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Derived autoequivalences and a weighted Beilinson resolution

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Abstract

Given a smooth stacky Calabi–Yau hypersurface X in a weighted projective space, we consider the functor G which is the composition of the following two autoequivalences of $\mathrm{D}^b(X)$: the first one is induced by the spherical object $\mathcal{O}_X$, while the second one is tensoring with $\mathcal{O}_X(1)$. The main result of the paper is that the composition of G with itself w times, where w is the sum of the weights of the weighted projective space, is isomorphic to the autoequivalence "shift by 2". The proof also involves the construction of a Beilinson type resolution of the diagonal for weighted projective spaces, viewed as smooth stacks.

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1. Introduction

Recent years have seen an increased activity in the study of algebraic varieties via their derived categories of coherent sheaves. Although this algebraic approach is indirect compared to the geometric investigations involving divisors, curves, or branched covers, just to name a few, it is nevertheless quite promising, as in some cases it allows for a deeper understanding. This is the case of varieties with interesting groups of derived autoequivalences, and in particular those with Kodaira dimension 0, where the autoequivalences are symmetries of the variety not visible in the geometrical presentation.

Despite significant progress, our understanding of the derived categories of coherent sheaves and their autoequivalences is limited (for a recent review we refer to [7]). In the present paper we hope to further this understanding by proving certain identities involving Fourier–Mukai functors on quasi-smooth Calabi–Yau varieties, viewed as smooth Deligne–Mumford stacks. The origin of these identities is closely tied with mirror symmetry. We first present our main result, then describe the setting in which it arises.

Let $\mathbb{P}^n(\mathbf{w})$ be an n-dimensional weighted projective space over a field $\Bbbk$, regarded as a smooth proper Deligne–Mumford stack, with weight vector $\mathbf{w} = (w_0, w_1, \ldots, w_n)$, and let $w = \sum_{i=0}^{n} w_i$ denote the sum of all

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the weights. We have several equivalent ways to think about $\mathbb{P}^n(\mathbf{w})$: graded scheme [8], toric stack [5], or quotient stack [3].

Let X be an anti-canonical hypersurface in $\mathbb{P}^n(\mathbf{w})$. By the stacky version of Bertini's theorem the generic member of the linear system $|-K_{\mathbb{P}^n(\mathbf{w})}| = |\mathcal{O}_{\mathbb{P}^n(\mathbf{w})}(w)|$ is a proper smooth Deligne–Mumford stack. Let $\mathrm{D}^b(X)$ denote the bounded derived category of coherent sheaves on the stack X. We have two functors naturally associated with these data:

$$\mathsf{L}\colon \mathrm{D}^b(X) \to \mathrm{D}^b(X) \qquad \mathsf{L}(\mathcal{F}) = \mathcal{F} \otimes \mathcal{O}_X(1), \tag{1}$$

and

$$\mathsf{K}\colon \mathrm{D}^b(X) \to \mathrm{D}^b(X) \qquad \mathsf{K}(\mathcal{F}) = \mathsf{C}(\mathbf{R}\mathrm{Hom}_X(\mathcal{O}_X, \mathcal{F}) \otimes_{\Bbbk} \mathcal{O}_X \longrightarrow \mathcal{F}), \tag{2}$$

for all $\mathcal{F} \in \mathrm{D}^b(X)$. The morphism in K is the evaluation map, and C is its cone. For an integer m we also have the autoequivalence of $\mathrm{D}^b(X)$ given by the translation functor $(-)[m]$; its action is "shift by m", i.e. $\mathcal{F} \mapsto \mathcal{F}[m]$. In fact also L is clearly an equivalence, and the same is true for K thanks to [24], where it is proved more generally that the functor defined by $\mathcal{F} \mapsto \mathsf{C}(\mathbf{R}\mathrm{Hom}_X(\mathcal{E}, \mathcal{F}) \otimes_{\Bbbk} \mathcal{E} \longrightarrow \mathcal{F})$ is an autoequivalence of $\mathrm{D}^b(X)$ whenever $\mathcal{E}$ is a *spherical object*, i.e., an object of $\mathrm{D}^b(X)$ such that $\mathcal{E} \otimes \omega_X \cong \mathcal{E}$ and

$$\mathrm{Hom}_{\mathrm{D}^b(X)}(\mathcal{E}, \mathcal{E}[i]) \cong \begin{cases} \Bbbk & \text{if } i = 0, \dim(X) \\ 0 & \text{otherwise.} \end{cases}$$

Note that X being Calabi–Yau immediately implies that $\mathcal{O}_X$ is a spherical object. Our main result is then the following nontrivial relation in the group $\mathrm{Aut}(\mathrm{D}^b(X))$ of (isomorphism classes of) autoequivalences of $\mathrm{D}^b(X)$:

Theorem 1.1. *Let X be a smooth anti-canonical stacky hypersurface in the Deligne–Mumford stack $\mathbb{P}^n(\mathbf{w})$, and let $\mathsf{G} = \mathsf{L} \circ \mathsf{K}$, where L and K are the autoequivalences of $\mathrm{D}^b(X)$ defined in Eqs. (1) and (2). Then there is an isomorphism of functors*

$$\underbrace{\mathsf{G} \circ \cdots \circ \mathsf{G}}_{w\text{-times}} \cong (-)[2]. \tag{3}$$

Let us spend some time trying to understand the origin of this statement. At first sight it might seem surprising that physics has anything to do with such an abstract branch of pure mathematics. But one should remember that historically some of the most interesting mathematical problems came from the real world, and primarily from physics. Recently, with the advent of string theory, the bridge of interactions between abstract mathematics and theoretical physics has entered an era of renaissance, with mirror symmetry the most prominent example of the interaction.

From the point of view of *strings* in string theory the appearance of the derived category is quite intriguing, but recent developments showed that D-branes mandate a categorical approach. In particular, Douglas argued that B-type topological D-branes are objects in the bounded derived category of coherent sheaves [11]. His work was subsequently axiomatized by Bridgeland [6], and has since been subject to active investigations. The A-type D-branes have a very different description, involving the derived Fukaya category. Mirror symmetry exchanges the A and B branes, and naturally leads to Kontsevich's *homological mirror symmetry* (HMS) conjecture. For a detailed exposition of these ideas we refer the reader to recent book [10].

To motivate our result we need to start with mirror symmetry in its pre-HMS phase. In this form mirror symmetry is an isomorphism between the (complexified) Kahler moduli space $\mathcal{M}_K(X)$ of a Calabi–Yau variety X and the moduli space of complex deformations $\mathcal{M}_c(\widetilde{X})$ of its mirror $\widetilde{X}$. For the precise definitions we refer to the book by Cox and Katz [9]. We will follow their terminology in this introduction.

We note at this point that the moduli spaces in question are only coarse moduli spaces; the fine moduli spaces are necessarily stacks. This fact complicates any existing intuitive mathematical understanding, but the conformal field theory (CFT) techniques that underlie the Cox and Katz exposition give us an alternative view, which we now elaborate on.

Mirror symmetry also suggested a natural way to complexify the Kahler moduli space and how to compactify it. The complexified Kahler moduli space $\mathcal{M}_K(X)$ in general is an intricate object, but for X a hypersurface in a toric variety it has a rich combinatorial structure and is relatively well understood. In particular, the fundamental group of

$\mathcal{M}_K(X)$ in general is nontrivial, and one can talk about various monodromy representations. More concretely, there are two types of boundary divisors in $\mathcal{M}_K(X)$: “large radius divisor” and the “discriminant” (some authors refer to both as discriminant, but for us the distinction is important). Both of these are reducible in general. At the large radius divisor certain cycles of X, viewed as a Kahler manifold, acquire infinite volume. The discriminant is somewhat harder to describe. The original definition is that the CFT associated to a string probing X becomes singular at such a point in moduli space. Generically this happens because some D-brane (or several of them, even infinitely many) becomes massless, and therefore the effective CFT description provided by the string fails. A consequence of this fact is that, by using the mirror map isomorphism of the moduli spaces, as one approaches the discriminant in $\mathcal{M}_K(X)$ one is moving in $\mathcal{M}_c(\widetilde{X})$ to a point where the mirror $\widetilde{X}$ is developing a singularity.

Armed with this picture of $\mathcal{M}_K(X)$, we can fix a basepoint O, and look at loops in $\mathcal{M}_K(X)$ based at O. The CFT description of string theory shows that traversing such a loop gives in general a non-trivial functor $\mathrm{D}^b(X) \to \mathrm{D}^b(X)$, which moreover has to be an equivalence (string theory does not seem to able to distinguish between isomorphism and equivalence). Therefore we arrive at a group homomorphism, first suggested by Kontsevich [20]:

$$\mu\colon \pi_1(\mathcal{M}_K(X)) \longrightarrow \mathrm{Aut}(\mathrm{D}^b(X)).$$

At present writing very little is known about μ. Kontsevich’s ideas were generalized by Horja [13] and Morrison [23]. The question at hand is: given a pointed loop in $\mathcal{M}_K(X)$, what is the associated autoequivalence of $\mathrm{D}^b(X)$? Progress in this direction was made in [2], where this question is answered for the EZ-degenerations introduced in [14].

It is clear now that given a presentation of $\pi_1(\mathcal{M}_K(X))$ where we know the images under μ of the generators, the relations in the presentation will determine interesting identities in $\mathrm{Aut}(\mathrm{D}^b(X))$. We now turn to an example of this sort. Let X be a smooth degree $w = \sum_{i=0}^n w_i$ variety in $\mathbb{P}^n(\mathbf{w})$, in other words let X be such that it does not meet the singularities of $\mathbb{P}^n(\mathbf{w})$. In this case the compactification of $\mathcal{M}_K(X)$ is isomorphic to $\mathbb{P}^1$, and we have three distinguished points P_{LC}, P_0 and P_F. It is easier to describe them in terms of $\mathcal{M}_c(\widetilde{X})$: P_{LC} is a large complex structure limit point (with maximally unipotent monodromy), at P_0 the family $\widetilde{X}$ has rational double points, while at P_F it has additional automorphisms. If $\widetilde{X}$ has a Fermat form, i.e., w_i divides w, then we are talking about an additional cyclic symmetry $\mathbb{Z}_w$.

Let M_P denote the monodromy associated to a loop around the point P. Since P_{LC} and P_0 are the only limit points of $\mathcal{M}_K(X)$, and the compactification of this is isomorphic to $\mathbb{P}^1$ (see [9]), with $\pi_1(\mathbb{P}^1 - \{2 \text{ points}\}) = \mathbb{Z}$, one would want to conclude, incorrectly, that $M_{P_{LC}}$ and M_{P_0} are related. On the other hand, the extra automorphisms indicates that P_F is a stacky point in the moduli space, with finite stabilizer, and so, at best, the w-th power of $M_{P_F} \cong M_{P_{LC}} \circ M_{P_0}$ is the identity. In the case of $\mathbb{P}^n(\mathbf{w}) = \mathbb{P}^4$ Kontsevich proposed that $M_{P_0} = \mathsf{K}$ from Eq. (2), $M_{P_{LC}} = \mathsf{L}$, and checked that indeed $M_{P_F}^5 = \mathrm{id}$ in K-theory. Later on Aspinwall realized that in fact $M_{P_F}^5 \cong (-)[2]$. Based on physical considerations it was clear to us that the Kontsevich–Aspinwall result should hold in the weighted case as well, which eventually led us to Theorem 1.1. Cases where $\mathcal{M}_K(X)$ is higher dimensional were investigated in [16,17].

Our proof is inspired by [1, Sec. 7.1.4], where the case $\mathbb{P}^n(\mathbf{w}) = \mathbb{P}^4$ is outlined. Actually with the same technique a more general result was independently obtained by Kuznetsov in [21, Section 4], where smooth Fano (but the argument applies to the Calabi–Yau case as well) hypersurfaces in $\mathbb{P}^n$ were considered (in Remark 4.7 we explain how Kuznetsov’s result extends to the weighted case, as suggested to us by the author after the first version of this paper was made public). However, the proof of Theorem 1.1 is much harder, and a different approach is needed overall. Still, the idea to trade the composition of the functors for the composition of their kernels, and then use a resolution of the diagonal of the (weighted in our case) projective space proved very useful to us. In fact a good part of the paper is devoted to the construction of a resolution of the diagonal for $\mathbb{P}^n(\mathbf{w})$, which is similar but still very different from the well-known Beilinson resolution of $\mathbb{P}^n$.

Notation. A complex A of some abelian category $\mathfrak{A}$ is given by a collection of objects A^i of $\mathfrak{A}$, together with morphisms $d_A^i\colon A^i \to A^{i+1}$ such that $d_A^{i+1} \circ d_A^i = 0$ for every $i \in \mathbb{Z}$. When A is just an object of $\mathfrak{A}$, it is viewed as a bounded complex with $A^0 = A$ and $A^i = 0$ for $i \neq 0$. A morphism of complexes $f\colon A \to B$ is given by a collection of maps $f^i\colon A^i \to B^i$ such that $d_B^i \circ f^i = f^{i+1} \circ d_A^i$. For $k \in \mathbb{Z}$ the shifted complex $A[k]$ is defined by $A[k]^i := A^{i+k}$ and $d_{A[k]}^i := (-1)^k d_A^{i+k}$; similarly, $f[k]$ is defined by $f[k]^i := f^{i+k}$.

The dual of a locally free sheaf $\mathcal{L}$ (or more generally of a complex of locally free sheaves) will be denoted by $\mathcal{L}^\vee$; the same notation will be used for the dual of a vector space.

2. Preliminaries on Fourier–Mukai functors

In order to fix notations and conventions that will be used throughout the paper, we start recalling some definitions and basic facts about triangulated categories, derived categories and derived functors. For a thorough treatment of the subject we refer to [12] or [15].

If $f: A \to B$ is a morphism in a triangulated category, then a *cone* of f is an object $\mathsf{C}(f)$ (defined up to isomorphism), which fits into a distinguished triangle

$$A \xrightarrow{f} B \longrightarrow \mathsf{C}(f) \longrightarrow A[1].$$

For an abelian category $\mathfrak{A}$, $\mathrm{C}^b(\mathfrak{A})$ denotes the abelian category of bounded complexes of $\mathfrak{A}$ (its objects are complexes A such that $A^i = 0$ for $|i| \gg 0$). The *mapping cone* of a morphism $f: A \to B$ of $\mathrm{C}^b(\mathfrak{A})$ is the complex $\mathsf{MC}(f)$ defined by

$$\mathsf{MC}(f)^i := A^{i+1} \oplus B^i, \qquad d^i_{\mathsf{MC}(f)} := \begin{pmatrix} -d_A^{i+1} & 0 \\ f^{i+1} & d_B^i \end{pmatrix}. \tag{4}$$

$\mathrm{K}^b(\mathfrak{A})$ (respectively $\mathrm{D}^b(\mathfrak{A})$) will be the bounded homotopy (respectively derived) category of $\mathfrak{A}$. We recall that $\mathrm{K}^b(\mathfrak{A})$ and $\mathrm{D}^b(\mathfrak{A})$ are triangulated categories and have the same objects as $\mathrm{C}^b(\mathfrak{A})$. For a morphism f in $\mathrm{C}^b(\mathfrak{A})$, f will also denote its image in $\mathrm{K}^b(\mathfrak{A})$, or in $\mathrm{D}^b(\mathfrak{A})$. In both categories $\mathsf{C}(f) \cong \mathsf{MC}(f)$, but $\mathsf{MC}(f)$ will be used only when the specific form of the resulting complex is needed.

If $\mathfrak{B}$ is another abelian category, a left exact functor $F: \mathfrak{A} \to \mathfrak{B}$ trivially extends to an exact functor again denoted by $F: \mathrm{K}^b(\mathfrak{A}) \to \mathrm{K}^b(\mathfrak{B})$. When it exists, its right derived functor will be denoted by $\mathbf{R}F: \mathrm{D}^b(\mathfrak{A}) \to \mathrm{D}^b(\mathfrak{B})$; we set $R^iF := H^i \circ \mathbf{R}F$ for $i \in \mathbb{Z}$. If A is an object of $\mathrm{D}^b(\mathfrak{A})$ such that each A^i is F-acyclic (i.e., $R^jF(A^i) = 0$ for $j > 0$), then $\mathbf{R}F(A) \cong F(A)$ in $\mathrm{D}^b(\mathfrak{B})$. Similar considerations hold if F is right exact, in which case its left derived functor will be denoted by $\mathbf{L}F$.

For simplicity in the following we will call *stack* a Deligne–Mumford stack which is proper and smooth over the base field $\Bbbk$, and such that every coherent sheaf is a quotient of a locally free sheaf of finite rank. In fact all stacks we will consider in the rest of the paper will be stacks associated to normal projective varieties with only quotient singularities (namely, weighted projective spaces, quasi-smooth hypersurfaces in them and products of such varieties), and those satisfy our condition thanks to [18, Theorem 4.2]. When the proofs remain essentially the same, we will use results stated in the literature only for schemes for stacks as well, but most of the time we point this out.

If Y is a stack, $\mathfrak{Coh}(Y)$ will denote the abelian category of coherent sheaves on Y, and we set for brevity $\mathrm{C}^b(Y) := \mathrm{C}^b(\mathfrak{Coh}(Y))$, $\mathrm{K}^b(Y) := \mathrm{K}^b(\mathfrak{Coh}(Y))$ and $\mathrm{D}^b(Y) := \mathrm{D}^b(\mathfrak{Coh}(Y))$. If $f: Y \to Z$ is a morphism of stacks, there are derived functors $\mathbf{R}f_*: \mathrm{D}^b(Y) \to \mathrm{D}^b(Z)$ and $\mathbf{L}f^*: \mathrm{D}^b(Z) \to \mathrm{D}^b(Y)$. Notice that $\mathbf{R}f_* \cong f_*$ if f is finite and $\mathbf{L}f^* \cong f^*$ if f is flat. When Z is a point f_* can be identified with $\Gamma(Y, -)$, and $R^i\Gamma(Y, -)$ will be denoted by $H^i(Y, -)$. Our definition of stack also implies that there is a left derived functor for the tensor product, denoted by $- \overset{\mathbf{L}}{\otimes} -: \mathrm{D}^b(Y) \times \mathrm{D}^b(Y) \to \mathrm{D}^b(Y)$. Clearly $\mathcal{F} \overset{\mathbf{L}}{\otimes} \mathcal{G} \cong \mathcal{F} \otimes \mathcal{G}$ if, either, each $\mathcal{F}^i$, or each $\mathcal{G}^i$, is locally free.

Given $\mathcal{E} \in \mathrm{D}^b(Y \times Z)$, and denoting by $p: Y \times Z \to Y$ and $q: Y \times Z \to Z$ the projections, the exact functor $\Phi_{\mathcal{E}} = \Phi_{\mathcal{E}}^{Z \to Y}: \mathrm{D}^b(Z) \to \mathrm{D}^b(Y)$ defined by

$$\Phi_{\mathcal{E}}(\mathcal{F}) := \mathbf{R}p_*(\mathcal{E} \overset{\mathbf{L}}{\otimes} q^*\mathcal{F})$$

for $\mathcal{F} \in \mathrm{D}^b(Z)$, is called a *Fourier–Mukai functor* with kernel $\mathcal{E}$.

Lemma 2.1. *If $\mathcal{E} \in \mathrm{D}^b(Y \times Z)$ is such that $\Phi_{\mathcal{E}} \cong 0$, then $\mathcal{E} \cong 0$.*

Proof. Assume on the contrary that $\mathcal{E} \ncong 0$, and let m be the least integer such that $H^m(\mathcal{E}) \neq 0$. Setting $\mathcal{F} := H^m(\mathcal{E}) \in \mathfrak{Coh}(Y \times Z)$, we claim that it is enough to prove that

$$p_*(\mathcal{F} \otimes q^*\mathcal{L}) \neq 0 \qquad \text{for some } \mathcal{L} \in \mathfrak{Coh}(Z) \text{ locally free.} \tag{5}$$

Indeed, assuming this, the definition of $\mathcal{F}$ implies that there is a distinguished triangle in $\mathrm{D}^b(Y \times Z)$

$$\mathcal{E}'[-1] \longrightarrow \mathcal{F}[-m] \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}'$$

with $H^i(\mathcal{E}') = 0$ for $i \le m$, from which it is easy to deduce that $R^i p_*(\mathcal{E}' \otimes q^*\mathcal{L}) = 0$ for $i \le m$. Then, applying the exact functor $\mathbf{R}p_*(- \otimes q^*\mathcal{L})$ to the above triangle, and taking the associated cohomology sequence, we obtain

$$0 \ne p_*(\mathcal{F} \otimes q^*\mathcal{L}) \cong R^m p_*(\mathcal{F}[-m] \otimes q^*\mathcal{L}) \cong R^m p_*(\mathcal{E} \otimes q^*\mathcal{L}) \cong H^m(\Phi_{\mathcal{E}}(\mathcal{L})),$$

which contradicts the hypothesis $\Phi_{\mathcal{E}} \cong 0$.

In order to prove (5) it is obviously enough to find a locally free sheaf $\mathcal{L}$ such that $f^* p_*(\mathcal{F} \otimes q^*\mathcal{L}) \ne 0$, where $f\colon V \to Y$ is an étale and surjective morphism and V an affine scheme. Applying the "flat base change" theorem (for stacks this is [22, Prop. 13.1.9]) to the Cartesian square

$$\begin{tikzcd} V \times Z \arrow[r, "\bar{f}"] \arrow[d, "\bar{p}"'] & Y \times Z \arrow[d, "p"] \\ V \arrow[r, "f"] & Y \end{tikzcd}$$

and setting $\overline{\mathcal{F}} := \overline{f}^*\mathcal{F} \in \mathfrak{Coh}(V \times Z)$, we see that $f^* p_*(\mathcal{F} \otimes q^*\mathcal{L}) \cong \overline{p}_*(\overline{\mathcal{F}} \otimes \overline{q}^*\mathcal{L})$, where $\overline{q} = q \circ \overline{f}\colon V \times Z \to Z$ is the projection. Hence it is enough to show that

$$0 \ne \mathrm{Hom}_V(\mathcal{O}_V, \overline{p}_*(\overline{\mathcal{F}} \otimes \overline{q}^*\mathcal{L})) \cong \mathrm{Hom}_{V\times Z}(\mathcal{O}_{V\times Z}, \overline{\mathcal{F}} \otimes \overline{q}^*\mathcal{L}) \cong \mathrm{Hom}_{V\times Z}(\overline{q}^*\mathcal{L}^\vee, \overline{\mathcal{F}}) \cong \mathrm{Hom}_Z(\mathcal{L}^\vee, \overline{q}_*\overline{\mathcal{F}}).$$

Taking into account that $\overline{q}_*\overline{\mathcal{F}} = \overline{q}_*\overline{f}^*\mathcal{F} \ne 0$ (because $\overline{q}$ is affine and $\overline{f}$ is étale and surjective) and the fact that the quasi-coherent sheaf $\overline{q}_*\overline{\mathcal{F}}$ is the inductive limit of its coherent subsheaves (by [22, Prop. 15.4]), the existence of $\mathcal{L} \in \mathfrak{Coh}(Z)$ locally free such that $\mathrm{Hom}_Z(\mathcal{L}^\vee, \overline{q}_*\overline{\mathcal{F}}) \ne 0$ follows from the fact that every coherent sheaf on Z is a quotient of a locally free sheaf. $\square$

Now we specialize to the case $Y = Z$, although much of what we are going to say can be extended to the general case, with obvious modifications.

Given $\mathcal{F}, \mathcal{G} \in \mathrm{K}^b(Y)$, their *exterior* tensor product is defined as

$$\mathcal{F} \boxtimes \mathcal{G} := \pi_2^*\mathcal{F} \otimes \pi_1^*\mathcal{G} \in \mathrm{K}^b(Y \times Y),$$

where π_i is the natural projection to the ith factor of $Y \times Y$. The symbol $\overset{\mathbf{L}}{\boxtimes}$ will be used for the derived functor of exterior tensor product (again, $\mathcal{F} \overset{\mathbf{L}}{\boxtimes} \mathcal{G} \cong \mathcal{F} \boxtimes \mathcal{G}$ if either each $\mathcal{F}^i$ or each $\mathcal{G}^i$ is locally free).

Given $\mathcal{E} \in \mathrm{D}^b(Y \times Y)$, we define two Fourier–Mukai functors $\Phi^i_{\mathcal{E}}\colon \mathrm{D}^b(Y) \to \mathrm{D}^b(Y)$:

$$\Phi^i_{\mathcal{E}}(\mathcal{F}) := \mathbf{R}\pi_{3-i\,*}(\mathcal{E} \overset{\mathbf{L}}{\otimes} \pi_i^*\mathcal{F}), \quad \text{for } i = 1, 2.$$

The composition of Fourier–Mukai functors is again a Fourier–Mukai functor (see [15, Prop. 5.10]). In particular,

$$\Phi^1_{\mathcal{F}\star\mathcal{E}} \cong \Phi^1_{\mathcal{F}} \circ \Phi^1_{\mathcal{E}},$$

where $\star$ is the composition of kernels, and for $\mathcal{E}, \mathcal{F} \in \mathrm{D}^b(Y \times Y)$ it is defined by

$$\mathcal{F} \star \mathcal{E} = \mathbf{R}\pi_{1,3\,*}(\pi_{2,3}^*\mathcal{F} \overset{\mathbf{L}}{\otimes} \pi_{1,2}^*\mathcal{E}) \in \mathrm{D}^b(Y \times Y).$$

$\pi_{i,j}$ is projection to the ith times jth factor in the product $Y \times Y \times Y$.

Lemma 2.2. *Given $\mathcal{F}, \mathcal{E}_i, \mathcal{F}_i \in \mathrm{D}^b(Y)$, for $i = 1, 2$, and $\mathcal{W} \in \mathrm{D}^b(Y \times Y)$, we have the following isomorphisms:*

(1) $\Phi^i_{\mathcal{E}_2 \overset{\mathbf{L}}{\boxtimes} \mathcal{E}_1}(\mathcal{F}) \cong \mathcal{E}_{3-i} \otimes_{\Bbbk} \mathbf{R}\Gamma(Y, \mathcal{E}_i \overset{\mathbf{L}}{\otimes} \mathcal{F})$, *for $i = 1, 2$;*

(2) $(\mathcal{F}_2 \overset{\mathbf{L}}{\boxtimes} \mathcal{F}_1) \star \mathcal{W} \cong \mathcal{F}_2 \overset{\mathbf{L}}{\boxtimes} \Phi^2_{\mathcal{W}}(\mathcal{F}_1)$*;*

(3) $\mathcal{W} \star (\mathcal{F}_2 \overset{\mathbf{L}}{\boxtimes} \mathcal{F}_1) \cong \Phi^1_{\mathcal{W}}(\mathcal{F}_2) \overset{\mathbf{L}}{\boxtimes} \mathcal{F}_1$;

(4) $(\mathcal{F}_2 \overset{\mathbf{L}}{\boxtimes} \mathcal{F}_1) \star (\mathcal{E}_2 \overset{\mathbf{L}}{\boxtimes} \mathcal{E}_1) \cong \mathcal{F}_2 \overset{\mathbf{L}}{\boxtimes} \mathcal{E}_1 \otimes_{\Bbbk} \mathbf{R}\Gamma(Y, \mathcal{E}_2 \overset{\mathbf{L}}{\otimes} \mathcal{F}_1)$.

Proof. (1) By definition we have

$$\Phi^i_{\mathcal{E}_2 \overset{\mathbf{L}}{\boxtimes} \mathcal{E}_1}(\mathcal{F}) \cong \mathbf{R}\pi_{3-i\,*}\left(\pi^*_{3-i}\mathcal{E}_{3-i} \overset{\mathbf{L}}{\otimes} \pi^*_i(\mathcal{E}_i \overset{\mathbf{L}}{\otimes} \mathcal{F})\right)$$

and using the projection formula

$$\mathbf{R}\pi_{3-i\,*}\left(\pi^*_{3-i}\mathcal{E}_{3-i} \overset{\mathbf{L}}{\otimes} \pi^*_i(\mathcal{E}_i \overset{\mathbf{L}}{\otimes} \mathcal{F})\right) \cong \mathcal{E}_{3-i} \overset{\mathbf{L}}{\otimes} \mathbf{R}\pi_{3-i\,*}\pi^*_i(\mathcal{E}_i \overset{\mathbf{L}}{\otimes} \mathcal{F}).$$

Finally, $\mathbf{R}\pi_{3-i\,*}\pi^*_i(\mathcal{E}_i \overset{\mathbf{L}}{\otimes} \mathcal{F}) \cong \mathcal{O}_Y \otimes_{\Bbbk} \mathbf{R}\Gamma(Y, \mathcal{E}_i \overset{\mathbf{L}}{\otimes} \mathcal{F})$ by the "flat base change" theorem applied to the Cartesian square

$$\begin{tikzcd} Y \times Y \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] & Y \arrow[d] \\ Y \arrow[r] & \operatorname{Spec} \Bbbk. \end{tikzcd} \tag{6}$$

(2) By definition

$$(\mathcal{F}_2 \overset{\mathbf{L}}{\boxtimes} \mathcal{F}_1) \star \mathcal{W} = \mathbf{R}\pi_{1,3*}\left(\pi^*_{2,3}(\pi^*_2\mathcal{F}_2 \overset{\mathbf{L}}{\otimes} \pi^*_1\mathcal{F}_1) \overset{\mathbf{L}}{\otimes} \pi^*_{1,2}\mathcal{W}\right).$$

Since $\pi_2 \circ \pi_{2,3} = \pi_2 \circ \pi_{1,3}$ and $\pi_1 \circ \pi_{2,3} = \pi_2 \circ \pi_{1,2}$, the last expression can be rewritten as

$$\mathbf{R}\pi_{1,3*}\left(\pi^*_{1,3}\pi^*_2\mathcal{F}_2 \overset{\mathbf{L}}{\otimes} \pi^*_{1,2}(\pi^*_2\mathcal{F}_1 \overset{\mathbf{L}}{\otimes} \mathcal{W})\right).$$

Using the projection formula and the "flat base change" theorem for the Cartesian square

$$\begin{tikzcd} Y \times Y \times Y \arrow[r, "\pi_{1,2}"] \arrow[d, "\pi_{1,3}"'] & Y \times Y \arrow[d, "\pi_1"] \\ Y \times Y \arrow[r, "\pi_1"] & Y \end{tikzcd} \tag{7}$$

we arrive at

$$\pi^*_2\mathcal{F}_2 \overset{\mathbf{L}}{\otimes} \mathbf{R}\pi_{1,3*}\pi^*_{1,2}(\pi^*_2\mathcal{F}_1 \overset{\mathbf{L}}{\otimes} \mathcal{W}) \cong \pi^*_2\mathcal{F}_2 \overset{\mathbf{L}}{\otimes} \pi^*_1\mathbf{R}\pi_{1*}(\pi^*_2\mathcal{F}_1 \overset{\mathbf{L}}{\otimes} \mathcal{W}) = \mathcal{F}_2 \overset{\mathbf{L}}{\boxtimes} \Phi^2_{\mathcal{W}}(\mathcal{F}_1).$$

This proves (2).

The proof of (3) is completely similar to the proof of (2), while (4) follows immediately from the first two statements. □

Let $\delta \colon Y \to Y \times Y$ be the diagonal morphism. We will write $\mathcal{O}_{\Delta_Y}$ (or simply $\mathcal{O}_\Delta$) for $\delta_*\mathcal{O}_Y$. It is well known that $\Phi^i_{\mathcal{O}_\Delta} \cong \mathrm{id}_{\mathrm{D}^b(Y)}$ for $i = 1, 2$, and more generally, if $\mathcal{L}$ is a line bundle on Y, $\Phi^i_{\delta_*\mathcal{L}}$ is isomorphic to the autoequivalence of $\mathrm{D}^b(Y)$ defined by $\mathcal{F} \mapsto \mathcal{F} \otimes \mathcal{L}$ (see, e.g., [15, Ex. 5.4]).

3. The resolution of the diagonal for weighted projective spaces

Let $\mathbb{P} := \mathbb{P}^n(\mathbf{w})$ be a weighted projective space (regarded as a stack)[1] with weight vector $\mathbf{w} = (w_0, w_1, \ldots, w_n)$, and $w_i > 0$ for all $i = 0, \ldots, n$. We introduce some notations that will be used throughout the paper. For a subset

[1] We will often refer to [8], where weighted projective spaces are regarded as graded schemes. The equivalence with the point of view of stacks is explained in [8, Section 1.6]

$I \subseteq \{0, 1, \ldots, n\}$ the symbol $|I|$ denotes the cardinality of I. Similarly we introduce the following sums of weights:

$$w_I = \sum_{i \in I} w_i; \quad w = w_{\{0,1,\ldots,n\}} = \sum_{i=0}^{n} w_i.$$

Let $\mathrm{P} = \Bbbk[x_0, x_1, \ldots, x_n]$ be the graded polynomial ring where the generators have $\deg(x_i) = w_i$. First, recall the Koszul complex $\mathcal{K}$ on $\mathbb{P} = \mathbb{P}^n(\mathbf{w})$ associated to the regular sequence $(x_0, x_1, \ldots, x_n)$, whose jth term is given by

$$\mathcal{K}^j = \bigoplus_{|I|=-j} \mathcal{O}(-w_I),$$

where for a given $i \in \mathbb{Z}$ we abbreviated $\mathcal{O}_{\mathbb{P}}(i)$ by $\mathcal{O}(i)$. The summation is over all subsets of $\{0, 1, \ldots, n\}$ of cardinality $-j$. Obviously, $\mathcal{K}^j$ is nonzero only for $-n-1 \leq j \leq 0$. The components of the jth differential of $\mathcal{K}$

$$\mathcal{K}^j = \bigoplus_{|I|=-j} \mathcal{O}(-w_I) \xrightarrow{d^j_{\mathcal{K}}} \mathcal{K}^{j+1} = \bigoplus_{|I'|=-j-1} \mathcal{O}(-w_{I'})$$

are given by

$$\left(d^j_{\mathcal{K}}\right)^I_{I'} = \begin{cases} (-1)^{N^i_I}\, x_i \in \mathrm{P}_{w_i} & \text{if } I = I' \cup \{i\} \\ 0 & \text{otherwise} \end{cases}$$

where the integer N^i_I is defined as the cardinality

$$N^i_I = |\{j \in I : j < i\}|.$$

The notation P_a, for $a \in \mathbb{Z}$, refers to the degree a subspace of $\mathrm{P} = \Bbbk[x_0, x_1, \ldots, x_n]$; P_a will be viewed as a $\Bbbk$-vector space. Since we chose the weights of the x_i's to be positive, P_a is the zero vector space whenever $a < 0$.

Following [8, Definition 2.5.2] for $-w < l \leq 0$ we introduce the subcomplex $\mathcal{M}_l$ of the twisted Koszul complex $\mathcal{K}(-l)$:

$$\mathcal{M}^j_l = \bigoplus_{|I|=-j, w_I \leq -l} \mathcal{O}(-l - w_I) \subseteq \bigoplus_{|I|=-j} \mathcal{O}(-l - w_I) = \mathcal{K}^j(-l).$$

Note that $\mathcal{M}_0$ is the complex $\mathcal{O}$ concentrated in degree 0.

The $\mathcal{M}_l$'s have a natural interpretation: the left dual of the full and strong exceptional sequence $(\mathcal{O}, \mathcal{O}(1), \ldots, \mathcal{O}(w-1))$ in $\mathrm{D}^b(\mathbb{P})$ is the full exceptional sequence $(\mathcal{M}_{1-w}[1-w], \ldots, \mathcal{M}_{-1}[-1], \mathcal{M}_0)$ [8, Proposition 2.5.11]. We will use the complexes $\mathcal{M}_l$ to give a generalization of Beilinson resolution of the diagonal for the stack $\mathbb{P}$.

For every $-w < k \leq 0$ we define the complex $\mathcal{R}_k \in \mathrm{C}^b(\mathbb{P} \times \mathbb{P})$ inductively. The starting point is

$$\mathcal{R}_0 := \mathcal{O}_{\mathbb{P}} \boxtimes \mathcal{M}_0 \cong \mathcal{O}_{\mathbb{P}\times\mathbb{P}}.$$

For $-w < k < 0$ we set

$$\mathcal{R}_k := \mathsf{MC}(\alpha_k : \mathcal{O}(k) \boxtimes \mathcal{M}_k[-1] \longrightarrow \mathcal{R}_{k+1}), \tag{8}$$

where α_k is a natural map that will be defined below. Observe that $\mathcal{R}_k$ is defined in terms of $\mathcal{R}_{k+1}$ since k is increasingly more negative. This convention ties well with the fact that all of our complexes will be nontrivial *only in negative* degree.

Given the definition of the mapping cone in (4), for a fixed k, the components of the complex $\mathcal{R}_k$ are immediate: for $j \in \mathbb{Z}$ they are

$$\mathcal{R}^j_k = \bigoplus_{k \leq l \leq 0} \mathcal{O}(l) \boxtimes \mathcal{M}^j_l = \bigoplus_{\substack{k \leq l \leq 0 \\ |I|=-j, w_I \leq -l}} \mathcal{O}(l, -l - w_I). \tag{9}$$

Here we use the shorthand notation

$$\mathcal{O}(i, j) = \mathcal{O}_{\mathbb{P}\times\mathbb{P}}(i, j) := \mathcal{O}_{\mathbb{P}}(i) \boxtimes \mathcal{O}_{\mathbb{P}}(j), \quad \text{for } i, j \in \mathbb{Z}.$$

As a result, the jth component of α_k in (8) has to be a map between

$$\alpha_k^j\colon \bigoplus_{|I|=1-j, w_I\le -k} \mathcal{O}(k,-k-w_I) \longrightarrow \bigoplus_{\substack{k<l\le 0\\ |I|=-j, w_I\le -l}} \mathcal{O}(l,-l-w_I). \tag{10}$$

Now each component of α_k^j

$$\left(\alpha_k^j\right)_{l,I'}^{I} \in \mathrm{Hom}_{\mathbb{P}\times\mathbb{P}}(\mathcal{O}(k,-k-w_I),\mathcal{O}(l,-l-w_{I'})) \cong \mathrm{P}_{l-k}\otimes_{\Bbbk}\mathrm{P}_{k-l+w_I-w_{I'}}$$

for $k<l\le 0$, $|I|=1-j$, $|I'|=-j$, $w_I\le -k$ and $w_{I'}\le -l$, is defined to be

$$\left(\alpha_k^j\right)_{l,I'}^{I} := \begin{cases} -(-1)^{N_I^i}\, x_i\otimes 1\in \mathrm{P}_{w_i}\otimes_{\Bbbk}\mathrm{P}_0 & \text{if } l=k+w_i,\ I=I'\cup\{i\}\\ 0 & \text{otherwise.}\end{cases} \tag{11}$$

Note that the map $x_i\otimes 1$ imposes the conditions $l=k+w_i$ and $-k-w_I=-l-w_{I'}$, which lead to $w_I=w_{I'}+w_i$, and is automatically satisfied by the condition $I=I'\cup\{i\}$.

For the inductive definition in (8) to make sense we need to make sure that for any $-w<k<0$

(i) $\mathcal{R}_{k+1}$ is a complex,
(ii) α_k is a morphism of complexes.

We can prove these simultaneously by induction: $\mathcal{R}_0=\mathcal{O}_{\mathbb{P}}\boxtimes\mathcal{M}_0$ is clearly a complex; by assuming that $\mathcal{R}_{k+1}$ is a complex and that α_k is a map of complexes, the mapping cone construction in (8) guarantees that $\mathcal{R}_k$ is a complex, i.e. $d_{\mathcal{R}_k}$ is a differential. Therefore, the key point is to prove (ii), and then (i) follows automatically.

In the light of (9) and (11) and the definition (8), it is immediate to write down explicitly the candidate differentials of $\mathcal{R}_k$ (so far these are only maps, since we have not yet proven that $d_{\mathcal{R}_k}^2=0$). Each component

$$\left(d_{\mathcal{R}_k}^j\right)_{l',I'}^{l,I} \in \mathrm{Hom}_{\mathbb{P}\times\mathbb{P}}(\mathcal{O}(l,-l-w_I),\mathcal{O}(l',-l'-w_{I'})) \cong \mathrm{P}_{l'-l}\otimes_{\Bbbk}\mathrm{P}_{l-l'+w_I-w_{I'}}$$

of $d_{\mathcal{R}_k}^j\colon \mathcal{R}_k^j\to\mathcal{R}_k^{j+1}$, for $k\le l,l'\le 0$, $|I|=-j$, $|I'|=-j-1$, $w_I\le -l$ and $w_{I'}\le -l'$, is given by

$$\left(d_{\mathcal{R}_k}^j\right)_{l',I'}^{l,I} = \begin{cases} (-1)^{N_I^i}\, 1\otimes x_i\in \mathrm{P}_0\otimes_{\Bbbk}\mathrm{P}_{w_i} & \text{if } l'=l,\ I=I'\cup\{i\}\\ -(-1)^{N_I^i}\, x_i\otimes 1\in \mathrm{P}_{w_i}\otimes_{\Bbbk}\mathrm{P}_0 & \text{if } l'=l+w_i,\ I=I'\cup\{i\}\\ 0 & \text{otherwise.}\end{cases} \tag{12}$$

This is straightforward to check using the definition of mapping cone and assuming inductively that it holds for $k+1$. Thus everything will be well defined once we prove the following:

Lemma 3.1. $\alpha_k\colon \mathcal{O}(k)\boxtimes\mathcal{M}_k[-1]\longrightarrow\mathcal{R}_{k+1}$ *as defined above is a map of complexes.*

Proof. By the inductive hypothesis discussed above, we can assume that $\mathcal{R}_{k+1}$ is a complex, and that $d_{\mathcal{R}_{k+1}}$ is given by (12). Glancing at the definition (8), we need to show that

$$\alpha_k^{j+1}\circ d^j_{\mathcal{O}(k)\boxtimes\mathcal{M}_k[-1]} = d^j_{\mathcal{R}_{k+1}}\circ\alpha_k^j.$$

Using the fact that $d^j_{\mathcal{O}(k)\boxtimes\mathcal{M}_k[-1]} = -\,\mathrm{id}_{\mathcal{O}(k)}\boxtimes d^{j-1}_{\mathcal{M}_k}$, this is equivalent to the following square being commutative:

$$\begin{tikzcd} \mathcal{O}(k)\boxtimes\mathcal{M}_k^{j-1} \arrow[r, "-\,\mathrm{id}_{\mathcal{O}(k)}\boxtimes d^{j-1}_{\mathcal{M}_k}"] \arrow[d, "\alpha_k^j"] & \mathcal{O}(k)\boxtimes\mathcal{M}_k^{j} \arrow[d, "\alpha_k^{j+1}"] \\ \mathcal{R}_{k+1}^j \arrow[r, "d^j_{\mathcal{R}_{k+1}}"] & \mathcal{R}_{k+1}^{j+1}. \end{tikzcd} \tag{13}$$

Let us restrict to one of the direct summands of $\mathcal{O}(k) \boxtimes \mathcal{M}_k^{j-1}$, say $\mathcal{O}(k, -k - w_I)$. The square equation (13) involves two mappings: the "horizontal then vertical" map is

$$\mathcal{O}(k, -k - w_I) \xrightarrow{\bigoplus_{\substack{i \in I \\ i' \in I_i}} (-1)^{N_I^i + N_{I_i}^{i'}} (x_{i'} \otimes x_i)} \bigoplus_{\substack{i, i' \in I \\ i \neq i'}} \mathcal{O}(k + w_{i'}, w_i - k - w_I)$$

where $I_i = I - \{i\}$; while the "vertical then horizontal" map is

$$\mathcal{O}(k, -k - w_I) \xrightarrow{\bigoplus_{\substack{i \in I \\ i' \in I_i}} -(-1)^{N_I^i + N_{I_i}^{i'}} \begin{matrix} (x_i \otimes x_{i'}) \oplus \\ (-x_i x_{i'} \otimes 1) \end{matrix}} \bigoplus_{\substack{i, i' \in I \\ i \neq i'}} \begin{matrix} \mathcal{O}(k + w_i, w_{i'} - k - w_I) \oplus \\ \mathcal{O}(k + w_i + w_{i'}, -k - w_I) \end{matrix}.$$

Let us focus on the second map. Writing $\bigoplus_{i \in I, i' \in I_i}$ artificially singled out one element of the set $\{i, i'\}$, since the summation is over pairs $i \neq i'$. Changing variables $i \leftrightharpoons i'$ and observing that

$$(-1)^{N_I^i + N_{I_i}^{i'}} = -(-1)^{N_I^{i'} + N_{I_{i'}}^i}$$

we see immediately that the second component, $(-x_i x_{i'} \otimes 1)$, is in fact the zero map; whereas after the exchange $i \leftrightharpoons i'$ the first component is identical to the "horizontal then vertical" map. This proves that the square indeed commutes. $\square$

The main benefit of these definitions is the following generalization to weighted projective spaces of Beilinson's resolution of the diagonal for $\mathbb{P}^n$:

Proposition 3.2. *There is a natural morphism of complexes $\nu \colon \mathcal{R}_{1-w} \to \mathcal{O}_\Delta$, which descends to an isomorphism in* $\mathrm{D}^b(\mathbb{P} \times \mathbb{P})$.

Proof. Let us set for brevity $\mathcal{R} := \mathcal{R}_{1-w}$. In order to construct the natural morphism of complexes $\nu \colon \mathcal{R} \to \mathcal{O}_\Delta$, start with the adjunction $\delta^* \dashv \delta_*$ and observe that

$$\mathrm{Hom}_{\mathbb{P} \times \mathbb{P}}(\mathcal{O}(l, -l), \mathcal{O}_\Delta = \delta_* \mathcal{O}_{\mathbb{P}}) \cong \mathrm{Hom}_{\mathbb{P}}(\delta^* \mathcal{O}(l, -l), \mathcal{O}_{\mathbb{P}}) \cong \mathrm{Hom}_{\mathbb{P}}(\mathcal{O}_{\mathbb{P}}, \mathcal{O}_{\mathbb{P}})$$

for any integer l. Let $f_l \colon \mathcal{O}(l, -l) \to \mathcal{O}_\Delta$ be the morphism corresponding to the identity under this isomorphism. Equivalently, f_l is induced by "multiplication". This follows from the fact that

$$\mathcal{O}(l, -l) = \pi_2^* \mathcal{O}(l) \otimes \pi_1^* \mathcal{O}(-l) \cong \pi_2^* \mathcal{O}(l) \otimes \pi_1^* \mathcal{O}(l)^\vee$$

and we can use the natural pairing between $\mathcal{O}(l)$ and $\mathcal{O}(l)^\vee$ to map $\mathcal{O}(l, -l)$ into $\mathcal{O}_\Delta$.

Since $\mathcal{R}^0 = \bigoplus_{-w < l \leq 0} \mathcal{O}(l, -l)$, we define $\nu^0 := \bigoplus_{-w < l \leq 0} f_l$; while of course for $i \neq 0$ we let $\nu^i = 0$. To check that $\nu \colon \mathcal{R} \to \mathcal{O}_\Delta$ is a morphism of complexes, we only need to show that $\nu^0 \circ d_{\mathcal{R}}^{-1} = 0$. The relevant diagram is

$$\begin{array}{c} \mathcal{R} \\ \Big\downarrow \nu \\ \mathcal{O}_\Delta \end{array} \quad : \quad \begin{tikzcd} \bigoplus_{-w < l \leq 0, w_i \leq -l} \mathcal{O}(l, -l - w_i) \arrow[r, "d_{\mathcal{R}}^{-1}"] \arrow[d, "\nu^{-1} = 0"] & \bigoplus_{-w < l \leq 0} \mathcal{O}(l, -l) \arrow[d, "\nu^0"] \\ 0 \arrow[r] & \mathcal{O}_\Delta. \end{tikzcd}$$

Pick a component in $\mathcal{R}^{-1}$, say $\mathcal{O}(l, -l - w_i)$. There are two nontrivial components of $d_{\mathcal{R}}^{-1}$ emanating from it: $\left(d_{\mathcal{R}}^{-1}\right)_{l, \emptyset}^{l, \{i\}}$ and $\left(d_{\mathcal{R}}^{-1}\right)_{l + w_i, \emptyset}^{l, \{i\}}$; and it is clear from (12) that

$$f_l \circ \left(d_{\mathcal{R}}^{-1}\right)_{l, \emptyset}^{l, \{i\}} + f_{l + w_i} \circ \left(d_{\mathcal{R}}^{-1}\right)_{l + w_i, \emptyset}^{l, \{i\}} = 0.$$

This proves that $\nu \colon \mathcal{R} \to \mathcal{O}_\Delta$ is indeed a morphism of complexes.

To show that ν is an isomorphism in $\mathrm{D}^b(\mathbb{P}\times\mathbb{P})$ it suffices to prove that $\Phi^1_\nu\colon \Phi^1_{\mathcal{R}}\to\Phi^1_{\mathcal{O}_\Delta}$ is an isomorphism of functors. Indeed, assuming that Φ^1_ν is an isomorphism, we immediately deduce the isomorphism of functors $\Phi^1_{\mathsf{C}(\nu)}\cong 0$, which implies that $\mathsf{C}(\nu)\cong 0$ (hence ν is an isomorphism) in $\mathrm{D}^b(\mathbb{P}\times\mathbb{P})$ by Lemma 2.1.

Now recall that $(\mathcal{O}(1-w),\dots,\mathcal{O}(-1),\mathcal{O})$ is a full and strong exceptional sequence ([8, Remark 2.2.6], [3, Theorem 2.12]), and hence, in particular, it generates $\mathrm{D}^b(\mathbb{P})$ as triangulated category. Therefore, in order to prove that Φ^1_ν is an isomorphism it is enough to show that

$$\Phi^1_\nu(\mathcal{O}(k))\colon\ \Phi^1_{\mathcal{R}}(\mathcal{O}(k))\longrightarrow\Phi^1_{\mathcal{O}_\Delta}(\mathcal{O}(k))$$

is an isomorphism for $-w<k\le 0$.

It is immediate that $\Phi^1_{\mathcal{O}_\Delta}(\mathcal{O}(k))\cong\mathcal{O}(k)$. Next we compute $\Phi^1_{\mathcal{R}}(\mathcal{O}(k))$ in two different ways: first using the recursion (8), and then using (9). For the first computation notice that by part (1) of Lemma 2.2

$$\Phi^1_{\mathcal{O}(l)\boxtimes\mathcal{M}_l}(\mathcal{O}(k))\cong\mathcal{O}(l)\otimes_{\Bbbk}\mathbf{R}\Gamma(\mathbb{P},\mathcal{M}_l\otimes\mathcal{O}(k)).$$

On the other hand, for $-w<k,l\le 0$, $\dim H^i(\mathbb{P},\mathcal{M}_l(k))=\delta_{k,l}\delta_{i,0}$, as shown in the proof of [8, Theorem 2.5.8]; therefore

$$\Phi^1_{\mathcal{O}(l)\boxtimes\mathcal{M}_l}(\mathcal{O}(k))\cong\begin{cases}\mathcal{O}(k) & \text{if } k=l\\ 0 & \text{if } k\neq l.\end{cases}$$

It is also immediate that

$$\Phi^1_{\mathsf{MC}(f\colon\mathcal{A}\to\mathcal{B})}(\mathcal{F})\cong\mathsf{C}\Big(\Phi^1_f(\mathcal{F})\colon\ \Phi^1_{\mathcal{A}}(\mathcal{F})\longrightarrow\Phi^1_{\mathcal{B}}(\mathcal{F})\Big).$$

Using these two facts and the defining equation (8), it is easy to deduce that

$$\Phi^1_{\mathcal{R}}(\mathcal{O}(k))\cong\mathcal{O}(k). \tag{14}$$

At this point it is clear that $\Phi^1_\nu(\mathcal{O}(k))$ can be identified with a map $\mathcal{O}(k)\to\mathcal{O}(k)$, but it is not evident which map it is. To settle this question we can also compute $\Phi^1_{\mathcal{R}}(\mathcal{O}(k))$ directly from (9) and (12). Observe that by the projection formula

$$\mathbf{R}\pi_{2*}\mathcal{O}(i,j)=\mathbf{R}\pi_{2*}(\pi_2^*\mathcal{O}(i)\otimes\pi_1^*\mathcal{O}(j))\cong\mathcal{O}(i)\otimes\mathbf{R}\pi_{2*}\pi_1^*\mathcal{O}(j).$$

On the other hand, for any $j>-w$, from (6) we know that $\mathbf{R}\pi_{2*}\pi_1^*\mathcal{O}(j)=\mathcal{O}\otimes_{\Bbbk}\mathrm{P}_j$; and as a result

$$\mathbf{R}\pi_{2*}\mathcal{O}(i,j)\cong\pi_{2*}\mathcal{O}(i,j)\cong\mathcal{O}(i)\otimes_{\Bbbk}\mathrm{P}_j. \tag{15}$$

Therefore, for any $-w<k\le 0$, every term of $\mathcal{R}\otimes\pi_2^*\mathcal{O}(k)$ is π_{1*}-acyclic, and we have a natural isomorphism $\Phi^1_{\mathcal{R}}(\mathcal{O}(k))\cong\pi_{1*}(\mathcal{R}\otimes\pi_2^*\mathcal{O}(k))\cong\mathcal{C}_k$, where

$$\mathcal{C}_k^j=\bigoplus_{-w<l\le 0,|I|=-j,w_I\le -l}\mathcal{O}(l)\otimes_{\Bbbk}\mathrm{P}_{k-l-w_I}$$

and $d^j_{\mathcal{C}_k}$ is induced by $d^j_{\mathcal{R}}$, and in fact is given by exactly the same formal expression as (12), although the two act on different complexes. Also note that most terms in $\mathcal{C}_k^j$ are zero, except for those that satisfy the condition $k-l-w_I\ge 0$.

In this presentation $\Phi^1_\nu(\mathcal{O}(k))$ can be identified with the natural morphism $\nu_k\colon\mathcal{C}_k\to\mathcal{O}(k)$, where the only nonzero component of ν_k is

$$\nu_k^0\colon\bigoplus_{-w<l\le 0}\mathcal{O}(l)\otimes_{\Bbbk}\mathrm{P}_{k-l}\longrightarrow\mathcal{O}(k),$$

and this is such that each $\nu_k^0|_{\mathcal{O}(l)\otimes_{\Bbbk}\mathrm{P}_{k-l}}$ is given by the multiplication map. This follows readily from the definition of $\nu\colon\mathcal{R}\to\mathcal{O}_\Delta$ and the isomorphisms that led to $\Phi^1_{\mathcal{R}}(\mathcal{O}(k))\cong\mathcal{C}_k$.

Clearly ν_k^0 is surjective, hence

$$H^0(\nu_k)\colon\ H^0(\mathcal{C}_k)=\mathcal{C}_k^0/\operatorname{im}d^{-1}_{\mathcal{C}_k}\longrightarrow\mathcal{O}(k)$$

is also surjective. We have already seen in Eq. (14) that $H^0(\mathcal{C}_k) \cong \mathcal{O}(k)$ and $H^i(\mathcal{C}_k) = 0$ for $i \neq 0$, and thus $H^0(\nu_k)$ is an isomorphism; and consequently ν_k is a quasi-isomorphism.[2] $\square$

Remark 3.3. If $w_0 = \cdots = w_n = 1$, i.e. $\mathbb{P}$ is the ordinary projective space $\mathbb{P}^n$, then the complex $\mathcal{R} = \mathcal{R}_{-n}$ defined in (8) does not coincide with the well-known resolution of the diagonal first considered by Beilinson [4] (for weighted projective planes, i.e. $n = 2$, the resolution coincides with the one considered in the context of quivers by King [19]). Beilinson's resolution $\mathcal{B} \in \mathrm{C}^b(\mathbb{P}^n \times \mathbb{P}^n)$ is defined by $\mathcal{B}^j := \mathcal{O}(j) \boxtimes \Omega^{-j}(-j)$ (again, $\mathcal{B}^j \neq 0$ only for $-n \leq j \leq 0$) with differential given (for $-n \leq j < 0$) by the natural morphism

$$\begin{aligned} d^j_{\mathcal{B}} &\in \mathrm{Hom}_{\mathbb{P}^n \times \mathbb{P}^n}(\mathcal{O}(j) \boxtimes \Omega^{-j}(-j), \mathcal{O}(j+1) \boxtimes \Omega^{-j-1}(-j-1)) \\ &\cong \mathrm{Hom}_{\mathbb{P}^n}(\mathcal{O}(j), \mathcal{O}(j+1)) \otimes_{\Bbbk} \mathrm{Hom}_{\mathbb{P}^n}(\Omega^{-j}(-j), \Omega^{-j-1}(-j-1)) \cong \mathrm{P}_1 \otimes_{\Bbbk} \mathrm{P}_1^{\vee} \cong \mathrm{Hom}_{\Bbbk}(\mathrm{P}_1, \mathrm{P}_1) \end{aligned}$$

corresponding to id. Observe that if we define complexes $\mathcal{B}_k$ by

$$\mathcal{B}^j_k := \begin{cases} \mathcal{B}^j & \text{if } j \geq k \\ 0 & \text{if } j < k \end{cases}$$

with $d^j_{\mathcal{B}_k} := d^j_{\mathcal{B}}$ for $j \geq k$ (hence $\mathcal{B} = \mathcal{B}_{-n}$), then $\mathcal{B}_0 = \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^n} = \mathcal{R}_0$, while for $-n \leq k < 0$

$$\mathcal{B}_k = \mathsf{MC}\Big(\mathcal{O}(k) \boxtimes \Omega^{-k}(-k)[-k-1] \longrightarrow \mathcal{B}_{k+1}\Big)$$

(where the morphism is induced by $d^k_{\mathcal{B}}$), in analogy with (8). Actually, using the fact that $\mathcal{M}_k \cong \Omega^{-k}(-k)[-k]$ in $\mathrm{D}^b(\mathbb{P}^n)$ (see [8, Remark 2.5.9]), it can be easily proved by descending induction on k that $\mathcal{B}_k \cong \mathcal{R}_k$ in $\mathrm{D}^b(\mathbb{P}^n \times \mathbb{P}^n)$ (but not in $\mathrm{K}^b(\mathbb{P}^n \times \mathbb{P}^n)$). Clearly the complex $\mathcal{B}$ is simpler than $\mathcal{R}$, but it does not seem possible to extend it to the weighted case, the problem coming from the fact that the complexes $\mathcal{M}_k$ are not quasi-isomorphic to shifts of ordinary sheaves in general.

4. Proof of the theorem

We start out by expressing the functors K and L appearing in Theorem 1.1, and defined in (1) and (2), as Fourier–Mukai functors. Lemma 3.2 of [24] shows that the kernel of K is $\mathcal{K} := \mathsf{C}\Big(\mathcal{O}_{X \times X} \xrightarrow{\delta^\sharp} \delta_* \mathcal{O}_X\Big)$, where $\delta^\sharp$ is the natural map associated to $\delta \colon X \hookrightarrow X \times X$ (clearly $\mathcal{K} \cong \mathcal{I}_\Delta[1]$, where $\mathcal{I}_\Delta$ is the ideal sheaf of the diagonal in $X \times X$, but this observation is of no use in our context). Similarly, the kernel of L is $\mathcal{L} := \delta_* \mathcal{O}_X(1)$. Therefore $\mathsf{G} = \mathsf{L} \circ \mathsf{K} \cong \Phi^1_{\mathcal{L} \star \mathcal{K}}$. Using (3) of Lemma 2.2 it is easy to show that

$$\mathcal{L} \star \mathcal{K} = \delta_* \mathcal{O}_X(1) \star \mathsf{C}\Big(\mathcal{O}_{X \times X} \xrightarrow{\delta^\sharp} \delta_* \mathcal{O}_X\Big) \cong \mathsf{C}\Big(\mathcal{O}_{X \times X}(1,0) \xrightarrow{g} \delta_* \mathcal{O}_X(1)\Big),$$

where $g \colon \mathcal{O}_{X \times X}(1,0) \to \delta_* \mathcal{O}_X(1)$ is the natural morphism. Using this fact, Theorem 1.1 is a consequence of the following proposition:

Proposition 4.1. *Let X be a smooth anti-canonical stacky hypersurface in $\mathbb{P}$. For the natural morphism $g \colon \mathcal{O}_{X \times X}(1,0) \to \delta_* \mathcal{O}_X(1)$ let $\mathcal{G}$ denote its mapping cone; i.e. $\mathcal{G} := \mathsf{MC}(g)$. Then $(\mathcal{G})^{\star w} \cong \mathcal{O}_{\Delta_X}[2]$ in $\mathrm{D}^b(X \times X)$.*

The rest of the section is dedicated to the proof of the proposition.

Let $\iota \colon X \hookrightarrow \mathbb{P}$ denote the inclusion. For a complex $\mathcal{E} \in \mathrm{K}^b(\mathbb{P})$ we will write $\widehat{\mathcal{E}}$ for the restriction $\iota^* \mathcal{E} \in \mathrm{K}^b(X)$ and similarly for morphisms in $\mathrm{K}^b(\mathbb{P})$. The same notation will be used for the inclusion $\iota \times \iota \colon X \times X \hookrightarrow \mathbb{P} \times \mathbb{P}$, but it will be clear from the context which one is meant. Note that if each $\mathcal{E}^j$ is locally free, then $\widehat{\mathcal{E}} \cong \mathbf{L}\iota^* \mathcal{E}$ in $\mathrm{D}^b(X)$.

For $0 < m \leq w$ we define the complex

$$\mathcal{G}_m := \mathsf{MC}\Big(\widehat{\mathcal{R}}_{1-m} \xrightarrow{\zeta_m} \delta_* \mathcal{O}_X\Big) \tag{16}$$

[2] In fact ν_k is an isomorphism in $\mathrm{K}^b(\mathbb{P})$, with inverse given by the natural morphism $\mathcal{O}(k) \hookrightarrow \mathcal{C}^0_k$, but this statement is of marginal interest to us.

where each component of the morphism

$$\zeta_m^0\colon \widehat{\mathcal{R}}_{1-m}^0 = \bigoplus_{-m<l\le 0} \mathcal{O}_{X\times X}(l,-l) \to \delta_*\mathcal{O}_X \tag{17}$$

is the natural map induced by multiplication (the argument in the proof of Proposition 3.2 showing that ν is a morphism of complexes also shows that ζ_m is a morphism of complexes).

We claim that it suffices to prove that

Claim 4.2. $(\mathcal{G})^{\star m} \cong \mathcal{G}_m(m,0)$ *for all* $0 < m \le w$.

Indeed, assuming this, and taking into account that $(\delta_*\mathcal{O}_X)(m,0) \cong \delta_*\mathcal{O}_X(m)$, the proposition follows from the following:

Lemma 4.3. $\mathcal{G}_w \cong \delta_*\mathcal{O}_X(-w)[2]$ *in* $\mathrm{D}^b(X\times X)$.

Proof. Since $\iota\times\iota$ is a closed immersion, it is sufficient to prove that

$$(\iota\times\iota)_*\mathcal{G}_w \cong (\iota\times\iota)_*\delta_*\mathcal{O}_X(-w)[2] \tag{18}$$

in $\mathrm{D}^b(\mathbb{P}\times\mathbb{P})$. By the projection formula and using the fact that $\delta\circ\iota = (\iota\times\iota)\circ\delta$ (the first δ is the diagonal of $\mathbb{P}$, while the second is the diagonal of X),

$$(\iota\times\iota)_*\mathcal{G}_w \cong \mathsf{C}\big((\iota\times\iota)_*\widehat{\mathcal{R}}_{1-w} \longrightarrow (\iota\times\iota)_*\delta_*\mathcal{O}_X\big) \cong \mathsf{C}(\mathcal{R}_{1-w}\otimes(\iota\times\iota)_*\mathcal{O}_{X\times X} \longrightarrow \delta_*\iota_*\mathcal{O}_X)\,.$$

It is useful to replace $(\iota\times\iota)_*\mathcal{O}_{X\times X}$ with a locally free resolution. To this purpose, let us start with the short exact sequence defining X

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}}(-w) \xrightarrow{s} \mathcal{O}_{\mathbb{P}} \xrightarrow{\iota^\sharp} \iota_*\mathcal{O}_X \longrightarrow 0. \tag{19}$$

Since we want a resolution for $(\iota\times\iota)_*\mathcal{O}_{X\times X}$ we consider the complex $\mathcal{S}$:

$$\mathcal{S}:\qquad 0\to\mathcal{O}_{\mathbb{P}\times\mathbb{P}}(-w,-w) \xrightarrow{\begin{pmatrix}-1\otimes s\\ s\otimes 1\end{pmatrix}} \begin{matrix}\mathcal{O}_{\mathbb{P}\times\mathbb{P}}(-w,0)\\ \oplus\\ \mathcal{O}_{\mathbb{P}\times\mathbb{P}}(0,-w)\end{matrix} \xrightarrow{\begin{pmatrix}s\otimes 1 & 1\otimes s\end{pmatrix}} \mathcal{O}_{\mathbb{P}\times\mathbb{P}}\to 0$$

where $\mathcal{O}_{\mathbb{P}\times\mathbb{P}}$ is sitting in degree 0.

As $X\times X$ is a codimension two complete intersection in $\mathbb{P}\times\mathbb{P}$, defined precisely by $s\otimes 1$ and $1\otimes s$, it is clear that the canonical map $(\iota\times\iota)^\sharp\colon \mathcal{S}^0 = \mathcal{O}_{\mathbb{P}\times\mathbb{P}} \to (\iota\times\iota)_*\mathcal{O}_{X\times X}$ induces a quasi-isomorphism between $\mathcal{S}$ and $(\iota\times\iota)_*\mathcal{O}_{X\times X}$. Thus

$$(\iota\times\iota)_*\mathcal{G}_w \cong \mathsf{C}(f\colon \mathcal{R}_{1-w}\otimes\mathcal{S} \longrightarrow \delta_*\iota_*\mathcal{O}_X)\,.$$

In the light of (17) each component of $f^0\colon (\mathcal{R}_{1-w}\otimes\mathcal{S})^0 \cong \bigoplus_{-w<l\le 0}\mathcal{O}_{\mathbb{P}\times\mathbb{P}}(l,-l) \to \delta_*\iota_*\mathcal{O}_X$ is the natural one (corresponding to $\iota^\sharp$ under adjunction).

Observing that $\delta^*\mathcal{S}$ can be identified with the complex

$$0\to\mathcal{O}_{\mathbb{P}}(-2w) \xrightarrow{\begin{pmatrix}-s\\ s\end{pmatrix}} \begin{matrix}\mathcal{O}_{\mathbb{P}}(-w)\\ \oplus\\ \mathcal{O}_{\mathbb{P}}(-w)\end{matrix} \xrightarrow{\begin{pmatrix}s & s\end{pmatrix}} \mathcal{O}_{\mathbb{P}}\to 0$$

and denoting by $h\colon \delta^*\mathcal{S}\to\iota_*\mathcal{O}_X$ the natural morphism defined by $h^0 = \iota^\sharp\colon (\delta^*\mathcal{S})^0 \cong \mathcal{O}_{\mathbb{P}} \to \iota_*\mathcal{O}_X$, it is also straightforward to check that the diagram

$$\begin{tikzcd}
\mathcal{R}_{1-w}\otimes\mathcal{S} \arrow[r, "f"] \arrow[d, "\nu\otimes\mathrm{id}"'] & \delta_*\iota_*\mathcal{O}_X \\
(\delta_*\mathcal{O}_{\mathbb{P}})\otimes\mathcal{S} \arrow[r, "\varpi", "\cong"'] & \delta_*\delta^*\mathcal{S} \arrow[u, "\delta_*h"']
\end{tikzcd}$$

commutes ($\nu\colon \mathcal{R}_{1-w} \to \mathcal{O}_\Delta$ is the morphism from Proposition 3.2). Using this and Proposition 3.2 we have that

$$(\iota \times \iota)_* \mathcal{G}_w \cong \mathsf{C}(f) \cong \mathsf{C}((\delta_* h) \circ \varpi \circ (\nu \otimes \mathrm{id})) \cong \mathsf{C}(\delta_* h) \cong \delta_* \mathsf{C}(h)\,.$$

On the other hand, $\delta^*\mathcal{S}$ is isomorphic in $\mathrm{C}^b(\mathbb{P})$ to the complex

$$\delta^*\mathcal{S}: \qquad 0 \to \mathcal{O}_{\mathbb{P}}(-2w) \xrightarrow{\begin{pmatrix}0\\ s\end{pmatrix}} \begin{matrix}\mathcal{O}_{\mathbb{P}}(-w)\\ \oplus \\ \mathcal{O}_{\mathbb{P}}(-w)\end{matrix} \xrightarrow{\begin{pmatrix}s & 0\end{pmatrix}} \mathcal{O}_{\mathbb{P}} \to 0,$$

which clearly splits as $\mathsf{MC}\Big(\mathcal{O}_{\mathbb{P}}(-w) \xrightarrow{s} \mathcal{O}_{\mathbb{P}}\Big) \oplus \mathsf{MC}\Big(\mathcal{O}_{\mathbb{P}}(-2w) \xrightarrow{s} \mathcal{O}_{\mathbb{P}}(-w)\Big)[1]$. Since we defined h^0 to be $\iota^\sharp\colon \mathcal{O}_{\mathbb{P}} \to \iota_*\mathcal{O}_X$ we have that

$$\mathsf{C}(h) \cong \Big(\mathcal{O}_{\mathbb{P}}(-w) \xrightarrow{s} \mathcal{O}_{\mathbb{P}} \xrightarrow{\iota^\sharp} \iota_*\mathcal{O}_X\Big) \oplus \mathsf{MC}\Big(\mathcal{O}_{\mathbb{P}}(-2w) \xrightarrow{s} \mathcal{O}_{\mathbb{P}}(-w)\Big)[2].$$

Taking into account (19) it is clear that $\mathsf{C}(h) \cong \iota_*\mathcal{O}_X(-w)[2]$ in $\mathrm{D}^b(\mathbb{P})$. Therefore

$$(\iota \times \iota)_* \mathcal{G}_w \cong \delta_* \mathsf{C}(h) \cong \delta_*\iota_*\mathcal{O}_X(-w)[2] \cong (\iota \times \iota)_*\delta_*\mathcal{O}_X(-w)[2],$$

which proves (18), and concludes the proof of the lemma. □

Returning to the proof of the proposition, we will prove Claim 4.2 by induction on m. For $m = 1$ the isomorphism $\mathcal{G} \cong \mathcal{G}_1(1, 0)$ immediately follows from the definitions of $\mathcal{G}$ and $\mathcal{G}_1$. Therefore we assume Claim 4.2 to hold for some $0 < m < w$, and prove it for $m + 1$.

Noticing that $\mathcal{G} \cong \delta_*\mathcal{O}_X(1) \star \mathcal{G}_1$, what we need to show is equivalent to

$$\mathcal{G}_1 \star \mathcal{G}_m(m, 0) \cong \mathcal{G}_{m+1}(m, 0). \tag{20}$$

In order to prove this, we start with the following lemma:

Lemma 4.4. *For any $\mathcal{E} \in \mathrm{D}^b(X \times X)$ we have a natural isomorphism*

$$\mathcal{G}_1 \star \mathcal{E} \cong \mathsf{C}(\mathcal{O}_X \boxtimes \mathbf{R}\pi_{1*}\mathcal{E} \longrightarrow \mathcal{E}) \cong \mathsf{C}\big(\pi_1^*\mathbf{R}\pi_{1*}\mathcal{E} \longrightarrow \mathcal{E}\big)\,,$$

where the morphism in the right hand side is the natural one (corresponding, under adjunction, to $\mathrm{id}_{\mathbf{R}\pi_{1*}\mathcal{E}}$).

Proof. If $f\colon Y \to Z$ is a morphism of stacks, $\beta_f\colon \mathbf{L}f^* \circ \mathbf{R}f_* \to \mathrm{id}_{\mathrm{D}^b(Y)}$ and $\gamma_f\colon \mathrm{id}_{\mathrm{D}^b(Z)} \to \mathbf{R}f_* \circ \mathbf{L}f^*$ will denote the adjunction morphisms; then we need to prove that $\mathcal{G}_1 \star \mathcal{E} \cong \mathsf{C}\big(\beta_{\pi_1}(\mathcal{E})\big)$. Since $(-) \star \mathcal{E}$ is an exact functor we have $\mathcal{G}_1 \star \mathcal{E} \cong \mathsf{C}\big(\delta^\sharp \star \mathrm{id}_{\mathcal{E}}\colon \mathcal{O}_{X\times X} \star \mathcal{E} \to \delta_*\mathcal{O}_X \star \mathcal{E}\big)$. Applying the "flat base change" theorem to the Cartesian square

$$\begin{tikzcd} X \times X \arrow[r, "\tilde{\delta}"] \arrow[d, "\pi_2"'] & X \times X \times X \arrow[d, "\pi_{2,3}"] \\ X \arrow[r, "\delta"] & X \times X \end{tikzcd}$$

and using the projection formula, we find a natural isomorphism

$$\delta_*\mathcal{O}_X \star \mathcal{E} = \mathbf{R}\pi_{1,3_*}(\pi_{2,3}^*\delta_*\mathcal{O}_X \overset{\mathbf{L}}{\otimes} \pi_{1,2}^*\mathcal{E}) \cong \mathbf{R}\pi_{1,3_*}(\tilde{\delta}_*\mathcal{O}_{X\times X} \overset{\mathbf{L}}{\otimes} \pi_{1,2}^*\mathcal{E}) \cong \mathbf{R}\pi_{1,3_*}\tilde{\delta}_*\mathbf{L}\tilde{\delta}^*\pi_{1,2}^*\mathcal{E}.$$

On the other hand, $\mathcal{O}_{X\times X} \star \mathcal{E} \cong \mathbf{R}\pi_{1,3_*}\pi_{1,2}^*\mathcal{E}$, and it is clear that, with these identifications, the morphism $\delta^\sharp \star \mathrm{id}_{\mathcal{E}}$ corresponds to

$$\mathbf{R}\pi_{1,3_*}\gamma_{\tilde{\delta}}(\pi_{1,2}^*\mathcal{E})\colon \mathbf{R}\pi_{1,3_*}\pi_{1,2}^*\mathcal{E} \to \mathbf{R}\pi_{1,3_*}\tilde{\delta}_*\mathbf{L}\tilde{\delta}^*\pi_{1,2}^*\mathcal{E}.$$

Now, $\mathbf{R}\pi_{1,3_*}\tilde{\delta}_*\mathbf{L}\tilde{\delta}^*\pi_{1,2}^*\mathcal{E} \cong \mathcal{E}$ (because $\pi_{1,3} \circ \tilde{\delta} = \pi_{1,2} \circ \tilde{\delta} = \mathrm{id}_{X\times X}$), whereas $\mathbf{R}\pi_{1,3_*}\pi_{1,2}^*\mathcal{E} \cong \pi_1^*\mathbf{R}\pi_{1*}\mathcal{E}$ (by the "flat base change" theorem for the Cartesian square (7), with X in place of Y). To be more precise, the latter isomorphism can be expressed as the composition

$$\pi_1^*\mathbf{R}\pi_{1*}\mathcal{E} \xrightarrow{\pi_1^*\mathbf{R}\pi_{1*}\gamma_{\pi_{1,2}}(\mathcal{E})} \pi_1^*\mathbf{R}\pi_{1*}\mathbf{R}\pi_{1,2*}\pi_{1,2}^*\mathcal{E} \xrightarrow{\sim} \pi_1^*\mathbf{R}\pi_{1*}\mathbf{R}\pi_{1,3*}\pi_{1,2}^*\mathcal{E} \xrightarrow{\beta_{\pi_1}(\mathbf{R}\pi_{1,3*}\pi_{1,2}^*\mathcal{E})} \mathbf{R}\pi_{1,3*}\pi_{1,2}^*\mathcal{E},$$

where the middle map is the natural isomorphism (due to the fact that $\pi_1 \circ \pi_{1,2} = \pi_1 \circ \pi_{1,3}$). Thus we can conclude that $\mathcal{G}_1 \star \mathcal{E} \cong \mathsf{C}\Big(\mathbf{R}\pi_{1,3*}\gamma_{\tilde{\delta}}(\pi_{1,2}^*\mathcal{E})\Big) \cong \mathsf{C}\big(\beta_{\pi_1}(\mathcal{E})\big)$, provided we show that in the diagram (where the unnamed arrows denote the natural isomorphisms)

$$\begin{tikzcd}
\pi_1^*\mathbf{R}\pi_{1*}\mathcal{E} \arrow[r, "\pi_1^*\mathbf{R}\pi_{1*}\gamma_{\pi_{1,2}}(\mathcal{E})"] \arrow[dr] \arrow[dd, "\beta_{\pi_1}(\mathcal{E})"'] & \pi_1^*\mathbf{R}\pi_{1*}\mathbf{R}\pi_{1,2*}\pi_{1,2}^*\mathcal{E} \arrow[r] & \pi_1^*\mathbf{R}\pi_{1*}\mathbf{R}\pi_{1,3*}\pi_{1,2}^*\mathcal{E} \arrow[dl, "\pi_1^*\mathbf{R}\pi_{1*}\mathbf{R}\pi_{1,3*}\gamma_{\tilde{\delta}}(\pi_{1,2}^*\mathcal{E})"'] \arrow[d, "\beta_{\pi_1}(\mathbf{R}\pi_{1,3*}\pi_{1,2}^*\mathcal{E})"] \\
 & \pi_1^*\mathbf{R}\pi_{1*}\mathbf{R}\pi_{1,3*}\tilde{\delta}_*\mathbf{L}\tilde{\delta}^*\pi_{1,2}^*\mathcal{E} \arrow[dr, "\beta_{\pi_1}(\mathbf{R}\pi_{1,3*}\tilde{\delta}_*\mathbf{L}\tilde{\delta}^*\pi_{1,2}^*\mathcal{E})"'] & \mathbf{R}\pi_{1,3*}\pi_{1,2}^*\mathcal{E} \arrow[d, "\mathbf{R}\pi_{1,3*}\gamma_{\tilde{\delta}}(\pi_{1,2}^*\mathcal{E})"] \\
\mathcal{E} & & \mathbf{R}\pi_{1,3*}\tilde{\delta}_*\mathbf{L}\tilde{\delta}^*\pi_{1,2}^*\mathcal{E} \arrow[ll]
\end{tikzcd}$$

the outer square commutes. This follows from the fact that the three inner triangles commute, as it can be easily checked using well-known compatibilities between adjunction morphisms. $\square$

In order to apply the lemma for $\mathcal{E} = \mathcal{G}_m(m,0)$ we need to compute $\mathbf{R}\pi_{1*}\mathcal{G}_m(m,0)$. From the definition (16) $\mathcal{G}_m^0(m,0) \cong \delta_*\mathcal{O}_X(m)$, whereas for $j \neq 0$

$$\mathcal{G}_m^j(m,0) = \bigoplus_{-m<l\leq 0,|I|=-j-1,w_I\leq -l} \mathcal{O}_{X\times X}(l+m, -l-w_I). \tag{21}$$

To evaluate $\mathbf{R}\pi_{1*}\mathcal{G}_m(m,0)$ we use the following lemma, whose proof is straightforward:

Lemma 4.5. *With X as above, and for two integers p and p', with $0 < p < w$, we have the isomorphism*

$$R^k\pi_{1*}\mathcal{O}_{X\times X}(p,p') \cong H^k(X, \mathcal{O}_X(p)) \otimes_{\Bbbk} \mathcal{O}_X(p') \cong \begin{cases} \mathrm{P}_p \otimes_{\Bbbk} \mathcal{O}_X(p') & \textit{if } k=0 \\ 0 & \textit{if } k \neq 0. \end{cases}$$

Going back to (21), we see that $0 < l+m \leq m$. The inductive hypothesis also assumed that $0 < m < w$, thus $0 < l+m < w$ and the lemma applies. Noting that

$$\mathbf{R}\pi_{1*}\mathcal{G}_m^0(m,0) \cong \mathbf{R}\pi_{1*}\delta_*\mathcal{O}_X(m) \cong \pi_{1*}\delta_*\mathcal{O}_X(m) \cong \mathcal{O}_X(m),$$

and applying Lemma 4.5 we see that every term of $\mathcal{G}_m(m,0)$ is π_{1*}-acyclic, hence $\mathbf{R}\pi_{1*}\mathcal{G}_m(m,0) \cong \pi_{1*}\mathcal{G}_m(m,0)$. Inspecting the complex $\pi_{1*}\mathcal{G}_m(m,0)$ one observes that

$$\mathbf{R}\pi_{1*}\mathcal{G}_m(m,0) \cong \pi_{1*}\mathcal{G}_m(m,0) \cong \widehat{\mathcal{F}}_m, \tag{22}$$

where for $0 < m < w$ we defined

$$\mathcal{F}_m := \mathsf{MC}\Big(\pi_{1*}\mathcal{R}_{1-m}(m,0) \xrightarrow{\rho_m} \mathcal{O}_{\mathbb{P}}(m)\Big), \tag{23}$$

with each component of the morphism

$$\rho_m^0 : \pi_{1*}\mathcal{R}_{1-m}^0(m,0) \cong \bigoplus_{-m<l\leq 0} \mathrm{P}_{l+m} \otimes_{\Bbbk} \mathcal{O}_{\mathbb{P}}(-l) \longrightarrow \mathcal{O}_{\mathbb{P}}(m)$$

given as usual by the multiplication map.

Combining Lemma 4.4 and (22) we obtain

$$\mathcal{G}_1 \star \mathcal{G}_m(m,0) \cong \mathsf{MC}\big(\eta_m : \mathcal{O}_X \boxtimes \widehat{\mathcal{F}}_m \to \mathcal{G}_m(m,0)\big), \tag{24}$$

where $\eta_m^0 : \mathcal{O}_{X\times X}(0,m) \to \delta_*\mathcal{O}_X(m)$ is the natural map, and for $j<0$

$$\eta_m^j : \bigoplus_{\substack{-m<l\le 0\\ |I|=-j-1, w_I\le -l}} \mathrm{P}_{l+m}\otimes_{\Bbbk}\mathcal{O}_{X\times X}(0,-l-w_I) \longrightarrow \bigoplus_{\substack{-m<l\le 0\\ |I|=-j-1, w_I\le -l}} \mathcal{O}_{X\times X}(l+m,-l-w_I)$$

is induced by the multiplication maps $\mathrm{P}_{l+m}\otimes_{\Bbbk}\mathcal{O}_X \to \mathcal{O}_X(l+m)$.

It is easy to check that there is a natural morphism of complexes $\epsilon_m : \mathcal{M}_{-m} \to \mathcal{F}_m$, where $\epsilon_m^0 = \mathrm{id}_{\mathcal{O}(m)}$ and for $j<0$ each component of ϵ_m^j

$$\left(\epsilon_m^j\right)_{l,I'}^I \in \mathrm{Hom}_{\mathbb{P}}(\mathcal{O}(m-w_I), \mathrm{P}_{l+m}\otimes_{\Bbbk}\mathcal{O}(-l-w_{I'})) \cong \mathrm{P}_{l+m}\otimes_{\Bbbk}\mathrm{P}_{-m-l+w_I-w_{I'}}$$

(for $-m<l\le 0$, $|I|=-j$, $|I'|=-j-1$, $w_I\le m$ and $w_{I'}\le -l$) is given by

$$\left(\epsilon_m^j\right)_{l,I'}^I := \begin{cases} -(-1)^{N_I^i} x_i\otimes 1 \in \mathrm{P}_{w_i}\otimes_{\Bbbk}\mathrm{P}_0 & \text{if } l+m=w_i,\ I=I'\cup\{i\}\\ 0 & \text{otherwise.}\end{cases} \tag{25}$$

Lemma 4.6. *$\epsilon_m : \mathcal{M}_{-m} \to \mathcal{F}_m$ is an isomorphism in $\mathrm{D}^b(\mathbb{P})$ for any $0<m<w$.*

We will prove the lemma shortly, but first we look at its implications. Setting

$$\tilde{\eta}_m = \eta_m \circ (\mathrm{id}_{\mathcal{O}_X}\boxtimes\widehat{\epsilon}_m) : \mathcal{O}_X\boxtimes\widehat{\mathcal{M}}_{-m} \longrightarrow \mathcal{G}_m(m,0),$$

Lemma 4.6 and (24) imply that

$$\mathcal{G}_1 \star \mathcal{G}_m(m,0) \cong \mathsf{MC}(\eta_m) \cong \mathsf{MC}(\tilde{\eta}_m) \in \mathrm{D}^b(X\times X).$$

The components of $\tilde{\eta}_m$ can be calculated by composing η_m and $\mathrm{id}_{\mathcal{O}_X}\boxtimes\widehat{\epsilon}_m$. After an explicit calculation it turns out that $\tilde{\eta}_m^0 : \mathcal{O}_{X\times X}(0,m) \to \delta_*\mathcal{O}_X(m)$ is the natural map; while for $j<0$

$$\tilde{\eta}_m^j : (\mathcal{O}_X\boxtimes\widehat{\mathcal{M}}_{-m})^j \cong \mathcal{O}_X\boxtimes\widehat{\mathcal{M}}_{-m}^j \longrightarrow (\mathcal{G}_m(m,0))^j \cong \widehat{\mathcal{R}}_{1-m}^{j+1}(m,0)$$

can be identified with $\widehat{\alpha}_{-m}^{j+1}(m,0)$. From the defining equation (16) of $\mathcal{G}_m$ we see that

$$\mathsf{MC}(\tilde{\eta}_m)^0 \cong \mathcal{G}_m^0(m,0) \cong \delta_*\mathcal{O}_X(m) \cong \mathcal{G}_{m+1}^0(m,0),$$

while for $j<0$, recalling (8),

$$\mathsf{MC}(\tilde{\eta}_m)^j \cong \mathcal{G}_m^j(m,0) \oplus (\mathcal{O}_X\boxtimes\widehat{\mathcal{M}}_{-m})^{j+1} \cong \widehat{\mathcal{R}}_{1-m}^{j+1}(m,0)\oplus\mathcal{O}_X\boxtimes\widehat{\mathcal{M}}_{-m}^{j+1} \cong \widehat{\mathcal{R}}_{-m}^{j+1}(m,0) \cong \mathcal{G}_{m+1}^j(m,0).$$

Using the explicit form of $\tilde{\eta}_m$ it is also immediate to check that the differential of $\mathsf{MC}(\tilde{\eta}_m)$ can be identified with that of $\mathcal{G}_{m+1}(m,0)$. Hence $\mathsf{MC}(\tilde{\eta}_m) \cong \mathcal{G}_{m+1}(m,0)$ in $\mathrm{C}^b(X\times X)$, thereby proving (20), and finishing the proof of the proposition.

Proof (*Proof of* Lemma 4.6). We will show that $\mathsf{C}(\epsilon_m)\cong 0$, which immediately implies that ϵ_m is an isomorphism in $\mathrm{D}^b(\mathbb{P})$. From the defining equation (23) we have a distinguished triangle

$$\mathcal{F}_m[-1] \xrightarrow{p} \pi_{1*}\mathcal{R}_{1-m}(m,0) \xrightarrow{\rho_m} \mathcal{O}(m) \longrightarrow \mathcal{F}_m,$$

where p is the natural projection. Using the octahedral axiom (TR4) of triangulated categories we have

$$\begin{aligned}\mathsf{C}(\epsilon_m) &= \mathsf{C}\Big(\mathcal{M}_{-m} \xrightarrow{\epsilon_m} \mathsf{MC}\Big(\pi_{1*}\mathcal{R}_{1-m}(m,0) \xrightarrow{\rho_m} \mathcal{O}(m)\Big)\Big)\\ &\cong \mathsf{C}\Big(\mathsf{C}\Big(\mathcal{M}_{-m}[-1] \xrightarrow{p\circ\epsilon_m[-1]} \pi_{1*}\mathcal{R}_{1-m}(m,0)\Big) \longrightarrow \mathcal{O}(m)\Big).\end{aligned} \tag{26}$$

Now, by (8), for any $-w<k<0$ there is a distinguished triangle

$$\mathcal{O}(k)\boxtimes\mathcal{M}_k[-1] \xrightarrow{\alpha_k} \mathcal{R}_{k+1} \longrightarrow \mathcal{R}_k \longrightarrow \mathcal{O}(k)\boxtimes\mathcal{M}_k$$

in $\mathrm{K}^b(\mathbb{P}\times\mathbb{P})$. Applying the exact functor $\mathbf{R}\pi_{1*}(\mathcal{O}(m,0)\otimes -)$ and reasoning as in the proof of Lemma 4.5 gives another distinguished triangle in $\mathrm{K}^b(\mathbb{P})$:

$$\mathrm{P}_{k+m}\otimes_{\Bbbk}\mathcal{M}_k[-1]\xrightarrow{\pi_{1*}\alpha_k(m,0)}\pi_{1*}\mathcal{R}_{k+1}(m,0)\longrightarrow\pi_{1*}\mathcal{R}_k(m,0)\longrightarrow\mathrm{P}_{k+m}\otimes_{\Bbbk}\mathcal{M}_k.$$

From this we can deduce two things:

(i) For $-w<k<-m$, since $\mathrm{P}_{k+m}=0$, it follows that $\pi_{1*}\mathcal{R}_{k+1}(m,0)\cong\pi_{1*}\mathcal{R}_k(m,0)$. In other words

$$\pi_{1*}\mathcal{R}_{-m}(m,0)\cong\pi_{1*}\mathcal{R}_{-1-m}(m,0)\cong\cdots\cong\pi_{1*}\mathcal{R}_{1-w}(m,0).$$

(ii) For $k=-m$ we have that

$$\mathsf{C}(\pi_{1*}\alpha_{-m}(m,0))\cong\pi_{1*}\mathcal{R}_{-m}(m,0).$$

From (i) and (ii) it follows that in fact

$$\mathsf{C}(\pi_{1*}\alpha_{-m}(m,0))\cong\pi_{1*}\mathcal{R}_{1-w}(m,0).$$

On the other hand, it is easy to check that $p\circ\epsilon_m[-1]\colon\mathcal{M}_{-m}[-1]\to\pi_{1*}\mathcal{R}_{1-m}(m,0)$ can be identified with $\pi_{1*}\alpha_{-m}(m,0)\colon\mathrm{P}_0\otimes_{\Bbbk}\mathcal{M}_{-m}[-1]\to\pi_{1*}\mathcal{R}_{1-m}(m,0)$. Therefore

$$\mathsf{C}(p\circ\epsilon_m[-1])\cong\pi_{1*}\mathcal{R}_{1-w}(m,0).$$

An argument similar to the one used in the proof of Lemma 4.5 also shows that $\Phi^2_{\mathcal{R}_{1-w}}(\mathcal{O}(m))\cong\pi_{1*}\mathcal{R}_{1-w}(m,0)$. Combining this with the previous equation, and using Proposition 3.2, gives

$$\mathsf{C}(p\circ\epsilon_m[-1])\cong\Phi^2_{\mathcal{R}_{1-w}}(\mathcal{O}(m))\cong\Phi^2_{\mathcal{O}_\Delta}(\mathcal{O}(m))\cong\mathcal{O}(m).$$

By (26) this means that $\mathsf{C}(\epsilon_m)\cong\mathsf{C}\Big(\mathcal{O}(m)\xrightarrow{\lambda}\mathcal{O}(m)\Big)$ for some $\lambda\in\Bbbk$, and we claim that $\lambda\neq 0$. To prove this, observe that by [8, Cor. 2.4.6] there exists unique up to isomorphism $\mathcal{E}\in\mathrm{C}^b(\mathbb{P})$ such that $\mathcal{E}\cong\mathsf{C}(\epsilon_m)$ in $\mathrm{D}^b(\mathbb{P})$, each $\mathcal{E}^j$ is a sum of terms of the form $\mathcal{O}(l)$ with $0\le l<w$ and $\mathcal{E}$ is *minimal*, meaning that the components of each $d^j_{\mathcal{E}}$ from $\mathcal{O}(l)$ to $\mathcal{O}(l)$ are 0 for every l. On the other hand, since $\epsilon^0_m=\mathrm{id}_{\mathcal{O}(m)}$ and for $j\neq 0$ both $\mathcal{M}^j_{-m}$ and $\mathcal{F}^j_m$ are sums of terms of the form $\mathcal{O}(l)$ with $0\le l<m$, it is easy to deduce that also $\mathcal{E}^j$ for every $j\in\mathbb{Z}$ is a sum of terms of the form $\mathcal{O}(l)$ with $0\le l<m$. So we must have $\lambda\neq 0$ (otherwise obviously $\mathcal{E}=\mathcal{O}(m)\oplus\mathcal{O}(m)[1]$), which proves that $\mathsf{C}(\epsilon_m)\cong 0$. $\square$

Remark 4.7. Theorem 1.1 admits the following generalization. Let X be a smooth stacky hypersurface of degree $d\le w$ in $\mathbb{P}$. Then one can still define the functors L and K as in (1) and (2), although K is not an equivalence when $d<w$. Denoting by $\langle\mathcal{O}_X(1),\ldots,\mathcal{O}_X(w-d)\rangle$ the full triangulated subcategory of $\mathrm{D}^b(X)$ generated by the exceptional sequence $(\mathcal{O}_X(1),\ldots,\mathcal{O}_X(w-d))$, we consider its right orthogonal $\mathfrak{D}:=\langle\mathcal{O}_X(1),\ldots,\mathcal{O}_X(w-d)\rangle^{\perp}$. Explicitly, $\mathfrak{D}$ is the full triangulated subcategory of $\mathrm{D}^b(X)$ with objects the complexes $\mathcal{F}$ such that $\mathbf{R}\mathrm{Hom}_X(\mathcal{O}_X(i),\mathcal{F})=0$ for $0<i\le w-d$ (obviously $\mathfrak{D}=\mathrm{D}^b(X)$ if and only if $d=w$). It is easy to see that $\mathsf{G}=\mathsf{L}\circ\mathsf{K}\colon\mathrm{D}^b(X)\to\mathrm{D}^b(X)$ sends $\mathfrak{D}$ to itself, and one proves that

$$\underbrace{(\mathsf{G}|_{\mathfrak{D}})\circ\cdots\circ(\mathsf{G}|_{\mathfrak{D}})}_{d\text{-times}}\cong(-)[2] \tag{27}$$

(clearly this generalizes (3), which is the particular case $d=w$). When $\mathbb{P}=\mathbb{P}^{n+1}$ and $d<w=n+2$, (27) is equivalent to [21, Lemma 4.2], where G is replaced by $\mathsf{O}[1]=\mathsf{K}\circ\mathsf{L}$ and $\mathfrak{D}$ by $\mathfrak{A}=\langle\mathcal{O}_X,\ldots,\mathcal{O}_X(n+1-d)\rangle^{\perp}$ (the equivalence of the two statements follows from the fact that $\mathsf{G}\cong\mathsf{L}\circ\mathsf{O}[1]\circ\mathsf{L}^{-1}$ and $\mathfrak{D}=\mathsf{L}(\mathfrak{A})$).

The proof of (27) is quite similar to that of Theorem 1.1, so here we just sketch the main steps, emphasizing what needs to be added or changed. Of course, we still have $\mathsf{G}\cong\Phi^1_{\mathcal{G}}$ with $\mathcal{G}$ defined as in Proposition 4.1, but now Claim 4.2 holds only for $0<m\le d$ (essentially with the same proof, and replacing w with d in Lemma 4.5). On the other hand, Lemma 4.3 must be changed to $\mathcal{G}_w\cong\delta_*\mathcal{O}_X(-d)[2]$ (with the same proof). Therefore (27) will follow if we prove that

$$\Phi^1_{\mathcal{G}_d(d,0)}|_{\mathfrak{D}}\cong\Phi^1_{\mathcal{G}_w(d,0)}|_{\mathfrak{D}}. \tag{28}$$

To this purpose observe that by what we have shown at the end of the proof of Proposition 4.1, there are distinguished triangles in $\mathrm{K}^b(X \times X)$ for $d \leq m < w$

$$\mathcal{O}_X(d-m) \boxtimes \widehat{\mathcal{M}}_{-m} \xrightarrow{\tilde{\eta}_m(d-m,0)} \mathcal{G}_m(d,0) \longrightarrow \mathcal{G}_{m+1}(d,0) \longrightarrow \mathcal{O}_X(d-m) \boxtimes \widehat{\mathcal{M}}_{-m}[1].$$

In order to conclude that (28) holds, it is then enough to prove that

$$\Phi^1_{\mathcal{O}_X(d-m)\boxtimes\widehat{\mathcal{M}}_{-m}}(\mathfrak{D}) = 0 \tag{29}$$

for $d \leq m < w$. Now, by part (1) of Lemma 2.2

$$\Phi^1_{\mathcal{O}_X(d-m)\boxtimes\widehat{\mathcal{M}}_{-m}}(\mathcal{F}) \cong \mathcal{O}_X(d-m) \otimes_{\Bbbk} \mathbf{R}\Gamma(X, \widehat{\mathcal{M}}_{-m} \otimes \mathcal{F})$$

for every $\mathcal{F} \in \mathrm{D}^b(X)$. Since $\mathcal{M}_{-m} \cong \tilde{\mathcal{M}}_{-m}$ in $\mathrm{D}^b(\mathbb{P})$, where each $\tilde{\mathcal{M}}^j_{-m}$ is a finite direct sum of terms of the form $\mathcal{O}_{\mathbb{P}}(i)$ with $m - w \leq i < 0$ (see [8, Remarks 2.5.4 and 2.5.5]), we have that

$$\mathbf{R}\Gamma(X, \widehat{\mathcal{M}}_{-m} \otimes \mathcal{F}) \cong \mathbf{R}\Gamma(X, \widehat{\tilde{\mathcal{M}}}_{-m} \otimes \mathcal{F}) \cong \mathbf{R}\mathrm{Hom}_X(\widehat{\tilde{\mathcal{M}}}{}^{\vee}_{-m}, \mathcal{F}),$$

and the last term is 0 if $d \leq m < w$ (in which case clearly $\widehat{\tilde{\mathcal{M}}}{}^{\vee}_{-m} \in \langle \mathcal{O}_X(1), \ldots, \mathcal{O}_X(w-d) \rangle$) and $\mathcal{F} \in \mathfrak{D}$, thereby proving (29).

In analogy with [21, Lemma 4.1] it can also be proved that the Serre functor of $\mathfrak{D}$ is isomorphic to $(\mathsf{G}|_{\mathfrak{D}})^{\circ d-w}[n-1]$ (for $d = w$ this is just $(-)[n-1]$).

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