \mathrm{Aut}_g(S)$ is its maximal torus.*

(ii) *The unipotent radical R_u of $\mathrm{Aut}_g(S)$ is isomorphic as a variety to an affine space of dimension $\sum_{i=1}^{s} |\Delta_i|(\dim_{\mathbb{C}} S_{\alpha_i} - |\Delta_i|)$.*

(iii) *$\mathrm{Aut}_g(S)$ contains a subgroup G_s isomorphic to the reductive group $\prod_{i=1}^{s} GL(S'_{\alpha_i})$ of dimension $\sum_{i=1}^{s} |\Delta_i|^2$.*

(iv) *$\mathrm{Aut}_g(S)$ is isomorphic to the semidirect product $R_u \rtimes G_s$.*

Proof. We first study the structure of the $\mathbb{C}$-algebra $\mathrm{End}_g(S)$. We have a natural vector space isomorphism

$$\mathrm{End}_g(S) \simeq \oplus_{i=1}^{s} \mathrm{Hom}_{\mathbb{C}}(S'_{\alpha_i}, S_{\alpha_i})$$

since S'_{α_i} is spanned by x_ρ for $\rho \in \Delta_i$ and a graded homomorphism $\phi : S \to S$ is uniquely determined by $\phi(x_\rho) \in S_{\alpha_i}$. Then the direct sum decomposition (7) gives an exact sequence

$$(8) \qquad 0 \longrightarrow \oplus_{i=1}^{s} \mathrm{Hom}_{\mathbb{C}}(S'_{\alpha_i}, S''_{\alpha_i}) \longrightarrow \mathrm{End}_g(S) \xrightarrow{\Psi} \oplus_{i=1}^{s} \mathrm{End}_{\mathbb{C}}(S'_{\alpha_i}) \longrightarrow 0 .$$

Notice that Ψ is a ring homomorphism. This follows immediately from the observation that $\phi(S''_{\alpha_i}) \subset S''_{\alpha_i}$ for any $\phi \in \mathrm{End}_g(S)$. Thus we can regard $\mathcal{N} = \oplus_{i=1}^{s} \mathrm{Hom}_{\mathbb{C}}(S'_{\alpha_i}, S''_{\alpha_i})$ as an ideal of $\mathrm{End}_g(S)$. Note that $\phi \in \mathcal{N}$ if and only if $\phi(x_\rho) \in S''_{\alpha_i}$ for $\rho \in \Delta_i$.

We claim that every element of $\mathcal{N}$ is nilpotent. To prove this, we will use the order relation on the monomials of S defined in Lemma 1.3. For variables, the relation is defined by $x_{\rho'} < x_\rho$ if and only if there is a monomial x^D with $\deg(x^D) = \deg(x_\rho)$, $x_{\rho'}|x^D$, and $x_{\rho'} \neq x^D$. If $\deg(x_\rho) = \alpha_i$, this is equivalent to the existence of $x^D \in S''_{\alpha_i}$ with $x_{\rho'}|x^D$.

Given $\phi \in \mathcal{N}$, let $N(\phi)$ the be number of variables x_ρ for which $\phi(x_\rho) = 0$. If $N(\phi) = |\Delta(1)|$, then $\phi = 0$ and we are done. If $N(\phi) < |\Delta(1)|$, then look at the variables where ϕ doesn't vanish and pick one, say x_ρ, which is *minimal* with respect to $<$. We can do this because X is complete and hence $<$ is antisymmetric by Lemma 1.3. If $\deg(x_\rho) = \alpha_i$, then as we noted above, $x_{\rho'} < x_\rho$ for any variable $x_{\rho'}$ dividing a monomial in S''_{α_i}. By the minimality of x_ρ, it follows that $\phi(x_{\rho'}) = 0$, and consequently $\phi(S''_{\alpha_i}) = 0$. This implies that $\phi^2(x_\rho) = 0$ since $\phi(x_\rho) \in S''_{\alpha_i}$. We conclude that $N(\phi^2) > N(\phi)$ since ϕ^2 also vanishes on every variable killed by ϕ. The same argument applies to $\phi^2, \phi^4, \phi^8, \ldots \in \mathcal{N}$, and it follows that some power of ϕ must be zero. Thus ϕ is nilpotent as claimed.

Once we know that $\mathcal{N}$ is a nilpotent ideal, it follows that every element of $1 + \mathcal{N}$ is invertible, so that $1 + \mathcal{N}$ is a unipotent subgroup of $\mathrm{Aut}_g(S)$. From the exact sequence (8), we also see that a element $\phi \in \mathrm{End}_g(S)$ is invertible if and only if its image in $\prod_{i=1}^{s} GL(S'_{\alpha_i})$ is invertible. From this, it follows that we get an exact sequence

$$(9) \qquad 1 \longrightarrow 1 + \mathcal{N} \longrightarrow \mathrm{Aut}_g(S) \longrightarrow \prod_{i=1}^{s} GL(S'_{\alpha_i}) \longrightarrow 1.$$

Since $\prod_{i=1}^{s} GL(S'_{\alpha_i})$ is a reductive group, it follows that $1+\mathcal{N}$ is the unipotent radical of $\mathrm{Aut}_g(S)$. Then part (ii) of the proposition follows because as a variety, we have $1+\mathcal{N} \simeq \mathcal{N}$, which is an affine space of the required dimension.

To prove part (iii), observe that we have a map

$$s : \prod_{i=1}^{s} \mathrm{End}_{\mathbb{C}}(S'_{\alpha_i}) \to \mathrm{End}_g(S)$$

where $(\phi_i) \in \prod_{i=1}^{s} \mathrm{End}_{\mathbb{C}}(S'_{\alpha_i})$ maps to the homomorphism ϕ defined by $\phi(x_\rho) = \phi_i(x_\rho)$ when $\rho \in \Delta_i$. It is easy to check that s is a ring homomorphism which is a section of the map Ψ of (8). It follows that s induces a map

$$s^* : \prod_{i=1}^{s} GL(S'_{\alpha_i}) \longrightarrow \mathrm{Aut}_g(S)$$

which is a section of the exact sequence (9). The image G_s of s^* is a subgroup of $\mathrm{Aut}_g(S)$ isomorphic to $\prod_{i=1}^{s} GL(S'_{\alpha_i})$, and thus $\mathrm{Aut}_g(S)$ is the semidirect product of $(1+\mathcal{N}) \rtimes G_s$. This proves parts (iii) and (iv), and then part (i) now follows immediately. $\square$

We will next study the *roots* of $\mathrm{Aut}(X)$ as defined by Demazure. Recall that the roots of the complete toric variety X are the set

$$R(N,\Delta) = \{m \in M : \exists \rho \in \Delta(1) \text{ with } \langle m, n_\rho \rangle = 1 \text{ and } \langle m, n_{\rho'} \rangle \le 0 \text{ for } \rho' \ne \rho\}$$

(see [Demazure, §3.1] or [Oda, §3.4]). We divide the roots $R(N,\Delta)$ into two classes as follows:

$$R_s(N,\Delta) = R(N,\Delta) \cap (-R(N,\Delta)) \quad R_u(N,\Delta) = R(N,\Delta) - R_s(N,\Delta) .$$

The roots in $R_s(N,\Delta)$ are called the semisimple roots.

We can describe $R(N,\Delta)$ and $R_s(N,\Delta)$ in terms of the graded ring S as follows:

Lemma 4.4. *There is are one-to-one correspondences*

$$R(N,\Delta) \longleftrightarrow \{(x_\rho, x^D) : x^D \in S \text{ is a monomial}, x^D \ne x_\rho, \deg(x^D) = \deg(x_\rho)\}$$
$$R_s(N,\Delta) \longleftrightarrow \{(x_\rho, x_{\rho'}) : x_\rho \ne x_{\rho'}, \deg(x_\rho) = \deg(x_{\rho'})\} .$$

Proof. Given $m \in R(N,\Delta)$, we have ρ with $\langle m, n_\rho \rangle = 1$ and $\langle m, n_{\rho'} \rangle \le 0$ for $\rho' \ne \rho$. Then consider the map

$$m \mapsto \left(x_\rho, \prod_{\rho' \ne \rho} x_{\rho'}^{\langle -m, n_{\rho'} \rangle} \right) .$$

Since m is a root, the second component has nonnegative exponents and hence is a monomial $x^D \in S$. It is also clear that x_ρ and x^D are distinct and have the same degree.

This map is injective because $M \to \mathbb{Z}^{\Delta(1)}$ is injective. To see that it is surjective, suppose we have $x_\rho \ne x^D$ of the same degree in S. A first observation is that $x_\rho \nmid x^D$,

for otherwise we could write $x^D = x^\rho x^E$, and it would follow that $x^E \in S$ would have degree zero. Yet X complete implies $S_0 = \mathbb{C}$, so that x^E must be constant. This would force $x_\rho = x^D$, which is impossible. Then, writing $x^D = \prod_{\rho' \neq \rho} x_{\rho'}^{a_{\rho'}}$, the definition of $\deg(x_\rho) = \deg(x^D)$ implies there is $m \in M$ with

$$1 = \langle m, n_\rho \rangle + 0 \ , \quad 0 = \langle m, n_{\rho'} \rangle + a_{\rho'} \text{ for } \rho' \neq \rho \ .$$

Thus m is a root mapping to the pair (x_ρ, x^D).

The correspondence for semisimple roots follows easily from the observation that if $m \leftrightarrow (x_\rho, x^D)$ and $-m \leftrightarrow (x_{\rho'}, x^{D'})$, then $x^D = x_{\rho'}$. $\qquad\square$

This lemma allows us to think of semisimple roots as pairs of distinct variables of the same degree.

Given a root $m \in R(N, \Delta)$, we have the corresponding pair (x_ρ, x^D). Then, for $\lambda \in \mathbb{C}$, consider the map

$$y_m(\lambda) : S \longrightarrow S$$

defined by $y_m(\lambda)(x_\rho) = x_\rho + \lambda x^D$ and $y_m(\lambda)(x_{\rho'}) = x_{\rho'}$ for $\rho' \neq \rho$. It is easy to see that $y_m(\lambda)$ gives a one parameter group of graded automorphisms of S, so that $y_m(\lambda) \in \mathrm{Aut}_g(S)$. We can relate the $y_m(\lambda)$ to $\mathrm{Aut}_g(S)$ as follows:

Proposition 4.5. *Let X be a complete toric variety. Then $\mathrm{Aut}_g(S)$ is generated by the torus $(\mathbb{C}^*)^{\Delta(1)}$ and the one parameter subgroups $y_m(\lambda)$ for $m \in R(N, \Delta)$.*

Proof. In Proposition 4.3, we gave showed that $\mathrm{Aut}_g(S)$ was the semidirect product

$$\mathrm{Aut}_g(S) \simeq (1 + \mathcal{N}) \rtimes G_s \ , \quad \text{where } G_s \simeq \oplus_{i=1}^s GL(S'_{\alpha_i}) \ .$$

In this decomposition, we know from Proposition 4.3 that $(\mathbb{C}^*)^{\Delta(1)} \subset \mathrm{Aut}_g(S)$ is the maximal torus and hence lies in G_s.

First suppose that $m \leftrightarrow (x_\rho, x_{\rho'})$ is in $R_s(N, \Delta)$. Then, letting $\alpha_i = \deg(x_\rho)$, we have $x_\rho, x_{\rho'} \in S'_{\alpha_i}$ in the direct sum (7). Since $y_m(\lambda)(x_\rho) = x_\rho + \lambda x_{\rho'}$ and $y_m(\lambda)(x_{\rho''}) = x_{\rho''}$ for $\rho'' \neq \rho$, it follows that $y_m(\lambda) \in G_s$, and in fact, the $y_m(\lambda)$ correspond to the roots of $G_s = \oplus_{i=1}^s GL(S'_{\alpha_i})$ in the usual sense of algebraic groups. Since any reductive group is generated by its maximal torus and the one parameter subgroups given by its roots, it follows that G_s is generated by $(\mathbb{C}^*)^{\Delta(1)}$ and the $y_m(\lambda)$ for $m \in R_s(N, \Delta)$.

Next, suppose that $m \leftrightarrow (x_\rho, x^D)$ is a root in $R_u(N, \Delta)$. If we set $\alpha_i = \deg(x_\rho)$, then $x^D \in S''_{\alpha_i}$ in the direct sum (7). It follows that $y_m(\lambda) \in 1 + \mathcal{N}$. We claim that $1 + \mathcal{N}$ is generated by the $y_m(\lambda)$ for $m \in R_u(N, \Delta)$. To prove this, suppose that $\phi \in 1 + \mathcal{N}$. If $\phi(x_\rho) = x_\rho + u_\rho$, where $u_\rho \in S''_{\alpha_i}$, then define the *weight* of ϕ to be the sum

$$w(\phi) = \sum_\rho \text{number of nonzero coefficients of } u_\rho \ .$$

Note the ϕ is the identity if and only if $w(\phi) = 0$. If ϕ is not the identity, then we can use the ordering of variables from the proof of Proposition 4.3 to pick a x_ρ with $u_\rho \neq 0$ such that x_ρ is *maximal* with respect to this property. Let λx^D is a nonzero term of u_ρ, so that $u_\rho = \lambda x^D + u'_\rho$. Then (x_ρ, x^D) corresponds to a root $m \in R_u(N, \Delta)$, and we have

$$(10) \qquad y_m(-\lambda) \circ \phi(x_\rho) = y_m(-\lambda)(x_\rho + \lambda x^D + u'_\rho) = x_\rho + u'_\rho$$

since none of the monomials in $u_\rho = \lambda x^D + u'_\rho$ involve x_ρ. Next suppose that $\rho' \neq \rho$. Then x_ρ doesn't appear in any monomial in $u_{\rho'}$, for if it did, we would have $x_\rho < x_{\rho'}$, which would contradict the maximality of x_ρ. Thus $\phi(x_{\rho'}) = x_{\rho'} + u_{\rho'}$ only involves variables different from x_ρ, and hence

$$(11) \qquad y_m(-\lambda) \circ \phi(x_{\rho'}) = \phi(x_{\rho'}) \quad \text{for } \rho' \neq \rho .$$

From (10) and (11) we conclude that $w(y_m(-\lambda) \circ \phi) < w(\phi)$, and from here it is easy to see ϕ can be written in terms of the $y_m(\lambda)$. This shows that $1 + \mathcal{N}$ is generated by the $y_m(\lambda)$ for $m \in R_u(N, \Delta)$. Given the generators we've found for G_s and $1 + \mathcal{N}$, the proposition follows now immediately. $\qquad \square$

Our next task is to relate $\text{Aut}_g(S)$ to the group $\widetilde{\text{Aut}}^0(X)$ of Definition 4.1. Every $\phi \in \text{Aut}_g(S)$ gives an automorphism $\phi_* : \mathbb{C}^{\Delta(1)} \to \mathbb{C}^{\Delta(1)}$ which sends $t = (t_\rho) \in \mathbb{C}^{\Delta(1)}$ to $\phi_*(t) = (\phi(x_\rho)(t))$ (the notation $\phi(x_\rho)(t)$ means the polynomial $\phi(x_\rho)$ evaluated at the point t). It is easy to check that ϕ_* which commutes with the action of G. Our next step is to show that ϕ_* preserves $\mathbb{C}^{\Delta(1)} - Z \subset \mathbb{C}^{\Delta(1)}$:

Proposition 4.6. *Assume that X is a complete simplicial toric variety, and let $\phi_* : \mathbb{C}^{\Delta(1)} \to \mathbb{C}^{\Delta(1)}$ is the automorphism determined by $\phi \in \text{Aut}_g(S)$. Then ϕ_* preserves $\mathbb{C}^{\Delta(1)} - \Sigma$, commutes with G, and hence lies in $\widetilde{\text{Aut}}^0(X)$.*

Proof. Every element of $(\mathbb{C}^*)^{\Delta(1)}$ clearly preserves $\mathbb{C}^{\Delta(1)} - Z$, so that by Proposition 4.5, it suffices to prove the proposition when $\phi = y_m(\lambda)$, where m is a root in $R(N, \Delta)$. Also, it suffices to show that $y_m(\lambda)_*(\mathbb{C}^{\Delta(1)} - Z) \subset \mathbb{C}^{\Delta(1)} - Z$. To prove this, suppose $m \leftrightarrow (x_\rho, x^D)$. We will write a point of $\mathbb{C}^{\Delta(1)}$ as $t = (t_\rho, \mathbf{t})$, where $\mathbf{t}$ is a vector indexed by $\Delta(1) - \{\rho\}$. Then $y_m(\lambda)_*$ is given by

$$y_m(\lambda)_*(t_\rho, \mathbf{t}) = (t_\rho + \lambda \mathbf{t}^D, \mathbf{t}) ,$$

where $\mathbf{t}^D$ is the monomial x^D evaluated at $\mathbf{t}$ (this makes sense since $x_\rho \nmid x^D$).

As in the proof of Theorem 2.1, write $\mathbb{C}^{\Delta(1)} - Z = \cup_\sigma U_\sigma$, where $U_\sigma = \{t \in \mathbb{C}^{\Delta(1)} : t^{\hat\sigma} \neq 0\}$. Recall that $x^{\hat\sigma} = \prod_{\rho \notin \sigma(1)} x_\rho$. Now let $t = (t_\rho, \mathbf{t}) \in U_\sigma$, and set $u = y_m(\lambda)_*(t) = (t_\rho + \lambda \mathbf{t}^D, \mathbf{t})$. We will show that $u \in U_\tau$ for some $\tau \in \Delta$ by an argument similar to Proposition 3.14 of [Oda]. When $\rho \in \sigma(1)$, we have $u^{\hat\sigma} = t^{\hat\sigma} \neq 0$, so that $u \in U_\sigma$. So suppose that $\rho \notin \sigma(1)$. Since $\langle m, n_\rho \rangle = 1$ and $\langle m, n_{\rho'} \rangle \leq 0$ for $\rho' \neq \rho$, it follows that

$\check{\sigma} \cap \{m\}^\perp$ is a face of σ such that ρ is on one side of $\{m\}^\perp$ and all of the other one-dimensional cones of Δ are on the other side. But Δ is complete, so ρ and $\hat{\sigma} \cap \{m\}^\perp$ must be faces of some cone of Δ which is on the same side of $\{m\}^\perp$ as ρ. Since Δ is simplicial, it follows that $\tau = \rho + \hat{\sigma} \cap \{m\}^\perp$ must be a cone of Δ. There are now two cases to consider:

Case 1: If $t^{\hat{\tau}} \neq 0$, then since $\rho \in \tau(1)$, we have $u^{\hat{\tau}} = t^{\hat{\tau}} \neq 0$. Thus $u \in U_\tau$.

Case 2: If $t^{\hat{\tau}} = 0$, then $t_{\rho_0} = 0$ for some $\rho_0 \notin \tau(1)$. Since $\tau = \rho + \hat{\sigma} \cap \{m\}^\perp$, this means $\langle -m, n_{\rho_0} \rangle > 0$. Then $\prod_{\rho' \neq \rho} t_{\rho'}^{\langle -m, n_{\rho'} \rangle} = 0$, which implies

$$u^{\hat{\sigma}} = \left(t_\rho + \lambda \prod_{\rho' \neq \rho} t_{\rho'}^{\langle -m, n_{\rho'} \rangle} \right) \prod_{\rho' \notin \sigma(1) \cup \{\rho\}} t_{\rho'} = (t_\rho + 0) \prod_{\rho' \notin \sigma(1) \cup \{\rho\}} t_{\rho'} = t^{\hat{\sigma}} \neq 0 \ .$$

It follows that $u \in U_\sigma$.

Thus $y_m(\lambda)_*$ preserves $\mathbb{C}^{\Delta(1)} - Z$, and the proposition is proved. $\square$

We can now begin the proof of Theorem 4.2:

Proof of Theorem 4.2. We first prove part (iii) of the theorem, which asserts that $\mathrm{Aut}_g(S)$ is naturally isomorphic to $\widetilde{\mathrm{Aut}}^0(X)$. From Proposition 4.6, the map sending ϕ to ϕ_*^{-1} defines a group homomorphism

$$(12) \qquad\qquad\qquad \mathrm{Aut}_g(S) \longrightarrow \widetilde{\mathrm{Aut}}^0(X) \ .$$

This map is easily seen to be injective. To see that it is surjective, suppose that $\psi : \mathbb{C}^{\Delta(1)} - Z \to \mathbb{C}^{\Delta(1)} - Z$ is G-equivariant. Then for any ρ we get $x_\rho \circ \psi : \mathbb{C}^{\Delta(1)} - Z \to \mathbb{C}$, and since Σ has codimension ≥ 2 by Lemma 1.4, purity of the branch locus implies that $x_\rho \circ \psi$ extends to a polynomial u_ρ defined on $\mathbb{C}^{\Delta(1)}$. Since ψ commutes with the action of G, it follows that u_ρ is homogeneous and $\deg(u_\rho) = \deg(x_\rho)$. Then $\phi(x_\rho) = u_\rho$ defines a graded homomorphism $\phi : S \to S$ with the property that $\phi_* = \psi$. To see that ϕ is an isomorphism, apply the above argument to ψ^{-1}. This shows that (12) is an isomorphism, and part (iii) of Theorem 4.2 is done.

We now turn to part (ii) of the theorem, which asserts that we have an exact sequence

$$1 \longrightarrow G \longrightarrow \widetilde{\mathrm{Aut}}(X) \longrightarrow \mathrm{Aut}(X) \longrightarrow 1 \ .$$

It is obvious that every element of G gives the identity automorphism of X. Conversely, suppose that $\phi \in \widetilde{\mathrm{Aut}}(X)$ induces the identity automorphism on X. We need to show that $\phi \in G$. Our assumption implies that $\phi(Y) = Y$ for every G-invariant $Y \subset \mathbb{C}^{\Delta(1)} - Z$, so that in particular, $\phi(U_\rho) = U_\rho$ for every $\rho \in \Delta(1)$. Since $U_\rho = \mathrm{Spec}(S_\rho)$, we get an induced map $\phi^* : S_\rho \to S_\rho$, and hence $\phi^*(S) \subset \cap_\rho S_\rho$. It is easy to see that $\cap_\rho S_\rho = S$, and it follows that $\phi^*(S) \subset S$. Thus, for every ρ, we have $\phi^*(x_\rho) = A_\rho \in S$. Note, however, that A_ρ need not be homogeneous of degree $\deg(x_\rho)$.

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Since $\mathbf{V}(x_\rho) - Z \subset \mathbb{C}^{\Delta(1)} - Z$ is also G-invariant, it is stable under ϕ. But the ρth coordinate of $\phi(t)$ is $A_\rho(t)$, so that A_ρ vanishes on $\mathbf{V}(x_\rho) - Z$. Since Z has codimension ≥ 2, we conclude that A_ρ vanishes on $\mathbf{V}(x_\rho)$. The Nullstellensatz implies that $x_\rho | A_\rho$, and hence $\phi^*(x_\rho) = x_\rho B_\rho$ for some $B_\rho \in S$. The same argument applied to ϕ^{-1} shows that $\phi^{-1*}(x_\rho) = x_\rho C_\rho$ for some $C_\rho \in S$. Then

$$x_\rho = \phi^* \circ \phi^{-1*}(x_\rho) = \phi^*(x_\rho C_\rho) = x_\rho B_\rho \phi^*(C_\rho) \ .$$

This implies that $1 = B_\rho \phi^*(C_\rho)$ in the polynomial ring $S = \mathbb{C}[x_\rho]$, so that B_ρ is a nonzero constant. Thus $\phi = (B_\rho) \in (\mathbb{C}^*)^{\Delta(1)}$. Since ϕ induces the identity on $T \subset X$, the exact sequence $1 \to G \to (\mathbb{C}^*)^{\Delta(1)} \to T \to 1$ shows that $\phi \in G$ as claimed. This proves that G is the kernel of $\widetilde{\mathrm{Aut}}(X) \to \mathrm{Aut}(X)$.

We still need to prove that $\widetilde{\mathrm{Aut}}(X) \to \mathrm{Aut}(X)$ is onto. We will first show that there is an exact sequence

(13) $$1 \longrightarrow G \longrightarrow \widetilde{\mathrm{Aut}}^0(X) \longrightarrow \mathrm{Aut}(X) \longrightarrow \mathrm{Aut}(A_{n-1}(X)) \ .$$

The map $\mathrm{Aut}(X) \to \mathrm{Aut}(A_{n-1}(X))$ is induced by direct image of divisors: given $\phi \in \mathrm{Aut}(X)$, a divisor class $[\sum_i a_i D_i]$ maps to $[\sum_i a_i \phi(D)]$.

To show exactness, let $\psi \in \widetilde{\mathrm{Aut}}^0(X)$. We need to show that the induced automorphism of X, which will also be denoted ψ, is the identity on $A_{n-1}(X)$. By part (iii) of the theorem, we can write $\psi(t) = (u_\rho(t))$ where $u_\rho \in S_{\deg(x_\rho)}$. If $D'_\rho = \mathbf{V}(x_\rho) - Z$, then

$$\mathrm{div}(u_\rho/x_\rho) = \psi^{-1}(D'_\rho) - D'_\rho$$

in $\mathbb{C}^{\Delta(1)} - Z$. Since u_ρ and x_ρ have the same degree, the quotient u_ρ/x_ρ is G-invariant and descends to a rational function on X. Then it follows immediately that

$$\mathrm{div}(u_\rho/x_\rho) = \psi^{-1}(D_\rho) - D_\rho$$

in X. This shows that $[\psi^{-1}(D_\rho)] = [D_\rho]$, so that ψ induces the identity on $A_{n-1}(X)$.

Next, assume that $\phi \in \mathrm{Aut}(X)$ induces the identity on $A_{n-1}(X)$. We need to show that ϕ comes from an element of $\widetilde{\mathrm{Aut}}^0(X)$. Fix $\rho \in \Delta(1)$, and let $\alpha_i = \deg(x_\rho) = [D_\rho]$. Since $[\phi^{-1}(D_\rho)] = [D_\rho]$ in $A_{n-1}(X)$, we can find a rational function g_ρ on X such that

$$\mathrm{div}(g_\rho) = \phi^{-1}(D_\rho) - D_\rho \ .$$

Then $\mathrm{div}(g_\rho) + D_\rho \geq 0$, which implies $g_\rho \in H^0(X, \mathcal{O}_X(D_\rho))$. Using the isomorphism $H^0(X, \mathcal{O}_X(D_\rho)) \simeq S_{\alpha_i}$ from Proposition 1.1, we can write $g_\rho = u_\rho/x_\rho$ for some $u_\rho \in S_{\alpha_i}$. Note that u_ρ is only determined up to a constant. We will choose the constant as follows.

Given $m \in M$, $\prod_\rho x_\rho^{\langle m, n_\rho \rangle}$ and $\prod_\rho u_\rho^{\langle m, n_\rho \rangle}$ are rational functions on X. Then

$$
\begin{aligned}
\mathrm{div}\big(\textstyle\prod_\rho u_\rho^{\langle m,n_\rho\rangle}\big) &= \mathrm{div}\big(\textstyle\prod_\rho (u_\rho/x_\rho)^{\langle m,n_\rho\rangle}\big) + \mathrm{div}\big(\textstyle\prod_\rho x_\rho^{\langle m,n_\rho\rangle}\big) \\
&= \textstyle\sum_\rho \langle m,n_\rho\rangle (\phi^{-1}(D_\rho) - D_\rho) + \sum_\rho \langle m,n_\rho\rangle D_\rho \\
&= \phi^{-1}\big(\textstyle\sum_\rho \langle m,n_\rho\rangle D_\rho\big) = \mathrm{div}\Big(\big(\textstyle\prod_\rho x_\rho^{\langle m,n_\rho\rangle}\big) \circ \phi\Big) .
\end{aligned}
$$

This implies that there is $\gamma(m) \in \mathbb{C}^*$ such that

$$
\gamma(m)\textstyle\prod_\rho u_\rho^{\langle m,n_\rho\rangle} = \big(\textstyle\prod_\rho x_\rho^{\langle m,n_\rho\rangle}\big) \circ \phi .
$$

It is easy to see that $\gamma : M \to \mathbb{C}^*$ is a homomorphism, which implies that $\gamma \in T$. The map $(\mathbb{C}^*)^{\Delta(1)} \to T$ is surjective, so that there is $t = (t_\rho) \in (\mathbb{C}^*)^{\Delta(1)}$ such that $\gamma(m) = \prod_\rho t_\rho^{\langle m,n_\rho\rangle}$ for all $m \in M$. Thus, if we replace u_ρ by $t_\rho u_\rho$, then for $m \in M$, we have

(14)
$$
\textstyle\prod_\rho u_\rho^{\langle m,n_\rho\rangle} = \big(\textstyle\prod_\rho x_\rho^{\langle m,n_\rho\rangle}\big) \circ \phi .
$$

Now define $\psi : S \to S$ by $\psi(x_\rho) = u_\rho$, and consider the diagram:

$$
\begin{array}{ccccc}
S_{\alpha_i} & & \xrightarrow{\;\psi\;} & & S_{\alpha_i} \\
\downarrow{\scriptstyle x_\rho^{-1}} & & & & \downarrow{\scriptstyle x_\rho^{-1}} \\
H^0(X,\mathcal{O}_X(D_\rho)) & \xrightarrow{\;\phi^*\;} & H^0(X,\mathcal{O}_X(\phi^{-1}(D_\rho))) & \xrightarrow{\;u_\rho/x_\rho\;} & H^0(X,\mathcal{O}_X(D_\rho))
\end{array}
$$

Here, the vertical arrows are the isomorphisms given by Proposition 1.1. Also, ϕ^* maps $g \in H^0(X,\mathcal{O}_X(D_\rho))$ to $g \circ \phi \in H^0(X,\mathcal{O}_X(\phi^{-1}(D_\rho)))$, and the final map on the bottom takes $g \in H^0(X,\mathcal{O}_X(\phi^{-1}(D_\rho)))$ to $g \cdot (u_\rho/x_\rho) \in H^0(X,\mathcal{O}_X(D_\rho))$. To see that the diagram commutes, let $u \in S_{\alpha_i}$ be a monomial. Using the left and bottom maps, we get

$$
\frac{u}{x_\rho} \circ \phi \cdot \frac{u_\rho}{x_\rho} \in H^0(X,\mathcal{O}_X(D_\rho)) .
$$

However, since u and x_ρ have the same degree, we can write $u/x_\rho = \prod_\rho x_\rho^{\langle m,n_\rho\rangle}$ for some $m \in M$. Then (14) easily implies that $(u/x_\rho) \circ \phi = \psi(u)/u_\rho$, so that

$$
\frac{u}{x_\rho} \circ \phi \cdot \frac{u_\rho}{x_\rho} = \frac{\psi(u)}{u_\rho} \cdot \frac{u_\rho}{x_\rho} = \frac{\psi(u)}{x_\rho} .
$$

This shows that the above diagram commutes. Since the maps on the bottom are clearly isomorphisms, it follows that $\psi_{|S_{\alpha_i}}$ is an isomorphism. This shows that $\psi \in \mathrm{Aut}_g(S)$, and then Proposition 4.6 gives $\psi_* \in \widetilde{\mathrm{Aut}}^0(X)$.

From ψ_* we get at automorphism of X which we will also denote by ψ_*. Since $u_\rho = x_\rho \circ \psi_*$ on $\mathbb{C}^{\Delta(1)} - Z$, it follows that for any $m \in M$, we have

$$
\textstyle\prod_\rho u_\rho^{\langle m,n_\rho\rangle} = \big(\textstyle\prod_\rho x_\rho^{\langle m,n_\rho\rangle}\big) \circ \psi_* .
$$

From (14), we see that ϕ and ψ_* agree as birational automorphisms of T, which implies $\psi = \psi_*$ as automorphisms of X. This completes the proof that the sequence (13) is exact.

The next step in showing that $\widetilde{\mathrm{Aut}}(X) \to \mathrm{Aut}(X)$ is onto is to study the role played by the automorphisms of the fan Δ. Let $\mathrm{Aut}(N, \Delta)$ be the group of lattice isomorphisms φ of N which preserve the fan Δ. Such a φ permutes $\Delta(1)$ and this permutation determines φ uniquely since $\Delta(1)$ spans $N_\mathbb{R}$. It follows that $\mathrm{Aut}(N, \Delta)$ is a finite group.

We first relate $\mathrm{Aut}(N, \Delta)$ to $\widetilde{\mathrm{Aut}}(X)$. Given $\varphi \in \mathrm{Aut}(N, \Delta)$, the induced permutation of $\Delta(1)$ can be regarded as a permutation matrix $P_\varphi \in GL(\mathbb{C}^{\Delta(1)})$. We claim that φ induces an automorphism $\widehat{\varphi}^* : G \to G$ such that for any $g \in G$, we have

$$(15) \qquad\qquad g \circ P_\varphi = P_\varphi \circ \widehat{\varphi}^*(g) \ .$$

To prove this, first note that φ induces a dual automorphism $\varphi^* : M \to M$. Consider the following diagram:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & \mathbb{Z}^{\Delta(1)} & \longrightarrow & A_{n-1}(X) & \longrightarrow & 0 \\
 & & \downarrow{\varphi^{*-1}} & & \downarrow{P_\varphi} & & & & \\
0 & \longrightarrow & M & \longrightarrow & \mathbb{Z}^{\Delta(1)} & \longrightarrow & A_{n-1}(X) & \longrightarrow & 0
\end{array}
$$

It is straightforward to check that the diagram commutes, which shows that we get an induced isomorphism $\widehat{\varphi} : A_{n-1}(X) \to A_{n-1}(X)$. This in turn gives $\widehat{\varphi}^* : G \to G$ since $G = \mathrm{Hom}_\mathbb{Z}(A_{n-1}(X), \mathbb{C}^*)$, and (15) now follows easily.

From (15), we see that P_φ normalizes G, and notice also that P_φ preserves Z because φ preserves Δ. Hence $P_\varphi \in \widetilde{\mathrm{Aut}}(X)$ for all $\varphi \in \mathrm{Aut}(N, \Delta)$. Furthermore, it is easy to check that the induced automorphism of X is the usual way that $\varphi \in \mathrm{Aut}(N, \Delta)$ acts on X. By abuse of notation, we will write $P_\varphi \in \mathrm{Aut}(X)$.

Let $H \subset \mathrm{Aut}(X)$ be the image of $\widetilde{\mathrm{Aut}}^0(X) \to \mathrm{Aut}(X)$. By (13) and Proposition 4.3, it follows that $H \simeq \widetilde{\mathrm{Aut}}^0(X)/G \simeq \mathrm{Aut}_g(S)/G$ is a connected affine algebraic group with $(\mathbb{C}^*)^{\Delta(1)}/G \simeq T$ as maximal torus. Also, (13) implies that H is normal in $\mathrm{Aut}(X)$.

Now we can finally show that $\widetilde{\mathrm{Aut}}(X) \to \mathrm{Aut}(X)$ is onto. Suppose that $\phi \in \mathrm{Aut}(X)$. Then we have $\phi T \phi^{-1} \subset \phi H \phi^{-1} = H$, so that $\phi T \phi^{-1}$ is a maximal torus in H. Since all maximal tori of H are conjugate, there is some $\psi \in H$ such that $\phi T \phi^{-1} = \psi T \psi^{-1}$. If we set $\mu = \psi^{-1} \circ \phi$, it follows that $\mu T \mu^{-1} = T$ in $\mathrm{Aut}(X)$. This easily implies that $\mu(T) = T$ in X. Thus $\mu_{|T}$ is an automorphism of T. But the automorphism group of a torus is generated by translations and group automorphisms. Thus $\mu_{|T} = g \circ t$, where g is a group automorphism and $t \in T$. We claim that $\mu \circ t^{-1} : X \to X$ is T-equivariant. This is certainly true on T since $(\mu \circ t^{-1})_{|T} = g$, and the claim follows since T is dense in X. By Theorem 1.13 of [Oda], this implies $\mu \circ t^{-1} = P_\varphi$ for some $\varphi \in \mathrm{Aut}(N, \Delta)$.

It follows that $\phi = \psi \circ \mu = \psi \circ P_\varphi \circ t$ in $\mathrm{Aut}(X)$. Thus $\mathrm{Aut}(X)$ is generated by H and the P_φ for $\varphi \in \mathrm{Aut}(N, \Delta)$. Since every P_φ comes from an element of $\widetilde{\mathrm{Aut}}(X)$, $\widetilde{\mathrm{Aut}}(X) \to \mathrm{Aut}(X)$ is surjective. This completes the proof of part (ii) of Theorem 4.2.

Finally, consider part (i) of the theorem, which asserts that $\widetilde{\mathrm{Aut}}(X)$ is an affine algebraic group with $\widetilde{\mathrm{Aut}}^0(X)$ as the connected component of the identity. To show this, note that since $\mathrm{Aut}(X)$ is generated by the image of $\widetilde{\mathrm{Aut}}^0(X)$ and the P_φ, it follows from part (ii) of the theorem that $\widetilde{\mathrm{Aut}}(X)$ is generated by $\widetilde{\mathrm{Aut}}^0(X)$ and P_φ for $\varphi \in \mathrm{Aut}(N, \Delta)$. Since $\mathrm{Aut}(N, \Delta)$ is finite, $\widetilde{\mathrm{Aut}}^0(X)$ has finite index in $\widetilde{\mathrm{Aut}}(X)$. We already know that $\widetilde{\mathrm{Aut}}^0(X)$ is a connected affine algebraic group, and hence $\widetilde{\mathrm{Aut}}(X)$ is an affine algebraic group with $\widetilde{\mathrm{Aut}}^0(X)$ as the connected component of the identity. Theorem 4.2 is now proved. $\qquad\square$

From the above proof, we can extract some additional information about $\widetilde{\mathrm{Aut}}(X)$ and $\mathrm{Aut}(X)$. For example, $\widetilde{\mathrm{Aut}}(X)$ is generated by $(\mathbb{C}^*)^{\Delta(1)}$, the $y_m(\lambda)$ for $m \in R(N, \Delta)$, and the P_φ for $\varphi \in \mathrm{Aut}(N, \Delta)$. To discuss $\mathrm{Aut}(X)$ in more detail, we need some notation: for a root $m \in R(N, \Delta)$, we have $y_m(\lambda)_* \in \widetilde{\mathrm{Aut}}^0(X)$, and we let $x_m(\lambda)$ denote the resulting automorphism of X. It is easy to check that this is the same automorphism $x_m(\lambda)$ as defined by [Demazure, §2.3] or [Oda, §3.4]. Then we can describe $\mathrm{Aut}(X)$ as follows:

Corollary 4.7. *Let X be a complete simplicial toric variety. Then:*
 (i) $\mathrm{Aut}(X)$ *is an affine algebraic group with T as maximal torus.*
 (ii) $\mathrm{Aut}(X)$ *is generated by T, the $x_m(\lambda)$ for $m \in R(N, \Delta)$, and the P_φ for $\varphi \in \mathrm{Aut}(N, \Delta)$.*
 (iii) *If $\mathrm{Aut}^0(X)$ is the connected component of the identity of $\mathrm{Aut}(X)$, then*

$$\mathrm{Aut}^0(X) \simeq \widetilde{\mathrm{Aut}}^0(X)/G \simeq \mathrm{Aut}_g(S)/G \ .$$

 (iv) $\mathrm{Aut}^0(X)$ *is the semidirect product of its unipotent radical with a reductive group G_s whose root system consists of components of type $\mathbf{A}$.*
 (v) *From the partition $\Delta(1) = \Delta_1 \cup \cdots \cup \Delta_s$ (where the x_ρ have the same degree for $\rho \in \Delta_i$), let Σ_{Δ_i} denotes the symmetric group on Δ_i. Then $\prod_{i=1}^s \Sigma_{\Delta_i}$ is the Weyl group of G_s. Furthermore, we can regard this group as a normal subgroup of $\mathrm{Aut}(N, \Delta)$, and there are natural isomorphisms*

$$\mathrm{Aut}(X)/\mathrm{Aut}^0(X) \simeq \widetilde{\mathrm{Aut}}(X)/\widetilde{\mathrm{Aut}}^0(X) \simeq \mathrm{Aut}(N, \Delta)/\left(\prod_{i=1}^s \Sigma_{\Delta_i}\right) \ .$$

Proof. Parts (i) – (iv) of the corollary follow easily from the proof of Theorem 4.2 and the other results of this section. As for part (v), it is clear that we have an isomorphism $\mathrm{Aut}(X)/\mathrm{Aut}^0(X) \simeq \widetilde{\mathrm{Aut}}(X)/\widetilde{\mathrm{Aut}}^0(X)$. Furthermore, we can regard $\mathrm{Aut}(N, \Delta)$ as a subgroup of $\widetilde{\mathrm{Aut}}(X)$, and then we have an isomorphism

$$\widetilde{\mathrm{Aut}}(X)/\widetilde{\mathrm{Aut}}^0(X) \simeq \mathrm{Aut}(N, \Delta)/(\mathrm{Aut}(N, \Delta) \cap \widetilde{\mathrm{Aut}}^0(X)) \ .$$

Thus, regarding elements of $\prod_{i=1}^s \Sigma_{\Delta_i}$ as permutation matrices, it suffices to prove that

$$\mathrm{Aut}(N, \Delta) \cap \widetilde{\mathrm{Aut}}^0(X) = \prod_{i=1}^s \Sigma_{\Delta_i} \ .$$

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One direction is easy, for if $P_\varphi \in \mathrm{Aut}(N, \Delta) \cap \widetilde{\mathrm{Aut}}^0(X)$, then by (13), we know that P_φ induces the identity automorphism on $A_{n-1}(X)$. This shows that as a permutation of the $\rho's$, P_φ preserves the degree of x_ρ, and it follows immediately that $P_\varphi \in \prod_{i=1}^{s} \Sigma_{\Delta_i}$.

Going the other way, suppose that $P \in \prod_{i=1}^{s} \Sigma_{\Delta_i}$. Since $\prod_{i=1}^{s} \Sigma_{\Delta_i} \subset \prod_{i=1}^{s} GL(S_{\alpha_i}) \subset \widetilde{\mathrm{Aut}}^0(X)$, P induces an automorphism ϕ of X. On $\mathbb{C}^{\Delta(1)} - Z$, P permutes the variables, so that on X, ϕ permutes the D_ρ. Since $T = X - \cup_\rho D_\rho$, it follows that $\phi(T) = T$. As we saw in the proof of Theorem 4.2, this implies that $\phi = P_\varphi \circ t$ for some $\varphi \in \mathrm{Aut}(N, \Delta)$ and $t \in T$. Back in $\widetilde{\mathrm{Aut}}(X)$, we then have $P = P_\varphi \circ u$ for some $u \in (\mathbb{C}^*)^{\Delta(1)}$. Since P and P_φ are permutation matrices, we conclude that $P = P_\varphi$, and it follows that $P \in \mathrm{Aut}(N, \Delta) \cap \widetilde{\mathrm{Aut}}^0(X)$. This completes the proof of the corollary. $\square$

When X is smooth, this corollary is due to Demazure (see [Demazure] or [Oda, §3.4]).

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