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Frobenius Direct Images of Line Bundles on Toric Varieties

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1. INTRODUCTION

A toric variety over an algebraically closed field k is a normal variety X containing the algebraic group $T = (k^*)^n$ as an open dense subset, and with a group action $T \times X \to X$ extending the group law of T. If k has characteristic $p > 0$ we can define the absolute Frobenius morphism on X. Remember that the absolute Frobenius morphism on a variety X, over a field of characteristic $p > 0$, is the morphism of schemes F: $X \to X$, which is the identity on the underlying set of X and the pth power map on sheaf level. An old result of Hartshorne [3, Corollary 6.4, p. 138] tells us that if $X = \mathbb{P}_k^n$ and L is a line bundle on X, then the direct image of L via F_*, that is, F_*L, splits into a direct sum of line bundles. In this paper we generalize this result to smooth toric varieties. Our proof will be constructive and will give us an algorithm for computing a decomposition explicitly. We will consider toric varieties as constructed from convex bodies called fans as described in [2]; in particular we will assume that the reader is familiar with the first chapter of [2].

Using results on Grothendieck differential operators and T-linearized sheaves Bøgvad [1] later gave nonconstructive generalizations of the above. One generalization is that F_*M, when M is a T-linearized vector bundle on X, splits into a sum of bundles of the same rank as M.

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2. IDEA BEHIND THE PROOF

This section should be regarded as an example illustrating the idea behind the proof.

Consider the absolute Frobenius morphism $F\colon \mathbb{P}^1 \to \mathbb{P}^1$ on the projective space $\mathbb{P}^1$ over an algebraically closed field k of characteristic p. By [3, Corollary 6.4, p. 138] we know that $F_*\mathcal{O}_{\mathbb{P}^1}$ splits into a direct sum of p line bundles. In the following we will show how we may choose such line bundles $L_1, L_2, \ldots, L_p$ and an isomorphism

$$\phi\colon L_1 \oplus L_2 \oplus \cdots \oplus L_p \to F_*\mathcal{O}_{\mathbb{P}^1}.$$

For this we first need to define some notation. Cover $\mathbb{P}^1$ by the standard open affine varieties $U_0 = \operatorname{Spec}(k[X])$ and $U_1 = \operatorname{Spec}(k[Y])$ (so with this notation we should regard X as being equal to Y^{-1}). Let $F_*k[X]$ (resp. $F_*k[Y]$) denote the $k[X]$-module (resp. $k[Y]$-module) which as an abelian group is $k[X]$ (resp. $k[Y]$) but where the $k[X]$-module (resp. $k[Y]$-module) structure is twisted by the pth power map. Then $F_*\mathcal{O}_{\mathbb{P}^1}(U_0) = F_*k[X]$ and $F_*\mathcal{O}_{\mathbb{P}^1}(U_1) = F_*k[Y]$. A basis for $F_*\mathcal{O}_{\mathbb{P}^1}(U_0)$ (resp. $F_*\mathcal{O}_{\mathbb{P}^1}(U_1)$) as a $\mathcal{O}_{\mathbb{P}^1}(U_0)$-module (resp. $\mathcal{O}_{\mathbb{P}^1}(U_1)$-module) is $e_i = X^i$, $i = 0, 1, \ldots, p-1$ (resp. $f_j = Y^j$, $j = 0, 1, \ldots, p-1$).

The line bundles L_r will be constructed as subsheaves of the constant sheaf $\mathcal{K}$ corresponding to the field of rational functions on $\mathbb{P}^1$. More precisely L_r will be regarded as the subsheaf of $\mathcal{K}$ corresponding to a Cartier divisor $\{(U_0, 1), (U_1, Y^{h(r)})\}$ determined by an integer $h(r)$. Clearly $h(r)$ is the unique integer such that L_r is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(h(r))$.

The choices above enable us to put some simple conditions on ϕ, which will turn out to determine ϕ and the integers $h(r)$ (up to permutation) uniquely. More precisely we want ϕ to be a map such that the image of $1 \in L_r(U_0)$ (resp. $Y^{-h(r)} \in L_r(U_1)$) is some $e_{i(r)}$ (resp. $f_{j(r)}$). Notice that this condition forces $i(\cdot)$ and $j(\cdot)$ to be permutations of the set $\{0, 1, \ldots, p-1\}$.

The fact that the local describing conditions on ϕ should coincide on $U_0 \cap U_1$ forces $i(r)$, $j(r)$, and $h(r)$ to satisfy

$$Y^{-i(r)} = (Y^{h(r)})^p Y^{j(r)} \Leftrightarrow -i(r) = h(r)p + j(r).$$

As $j(r)$ is an integer between 0 and $p-1$ this means that $h(r)$ and $j(r)$ are determined from $i(r)$ simply by division by p with remainder. Let us permute the indices such that $i(r) = r$. Then

$$(j(0), h(0)) = (0, 0) \qquad \text{and} \qquad (j(r), h(r)) = (p - r, -1), \qquad r = 1, 2, \ldots, p-1.$$

The uniqueness of the line bundles L_r follows as they only depend on the integers $h(r)$, and also the map ϕ is determined as it only depends on $i(r)$ and $j(r)$.

The conditions which we put on ϕ have determined ϕ for us. That a ϕ with the given conditions exists is now an easy exercise. In fact this only amounts to saying that $i(\cdot)$ and $j(\cdot)$ are permutations. We conclude that

$$\mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(-1)^{\oplus(p-1)} \cong F_*\mathscr{O}_{\mathbb{P}^1}.$$

This ends our example.

3. BASIC DEFINITIONS AND NOTATIONS

In this section we will introduce the notation that will be used throughout the paper. The varieties we consider will be defined over a fixed algebraically closed field k of positive or zero characteristic.

Let $N = \mathbb{Z}^n$ be a lattice of rank n and let M be its dual. A fan $\Delta = \{\sigma_0, \sigma_1, \ldots, \sigma_q\}$ consisting of rational strongly convex polyhedral cones in $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ determines an n-dimensional toric variety $X(\Delta)$. In the following we will assume that such an n-dimensional smooth toric variety $X(\Delta)$ has been chosen. Choose an ordered $\mathbb{Z}$-basis $(e_1, e_2, \ldots, e_n)$ for N and let $(\hat{e}_1, \hat{e}_2, \ldots, \hat{e}_n)$ be the dual basis in M; that is, $\hat{e}_i(e_j) = \delta_{ij}$. Since $X(\Delta)$ is smooth every cone $\sigma_i \in \Delta$ is generated by part of a $\mathbb{Z}$-basis [2, p. 29] for N,

$$\sigma_i = \operatorname{Span}_{\mathbb{R}_{\geq 0}}\{v_{i1}, v_{i2}, \ldots, v_{id_i}\}, \qquad d_i = \dim(\sigma_i).$$

For each i expand the set $\{v_{i1}, v_{i2}, \ldots, v_{id_i}\}$ to a $\mathbb{Z}$-basis for N: $\{v_{i1}, v_{i2}, \ldots, v_{in}\}$, and form the matrix $A_i \in \mathrm{GL}_n(\mathbb{Z})$, having as the jth row the coordinates of v_{ij} expressed in the basis $(e_1, e_2, \ldots, e_n)$. Let $B_i = A_i^{-1} \in \mathrm{GL}_n(\mathbb{Z})$ and denote the jth column vector in B_i by w_{ij}. Introducing the symbols $X^{\hat{e}_1}, X^{\hat{e}_2}, \ldots, X^{\hat{e}_n}$ we can form the ring $R = k[(X^{\hat{e}_1})^{\pm 1}, (X^{\hat{e}_2})^{\pm 1}, \ldots, (X^{\hat{e}_n})^{\pm 1}]$, which is the affine coordinate ring of the torus $T \subset X(\Delta)$. The coordinate ring of the open affine subvariety U_{σ_i} of $X(\Delta)$, corresponding to the cone σ_i in Δ, is then equal to the subring

$$R_i = k[X^{w_{i1}}, \ldots, X^{w_{id_i}}, X^{\pm w_{i(d_i+1)}}, \ldots, X^{\pm w_{in}}] \subseteq R$$

where we use the multinomial notation $X^w = (X^{\hat{e}_1})^{w_1} \cdots (X^{\hat{e}_n})^{w_n}$, when w is a vector of the form $w = (w_1, \ldots, w_n)$. For convenience we will in the following also write $X_{ij} = X^{w_{ij}}$ such that

$$R_i = k[X_{i1}, \ldots, X_{id_i}, X_{i(d_i+1)}^{\pm 1}, \ldots, X_{in}^{\pm 1}].$$

For each i and j we have that $\sigma_i \cap \sigma_j$ is a face of σ_i. This implies [2, Proposition 2, p. 13] that there exists a monomial M_{ij} in $X_{i1}, \ldots, X_{in}$ such that the localized ring $(R_i)_{M_{ij}}$ is the coordinate ring of $\sigma_i \cap \sigma_j$. Choose such monomials for all i and j, and notice that $(R_i)_{M_{ij}} = (R_j)_{M_{ji}}$. For every i and j define

$$I_{ij} = \{v \in \mathrm{Mat}_{n,1}(\mathbb{Z}) \mid X_i^v \text{ is a unit in } (R_i)_{M_{ij}}\}$$

where we use the multinomial notation $X_i^v = (X_{i1})^{v_1} \cdots (X_{in})^{v_n}$ when v is a column vector with entries $v_1, \ldots, v_n$. Defining $C_{ij} = B_j^{-1}B_i \in \mathrm{GL}_n(\mathbb{Z})$ and letting f_s, $s = 1, \ldots, n$, denote the element in $\mathrm{Mat}_{n,1}(\mathbb{Z})$ with a single 1 in position s and zeroes elsewhere, we can formulate and prove our first lemma.

LEMMA 1. *With the notations above we have*

1. $X_i^v = X_j^{C_{ij}v}$, $v \in \mathrm{Mat}_{n,1}(\mathbb{Z})$.
2. I_{ij} *is a subgroup of* $\mathrm{Mat}_{n,1}(\mathbb{Z})$.
3. $C_{ij}I_{ij} = I_{ji}$.
4. $v \in \mathrm{Mat}_{n,1}(\mathbb{Z}) : X_i^v \in (R_i)_{M_{ij}}$ *and* $v_l < 0 \Rightarrow f_l \in I_{ij}$.
5. $v \in I_{ij}$ *and* $v_l \neq 0 \Rightarrow f_l \in I_{ij}$.
6. $(C_{ji})_{kl} < 0 \Rightarrow f_k \in I_{ij}$.

Proof. Points (1) to (5) are straightforward by definition. Point (6) follows from (1) and (4) using that $X_i^{C_{ji}f_l} = X_{jl} \in (R_j)_{M_{ji}} = (R_i)_{M_{ij}}$. ∎

For every positive integer m define $P_m = \{v \in \mathrm{Mat}_{n,1}(\mathbb{Z}) \mid 0 \leq v_i < m\}$. Our main lemma then says

LEMMA 2. *Let $v \in P_m$ and $w \in I_{ji}$. If $h \in \mathrm{Mat}_{n,1}(\mathbb{Z})$ and $r \in P_m$ satisfies*

$$C_{ij}v + w = mh + r$$

then $h \in I_{ji}$.

Proof. By using Lemma 1 (2) it is enough to show that $f_l \in I_{ji}$ whenever the lth entry h_l in h is nonzero. So suppose $h_l \neq 0$. By Lemma 1 (5) we may assume that $w_l = 0$. Furthermore Lemma 1 (6) tells us that we may assume $(C_{ij})_{lk} \geq 0$ for all k.

Now look at the matrix C_{ji}. If $(C_{ji})_{st} < 0$, then Lemma 1 (6) implies that $f_s \in I_{ij}$ and further Lemma 1 (3) gives $C_{ij}f_s \in I_{ji}$. Lemma 1 (5) now tells us that we may assume that the lth entry of $C_{ij}f_s$ is zero; that is, $(C_{ij})_{ls} = 0$. The conclusion is that we may assume the following implication to be true

$$(C_{ji})_{st} < 0 \quad \Rightarrow \quad (C_{ij})_{ls} = 0, \qquad s, t \in \{1, 2, \ldots, n\}.$$

As C_{ij} and C_{ji} are inverse to each other

$$\sum_{s=1}^{n} (C_{ij})_{ls}(C_{ji})_{sl} = 1.$$

Our assumptions now imply that there exists an integer a such that

$$(C_{ij})_{la} = (C_{ji})_{al} = 1.$$

$$(C_{ij})_{ls}(C_{ji})_{sl} = 0, \qquad s \neq a.$$

Looking at the equality $C_{ij}v + w = mh + r$ tells us that there must exist an integer $b \neq a$ such that $(C_{ij})_{lb} \neq 0$ (else $h_l = 0$). Now consider the relation

$$\sum_{s=1}^{n} (C_{ij})_{ls}(C_{ji})_{sz} = 0, \qquad z \neq l.$$

This implies

$$(C_{ij})_{ls}(C_{ji})_{sz} = 0, \qquad z \neq l,\ s \in \{1, 2, \ldots, n\}.$$

As $(C_{ij})_{lb} \neq 0$ we get $(C_{ji})_{bz} = 0$, $z \neq l$. But then the bth row of C_{ji} is zero, which is a contradiction. ∎

Using Lemma 2 we can now, for every $m \in \mathbb{N}$ and $w \in I_{ji}$, define maps

$$h^w_{ijm} : P_m \to I_{ji}$$

$$r^w_{ijm} : P_m \to P_m$$

which are determined by the equality

$$v \in P_m : C_{ij}v + w = mh^w_{ijm}(v) + r^w_{ijm}(v).$$

LEMMA 3. *The map $r^w_{ijm} : P_m \to P_m$ is a bijection.*

Proof. Suppose that $r^w_{ijm}(v_1) = r^w_{ijm}(v_2)$ for $v_1, v_2 \in P_m$. From the definitions it then follows that

$$v_1 - v_2 = m(C_{ji}(h^w_{ijm}(v_1) - h^w_{ijm}(v_2))) \in m \cdot \mathrm{Mat}_{n,1}(\mathbb{Z}).$$

which can only be the case if $v_1 = v_2$. ∎

From now on we will by a toric variety understand a smooth toric variety with notations and choices fixed as above.

4. CONSTRUCTIONS OF CERTAIN VECTOR BUNDLES

Given a line bundle L on an n-dimensional toric variety $X = X(\Delta)$ and a positive integer m, we will in this section construct a vector bundle F_mL of rank m^n. The vector bundle F_mL will by definition be a direct sum of line bundles.

Let notation and choices be fixed as in Section 3. Let K denote the field of rational functions on X, and let $\mathcal{K}$ be the constant sheaf corresponding to K. Recall that a Cartier divisor on X can be represented by a collection of pairs $\{(U_i, f_i)\}$ where

- The U_i's form an open affine cover of X.
- $f_i \in K^*$ and f_i/f_j is a unit in $\Gamma(U_i \cap U_j, \mathcal{O}_X)$.

Denote the group of Cartier divisors on X by $\mathrm{Div}(X)$. Given a Cartier divisor D on X we can form the associated line bundle $\mathcal{O}(D)$, which is the subsheaf of $\mathcal{K}$ generated by $1/f_i$ over U_i. In this way we get a surjective map from $\mathrm{Div}(X)$ onto the group $\mathrm{Pic}(X)$ of line bundles on X modulo isomorphism. The kernel of this map is the group of principal Cartier divisors, that is, Cartier divisors which can be represented by a set $\{(U_i, f)\}$, $f \in K^*$. This way we get the well-known short exact sequence

$$0 \to K^* \to \mathrm{Div}(X) \to \mathrm{Pic}(X) \to 0.$$

Now the group $\mathrm{Div}(X)$ contains the subgroup $\mathrm{Div}_T(X)$ of T-Cartier divisors. By a T-Cartier divisor we will understand a Cartier divisor D which can be represented in the form [2, Chapter 3.3]

$$\{(U_{\sigma_i}, X_i^{u_i})\}_{\sigma_i \in \Delta}, \qquad u_i \in \mathrm{Mat}_{n,1}(\mathbb{Z}).$$

Notice that the condition for such a set to represent a Cartier divisor can be expressed as

$$u_j - C_{ij} u_i \in I_{ji}, \qquad \text{for all } i \text{ and } j.$$

A general fact about toric varieties [2, p. 63] tells us that the induced map from $\mathrm{Div}_T(X)$ to $\mathrm{Pic}(X)$ is surjective. We arrive at the short exact sequence

$$0 \to K^* \cap \mathrm{Div}_T(X) \to \mathrm{Div}_T(X) \to \mathrm{Pic}(X) \to 0.$$

Notice that an element $D \in \mathrm{Div}_T(X)$ is in $K^* \cap \mathrm{Div}_T(X)$ exactly when it is represented by a set $\{(U_{\sigma_i}, X_i^{u_i})\}$ with $u_j - C_{ij} u_i = 0$.

Let now D be a T-Cartier divisor and $m \in \mathbb{N}$. Fix a set $\{(U_{\sigma_i}, X_i^{u_i})\}$ which represents D and define $u_{ij} = u_j - C_{ij} u_i$. Then

LEMMA 4. *Fix $v \in P_m$ and a cone $\sigma_l \in \Delta$. If $v_i = h_{lim}^{u_{li}}(v)$ then $\{(u_{\sigma_i}, X_i^{v_i})\}$ represents a T-Cartier divisor.*

Proof. As noticed above it is enough to show that $v_j - C_{ij} v_i \in I_{ji}$ for all i and j. For this purpose consider the equality $C_{li} v + u_{li} = m v_i + r_{lim}^{u_{li}}(v)$. Multiplying by C_{ij} and using that $C_{ij} u_{li} = u_{lj} - u_{ij}$ and $C_{lj} v + u_{lj} = m v_j + r_{ljm}^{u_{lj}}(v)$ tells us

$$C_{ij} r_{lim}^{u_{li}}(v) + u_{ij} = m(v_j - C_{ij} v_i) + r_{ljm}^{u_{lj}}(v).$$

This means

$$r_{ljm}^{u_{lj}}(v) = r_{ijm}^{u_{ij}}(r_{lim}^{u_{li}}(v)),$$

$$v_j - C_{ij} v_i = h_{ijm}^{u_{ij}}(r_{lim}^{u_{li}}(v)) \in I_{ji}$$

by Lemma 2. ∎

Addendum 1. With the notation above we have

$$r^{u_{lj}}_{ljm}(v) = r^{u_{ij}}_{ijm}(r^{u_{li}}_{lim}(v)),$$

$$h^{u_{lj}}_{ljm}(v) - C_{ij} h^{u_{li}}_{lim}(v) = h^{u_{ij}}_{ijm}(r^{u_{li}}_{lim}(v))$$

for all i,j, and l.

Lemma 4 enables us for each $m \in \mathbb{Z}$ to define a map

$$F_m : \mathrm{Pic}(X) \to \mathrm{Vect}^m(X),$$

where we denote by $\mathrm{Vect}^m(X)$ the set of vector bundles of rank m^n modulo isomorphism. Given $[L] \in \mathrm{Pic}(X)$ we define the map in the following way:

Let $\{(U_{\sigma_i}, X_i^{u_i})\}$ represent a T-Cartier divisor D such that $\mathcal{O}(D) \cong L$, and choose an element $l \in \{0, 1, \ldots, q\}$ (remember that $q+1$ is the number of cones in Δ). Let D_v, $v \in P_m$ denote the Cartier divisor corresponding to $\{(U_{\sigma_i}, X_i^{v_i})\}$ where $v_i = h^{u_{li}}_{lim}(v)$. We define

$$F_m([L]) = \left[\bigoplus_{v \in P_m} \mathcal{O}(D_v)\right].$$

4.1. F_m is well defined. We will now show that F_m is well defined. Let us first show that F_m is independent of the choice of l. So suppose $[L]$, D, and $\{(U_{\sigma_i}, X_i^{u_i})\}$ are chosen as in the definition of F_m, and let l and l' be two elements in $\{0, 1, \ldots, q\}$. If $v \in P_m$ we denote the Cartier divisors corresponding to l and l' by D_v and D'_v, respectively. Referring to Lemma 3 it is enough to show that

$$\forall v \in P_m : \mathcal{O}(D_v) \cong \mathcal{O}(D'_w), \qquad w = r^{u_{ll'}}_{ll'm}(v)$$

which is equivalent to

$$\{(U_{\sigma_i}, X_i^{z_i})\}, \qquad z_i = h^{u_{li}}_{lim}(v) - h^{u_{l'i}}_{l'im}(w)$$

being a principal Cartier divisor. But by Addendum 1 we know that

$$z_i = C_{l'i} h^{u_{ll'}}_{ll'm}(v)$$

from which we conclude that

$$z_{ij} = z_j - C_{ij} z_i = 0.$$

It follows that $\{(U_{\sigma_i}, X_i^{z_i})\}$ is principal as desired.

Let us next show that the definition is independent of the choice of $\{(U_{\sigma_i}, X_i^{u_i})\}$. Suppose $\{(U_{\sigma_i}, X_i^{u'_i})\}$ represents a Cartier divisor D' such that $\mathcal{O}(D') \cong L$. Then $u_{ij} = u'_{ij}$ for all i and j. But then

$$v_i = h^{u_{li}}_{lim}(v) = h^{u'_{li}}_{lim}(v) = v'_i.$$

We conclude that $\mathcal{O}(D_v)$ are equal to $\mathcal{O}(D_{v'})$. It is thereby shown that F_m is well defined. Notice however that we do not claim that F_m is independent of the definitions and choices made in Section 3 (even though it might be).

5. SPLITTING OF F_*L

Let X be a variety over an algebraically closed field k of characteristic $p > 0$. The absolute Frobenius morphism on X is the morphism $F\colon X \to X$ of schemes which is the identity on the underlying set of X, and the pth power map on sheaf level $F^{\#}\colon \mathcal{O}_X \to \mathcal{O}_X$. When X is a smooth variety and E is a locally free sheaf on X of rank d, the direct image F_*E is also a locally free sheaf, but of rank $dp^{\dim(X)}$. We will now prove our main result.

THEOREM 1. *Let $X = X(\Delta)$ be a smooth toric variety over a field k of positive characteristic and let $F\colon X \to X$ be the absolute Frobenius morphism. Taking direct images via F_* induces for each $r \in \mathbb{N}$ a map*

$$(F_*)^r\colon \operatorname{Pic}(X) \to \operatorname{Vect}^m(X), \qquad m = p^r$$

which coincides with F_m. In particular F_*L splits into a direct sum of line bundles when L is a line bundle on X.

Proof. Let $[L] \in \operatorname{Pic}(X)$ and choose a T-Cartier divisor D such that $\mathcal{O}(D) \cong L$ and a set $\{(U_{\sigma_i}, X_i^{u_i})\}$ representing D. For each $v \in P_m$ let D_v be the Cartier divisor associated to $\{(U_{\sigma_i}, X_i^{v_i})\}$ where $v_i = h_{0im}^{u_{0i}}(v)$. For a given $v \in P_m$ we will now construct a map

$$\pi^v\colon \mathcal{O}(D_v) \to (F_*)^r(\mathcal{O}(D)).$$

This will be done by defining it locally over U_{σ_i} for each $i \in \{0, 1, \dots, q\}$

$$\pi_i^v\colon \mathcal{O}(D_v)(U_{\sigma_i}) \to (F_*^r(\mathcal{O}(D)))(U_{\sigma_i}).$$

For this purpose denote by F_*^rK the $\mathcal{O}_X(U_{\sigma_i})$-module which as an abelian group is the field of rational functions K on X, but where the $\mathcal{O}_X(U_{\sigma_i})$-module structure is twisted by the p^rth power map on $\mathcal{O}_X(U_{\sigma_i})$. Then $(F_*^r(\mathcal{O}(D)))(U_{\sigma_i})$ is the free submodule of F_*^rK with basis $\{X_i^w X_i^{-u_i}\}_{w \in P_m}$. On the other hand $\mathcal{O}(D_v)(U_{\sigma_i})$ is the free submodule of K with generator $X_i^{-v_i}$. We define the map π_i^v to be the unique $\mathcal{O}_X(U_{\sigma_i})$-linear map satisfying

$$\pi_i^v(1 \cdot X_i^{-v_i}) = X_i^{r_{0im}^{u_{0i}}(v)} \cdot X_i^{-u_i}.$$

We claim that these local morphisms glue together. This is so because over $U_{\sigma_i} \cap U_{\sigma_j}$ we have by Lemma 1 (1) that

$$1 \cdot X_i^{-v_i} = X_j^{v_j - C_{ij}v_i} \cdot X_j^{-v_j} = X_j^{v_{ij}} \cdot X_j^{-v_j}$$

which by π_j^v maps to

$$X_j^{mv_{ij} + r_{0jm}^{u_{0j}}(v)} \cdot X_j^{-u_j}.$$

Further, this last expression is by Lemma 1 and Addendum 1 easily seen to be equal to $\pi_i^v(1 \cdot X_i^{-v_i})$. As the local morphisms glue together we get the desired morphism π^v: $\mathcal{O}(D_v) \to (F_*)^r(\mathcal{O}(D))$. The collection of maps π^v now induces a morphism

$$\pi : \bigoplus_{v \in P_m} \mathcal{O}(D_v) \to (F_*)^r(\mathcal{O}(D))$$

and Lemma 3 tells us that this is an isomorphism. ∎

6. RESULTS

In this section we will state and prove a few results about F_m. In the characteristic $p > 0$ situation these results translate into statements about F_*^r, referring to Theorem 1.

PROPOSITION 1. *Let L be a fixed line bundle on a smooth toric variety. Then the set of line bundles occurring in the definition of F_mL, as m runs through $\mathbb{N}$ is finite.*

Proof. We only have to notice that $h_{ijm}^{u_{ij}}(v)$ in the definition of F_mL is bounded for $m \in \mathbb{Z}$. This is an easy exercise. ∎

PROPOSITION 2. *Let L be a line bundle on a smooth toric variety. The line bundle* $\det(F_mL)$ (*i.e., the top exterior power of F_mL*) *is isomorphic to*

$$\det(F_mL) \cong \omega_X^{m^{n-1}(m-1)/2} \otimes L^{m^{n-1}}$$

where ω_X is the dualizing sheaf on X.

Proof. Choose a T-Cartier divisor D so that $\mathcal{O}(D) \cong L$ and a set $\{(U_{\sigma_i}, X_i^{u_i})\}$ representing D. As D_v, $v \in P_m$, is represented by $\{(U_{\sigma_i}, X_i^{h_{0im}^{u_{0i}}(v)})\}$ (we use $l = 0$ in the definition of F_mL) we get that $\det(\sum_{v \in P_m} D_v)$ is represented by $\{(U_{\sigma_i}, X_i^{h_i})\}$, where $h_i = \sum_{v \in P_m} h_{0im}^{u_{0i}}(v)$. Using the equality

$$C_{0i}v + u_{0i} = mh_{0im}^{u_{0i}}(v) + r_{0im}^{u_{0i}}(v)$$

and summing over $v \in P_m$ we find

$$C_{0i}\left(\sum_{v \in P_M} v\right) + m^n u_{0i} = mh_i + \sum_{v \in P_m} r_{0im}^{u_{0i}}(v).$$

Now Lemma 3 tells us that

$$\sum_{v \in P_m} r_{0im}^{u_{0i}}(v) = \sum_{v \in P_m} v = \frac{1}{2} m^n(m-1)e$$

where $e = (1, 1, \ldots, 1)$, so that

$$h_i = m^{n-1}u_{0i} + \frac{1}{2}m^{n-1}(m-1)(C_{0i}e - e).$$

By recognizing $\{(U_{\sigma_i}, X_i^{-e})\}$ as representing the canonical divisor [2, p. 85–86] the lemma follows. ∎

In the characteristic $p > 0$ situation the absolute Frobenius morphism F is an affine morphism. This means that the cohomology groups of L and F_*L are isomorphic. One can therefore ask if a similar result is true when F_*L is replaced by F_mL. The answer will follow from the next proposition.

PROPOSITION 3. *Let L be a line bundle on a smooth toric variety and $m \in \mathbb{N}$. As sheaves of abelian groups L and F_mL are isomorphic. In particular their cohomologies agree.*

Proof. Regard L as the sheaf $\mathcal{O}(D)$ where D is represented by $\{(U_{\sigma_i}, X_i^{u_i})\}$. Let $v \in P_m$ and define $v_i = h_{0im}^{u_{0i}}(v)$. We will then construct a map of sheaves of abelian groups

$$\psi^v\colon \mathcal{O}(D_v) \to L.$$

This will be done by defining it locally over U_{σ_i};

$$\psi_i^v\colon \mathcal{O}(D_v)(U_{\sigma_i}) \to L(U_{\sigma_i}).$$

As submodules of K we have that $\mathcal{O}(D_v)(U_{\sigma_i})$ and $L(U_{\sigma_i})$ are the free $\mathcal{O}_X(U_{\sigma_i})$-modules with generators $1 \cdot X_i^{-v_i}$ and $1 \cdot X_i^{-u_i}$, respectively. If $w \in \mathrm{Mat}_{n,1}(\mathbb{Z})$ is a vector with $X_i^w \in R_i$ we define

$$\psi_i^v(aX_i^w \cdot {}_i^{-v_i}) = aX_i^{mw + r_{0im}^{u_{0i}}(v)} \cdot X_i^{-u_i}, \qquad a \in k$$

and expand this to a morphism of groups. This defines the map ψ_i^v of abelian groups. It is now an easy exercise to check that these local morphisms glue together and give us the desired map $\psi^v\colon \mathcal{O}(D_v) \to L$. The collection of maps ψ^v now induces a map of sheaves of abelian groups

$$\psi : \bigoplus_{v \in P_m} \mathcal{O}(D_v) \to L$$

which by Lemma 3 can be seen to be an isomorphism. ∎

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