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Koszul Modules

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Dedicated to Prof. Wolmer Vasconcelos on the occasion of his 65th birthday

Abstract

This paper studies a new class of modules over noetherian local rings, called Koszul modules. It is proved that when a Koszul local ring R is a complete intersection, all high syzygies of finitely generated R-modules are Koszul. A stronger result is obtained when R is Golod and of embedding dimension d: the $2d$th syzygy of every R-module is Koszul. In addition, results are established that demonstrate that Koszul modules possess good homological properties; for instance, their Poincaré series is a rational function.

MSC: 13D02; 13H10; 13D07; 13D40

0. Introduction

Consider a noetherian local ring R with maximal ideal $\mathfrak{m}$ and residue field k. Given a finitely generated R-module M and F its minimal free resolution, denote $\mathrm{lin}^R(F)$ the associated graded complex of F arising from the standard $\mathfrak{m}$-adic filtration; the details are given in Section 1. This is a complex of free modules over the associated graded ring $\mathrm{gr}_{\mathfrak{m}}(R)$ of R and its differentials depend only on the *linear part*—in a sense made precise in Section 4—of the corresponding differentials on F; whence the notation.

We say that M is *Koszul* if $\mathrm{lin}^R(F)$ is acyclic. Such modules are module theoretic analogues of Koszul algebras introduced by Priddy [19]. Many of the results in this paper

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support this claim. Koszul modules have appeared previously in literature under the name: *modules with linear resolution*; confer [24]. We prefer to call them 'Koszul modules' because the latter terminology is misleading when dealing with graded modules over graded rings: in that context there are more Koszul modules than those for which the differentials in the minimal resolution can be represented by matrices of linear forms; 1.9 gives one such example. The local ring R is said to be a Koszul *ring* if the R-module k is Koszul. This is tantamount to requiring that the graded k-algebra $\mathrm{gr}_{\mathfrak{m}}(R)$ be Koszul in the classical sense of the word; see 1.10.

In studying Koszul modules, it is profitable to consider the larger class of modules M for which the number $\mathrm{ld}_R\,(M) = \sup\{i \,|\, \mathrm{H}_i(\mathrm{lin}^R(F)) \neq 0\}$, called the *linearity defect of M*, is finite. These are the modules in whose minimal free resolutions the linear part predominates. Observe that $\mathrm{ld}_R\,(M) \leqslant d$ is finite if and only if $\Omega^d\,(M)$, the dth syzygy of M, is Koszul.

A basic issue in this context is that of existence of modules of finite linearity defect. Here one encounters a rather surprising phenomenon: if $\mathrm{ld}_R\,(k) < \infty$, then R is Koszul, at least when R is a standard graded k-algebra. Nevertheless, even when R is Koszul there can be modules M with $\mathrm{ld}_R\,(M) = \infty$. Indeed, if $\mathrm{ld}_R\,(M)$ is finite, then by Proposition 1.8 the Poincaré series of M is a rational function, and its denominator term depends only on the Hilbert series of $\mathrm{gr}_{\mathfrak{m}}(R)$. However, even over the seemingly innocuous Koszul algebra $k[x_1, x_2, x_3]/(x_1, x_2, x_3)^2 \otimes_k k[y_1, y_2, y_3]/(y_1, y_2, y_3)^2$ there are finitely generated modules with transcendental Poincaré series; see [12], and also [23]. Furthermore, Roos [22] has constructed a Koszul algebra R and a family of R-modules $\{M_n\}_{n\in\mathbb{N}}$ such that the Poincaré series of each M_n is rational, but these do not share a common denominator. These examples lead one to consider the problem of classifying the rings over which each module is of finite (or bounded) linearity defect.

We prove that if a Koszul local ring R is either a complete intersection or Golod, then $\mathrm{ld}_R\,(M)$ is finite for every finitely generated R-module M; this is the content of Corollary 5.10. There is a dichotomy present here, at least in the graded case: when R is a complete intersection, there is no global upper bound on $\mathrm{ld}_R\,(M)$, unless R is a hypersurface. This follows from Proposition 6.7 which says that for a Gorenstein standard graded k-algebra R, if there is a global bound on the linearity defect, then R is a hypersurface. On the other hand, when R is Golod, Corollary 6.2 yields that $\mathrm{ld}_R\,(M) \leqslant 2\,\mathrm{emb\ dim}\,R$ for every finitely generated R-module M.

Modules of finite linear defect are particularly well behaved with regards to their homological properties. One example of this phenomenon is the rationality of Poincaré series mentioned above. More sophisticated manifestations concern the left module structure of $\mathrm{Ext}_R(M,k)$ over the k-subalgebra $R^!$ of $\mathrm{Ext}_R(k,k)$ generated by $\mathrm{Ext}^1_R(k,k)$. For instance, we prove in Theorem 2.2 that if $\mathrm{ld}_R\,(M)$ is finite, then $\mathrm{Ext}_R(M,k)$ is finitely generated over $R^!$ by elements with degrees $\leqslant \mathrm{ld}_R\,(M)$. In particular, when the ring R is Koszul, $R^! = \mathrm{Ext}_R(k,k)$; this last result is well known when R is a standard graded k-algebra.

Over Koszul rings Theorem 2.2 admits a vast generalization, best captured by the formula

$$\mathrm{ld}_R\,(M) = \mathrm{reg}_{R^!}\ \mathrm{Ext}_R(M,k).$$

It is this equality, in conjunction with work of Backelin and Roos [5], that allows us to deduce the finiteness of linearity defect of modules over complete intersections and Golod rings, discussed in the preceding paragraphs. It implies also that when $\mathrm{ld}_R\,(M)$ is finite, the

$R^!$-module $\operatorname{Ext}_R(M,k)$ and all its higher syzygies are finitely generated. The term on the right in the formula above is the *regularity* of $\operatorname{Ext}_R(M,k)$ as an $R^!$-module; the relevant definition and the proofs of the preceding results are the contents of Section 5.

Now for an overview of the paper: Section 1 contains the basic definitions and first results about Koszul modules, and more generally, about modules with finite linear defect. While the bulk of the results in this paper are stated and proved for modules over local rings, they are applicable also to graded modules over graded k-algebras. For a graded module M over a standard graded ring R, the invariant $\operatorname{ld}_R(M)$ is closely linked to the regularity of M over R. Section 1 explores also some of these issues.

The main goal in Section 2 is the proof of the result concerning the finite generation of $\operatorname{Ext}_R(M,k)$ over $R^!$. This requires detailed calculations involving products on $\operatorname{Ext}_R(k,k)$ and its action on $\operatorname{Ext}_R(M,k)$. These play a key role also in the subsequent development, as do the results in the short Section 3 that is devoted to dealing with certain delicate issues pertaining to left and right module structures on $\operatorname{Ext}_R(k,k)$.

Section 4 is the heart of the paper. It is devoted entirely to the proof of Theorem 4.1 that expresses the homology of the linear part of the minimal resolution F of M, over a Koszul ring R, in terms of the (co)homology of $\operatorname{Tor}^R(k,M)$ when viewed as module over $R^!$. A crucial ingredient in its proof is Theorem 4.5 that provides an explicit description of $\operatorname{lin}^R(F)$ in terms of the action of $\operatorname{Ext}^1_R(k,k)$ on $\operatorname{Tor}^R(k,M)$. The material in Section 5 has already been described. One of the main contributions of Section 6 is Theorem 6.1 which states that given a surjective homomorphism of Koszul local rings $\varphi\colon S \to R$ and M a finitely generated R-module, one has

$$\operatorname{ld}_S(M) = \operatorname{ld}_R(M) \quad \text{whenever } M \text{ is } \varphi\text{-Golod.}$$

Its proof utilizes a new characterization of φ-Golod modules whose applicability extends beyond the task on hand. As corollary, we deduce the global bound for $\operatorname{ld}_R(M)$, when R is a Golod ring.

This research is inspired by the paper of Eisenbud et al. [9] concerning modules over the exterior algebra: many of the results presented here are either analogous to, or extensions of, those in [9]. For instance, the part of Corollary 5.10 concerning complete intersections is akin to [9, (3.1)] which asserts that, for finitely generated graded modules over the exterior algebra, the linearity defect is finite. This is no coincidence since homological algebra over complete intersections is similar to that over exterior algebras. Furthermore, Theorem 5.4, relating linearity defect to regularity over the Ext algebra, also has its counterpart in [9, (2.4)]. Having said this, it should be noted that Eisenbud et al. [9] makes critical use of the Bernstein–Gelfand–Gelfand correspondence. This is not available in our framework, which encompasses both local and graded rings, and, more to the point, a good number of our results do not require the ring to be Koszul.

After this article was accepted for publication, we discovered that recently Martinez and Zacharia [17] have studied the analogue of Koszul modules over finite dimensional algebras, albeit under the name ‘weakly Koszul modules’. They reserve the name ‘Koszul modules’ for those with modules linear resolutions. The reader may consult the references in that article, and also in Beilinson et al. [6], for related work in the setting of finite dimensional algebras.

1. Koszul modules

In this article, one has to deal with graded objects (rings, modules, etc.) that, depending on the situation on hand, are best viewed as being graded homologically or as being graded cohomologically. In order to streamline the discussion, we adopt the following convention.

1.1. Grading. Every object V graded by $\mathbb{Z}$ has both a lower grading and an upper grading, with the identification

$$V_i = V^{-i} \quad \text{for each } i \in \mathbb{Z}.$$

This does lead to ambiguity in the meaning of the word "degree", as applied to an element in V. We circumvent this difficulty by specifying, unless it is clear from the context, whether the degree in question is lower or upper.

1.2. The linear part of a complex. Let $(R, \mathfrak{m}, k)$ be a local ring. A complex of R-modules

$$C\colon \cdots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \cdots$$

is said to be *minimal* if $\partial_n(C_n) \subseteq \mathfrak{m}C_{n-1}$. The *standard filtration* $\mathscr{F}$ of a minimal complex C is defined by subcomplexes $\{\mathscr{F}^i\mathscr{C}\}_{i\geqslant 0}$, where

$$(\mathscr{F}^i C)_n = \mathfrak{m}^{i-n}C_n \quad \text{for all} \quad n \in \mathbb{Z}$$

with the convention that $\mathfrak{m}^j = R$ for $j \leqslant 0$. The minimality of the complex C ensures that $\partial(\mathscr{F}^i C) \subseteq \mathscr{F}^i C$ for each integer i so that $\mathscr{F}$ *is* a filtration of C. The associated graded complex with respect to this filtration is denoted $\mathrm{lin}^R(C)$, and called the *linear part of* C. By construction, $\mathrm{lin}^R(C)$ is a complex of graded modules over the graded ring $\mathrm{gr}_{\mathfrak{m}}(R)$ and has the property that

$$\mathrm{lin}^R_n(C) = \mathrm{gr}_{\mathfrak{m}}(C_n)(-n),$$

where, for any graded object V, we denote $V(n)$ the graded object with component in homology degree i being V_{i+n}. In particular, when C is a complex of free modules the complex of $\mathrm{gr}_{\mathfrak{m}}(R)$-modules $\mathrm{lin}^R(C)$ is free and, by construction, the differentials on it can be described by matrices of *linear forms*; this partly explains our terminology.

Now we introduce the main objects of interest. Recall that over local rings, each finitely generated module has a minimal free resolution, and that this is unique (up to isomorphism). Thus, one may speak of *the* minimal free resolution of such a module.

1.3. Definition. A finitely generated R-module M is said to be *Koszul* if $\mathrm{lin}^R(F)$ is acyclic, where F is the minimal free resolution of M.

Koszul modules are precisely those modules that have *linear resolutions*, in the terminology of Şega [24, Section 2]; this is justified by Proposition 1.5. In order to state this result, we introduce certain families of maps akin to those in [24, (2.2)].

1.4. Let M be finitely generated R-module and F its minimal R-free resolution. Set $A = \mathrm{gr}_{\mathfrak{m}}(R)$. Let U be a graded A-free resolution of $\mathrm{gr}_{\mathfrak{m}}(M)$. The augmentation $F \to M$ gives rise to an A-linear map $\mathrm{lin}^R(F) \to \mathrm{gr}_{\mathfrak{m}}(M)$. Since the complex of A-modules $\mathrm{lin}^R(F)$ is free, this map lifts to a morphism of complexes of A-modules

$$\kappa\colon \mathrm{lin}^R(F) \to U,$$

and such a lift is unique up to homotopy.

For each $z \in F_n$, denote $\bar{z}$ its residue class in $\mathrm{lin}^R_n(F)_n = F_n/\mathfrak{m}F_n$. For each integer n, set

$$\chi^n_M\colon \mathrm{Ext}^n_A(\mathrm{gr}_{\mathfrak{m}}(M), k) = \mathrm{H}_{-n}(\mathrm{Hom}_A(U, k)) \longrightarrow \mathrm{Hom}_R(F_n, k) = \mathrm{Ext}^n_R(M, k),$$
$$\chi^n_M(\mathrm{cls}\,(\alpha))(z) = \alpha(\kappa(\bar{z})).$$

A routine calculation reveals that χ^n_M is well defined and k-linear.

Now, since A and $\mathrm{gr}_{\mathfrak{m}}(M)$ are graded, so is $\mathrm{Ext}^n_A(\mathrm{gr}_{\mathfrak{m}}(M), k)$. Note that each χ^n_M factors through $\mathrm{Hom}_A(\mathrm{lin}^R_n(F), k)$, which is concentrated in lower degree $-n$, so χ^n_M decomposes as

$$\mathrm{Ext}^n_A(\mathrm{gr}_{\mathfrak{m}}(M), k) \overset{\pi^n_M}{\twoheadrightarrow} \mathrm{Ext}^n_A(\mathrm{gr}_{\mathfrak{m}}(M), k)_{-n} \xrightarrow{\nu^n_M} \mathrm{Ext}^n_R(M, k).$$

We remark that χ^n_M and ν^n_M are the k-vector space duals of the maps τ^M_n and λ^M_n, respectively, defined in [24, (2.2)]; this is immediate from the corresponding definitions.

It turns that χ_k is a homomorphism of k-algebras, and χ_M is χ_k-equivariant; see Proposition 2.14. These properties are not used in this section.

The next item provides diverse characterizations of Koszul modules. Mostly, it is a repackaging of well-known results; cf. [24, (2.3)]. Implication (1) $\Rightarrow$ (2) uses an elementary lemma on filtered exact sequences; a detailed proof of this latter result given for completeness, is relegated to the end of this section.

1.5. Proposition. *Let M be a finitely generated R-module and F its minimal free resolution. The following conditions are equivalent*:

(1) *M is Koszul.*
(2) $\mathrm{lin}^R(F)$ *is a minimal free resolution of* $\mathrm{gr}_{\mathfrak{m}}(M)$ *over* $\mathrm{gr}_{\mathfrak{m}}(R)$.
(3) *The* $\mathrm{gr}_{\mathfrak{m}}(R)$*-module* $\mathrm{gr}_{\mathfrak{m}}(M)$ *has a linear resolution.*
(4) *The maps π_M, ν_M, and χ_M are bijective.*
(5) *The map χ_M is injective.*

Proof (*Implication*). (2) $\Rightarrow$ (1) is a tautology. As to the reverse implication: $\mathrm{lin}^R(F)$ is minimal, by construction, and acyclic, since M is Koszul. Hence Lemma 1.16 yields that $\mathrm{H}_0(\mathrm{lin}^R(F)) = \mathrm{gr}_{\mathfrak{m}}(M)$.

The remaining implications can be read off [24, (2.3)], keeping in mind that χ^M and ν^n_M are the k-vector space duals of the maps τ^M_n and λ^M_n, respectively. □

Over any local ring $(R, \mathfrak{m}, k)$, there are two canonical 'test' modules: R itself and k. The former is always Koszul; however, requiring that k be Koszul is a severe restriction on R.

1.6. Definition. The local ring $(R, \mathfrak{m}, k)$ is *Koszul* if the R-module k is Koszul.

This notion of a Koszul ring coincides with the one due to Herzog et al. [11] who defined it by the property that the associated graded algebra is Koszul, in the usual sense of the word; see Remark 1.10.

Koszul modules over Koszul rings are the focus of Sections 5 and 6. For the moment though, we concentrate on Koszul modules in general. To begin with, we introduce an invariant that is one measure of how far a module is from being Koszul.

1.7. Definition. Let M be a finitely generated R-module and F its minimal free resolution. Define

$$\operatorname{ld}_R(M) = \sup\{i \mid \mathrm{H}_i(\operatorname{lin}^R(F)) \neq 0\}.$$

This number is the *linearity defect* of M. When $\operatorname{ld}_R(M)$ is finite, we say that for the complex F the linear part predominates; this is keeping in line with the terminology of Eisenbud et al. [9]. Note that $\operatorname{ld}_R(M) = 0$ if and only if M is Koszul. More generally, for any integer d, one has $\operatorname{ld}_R(M) \leqslant d$ if and only if the *dth syzygy module* $\Omega^d(M)$ of M is Koszul.

1.8. Poincaré series. Let M be a finitely generated R-module. Recall that the *nth Betti number of M*, denoted $b_n(M)$, is the integer $\operatorname{rank}_R F_n$, where F is the minimal free resolution of M. As usual, the *Poincaré series of M* is the formal power series

$$\mathrm{P}_M^R(t) = \sum_{n \geqslant 0} b_n(M) t^n.$$

It is known that $\mathrm{P}_M^R(t)$ need not be a rational; not even for $M = k$; see the discussion surrounding [2, (4.3.10)]. Moreover, in [22], Roos constructs a ring R and a family of finitely generated modules $\{M_n\}_{n \in \mathbb{N}}$ such that $\mathrm{P}_{M_n}^R(t)$ is rational for each n, but these do not have a common denominator. These comments may serve to highlight the relevance of the following result. In what follows, for any graded k-vector space V, its Hilbert series $\sum_{n \in \mathbb{Z}} \operatorname{rank}_k(V_n) t^n$ is denoted $\operatorname{Hilb}_V(t)$.

Proposition. *Let $(R, \mathfrak{m}, k)$ be a local ring. If $\operatorname{ld}_R(M)$ is finite, then there exists a polynomial $q(t)$ with integer coefficients such that*

$$\mathrm{P}_M^R(t) = \frac{q(t)}{D(t)} \quad \text{where } D(t) = \operatorname{Hilb}_{\operatorname{gr}_{\mathfrak{m}}(R)}(-t)(1+t)^{\dim R}.$$

In particular, the denominator of $\mathrm{P}_M^R(t)$ depends only on the Hilbert series of $\operatorname{gr}_{\mathfrak{m}}(R)$.

Proof. Set $d = \operatorname{ld}_R(M)$. Then $\mathrm{P}_M^R(t) = t^d \mathrm{P}_{\Omega^d(M)}^R(t) + p(t)$, where $p(t) = \sum_{n=0}^{d-1} b_n(M) t^n$. Thus, $\mathrm{P}_M^R(t)$ has the desired form if and only if $\mathrm{P}_{\Omega^d(M)}^R(t)$ does. Furthermore $\Omega^d(M)$ is Koszul. Hence, by passing to $\Omega^d(M)$ one may assume that M is Koszul.

Let F be the minimal free resolution of M. By Proposition 1.5, the complex of free $\mathrm{gr}_{\mathfrak{m}}(R)$-modules $\mathrm{lin}^R(F)$ is a free resolution of $\mathrm{gr}_{\mathfrak{m}}(M)$; this yields the first of the following identities. The second holds because this resolution is, by construction, linear and Hilbert series are additive on short exact sequences.

$$\mathrm{P}^R_M(t) = \mathrm{P}^{\mathrm{gr}_{\mathfrak{m}}(R)}_{\mathrm{gr}_{\mathfrak{m}}(M)}(t) \quad \text{and} \quad \mathrm{Hilb}_{\mathrm{gr}_{\mathfrak{m}}(M)}(t) = \mathrm{Hilb}_{\mathrm{gr}_{\mathfrak{m}}(R)}(t)\mathrm{P}^{\mathrm{gr}_{\mathfrak{m}}(R)}_{\mathrm{gr}_{\mathfrak{m}}(M)}(-t).$$

Hence $\mathrm{P}^R_M(t) = \mathrm{Hilb}_{\mathrm{gr}_{\mathfrak{m}}(M)}(-t)\mathrm{Hilb}_{\mathrm{gr}_{\mathfrak{m}}(R)}(-t)^{-1}$. It remains to note that the Hilbert series of $\mathrm{gr}_{\mathfrak{m}}(M)$ and $\mathrm{gr}_{\mathfrak{m}}(R)$ are both rational functions with denominator $(1-t)^{\dim R}$. $\square$

The graded case. Let k be a field. By a *standard graded k-algebra*, we mean a graded k-algebra $R = \{R_i\}_{i\geqslant 0}$ such that $R_0 = k$, $\mathrm{rank}_k(R_1)$ is finite, and R_1 generates R as a k-algebra. In such a k-algebra the ideal $R_{\geqslant 1}$, which we denote $\mathfrak{m}$, is the unique homogeneous maximal ideal.

Let R be a standard graded k-algebra.

Let C be a minimal complex of graded R-modules; this includes the assumption that the differentials are homogeneous of degree 0. As in the local case one can define its standard filtration with respect to $\mathfrak{m}$ and the associated graded complex $\mathrm{lin}^R(C)$. Observe that since R is standard graded, the k-algebras R and $\mathrm{gr}_{\mathfrak{m}}(R)$ are naturally isomorphic. Moreover, for each integer n, the R-modules $R(n)$ and $\mathrm{gr}_{\mathfrak{m}}(R(n))$ are isomorphic. In particular, if C is a complex of free R-modules, then so is $\mathrm{lin}^R(C)$.

Given these considerations, one can speak of Koszul modules, and also of the invariant $\mathrm{ld}_R\,(M)$ associated to any graded finitely generated R-module M. In keeping with tradition, when the R-module k is Koszul, we say that R is a *Koszul algebra*, rather than a Koszul ring.

1.9. Example. Suppose that M has a linear resolution, that is to say, the differentials in the minimal graded free resolution F of M can be represented by matrices of linear forms. In this case M is Koszul, for then $\mathrm{lin}^R(F)$ is isomorphic to F.

However, these are not the only Koszul modules. Consider, for example, the polynomial ring $k[x, y]$, and the module M whose minimal resolution is

$$F\colon 0 \longrightarrow R \xrightarrow{\begin{pmatrix} x^2 \\ y \end{pmatrix}} R^2 \longrightarrow 0.$$

It is not hard to check that the complex $\mathrm{lin}^R(F)$ is $0 \to R \xrightarrow{\binom{0}{y}} R^2 \to 0$, and hence acyclic.

1.10. Remark. The k-algebra R is Koszul if and only if k has linear resolution; this is contained in (the graded analogue of) Proposition 1.5, since $\mathrm{gr}_{\mathfrak{m}}(k) = k$ and $\mathrm{gr}_{\mathfrak{m}}(R) = R$. Thus, the class of Koszul algebras as introduced above coincides with the classical one. This observation along with 1.5 implies also that a *local ring* is Koszul if and only if its associated graded algebra is Koszul.

When one speaks of the Koszul property, the invariant that springs to mind is regularity; the next item compares it to linearity defect.

1.11. Regularity. Let R be a standard graded k-algebra and M a graded finitely generated R-module. Let F be the minimal graded free resolution of M; in particular, $F_n=\bigoplus_{i\in\mathbb{Z}} R(-i)^{b_{n,i}}$ for each integer n. The integer $b_{n,i}$ is the (n,i)*th graded Betti number of M*.

The *regularity of M* is the integer

$$\operatorname{reg}_R M = \sup\{r \in \mathbb{Z} | b_{n,n+r} \neq 0 \text{ for some integer } n\}.$$

The standard filtration of F is compatible with internal gradings so $\mathscr{F}^i F$ is a graded subcomplex of F for each integer i. Thus, the associated graded $\operatorname{lin}^R(F)$ is a complex of *bigraded* R-modules, where R is viewed as a bigraded k-algebra with components

$$R_{p,q} = \begin{cases} 0 & \text{if } p \neq q, \\ R_p & \text{if } p = q. \end{cases}$$

Observe that $\operatorname{lin}^R_n(F) \cong \bigoplus_{i\in\mathbb{Z}} R(-n,-i)^{b_{n,i}}$. In detail, an element $f \in \operatorname{lin}^R_n(F)$ is bihomogenous of degree (p,q) if $f \equiv g \bmod \mathfrak{m}^{p+1}F_n$, where g is homogeneous of degree q and lies in $\mathfrak{m}^p F_n \backslash \mathfrak{m}^{p+1}F_n$. The differential in $\operatorname{lin}^R(F)$ can be described by matrices of bihomogenous forms of degree $(1,1)$, so it decomposes as a direct sum of subcomplexes of graded R-modules

$$\operatorname{lin}^R(F) = \bigoplus_{r\in\mathbb{Z}} F^r \quad \text{with } F^r_n = R(-n,-n-r)^{b_{n,n+r}}.$$

The subcomplex F^r is called the *rth linear strand of F*. Keeping track of the degrees leads us to the realization that

Lemma. $\operatorname{reg}_R M = \sup\{r \in \mathbb{Z} | F^r \neq 0\}$.

This completes the preparation for the result below. It can been deduced also from Theorem 2.2, which throws a different light on the matter on hand. We provide a direct, and elementary, proof because justifying the latter calls for some fairly sophisticated technology.

It should be noted that its converse need not hold. Indeed, Eisenbud and Avramov [3, (1)] have proved that every finitely generated module over a Koszul algebra has finite regularity. However, not every such module has finite linearity defect; see 1.8.

1.12. Proposition. *If* $\operatorname{ld}_R(M)$ *is finite, then so is* $\operatorname{reg}_R M$.

Proof. Suppose that $\operatorname{reg}_R M$ is infinite. Let $\mathscr{S} = \{r \in \mathbb{Z} | F^r \neq 0\}$; here we are using the notation from 1.11. The preceding lemma implies that the set $\mathscr{S}$ is infinite. For each $r \in \mathscr{S}$, let $n(r)$ be the least integer such that $F^r_{n(r)} \neq 0$. For every integer n there are only finitely many $r \in \mathscr{S}$ with $n(r) = n$, since F_n is finitely generated, so that in $\mathscr{S}$ there is an infinite strictly increasing sequence $r_1 < r_2 < \cdots$ with $n(r_1) < n(r_2) < \cdots$. By definition $F^{r_k}_n = 0$ for $n < n(r_k)$; moreover, F^{r_k} is minimal. Thus, $\mathrm{H}_{n(r_k)}(F^{r_k}) \neq 0$ for each r_k, hence $H_{n(r_k)}(\operatorname{lin}^R(F)) \neq 0$, since F^{r_k} is a direct summand of $\operatorname{lin}^R(F)$. In particular, $\operatorname{ld}_R(M) = \infty$, as desired. $\square$

Here is one corollary of the preceding result.

1.13. Proposition. *Let R be a standard graded k-algebra. If* $\operatorname{ld}_R(k) < \infty$, *then R is Koszul.*

Proof. If $\operatorname{ld}_R(k)$ is finite, then Proposition 1.12 yields that $\operatorname{reg}_R k$ is finite, and then, according to Avramov and Peeva [4, (2)], R is Koszul. $\square$

The proposition can be rephrased as: In the graded case, $\operatorname{ld}_R(k) < \infty$ implies $\operatorname{ld}_R(k) = 0$.

1.14. Question. Let $(R, \mathfrak{m}, k)$ be a local ring. If $\operatorname{ld}_R(k) < \infty$, then is $\operatorname{ld}_R(k) = 0$?

Theorem 2.2 provides some input on this problem—see 2.4—but not enough to solve it.

The precise relation between regularity and linear defect remains mysterious; however, see the remarks after Theorem 2.2. For the special case of Koszul modules though a result of Römer [20] provides a complete and elegant description.

Theorem (*Römer*). *Let M be a finitely generated graded R-module. Then M is Koszul if and only if, for each integer i, the R-submodule of M generated by M_i has a linear resolution.*

1.15. Bigraded Poincaré series. Let R be a standard graded k-algebra and M a graded finitely generated R-module. In the notation of 1.11, the *bigraded Poincaré series* of M is the formal power series

$$\mathrm{P}_M^R(t,s) = \sum_{n \geqslant 0, i \in \mathbb{Z}} b_{n,i} t^n s^i \quad \text{in } \mathbb{Z}[s][\![t]\!].$$

The additional grading allows one to refine Proposition 1.8 as follows.

Proposition. *If* $\operatorname{ld}_R(M)$ *is finite, then, with F^r the rth linear strand of the minimal resolution F of M, there is an equality of formal power series*:

$$\mathrm{P}_M^R(t,s) = \frac{1}{\mathrm{Hilb}_R(-st)} \sum_{r,n \in \mathbb{Z}} (-1)^n \frac{\mathrm{Hilb}_{\mathrm{H}_n(F^r)}(-st)}{(-t)^r}.$$

In particular, the bigraded Poincaré series of M is rational.

Remark. Note that the summation above is finite since there are only finitely many nonzero linear strands, by Lemma 1.11 and Proposition 1.12, and each strand has only finitely many nonzero homology modules, since $\operatorname{ld}_R(M)$ is finite.

Proof. Since each linear strand F^r has only finitely many homology modules, the additivity of Hilbert series on short exact sequences yields

$$\sum_{n \geqslant 0} (-1)^n b_{n,n+r}\, s^{n+r} \mathrm{Hilb}_R(s) = \sum_{n \geqslant 0} (-1)^n \mathrm{Hilb}_{\mathrm{H}_n(F^r)}(s).$$

On the other hand, by reindexing the formula defining the Poincaré series one obtains that

$$\mathrm{P}_M^R(t,s) = \sum_{n \geqslant 0, r \in \mathbb{Z}} b_{n,n+r}\, t^n s^{n+r}.$$

An elementary calculation now yields the desired expression for the Poincaré series of M. $\square$

The last item in this section is a simple lemma on filtered modules; it was used to prove 1.5.

1.16. Filtered modules. A *filtered module* is an R-module U with a filtration $\{U^i\}_{i\geqslant 0}$ such that $U^0 = U$ and $U^{n+1} \subseteq U^n$ for each $n \geqslant 0$. The associated graded module of a filtered module U is the graded R-module $\mathrm{gr}(U)$ with degree n component U^n/U^{n+1}.

A *homomorphism of filtered modules* is an R-module homomorphism $\varphi: U \to V$ such that $\varphi(U^n) \subseteq V^n$. In this case, $\mathrm{Ker}\varphi$ is filtered with $(\mathrm{Ker}\varphi)^n = (\mathrm{Ker}\varphi) \cap U^n$. Similarly, $\mathrm{Coker}\,\varphi$ is filtered with $(\mathrm{Coker}\,\varphi)^n = (V^n + \varphi(U))/\varphi(U)$.

Lemma. *Let* $U \xrightarrow{\varphi} V \xrightarrow{\psi} W$ *be homomorphisms of filtered R-modules. If the associated graded sequence is exact, then the canonical homomorphism* $\mathrm{Coker}(\mathrm{gr}(\psi)) \to \mathrm{gr}(\mathrm{Coker}\,\psi)$ *is bijective*.

Proof. Fix an integer $n \geqslant 0$. The degree n part of the homomorphism in question is the surjection

$$\mathrm{Coker}(\mathrm{gr}(\psi))_n = \frac{W^n}{W^{n+1} + \psi(V^n)} \longrightarrow \frac{W^n + \psi(V)}{W^{n+1} + \psi(V)} = \mathrm{gr}(\mathrm{Coker}\,\psi)_n.$$

This map is injective if and only if $W^n \cap (W^{n+1} + \psi(V)) \subseteq W^{n+1} + \psi(V^n)$. Thus, it suffices to prove that $W^n \cap \psi(V) \subseteq \psi(V^n)$. To this end we establish by a descending induction on s that

$$W^n \cap \psi(V^s) \subseteq \psi(V^n) \quad \text{for all} \quad 0 \leqslant s \leqslant n.$$

The basis step $s = n$ is tautological. Suppose that the inclusion above holds for $s+1$, for some integer s where $0 \leqslant s \leqslant n-1$. The exactness of the sequence

$$\frac{U^s}{U^{s+1}} \xrightarrow{\mathrm{gr}(\varphi)_s} \frac{V^s}{V^{s+1}} \xrightarrow{\mathrm{gr}(\psi)_s} \frac{W^s}{W^{s+1}}$$

yields $\psi^{-1}(W^{s+1}) \cap V^s \subseteq \varphi(U^s) + V^{s+1}$. Thus, $W^{s+1} \cap \psi(V^s) \subseteq \psi(V^{s+1})$. This explains the inclusion on the right in the computation below, while the one of the left holds because $n \geqslant s+1$,

$$W^n \cap \psi(V^s) \subseteq W^{s+1} \cap \psi(V^s) \subseteq \psi(V^{s+1}).$$

Therefore, $W^n \cap \psi(V^s) \subseteq W^n \cap \psi(V^{s+1})$. It remains to invoke the induction hypothesis. $\square$

2. Cohomology of modules of finite linearity defect

The highlight of this section is Theorem 2.2 which reveals a striking property of modules of finite linear defect. First, some terminology.

2.1. The Koszul dual. Recall that the graded k-vector space $\mathrm{Ext}_R(k,k)$ has the structure of an associative k-algebra, and that, for any R-module M, the k-vector space $\mathrm{Ext}_R(M,k)$ has the structure of a left $\mathrm{Ext}_R(k,k)$-module; see 2.8. The k-subalgebra of $\mathrm{Ext}_R(k,k)$ generated its elements of degree 1 is called the *Koszul dual* of R, and often denoted $R^!$. Thus

$$R^! = \langle \mathrm{Ext}^1_R(k,k)\rangle.$$

Since $R^!$ is a subalgebra of $\mathrm{Ext}_R(k,k)$, there is a structure of a left $R^!$-module on $\mathrm{Ext}_R(M,k)$.

Here is the our result. A far stronger statement can be established when R is Koszul; cf. 5.5.

2.2. Theorem. *Let $(R,\mathfrak{m},k)$ be a local ring and M a finitely generated R-module. If $\mathrm{ld}_R\,(M)$ is finite, then as a left $R^!$-module $\mathrm{Ext}_R(M,k)$ has a finite generating set consisting of elements of degree $\leqslant \mathrm{ld}_R\,(M)$.*

2.3. Remark. There is a precise analogue of this result for standard graded k-algebras. Now, for such a k-algebra R, the bigraded k-algebra $R^!$ lies on the 'diagonal': each (homogeneous) element has degree $(n,-n)$, for some integer n. Consequently, when $\mathrm{Ext}_R(M,k)$ is finitely generated, $\mathrm{reg}_R\, M$ is finite. In this way, one recovers Proposition 1.12 from the theorem above.

2.4. Remark. The preceding result might be a first step to solving Question 1.14. Indeed, it implies that when $\mathrm{ld}_R\,(k) < \infty$ the inclusion of k-algebras $R^! \subseteq \mathrm{Ext}_R(k,k)$ is a module finite extension. This is even a free extension as the inclusion is one of Hopf algebras. However, these properties in themselves do not imply that R is Koszul; see Example 2.6.

The proof of Theorem 2.2 is given in 2.15. It is worth spelling out the following corollary, which was first proved by Şega [24, (2.3)].

2.5. Corollary. *If the local ring R is Koszul, then $R^! = \mathrm{Ext}_R(k,k)$.*

This result is classical for standard graded k-algebras; moreover, for such algebras the converse is also true: the identity $R^! = \mathrm{Ext}_R(k,k)$ captures the Koszul property; cf. [16, Theorem 1.2]. This is not so in general, as the example below from [16] illustrates.

2.6. Example. Let $R = k[x,y]/(x^2+y^3, xy)$. Since R is a complete intersection, a computation due to Sjödin [25, Theorem 5] gives

$$\mathrm{Ext}_R(k,k) = \mathrm{T}\,(u,v)\,/(v^2,[u^2,v]) \quad \text{where } |u| = 1 = |v|.$$

Here $\mathrm{T}\,(u,v)$ denotes the tensor algebra on variables u, v. In particular $\mathrm{Ext}_R(k,k)$ is generated by $\mathrm{Ext}^1_R(k,k)$. However, R is not Koszul since the associated graded algebra is $k[x,y]/(x^2,xy,y^4)$ and this is not Koszul.

As is clear by now, one of the protagonists in this work is $\mathrm{Ext}_R(k,k)$. We begin by a review of basic computations concerning its algebra structure and its action on $\mathrm{Tor}^R\,(k,M)$

and $\mathrm{Ext}_R(M,k)$; these play a key role in the sequel. No doubt all this is old hat to the experts, but suitable references are difficult to find.

The following notation that will be operative throughout Sections 2–4.

2.7. Notation. Let $(R,\mathfrak{m},k)$ be a local ring and M a complex of R-modules. For any (graded) k-vector space V denote $V^\vee$ the (graded) dual vector space $\mathrm{Hom}_k(V,k)$. Moreover

(a) let $r_1,\dots,r_e$ be a minimal generating set of $\mathfrak{m}$;
(b) let x_i be the residue class of r_i in $\mathfrak{m}/\mathfrak{m}^2$. Thus, $x_1,\dots,x_e$ is a basis for $\mathfrak{m}/\mathfrak{m}^2$;
(c) let $y_1,\dots,y_e$ be the basis of $(\mathfrak{m}/\mathfrak{m}^2)^\vee$ dual to $x_1,\dots,x_e$;
(d) let P be an R-free resolution of k such that $P_0=R$ and $P_1=\bigoplus_{i=1}^e Re_i$ with $\partial(e_i)=r_i$, and let $\varepsilon\colon P\to k$ be the augmentation.

2.8. The Ext-algebra. Since P is an R-free resolution of k, one may identify the k-vector space $\mathrm{Ext}_R^n(k,k)$ with $\mathrm{H}_{-n}(\mathrm{Hom}_R(P,P))$. Composition endows the complex $\mathrm{Hom}_R(P,P)$ with a structure of a differential graded (DG) algebra. This induces on its homology, that is to say, on $\mathrm{Ext}_R(k,k)$, a structure of a graded k-algebra. In the sequel, this is taken to be the product on $\mathrm{Ext}_R(k,k)$. It differs from the classical Yoneda product only by sign; cf. [25, Lemma 1].

Let F be a free resolution of M. Then $\mathrm{Ext}_R^n(M,k)=\mathrm{H}_{-n}(\mathrm{Hom}_R(F,P))$. Akin to the situation described above, composition makes $\mathrm{Hom}_R(F,P)$ a left DG $\mathrm{Hom}_R(P,P)$-module, and passing to homology gives a left $\mathrm{Ext}_R(k,k)$-module structure on $\mathrm{Ext}_R(M,k)$.

Similarly, $\mathrm{Tor}_n^R(k,M)=\mathrm{H}_n(P\otimes_R F)$ and $\mathrm{Tor}^R(k,M)$ is a left $\mathrm{Ext}_R(k,k)$-module: $\mathrm{Hom}_R(P,P)$ acts on $P\otimes_R M$ by evaluation on the left-hand factor of the tensor product. This makes $P\otimes_R M$ a DG left module over the DG algebra $\mathrm{Hom}_R(P,P)$. Passing to homology yields the desired structure. Now, $\mathrm{Hom}_R(P,P)$ acts also on $P\otimes_R M$ and the quasiisomorphism $P\otimes_R F\twoheadrightarrow P\otimes_R M$ is a homomorphism of DG $\mathrm{Hom}_R(P,P)$ modules. Thus, one can compute the $\mathrm{Ext}_R(k,k)$ action on $\mathrm{Tor}^R(k,M)$ via the complex $P\otimes_R M$.

Next we delve deeper into the k-algebra $R^!$. Since $R^!$ is a subalgebra of $\mathrm{Ext}_R(k,k)$, there is a structure of a left $R^!$-module on $\mathrm{Tor}^R(k,M)$ and on $\mathrm{Ext}_R(M,k)$. These structures are captured by the linear part of the differential in the minimal resolution, when it exists, of M. This is the content of the following paragraphs.

To begin with, recall that $\mathrm{Ext}_R^1(k,k)=(\mathfrak{m}/\mathfrak{m}^2)^\vee$. Its basis $y_1,\dots,y_e$ can be described quite explicitly in terms of cycles in $\mathrm{Hom}_R(P,P)$:

2.9. *There are degree* -1 *cycles* $\theta_1,\dots,\theta_e$ *in* $\mathrm{Hom}_R(P,P)$ *with* $y_i=\mathrm{cls}\left(\theta_i\right)$ *and such that*

$$\theta_i(e_j)=\begin{cases}1 & \text{if } i=j\\ 0 & \text{if } i\neq j\end{cases}\qquad\text{for all}\quad 1\leqslant i,\ j\leqslant e.$$

A proof of this result is contained in [25, Theorem 1]; see also [2, (10.2.1)].

2.10. Products in $R^!$. Suppose that the local ring R is $\mathfrak{m}$-adically complete. Cohen's structure theorem provides a surjective local homomorphism $\pi\colon (Q, \mathfrak{n}, k) \to R$ with Q regular; one can ensure that $\operatorname{Ker}(\pi) \subseteq \mathfrak{n}^2$. Let $\widetilde{r}_1, \ldots, \widetilde{r}_e$ be elements in Q such that $\pi(\widetilde{r}_i) = r_i$ for each integer i. Then $(\widetilde{r}_1, \ldots, \widetilde{r}_e) = \mathfrak{n}$ since $\operatorname{Ker}(\pi) \subseteq \mathfrak{n}^2$. Let $s_1, \ldots, s_c$ be a minimal generating set for $\operatorname{Ker}(\pi)$. For each integer k, write

$$s_k = \sum_{1 \leqslant i \leqslant j \leqslant e} a_{ijk}\widetilde{r}_i\widetilde{r}_j \quad \text{with} \quad a_{ijk} \in Q.$$

In what follows, given an element $a \in Q$, its residue class in $k = Q/\mathfrak{n}$ is denoted $\overline{a}$.

It is established in [25, Section 3], see also [2, (10.2.2)], that there is a basis $u_1, \ldots, u_c$ of $\operatorname{Ext}^2_R(k, k)$ such that for all integers i, j, p with $1 \leqslant i < j \leqslant e$ and $1 \leqslant p \leqslant e$, one has

$$[y_i, y_j] = -\sum_{k=1}^{c} \overline{a}_{ijk} u_k \quad \text{and} \quad y_p^2 = -\sum_{k=1}^{c} \overline{a}_{ppk} u_k.$$

Here $[y_i, y_j]$ denotes the (graded) commutator $y_i y_j + y_j y_i$ of the elements y_i and y_j.

We say that a graded module U over a graded ring A is *bounded to the right* if $U_i = 0$ for $i \ll 0$, and *degreewise finite* if the A_0-module U_i is finitely generated for each integer i. These notions apply also to graded modules over our local ring R, which we view as a graded ring concentrated in degree 0.

2.11. The action of $R^!$ on $\operatorname{Tor}^R(k, M)$**.** Let M be a complex of R modules whose homology is bounded to the right and degreewise finite. Such a complex has a minimal free resolution F, and

$$\operatorname{Tor}^R(k, M) = \mathrm{H}(k \otimes_R F) = k \otimes_R F.$$

As has been described in 2.8, the k-algebra $\operatorname{Ext}_R(k, k)$ acts on $\operatorname{Tor}^R(k, M)$; hence, via the identification above, it acts on $k \otimes_R F$. In turns out that the action of $\operatorname{Ext}^1_R(k, k)$ on $k \otimes_R F$ is determined completely by the linear part of the differential on F. This is the gist of the following result. A piece of notation: for any element $z \in F$, we write $\overline{z}$ for its residue class in $k \otimes_R F$.

Lemma. *Let z be an element in F_n and write $\partial(z) = \sum_{j=1}^{e} r_j f_j$ with $f_j \in F_{n-1}$. Then $y_i \cdot \overline{z} = -\overline{f_i}$ for each integer $1 \leqslant i \leqslant e$.*

Proof. The complex of R-modules F is free so the canonical surjection $\varepsilon \otimes_R F\colon P \otimes_R F \twoheadrightarrow k \otimes_R F$, where ε is the quasiisomorphism defined in 2.7(d), is a quasiisomorphism. Since

$$(P \otimes_R F)_n = F_n \oplus (P_1 \otimes F_{n-1}) \oplus (P_{\geqslant 2} \otimes_R F)_n,$$

the cycle $\overline{z}$ lifts to a cycle in $P \otimes_R F$ of the form

$$\widetilde{z} = z - \sum_{j=1}^{e} (e_j \otimes_R f_j) + w \quad \text{with} \quad w \in (P_{\geqslant 2} \otimes_R F).$$

Let $\theta_i \in \mathrm{Hom}_R(P, P)$ be the degree -1 cycle corresponding to y_i that is provided by 2.9. Then $\theta_i(\widetilde{z}) = -f_i + \theta_i(w)$ so $y_i \cdot \mathrm{cls}\,(\widetilde{z}) = \mathrm{cls}\,\big(-f_i + \theta_i(w)\big)$ in $\mathrm{H}(P \otimes_R F)$. Thus, in $k \otimes_R F$ one has that

$$y_i \cdot \overline{z} = \varepsilon(y_i \cdot \mathrm{cls}\,(\widetilde{z})) = \varepsilon(\mathrm{cls}\,\big(-f_i + \theta_i(w)\big)) = \mathrm{cls}\,\big(\varepsilon(-f_i + \theta_i(w))\big) = -\overline{f_i},$$

where the last equality holds because $\theta_i(w) \in (P_{\geqslant 1} \otimes_R F) = \mathrm{Ker}(\varepsilon)$. □

Here is the cohomological counterpart of 2.11.

2.12. The action of $R^!$ on $\mathrm{Ext}_R(M, k)$. As before, let M be a complex of R-modules whose homology is bounded to the right and degreewise finite, and let F be its minimal free resolution. By definition, $\mathrm{Ext}_R(M, k) = \mathrm{H}(\mathrm{Hom}_R(F, P))$; see 2.8. However, the complex of R-modules F is free, so the canonical surjection $\mathrm{Hom}_R(F, \varepsilon)\colon \mathrm{Hom}_R(F, P) \to \mathrm{Hom}_R(F, k)$ is a quasiisomorphism. The minimality of F ensures that the differential on $\mathrm{Hom}_R(F, k)$ is trivial, so that

$$\mathrm{Ext}_R(M, k) = \mathrm{Hom}_R(F, k).$$

Via this identification, $\mathrm{Hom}_R(F, k)$ acquires a left $\mathrm{Ext}_R(k, k)$-module structure.

Lemma. *Let α be a homomorphism in $\mathrm{Hom}_R(F, k)$ of degree $1 - n$, and let $z \in F_n$. Write $\partial(z) = \sum_{j=1}^{e} r_j f_j$ with $f_j \in F_{n-1}$. Then $(y_i \cdot \alpha)(z) = (-1)^{1-n}\alpha(f_i)$ for each integer $1 \leqslant i \leqslant e$.*

Proof. Under the quasiisomorphism $\mathrm{Hom}_R(F, \varepsilon)$, the element α lifts to a cycle $\widetilde{\alpha}$ in $\mathrm{Hom}_R(F, P)$, of degree $1 - n$. Now $\widetilde{\alpha}(z) \in P_1$, since its degree is 1. Say $\widetilde{\alpha}(z) = \sum_{j=1}^{e} c_j e_j$, with $c_j \in R$. Then

$$\partial\widetilde{\alpha}(z) = \sum_{j=1}^{e} c_j r_j \quad \text{and} \quad \widetilde{\alpha}\partial(z) = \sum_{j=1}^{e} r_j \widetilde{\alpha}(f_j).$$

Since $\partial\widetilde{\alpha} = (-1)^{1-n}\widetilde{\alpha}\partial$, we obtain that $\sum_{j=}^{e} (c_j - (-1)^{1-n}\widetilde{\alpha}(f_j))r_j = 0$ in P_0; recall that $P_0 = R$. As $r_1, \ldots, r_e$ is a minimal generating set for $\mathfrak{m}$, the element $c_j - (-1)^{1-n}\widetilde{\alpha}(f_j)$ is in $\mathfrak{m}$, for each integer j. Now $\varepsilon\widetilde{\alpha} = \alpha$ and $\mathfrak{m} \subset \mathrm{Ker}(\varepsilon)$, consequently $\varepsilon(c_j) = (-1)^{1-n}\alpha(f_j)$ for each integer j. Therefore, with θ_i the cycle in $\mathrm{Hom}_R(P, P)$ corresponding to y_i, given by 2.9, one obtains

$$(y_i \cdot \alpha)(z) = \varepsilon(\theta_i(\widetilde{\alpha}(z))) = \varepsilon\left(\theta_i\left(\sum_{j=1}^{e} c_j e_j\right)\right) = \varepsilon(c_i) = (-1)^{1-n}\alpha(f_i).$$

This is the equality we seek. □

Two more ingredients are required before one can deal with Theorem 2.2. Here is the first.

2.13. Let A be a standard graded k-algebra and G a graded degreewise finite A-module with $G_i = 0$ for $i \ll 0$. Set $s = \inf\{i \mid G_i \neq 0\}$, and

$$\mathscr{D} = \bigoplus_{n\in\mathbb{N}} \operatorname{Ext}^n_A(k,k)_{-n} \quad \text{and} \quad \mathscr{U} = \bigoplus_{n\in\mathbb{N}} \operatorname{Ext}^n_A(G,k)_{-n-s}.$$

Then $\mathscr{D}$ is a k-subalgebra, called the *diagonal subalgebra*, of $\operatorname{Ext}_A(k,k)$. In particular, $\mathscr{D}$ acts on $\operatorname{Ext}_A(G,k)$ giving it a structure of a left $\mathscr{D}$-module. Since $\mathscr{D}$ is situated on the diagonal, the subspace $\mathscr{U}$ of $\operatorname{Ext}_A(G,k)$ inherits this structure; we call it the *diagonal submodule* of $\operatorname{Ext}_A(G,k)$. It should be emphasized that this is a $\mathscr{D}$-submodule, but not necessarily an $\operatorname{Ext}_A(k,k)$-submodule.

It turns out that $\mathscr{D}$ coincides with $A^!$, the Koszul dual of A; see 2.1. Moreover, $\mathscr{U}$ is generated as a left $\mathscr{D}$-module by $\mathscr{U}^0$, which is to say, by $\operatorname{Hom}_k(G_s,k)$. These statements, and more, are contained in the next result; in it $\mathrm{T}\left(A^\vee_{-1}\right)$ denotes the tensor algebra on the k-vector space $A^\vee_{-1}$ concentrated in degree -1, and $(\operatorname{Im}(\mu_R^\vee))$ denotes its 2-sided ideal generated by $\operatorname{Im}(\mu_R^\vee)$.

Proposition. *Let $\mu_R\colon A_1 \otimes_k A_1 \to A_2$ and $\mu_M\colon A_1 \otimes_k G_s \to G_{s+1}$ be the respective product maps. Then $\mathscr{D} = A^!$, and there are canonical isomorphisms*:

$$\mathscr{D} \cong \frac{\mathrm{T}\left(A^\vee_{-1}\right)}{(\operatorname{Im}(\mu_R^\vee))} \quad \textit{and} \quad \mathscr{U} \cong \frac{\mathscr{D} \otimes_k G^\vee_{-s}}{\mathscr{D}\cdot \operatorname{Im}(\mu_M^\vee)},$$

where the one on the left is of k-algebras, and the one on the right is of left $\mathscr{D}$-modules.

Proof. The assertions concerning $\mathscr{D}$ are special cases of a result of Löfwall [16, Corollary 1.1]. His arguments can be extended without any ado to obtain the desired formula for $\mathscr{U}$. $\square$

The next item establishes some structural properties of the maps defined in 1.4.

2.14. Let $(R,\mathfrak{m},k)$ be a local ring and M a finitely generated R-module. Set $A = \operatorname{gr}_{\mathfrak{m}}(R)$. The construction in 1.4 yields diagrams

$$\operatorname{Ext}_A(k,k) \xrightarrow{\pi_k} \bigoplus_{n\in\mathbb{N}} \operatorname{Ext}^n_A(k,k)_{-n} \xrightarrow{\nu_k} \operatorname{Ext}_R(k,k),$$

$$\operatorname{Ext}_A(\operatorname{gr}_{\mathfrak{m}}(M),k) \xrightarrow{\pi_M} \bigoplus_{n\in\mathbb{N}} \operatorname{Ext}^n_A(\operatorname{gr}_{\mathfrak{m}}(M),k)_{-n} \xrightarrow{\nu_M} \operatorname{Ext}_R(M,k),$$

such that the composed map in the first diagram is χ_k, and in the second it is χ_M.

Proposition. *The maps π_k, ν_k, and χ_k are homomorphisms of k-algebras. Moreover, π_M is π_k-equivariant, ν_M is ν_k-equivariant, and χ_M is χ_k-equivariant.*

Proof. First we tackle χ_k and χ_M. Since these maps are k-linear, by construction, it suffices to prove that for $\varepsilon \in \operatorname{Ext}_A(k,k)$ and $\alpha \in \operatorname{Ext}_A(\operatorname{gr}_{\mathfrak{m}}(M),k)$ one has

(†)

$$\chi_M(\varepsilon \cdot \alpha) = \chi_k(\varepsilon) \cdot \chi_M(\alpha).$$

This would settle the equivariance of χ_M, and specializing M to k would yield also that χ_k is an algebra homomorphism. There are further reductions: note that

(‡)

$$\operatorname{Ker}(\pi_k^n) = \operatorname{Ext}_A^n(k,k)_{i<-n} \quad \text{and} \quad \operatorname{Ker}(\pi_M^n) = \operatorname{Ext}_A^n(\operatorname{gr}_{\mathfrak{m}}(M),k)_{i<-n}.$$

In particular, when checking (†), one may assume that ε lies in $\bigoplus_{n\in\mathbb{N}} \operatorname{Ext}_A^n(k,k)_{-n}$, the diagonal subalgebra of $\operatorname{Ext}_A(k,k)$. Then, it suffices to verify (†) for $\varepsilon = y_i$, where $y_1, \ldots, y_e$ are the elements defined in 2.7. This is because, by Proposition 2.13, the diagonal subalgebra is generated by $\operatorname{Ext}_A^1(k,k)$, that is to say, by $(\mathfrak{m}/\mathfrak{m}^2)^\vee$. Now $\operatorname{Ext}_R^1(k,k) = (\mathfrak{m}/\mathfrak{m}^2)^\vee$ as well; a direct computation establishes that χ_k^1 is the identity. The long and short of this discussion is that the desired results on χ_k and χ_M follow once we check (†) for $\varepsilon = y_i$ for each integer $1 \leqslant i \leqslant e$.

Let F be the minimal R-free resolution of M, let U be the minimal graded A-free resolution of $\operatorname{gr}_{\mathfrak{m}}(M)$, and $\kappa\colon \operatorname{lin}^R(F) \to U$ the comparison map; see 1.4. Then $\operatorname{Ext}_R^n(M,k) = \operatorname{Hom}_R(F_n,k)$ and $\operatorname{Ext}_A^n(\operatorname{gr}_{\mathfrak{m}}(M),k) = \operatorname{Hom}_A(U_n,k)$. Fix a homomorphism α in $\operatorname{Hom}_A(U,k)$ of degree $1-n$ and an element $z \in F_n$. By definition

$$\chi_M(\alpha)(z) = \alpha(\kappa(\overline{z})),$$

where $\overline{z}$ denotes the residue class of z in $F/\mathfrak{m}F$. Write $\partial(z) = \sum_{j=1}^e r_j f_j$. Then, in $\operatorname{lin}^R(F)$ one has $\partial(\overline{z}) = \sum_{j=1}^e x_j \overline{f}_j$, where x_j is the residue class in $\mathfrak{m}/\mathfrak{m}^2$ of the element r_j. Since κ is a morphism of complexes one obtains $\partial(\kappa(\overline{z})) = \kappa(\partial(\overline{z})) = \sum_{j=1}^e x_j \kappa(\overline{f}_j)$. The formula above and 2.12 yield

$$\chi_M(y_i \cdot \alpha)(z) = (y_i \cdot \alpha)(\kappa(\overline{z})) = (-1)^{1-n}\alpha(\kappa(\overline{f}_i)),$$

$$(y_i \cdot \chi_M(\alpha))(z) = (-1)^{1-n}\chi_M(\alpha)(f_i) = (-1)^{1-n}\alpha(\kappa(\overline{f}_i)).$$

Since z was chosen arbitrarily, this yields: $\chi_M(y_i \cdot \alpha) = y_i \cdot \chi_M(\alpha)$, for each integer i, as desired.

This completes the proof that χ_k is a homomorphism of k-algebras and χ_M is χ_k-equivariant.

As to the remaining assertions: Since each graded vector space $\operatorname{Ext}_A^n(k,k)$ is concentrated in degrees $\leqslant -n$; one obtains from (‡) that $\operatorname{Ker}(\pi_k)$ is a 2-sided ideal in $\operatorname{Ext}_A(k,k)$ and that $\operatorname{Ker}(\pi_M)$ is a (left) $\operatorname{Ext}_A(k,k)$-submodule of $\operatorname{Ext}_A(\operatorname{gr}_{\mathfrak{m}}(M),k)$. Therefore, π_k is a k-algebra homomorphism and π_M is π_k-equivariant. The surjectivity of these homomorphisms and the identities $\chi_k = \nu_k \circ \pi_k$ and $\chi_M = \nu_M \circ \pi_k$ entail the stated properties of ν_k and ν_M. □

2.15. Proof of Theorem 2.2. As usual, we write $R^!$ for $\langle \operatorname{Ext}_R^1(k,k)\rangle$ and A for $\operatorname{gr}_{\mathfrak{m}}(R)$.

Suppose $\mathrm{ld}_R\,(M)=0$, so that M is Koszul. Then Proposition 1.5 provides the isomorphism

$$\chi_M \colon \bigoplus_{n\in\mathbb{N}} \mathrm{Ext}^n_A(\mathrm{gr}_{\mathfrak{m}}(M), k)_{-n} \xrightarrow{\cong} \mathrm{Ext}_R(M, k).$$

Proposition 2.13 yields that the module on the left is generated as an $A^!$-module by the k-vector subspace $\mathrm{Ext}^0_A(\mathrm{gr}_{\mathfrak{m}}(M), k)_0$. Therefore, one obtains that the $R^!$-module $\mathrm{Ext}_R(M, k)$ is generated by $\mathrm{Ext}^0_R(M, k)$, since χ_M is χ_k-equivariant; see Proposition 2.14. This is the result we seek.

Suppose $d = \mathrm{ld}_R\,(M) \geqslant 1$. The first syzygy $\Omega\,(M)$ of M is defined by the exact sequence $0 \to \Omega\,(M) \to F \to M \to 0$, where F is a free module and $\Omega\,(M) \subseteq \mathfrak{m}F$. When $\mathrm{Hom}_R(-, k)$ is applied to it, it unwraps into the long exact sequence

$$\cdots \to \mathrm{Ext}^i_R(\Omega\,(M)\,, k) \xrightarrow{\eth^i} \mathrm{Ext}^{i+1}_R(M, k) \to \mathrm{Ext}^{i+1}_R(F, k) \to \mathrm{Ext}^{i+1}_R(\Omega\,(M)\,, k) \\ \to \cdots.$$

Due to the choice of F, one obtains that $\eth^i$ is an isomorphism for each $i \geqslant 0$. Furthermore, it is well known, and not difficult to prove, that the $\eth^i$ are compatible with the corresponding $\mathrm{Ext}_R(k, k)$-module structures. Thus, the long exact sequence above can be packaged into the short exact sequence of graded left $\mathrm{Ext}_R(k, k)$-modules

$$0 \to \mathrm{Ext}_R(\Omega\,(M)\,, k)(1) \to \mathrm{Ext}_R(M, k) \to \mathrm{Ext}^0_R(M, k) \to 0.$$

Here $\mathrm{Ext}^0_R(M, k)$ has the trivial left $\mathrm{Ext}_R(k, k)$-action. The exact sequence above is also one of left $R^!$-modules, since the latter is a subalgebra of $\mathrm{Ext}_R(k, k)$. Now, $\mathrm{ld}_R\,(\Omega\,(M)) = \mathrm{ld}_R\,(M) - 1$; thus, by an iteration, one can assume that $\mathrm{Ext}_R(\Omega\,(M)\,, k)$ is generated over $R^!$ by elements in degrees $\leqslant d - 1$. Therefore, in the exact sequence above, the module on the left, and hence also the one in the middle, is generated by elements in degrees $\leqslant d$. $\square$

3. Conjugation

Let $(R, \mathfrak{m}, k)$ be a local ring. The first item in this section describes a natural procedure for converting each left $\mathrm{Ext}_R(k, k)$-module into a right $\mathrm{Ext}_R(k, k)$-module with the same underlying k-vector space, and vice versa. Its relevance to the topic on hand is explained by 3.3.

3.1. Conjugation. Set $E = \mathrm{Ext}_R(k, k)$. This graded k-algebra can be enriched to a graded Hopf algebra, and part of that structure is an anti-automorphism $\sigma \colon E \to E$, called the *conjugation*; see [10, Chapter II, Section 2], and the paper of Milnor and Moore [18, (8.4)]. For each graded left E-module U, the graded k-vector space underlying it has the structure of right E-module, with product

$$u * \varepsilon = (-1)^{|\varepsilon||u|}\sigma(\varepsilon)u \quad \text{for } u \in U \text{ and } \varepsilon \in E.$$

Denote U^σ the right E-module defined thus. This procedure is functorial: Each homomorphism $f \colon U \to V$ of graded left E-modules transforms into a homomorphism of graded right E-modules: $f\colon U^\sigma \to V^\sigma$. These remarks may be summed up as conjugation is an equivalence between the categories of left E-modules and right E-modules.

The remark below is used in Section 4; it holds, more generally, for modules over Hopf algebras.

Lemma. *Let U and V be graded left E-modules. There is an natural isomorphism of bigraded k-vector spaces*

$$\mathrm{Tor}^E\left(U^\sigma, V\right) \cong \mathrm{Tor}^E\left(V^\sigma, U\right).$$

Proof. One has the natural 'twisting' homomorphism of graded k-vector spaces

$$U^\sigma \otimes_k V \to V^\sigma \otimes_k U, \quad \text{where} \quad u \otimes_k v \mapsto (-1)^{|u||v|} v \otimes_k u.$$

A simple, and gratifying, computation establishes that this map descends to a homomorphism $U^\sigma \otimes_E V \to V^\sigma \otimes_E U$; it is bijective for it has an obvious inverse. The derived avatar of this isomorphism is the one we seek. $\square$

3.2. Duality. Set $E = \mathrm{Ext}_R(k,k)$. Let M be a complex of R-modules whose homology is bounded to the right and, degreewise finite. Then Hom-Tensor adjunction yields a natural isomorphism of graded k-vector spaces:

($\star$)

$$\mathrm{Ext}_R(M,k) \cong \mathrm{Hom}_k(\mathrm{Tor}^R(k,M), k).$$

Via this isomorphism, the left action of E on $\mathrm{Tor}^R(k,M)$ passes to a *right* action on $\mathrm{Ext}_R(M,k)$. Now $\mathrm{Ext}_R(M,k)$ has a second right E-module structure arising via conjugation—see 3.1—from its usual *left* E-module structure.

The following proposition tells us that these two structures on $\mathrm{Ext}_R(M,k)$ coincide. This coincidence is exploited in the proofs of Theorems 4.1 and 6.1.

3.3. Proposition. *Suppose that R is Koszul. Then the isomorphism in* ($\star$) *is compatible with the corresponding right E-module structures*

$$\mathrm{Ext}_R(M,k)^\sigma \cong \mathrm{Hom}_k(\mathrm{Tor}^R(k,M), k).$$

Dually the isomorphism $\mathrm{Hom}_k(\mathrm{Ext}_R(M,k)^\sigma, k) \cong \mathrm{Tor}^R(k,M)$ *is one of left E-modules.*

Remark. This result is basic, but there seems to be no good reference to it in the literature, except for the special case $M = k$; see [10, (2.3.3)]. It appears that (2.1) [14], which deals with the comodule action of $\mathrm{Tor}^R(k,k)$ on $\mathrm{Tor}^R(M,k)$, would settle the issue, at least when M is a module; however, deriving Proposition 3.3 from the latter result would be a technical digression.

Proof. The ring R is Koszul, so $E = \langle \mathrm{Ext}^1_R(k,k)\rangle$; see 2.5. This means that the elements $y_1, \ldots, y_e$, from 2.7, generate the k-algebra E, since $\mathrm{Ext}^1_R(k,k) = (\mathfrak{m}/\mathfrak{m}^2)^\vee$. Consequently, it suffices to verify the required compatibility for the action of each of the y_i. The effect of conjugating the y_i is easy to describe

(⋆⋆)

$$\sigma(y_i) = -y_i \quad \text{for all} \quad 1 \leqslant i \leqslant e.$$

This is immediate from the proof of [18, (8.2)].

Let F be a minimal free resolution of M. Then $\mathrm{Tor}^R(k, M) = k \otimes_R F$ and $\mathrm{Ext}_R(M, k) = \mathrm{Hom}_R(F, k)$; the induced left action of $\mathrm{Ext}^1_R(k, k)$ on $k \otimes_R F$ and $\mathrm{Hom}_R(F, k)$ is described in 2.11 and 2.12, respectively. The isomorphism in question is then the usual adjunction:

$$\mathrm{Hom}_R(F, k) \cong \mathrm{Hom}_k(k \otimes_R F, k).$$

Let $\alpha \in \mathrm{Hom}_R(F, k)$ be an element of degree $1 - n$. Let z be an element in F_n; write $\partial(z) = \sum_{j=1}^{e} r_j f_j$. By 2.11, the right action of y_i on α induced by the isomorphism above is given by

$$(\alpha \cdot y_i)(z) = -\alpha(f_i).$$

On the other hand, the right action on $\mathrm{Hom}_R(F, k)$ induced by conjugation is given by

$$(\alpha * y_i)(z) = (-1)^{1-n}(\sigma(y_i) \cdot \alpha)(z) = -(-1)^{1-n}(\sigma \cdot \alpha)(z) = -\alpha(f_i),$$

where second equality is by (⋆⋆) above and the last is by 2.12. Thus $\alpha \cdot y_i = \alpha * y_i$ on elements of degree n. Thus $\alpha \cdot y_i = \alpha * y_i$, since both are of degree $-n$. This settles the issue. □

4. The (co)homology of the linear part of a complex

This section is dedicated entirely to the proof of the theorem below. It generalizes, and is inspired by, the article of Eisenbud et al. [9, (3.4)], who have dealt with the case where R is either a polynomial ring or an exterior algebra. Some of the key ideas in the proof are contained already in work of Buchweitz [7].

4.1. Let $(R, \mathfrak{m}, k)$ be a local ring. As before, we write $R^!$ for the Koszul dual of R; see 2.1. Let M a complex of R-modules whose homology is bounded to the right and degreewise finite.

Theorem. *Let F be the minimal free resolution of M. If R is Koszul, then for all $n, i \in \mathbb{Z}$ one has isomorphisms of k-vector spaces*

$$\mathrm{H}_n(\mathrm{lin}^R(F))_i = \mathrm{Ext}^{i-n}_{R^!}(k, \mathrm{Tor}^R(k, M))^{-i},$$

$$\mathrm{Hom}_k(\mathrm{H}_n(\mathrm{lin}^R(F))_i, k) = \mathrm{Tor}^{R^!}_{i-n}\big(k, \mathrm{Ext}_R(M, k)\big)^i.$$

In the sequel, this result is invoked only for the special case where M is a module (concentrated in degree 0). The more general result is established with a view towards future applications. At any rate, nothing is gained by restricting oneself to modules.

Remark. The statement of the theorem above involves Tor and Ext functors simultaneously. The natural grading on the former is homological whilst on the latter it is cohomological. We should like to remind the reader of our convention, set out in 1.1, regarding gradings.

The proof of Theorem 4.1 requires considerable preparation and is given at the end of the section. An important intermediate step is Theorem 4.5 which describes $\operatorname{lin}^R(F)$ in terms of the action of $R^!$ on $\operatorname{Tor}^R(k, M)$; this result does not require R to be Koszul.

4.2. Notation. In what follows, the notation introduced in 2.7 will be used without specific comment. In addition, it is convenient to set $A = \operatorname{gr}_{\mathfrak{m}}(R)$, and to write $\mathscr{E}$ for the k-algebra $\operatorname{Ext}_R(k, k)$, viewed as being an object with a lower grading; see 1.1. Thus, A is concentrated in nonnegative degrees whilst $\mathscr{E}$ is concentrated in nonpositive degrees.

4.3. Consider the k-algebra $A \otimes_k \mathscr{E}$ with product defined by

$$(x' \otimes_k y') \cdot (x \otimes_k y) = x'x \otimes_k yy' \quad \text{for all} \quad x', x \in A \text{ and } y', y \in \mathscr{E}.$$

Observe that $(\mathfrak{m}/\mathfrak{m}^2) \otimes_k (\mathfrak{m}/\mathfrak{m}^2)^\vee$ is the subspace $A_1 \otimes_k \mathscr{E}_{-1}$ of $A \otimes_k \mathscr{E}$. The k-vector space $\mathfrak{m}/\mathfrak{m}^2$ is finite dimensional, so the canonical map

$$(\mathfrak{m}/\mathfrak{m}^2) \otimes_k (\mathfrak{m}/\mathfrak{m}^2)^\vee \longrightarrow \operatorname{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, \mathfrak{m}/\mathfrak{m}^2)$$

is an isomorphism. Denote Δ the element in $(\mathfrak{m}/\mathfrak{m}^2) \otimes_k (\mathfrak{m}/\mathfrak{m}^2)^\vee$ corresponding to the identity map on $\mathfrak{m}/\mathfrak{m}^2$. Observe that

$$\Delta = \sum_{i=1}^{e} (x_i \otimes_k y_i),$$

in terms of the chosen basis $x_1, \ldots, x_e$ of $\mathfrak{m}/\mathfrak{m}^2$ and its dual $y_1, \ldots, y_e$.

Lemma. $\Delta^2 = 0$, *when viewed as an element in* $A \otimes_k \mathscr{E}$.

Proof. Let $\widehat{R}$ denote the completion of R at its maximal ideal $\mathfrak{m}$. The canonical map $R \to \widehat{R}$ is flat and the maximal ideal $\widehat{\mathfrak{m}}$ of the local ring $\widehat{R}$ equals $\mathfrak{m}\widehat{R}$. Therefore, $\operatorname{gr}_{\mathfrak{m}}(R) \cong \operatorname{gr}_{\widehat{\mathfrak{m}}}(\widehat{R})$ and $\operatorname{Ext}_R(k, k) \cong \operatorname{Ext}_{\widehat{R}}(k, k)$. So by passing to $\widehat{R}$ one can assume that R is complete. Now one is in a position to utilize the results in 2.10 and 2.11; we adopt their notation.

The elements $x_1, \ldots, x_e$ form a basis for $\mathfrak{m}/\mathfrak{m}^2$, so they generate the k-algebra A. By construction, each x_i is the residue class of the element $\widetilde{r}_i$. Thus, for any integer k, one has that

(⋆)

$$\sum_{1 \leqslant i \leqslant j \leqslant e} \overline{a}_{ijk} x_i x_j = \sum_{1 \leqslant i \leqslant j \leqslant e} a_{ijk} \widetilde{r}_i \widetilde{r}_j \bmod (\mathfrak{n}^2 + \operatorname{Ker}(\pi)) = 0,$$

where the equality on the right holds because $\sum_{i,j} a_{ijk} \widetilde{r}_i \widetilde{r}_j = s_k$, which is an element in $\operatorname{Ker}(\pi)$.

Recall that $\Delta = \sum_{i=1}^{e} x_i \otimes_k y_i$. This leads to the following computation, where the second equality holds as A is commutative; the third is by 2.10, while the last is due to ($\star$) above:

$$\begin{aligned}
\Delta^2 &= \sum_{1 \leqslant i<j \leqslant e} (x_i x_j \otimes_k y_j y_i + x_j x_i \otimes_k y_i y_j) + \sum_{i=1}^{e} (x_i^2 \otimes_k y_i^2) \\
&= \sum_{1 \leqslant i<j \leqslant e} (x_i x_j \otimes_k [y_i, y_j]) + \sum_{i=1}^{e} (x_i^2 \otimes_k y_i^2) \\
&= \sum_{\substack{1 \leqslant i<j \leqslant e \\ 1 \leqslant p \leqslant c}} -(x_i x_j \otimes_k \overline{a}_{ijp} u_p) + \sum_{\substack{1 \leqslant i \leqslant e \\ 1 \leqslant p \leqslant c}} -(x_i^2 \otimes_k \overline{a}_{iip} u_p) \\
&= \sum_{\substack{1 \leqslant i \leqslant j \leqslant e \\ 1 \leqslant p \leqslant c}} -(x_i x_j \otimes_k \overline{a}_{ijp} u_p) \\
&= \sum_{p=1}^{c} -\left(\sum_{1 \leqslant i \leqslant j \leqslant e} \overline{a}_{ijp} x_i x_j \right) \otimes_k u_p \\
&= 0.
\end{aligned}$$

Thus $\Delta^2 = 0$; this is the result we seek. $\square$

4.4. Construction I. Let $U = \{U_i\}_{i \in \mathbb{Z}}$ be a graded left $\mathscr{E}$-module. For each integer n, let

$$\beta_n(U) = A(-n) \otimes_k U_n$$

be the graded free A-module, with A acting via the left-hand factor of the tensor product. Let $\delta_n : \beta_n(U) \to \beta_{n-1}(U)$ be the k-linear map with

$$\delta_n(s \otimes_k u) = \sum_{i=1}^{e} (s x_i \otimes_k y_i u) \quad \text{for all} \quad (s \otimes_k u) \in \beta_n(U).$$

Claim. *For each integer n, the map δ_n is degree* 0 *homomorphism of A-modules and $\delta_{n-1} \circ \delta_n = 0$. In other words, $(\beta_*(U), \delta_*)$ is a complex of graded free A-modules.*

Indeed, it is elementary to verify that δ_n is A-linear and that it preserves (internal) degrees. Thus, the only moot point is whether δ_n is a differential. Note that the k-vector space $\beta_*(U)$ is a right module over $A \otimes_k \mathscr{E}$ with product defined by $(s \otimes_k u) \cdot (x \otimes_k y) = (sx \otimes_k yu)$. In particular, $\delta(s \otimes_k u) = (s \otimes_k u) \cdot \Delta$, where Δ is the element in 4.3. It remains to invoke Lemma 4.3.

This brings us to one of the major landmarks on the road to Theorem 4.1.

4.5. Let F be a minimal free resolution of M; remember that $\mathrm{H}(M)$ is assumed to be a bounded to the right and degreewise finite. Recall that $\mathscr{E} = \mathrm{Ext}_R(k, k)$.

Since $\mathrm{Tor}^R(k, M)$ is a left module over $\mathscr{E}$—see 2.8—one may construct the complex of graded A-modules $\beta(\mathrm{Tor}^R(k, k))$, as in 4.4. The crux of the theorem below is that this complex is naturally isomorphic to $\mathrm{lin}^R(F)$. First, a description of the candidate for the isomorphism: Since F is minimal $\mathrm{Tor}_n^R(k, M) = k \otimes_R F_n = F_n/\mathfrak{m}F_n$. This latter vector space is exactly the degree n component of $\mathrm{lin}_n^R(F)$, so multiplication by $(-1)^n$ yields an inclusion of k-vector spaces: $\mathrm{Tor}_n^R(k, M) \hookrightarrow \mathrm{lin}_n^R(F)$. This extends to a degree 0 homomorphism of graded A-modules

$$\kappa_n : A(-n) \otimes_k \mathrm{Tor}_n^R(k, M) \longrightarrow \mathrm{lin}_n^R(F).$$

Combining these yields a homomorphism of graded A-modules: $\beta(\mathrm{Tor}^R(k, M)) \to \mathrm{lin}^R(F)$.

Theorem. *The homomorphism $\kappa : \beta(\mathrm{Tor}^R(k, M)) \to \mathrm{lin}^R(F)$ is an isomorphism of complexes.*

Remark. The definition of κ utilizes signed multiplication on $k \otimes_R F$ rather than the identity because the multiplication on Ext is the composition, and not the Yoneda, product.

Proof. We identify $(k \otimes_R F)$ with $\mathrm{Tor}^R(k, M)$, and adopt the notation in 2.11.

Fix an integer n. The A-module $\mathrm{lin}_n^R(F)$ is free and generated by the subspace of elements of degree n. Since this latter subspace is isomorphic to $(k \otimes_R F)_n$ via κ_n, one obtains that κ_n is an isomorphism. It remains to prove that κ is compatible with the differentials.

Let z be an element in $F_n \setminus \mathfrak{m}F_n$. Write $\partial(z) = \sum_{j=1}^{e} r_j f_j \in F_{n-1}$. Then $\partial(\overline{z}) = \sum_{j=1}^{e} x_j \overline{f}_j$ in $\mathrm{lin}^R(F)$; this explains the first row in the diagram below. As to the second: the equality is due to Lemma 2.11, while the rest are by definition.

$$(1 \otimes_k \overline{z}) \xrightarrow{\kappa} (-1)^n \overline{z} \xrightarrow{\partial} (-1)^n \sum_{j=1}^{e} x_j \overline{f}_j,$$

$$(1 \otimes_k \overline{z}) \xrightarrow{\delta} \sum_{j=1}^{e} (x_j \otimes_k y_j \overline{z}) = -\sum_{j=1}^{e} (x_j \otimes_k \overline{f}_j) \xrightarrow{\kappa} (-1)^n \sum_{j=1}^{e} x_j \overline{f}_j$$

Thus $\partial\kappa = \kappa\delta$ on elements of the form $1 \otimes_k \overline{z}$ with $z \in F_n \setminus \mathfrak{m}F_n$. The A-module $\beta_n(k \otimes_R F)$ is spanned by such elements and all the maps in question are A-linear, hence one deduces that κ commutes with the differentials. This completes the proof of the theorem. □

By definition, when $(R, \mathfrak{m}, k)$ is Koszul, the linear part of the minimal resolution of k is acyclic, with homology equal to k, and also minimal. Then the resolution of k over $\mathrm{gr}_{\mathfrak{m}}(R)$ provided by the theorem above coincides with one given by Priddy [19] and Löwall [16].

4.6. Corollary. *If R is Koszul, then $\beta(\mathrm{Tor}^R(k, k))$ is the minimal A-free resolution of k.*

The next item is a formula for the homology of the complex $\beta(U)$ constructed in 4.4. This calls for an alternate description of that complex.

4.7. Construction II. Let $U = \{U_i\}_{i \in \mathbb{Z}}$ be a graded degreewise finite left $\mathscr{E}$-module. As per our convention, $A^\vee$ denotes the graded k-dual of A, so $A^\vee_{-n} = \mathrm{Hom}_k(A_n, k)$, for each n. Let

$$X_n = \mathscr{E}(n) \otimes_k (A^\vee)_{-n}$$

be the graded free left $\mathscr{E}$-module with multiplication induced from the *left* action of $\mathscr{E}$ on itself. Let $\partial_n\colon X_n \to X_{n-1}$ be the k-linear map with

$$\partial_n(e \otimes_k a) = \sum_{i=1}^{e} (ey_i \otimes_k x_i a) \quad \text{for all} \quad (e \otimes_k a) \in X_n.$$

It is immediate that ∂_n is $\mathscr{E}$-linear and that it has degree 0. Arguments akin to those used in 4.4 establish that $\partial_{n-1} \circ \partial_n$ for each integer n. Thus, X is a complex of graded free $\mathscr{E}$-modules. In particular, one can construct $\mathrm{Hom}_{\mathscr{E}}(X, U)$; a priori, this is a complex of k-vector spaces. It turns out that this complex coincides with $\beta(U)$, up to shift in degrees.

To see this, note that up to a reordering of degrees the *bigraded* k-vector spaces underlying $\beta(U)$ and X are $A \otimes_k U$ and $\mathscr{E} \otimes_k A^\vee$ respectively. More precisely

$$\beta_n(U)_i = A_{i-n} \otimes_k U_n \quad \text{and} \quad X_{n,i} = \mathscr{E}_{i+n} \otimes_k (A^\vee)_{-n}.$$

Therefore, the canonical evaluation map: $A^\vee \otimes_k A \to k$ gives rise to the k-linear map

$$\Phi\colon \beta(U) \to \mathrm{Hom}_{\mathscr{E}}(X, U) \quad \text{with} \quad \Phi(s \otimes_k u)(e \otimes_k f) = ef(s)u$$

for all $(s \otimes_k u) \in \beta(U)$ and $(e \otimes_k f) \in X$. This is a bijection. Indeed, classical Hom-Tensor adjointness yields

$$\begin{aligned}\mathrm{Hom}_{\mathscr{E}}(X_n, U) &= \mathrm{Hom}_{\mathscr{E}}(\mathscr{E}(n) \otimes_k (A^\vee)_{-n}, U)\\ &\cong \mathrm{Hom}_k((A^\vee)_{-n}, \mathrm{Hom}_{\mathscr{E}}(\mathscr{E}(n), U))\\ &\cong \mathrm{Hom}_k((A^\vee)_{-n}, U(-n)).\end{aligned}$$

Under this identification Φ coincides, at the level of bigraded vector spaces, with the canonical map $A_{i-n} \otimes_k U_n \to \mathrm{Hom}_k((A^\vee)_{n-i}, U_n)$, so Φ is bijective. Moreover, while it is *not* a morphism of complexes, it is compatible with differentials; this can be verified easily by a direct computation. This discussion justifies the following:

Lemma. *As a k-vector space,* $\mathrm{H}_n(\beta(U))_i \cong \mathrm{H}_{n-i}(\mathrm{Hom}_{\mathscr{E}}(X, U))_i$ *for all integers* $n, i \in \mathbb{Z}$.

The lemma above and Theorem 4.5 lead to the following counterpart of Corollary 4.6. It describes the resolution of k over the $\mathrm{Ext}_R(k, k)$ when R is Koszul, since the Ext-algebra is $\mathscr{E}$, except for the grading convention.

4.8. Corollary. *If R is Koszul, then X is the minimal free resolution of k over $\mathscr{E}$.*

Proof. Let G be a minimal free resolution of k over R. Since R is Koszul, $\mathrm{lin}^R(G)$ is acyclic and $\mathrm{H}_0(\mathrm{lin}^R(G)) \cong k$; see 1.5. Hence, by Theorem 4.5, the same holds for $\beta(\mathrm{Tor}^R(k, k))$. Thus, Lemma 4.7 yields

$$\mathrm{H}(\mathrm{Hom}_{\mathscr{E}}(X, \mathrm{Tor}^R(k, k))) \cong k.$$

Now, $\mathrm{Tor}^R(k,k) = \mathrm{Hom}_k(\mathscr{E},k)$ as left $\mathscr{E}$-modules, by the special case $M=k$ of 3.3, so

$$\mathrm{Hom}_{\mathscr{E}}(X, \mathrm{Tor}^R(k,k)) \cong \mathrm{Hom}_{\mathscr{E}}(X, \mathrm{Hom}_k(\mathscr{E},k)) \cong \mathrm{Hom}_k(X,k).$$

Thus, $\mathrm{H}(\mathrm{Hom}_k(X,k)) \cong k$. Since k is a field, this is tantamount to: $\mathrm{H}(X) \cong k$, as desired. □

As a direct application of the calculation above and Lemma 4.7 one obtains a formula for the homology of the complex $\beta(U)$. It should be noted that it is contained in the work of Buchweitz [7], who establishes a similar result for nonnecessarily commutative Koszul k-algebras.

4.9. Proposition. *Let U be a graded degreewise finite left $\mathscr{E}$-module. If R is Koszul, then*

$$\mathrm{H}_n(\beta(U))_i = \mathrm{Ext}^{i-n}_{\mathscr{E}}(k,U)_i \quad \textit{for all} \quad n,i \in \mathbb{Z}.$$

The preceding results all culminate in the

Proof of Theorem 4.1. The first isomorphism follows from Theorem 4.5 and the special case $U = \mathrm{Tor}^R(k,M)$ of Proposition 4.9; only, one has to translate the lower grading in the latter result to upper grading.

By the procedure described in 3.1, the left $R^!$-module $\mathrm{Ext}_R(M,k)$ gives rise to the right module $\mathrm{Ext}_R(M,k)^\sigma$, with the same underlying graded k-vector space. Since $\mathrm{Ext}_R(M,k)$ is degreewise finite, the following natural homomorphism (of graded k-vector spaces) is bijective:

$$\mathrm{Tor}^R(k,M) \xrightarrow{\cong} \mathrm{Hom}_k(\mathrm{Ext}_R(M,k)^\sigma, k).$$

According to Proposition 3.3, this isomorphism is compatible with the corresponding left $R^!$-module structures. This explains the second of the following isomorphisms of graded k-vector spaces; the first is the usual one,

$$\begin{aligned}\mathrm{Hom}_k(\mathrm{Tor}^{R^!}_n\left(\mathrm{Ext}_R(M,k)^\sigma, k\right), k) &\cong \mathrm{Ext}^n_{R^!}(k, \mathrm{Hom}_k(\mathrm{Ext}_R(M,k)^\sigma, k))\\ &\cong \mathrm{Ext}^n_{R^!}(k, \mathrm{Tor}^R(k,M)).\end{aligned}$$

The k-vector space $\mathrm{Tor}^{R^!}_n\left(\mathrm{Ext}_R(M,k)^\sigma, k\right)$ is degreewise finite. This is because $(R^!)^0 = k$, and both $R^!$ and $\mathrm{Ext}_R(M,k)$ are degreewise finite and concentrated in nonnegative degrees; see 5.2. Thus, the composed isomorphism above dualizes to the first isomorphism below, while Lemma 3.1 provides the next one,

$$\begin{aligned}\mathrm{Hom}_k(\mathrm{Ext}^n_{R^!}(k, \mathrm{Tor}^R(k,M)), k) &\cong \mathrm{Tor}^{R^!}_n\left(\mathrm{Ext}_R(M,k)^\sigma, k\right)\\ &\cong \mathrm{Tor}^{R^!}_n\left(k, \mathrm{Ext}_R(M,k)\right).\end{aligned}$$

Combining this with the first isomorphism in the theorem completes the proof. □

5. Linearity defect as regularity

In this section we establish that the linearity defect of finitely generated modules over local rings that are either complete intersection or Golod is finite. We deduce these results by interpreting the linearity defect of a module as the "regularity" of an appropriate module over the Ext-algebra of the local ring.

To this end, it is expedient to introduce a notion of regularity that is applicable to rings that may not be commutative or even noetherian. Our aim is to develop this notion only to a point which allows us to state (and prove!) the result we seek. Also, since the theory is intended to be applied to Ext-algebras of local rings, the algebras and modules are assumed to be upper graded.

5.1. Regularity. Let $E = \{E^i\}_{i \geqslant 0}$ be a graded degreewise finite k-algebra with $E^0 = k$, and let $U = \{U^i\}_{i \in \mathbb{Z}}$ be a graded degreewise finite left E-module.

We define the *regularity* of U to be the integer

$$\operatorname{reg}_E U = \sup\{r \mid \operatorname{Ext}_E^n(U,k)^{-n-r} \neq 0 \text{ for some integer } n\}.$$

When E is commutative (and noetherian), this notion of regularity coincides with the classical one. The isomorphism of graded k-vector spaces: $\operatorname{Hom}_k(\operatorname{Tor}_n^E(k,U),k) \cong \operatorname{Ext}_E^n(U,k)$ implies that regularity may be read off from appropriate Tor functors:

($\star$)

$$\operatorname{reg}_E U = \sup\{r \mid \operatorname{Tor}_n^E(k,U)^{n+r} \neq 0 \text{ for some integer } n\}.$$

For our purposes, the Ext-definition of regularity is more convenient. One reason for this is the presence of the following test to detect modules of finite regularity.

Lemma. *If* $\operatorname{reg}_E k$ *is finite and the* $\operatorname{Ext}_E(k,k)$*-module* $\operatorname{Ext}_E(U,k)$ *is finitely generated, then* $\operatorname{reg}_E U$ *is finite as well.*

Proof. The hypothesis yields a surjective degree 0 homomorphism

$$\bigoplus_{i \in \mathbb{Z}} \operatorname{Ext}_E(k,k)(-i)^{b_i} \twoheadrightarrow \operatorname{Ext}_E(U,k),$$

where all but finitely many of the b_i are zero. Then $\operatorname{reg}_E U \leqslant \operatorname{reg}_E k + \sup\{i \in \mathbb{Z} \mid b_i \neq 0\}$. $\square$

One nice thing about E is that many modules over it have minimal resolutions. Here is a brief resume of the basic facts concerning them.

5.2. Minimal resolutions. Suppose the E-module U, in addition to being degreewise finite, is bounded to the left; in other words, $U^i = 0$ for $i \ll 0$. Then U admits a graded free resolution X such that for each integer n, one has

$$X_n \cong \bigoplus_{i \in \mathbb{Z}} E(-i)^{b_{n,i}} \quad \text{with } b_{n,i} = 0 \text{ for } i \ll 0,$$

and $k \otimes_E X$ has trivial differentials. Such a *minimal free resolution* exists and is unique (up to isomorphism) because Nakayama's lemma applies to all E-modules which are bounded to the left. At any rate, it is guaranteed by results of Eilenberg [8]. Taking a cue from the local case, we call the left E-module $\operatorname{Coker}(\partial_{n+1})$ the *nth syzygy* of U.

The existence of such minimal resolution implies:

5.3. Lemma. *Let U be a degreewise finite left E-module which is bounded to the left. If $\operatorname{reg}_E U$ is finite, then U and all its higher syzygies are finitely generated.*

Proof. Let X be the minimal free resolution of U. Fix an integer n. We adopt the notation from 5.2. Since $\operatorname{Tor}_n^E(k, U) = k \otimes_E X_n$ and $\operatorname{reg}_E U$ is finite, one obtains from the equality $(\star)$ in 5.1 that $b_{n,i} = 0$ for all but finitely many i. Hence X_n is finitely generated. It remains to note that X_n maps surjectively onto the nth syzygy of U. □

Here is the advertised relation between linear defect and regularity; that such a theorem might exist was suggested by Eisenbud et al. [9] which contains an analogues statement in the case where R is an exterior algebra. Its proof is short only because most of the groundwork has already been laid. Observe that one cannot to do away with the hypothesis that R is Koszul as $\operatorname{reg}_{R^!} \operatorname{Ext}_R(k, k) = 0$.

5.4. Theorem. *Let $(R, \mathfrak{m}, k)$ be a Koszul local ring and let M be a finitely generated R-module. Set $R^! = \operatorname{Ext}_R(k, k)$. Then*

$$\operatorname{ld}_R(M) = \operatorname{reg}_{R^!} \operatorname{Ext}_R(M, k).$$

In particular, $\operatorname{ld}_R(M) < \infty$ *if the* $\operatorname{Ext}_{R^!}(k, k)$*-module* $\operatorname{Ext}_{R^!}(\operatorname{Ext}_R(M, k), k)$ *is finitely generated.*

Proof. The formula for $\operatorname{ld}_R(M)$ is justified by the following equalities, in which Theorem 4.1 implies the first and the second is due to $(\star)$ in 5.1 above,

$$\begin{aligned}\operatorname{ld}_R(M) &= \sup\{n \mid \operatorname{Tor}^E_{i-n}\big(k, \operatorname{Ext}_R(M, k)\big)^i \neq 0 \text{ for some integer } i\} \\ &= \operatorname{reg}_{R^!} \operatorname{Ext}_R(M, k).\end{aligned}$$

Specializing this equality to $M = R$ one obtains: $\operatorname{reg}_{R^!} k = \operatorname{ld}_R(R) = 0$. Thus, the finiteness of $\operatorname{ld}_R(M)$ is a consequence of Lemma 5.1. □

Combining the preceding result with Lemma 5.3 yields this next result. Over Koszul rings, it constitutes a vast generalization of Theorem 2.2.

5.5. Corollary. *If $\operatorname{ld}_R(M)$ is finite, then the left $R^!$-module $\operatorname{Ext}_R(M, k)$ and all its higher syzygies are finitely generated.*

We apply the preceding results to modules over complete intersections and Golod rings. The latter class may be less familiar than the former to many readers, so the next items recapitulate the relevant definitions and also some elementary results concerning those; see [10,13], or [2, Section 5].

5.6. Golod homomorphisms. Let $(R, \mathfrak{m}, k)$ be a local ring and M a finitely generated R-module.

Let K^R be the Koszul complex for R and set $K^M = K^R \otimes_R M$. There is a coefficientwise inequality of formal power series

(†)

$$\mathrm{P}_M^R(t) \preccurlyeq \frac{\sum_i \operatorname{rank}_k \mathrm{H}_i(K^M)t^i}{1 - \sum_j \operatorname{rank}_k \mathrm{H}_j(K^R)t^{j+1}}.$$

If equality holds, then the module M is *Golod*. One says that R is a *Golod ring* if k is Golod.

Here is a relative version of the Golod property: Let $\varphi\colon (S, \mathfrak{n}, k) \twoheadrightarrow (R, \mathfrak{m}, k)$ be a surjective local homomorphism. There is a coefficientwise inequality of formal power series

(‡)

$$\mathrm{P}_M^R(t) \preccurlyeq \frac{\mathrm{P}_M^S(t)}{1 - t\left(\mathrm{P}_R^S(t) - 1\right)}.$$

If equality holds, then M is φ*-Golod*. If k is φ-Golod, then φ is a *Golod homomorphism*.

Remark. Let S be regular, with emb dim $S =$ emb dim R, and let K^S be the Koszul complex for S. Then $\mathrm{H}_0(K^S) = k$ and, since S is regular, $\mathrm{H}_i(K^S) = 0$ for $i \geqslant 1$. Since emb dim $S =$ emb dim R, one has that $K^R = K^S \otimes_S R$ and $K^M = K^S \otimes_S M$; thus $\sum_i \operatorname{rank}_k \mathrm{H}_i(K^M)t^i = \mathrm{P}_M^S(t)$ and $\sum_j \operatorname{rank}_k \mathrm{H}_i(K^R)t^i = \mathrm{P}_R^S(t)$. Hence the R-module M is Golod if and only if it is φ-Golod.

The graded version of the Golod property has certain special features.

5.7. Golod homomorphisms: the graded case. Let k be a field, $\varphi\colon S \to R$ a homomorphism of standard graded k-algebras, and let M be a graded finitely generated R-module. The bigraded version of the inequality of formal power series (‡) in 5.6 reads

$$\mathrm{P}_M^R(s,t) \preccurlyeq \frac{\mathrm{P}_M^S(s,t)}{1 - t(\mathrm{P}_R^S(s,t) - 1)}.$$

As in the case of local rings, one says that M is φ-Golod if equality holds, and that the homomorphism φ is Golod if k is φ-Golod. There is an obvious bigraded version of (†) as well, and hence a notion of a Golod module and a Golod ring. Moreover, one has an analogue of Remark 5.6 for standard graded algebras.

One nice thing about standard graded algebras is that one can, in certain cases, detect Golod homomorphisms via appropriate graded Betti numbers. This is the gist of the next result. It is based on the following notion: an ideal I in the k-algebra S has a *2-linear resolution* if for any pair of integers n, i one has $b_{n,i}(I) = 0$ whenever $i \neq n + 2$; equivalently, if the S-module I is generated in degree 2 and has a linear resolution.

5.8. Proposition. *The following conditions are equivalent*:

(1) *R is Koszul and the homomorphism φ is Golod*.
(2) *S is Koszul and the S-module* $\operatorname{Ker}(\varphi)$ *has a* 2-*linear resolution*.

Proof. Set $I = \operatorname{Ker}(\varphi)$. Thus $\mathrm{P}^S_R(s,t) = t\mathrm{P}^S_I(s,t) + 1$ and the coefficientwise inequality 5.7 reads

(∗)

$$\mathrm{P}^R_k(s,t) \preccurlyeq \frac{\mathrm{P}^S_k(s,t)}{(1 - t^2\mathrm{P}^S_I(s,t))}.$$

(1) ⇒ (2): Since R is Koszul, $\mathrm{P}^R_k(s,t) = \sum_{n\geqslant 0} b_{n,n}(st)^n$, and since φ is Golod, equality holds in (∗) above. Thus, $t^2\mathrm{P}^S_I(s,t)$ is a formal power series in ts; that is to say, I has a 2-linear resolution.

(2) ⇒ (1): Since I is 2-linear, $t^2\mathrm{P}^S_I(s,t)$ is a formal power series in st. Thus, (∗) implies that $\operatorname{reg}_R k = 0$, so R is Koszul; see 1.10. As for the Golod property: let X be a differential graded algebra resolution of k over S and let Y denote the differential graded algebra $R\otimes_S X$; see [2, Section 2] for details. Since $\mathrm{H}(Y) = \operatorname{Tor}^S(R,k)$ and I has a 2-linear resolution, $\mathrm{H}_n(Y)_i = 0$ for each $n \geqslant 1$ and for each i with $i \neq n+1$. Therefore, for degree reasons, Y must have trivial Massey operations. Indeed, a Massey product $\mu(h_1,\ldots,h_p)$ with $h_j \in \mathrm{H}_{n_j}(Y)_{n_j+1}$ and $p \geqslant 2$ lives in $\mathrm{H}_n(Y)_{n+p}$ where $n = \sum_{j=1}^p n_j$; however, $\mathrm{H}_n(Y)_{n+p} = 0$. Hence R is Golod, by Levin [13, (1.5)]; also see [1]. □

Now for the principal result of this section.

5.9. Theorem. *Let R be a Koszul local ring. If there is a Golod homomorphism φ: $S \to R$ such that S is a complete intersection, then* $\operatorname{ld}_R(M) < \infty$ *for every finitely generated R-module M*.

Proof. Set $R^! = \operatorname{Ext}_R(k,k)$. A result of Backelin and Roos [5, Theorem 2.a] provides that the $\operatorname{Ext}_{R^!}(k,k)$-module $\operatorname{Ext}_{R^!}(\operatorname{Ext}_R(M,k),k)$ is finitely generated. Now apply Theorem 5.4. □

Here is a digestible special case of the preceding result.

5.10. Corollary. *Let R be a Koszul local ring. If R is either a complete intersection or Golod, then* $\operatorname{ld}_R(M) < \infty$ *for every finitely generated R-module M*.

Proof. The identity homomorphism $R \to R$ is Golod, so Theorem 5.9 yields the desired result when R is a complete intersection.

Suppose R is Golod. Let $\widehat{R}$ denote the completion of R at its maximal ideal. Then $\widehat{R}$ is Golod and $\operatorname{ld}_R(M) = \operatorname{lpd}_{\widehat{R}}(\widehat{M})$. Thus, by passing to $\widehat{R}$ one may assume that R is complete. Then Cohen's structure theorem provides a surjective local homomorphism φ: $S \to R$ with S a regular local ring; one can ensure also that $\operatorname{emb\,dim} S = \operatorname{emb\,dim} R$.

By Remark 5.6, the homomorphism φ is Golod, so the result we seek is once again contained in Theorem 5.9. $\square$

5.11. Remark. Let R be a multigraded ring, and M a finitely generated multigraded R-module. Set $R^! = \mathrm{Ext}_R(k,k)$. In [5, Section 4], the authors surmise that the $\mathrm{Ext}_{R^!}(k,k)$-module $\mathrm{Ext}_{R^!}(\mathrm{Ext}_R(M,k),k)$ is finitely generated. If this turns out to be the case, then when R is Koszul, Theorem 5.4 would imply that $\mathrm{ld}_R(M) < \infty$.

6. Golod modules

The main contribution of this section is a 'change of base' result for computing the linear defect.

6.1. Theorem. *Let $\varphi\colon S \twoheadrightarrow R$ be a surjective local homomorphism such that R and S are Koszul. Let M be a finitely generated R-module. If M is φ-Golod, then*

$$\mathrm{ld}_R(M) = \mathrm{ld}_S(M).$$

In the graded case it is not necessary to assume a priori that S is Koszul: Levin [15, (1.1)] yields that when M, assuming it is nonzero, is φ-Golod, so is φ. Proposition 5.8 takes care of the rest.

The theorem above allows for a quantitative strengthening of the part of Corollary 5.10 dealing with Golod rings. It is best described in terms of the following invariant, modeled on the global dimension of rings: For each local ring R, set

$$\mathrm{gl\ lpd}(R) = \sup\{\mathrm{ld}_R(M) \mid M \text{ is a finitely generated } R\text{-module}\}.$$

Here is the advertized result.

6.2. Corollary. *If a local ring R is Koszul and Golod, then* $\mathrm{gl\ lpd}(R) \leqslant 2\,\mathrm{emb\ dim}\,R$.

Proof. By completing R at its maximal ideal, one may assume that R is complete. Then Cohen's structure theorem provides a surjective homomorphism $\varphi\colon S \to R$ where S is a regular local ring with $\mathrm{emb\ dim}\,S = \mathrm{emb\ dim}\,R$. Since S is regular, φ is Golod; see the remark in 5.6. Let M be a finitely generated R-module and set $d = \mathrm{proj\,dim}_S M$. Then $\Omega^d(M)$, the dth syzygy of M over R, is φ-Golod; this is by Avramov [2, (5.3.2)]. Thus

$$\mathrm{ld}_R(M) \leqslant \mathrm{ld}_R\left(\Omega^d(M)\right) + d = \mathrm{ld}_S\left(\Omega^d(M)\right) + d \leqslant d + d = 2d,$$

where the equality is by Theorem 6.1. It remains to note that $d \leqslant \mathrm{emb\ dim}\,S$. $\square$

Remark. Suppose that R is a standard graded k-algebra. Assume that R, in addition to being Koszul and Golod, is Cohen–Macaulay. Fairly elementary arguments, involving reduction to the case where $\dim R = 0$, allow one to deduce that $\mathrm{gl\ lpd}(R) \leqslant \dim R$. This leads us to surmise that the upper bound in the corollary above can be improved considerably.

The proof of Theorem 6.1, given in 6.6, uses some ideas from [2,5,21] and the following remark concerning modules over tensor algebras.

6.3. Let $\Lambda = \{\Lambda^i\}_{i \geqslant 1}$ be a graded k-vector space. We denote $\mathrm{T}(\Lambda)$ the tensor algebra on Λ.

Lemma. *Let U be a graded left-$\mathrm{T}(\Lambda)$-module which is bounded to the left and degreewise finite. Let $\mu: \Lambda \otimes_k U \to U$ be the map with $\mu(\lambda \otimes_k u) = \lambda u$ for $\lambda \in \Lambda$ and $u \in U$. The following conditions are equivalent*:

(1) *μ is injective*.
(2) *U is free*.
(3) *for some (respectively, any) homogeneous k-linear section of the surjection $U \twoheadrightarrow \mathrm{Coker}(\mu)$, the induced homomorphism $\mathrm{T}(\Lambda) \otimes_k \mathrm{Coker}(\mu) \to U$ of left $\mathrm{T}(\Lambda)$-modules is bijective*.

Proof. The tensor algebra being what it is—a free algebra—one has the short exact sequence of right $\mathrm{T}(\Lambda)$-modules: $0 \to \Lambda \otimes_k \mathrm{T}(\Lambda) \xrightarrow{\iota} \mathrm{T}(\Lambda) \to k \to 0$, where $\iota(\lambda \otimes_k t) = \lambda t$.

Applying $-\otimes_{\mathrm{T}(\Lambda)} U$ to it leads to the exact sequence

$$0 \to \mathrm{Tor}_1^{\mathrm{T}(\Lambda)}(k, U) \to (\Lambda \otimes_k U) \xrightarrow{\mu} U \to k \otimes_{\mathrm{T}(\Lambda)} U \to 0.$$

In particular, μ is injective if and only if $\mathrm{Tor}_1^{\mathrm{T}(\Lambda)}(k, U) = 0$. Since U admits a minimal free resolution—see 5.2—this last condition is equivalent to the freeness of U; this settles (1) $\Leftarrow \Rightarrow$ (2). The remaining nontrivial implication (2) $\Rightarrow$ (3) follows by a standard argument based on Nakayama's lemma. $\square$

6.4. Let $\varphi: (S, \mathfrak{n}, k) \twoheadrightarrow (R, \mathfrak{m}, k)$ be a Golod homomorphism. By Levin [13, Theorem 1.5], there is an exact sequence of complexes of R-modules:

(†)

$$0 \to X \otimes_S R \to Y \to Y \otimes_R F \to 0,$$

where X is the minimal resolution of k as an S-module, Y is the minimal resolution of k as an R-module, and F is a complex of free R-modules with trivial differentials. Since the complexes are minimal, tensoring with k leads to the exact sequence of graded k-vector spaces

$$0 \to \mathrm{Tor}^S(k, k) \to \mathrm{Tor}^R(k, k) \to \mathrm{Tor}^R(k, k) \otimes_k V \to 0,$$

where $V = k \otimes_R F$. Dualizing this yields the exact sequence

(‡)

$$0 \to \Lambda \otimes_k \mathrm{Ext}_R(k, k) \to \mathrm{Ext}_R(k, k) \to \mathrm{Ext}_S(k, k) \to 0,$$

where we write Λ for $\mathrm{Hom}_k(V, k)$. In the sequel, we view Λ as a subset of $\mathrm{Ext}_R(k, k)$ by identifying it with (the image of) $\Lambda \otimes_k 1$. Via this inclusion, Λ acts on $\mathrm{Ext}_R(M, k)$ for each R-module M.

Lemma. *Suppose that the local ring R is Koszul. Let M be a finitely generated R-module. Then there is an exact sequence of graded k-vector spaces*

$$\Lambda \otimes_k \operatorname{Ext}_R(M,k) \xrightarrow{\mu} \operatorname{Ext}_R(M,k) \xrightarrow{\pi} \operatorname{Ext}_S(M,k),$$

where $\mu(\lambda \otimes_k w) = \lambda w$ *and the map* π *is the canonical one. Moreover, M is* φ*-Golod if and only if* μ *is injective if and only if* π *is surjective.*

Proof. Applying $-\otimes_R M$ to the exact sequence (†) above leads to the exact sequence

$$0 \to X \otimes_S M \to Y \otimes_R M \to (Y \otimes_R M) \otimes_R F \to 0.$$

Taking homology gives the exact sequence:

$$\operatorname{Tor}^S(k,M) \to \operatorname{Tor}^R(k,M) \to \operatorname{Tor}^R(k,M) \otimes_k V.$$

Dualizing this leads to the exact sequence of graded k-vector spaces:

$$\Lambda \otimes_k \operatorname{Ext}_R(M,k) \xrightarrow{\mu'} \operatorname{Ext}_R(M,k) \xrightarrow{\pi} \operatorname{Ext}_S(M,k).$$

There is a subtlety here: Since $\operatorname{Ext}_R(M,k)$ appears as the dual of $\operatorname{Tor}^R(k,M)$, one has $\mu'(\lambda \otimes_k w) = \lambda * w$, where the product $\star$ is the conjugate—in the sense of 3.1—of the *right* action of $\operatorname{Ext}_R(k,k)$ on $\operatorname{Ext}_R(M,k)$ that is induced from $\operatorname{Tor}^R(k,M)$. However, since R is Koszul and M is finitely generated, this coincides with the usual left action of $\operatorname{Ext}_R(k,k)$ on $\operatorname{Ext}_R(M,k)$; see Proposition 3.3. In other words, $\mu' = \mu$, and the exact sequence above is the one we seek.

As to the remaining assertion: μ and π arise from exact sequences of complexes, so μ is injective if and only if π is surjective. The exactness of the sequence above yields the coefficientwise inequality of formal power series

$$\mathrm{P}^R_M(t)(1 - \operatorname{Hilb}_\Lambda(t)) \preccurlyeq \mathrm{P}^S_M(t).$$

Moreover, equality holds if and only if μ is injective and π is surjective. It remains to note that $\operatorname{Hilb}_\Lambda(t) = t(\mathrm{P}^S_R(t) - 1)$, by (†), since the homomorphism φ is Golod. $\square$

6.5. Let $\varphi\colon (S,\mathfrak{n},k) \twoheadrightarrow (R,\mathfrak{m},k)$ be a Golod homomorphism. The inclusion $\Lambda \hookrightarrow \operatorname{Ext}_R(k,k)$, from 6.4, gives rise to a homomorphism of k-algebras $\mathrm{T}(\Lambda) \to \operatorname{Ext}_R(k,k)$. This map is injective: this is because $\operatorname{Ext}_R(k,k)$ is free as a left $\mathrm{T}(\Lambda)$-module, by the exact sequence (‡) in 6.4 and Lemma 6.3. Consequently, one can view $\mathrm{T}(\Lambda)$ as a k-subalgebra of $\operatorname{Ext}_R(k,k)$.

At any rate, $\operatorname{Ext}_R(M,k)$ acquires a structure of a left $\mathrm{T}(\Lambda)$-module, for any R-module M. The result below characterizes φ-Golod modules in terms of this structure; it appears to be new, although the basic ingredients in the proof are not; cf. [5,13,21]. Only implication (1) $\Rightarrow$ (5) is invoked in the sequel, but the argument is more transparent with all five conditions on hand.

Theorem. *Let* $\varphi\colon (S,\mathfrak{n},k) \twoheadrightarrow (R,\mathfrak{m},k)$ *be a Golod homomorphism, and let M be a finitely generated R-module. Suppose that R is Koszul. Then the following conditions are* *equivalent*:

(1) *M is φ-Golod.*
(2) *The left* $\mathrm{T}(\Lambda)$*-module* $\mathrm{Ext}_R(M,k)$ *is free.*
(3) *There is an isomorphism of left* $\mathrm{T}(\Lambda)$*-modules*:

$$\mathrm{Ext}_R(M,k) \cong \mathrm{T}(\Lambda) \otimes_k \mathrm{Ext}_S(M,k),$$

and the map $\mathrm{Ext}_R(M,k) \to \mathrm{Ext}_S(M,k)$ *induced by* $\mathrm{T}(\Lambda) \to k$ *is the canonical one.*
(4) *The following canonical map of bigraded k-algebras is bijective*:

$$\mathrm{Ext}_{\mathrm{Ext}_S(k,k)}(\mathrm{Ext}_S(M,k),k) \to \mathrm{Ext}_{\mathrm{Ext}_R(k,k)}(\mathrm{Ext}_R(M,k),k).$$

(5) *There is an isomorphism of bigraded vector spaces*

$$\mathrm{Ext}_{\mathrm{Ext}_S(k,k)}(\mathrm{Ext}_S(M,k),k) \cong \mathrm{Ext}_{\mathrm{Ext}_R(k,k)}(\mathrm{Ext}_R(M,k),k).$$

Remark. We ask that R be Koszul only because of Lemma 6.4. In its turn, the latter result required that hypothesis because its proof invokes Proposition 3.3. Thus, in the light of the remarks following Proposition 3.3, it is to be expected that the equivalences above hold for all local rings.

Proof. Set $\mathscr{E}_R = \mathrm{Ext}_R(k,k)$ and $\mathscr{E}_S = \mathrm{Ext}_S(k,k)$.

(1) $\iff$ (2) $\iff$ (3): This is immediate from 6.4 and 6.3.

(3) $\Rightarrow$ (4): By definition, the Golod property for φ translates to the R-module k being φ-Golod. Therefore, the special case $M=k$ of the already established implication (1) $\Rightarrow$ (3) yields that

(a) $\mathscr{E}_R$ is free when viewed as a left $\mathrm{T}(\Lambda)$-module, and
(b) the canonical map $k \otimes_{\mathrm{T}(\Lambda)} \mathscr{E}_R \to \mathscr{E}_S$ is bijective.

Observe that the map in (b) is one of graded k-algebras and also of right $\mathscr{E}_R$-modules. In the isomorphisms below (b) gives the one on the left and (a) the other:

$$\begin{aligned}\mathrm{Tor}^{\mathscr{E}_R}\left(\mathscr{E}_S, \mathrm{Ext}_R(M,k)\right) &\cong \mathrm{Tor}^{\mathscr{E}_R}\left(k \otimes_{\mathrm{T}(\Lambda)} \mathscr{E}_R, \mathrm{Ext}_R(M,k)\right)\\ &\cong \mathrm{Tor}^{\mathrm{T}(\Lambda)}\left(k, \mathrm{Ext}_R(M,k)\right).\end{aligned}$$

These isomorphisms are compatible with the left $\mathscr{E}_S$-module structures, so the hypotheses imply

($\star\star$)

$$\mathrm{Tor}_q^{\mathscr{E}_R}\left(\mathscr{E}_S, \mathrm{Ext}_R(M,k)\right) = \begin{cases}\mathrm{Ext}_S(M,k) & \text{for } q=0,\\ 0 & \text{otherwise.}\end{cases}$$

A standard change of rings spectral sequence arising from the ring homomorphism $\mathscr{E}_R \to \mathscr{E}_S$ reads

$$\mathrm{E}_2^{p,q} = \mathrm{Ext}^p_{\mathscr{E}_S}(\mathrm{Tor}_q^{\mathscr{E}_R}\left(\mathscr{E}_S, \mathrm{Ext}_R(M,k)\right), k) \Longrightarrow \mathrm{Ext}^{p+q}_{\mathscr{E}_R}(\mathrm{Ext}_R(M,k),k).$$

Due to the vanishing of Tor in ($\star\star$), this spectral sequence collapses, so the edge homomorphism

$$\mathrm{Ext}^p_{\mathscr{E}_S}(\mathrm{Tor}_0^{\mathscr{E}_R}\left(\mathscr{E}_S, \mathrm{Ext}_R(M,k)\right), k) \xrightarrow{\cong} \mathrm{Ext}^p_{\mathscr{E}_R}(\mathrm{Ext}_R(M,k), k)$$

is bijective. Another invocation of ($\star\star$) settles the claim.

(4) $\Rightarrow$ (5): This is a triviality.

(5) $\Rightarrow$ (1): Since $\mathrm{Ext}_S(M,k)$ is bounded to the left and degreewise finite, the rank of the k-vector space $\mathrm{Ext}^n_{\mathscr{E}_S}(\mathrm{Ext}_S(M,k),k)^i$ is finite for each pair of integers n, i; see 5.2. The same remark applies to the k-vector space $\mathrm{Ext}^n_{\mathscr{E}_R}(\mathrm{Ext}_R(M,k),k)^i$. Thus, the given isomorphism of bigraded vector spaces results in the coefficientwise equality of bigraded Poincaré series:

$$\mathrm{P}^{\mathscr{E}_S}_{\mathrm{Ext}_S(M,k)}(t,s) = \mathrm{P}^{\mathscr{E}_R}_{\mathrm{Ext}_R(M,k)}(t,s).$$

On the other hand, one has always that

$$\mathrm{Hilb}_{\mathscr{E}_S}(s)\mathrm{P}^{\mathscr{E}_S}_{\mathrm{Ext}_S(M,k)}(-t,s) = \mathrm{Hilb}_{\mathrm{Ext}_S(M,k)}(s),$$

$$\mathrm{Hilb}_{\mathscr{E}_R}(s)\mathrm{P}^{\mathscr{E}_R}_{\mathrm{Ext}_R(M,k)}(-t,s) = \mathrm{Hilb}_{\mathrm{Ext}_R(M,k)}(s).$$

From the preceding equalities one deduces that $\mathrm{P}^R_k(s)\mathrm{P}^S_M(s) = \mathrm{P}^S_k(s)\mathrm{P}^R_M(s)$. Therefore

$$\mathrm{P}^R_M(s) = \frac{\mathrm{P}^R_k(s)\mathrm{P}^S_M(s)}{\mathrm{P}^S_k(s)} = \frac{\mathrm{P}^S_M(s)}{1 - t(\mathrm{P}^S_R(s) - 1)},$$

where the last equality is due to φ being a Golod homomorphism. Thus, M is φ-Golod. $\square$

Theorem 6.5 prepares us—admirably—for the:

6.6. Proof of Theorem 6.1. One may assume that $M \neq 0$. Then, φ is Golod, by Levin [15, (1.1)], so implication (1) $\Rightarrow$ (5) of Theorem 6.5 yields $\mathrm{reg}_{S^!}\,\mathrm{Ext}_S(M,k) = \mathrm{reg}_{R^!}\,\mathrm{Ext}_R(M,k)$. It remains to invoke Theorem 5.4. $\square$

The analogue of Corollary 6.2 for standard graded k-algebras has a partial converse:

6.7. Theorem. *Let R be a standard graded k-algebra. If R is Gorenstein, then the following conditions are equivalent*:

(1) gl lpd $(R) < \infty$;
(2) *R is a hypersurface defined by a quadratic form.*

Proof. To begin with, note that a hypersurface is Koszul if and only if it is defined by a quadratic form. Since a hypersurface is always Golod, Corollary 6.2 yields implication (2) $\Rightarrow$ (1).

Conversely, gl lpd (R) finite implies $\mathrm{ld}_R\,(k)$ finite, so R is Koszul, by Proposition 1.13. Suppose that R is not a hypersurface. Let $d = \dim R$. One may assume that k is infinite, and choose an R-sequence $x_1, \ldots, x_d$ with $x_i \in R_1$ for $i = 1, \ldots, d$. Set $\bar{R} = R/(x_1, \ldots x_d)$. Then $\mathrm{Ext}^d_R(k, R)(-d) \cong \mathrm{Hom}_R(k, \bar{R}) \cong k(-s)$ with $s \geqslant 2$, since R is not a hypersurface ring. Let F be a minimal graded free R-resolution of k, and set $F^* = \mathrm{Hom}_R(F, R)$. Then

$$H_i(F^*) \cong \begin{cases} 0 & \text{if } i \neq -d, \\ k(d-s) & \text{if } i = -d. \end{cases}$$

We denote G the complex $\cdots \to F_{i+1}(d-s) \to F_i(d-s) \to F_{i-1}(d-s) \to \cdots$ shifted d steps to the right. Then

$$H_i(G) \cong \begin{cases} 0 & \text{if } i \neq -d, \\ k(d-s) & \text{if } i = -d. \end{cases}$$

The isomorphism $H_{-d}(G) \cong k(d-s) \cong H_{-d}(F)$ lifts to a morphism $\alpha\colon G \to F^*$ of graded complexes. Now, the differentials in F can be expressed by matrices of linear forms, since R is Koszul, so an elementary induction argument allows one to deduce that $\alpha(G) \subseteq \mathfrak{m}F^*$. In other words, the mapping cone C of α is a minimal complex.

Choose any integer $n \geqslant d$. Let D^n be the truncated complex

$$\cdots \to C_i \to \cdots \to C_{-n+1} \to C_{-n} \to 0$$

shifted n steps to the left. The complex C is exact, by construction, so D^n is a graded minimal free R-resolution of $M_n = \mathrm{Coker}(C_{-n+1} \to C_{-n})$. Since $s \geqslant 2$, the complex D^n has two linear strands, namely the complexes $F^*_{i \geqslant -n}$ and G, both shifted n steps to the left. Thus the linear strands of M_n have nontrivial homology in homological degree $n - d$, and are exact for $i > n - d$. This implies that $\mathrm{ld}_R\,(M_n) = n - d$. Since n can be chosen to be arbitrarily large, one obtains that $\mathrm{gl\ lpd}\,(R) = \infty$. This contradicts the hypothesis. Therefore, R is a hypersurface. □

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