

CELLULAR FREE RESOLUTIONS FOR NORMALIZATIONS OF TORIC IDEALS

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ABSTRACT. For any toric ideal I in a polynomial ring S , we provide a combinatorial description of a free resolution of the integral closure of the S -module S/I . These new complexes arise from an extension of Bayer–Sturmfels’ theory of cellular free resolutions. As applications, we unify several constructions for a resolution of the diagonal embedding of a toric variety, and compare the locally free resolutions for toric subvarieties introduced by Hanlon–Hicks–Lazarev and Brown–Erman.

1. OVERVIEW

Explicit free resolutions not only hold a foundational place in commutative algebra [MS05, Pee11], but also play an outsized role in algebraic combinatorics [OW07], algebraic geometry [Eis05], and representation theory [CG10]. The Koszul complex, which resolves ideals generated by regular sequences, is the archetypal example. Extending this paradigm, the Eagon–Northcott, Buchsbaum–Rim, and Gulliksen–Negård complexes resolve certain ideals generated by minors of matrices; see [Wey03]. The theory of cellular free resolutions [BPS98, BS98], generalizing the Taylor and Eliahou–Kervaire resolutions, offers the most comprehensive approach for constructing combinatorial free resolutions of monomial ideals by linking them to cell complexes. While [BS98, PS98] formulate an analog for binomial ideals, the theory is much less well behaved or fully developed. The primary goal of this article is to significantly expand the framework for cellular free resolutions of binomial ideals. In particular, our construction yields a cellular free resolution for the integral closure of the quotient of a polynomial ring by a toric ideal.

To elaborate, fix a positive integer n and let $S := \mathbb{k}[x_1, x_2, \dots, x_n]$, where \mathbb{k} is a field. A saturated lattice $L \subseteq \mathbb{Z}^n$ is a free abelian subgroup containing all integral points in its rational span. Such saturated lattices are in bijection with toric ideals in the polynomial ring S :

$$L \subseteq \mathbb{Z}^n \quad \longleftrightarrow \quad I_L := \langle \mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \in S \mid \mathbf{u} - \mathbf{v} \in L \text{ for all } \mathbf{u}, \mathbf{v} \in \mathbb{N}^n \rangle.$$

Endow the real vector space $L_{\mathbb{R}} := L \otimes_{\mathbb{Z}} \mathbb{R}$ with the structure of an L -equivariant cell complex. A *compatible \mathbb{Z}^n -stratification* is a map $\psi: L_{\mathbb{R}} \rightarrow \mathbb{Z}^n$ that is constant on the relative interior of each cell in $L_{\mathbb{R}}$ and compatible with translations by elements of L ; see Definition 2.6. Homogenizing the cellular \mathbb{k} -complex of $L_{\mathbb{R}}$ using ψ yields a cellular free $S[L]$ -complex F_{ψ} whose i th term is

$$(F_{\psi})_i = \bigoplus_{\substack{\sigma \subset L_{\mathbb{R}} \\ \dim \sigma = i}} S(-\psi(\sigma))$$

and whose differentials are detailed in Definition 2.10.

The algebraic counterpart of passing from the universal cover $L_{\mathbb{R}}$ to the real torus $L_{\mathbb{R}}/L$ is realized via the extension-of-scalars functor $M \mapsto M \otimes_{S[L]} S$, which transforms any \mathbb{Z}^n -graded $S[L]$ -module into a (\mathbb{Z}^n/L) -graded S -module; see Remark 2.14. Our main contribution is the following theorem.

Theorem 1.1. *When $\psi: L_{\mathbb{R}} \rightarrow \mathbb{Z}^n$ is defined by $\psi(\mathbf{p}) := \lceil \mathbf{p} \rceil$ for all $\mathbf{p} \in L_{\mathbb{R}} \subseteq \mathbb{R}^n$, the cellular free S -complex $F_{\psi} \otimes_{S[L]} S$ is a (\mathbb{Z}^n/L) -graded resolution of the integral closure of the S -module S/I_L .*

A uniform combinatorial description of a free resolution of the integral closure of S/I_L for an arbitrary toric ideal I_L is new. The key innovation is the introduction of compatible \mathbb{Z}^n -stratifications,

which allow nonintegral vertices (or 0-cells) in $L_{\mathbb{R}}$ to be labeled by monomials in S . When the vertices lie in the lattice L , this framework recovers the constructions in [BS98, §3] and [MS05, §9.2]. In particular, the minimal free resolution of a unimodular Lawrence ideal described in [BPS01, Corollary 3.6] arises as a special case.

By decoupling vertices from lattice points, we expand the range of viable cell structures on the space $L_{\mathbb{R}}$. Both Example 5.8 and Lemma 5.10 illustrate this flexibility. For any compatible \mathbb{Z}^n -stratification, Proposition 2.13, Corollary 2.15, and Lemma 4.7 develop the fundamental theory of cellular free S -complexes. Furthermore, Theorem 1.1 implies that the integral closure of S/I_L is Cohen–Macaulay, thereby providing an alternative proof of [Hoc72, Theorem 1]. Taken together, these advances substantially enhance the power and scope of the theory of cellular free resolutions.

Primary geometric application. Our original motivation was to recast the locally free resolutions of [HHL24] in the language of combinatorial commutative algebra. To this end, consider a torus-equivariant closed embedding $\varphi: Y \rightarrow X$ of a normal toric variety Y into a smooth toric variety X with no torus factors. In this setting, the polynomial ring S is the Cox ring of X , the image $\varphi(Y)$ is cut out by a toric ideal in X , and $L \subseteq \mathbb{Z}^n$ is the corresponding lattice; see Remark 2.3. Let $\psi: L_{\mathbb{R}} \rightarrow \mathbb{Z}^n$ be the ceiling stratification; see Definition 3.1.

The main result [HHL24, Theorem A] constructs an explicit resolution by line bundles of the \mathcal{O}_X -module $\varphi_* \mathcal{O}_Y$. For the ceiling stratification, Theorem 4.2 identifies this resolution with the \mathcal{O}_X -complex associated to the cellular free S -complex $F_{\psi} \otimes_{S[L]} S$. This perspective circumvents the inductive arguments and discrete Morse theory used in [HHL24, §1.3]. For similar reasons, the preprint [BH25], which acknowledges that it was “inspired by an Oberwolfach talk by [the first author]” (see [Ber25]), employs essentially the same cellular techniques.

Beyond this reinterpretation, our algebraic approach demonstrates that the locally free resolution corresponds to an actual resolution of S -modules, rather than merely a virtual resolution in the sense of [BES20, Definition 1.1]. Although [HHL24, §2.3] already emphasizes the ceiling stratification, Proposition 3.3 and Theorem 3.4 further reveal its strong connection to integral closures. In this way, the cellular free S -complexes provide a remarkably simple and concise source for all of these locally free resolutions.

Understanding virtual resolutions. More significantly, the cellular free complexes arising from a ceiling stratification provide a framework for comparing the locally free resolutions in [HHL24, Theorem 3.5] and [BE24, Theorem 1.2]. Specifically, Corollary 4.6 shows that the resolution in [BE24, Theorem 1.2] is isomorphic to a direct summand of that in [HHL24, Theorem 3.5].

For the diagonal embedding $\Delta: Y \rightarrow X := Y \times Y$ of a smooth toric variety Y with no torus factors, Theorem 4.9 and Corollary 4.10 identify the resolutions by line bundles in both [BE24, Theorem 1.2] and [HHL24, Theorem 3.5] with the \mathcal{O}_X -complex associated to the minimal cellular free resolution of the integral closure of S/I_L . This corollary justifies the belief expressed in [BE24, p.3] and provides an explicit description of the terms; see also [BE24, Remark 1.6]. In essence, these cellular free S -complexes completely explain the relationship between the virtual resolutions in [BE24, Theorem 1.2] and [HHL24, Theorem 3.5].

Further geometric applications. By exploiting compatible \mathbb{Z}^n -stratifications other than the ceiling stratification, we incorporate additional resolutions of the diagonal for the toric variety Y within our framework. For the construction in [And24, Theorem 1.1], this is accomplished by introducing

an appropriate compatible \mathbb{Z}^n -stratification; see Definition 5.5. Example 5.6 underscores some of the subtleties connected with the associated cellular free complexes. For the resolution of the diagonal in [FH25, Example 3.15], the primary challenge is identifying the relevant cell structure, and Proposition 5.12 clarifies the case $Y = \mathbb{P}^2$.

Conventions. Throughout the document, \mathbb{N} denotes the set of nonnegative integers, and \mathbb{k} denotes a field. The phrase ‘cell complex’ is synonymous with ‘CW complex.’

Outline. Section 2 introduces compatible \mathbb{Z}^n -stratifications and extends the theory of cellular free resolutions to this broader setting. Section 3 shows that the componentwise ceiling function is a compatible \mathbb{Z}^n -stratification which yields a free resolution of the integral closure of the comodule of a toric ideal. Section 4 explores applications of this ceiling stratification, demonstrating that our construction recovers the virtual resolutions in [HHL24, Theorem 3.5] and elucidates the locally free resolutions in [BE24, Theorem 1.2]. Section 5 situates other resolutions of the diagonal—namely those by [And24, Theorem 1.1] and [FH25, Example 3.15]—within this unified framework.

2. CELLULAR FREE RESOLUTIONS FROM STRATIFICATIONS

In this section, we construct free resolutions of monomial modules by extending the cellular techniques from [BS98, §3] and [MS05, §4]. Chiefly, we allow for more general monomial labelings of the cells by introducing a stratification. We also remove the regularity assumption on the underlying cell complex and the positivity assumption on the lattice L , allowing $L \cap \mathbb{N}^n \neq \{0\}$.

Topological context. Fix a positive integer n . The *lattice* L is a subgroup of the free abelian group \mathbb{Z}^n . Let d be the rank of L , and let $L_{\mathbb{R}} := L \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^d$ be the associated real vector space endowed with the Euclidean topology. The quotient space $L_{\mathbb{R}}/L$ is a real torus. The inclusion $L \subseteq \mathbb{Z}^n$ gives rise to an inclusion of real vector spaces $L_{\mathbb{R}} \subseteq \mathbb{R}^n$. We tacitly regard the elements of L or $L_{\mathbb{R}}$ as elements of \mathbb{Z}^n or \mathbb{R}^n respectively.

A cell complex in $L_{\mathbb{R}}$ is *L -equivariant* if, for each open cell σ and each lattice point $v \in L$, the translate $\sigma + v$ is also an open cell. Assume that the topological space $L_{\mathbb{R}}$ is endowed with the structure of an L -equivariant cell complex; see [Hat02, Appendix] for more on cell complexes. Via the quotient map $\pi: L_{\mathbb{R}} \rightarrow L_{\mathbb{R}}/L$, the L -equivariant cell complex on $L_{\mathbb{R}}$ equips the real torus $L_{\mathbb{R}}/L$ with the structure of a cell complex.

The following is our favorite source for L -equivariant cell complex structures on the space $L_{\mathbb{R}}$.

Remark 2.1 (Cell complex arising from a periodic arrangement). As in [DP11, §2.1], a central hyperplane arrangement in the space $L_{\mathbb{R}}$ is the set of hyperplanes $\{x \in L_{\mathbb{R}} \mid u_i \cdot x = 0\}$ for some finite set of vectors u_1, u_2, \dots, u_m in $\text{Hom}(L, \mathbb{Z})$. The *periodic arrangement* \mathcal{A} consists of all affine hyperplanes of the form $H := \{p \in L_{\mathbb{R}} \mid u_i \cdot p = j\}$, where $1 \leq i \leq m$ and j is an integer. A *chamber* σ of the arrangement \mathcal{A} is a connected component of the complement $L_{\mathbb{R}} \setminus \bigcup_{H \in \mathcal{A}} H$. Hence, the closure $\overline{\sigma}$ is a convex polyhedron. A *face* of \mathcal{A} is the face of the polyhedron $\overline{\sigma}$: the intersection of $\overline{\sigma}$ with a supporting hyperplane. The faces of \mathcal{A} endow the space $L_{\mathbb{R}}$ with the structure of a polyhedral cell complex; see [MS05, Definition 4.1]. Unlike in [BPS01, §3], the vertices (or 0-cells) of this polyhedral cell complex need not be lattice points. Under the quotient map $\pi: L_{\mathbb{R}} \rightarrow L_{\mathbb{R}}/L$, this periodic arrangement defines a *toric arrangement* in $L_{\mathbb{R}}/L$; see [DP11, Definition 14.2].

A specific instance of this construction is significant enough to warrant a name.

Definition 2.2. The *standard cell structure* on the space $L_{\mathbb{R}} \subseteq \mathbb{R}^n$ is the polyhedral cell structure determined by the periodic arrangement arising from the standard basis vectors in \mathbb{R}^n .

We exploit the standard cell structure in toric geometry. For clarity and brevity, we focus on smooth toric varieties, with extensions to singular toric varieties and toric stacks left to the reader.

Remark 2.3 (Cell complex arising from a toric embedding). Fix a smooth toric variety X with no torus factors; see [CLS11, Corollary 3.3.10]. Let $\varphi: Y \rightarrow X$ be a torus-equivariant embedding of a normal toric variety Y . The toric morphism φ corresponds to an injective \mathbb{Z} -linear map $\bar{\varphi}: N_Y \rightarrow N_X$ between the lattices of one-parameter subgroups of Y and X ; see [CLS11, Theorem 3.3.4]. Dualizing the map $\bar{\varphi}$ gives the surjective \mathbb{Z} -linear map $\bar{\varphi}^*: M_X \rightarrow M_Y$ between the character lattices of X and Y . Set m and n to be the number of rays in the fans of Y and X , respectively. The groups of torus-invariant Weil divisors on Y and X are identified with \mathbb{Z}^m and \mathbb{Z}^n , respectively. Thus, we obtain the commutative diagram

$$\begin{array}{ccccccc}
 & & 0 & & & & \\
 & & \downarrow & & & & \\
 & & L & & & & \\
 & & \kappa \downarrow & \searrow \iota & & & \\
 0 & \longrightarrow & M_X & \xrightarrow{v} & \mathbb{Z}^n & \longrightarrow & \text{Pic}(X) \longrightarrow 0 \\
 & & \bar{\varphi}^* \downarrow & \downarrow & \downarrow & & \downarrow \\
 0 & \longrightarrow & M_Y & \longrightarrow & \mathbb{Z}^m & \longrightarrow & \text{Pic}(Y) \longrightarrow 0,
 \end{array}$$

where $L := \text{Ker}(\bar{\varphi}^*)$, the injection κ is the canonical inclusion, and the rows in the matrix of the map v are the primitive lattice points generating the rays in the fan of X ; see [CLS11, Theorem 4.2.1]. The image of the composite map $\iota := v \circ \kappa$ realizes the lattice L as a subgroup of \mathbb{Z}^n with rank $d := \dim X - \dim Y$. In this case, we always equip the space $L_{\mathbb{R}}$ with its standard cell structure.

We illustrate this toric construction with a concrete example.

Example 2.4 (Cell complex arising from the inclusion of a point into Hirzebruch surface). Let $X := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2))$ denote the second Hirzebruch surface. The rays in the fan of the toric variety X are generated by the primitive lattice points $(1, 0)$, $(0, 1)$, $(-1, 2)$, and $(0, -1)$; see [CLS11, Example 3.1.16]. Consider the inclusion of the identity point $Y := \{[1 : 1 : 1 : 1]\}$ in the dense torus of X . This corresponds to the unique \mathbb{Z} -linear map $\bar{\varphi}: \mathbb{Z}^0 \rightarrow \mathbb{Z}^2$. The lattice L has rank 2, and the matrix of the inclusion $\iota: \mathbb{Z}^2 \cong L \rightarrow \mathbb{Z}^4$, with respect to the standard bases, is

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 2 \\ 0 & -1 \end{bmatrix}.$$

The standard cell structure on the real vector space $L_{\mathbb{R}} \cong \mathbb{R}^2$ is induced by the standard basis in \mathbb{R}^4 . Identifying $L_{\mathbb{R}}$ with the real vector space \mathbb{R}^2 , the polyhedral cell structure arises from the rows of

this matrix under the standard basis. Hence, a fundamental domain for the L -action on $L_{\mathbb{R}}$ contains two 0-cells, five 1-cells, and three 2-cells, as depicted in Figure 2.5.

As the hyperplanes with normal vectors $(0, 1)$ and $(0, -1)$ coincide, only three colors are needed to display the fundamental domain. The hyperplanes with normal vectors $(1, 0)$ and $(-1, 2)$ intersect at a point outside of L , so each gives rise to two distinct 1-cells in the fundamental domain. \diamond

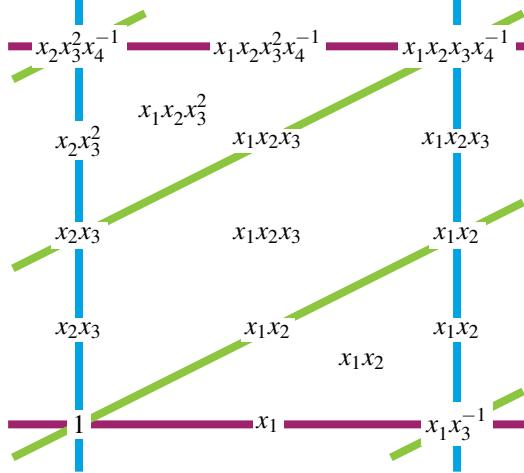


Figure 2.5. A compatible \mathbb{Z}^4 -stratification for the closed embedding of the identity point into the second Hirzebruch surface.

In addition to the cell structure on the space $L_{\mathbb{R}}$, we also regard the free abelian group \mathbb{Z}^n as a topological space. Since the set \mathbb{Z}^n is a poset under the componentwise order, it carries the *Alexandrov topology*: a subset U of \mathbb{Z}^n is open if, whenever $\mathbf{u} \leqslant \mathbf{v}$ and $\mathbf{u} \in U$, it follows that $\mathbf{v} \in U$. Building on [Lur17, Definition A.5.1], we make the following definition.

Definition 2.6. Let L be a lattice in \mathbb{Z}^n and assume that the space $L_{\mathbb{R}}$ is endowed with the structure of an L -equivariant cell complex. A continuous map $\psi: L_{\mathbb{R}} \rightarrow \mathbb{Z}^n$ is a *compatible \mathbb{Z}^n -stratification* if it is constant on each open cell in $L_{\mathbb{R}}$ and satisfies $\psi(\mathbf{p} + \mathbf{v}) = \psi(\mathbf{p}) + \mathbf{v}$ for all points $\mathbf{p} \in L_{\mathbb{R}}$ and all lattice points $\mathbf{v} \in L$. For each open cell $\sigma \subset L_{\mathbb{R}}$, define $\psi(\sigma) := \psi(\mathbf{p})$ for any point $\mathbf{p} \in \sigma$; this is well-defined because ψ is constant on each cell.

Since \mathbb{Z}^n is endowed the Alexandrov topology, continuity of ψ is equivalent to requiring that, for each lattice point $\mathbf{u} \in \mathbb{Z}^n$, the subset $(L_{\mathbb{R}})_{\leqslant \mathbf{u}} := \{\mathbf{p} \in L_{\mathbb{R}} \mid \psi(\mathbf{p}) \leqslant \mathbf{u}\}$ is closed. The second condition defining ψ implies that it is compatible with lattice translation. By design, a compatible \mathbb{Z}^n -stratification $\psi: L_{\mathbb{R}} \rightarrow \mathbb{Z}^n$ is determined by its values on a fundamental domain for the translation action of the lattice L on $L_{\mathbb{R}}$. In our applications, the topological space $L_{\mathbb{R}}$ typically has the coarsest cell structure that allows a given map ψ to define a compatible \mathbb{Z}^n -stratification.

We recover the labeled cell complexes in [MS05, Definition 4.2] as a special case.

Example 2.7 (Least common multiple stratifications). A compatible \mathbb{Z}^n -stratification $\psi: L_{\mathbb{R}} \rightarrow \mathbb{Z}^n$ is defined by assigning a lattice vector in \mathbb{Z}^n for each vertex in a fundamental domain. Since a compatible stratification commutes with translation by lattice points in L , the value ψ at every vertex in $L_{\mathbb{R}}$ is determined by this assignment. For any higher-dimensional open cell σ , the value $\psi(\sigma)$ is the componentwise maximum of the lattice vectors assigned to the vertices in the closure of σ . \diamond

The following tangible geometric example serves to further highlight Definition 2.6.

Example 2.8 (Compatible \mathbb{Z}^4 -stratification for an embedding into toric surface). As in Example 2.4, consider the closed embedding $\varphi: Y \rightarrow X$ of the identity point Y into the second Hirzebruch surface X . The map $\psi: L_{\mathbb{R}} \rightarrow \mathbb{Z}^4$ defined by $\psi(p) := [p]$ for all $p \in L_{\mathbb{R}} \subset \mathbb{R}^4$ is a compatible \mathbb{Z}^4 -stratification. Let $\iota_{\mathbb{R}}: \mathbb{R}^2 \rightarrow \mathbb{R}^4$ be the linear map induced by the inclusion $\iota: \mathbb{Z}^2 \rightarrow \mathbb{Z}^4$. For all $(q_1, q_2) \in \mathbb{R}^2 \cong L_{\mathbb{R}}$, we have

$$\psi(\iota_{\mathbb{R}}(q_1, q_2)) = [(q_1, q_2, -q_1 + 2q_2, -q_2)] = ([q_1], [q_2], [-q_1 + 2q_2], [-q_2]).$$

In Figure 2.5, we display the monomial with exponent vector $\psi(\sigma)$ at the center of the cell σ . \diamond

We record one technical feature of compatible \mathbb{Z}^n -stratifications.

Lemma 2.9. *Let $\psi: L_{\mathbb{R}} \rightarrow \mathbb{Z}^n$ be a compatible \mathbb{Z}^n -stratification. For any open cells σ and τ in the space $L_{\mathbb{R}}$, the inclusion $\tau \subseteq \overline{\sigma}$ implies that $\psi(\tau) \leq \psi(\sigma)$.*

Proof. In the Alexandrov topology, the subset $\psi(\sigma) - \mathbb{N}^n := \{\psi(\sigma) - u \in \mathbb{Z}^n \mid u \in \mathbb{N}^n\}$ is a closed set. As the map ψ is continuous, the preimage $\psi^{-1}(\psi(\sigma) - \mathbb{N}^n)$ is closed and contains σ . The hypothesis $\tau \subseteq \overline{\sigma}$ shows that the preimage also contains τ . We deduce that $\psi(\tau) \leq \psi(\sigma)$. \square

Homological algebra from the universal cover. In algebraic topology, the cell structure on the space $L_{\mathbb{R}}$ leads to a complex of vector spaces over the field \mathbb{k} . The map $\psi: L_{\mathbb{R}} \rightarrow \mathbb{Z}^n$ allows one to transform this into a graded complex of modules over a polynomial ring. Recall that $d := \dim L_{\mathbb{R}}$.

For any open cells $\sigma, \tau \subset L_{\mathbb{R}}$, let $\varepsilon(\sigma, \tau)$ be the *incidence number* (also known as the *topological degree*) of the composition of the attaching map of σ with the quotient map collapsing the complement of τ to a point; see [Hat02, p. 140]. The resulting map is a continuous map between spheres, and its topological degree measures how many times the boundary sphere of σ wraps around the sphere corresponding to τ ; see [Hat02, p. 134]. The integer $\varepsilon(\sigma, \tau)$ depends implicitly on the orientations chosen for the cells. In particular, we have $\varepsilon(\sigma, \tau) = 0$ unless $\dim \sigma = 1 + \dim \tau$ and τ is contained in the closure $\overline{\sigma}$. The L -equivariance of the cell structure implies that $\varepsilon(\sigma + v, \tau + v) = \varepsilon(\sigma, \tau)$ for any lattice point $v \in L$. When the cell structure on $L_{\mathbb{R}}$ is regular, the incidence numbers take values only in $\{-1, 0, 1\}$; see [MS05, §4.1].

The Cellular Boundary Formula in [Hat02, §2.2] establishes that the *cellular \mathbb{k} -complex* of $L_{\mathbb{R}}$ is

$$0 \leftarrow C_0(L_{\mathbb{R}}; \mathbb{k}) \xleftarrow{\partial_1} C_1(L_{\mathbb{R}}; \mathbb{k}) \xleftarrow{\partial_2} \cdots \xleftarrow{\partial_d} C_d(L_{\mathbb{R}}; \mathbb{k}) \leftarrow 0$$

with

$$C_i(L_{\mathbb{R}}; \mathbb{k}) := \bigoplus_{\substack{\sigma \subset L_{\mathbb{R}} \\ \dim \sigma = i}} \mathbb{k}\sigma, \quad \text{and} \quad \partial\sigma = \sum_{\tau \subset L_{\mathbb{R}}} \varepsilon(\sigma, \tau)\tau,$$

where each the open cell σ (or τ) is identified with a generator of the corresponding summand of the direct sum $C(L_{\mathbb{R}}; \mathbb{k})$.

To generalize the cellular free complexes in [BS98, §1] and [MS05, Definition 4.3], we use a compatible \mathbb{Z}^n -stratification $\psi: L_{\mathbb{R}} \rightarrow \mathbb{Z}^n$ to homogenize the cellular \mathbb{k} -complex $C(L_{\mathbb{R}}; \mathbb{k})$. Let $S := \mathbb{k}[x_1, x_2, \dots, x_n]$ denote the \mathbb{Z}^n -graded polynomial ring with $\deg(x^u) := u \in \mathbb{Z}^n$. For any open cell $\sigma \subset L_{\mathbb{R}}$, let $S\sigma \cong S(-\psi(\sigma))$ be the free \mathbb{Z}^n -graded S -module with a unique generator, also denoted by σ , in degree $\psi(\sigma)$.

Definition 2.10. Let L be a lattice in \mathbb{Z}^n . For any compatible \mathbb{Z}^n -stratification $\psi: L_{\mathbb{R}} \rightarrow \mathbb{Z}^n$, the *cellular free S-complex* F_{ψ} is the \mathbb{Z}^n -graded S -complex

$$0 \longleftarrow (F_{\psi})_0 \xleftarrow{\partial_1} (F_{\psi})_1 \xleftarrow{\partial_2} \cdots \xleftarrow{\partial_d} (F_{\psi})_d \longleftarrow 0$$

with

$$(F_{\psi})_i = \bigoplus_{\substack{\sigma \subset L_{\mathbb{R}} \\ \dim \sigma = i}} S\sigma, \quad \text{and} \quad \partial\sigma = \sum_{\tau \subset L_{\mathbb{R}}} \varepsilon(\sigma, \tau) x^{\psi(\sigma) - \psi(\tau)} \tau.$$

Moreover, the monomial S -module is $M_{\psi} := S \cdot \{x^{\psi(p)} \mid p \in L_{\mathbb{R}}\} \subseteq \mathbb{k}[x_1, x_1^{-1}, x_2, x_2^{-1}, \dots, x_n, x_n^{-1}]$.

Our earlier technical observation clarifies some aspects of this definition.

Remark 2.11. Lemma 2.9 confirms that all the entries $\varepsilon(\sigma, \tau) x^{\psi(\sigma) - \psi(\tau)}$ in the matrix representing the differential of F_{ψ} belong to S . Moreover, for any open cell σ in $L_{\mathbb{R}}$ and any vertex p in the closure $\overline{\sigma}$, Lemma 2.9 guarantees that $x^{\psi(\sigma) - \psi(p)} \in S$. Hence, the factorization $x^{\psi(\sigma)} = x^{\psi(\sigma) - \psi(p)} x^{\psi(p)}$ shows that the vertices in $L_{\mathbb{R}}$ form a generating set for the monomial S -module:

$$M_{\psi} = S \cdot \{x^{\psi(p)} \mid \text{the point } p \text{ is a vertex in cell complex } L_{\mathbb{R}}\}.$$

In other words, the map that sends a vertex $p \in L_{\mathbb{R}}$ to the Laurent monomial $x^{\psi(p)}$ defines a surjective \mathbb{Z}^n -graded S -module homomorphism from $(F_{\psi})_0$ onto M_{ψ} .

Example 2.12 (Monomial module for an embedding into a toric surface). Following Example 2.8, the map $\varphi: Y \rightarrow X$ is the inclusion of the identity point Y in the dense torus into the second Hirzebruch surface X . A compatible \mathbb{Z}^4 -stratification $\psi: L_{\mathbb{R}} \rightarrow \mathbb{Z}^4$ is defined, for all (q_1, q_2) in $\mathbb{R}^2 \cong L_{\mathbb{R}}$, by $\psi(\iota_{\mathbb{R}}(q_1, q_2)) = (\lceil q_1 \rceil, \lceil q_2 \rceil, \lceil -q_1 + 2q_2 \rceil, \lceil -q_2 \rceil)$. Since applying ψ to the vertices gives $\psi(\iota_{\mathbb{R}}(0, 0)) = (0, 0, 0, 0)$ and $\psi(\iota_{\mathbb{R}}(0, 0.5)) = (0, 1, 1, 0)$, the monomial S -module M_{ψ} is generated by the set

$$\{x_1^k x_2^\ell x_3^{-k+2\ell} x_4^{-\ell}, x_1^k x_2^{1+\ell} x_3^{1-k+2\ell} x_4^{-\ell} \mid k, \ell \in \mathbb{Z}\}.$$

◇

Under suitable hypotheses, the cellular S -complex F_{ψ} provides a free resolution of the monomial S -module M_{ψ} ; compare with [BS98, Proposition 1.2] and [MS05, Proposition 4.5].

Proposition 2.13. Let $\psi: L_{\mathbb{R}} \rightarrow \mathbb{Z}^n$ be a compatible \mathbb{Z}^n -stratification. The cellular free S -complex F_{ψ} is a \mathbb{Z}^n -graded free resolution of M_{ψ} if and only if, for every lattice point $u \in \mathbb{Z}^n$, the closed subset $(L_{\mathbb{R}})_{\leq u} := \{p \in L_{\mathbb{R}} \mid \psi(p) \leq u\}$ is acyclic.

Proposition 3.2 exhibits a compatible \mathbb{Z}^n -stratification that satisfies this topological condition.

Proof. Fix a lattice point $u \in \mathbb{Z}^n$. Consider the subset $(L_{\mathbb{R}})_{\leq u} := \{p \in L_{\mathbb{R}} \mid \psi(p) \leq u\}$ of $L_{\mathbb{R}}$. By construction, the degree u part of the cellular free S -complex F_{ψ} is precisely the cellular \mathbb{k} -complex of the subcomplex $(L_{\mathbb{R}})_{\leq u}$. It follows that F_{ψ} is a free resolution of the monomial S -module M_{ψ} if and only if $H_0((L_{\mathbb{R}})_{\leq u}; \mathbb{k}) \cong \mathbb{k}$ when $x^u \in M_{\psi}$, and $H_i((L_{\mathbb{R}})_{\leq u}; \mathbb{k}) = 0$ when $i > 0$ or $x^u \notin M_{\psi}$. Remark 2.11 establishes that $x^u \in M_{\psi}$ if and only if there exists a vertex $p \in L_{\mathbb{R}}$ such that $\psi(p) \leq u$. Therefore, the S -complex F_{ψ} is a free resolution of the S -module M_{ψ} if and only if every nonempty subcomplex $(L_{\mathbb{R}})_{\leq u}$ is acyclic. □

Homological algebra from the topological torus. Following [BS98, §3] and [MS05, §9.3], we develop an algebraic counterpart to the topological relationship between the universal cover $L_{\mathbb{R}}$ and the real torus $L_{\mathbb{R}}/L$. Consider the \mathbb{Z}^n -graded group algebra $S[L]$ of the lattice L over the polynomial ring $S := \mathbb{k}[x_1, x_2, \dots, x_n] = \mathbb{k}[\mathbb{N}^n]$; it is defined as

$$S[L] := S[z^v \mid v \in L] \subseteq S[z_1^{\pm 1}, z_2^{\pm 1}, \dots, z_n^{\pm 1}] = S[\mathbb{Z}^n],$$

where $\deg(x^u z^v) = u + v \in \mathbb{Z}^n$. The coefficient ring S of the group algebra $S[L]$ is itself an $S[L]$ -module via the isomorphism

$$S[L]/\langle z^v - 1 \mid v \in L \rangle \cong S.$$

Similarly, the \mathbb{Z}^n -graded map of \mathbb{k} -algebras from the group algebra $S[L]$ to the Laurent polynomial ring

$$R := \mathbb{k}[x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_n^{\pm 1}] = \mathbb{k}[\mathbb{Z}^n]$$

determined by $x^u z^v \mapsto x^{u+v}$ makes the ring R into a $S[L]$ -module. Furthermore, for any compatible \mathbb{Z}^n -stratification $\psi: L_{\mathbb{R}} \rightarrow \mathbb{Z}^n$, the monomial S -module M_{ψ} is an $S[L]$ -submodule of R because the cell complex structure on the space $L_{\mathbb{R}}$ is L -equivariant and the map ψ commutes with lattice translation. Although M_{ψ} is almost never a finitely generated S -module, it is a finitely generated $S[L]$ -module in all of our applications.

In contrast, the coefficient ring S is *not* an $S[L]$ -submodule of the Laurent polynomial ring R because it is not closed under multiplication by a Laurent monomial z^v for an arbitrary lattice point $v \in L$. Nevertheless, the cellular free S -complex F_{ψ} is a \mathbb{Z}^n -graded $S[L]$ -complex. For any homological index i and any lattice point $v \in L$, multiplication by $z^v \in S[L]$ permutes the summands in the direct sum $(F_{\psi})_i$ by sending the generator of the summand $S\sigma$ to the generator of $S\tau$, where $\tau := \sigma + v$; the L -equivariance of the underlying cell complex guarantees that the lattice translate τ of the open cell σ is also an open cell. Since $\psi(\tau) = \psi(\sigma) + v$, this permutation preserves the \mathbb{Z}^n -grading. Given the $S[L]$ -module structures on M_{ψ} and $(F_{\psi})_0$, Remark 2.11 shows that the canonical surjection from $(F_{\psi})_0$ onto M_{ψ} is automatically an $S[L]$ -module homomorphism. The category of \mathbb{Z}^n -graded $S[L]$ -modules is our algebraic analog for the universal cover $L_{\mathbb{R}}$.

On the other hand, the category of (\mathbb{Z}^n/L) -graded S -modules serves as our algebraic analog of the real topological torus $L_{\mathbb{R}}/L$. Let

$$\eta: \mathbb{Z}^n \rightarrow \mathbb{Z}^n/L$$

be the canonical surjective homomorphism of \mathbb{Z} -modules that sends the lattice point $u \in \mathbb{Z}^n$ to the coset $\eta(u) := \{u + v \mid v \in L\}$ in the quotient group \mathbb{Z}^n/L . By definition, the (\mathbb{Z}^n/L) -grading on the polynomial ring S is the coarsening of its \mathbb{Z}^n -grading:

$$\deg(x^u) := \eta(u) \in \mathbb{Z}^n/L.$$

Likewise, the (\mathbb{Z}^n/L) -grading on $S[L]$ is given by $\deg(x^u z^v) = \eta(u + v) = \eta(u) \in \mathbb{Z}^n/L$. Since the ideal $\langle z^v - 1 \mid v \in L \rangle$ in $S[L]$ is homogeneous with respect to the (\mathbb{Z}^n/L) -grading (but not the \mathbb{Z}^n -grading), the \mathbb{k} -algebra homomorphism $S[L] \rightarrow S[L]/\langle z^v - 1 \mid v \in L \rangle \cong S$ is homogeneous and has degree 0 in the (\mathbb{Z}^n/L) -grading. Hence, the corresponding extension-of-scalars functor

$$M \mapsto M \otimes_{S[L]} S$$

converts every \mathbb{Z}^n -graded $S[L]$ -module into a (\mathbb{Z}^n/L) -graded S -module.

The image of the cellular free $S[L]$ -complex F_{ψ} under this extension-of-scalars functor has a straightforward description.

Remark 2.14. For each homological index i , the summand of the direct sum $(F_\psi)_i$ corresponding to the i -dimensional cell $\sigma \subset L_{\mathbb{R}}$ maps to the free (\mathbb{Z}^n/L) -graded S -module $S\sigma' \cong S(-(\eta \circ \psi)(\sigma))$; the generator of $S\sigma'$ corresponds to the coset $\sigma' := \{\tau \subset L_{\mathbb{R}} \mid \tau = \sigma + v \text{ for some } v \in L\}$ in $L_{\mathbb{R}}/L$ and the degree of this generator is $\eta(\psi(\sigma))$. It follows that

$$(F_\psi)_i \otimes_{S[L]} S = \bigoplus_{\substack{\sigma' \subset L_{\mathbb{R}}/L \\ \dim \sigma' = i}} S\sigma'.$$

To describe the differential on the free S -complex $(F_\psi)_i \otimes_{S[L]} S$, choose representatives σ and τ in $L_{\mathbb{R}}$ for the cosets σ' and τ' in $L_{\mathbb{R}}/L$ that lie in the same fundamental domain. It follows that

$$\partial \sigma' = \sum_{\tau' \subset L_{\mathbb{R}}/L} \left(\sum_{v \in L} \varepsilon(\sigma, \tau + v) x^{\psi(\sigma) - \psi(\tau) - v} \right) \tau'.$$

The inner sum is finite: the closed cell $\overline{\sigma}$ contains only finitely many open cells, so $\varepsilon(\sigma, \tau + v) \neq 0$ for only finitely many lattice points $v \in L$. Unlike the \mathbb{Z}^n -graded $S[L]$ -complex F_ψ , the differentials in the (\mathbb{Z}^n/L) -graded S -complex $F_\psi \otimes_{S[L]} S$ are not necessarily given by monomial matrices.

The ensuing corollary gives the desired relationship between our two categories of modules.

Corollary 2.15 ([BS98, Theorem 3.2]). *Let $\psi: L_{\mathbb{R}} \rightarrow \mathbb{Z}^n$ be a compatible \mathbb{Z}^n -stratification. When the cellular free $S[L]$ -complex F_ψ is a \mathbb{Z}^n -graded resolution of the $S[L]$ -module M_ψ , the S -complex $F_\psi \otimes_{S[L]} S$ is a (\mathbb{Z}^n/L) -graded free resolution of the S -module $M_\psi \otimes_{S[L]} S$.*

The categorical arguments in [BS98, Theorem 3.2] and [MS05, Theorem 9.17] carry over essentially verbatim to our setting. We now outline an alternative proof that implicitly relates cohomology on the universal cover to that on the torus.

Sketch of proof. Fix α in the grading group \mathbb{Z}^n/L . The degree α part of the S -module $M \otimes_{S[L]} S$ equals \mathbb{k} if there exists a lattice point $u \in \mathbb{Z}^n$ such that $x^u \in M_\psi$ and equals 0 otherwise. Similarly, Proposition 2.13 establishes that the subcomplex $(L_{\mathbb{R}})_{\leq u}$ is acyclic if $x^u \in M_\psi$ and is empty otherwise. These equivalences are independent of the choice of u because both M_ψ and $L_{\mathbb{R}}$ are L -equivariant. It suffices to show that the degree α part of the S -complex $F_\psi \otimes_{S[L]} S$ is isomorphic to the degree u part of the $S[L]$ -complex F_ψ . One verifies that the canonical map from $(F_\psi)_u$ to $(F_\psi \otimes_{S[L]} S)_\alpha$ is an isomorphism by describing the image and preimage of basis vectors. \square

To leverage Corollary 2.15, another characterization of the S -module $M_\psi \otimes_{S[L]} S$ is needed. The canonical surjective \mathbb{Z} -module homomorphism $\eta: \mathbb{Z}^n \rightarrow \mathbb{Z}^n/L$ induces a surjective homomorphism of \mathbb{k} -algebras from the Laurent polynomial ring $R = \mathbb{k}[\mathbb{Z}^n]$ to the group algebra $\mathbb{k}[\mathbb{Z}^n/L]$: for any lattice point $u \in \mathbb{Z}^n$, the monomial $x^u \in R$ maps to the monomial $x^{\eta(u)} \in \mathbb{k}[\mathbb{Z}^n/L]$. Expanding on this identification gives the following description of $M_\psi \otimes_{S[L]} S$.

Proposition 2.16. *Let $\psi: L_{\mathbb{R}} \rightarrow \mathbb{Z}^n$ be a compatible \mathbb{Z}^n -stratification. The module $M_\psi \otimes_{S[L]} S$ is the S -submodule of the group algebra $\mathbb{k}[\mathbb{Z}^n/L]$ generated by*

$$\{x^{\eta \circ \psi(p)} \in \mathbb{k}[\mathbb{Z}^n/L] \mid \text{the point } p \text{ is a vertex in the cell complex } L_{\mathbb{R}}\}.$$

Proof. Since the extension-by-scalars functor is right exact, there exists a surjective S -module homomorphism from $R \otimes_{S[L]} S$ to $\mathbb{k}[\mathbb{Z}^n/L] \otimes_{S[L]} S$. We start by focusing on the target module. Using the $S[L]$ -module structure on R , the surjective \mathbb{k} -algebra homomorphism from R to $\mathbb{k}[\mathbb{Z}^n/L]$ endows the group algebra with an $S[L]$ -module structure. In particular, for any lattice point $v \in L$, the product $z^v \cdot x^u = x^{u+v} \in R$ maps to $x^{\eta(u+v)} = x^{\eta(u)} \in \mathbb{k}[\mathbb{Z}^n/L]$. Hence, multiplication by the monomial $z^v \in S[L]$ on the group algebra $\mathbb{k}[\mathbb{Z}^n/L]$ is trivial. We deduce that $\mathbb{k}[\mathbb{Z}^n/L] \otimes_{S[L]} S \cong \mathbb{k}[\mathbb{Z}^n/L]$.

Returning to the source module, the ring S becomes an $S[L]$ -module via $S \cong S[L]/\langle z^v - 1 \mid v \in L \rangle$. For any lattice point $u \in \mathbb{Z}^n$ and any lattice point $v \in L$, it follows that

$$x^{u+v} \otimes 1_S = x^u z^v \otimes 1_S = x^u \otimes 1_S$$

in $R \otimes_{S[L]} S$. Hence, two monomials in R are equivalent in the S -module $R \otimes_{S[L]} S$ if and only if their exponent vectors belong to the same coset in \mathbb{Z}^n/L . Having identified their monomial bases (as \mathbb{k} -vector spaces), we also deduce that $R \otimes_{S[L]} S \cong \mathbb{k}[\mathbb{Z}^n/L]$.

Lastly, Remark 2.11 shows that the S -module M_ψ is the submodule generated by the monomials $x^{\psi(p)} \in R$ for all vertices p in the cell complex $L_{\mathbb{R}}$. Since the extension-by-scalars functor is exact (see [BS98, Theorem 3.2] or [MS05, Theorem 9.17]), it follows that $M_\psi \otimes_{S[L]} S$ is the submodule generated by the monomials $x^{\eta \circ \psi(p)} \in \mathbb{k}[\mathbb{Z}^n/L]$ for all vertices p in the cell complex $L_{\mathbb{R}}$. \square

To illustrate a cellular free resolution of $M_\psi \otimes_{S[L]} S$, we revisit the earlier toric example.

Example 2.17 (Cellular free resolution for an embedding into a toric surface). As in Example 2.12, consider the inclusion of the identity point Y of the dense torus into the second Hirzebruch surface X with the compatible \mathbb{Z}^4 -stratification arising from the ceiling function. Let $S := \mathbb{k}[x_1, x_2, x_3, x_4]$ denote the Cox ring of X and choose the isomorphism $\text{Pic}(X) \cong \mathbb{Z}^2$, so that the degree of the variable x_i is identified with the i th column in the matrix

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

Figure 2.19 displays the twists of S corresponding to each open cell and the induced maps on the basis. The (nonminimal) cellular free resolution of the S -module $M_\psi \otimes_{S[L]} S$ is

$$\begin{array}{ccccc} S(0,0) & \xleftarrow{\quad \left[\begin{smallmatrix} x_4 & x_4 & -x_2x_3 & -x_1x_2 & x_3-x_1 \\ -x_3 & -x_1 & 1 & 1 & 0 \end{smallmatrix} \right]} & S(0,-1)^2 & \xleftarrow{\quad \left[\begin{smallmatrix} -x_1 & 1 & 0 \\ x_3 & -1 & 0 \\ 0 & -x_1 & 1 \\ 0 & x_3 & -1 \\ -x_4 & 0 & x_2 \end{smallmatrix} \right]} & S(-1,-1) \\ \oplus \downarrow & & \oplus \downarrow & & \oplus \downarrow \\ S(1,-1) & & S(1,-1)^2 & \xleftarrow{\quad \left[\begin{smallmatrix} -x_1 & 1 & 0 \\ x_3 & -1 & 0 \\ 0 & -x_1 & 1 \\ 0 & x_3 & -1 \\ -x_4 & 0 & x_2 \end{smallmatrix} \right]} & S(0,-1) \xleftarrow{0} S(1,-1) \end{array} \quad \diamond$$

To further illustrate cellular free resolutions, we present a more substantial example.

Example 2.18 (Cellular free resolution for a diagonal embedding). Consider the second Hirzebruch surface $Y := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2))$ with its diagonal embedding $\Delta: Y \rightarrow X := Y \times Y$. Following Remark 2.3 or [BPS01, Equation 3.2], the lattice arising from Δ is the Lawrence lifting

$$L = \Lambda(M_Y) := \{(v, -v) \in \mathbb{Z}^4 \times \mathbb{Z}^4 \mid v \in M_Y \subset \mathbb{Z}^4\} \subset \mathbb{Z}^8.$$

The lattice $L \cong M_Y$ has rank 2, and the matrix of the inclusion $\iota: \mathbb{Z}^2 \cong L \rightarrow \mathbb{Z}^8$, with respect to the standard bases, is

$$\begin{bmatrix} 1 & 0 & -1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 & -1 & -2 & 1 \end{bmatrix}^\top.$$

Let $\iota_{\mathbb{R}}: \mathbb{R}^2 \rightarrow \mathbb{R}^8$ be the linear map induced by $\iota: \mathbb{Z}^2 \rightarrow \mathbb{Z}^8$. The map $\psi: L_{\mathbb{R}} \rightarrow \mathbb{Z}^8$ defined by $\psi(\mathbf{p}) := \lceil \mathbf{p} \rceil$ for all $\mathbf{p} \in L_{\mathbb{R}} \subset \mathbb{R}^8$ is a compatible \mathbb{Z}^8 -stratification; see Proposition 3.2. For all $(q_1, q_2) \in \mathbb{R}^2$, we have

$$\begin{aligned} \psi(\iota_{\mathbb{R}}(q_1, q_2)) &= \lceil (q_1, q_2, -q_1 + 2q_2, -q_2, -q_1, -q_2, q_1 - 2q_2, q_2) \rceil \\ &= (\lceil q_1 \rceil, \lceil q_2 \rceil, \lceil -q_1 + 2q_2 \rceil, \lceil -q_2 \rceil, \lceil -q_1 \rceil, \lceil -q_2 \rceil, \lceil q_1 - 2q_2 \rceil, \lceil q_2 \rceil). \end{aligned}$$

The Cox ring of X is $S := \mathbb{k}[x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4]$, where $\text{Pic}(X) = \text{Pic}(Y) \oplus \text{Pic}(Y) \cong \mathbb{Z}^2 \oplus \mathbb{Z}^2$; see Example 2.17. Hence, the (minimal) cellular free resolution of the S -module $M_{\psi} \otimes_{S[L]} S$ is

$$\begin{array}{ccccc} & S(0, -1, 1, -1)^2 & & S(-1, -1, 1, -1) & \\ S(0, 0, 0, 0) & \xleftarrow{\partial_1} & \stackrel{\oplus}{S(1, -1, 0, -1)^2} & \xleftarrow{\partial_2} & S(0, -1, 0, -1) \xleftarrow{\oplus} 0, \\ \stackrel{\oplus}{S(1, -1, 1, -1)} & & \stackrel{\oplus}{S(-1, 0, -1, 0)} & & S(1, -1, -1, -1) \end{array}$$

with

$$\partial_1 = \begin{bmatrix} x_4y_2y_3 & x_4y_1y_2 & -x_2x_3y_4 & -x_1x_2y_4 & x_3y_1 - x_1y_3 \\ -x_3 & -x_1 & y_3 & y_1 & 0 \end{bmatrix} \quad \text{and} \quad \partial_2 = \begin{bmatrix} -x_1 & y_1 & 0 \\ x_3 & -y_3 & 0 \\ 0 & -x_1 & y_1 \\ 0 & x_3 & -y_3 \\ -x_4 & 0 & x_2y_4 \end{bmatrix}.$$

Taking associated sheaves, one obtains a locally-free resolution of the \mathcal{O}_X -module $\Delta_* \mathcal{O}_Y$. ◊

3. THE CEILING STRATIFICATION

This section exhibits a combinatorial free resolution for the integral closure of the comodule of a toric ideal. More precisely, for any prime lattice ideal I_L in the polynomial ring $S := \mathbb{k}[x_1, x_2, \dots, x_n]$, we produce a compatible \mathbb{Z}^n -stratification $\psi: L_{\mathbb{R}} \rightarrow \mathbb{Z}^n$ such that the cellular free S -complex $F_{\psi} \otimes_{S[L]} S$ is a (\mathbb{Z}^n/L) -graded resolution of the integral closure of the S -module S/I_L .

The lattice L is still a subgroup of \mathbb{Z}^n . We further assume that L is *saturated*, which means $L = \{ \mathbf{u} \in \mathbb{Z}^n \mid k \mathbf{u} \in L \text{ for some } k \in \mathbb{N} \}$. This saturated hypothesis ensures that the *lattice ideal*

$$I_L := \langle \mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \mid \mathbf{u} - \mathbf{v} \in L \text{ for all } \mathbf{u}, \mathbf{v} \in \mathbb{N}^n \rangle$$

is a prime ideal in the polynomial ring S ; see [MS05, Theorem 7.4 and Definition 7.24]. A prime lattice ideal is also known as a *toric ideal*.

In this context, the function on the space $L_{\mathbb{R}}$ that assigns the least integer greater than or equal to each coordinate deserves a formal name.

Definition 3.1. For any lattice $L \in \mathbb{Z}^n$, the *ceiling stratification* is the map $\psi: L_{\mathbb{R}} \rightarrow \mathbb{Z}^n$ defined by $\psi(\mathbf{p}) := \lceil \mathbf{p} \rceil = (\lceil p_1 \rceil, \lceil p_2 \rceil, \dots, \lceil p_n \rceil)$ for any point $\mathbf{p} \in L_{\mathbb{R}} \subseteq \mathbb{R}^n$.

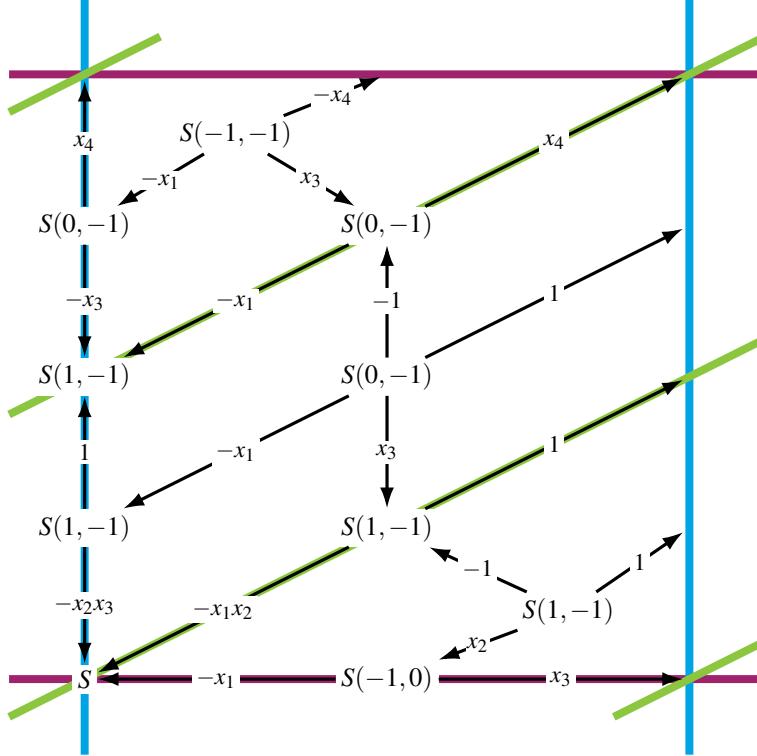


Figure 2.19. A diagram encoding the cellular free S -complex arising from the closed embedding of the identity point into the second Hirzebruch surface.

Example 2.8 presents a ceiling stratification. In the special case of toric varieties, this stratification appears implicitly in [Bon06] and explicitly in [HHL24, §2.3].

The standard cell structure is, in fact, the coarsest cell structure on the space $L_{\mathbb{R}}$ for which the coordinatewise ceiling function is constant on open cells.

Proposition 3.2. *Let $\psi: L_{\mathbb{R}} \rightarrow \mathbb{Z}^n$ be the ceiling stratification for a lattice $L \subseteq \mathbb{Z}^n$. When $L_{\mathbb{R}}$ has its standard cell structure, the map ψ is a compatible \mathbb{Z}^n -stratification, and the cellular free S -complex F_{ψ} is a \mathbb{Z}^n -graded free resolution of the \mathbb{Z}^n -graded S -module M_{ψ} .*

Proof. For each real number p and integer v , the ceiling function satisfies $\lceil p + v \rceil = \lceil p \rceil + v$. It follows that, for all points p in $L_{\mathbb{R}}$ and all lattice points $v \in L$, we have

$$\psi(p + v) = \lceil p + v \rceil = \lceil p \rceil + v = \psi(p) + v.$$

Thus, the ceiling stratification is compatible with lattice translation.

The set \mathbb{Z}^n is a poset under the componentwise order. Hence, for any lattice point $u \in \mathbb{Z}^n$, the subset $(L_{\mathbb{R}})_{\leq u} := \{p \in L_{\mathbb{R}} \mid \psi(p) = \lceil p \rceil \leq u\}$ is the intersection of $L_{\mathbb{R}}$ with $u + (\mathbb{R}_{\leq 0})^n$. Because the space $L_{\mathbb{R}}$ has the standard cell structure, the subset $(L_{\mathbb{R}})_{\leq u}$ is closed. It follows that the ceiling stratification is a continuous map that is constant on open cells. The subset $(L_{\mathbb{R}})_{\leq u}$ is a polyhedron and, thereby, contractible or empty. Hence, Proposition 2.13 establishes that the cellular free S -complex F_{ψ} is a resolution of the S -module M_{ψ} . \square

For a ceiling stratification, the monomial S -module $M_{\psi} \otimes_{S[L]} S$ has an appealing reinterpretation.

Proposition 3.3. *For the ceiling stratification $\psi: L_{\mathbb{R}} \rightarrow \mathbb{Z}^n$ of a saturated lattice $L \subseteq \mathbb{Z}^n$, the S -module $M_{\psi} \otimes_{S[L]} S$ is the integral closure of the S -module S/I_L .*

Proof. Consider the quotient semigroup $A := \mathbb{N}^n / \sim_L$, where the equivalence relation \sim_L on \mathbb{N}^n is defined by $\mathbf{u} \sim_L \mathbf{v}$ if and only if $\mathbf{u} - \mathbf{v} \in L$. The canonical surjection $\eta: \mathbb{Z}^n \rightarrow \mathbb{Z}^n / L$ induces a surjective \mathbb{k} -algebra map $S \rightarrow \mathbb{k}[A]$ with kernel I_L , so $S/I_L \cong \mathbb{k}[A]$; see [MS05, Theorem 7.3]. The integral closure of $\mathbb{k}[A]$ is generated, as a \mathbb{k} -algebra, by the saturation \bar{A} ; see [MS05, Theorem 7.25]. We claim that \bar{A} is generated over A by the elements $\eta([\mathbf{p}])$, where \mathbf{p} runs over the vertices of $L_{\mathbb{R}}$.

To see this, let $\mathbf{p} \in L_{\mathbb{R}}$ be a vertex. We first establish that $\eta([\mathbf{p}]) \in \bar{A} = (\mathbb{Z}^n / L) \cap (\mathbb{R}_{\geq 0} A)$. Since the target of η is \mathbb{Z}^n / L , it suffices to prove that $\eta([\mathbf{p}]) \in (\mathbb{R}_{\geq 0} A)$. Consider the \mathbb{R} -linear extension $\eta_{\mathbb{R}} := \eta \otimes_{\mathbb{Z}} \text{id}_{\mathbb{R}}: \mathbb{Z}^n \otimes_{\mathbb{Z}} \mathbb{R} \rightarrow (\mathbb{Z}^n / L) \otimes_{\mathbb{Z}} \mathbb{R}$. Since $\mathbf{p} \in L_{\mathbb{R}}$, we see that $\eta_{\mathbb{R}}(\mathbf{p}) = 0$. Moreover, the difference $[\mathbf{p}] - \mathbf{p}$ has nonnegative entries, so $\eta_{\mathbb{R}}([\mathbf{p}] - \mathbf{p}) \in (\mathbb{R}_{\geq 0} A)$. Thus, we obtain

$$\eta([\mathbf{p}]) = \eta([\mathbf{p}]) - \eta_{\mathbb{R}}(\mathbf{p}) = \eta_{\mathbb{R}}([\mathbf{p}] - \mathbf{p}) \in (\mathbb{R}_{\geq 0} A).$$

We next establish that the inclusion $\bar{A} \subseteq A + \{\eta([\mathbf{p}]) \mid \text{the point } \mathbf{p} \text{ is a vertex in } L_{\mathbb{R}}\}$. Let $\mathbf{a} \in \bar{A}$. By construction, there exist $\mathbf{q} \in \mathbb{R}_{\geq 0}^n$ and $\mathbf{u} \in \mathbb{Z}^n$ such that $\mathbf{a} = \eta_{\mathbb{R}}(\mathbf{q}) = \eta(\mathbf{u})$. Set $\mathbf{b} := \eta([\mathbf{q}]) \in A$ and $\mathbf{p} := \mathbf{u} - \mathbf{q}$. Since the kernel of \mathbb{R} -linear map $\eta_{\mathbb{R}}$ is $L_{\mathbb{R}}$ and

$$\eta_{\mathbb{R}}(\mathbf{p}) = \eta_{\mathbb{R}}(\mathbf{u} - \mathbf{q}) = \eta(\mathbf{u}) - \eta_{\mathbb{R}}(\mathbf{q}) = \mathbf{a} - \mathbf{b} = 0,$$

it follows that $\mathbf{p} \in L_{\mathbb{R}}$ and $\eta([\mathbf{p}]) = \eta([\mathbf{u} - \mathbf{q}]) = \eta(\mathbf{u}) + \eta([\mathbf{-q}]) = \eta(\mathbf{u}) - \eta([\mathbf{q}]) = \mathbf{a} - \mathbf{b}$. If $\mathbf{p} \in L_{\mathbb{R}}$ is not a vertex, then it is contained in the relative interior of a cell of $L_{\mathbb{R}}$. In particular, there exists a minimal-dimensional integral unit parallelepiped P in \mathbb{R}^n containing \mathbf{p} such that $P \cap L_{\mathbb{R}}$ coincides with the minimal-dimensional closed cell containing \mathbf{p} . For any vertex \mathbf{p}' with $\mathbf{p}' \in L_{\mathbb{R}} \cap P$, we have $[\mathbf{p}] - [\mathbf{p}'] \in \mathbb{N}^n$. It follows that $\mathbf{c} := \eta([\mathbf{p}] - [\mathbf{p}'])$ is in A and

$$\eta([\mathbf{p}']) = \eta([\mathbf{p}]) - \mathbf{c} = \mathbf{a} - \mathbf{b} - \mathbf{c}.$$

We deduce that $\mathbf{a} = \mathbf{b} + \mathbf{c} + \eta([\mathbf{p}'])$, where $\mathbf{b} + \mathbf{c} \in A$ and \mathbf{p}' is a vertex in $L_{\mathbb{R}}$.

Finally, Proposition 2.16 establishes that the S -module $M_{\psi} \otimes_{S[L]} S$ is generated in $\mathbb{k}[\mathbb{Z}^n / L]$ by the monomials $x^{\eta \circ \psi(\mathbf{p})} = x^{\eta([\mathbf{p}])}$, where \mathbf{p} runs over the vertices of $L_{\mathbb{R}}$. Since saturation of a semigroup corresponds to integral closure of its semigroup ring, we conclude that

$$M_{\psi} \otimes_{S[L]} S \cong \mathbb{k}[\bar{A}] = \overline{\mathbb{k}[A]} \cong \overline{S/I_L}. \quad \square$$

Combined with Corollary 2.15, this leads to the desired cellular free resolution for $\overline{S/I_L}$.

Theorem 3.4. *For the ceiling stratification $\psi: L_{\mathbb{R}} \rightarrow \mathbb{Z}^n$ of a saturated lattice $L \subseteq \mathbb{Z}^n$, the cellular free S -complex $F_{\psi} \otimes_{S[L]} S$ is a resolution of the integral closure of the S -module S/I_L .*

Proof. Corollary 2.15 and Proposition 3.2 establish that the cellular free S -complex $F_{\psi} \otimes_{S[L]} S$ is a (\mathbb{Z}^n / L) -graded resolution of the S -module $M_{\psi} \otimes_{S[L]} S$, and Proposition 3.3 shows that the S -module $M_{\psi} \otimes_{S[L]} S$ is isomorphic to the integral closure of the S -module S/I_L . \square

Proof of Theorem 1.1. This result is an informal version of Theorem 3.4. \square

Example 3.5 (Integral closure of a concrete toric ideal). As in Example 2.18, consider the diagonal embedding of the second Hirzebruch surface and the compatible stratification $\psi: L_{\mathbb{R}} \rightarrow \mathbb{Z}^8$ defined by $\psi(\mathbf{p}) = [\mathbf{p}]$. The corresponding toric ideal in $S := \mathbb{k}[x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4]$ is

$$I_L = \langle x_3y_1 - x_1y_3, x_4y_2y_3^2 - x_2x_3^2y_4, x_4y_1y_2y_3 - x_1x_2x_3y_4, x_4y_1^2y_2 - x_1^2x_2y_4 \rangle.$$

The element $x_2x_3y_3^{-1}y_4$ of the fraction field of S/I_L is a root of the monic polynomial $w^2 - x_2x_4y_2y_4$ and, thus, contained in the integral closure. This element corresponds to the half-integer vertex:

$$\psi(\iota_{\mathbb{R}}(0,0.5)) = \lceil(0,0.5,1,-0.5,0,-0.5,-1,0.5)\rceil = (0,1,1,0,0,0,-1,1). \quad \diamond$$

Remark 3.6 (Unimodular case). When the lattice $L \subseteq \mathbb{Z}^m$ is unimodular, the ceiling stratification on the Lawrence lift $\Lambda(L) \subseteq \mathbb{Z}^{2m}$ of L agrees with the labeling in [BPS01, §3]. Hence, Theorem 3.4 recovers, as a special case, the resolution of the diagonal via the Lawrence ideal for a unimodular toric variety in [BPS01, §6].

Remark 3.7 (Eliminating the saturated assumption). While the lattice L is assumed to be saturated in Theorem 3.4, the conclusion of the theorem remains valid even without this assumption. However, some care is required in formulating the correct generalization. When L is not saturated, the quotient ring S/I_L need not be a domain. Consequently, the integral closure of S/I_L should be taken within its total ring of fractions.

When L is not saturated, the quotient semigroup $A := \mathbb{N}^n / \sim_L$ is not affine, so one must refine the notion of saturation. The group completion \mathbb{Z}^n / L of A is a finitely generated abelian group, decomposing as the direct sum of a free abelian group and a torsion group: $\mathbb{Z}^n / L \cong \mathbb{Z}^r \oplus G_{\text{tor}}$. Projecting onto the torsion-free quotient, let A' denote the image of A in \mathbb{Z}^r . In this setting, the saturation of A is $\overline{A'} \oplus G_{\text{tor}}$. With these definitions, one verifies that the integral closure of the semigroup ring coincides the semigroup ring of its saturation. Moreover, the monomials corresponding to $\eta([\mathbf{p}]) \in \mathbb{Z}^n / L$, where \mathbf{p} runs over the vertices of $L_{\mathbb{R}}$, still generate the saturation.

4. APPLICATIONS OF THE CEILING STRATIFICATION

In this section, we use ceiling stratifications to describe resolutions of sheaves on a smooth toric variety. We relate the cellular free resolutions in Theorem 3.4 to the locally-free resolutions in [HHL24, Theorem A] and the minimal free resolutions in [BE24, Theorem 1.2]. In particular, we show that, for a smooth toric variety, the resolutions of the diagonal from [HHL24] and [BE24] coincide with the \mathcal{O}_X -complex associated to a cellular free resolution.

Throughout, the morphism $\varphi: Y \rightarrow X$ is a torus-equivariant embedding of a normal toric variety Y into a smooth toric variety X with no torus factors. The polynomial ring $S := \mathbb{k}[x_1, x_2, \dots, x_n]$ is the Cox ring of X . Following Remark 2.3, the lattice arising from the map φ is $L := \text{Ker}(\overline{\varphi}^*) \subseteq \mathbb{Z}^n$.

Recovering Hanlon–Hicks–Lazarev resolutions. Given $\varphi: Y \rightarrow X$, [HHL24, Theorem 3.5] produces a combinatorial resolution of the sheaf $\varphi_*\mathcal{O}_Y$ in terms of direct sums of line bundles on X . As explained in [HHL24, §3.4], the relevant chain complex of sheaves is obtained from a cell structure (also referred to as topological ‘stratification’) on the real torus corresponding to an ‘exit path category.’ Moreover, [HHL24, Proposition 3.9] introduces a functor from this exit path category to the category of coherent sheaves on X that assigns a line bundle to each open cell.

The subsequent lemma aligns the ceiling stratification in Proposition 3.2 with the categorical construction in [HHL24, §3]. As in Remark 2.3, we identify \mathbb{Z}^n with the group of torus-invariant divisors on X . For any lattice point $\mathbf{u} \in \mathbb{Z}^n$, we write $\mathcal{O}_X(\mathbf{u})$ for the corresponding line bundle.

Lemma 4.1. *The ceiling stratification $\psi: L_{\mathbb{R}} \rightarrow \mathbb{Z}^n$ and the construction in [HHL24, §3.4] agree: the induced cell structures on the real torus $L_{\mathbb{R}} / L$ coincide, and the line bundle assigned to the cell $\sigma' \subset L_{\mathbb{R}} / L$ is $\mathcal{O}_X(-\psi(\sigma))$, where the cell $\sigma \subset L_{\mathbb{R}}$ represents the coset σ' .*

Proof. As in Remark 2.3, the toric morphism $\varphi: Y \rightarrow X$ corresponds to a surjective \mathbb{Z} -linear map $\overline{\varphi}^*: M_X \rightarrow M_Y$ between the character lattices of X and Y , and $L := \text{Ker}(\overline{\varphi}^*)$. In [HHL24, §3.4], the real torus is defined to be the kernel of the induced map $(M_X \otimes_{\mathbb{Z}} \mathbb{R})/M_X \rightarrow (M_Y \otimes_{\mathbb{Z}} \mathbb{R})/M_Y$ on tori. Hence, this real torus is precisely the real topological torus $L_{\mathbb{R}}/L$.

We next analyze the cell structures on $L_{\mathbb{R}}/L$. Following Remark 2.3 or Proposition 3.2, the space $L_{\mathbb{R}}$ carries the standard cell structure. The inclusion $v: M_X \rightarrow \mathbb{Z}^n$ sends a character $m \in M_X$ to the lattice point in \mathbb{Z}^n whose i th entry is $u_i \cdot m$, where u_i is the primitive lattice point generating the i th ray in the fan of X ; see [CLS11, Proposition 4.1.2]. The composition of the canonical inclusion $\kappa: L \rightarrow M_X$ with v realizes the lattice L as a subgroup of \mathbb{Z}^n . By Definition 2.2, the standard cell structure on $L_{\mathbb{R}}$ is determined by restricting the periodic arrangement of the standard basis vectors in \mathbb{R}^n . As explained in Remark 2.1, this polyhedral cell structure on $L_{\mathbb{R}}$ arises from the hyperplanes of the form $\{p \in L_{\mathbb{R}} \mid u_i \cdot p = j\}$, where $1 \leq i \leq n$ and j is an integer. Under the quotient map $\pi: L_{\mathbb{R}} \rightarrow L_{\mathbb{R}}/L$, this periodic arrangement descends to a toric arrangement on the real torus $L_{\mathbb{R}}/L$. On the other hand, the topological stratification in [HHL24, §3.4] is also induced by the toric arrangement $L_{\mathbb{R}}/L$ consisting of the subtori determined by the linear forms u_i for all $1 \leq i \leq n$. We deduce that the two induced cell structures on the real torus $L_{\mathbb{R}}/L$ are identical.

It remains to relate the line bundles in [HHL24, Proposition 3.5] to the ceiling stratification $\psi: L_{\mathbb{R}} \rightarrow \mathbb{Z}^n$. For each point $q \in M_X \otimes_{\mathbb{Z}} \mathbb{R}$, [HHL24, Equation (6) in §2.3] assigns the line bundle

$$\mathcal{O}_X(([-u_1 \cdot q], [-u_2 \cdot q], \dots, [-u_n \cdot q])) = \mathcal{O}_X(-\lceil v_{\mathbb{R}}(q) \rceil),$$

where $v_{\mathbb{R}}: M_X \otimes_{\mathbb{Z}} \mathbb{R} \rightarrow \mathbb{R}^n$ is the linear extension of $v: M_X \rightarrow \mathbb{Z}^n$. Since $\iota := v \circ \kappa$ realizes the lattice L as a subgroup of \mathbb{Z}^n , its linear extension $\iota_{\mathbb{R}}$ realizes $L_{\mathbb{R}}$ as a linear subspace of \mathbb{R}^n . By construction, the ceiling function is constant on the open cells in $L_{\mathbb{R}}$, so the line bundle assigned to every point in σ is $\mathcal{O}_X(-\psi(\sigma))$. Passing to the real torus $L_{\mathbb{R}}/L$, we see that the ceiling stratification and the functor in [HHL24, Proposition 3.5] assign the same line bundle to each open cell. \square

With the relevant building blocks in place, we compare resolutions of the sheaf $\varphi_* \mathcal{O}_Y$. Following [CLS11, §5.3], each finitely generated S -module is associated to a quasi-coherent \mathcal{O}_X -module.

Theorem 4.2. *Let $\varphi: Y \rightarrow X$ be a torus-equivariant embedding of a normal toric variety Y into a smooth toric variety X with no torus factors, and let $L \subseteq \mathbb{Z}^n$ be the lattice arising from φ . For the ceiling stratification $\psi: L_{\mathbb{R}} \rightarrow \mathbb{Z}^n$, the \mathcal{O}_X -complex associated to the S -complex $F_{\psi} \otimes_{S[L]} S$ is isomorphic to the resolution of the \mathcal{O}_X -module $\varphi_* \mathcal{O}_Y$ in [HHL24, Theorem 3.5].*

Proof. For any lattice point $u \in \mathbb{Z}^n$, the coherent sheaf associated to the \mathbb{Z}^n -graded S -module $S(u)$ is the torus-equivariant line bundle $\mathcal{O}_X(u)$; see [CLS11, Proposition 5.3.7]. From the description of the S -complex $F_{\psi} \otimes_{S[L]} S$ in Remark 2.14, it follows that the i th term in its associated \mathcal{O}_X -complex is

$$\bigoplus_{\substack{\sigma' \subset L_{\mathbb{R}}/L \\ \dim \sigma' = i}} \mathcal{O}_X(-\psi(\sigma)),$$

where σ is a representative in $L_{\mathbb{R}}$ for the coset σ' in the quotient $L_{\mathbb{R}}/L$. Combining Lemma 4.1 with [HHL24, Equation (18) in §A.1] shows that this direct sum is also the i th term in the resolution in [HHL24, Theorem 3.5].

We next examine the differentials. Following Definition 2.10, the monomials $x^{\psi(\sigma)-\psi(\tau)} \in S$ appearing in the differentials determine the unique homomorphism from $S(-\psi(\sigma))$ to $S(-\psi(\tau))$

having degree $\mathbf{0} \in \mathbb{Z}^n$. Moreover, monomials in the Cox ring S are identified with global sections of line bundles; see [CLS11, Proposition 5.3.7]. Hence, the associated-sheaf functor sends the matrix $[x^{\psi(\sigma)-\psi(\tau)}]$ to the unique torus-equivariant morphism from $\mathcal{O}_X(-\psi(\sigma))$ to $\mathcal{O}_X(-\psi(\tau))$. On the other hand, [HHL24, Proposition 2.14] identifies the components of the differentials as the unique degree-0 maps between relevant summands of the direct sum. Therefore, the associated-sheaf functor applied to the differentials in Remark 2.14 agrees, up to scalars, with the differentials in [HHL24, Equation (19) in §A.1].

Finally, Remark 2.14 shows that the scalars in the differentials of the S -complex $F_\psi \otimes_{S[L]} S$ are determined by the incidence function ε . The construction in [HHL24, §3.5] allows for an arbitrary choice of ‘orientation.’ As observed in the paragraph following [HHL24, Example A.3], different choices of incidence functions produce isomorphic \mathcal{O}_X -complexes. \square

Remark 4.3. Since the construction of the S -complex $F_\psi \otimes_{S[L]} S$ applies in all characteristics, Theorem 4.2 implies that the construction of the \mathcal{O}_X -complex in [HHL24, §3.4] is also independent of the characteristic of the underlying field \mathbb{k} ; compare with [B+24, Remark 6.2].

Remark 4.4. Unlike Theorem 4.2, [HHL24, Theorem 3.5] relaxes the hypotheses on the ambient space X , allowing smooth toric stacks. For brevity, we again leave the variant of Theorem 4.2 for smooth toric stacks to the interested reader.

Reversing direction, the S -complex associated to the resolution in [HHL24, Theorem 3.5] yields a cellular free resolution. Following [CLS11, Definition 6.A.1], set

$$\Gamma_*(\mathcal{F}) := \bigoplus_{\mathbf{u} \in \text{Pic}(X)} H^0(X, \mathcal{F}(\mathbf{u})) = \bigoplus_{\mathbf{u} \in \text{Pic}(X)} H^0(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(\mathbf{u}))$$

for any sheaf \mathcal{F} of \mathcal{O}_X -modules on X .

Corollary 4.5. *The functor Γ_* applied to the resolution of $\varphi_* \mathcal{O}_Y$ in [HHL24, Theorem 3.5] is isomorphic to the S -complex $F_\psi \otimes_{S[L]} S$ associated to the ceiling stratification $\psi: L_{\mathbb{R}} \rightarrow \mathbb{Z}^n$.*

Sketch of proof. To capitalize on the torus action, assume that $S := \mathbb{k}[x_1, x_2, \dots, x_n]$ has the canonical \mathbb{Z}^n -grading and that all line bundles on X have a chosen torus-equivariant structure. For any lattice point $\mathbf{u} \in \mathbb{Z}^n$, [CLS11, Proposition 5.3.7] demonstrates that $\Gamma_*(\mathcal{O}_X(\mathbf{u})) = S(\mathbf{u})$, which implies that the terms of the complexes are identical. Since

$$\mathcal{H}\text{om}(\mathcal{O}_X(\mathbf{u}), \mathcal{O}_X(\mathbf{v})) \cong \mathcal{H}\text{om}(\mathcal{O}_X, \mathcal{O}_X(\mathbf{v} - \mathbf{u})) \cong \mathcal{O}_X(\mathbf{v} - \mathbf{u}),$$

torus-equivariant maps from $\mathcal{O}_X(\mathbf{u})$ to $\mathcal{O}_X(\mathbf{v})$ are the same as monomial maps from $S(\mathbf{u})$ to $S(\mathbf{v})$, which are unique up to scaling. Thus, the differentials agree up to a choice of incidence function. \square

Relation to Brown–Erman resolutions. Addressing a challenge raised in [BE24, p. 3], cellular free resolutions allow a comparison between the minimal free resolutions in [BE24, Theorem 1.2] and the locally-free resolutions in [HHL24, Theorem 3.5]. The next corollary shows that a resolution from [BE24, Theorem 1.2] is always isomorphic to a direct summand of one from [HHL24, Theorem 3.5].

Corollary 4.6. *Let $\varphi: Y \rightarrow X$ be a torus-equivariant embedding of a normal toric variety Y into a smooth toric variety X with no torus factors. When F is a minimal free resolution of the integral closure of the S -module S/I_L , the \mathcal{O}_X -complex associated to F is isomorphic to a direct summand of the resolution in [HHL24, Theorem 3.5].*

Proof. Let L be the sublattice of \mathbb{Z}^n in Remark 2.3, and let $\psi: L \rightarrow \mathbb{Z}^n$ denote the ceiling stratification. Theorem 3.4 establishes that the S -complex $F_\psi \otimes_{S[L]} S$ is a free resolution of the S -module S/I_L . Being a minimal free resolution of S/I_L , it follows that F is isomorphic to a direct summand of $F_\psi \otimes_{S[L]} S$. Theorem 4.2 shows that the \mathcal{O}_X -complex associated to $F_\psi \otimes_{S[L]} S$ is isomorphic to the resolution of the \mathcal{O}_X -module $\varphi_* \mathcal{O}_Y$ in [HHL24, Theorem 3.5]. Since the associated-sheaf functor commutes with finite direct sums, we conclude that the \mathcal{O}_X -complex associated to F is isomorphic to a direct summand of the resolution in [HHL24, Theorem 3.5]. \square

We also determine when a cellular free resolution is minimal. The criterion for minimality applies to every compatible stratification and is a minor variant of [BS98, Remark 1.4].

Lemma 4.7. *Let $\psi: L \rightarrow \mathbb{Z}^n$ be a compatible stratification. The cellular free S -complex F_ψ is minimal if and only if, for every lattice point $\mathbf{u} \in \mathbb{Z}^n$, each connected component of the preimage $\psi^{-1}(\mathbf{u})$ consists of a single open cell.*

Proof. A free S -complex is minimal if and only if no entry in the matrices of the differentials is a unit. From Definition 2.10, it follows that F_ψ is minimal if and only if, for all open cells $\sigma, \tau \subset L_{\mathbb{R}}$ such that τ lies in the closure of σ and $\dim \sigma = 1 + \dim \tau$, we have $\psi(\sigma) \neq \psi(\tau)$.

Fix a lattice point $\mathbf{u} \in \mathbb{Z}^n$. Suppose that the open cell σ is contained in the preimage $\psi^{-1}(\mathbf{u})$. When F_ψ is minimal, none of the open cells τ satisfying $\tau \subseteq \bar{\sigma}$ and $\dim \sigma = 1 + \dim \tau$ are contained in $\psi^{-1}(\mathbf{u})$. Hence, the connected component of this preimage that contains σ contains no other cells. Conversely, if each connected component of the preimage $\psi^{-1}(\mathbf{u})$ is a single open cell, then no open cell in the closure has the same value under ψ . \square

Remark 4.8. When each preimage of ψ lies in the closure of a single open cell, the criterion in Lemma 4.7 reduces to verifying that the preimages are relatively open.

One important family of embeddings always produces minimal free resolutions. Assume that Y is a smooth toric variety with no torus factors and let m be the number of rays in its fan. The product $X := Y \times Y$ is also a smooth toric variety with no torus factors and $n := 2m$ rays in its fan. As in Example 2.18, consider the diagonal embedding $\Delta: Y \rightarrow X$, where the lattice is

$$L := \Lambda(M_Y) := \{(\mathbf{v}, -\mathbf{v}) \in \mathbb{Z}^m \times \mathbb{Z}^m \mid \mathbf{v} \in M_Y\} \subset \mathbb{Z}^n.$$

The ceiling stratification $\psi: L_{\mathbb{R}} \rightarrow \mathbb{Z}^n$ is defined by $\psi(\mathbf{q}, -\mathbf{q}) = (\lceil \mathbf{q} \rceil, \lceil -\mathbf{q} \rceil) = (\lceil \mathbf{q} \rceil, -\lfloor \mathbf{q} \rfloor)$ for all $\mathbf{q} \in (M_Y \otimes_{\mathbb{Z}} \mathbb{R}) \subseteq \mathbb{R}^m$.

Theorem 4.9. *Let Y be a smooth toric variety with no torus factors, and let S denote the Cox ring of the product $Y \times Y$. When ψ is the ceiling stratification for the diagonal embedding $\Delta: Y \rightarrow Y \times Y$, the cellular S -complex $F_\psi \otimes_{S[L]} S$ is a minimal free resolution.*

Proof. Since Theorem 3.4 establishes that this cellular free S -complex is a resolution of the integral closure of the S -module S/I_L , it suffices to show that $F_\psi \otimes_{S[L]} S$ is minimal.

We first analyze the closed subset $(L_{\mathbb{R}})_{\leqslant(\mathbf{u}, \mathbf{v})}$, where $(\mathbf{u}, \mathbf{v}) \in \mathbb{Z}^{2m}$. Note that $\psi(\mathbf{q}, -\mathbf{q}) = (\mathbf{u}, \mathbf{v})$ if and only if $u_i = \lceil q_i \rceil$ and $v_i = \lceil -q_i \rceil$ for all $1 \leq i \leq m$. Since

$$\lceil q_i \rceil + \lceil -q_i \rceil = \begin{cases} 0 & \text{if } q_i \in \mathbb{Z} \\ 1 & \text{if } q_i \notin \mathbb{Z}, \end{cases}$$

it follows that either $u_i + v_i = 0$, $u_i + v_i = 1$, or the subset $(L_{\mathbb{R}})_{\leqslant(u,v)}$ is empty. Fixing the index i , the first case corresponds to the subset $\{p \in L_{\mathbb{R}} \mid q_i = u_i\}$ and the second corresponds to the subset $\{p \in L_{\mathbb{R}} \mid u_i - 1 < q_i < u_i\}$. Since each nonempty $(L_{\mathbb{R}})_{\leqslant(u,v)}$ is the finite intersection of such subsets, it is relatively open.

Combining Lemma 4.7 and Remark 4.8 establishes the minimality of the $S[L]$ -complex F_{ψ} . From the description of the differentials in Remark 2.14, we deduce that $F_{\psi} \otimes_{S[L]} S$ is also minimal. \square

Corollary 4.10. *Let Y be a smooth toric variety with no torus factors, and let S be the Cox ring of $X := Y \times Y$. When $\Delta: Y \rightarrow X$ is the diagonal embedding, the resolutions of the \mathcal{O}_X -module $\Delta_* \mathcal{O}_Y$ in [HHL24, Theorem 3.5] and [BE24, Theorem 1.2] are both isomorphic to the \mathcal{O}_X -complex associated to the minimal cellular free resolution of the integral closure of the S -module S/I_L .*

Proof. As in Remark 2.3, let $L \subseteq \mathbb{R}^n$ be the lattice arising from Δ , and let $\psi: L_{\mathbb{R}} \rightarrow \mathbb{Z}^n$ be the ceiling stratification. By Corollary 4.6, the resolution of the \mathcal{O}_X -module $\Delta_* \mathcal{O}_Y$ in [HHL24, Theorem 3.5] is isomorphic to the \mathcal{O}_X -complex associated to the S -complex $F_{\psi} \otimes_{S[L]} S$. Theorem 3.4 and Theorem 4.9 show that $F_{\psi} \otimes_{S[L]} S$ is a minimal free resolution of the integral closure of the S -module S/I_L . The resolution of $\Delta_* \mathcal{O}_Y$ in [BE24, Theorem 1.2] is the \mathcal{O}_X -module associated to the minimal free resolution of the integral closure of the S -module S/I_L , which completes the proof. \square

5. APPLICATIONS TO OTHER RESOLUTIONS OF THE DIAGONAL

By finding suitable compatible \mathbb{Z}^n -stratifications, this section reinterprets the resolutions of the diagonal in [And24, Theorem 1.1] and [FH25, Example 3.15] via cellular free resolutions.

Sheaves arising from nonceiling stratifications. We first demonstrate that Proposition 3.3 does not extend to all compatible \mathbb{Z}^m -stratifications. Precomposing a ceiling stratification with a translation by a real vector produces the desired counterexample.

Example 5.1 (A module derived from a nonceiling stratification). Let $S = \mathbb{k}[x_1, x_2, y_1, y_2]$ be the Cox ring of $\mathbb{P}^1 \times \mathbb{P}^1$. The lattice $L \subset \mathbb{Z}^4$ generated by the vector $(1, -1, -1, 1)$ determines the toric ideal $I_L = \langle x_1 y_2 - x_2 y_1 \rangle$, and this ideal cuts out the image of the diagonal embedding $\Delta: \mathbb{P}^1 \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$; compare with Example 2.18. Endow the space $L_{\mathbb{R}} \cong \mathbb{R}^1$ with an \mathbb{Z} -equivariant cell complex with a fundamental domain having the three vertices and the two 1-cells that correspond to the points 0, 0.5, and 1, and the open intervals $(0, 0.5)$ and $(0.5, 1)$ in \mathbb{R}^1 . Denote by $\iota_{\mathbb{R}}: \mathbb{R}^1 \rightarrow \mathbb{R}^4$ the inclusion with image $L_{\mathbb{R}}$.

One verifies that the map $\psi: L_{\mathbb{R}} \rightarrow \mathbb{Z}^4$ defined by $\psi(\iota_{\mathbb{R}}(q)) := \lceil (q - 0.5, -q, -q - 0.5, q) \rceil$ for all $q \in \mathbb{R}^1$ is a compatible \mathbb{Z}^4 -stratification. Proposition 2.16 shows that the S -module $M_{\psi} \otimes_{S[L]} S$ is generated by the monomials 1 and $y_1^{-1} y_2$ in the group algebra $\mathbb{k}[\mathbb{Z}^4/L]$. It follows that the sheaf on $\mathbb{P}^1 \times \mathbb{P}^1$ associated to $M_{\psi} \otimes_{S[L]} S$ coincides with the one associated to $(S/I_L)(0, 1)$. The latter is not the sheaf associated to S/I_L , which is the same as the sheaf associated to S/I_L . \diamond

Despite this example, the S -modules $M_{\psi} \otimes_{S[L]} S$ and S/I_L retain a geometric connection. For the remainder of this subsection, let $\varphi: Y \rightarrow X$ be a torus-equivariant embedding of a normal toric variety Y into a smooth toric variety X with no torus factors. The Cox ring of X is the polynomial ring $S := \mathbb{k}[x_1, x_2, \dots, x_n]$. As in Remark 2.3, $L \subseteq \mathbb{Z}^n$ is the lattice arising from φ .

Lemma 5.2. *For any compatible \mathbb{Z}^n -stratification $\psi: L_{\mathbb{R}} \rightarrow \mathbb{Z}^n$, the \mathcal{O}_X -modules associated to the S -modules $M_{\psi} \otimes_{S[L]} S$ and S/I_L coincide on the dense algebraic torus in X . Moreover, the support of the sheaf associated to $M_{\psi} \otimes_{S[L]} S$ equals the subvariety in X cut out by the toric ideal I_L .*

Proof. Since $R := \mathbb{k}[x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_n^{\pm 1}] = \mathbb{k}[\mathbb{Z}^n]$, [CLS11, Proposition 3.3.11 and Proposition 5.3.3] show that the coordinate ring of the dense algebraic torus in X is $(S_{x_1 x_2 \dots x_n})_0 = (R)_0$. Hence, it suffices to prove that the modules $(M_{\psi} \otimes_{S[L]} S) \otimes_S R$ and $(S/I_L) \otimes_S R$ are isomorphic. Both $M_{\psi} \otimes_{S[L]} S$ and S/I_L are S -submodules of $\mathbb{k}[\mathbb{Z}^n/L]$ containing a monomial. As any monomial generates $\mathbb{k}[\mathbb{Z}^n/L]$ as an R -module, tensoring with R yields the required isomorphisms

$$(M_{\psi} \otimes_{S[L]} S) \otimes_S R \cong \mathbb{k}[\mathbb{Z}^n/L] \cong (S/I_L) \otimes_S R.$$

Because every nonzero submodule of $\mathbb{k}[\mathbb{Z}^n/L]$ intersects $M_{\psi} \otimes_{S[L]} S$ nontrivially, the annihilator of the S -module $M_{\psi} \otimes_{S[L]} S$ equals the annihilator of the S -module $\mathbb{k}[\mathbb{Z}^n/L]$, which is the toric ideal I_L . Thus, the second assertion follows. \square

We also provide a sufficient condition to ensure that a compatible \mathbb{Z}^n -stratification $\psi: L_{\mathbb{R}} \rightarrow \mathbb{Z}^n$ gives rise to a locally-free resolution of the sheaf associated to the S -module S/I_L ; compare with [CLS11, Proposition 5.3.10]. As in [BS98, §0], the S -module arising from L is

$$M_L := S \cdot \{x^v \mid v \in L\} \subset R.$$

For any cone σ in the fan Σ_X of the toric variety X , the monomial $x^{\hat{\sigma}}$ is defined as the product of the variables in S corresponding to the rays not in σ .

Proposition 5.3. *Let $\psi: L_{\mathbb{R}} \rightarrow \mathbb{Z}^n$ be a compatible \mathbb{Z}^n -stratification such that $\psi(0) = 0$ and the cellular free $S[L]$ -complex F_{ψ} is a resolution. The \mathcal{O}_X -modules associated to the S -modules $M_{\psi} \otimes_{S[L]} S$ and S/I_L coincide if, for all $\sigma \in \Sigma_X$, there exists a nonnegative integer k such that*

$$(x^{\hat{\sigma}})^k \cdot M_{\psi} \subseteq M_L.$$

Proof. Fix a cone $\sigma \in \Sigma_X$. As $\psi(0) = 0$, there is an inclusion $M_L \subseteq M_{\psi}$ of $S[L]$ -modules. By hypothesis, there is a nonnegative integer k such that $(x^{\hat{\sigma}})^k \cdot M_{\psi} \subseteq M_L$. Combining these inclusions gives $(M_L)_{x^{\hat{\sigma}}} = (M_{\psi})_{x^{\hat{\sigma}}}$. Since [BS98, Lemma 3.1] establishes that $M_L \otimes_{S[L]} S = S/I_L$, we obtain

$$\begin{aligned} ((S/I_L)_{x^{\hat{\sigma}}})_0 &= ((M_L \otimes_{S[L]} S)_{x^{\hat{\sigma}}})_0 = ((M_L)_{x^{\hat{\sigma}}} \otimes_{S[L]} (S)_{x^{\hat{\sigma}}})_0 \\ &= ((M_{\psi})_{x^{\hat{\sigma}}} \otimes_{S[L]} (S)_{x^{\hat{\sigma}}})_0 = ((M_{\psi} \otimes_{S[L]} S)_{x^{\hat{\sigma}}})_0. \end{aligned}$$

Applying [CLS11, Proposition 5.3.3], the sheaves associated to the S -modules $M_{\psi} \otimes_{S[L]} S$ and S/I_L are identical on an open affine cover of X . Therefore, these \mathcal{O}_X -modules coincide. \square

Remark 5.4. One can strengthen Proposition 5.3 by reducing the set of cones that need to be considered. To see this, recall from Remark 2.3 that the toric morphism $\varphi: Y \rightarrow X$ corresponds to an injective \mathbb{Z} -linear map $\bar{\varphi}: N_Y \rightarrow N_X$ between the lattices of one-parameter subgroups of Y and X . By Lemma 5.2, the support of the sheaf associated to $M_{\psi} \otimes_{S[L]} S$ is the subvariety of X cut out by the toric ideal I_L . Thus, it suffices to check that these sheaves agree on the affine opens U_{σ} covering $\varphi(Y)$. More precisely, one must verify that there exists a nonnegative integer k such that

$$(x^{\hat{\sigma}})^k \cdot M_{\psi} \subseteq M_L$$

for all cones σ satisfying $\dim(\bar{\varphi}(N_Y) \cap \sigma) = \dim N_Y$.

Revisiting the Anderson resolution of the diagonal. We place the resolution of the diagonal in [And24, Theorem 1.1] within our framework. Again, let Y be a smooth toric variety with no torus factors, and let m be the number of rays in its fan. The diagonal embedding $\Delta: Y \rightarrow X := Y \times Y$ gives rise to the lattice

$$L = \Lambda(M_Y) := \{(\mathbf{v}, -\mathbf{v}) \in \mathbb{Z}^m \times \mathbb{Z}^m \mid \mathbf{v} \in M_Y \subset \mathbb{Z}^m\} \subset \mathbb{Z}^n;$$

see also Example 2.18. The toric ideal I_L cuts out the image of the diagonal embedding. Inspired by [And24, §3], we christen the following compatible \mathbb{Z}^n -stratification; compare with Example 2.7. We continue to assume that the space $L_{\mathbb{R}} \cong \mathbb{R}^m$ carries the standard cell structure.

Definition 5.5. For any Lawrence lattice $L \subset \mathbb{Z}^n$, the *Anderson stratification* is the map $\psi: L_{\mathbb{R}} \rightarrow \mathbb{Z}^n$ defined by $\psi(\mathbf{q}, -\mathbf{q}) = (\lfloor \mathbf{q} \rfloor, -\lfloor \mathbf{q} \rfloor)$ for each vertex $(\mathbf{q}, -\mathbf{q}) \in L_{\mathbb{R}} \subset \mathbb{R}^n$. For any higher-dimensional open cell σ , the value $\psi(\sigma)$ is the componentwise maximum of the lattice vectors assigned to the vertices in the closure of σ .

In general, the Anderson stratification $\psi: L_{\mathbb{R}} \rightarrow \mathbb{Z}^n$ does not satisfy the topological condition in Proposition 2.13, and the cellular free S -complex $F_{\psi} \otimes_{S[L]} S$ may fail to be a resolution. As the next example illustrates, even when this S -complex is a resolution, it need not resolve the S -module $M_{\psi} \otimes_{S[L]} S$. Nevertheless, [And24, Theorem 1.1] shows that, for the Anderson stratification, the \mathcal{O}_X -complex associated to $F_{\psi} \otimes_{S[L]} S$ is in fact a resolution of the \mathcal{O}_X -module $\Delta_* \mathcal{O}_Y$.

Example 5.6 (Cellular free resolution arising from an Anderson stratification). As in Example 2.18, consider the toric surface $Y := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2))$ with its diagonal embedding $\Delta: Y \rightarrow X := Y \times Y$, and write $S := \mathbb{k}[x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4]$ for the Cox ring of X . Let $\psi: L_{\mathbb{R}} \rightarrow \mathbb{Z}^8$ denote the Anderson stratification. Figure 5.7 displays a fundamental domain in which each vertex $\mathbf{q} \in L_{\mathbb{R}}$ is labeled with the monomial having exponent vector $\psi(\mathbf{q})$. The corresponding $\text{Pic}(X)$ -graded cellular free S -complex $F_{\psi} \otimes_{S[L]} S$ is

$$\begin{array}{ccccc} & S(0,0,1,-1)^2 & & S(-1,0,1,-1) & \\ S(0,0,0,0) \oplus & \xleftarrow{\partial_1} & S(-1,0,0,-1)^2 \oplus & \xleftarrow{\partial_2} & S(0,-1,0,-1) \oplus \\ S(-1,1,1,-1) & & S(-1,0,-1,0) & & S(-1,0,-1,-1) \\ & & & & \end{array} \longrightarrow 0,$$

where

$$\partial_1 = \begin{bmatrix} y_2y_3 & y_1y_2 & -x_3y_4 & -x_1y_4 & y_1x_3 - x_1y_3 \\ -x_2x_3 & -x_1x_2 & y_3x_4 & y_1x_4 & 0 \end{bmatrix} \quad \text{and} \quad \partial_2 = \begin{bmatrix} -x_1 & y_1x_4 & 0 \\ x_3 & -y_3x_4 & 0 \\ 0 & -x_1x_2 & y_1 \\ 0 & x_2x_3 & -y_3 \\ -y_2 & 0 & y_4 \end{bmatrix}.$$

Lifting up to the universal cover, the cellular free S -complex F_{ψ} is a resolution, but it does not resolve the monomial module M_{ψ} . Observe that the vertices $\mathbf{p}_1 := (0, 0.5, 1, -0.5, 0, -0.5, -1, 0.5)$ and $\mathbf{p}_2 := (1, 0.5, 0, -0.5, -1, -0.5, 0, -0.5)$ in $L_{\mathbb{R}}$ satisfy $\psi(\mathbf{p}_1) = (0, 0, 1, -1, 0, 0, -1, 1)$ and $\psi(\mathbf{p}_2) = (1, 0, 0, -1, -1, 0, 0, 1)$. Since $x_1x_3x_4^{-1}y_4 = \text{lcm}(x_3y_3^{-1}x_4^{-1}y_4, x_1y_1^{-1}x_4^{-1}y_4)$, the failure to resolve M_{ψ} in terms of Proposition 2.13 is witnessed by the closed subset $(L_{\mathbb{R}})_{\leqslant (1,0,1,-1,0,0,0,1)}$, which consists precisely of these two vertices and is thereby not contractible.

Alternatively, one certifies that the S -modules $H_0(F_{\psi} \otimes_{S[L]} S)$ and $M_{\psi} \otimes_{S[L]} S$ are different by showing that their annihilators are not equal. As in the proof of Lemma 5.2, the annihilator of

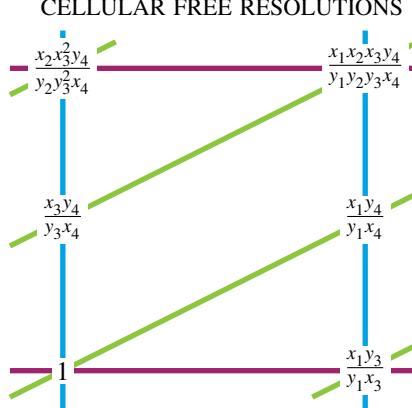


Figure 5.7. A fundamental domain of the Anderson stratification for the diagonal embedding of the second Hirzebruch surface.

$M_\psi \otimes_{S[L]} S$ is the toric ideal I_L . One verifies (for instance in *Macaulay2* [M2]) that the other annihilator is $I_L \cap \langle x_2, x_4 \rangle$. In particular, the homology module is supported on an ideal other than the toric ideal, but differs only on the irrelevant locus. \diamond

By introducing a small perturbation, [And24, §3] constructs a larger family of S -complexes. From our perspective, we recover this construction by making a small translation to the space $L_{\mathbb{R}}$ in the ambient space \mathbb{R}^n . The next example, which notably lies outside the realm of ceiling stratifications, illustrates this idea by adding a small parameter to Example 5.6.

Example 5.8 (Cellular free resolution arising from a small perturbation). As in Example 2.18 and Example 5.6, consider the diagonal embedding of the second Hirzebruch surface. Fix $0 < \varepsilon \ll 1$, and let $(L_{\mathbb{R}})_\varepsilon := L_{\mathbb{R}} + (0, 0, \varepsilon, 0) \subset \mathbb{R}^n$ have the standard cell structure, as shown in Figure 5.9. For the Anderson stratification $\psi: (L_{\mathbb{R}})_\varepsilon \rightarrow \mathbb{Z}^n$, the (nonminimal) $\text{Pic}(X)$ -graded cellular free S -complex $F_\psi \otimes_{S[L]} S$ is

$$\begin{array}{ccc}
 & S(0, 0, 0, 0) & \\
 & \oplus & \\
 S(0, 0, 0, 0)^2 & S(-1, 0, -1, 0) & S(-1, 0, -1, -1) \\
 \oplus & \oplus & \oplus \\
 S(-1, 1, 1, -1) & S(-1, 0, 0, -1)^2 & S(-2, 0, 0, -1) \\
 \oplus & \oplus & \oplus \\
 S(-2, 1, 2, -1) & S(-2, 1, 1, -1)^2 & S(-1, 0, 1, -1) \\
 & \oplus & \oplus \\
 & S(0, 0, 2, -1)^2 & S(0, 0, 2, -1)
 \end{array}$$

where

$$\partial_1 = \begin{bmatrix} -1 & y_1x_3 & 0 & -x_3y_4 & 0 & 0 & 0 & y_2 \\ 1 & -x_1y_3 & -x_1y_4 & 0 & 0 & 0 & y_2 & 0 \\ 0 & 0 & y_1x_4 & y_3x_4 & -x_1 & -x_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & y_1 & y_3 & -x_2 & -x_2 \end{bmatrix} \quad \text{and} \quad \partial_2 = \begin{bmatrix} 0 & x_1x_3y_4 & 0 & -y_2 \\ y_4 & 0 & -y_2 & 0 \\ -y_3 & x_3 & 0 & 0 \\ y_1 & -x_1 & 0 & 0 \\ 0 & -y_3x_4 & x_2x_3 & 0 \\ 0 & y_1x_4 & -x_1x_2 & 0 \\ 0 & 0 & -x_1y_3 & 1 \\ 0 & 0 & y_1x_3 & -1 \end{bmatrix}.$$

Combining Proposition 2.13 and Corollary 2.15 establishes that $F_\psi \otimes_{S[L]} S$ is a $\text{Pic}(X)$ -graded resolution of the S -module $M_\psi \otimes_{S[L]} S$.

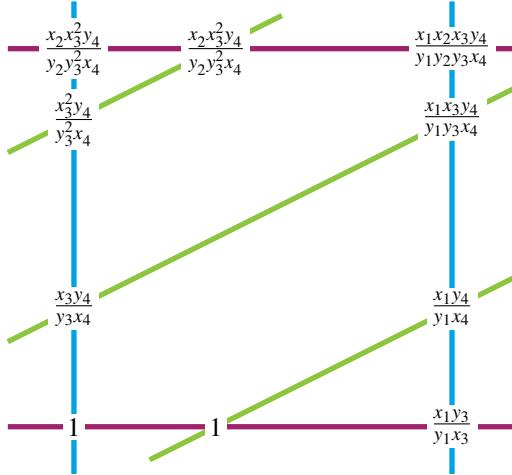


Figure 5.9. A fundamental domain of an ε -shifted stratification for the diagonal embedding of the second Hirzebruch surface.

We also claim that the \mathcal{O}_X -complex associated to $F_\psi \otimes_{S[L]} S$ is a resolution of the diagonal. The monomial S -module $M_\psi \otimes_{S[L]} S$ is generated by the Laurent monomials $1, x_3y_3^{-1}x_4^{-1}y_4$ and $x_3^2y_3^{-2}x_4^{-1}y_4$. Applying Proposition 5.3 and Remark 5.4, we check that $(x^\sigma)^k \cdot M_\psi \subseteq M_L$ for the relevant cones $\sigma \in \Sigma_X$ as follows:

$$\begin{aligned} x_1y_1x_2y_2 \cdot \frac{x_3y_4}{y_3x_4} &= y_1^2y_2^2 \cdot \frac{x_1x_2x_3y_4}{y_1y_2y_3x_4}, & x_2y_2x_3y_3 \cdot \frac{x_3y_4}{y_3x_4} &= y_2^2y_3^2 \cdot \frac{x_2x_3^2y_4}{y_2y_3^2x_4}, & x_1y_1x_4y_4 \cdot \frac{x_3y_4}{y_3x_4} &= x_1^2y_4^2 \cdot \frac{y_1x_3}{x_1y_3}, \\ x_1y_1x_2y_2 \cdot \frac{x_3^2y_4}{y_3^2x_4} &= x_1y_1y_2^2 \cdot \frac{x_2x_3^2y_4}{y_2y_3^2x_4}, & x_2y_2x_3y_3 \cdot \frac{x_3^2y_4}{y_3^2x_4} &= y_2^2x_3y_3 \cdot \frac{x_2x_3^2y_4}{y_2y_3^2x_4}, & (x_1y_1x_4y_4)^2 \cdot \frac{x_3^2y_4}{y_3^2x_4} &= x_1^4x_4y_4^3 \cdot \left(\frac{y_1x_3}{x_1y_3}\right)^2. \end{aligned}$$

Hence, the \mathcal{O}_X -module associated to $M_\psi \otimes_{S[L]} S$ is $\Delta_*\mathcal{O}_Y$. \diamond

Although these small perturbations often produce S -complexes that come closer to satisfying Proposition 2.13, it is not clear whether there is always a choice of ε that yields a free resolution.

Cellular interpretation of the Favero–Huang homotopy path algebra construction. We also discuss how the resolution of the diagonal in [FH25, §5] fits into our framework. To sidestep the technical details surrounding [FH25, Corollary 6.7], which describes a projective cellular resolution of the diagonal bimodule over a homotopy path algebra, we concentrate on the toric variety $Y = \mathbb{P}^2$.

Let $\Delta: \mathbb{P}^2 \rightarrow \mathbb{P}^2 \times \mathbb{P}^2$ denote the diagonal embedding, and let $S := \mathbb{k}[x_1, x_2, x_3, y_1, y_2, y_3]$ be the Cox ring of $\mathbb{P}^2 \times \mathbb{P}^2$. The corresponding lattice $L \subseteq \mathbb{R}^6$ is generated by the integral vectors $(1, -1, 0, -1, 1, 0)$ and $(0, -1, 1, 0, 1, -1)$. These vectors form a basis of L , which determines an isomorphism $\iota: \mathbb{Z}^2 \rightarrow L \subset \mathbb{Z}^6$. Extending the map ι linearly gives the map $\iota_{\mathbb{R}}: \mathbb{R}^2 \rightarrow \mathbb{R}^6$ whose image is exactly $L_{\mathbb{R}}$.

To describe the desired cell structure on the space $L_{\mathbb{R}}$, consider the closed subset

$$\mathcal{S} = \overline{\{p \in \mathbb{R}^3 \mid \lceil p_1 \rceil + \lceil p_2 \rceil + \lceil p_3 \rceil = 0 \text{ and at least one coordinate } p_i \in \mathbb{Z}\}}$$

in \mathbb{R}^3 (with its Euclidean topology). [MS05, §3] calls \mathcal{S} a “staircase surface.” We give \mathcal{S} the polyhedral cell structure determined by its intersection with the periodic arrangement arising from the standard basis in \mathbb{R}^3 ; compare with Definition 2.2. The maximal cells in \mathcal{S} are unit squares.

The linear projection of \mathcal{S} onto linear subspace $L_{\mathbb{R}} \subset \mathbb{R}^3$ along the vector $(1, 1, 1)$ endows the space $L_{\mathbb{R}}$ with a polyhedral cell structure.

The relevant compatible \mathbb{Z}^6 -stratification depends on an auxiliary function $\gamma: \mathbb{R}^2 \rightarrow \mathcal{S}$. This map is defined in two steps. First, include \mathbb{R}^2 into \mathbb{R}^3 via the linear map determined by the matrix

$$\begin{bmatrix} 1 & 0 \\ -1 & -1 \\ 0 & 1 \end{bmatrix}.$$

Next, project the point in \mathbb{R}^3 onto \mathcal{S} along the direction $(1, 1, 1)$. This composite map γ is a homeomorphism. Moreover, the image of the standard integral lattice $\mathbb{Z}^2 \subset \mathbb{R}^2$ under γ is $\{\mathbf{u} \in \mathcal{S} \mid u_1 + u_2 + u_3 = 0\} \cong \mathbb{Z}^2$. For any lattice point $\mathbf{u} \in \mathbb{Z}^2$, observe that $\iota(\mathbf{u}) = (\gamma(\mathbf{u}), -\gamma(\mathbf{u}))$.

Lemma 5.10. *The map $\psi: L_{\mathbb{R}} \rightarrow \mathbb{Z}^6$, defined by $\psi(\iota_{\mathbb{R}}(\mathbf{q})) = \lceil(\gamma(\mathbf{q}), -\gamma(\mathbf{q}))\rceil$ for all $\mathbf{q} \in \mathbb{R}^2 \cong L_{\mathbb{R}}$, is a compatible \mathbb{Z}^6 -stratification.*

Figure 5.11 displays part of the cell complex $L_{\mathbb{R}}$ in which each vertex $\mathbf{q} \in L_{\mathbb{R}}$ is labeled with the monomial having exponent vector $\psi(\mathbf{q})$. This agrees with the cell structure in [FH25, Example 3.15].

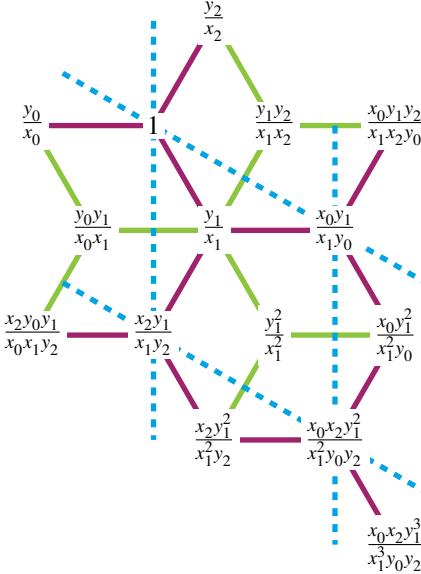


Figure 5.11. A portion of the labeled cell complex giving a Favero–Huang resolution of the diagonal for \mathbb{P}^2 . The dashed lines illustrate the underlying lattice.

Proof of Lemma 5.10. By construction, the map ψ is constant on the open cells in $L_{\mathbb{R}}$. Moreover, for any point $\mathbf{q} \in \mathbb{R}^2$ and any lattice point $\mathbf{u} \in \mathbb{Z}^2$, we have $\gamma(\mathbf{q} + \mathbf{u}) = \gamma(\mathbf{q}) + \gamma(\mathbf{u})$. It follows that

$$\begin{aligned} \psi(\iota_{\mathbb{R}}(\mathbf{q} + \mathbf{u})) &= \lceil(\gamma(\mathbf{q} + \mathbf{u}), -\gamma(\mathbf{q} + \mathbf{u}))\rceil = \lceil(\gamma(\mathbf{q}), -\gamma(\mathbf{q}))\rceil + \lceil(\gamma(\mathbf{u}), -\gamma(\mathbf{u}))\rceil \\ &= \psi(\iota_{\mathbb{R}}(\mathbf{q})) + (\gamma(\mathbf{u}), -\gamma(\mathbf{u})) = \psi(\iota_{\mathbb{R}}(\mathbf{q})) + \iota(\mathbf{u}), \end{aligned}$$

which shows that ψ is compatible with lattice translation.

To demonstrate that the map ψ is continuous, we must show that, for a fixed lattice point $\mathbf{u} \in \mathbb{Z}^6$, the subset $(L_{\mathbb{R}})_{\leqslant \mathbf{u}} = \{\mathbf{p} \in L_{\mathbb{R}} \mid \psi(\mathbf{p}) \leqslant \mathbf{u}\}$ is closed in $L_{\mathbb{R}}$. Lifting the defining inequality to the

ambient space \mathbb{R}^3 , we see that $(L_{\mathbb{R}})_{\leq u}$ is the projection from \mathbb{R}^3 of the intersection of the set \mathcal{S} with the closed rectangular cuboid $[-u_4, u_1] \times [-u_5, u_2] \times [-u_6, u_3]$. \square

Unlike the ceiling stratification, the subsets $(L_{\mathbb{R}})_{\leq u}$ are not always convex. For example, the subset corresponding to the lattice point $u := (0, 0, -1, 0, 0, -1)$ is a union of two quadrilaterals meeting at a single vertex. However, the cellular free S -complex $F_{\psi} \otimes_{S[L]} S$ is still a resolution.

Proposition 5.12. *When $\psi: L_{\mathbb{R}} \rightarrow \mathbb{Z}^6$ is the compatible \mathbb{Z}^6 -stratification defined in Lemma 5.10, the cellular free S -complex $F_{\psi} \otimes_{S[L]} S$ is a resolution of the monomial module $M_{\psi} \otimes_{S[L]} S$.*

Proof. The intersection of \mathcal{S} with a closed rectangular cuboid, whose faces are parallel with the coordinate hyperplanes, is contractible. We deduce that the subset $(L_{\mathbb{R}})_{\leq u}$ is contractible for all $u \in L_{\mathbb{R}}$. Thus, combining Proposition 2.13 and Corollary 2.15 shows that $F_{\psi} \otimes_{S[L]} S$ is a resolution. \square

For the compatible \mathbb{Z}^6 -stratification defined in Lemma 5.10, the monomial module $M_{\psi} \otimes_{S[L]} S$ is generated by the Laurent monomials 1, $x_1 y_1^{-1}$, and $x_1 y_1^{-1} x_2 y_2^{-1}$. The (minimal) \mathbb{Z}^2 -graded cellular free S -complex $F_{\psi} \otimes_{S[L]} S$ is

$$\begin{array}{ccccc} S(0,0) & \xleftarrow{\quad \oplus \quad} & S(1,-2)^3 & \xleftarrow{\quad \oplus \quad} & S(0,-2)^3 \xleftarrow{\quad} 0 \\ S(1,-1) & \xleftarrow{\left[\begin{array}{cccccc} 0 & 0 & 0 & -y_1 & -y_2 & -y_3 \\ -y_1 & -y_2 & -y_3 & x_1 & x_2 & x_3 \\ x_1 & x_2 & x_3 & 0 & 0 & 0 \end{array} \right]} & S(0,-1)^3 & \xleftarrow{\quad \oplus \quad} & \\ S(2,-2) & \xleftarrow{\quad \oplus \quad} & & & \end{array}$$

$$\begin{bmatrix} 0 & -x_3 & -x_2 \\ -x_3 & 0 & x_1 \\ x_2 & x_1 & 0 \\ 0 & y_3 & y_2 \\ y_3 & 0 & -y_1 \\ -y_2 & -y_1 & 0 \end{bmatrix}$$

Generalizing this construction recovers the resolutions of the diagonal for \mathbb{P}^n implicitly described in [FH25, Corollary 6.7]. It is unclear whether a similar approach can be used to construct resolutions of the diagonal for other toric varieties.

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