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Good and bad Koszul algebras and their Hochschild homology

Jan-Erik Roos

Department of Mathematics, Stockholm University, SE-106 91 Stockholm, Sweden

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Dedicated to Prof. Wolmer Vasconcelos on the occasion of his 65th birthday

Abstract

A systematic study of the homological behavior of finitely and linearly presented modules over Koszul algebras is started. In particular, it is shown that the generating series of the linear Betti numbers of these modules over some precise special Koszul algebras are rationally related to the generating series for the Betti numbers of *all* local commutative noetherian rings. It is also shown that there are very small commutative Koszul algebras (4 generators and 4, 5 or 6 relations) which are bad in the sense that rational generating series for Betti numbers of linearly presented modules over them cannot be put on a common denominator. Finally, we do some explicit calculations that indicate that if a Koszul algebra is *not* a local complete intersection, then the generating series for its Hochschild (and also cyclic homology) is an *irrational* function. This supports the conjecture that the answer to our Question 1 on pp. 185–186 of Roos (Varna (1986) 173–189; Lecture Notes in Mathematics, vol. 1352, Springer, Berlin, 1988) is positive.

MSC: 13D02; 13D03; 13D07; 16E30; 16E40; 16S37; 18G10

0. Introduction

Recall that if k is a field and $R = k[x_1, \ldots, x_n]/I$ is the quotient of a polynomial ring by a homogeneous ideal I, then R is said to be a *Koszul* algebra if the ith part of minimal

E-mail address: jeroos@math.su.se (J.-E. Roos).

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graded resolution of the R-module k is concentrated in degree i. Koszul algebras have often been thought as algebras with excellent homological properties. But, in the paper [14] by Jacobsson at the bottom of p. 252 [14] an example of a module M over a Koszul algebra is given such that the generating series of the Betti numbers of M is irrational.

In the first sections of the present paper we elaborate further on this theme, introducing the notions of *good* and *bad* Koszul algebras (definitions are given in Section 2).

Here is a good Koszul algebra:

$$R = k[x, y, z, u]/(x^2, xy, xz, xu) \tag{0.1}$$

and here is a bad one

$$S = k[x, y, z, u]/(x^2, xy, zu, u^2). \tag{0.2}$$

In the first case (0.1), for any finitely generated R-modules M and N of finite length the Poincaré–Betti series:

$$P_R^{M,N}(z) = \sum_{i \geqslant 0} \mathrm{length}_R(\mathrm{Tor}_i^R(M, N))z^i$$

are rational functions with a fixed denominator; more precisely

$$(1 - z - 3z^2 - 3z^3 - z^4)P_R^{M,N}(z)$$

is always a polynomial in z. In the second case (0.2), we will show in this paper that there are S-modules M and N of finite length for which $P_S^{M,N}(z)$ is an *irrational* function (one can even take M to be of length 3 and $N = M$). Furthermore, for any integer $\alpha \geqslant 2$ the matrices

$$\begin{pmatrix} y & x + \alpha u & u \\ z & -u & x + (\alpha - 2)u \end{pmatrix}$$

considered as maps $S^3 \longrightarrow S^2$ define S-modules M_α. We will in particular show (using [35] and the theory in Section 5 below) that the Poincaré–Betti series

$$P_R^{M_\alpha}(z) = P_R^{M_\alpha,k}(z) = \frac{2 - 5z + 5z^3 + z^4 - z^5}{(1-z)^2(1 - z - z^2)^2} + \frac{(1+z)z^\alpha}{(1-z)^2(1-z^\alpha)}. \tag{0.3}$$

In particular, if α goes to ∞ through the primes, one of the irreducible factors in the denominator of (0.3) will get a higher and higher degree. This answers negatively a question of Avramov [3]. The precise definitions of good and bad (in particular Koszul) algebras are given in Section 2.

In Section 1, we make a deeper analysis of a construction of Jacobsson [14] and we show that the series $P_R(z) = P_R^k(z)$ for *any* local commutative noetherian ring (R, m) where $k = R/m$ are "rationally related" to the series of modules defined by cokernels of (certain) matrices of linear forms over the special Koszul algebras ($n \geqslant 2$):

$$R_n = \frac{k[x_1, \ldots, x_n]}{(x_1, \ldots, x_n)^2} \otimes_k \frac{k[y_1, \ldots, y_n]}{(y_1, \ldots, y_n)^2}.$$

As a consequence we present some amusing examples, in particular an explicit 5×18 matrix of linear forms over R_5 whose corresponding module M has the property that $P^M_{R_5}(z)$ is a nice transcendental function. In particular, the Betti numbers of the linear part of the free R_5-resolution of M have a transcendental generating series which converges in the whole interior of the unit circle.

In Section 3, we study the preceding theory from the point of view of Ext-algebras and modules over them.

In Section 4, we prove/cite some general results about Ext-algebras of global dimension 3 and apply the theory to modules over R_n, defined by matrices of linear forms, and in Section 5 we specialize the theory to R_2 and its variants, one of them mentioned above, and in Section 6 we continue this study using results about Hochschild and cyclic homology that we present and deduce there. A useful irrationality Theorem 7.1 is presented in Section 7 and in Sections 6 and 8 we use that theorem to show that all Koszul algebras that are not complete intersections that we are able to treat have an irrational generating series for Hochschild and cyclic homology.

Finally, in Section 9 we present some open problems and conjectures.

1. A construction by Calle Jacobsson and its consequences

We start by recalling a very nice construction made by Jacobsson [14]: let $F_n = k\langle a_1, \ldots, a_n\rangle$ be the free associative algebra in n variables a_i of degree 1, and let

$$N = \frac{k\langle a_1, \ldots, a_n\rangle}{(f_1, \ldots, f_r)} \tag{1.1}$$

be a corresponding quotient algebra of F_n, where $(f_1, \ldots, f_r)$ denotes the 2-sided ideal in F_n, generated by r (non-commutative) quadratic forms f_i in the variables a_j. We say that $N = k \oplus N_1 \oplus N_2 \oplus N_3 \ldots$ is a (1, 2)-presented graded algebra. We denote by N_+ the augmentation ideal, i.e. the set of elements of strictly positive degree in N.

To the algebra N we will, following Jacobsson, construct a (1, 2)-presented Hopf algebra $U(g_N)$ as follows: let $L(N_1')$ and $L(N_1'')$ be the free graded Lie algebras on two copies of N_1 and the $L(N_+)$ be the free graded Lie algebra on N_+. Here, graded Lie algebras are take in the sense of algebraic topology, so that e.g. $[x, y] = -(-1)^{|x||y|}[y, x]$ where e.g. $|x|$ denotes the degree of x. Now, let g_N be the graded Lie algebra obtained by taking a semi-direct product of $L(N_1') \times L(N_1'')$ with $L(N_+)$ where we let $L(N_1') \times L(N_1'')$ operate on $L(N_+)$ by the formulas

$$a_i' \circ a = a_i a, \quad a \circ a_i'' = a a_i.$$

We obtain an exact split sequence of graded Lie algebras

$$0 \longrightarrow L(N_+) \longrightarrow g_N \longrightarrow L(N_1') \times L(N_1'') \longrightarrow 0 \tag{1.2}$$

and it is proved in [14] that the universal enveloping algebra $U(g_N)$ of g_N (which is a Hopf algebra) is a (1, 2)-presented algebra of the form:

$$\frac{k\langle a_1, \ldots, a_n, a_1', \ldots, a_n', a_1'', \ldots, a_n''\rangle}{([a_i', a_j''], [a_i', a_j] - [a_i, a_j''], 1 \leqslant i, j \leqslant n, F_1, \ldots F_r)}, \tag{1.3}$$

where all the a_i, a_i', a_i'' have degree 1 and where the F_s are defined as follows: suppose that $f_s = \sum_{1\leqslant i,j\leqslant n} c_{i,j}^s a_i a_j$. Then $F_s = \sum_{1\leqslant i,j\leqslant n} c_{i,j}^s [a_i', a_j]$. Furthermore, it follows from (1.2) that the Hilbert series of $U(g_N)$ is given by the formula

$$U(g_N)(z) = (1 - N_+(z))^{-1}(1 - |N_1|z)^{-2}, \tag{1.4}$$

where $|N_1| = n$ is the dimension of the k-vector space N_1 and where $N_+(z) = \sum_{i\geqslant 1} |N_i|z^i$ is the Hilbert series of the graded vector space N_+. For more details we refer to [14], where it is also noted that it follows from the Hochschild–Serre spectral sequence applied to (1.2) that the global homological dimension of $U(g_N)$ is $\leqslant 3$, so that the Koszul dual $R = U(g_N)^!$ (we use the notations of Manin [23,24]) is a commutative quadratic local ring (R, m), with $m^4 = 0$. For more details we refer again to [14], where it is furthermore proved that the Hilbert series $R(z) = 1 + 3nz + (2n^2 + r)z^2 + (|N_3| - n^3 + 2nr)z^3$ and that the Poincaré–Betti series $P_R(z)$ is given by the formula

$$1/P_R(z) = (1 + 1/z)/U(g_N)(z) - R(-z)/z. \tag{1.5}$$

This last formula follows from Lemma 2.4 of Jacobsson [14] which uses a result by Löfwall). For more details cf. Section 4 below. Now R can be completely reconstructed from the presentation (1.3) of $R^! = U(g_n)$. This is in principle done in [14] but we can go a little further and this will be very useful here: As in [14, p. 248] we let $(y_1, \ldots, y_n)$ be "dual variables" to $(a_1', \ldots, a_n')$, $(z_1, \ldots, z_n)$ be "dual variables" corresponding to $(a_1'', \ldots, a_n'')$ and $(x_1, \ldots, x_n)$ dual variables corresponding to $(a_1, \ldots, a_n)$. Let R_n be the commutative Koszul algebra

$$R_n = \frac{k[y_1, \ldots, y_n]}{(y_1, \ldots, y_n)^2} \otimes_k \frac{k[z_1, \ldots, z_n]}{(z_1, \ldots, z_n)^2}. \tag{1.6}$$

The $n^2 - r$ elements G_j of Jacobsson [14, p. 248] which are "dual" to the r elements F_i are all k-linear combinations of $y_i x_j + z_j x_i$ and the x_i all satisfy $x_i x_j = 0$. It follows that R can be presented as a trivial extension (for any graded module $M = \oplus M^i$ we denote by $s^{-1}M$ the graded module given by $s^{-1}M^i = M^{i-1}$)

$$R = R_n \propto s^{-1}M, \tag{1.7}$$

where M is the R_n-module defined as the cokernel of the map

$$s^{-1}R_n^{n^2-r} \longrightarrow R_n^n,$$

defined by the $n \times (n^2 - r)$ matrix of linear forms in the y_j, z_j, defined by the x-coefficients of the elements in the G_s. Now, use a theorem of Gulliksen [13] about the Poincaré–Betti series of a trivial extension on (1.7)! We obtain

$$P_R(z) = \frac{P_{R_n}(z)}{1 - zP_{R_n}^M(z)} \tag{1.8}$$

and since it follows from (1.6) that

$$P_{R_n}(z) = \frac{1}{(1 - nz)^2}, \tag{1.9}$$

we finally obtain, combining (1.8), (1.9), (1.5) and (1.4) that

$$P_{R_n}^M(z) = \frac{N^+(z)}{z} + \frac{N_{\geqslant 3}(z)}{z^2} + (n^2|N_2|z - |N_3|)z/(1-nz)^2, \tag{1.10}$$

where $N_{\geqslant 3}(z) = \sum_{i \geqslant 3} |N_i| z^i$. Let us say [1, p. 135] that two non-zero formal power series $A(z) = \sum_{i=0}^{i=\infty} a_i z^i$ and $B(z) = \sum_{i=0}^{i=\infty} b_i z^i$ are *rationally related* and let us write $A(z) \sim B(z)$ if there is a 2×2 matrix of polynomials $(a_{ij}(z))$ whose determinant is non-zero and such that

$$A(z) = \frac{a_{11}(z)B(z) + a_{12}(z)}{a_{21}(z)B(z) + a_{22}(z)}.$$

This is an equivalence relation and we summarize our results as follows:

Theorem 1.1. *Let N be any quadratic (not necessarily commutative) algebra of embedding dimension n presented with r quadratic relations. Then there exists a module $M = M_N$ over the commutative Koszul algebra*

$$R_n = \frac{k[y_1, \ldots, y_n]}{(y_1, \ldots, y_n)^2} \otimes_k \frac{k[z_1, \ldots, z_n]}{(z_1, \ldots, z_n)^2}$$

defined by a $r \times (n^2 - r)$ matrix of linear forms in the y_i, z_j such that $P_{R_n}^M(z)$ and $N(z)$ are rationally related. More precisely, we have the formula (1.10) *above, where M is the cokernel of the R_n-module map*

$$s^{-1} R_n^{n^2 - r} \longrightarrow R_n$$

defined by our matrix.

Before giving examples illustrating this theorem, we make it a little more precise. Since all our rings and algebras are graded, we have an extra grading on $\mathrm{Tor}_i^{R_n}(M, k)$, etc. leading to generating series of two variables. Inspection of the Calle Jacobsson proof, in particular the Lemma 2.4 (Calle Jacobsson proof which uses a result by C. Löfwall, cf. Section 4 below), leads to the following more precise formula:

$$P_{R_n}^M(x, y) = \frac{N^+(xy)}{xy} + y\left(\frac{N_{\geqslant 3}(xy)}{(xy)^2} + \frac{n^2|N_2|x^2y^2 - |N_3|xy}{(1-nxy)^2}\right). \tag{1.11}$$

In particular, it follows that in the display of Betti numbers of the computer programme Macaulay the module M has non-zero Betti numbers on two horizontal lines only: The linear part (the first horizontal line) whose generating series is $[N^+(xy)]/xy$, and the sublinear part (the second horizontal line), whose generating series is

$$y\left(\frac{N_{\geqslant 3}(xy)}{(xy)^2} + \frac{n^2|N_2|x^2y^2 - |N_3|xy}{(1-nxy)^2}\right). \tag{1.12}$$

Both the linear part and the sublinear part are rationally related to the series $N(z)$.

Remark 1.2. (A very simple recipe). If we analyze the reasoning leading up to the proof of Theorem 1.1 we see that there is a precise and very simple recipe for constructing the matrix M_N from the algebra N of Theorem 1.1, which shows that Theorem 1.1 is essentially a "sophisticated polarization procedure": Calculate the Koszul dual $N^!$ of N (cf. e.g. [20]). Write the Koszul dual $N^!$ as a quotient

$$\frac{k\langle x, y, z, \ldots\rangle}{(g_1, g_2, \ldots, g_{n^2-s})}, \tag{1.13}$$

where the g_j are quadratic (in general non-commutative) polynomials in (say) the variables $x, y, z, \ldots$. Take two new sets of variables $x', y', z', \ldots$ and $x'', y'', z'', \ldots$ and make the following "polarizations" in all the relations g_i: replace monomials of the type xy by $x'y + xy''$ and replace squares of the type z^2 by $z'z + zz''$, etc. everywhere. We get s new relations involving the $x, y, \ldots$, the $x', y', \ldots$ and the $x'', y'', \ldots$ linearly. To these s relations we now associate an $n \times (n^2 - s)$ matrix of linear forms in the variables $x', y', \ldots$ and the $x'', y'', \ldots$ only, as follows: if, say, the first of the $n^2 - s$ relations looks like $x'z + xz'' + y'y + yy''$ (corresponding to the relation $xz + y^2$ in $N^!$) then the first column in our matrix should be the transpose of the vector $(z'', y' + y'', x', 0, 0, \ldots)$. In general, we collect first the terms containing x (and replace x by 1), then the terms containing y (and replace y by 1), etc. This gives a row vector, whose transpose is the first column in a matrix M_N of linear forms in the $x', y', \ldots$ and $x'', y'', \ldots$ and which defines the module M_N over the commutative ring

$$R_n = \frac{k[x', y', \ldots]}{(x', y', \ldots)^2} \otimes_k \frac{k[x'', y'', \ldots]}{(x'', y'', \ldots)^2}.$$

We illustrate this procedure by two explicit and useful examples:

Example 1.3. In the paper by Fröberg–Gulliksen–Löfwall [11] the following quadratic (non-Hopf) algebra is studied:

$$A_\lambda = \frac{k\langle a, b, c\rangle}{(a^2, \lambda(bc - cb) - c^2, ac - ab, ba)}. \tag{1.14}$$

Let us suppose that k is of characteristic 0 and that λ is a parameter in k. It is proved in [11] that the Hilbert series of $A_\lambda(z)$ is given by

$$A_\lambda(z) = \begin{cases} \dfrac{1 + z - z^{\lambda+2}}{(1-z)^2(1-z^{\lambda+2})} & \text{if } \lambda \in \{1, 2, 3, \ldots\}, \\[2ex] \dfrac{1+z}{(1-z)^2} & \text{otherwise.} \end{cases}$$

Now, use Theorem 1.1 in the form of the recipe we have just described! The Koszul dual of the A_λ of (1.14) is easily calculated. We obtain

$$A_\lambda^! = \frac{k\langle A, B, C\rangle}{(B^2, \lambda C^2 + BC, CA, AB + AC, BC + CB)}. \tag{1.15}$$

If we "polarize" the relations in (1.15) according to the recipe just mentioned, we obtain the expressions:

$$B'B + BB'', \lambda(C'C + CC'') + B'C + BC'', C'A + CA'',$$

$$A'B + AB'' + A'C + AC'', B'C + BC'' + C'B + CB'',$$

which give, by the procedure we just mentioned, the following 3×5 matrix M_λ of linear terms in the variables $(A', B', C', A''.B'', C'')$:

$$\begin{pmatrix} 0 & 0 & C' & B''+C'' & 0 \\ B'+B'' & C'' & 0 & A' & C'+C'' \\ 0 & \lambda(C'+C'')+B' & A'' & A' & B'+B'' \end{pmatrix}$$

over the ring

$$R_3 = \frac{k[A', B', C']}{(A', B', C')^2} \otimes_k \frac{k[A'', B'', C'']}{(A, B'', C'')^2}.$$

We now want to use the two-variable version (1.11) of Theorem 1.1 above. We assume that $\lambda \in \{1, 2, 3, \ldots\}$ so that

$$N(z) = \frac{(1 + z - z^{\lambda+2})}{(1-z)^2(1-z^{\lambda+2})} = 1 + 3z + 5z^2 + 7z^3 + \ldots.$$

Therefore, in this case

$$\frac{N^+(z)}{z} = \frac{(2-z)}{(1-z)^2} + \frac{1}{(1-z)^2(1-z^{\lambda+2})} \tag{1.16}$$

and

$$\frac{N_{\geqslant 3}(z)}{z^2} + \frac{n^2|N_2|z^2 - |N_3|z}{(1-nz)^2} = \frac{4z^2\,(3-z)}{(1-z)^2(1-3z)^2} + \frac{z^{\lambda+1}}{(1-z)^2(1-z^{\lambda+2})}, \tag{1.17}$$

so that the R_3-module M_λ associated to N has according to (1.11) the following Poincaré–Betti series:

$$P_{R_3}^{M_\lambda}(x, y) = \varphi_0(xy) + y\varphi_1(xy), \tag{1.18}$$

where $\varphi_0(z)$, and $\varphi_1(z)$ are defined by the expressions on the right-hand side of (1.16) and (1.17), respectively. If we put $y = 1$ in (1.18) we obtain

$$P_{R_3}^{M}(x) = \frac{2 - 13x + 36x^2 - 13x^3}{(1-x)^2(1-3x)^2} + \frac{1 + x^{\lambda+1}}{(1-x)^2(1-x^{\lambda+2})}. \tag{1.19}$$

This gives a quick negative answer to a question by Avramov [3] (and this result of us was even cited in the internet additions/corrections to [3]). In Sections 5 and 6 below we will analyze (using our results in [35]) what we think is the smallest negative answer (the example mentioned in the introduction to the present paper) to this question.

Example 1.4. If N is a quadratic Hopf algebra, then its Koszul dual $N^!$ is a quadratic *commutative* ring R and the previous remark and simple recipe takes an even simpler form: first, we note that we can write any quadratic commutative ring $R = k[x_1, \dots, x_n]/(f_1, \dots, f_r)$ in the form:

$$R = \frac{k\langle x_1, \dots, x_n\rangle}{(x_i x_j - x_j x_i, (i < j), f_1, \dots, f_r)}, \tag{1.20}$$

where the f_i are "the same" as the relations in R, the only difference being that they now are considered to be relations in the (now) non-commutating variables x_i (to be quite specific, we can and will assume that the all products $x_i x_j$ occurring in the "commutative" f_s have $i \leqslant j$ and we keep that order when we write down (1.20)).

If we now follow the instructions in the general recipe above about "polarizations" of the relations in (1.20) we see that we will get a $n \times (\binom{n}{2} + r)$ matrix M which is formed by concatenating a "universal" $n \times \binom{n}{2}$ matrix M_U corresponding to the commutator relations $x_i x_j - x_j x_i$ and a special $n \times r$ matrix M_S corresponding to the f_s. Therefore, to each quadratic commutative ring R in n variables, we can associate a matrix M of linear forms over the Koszul algebra R_n whose Poincaré–Betti series over R_n is rationally related to the Hilbert series $R^!(z)$. We explicitly illustrate this procedure for one of the rings studied in [22], since this will also answer some other questions in the literature: let R be the commutative quadratic ring

$$R = \frac{k[x, y, z, u, v]}{(x^2, y^2, z^2, u^2, v^2, yz + xu, yu + xv, zu + yv)}. \tag{1.21}$$

It is proved in [22] that the Hilbert series of the Koszul dual $N = R^!$ of (1.21) is *irrational* and e.g. in characteristic 0 is given by the formula

$$\begin{aligned} N(z) &= \frac{(1 + z^3)^2}{(1 - z^2)^2(1 - z^4)} \prod_{n=1}^{\infty} \frac{(1 + z^{4n-3})^5}{(1 - z^{4n-2})^6} \frac{(1 + z^{4n-1})^3}{(1 - z^{4n})^3} \\ &= 1 + 5z + 18z^2 + 55z^3 + 150z^4 + 376z^5 + \dots . \end{aligned} \tag{1.22}$$

In particular, $N(z)$ is even transcendental and the construction of Theorem 1.1 in the form we have just described gives rise to a module M over the Koszul algebra:

$$R_5 = \frac{k[x, y, z, u, v]}{(x, y, z, u, v)^2} \otimes_k \frac{k[X, Y, Z, U, V]}{(X, Y, Z, U, V)^2},$$

defined by the cokernel of a 5×18 matrix $\mathscr{M}$ of linear forms in $x, y, z, u, v, X, Y, Z, U, V$ considered as a map

$$s^{-1} R_5^{18} \longrightarrow R_5^5.$$

The matrix $\mathscr{M}$ is the concatenation of a 5×10 universal matrix $\mathscr{M}_U$

$$\begin{pmatrix} Y-y & Z-z & U-u & V-v & 0 & 0 & 0 & 0 & 0 & 0 \\ x-X & 0 & 0 & 0 & Z-z & U-u & V-v & 0 & 0 & 0 \\ 0 & x-X & 0 & 0 & y-Y & 0 & 0 & U-u & V-v & 0 \\ 0 & 0 & x-X & 0 & 0 & y-Y & 0 & z-Z & 0 & V-v \\ 0 & 0 & 0 & x-X & 0 & 0 & y-Y & 0 & z-Z & u-U \end{pmatrix} \tag{1.23}$$

and a “specific” 5×8 matrix $\mathscr{M}_S$

$$\begin{pmatrix} x+X & 0 & 0 & 0 & 0 & U & V & 0 \\ 0 & y+Y & 0 & 0 & 0 & Z & U & V \\ 0 & 0 & z+Z & 0 & 0 & y & 0 & U \\ 0 & 0 & 0 & u+U & 0 & x & y & z \\ 0 & 0 & 0 & 0 & v+V & 0 & x & y \end{pmatrix}. \tag{1.24}$$

Finally, the two-variable Poincaré–Betti series of this module is according to the remark following Theorem 1.1 given by the formula:

$$P_{R_5}^M(x,y) = \frac{N^+(xy)}{xy} + y\left(\frac{N_{\geqslant 3}(xy)}{(xy)^2} + \frac{450x^2y^2 - 55xy}{(1-5xy)^2}\right), \tag{1.25}$$

where $N(z)$ is given by (1.22), and furthermore the Hilbert series of the R_5-module M is equal to $M(z) = 5 + 32z$. In particular $m^2M = 0$, M has finite regularity, and $P_{R_5}^M(x,y)$ is transcendental. This gives a positive answer to Problem 18, p. 18 of another paper by Avramov [2]. Furthermore, if k has characteristic $p \neq 0$ it is proved in [22] that even in this case the series $N(z)$ are also transcendental but different for each p (the precise formulae are given on p. 195 of Löfwall and Roos [22]). In particular, the $P_{R_5}^M(x,y)$ are also different for all p. An inspection of the formula (1.25) also shows that the generating function of the Betti numbers of the linear part of the R_5-resolution of M converges in the whole interior of the unit circle and determines a transcendental function there. According to (1.25) the generating function of the sublinear part of the same resolution is a transcendental function which has a double pole at $\frac{1}{5}$, but is meromorphic in the whole interior of the unit circle.

Remark 1.5. In the paper by Anick and Gulliksen [1] 17 different classes of generating series which are rationally related are studied. It follows from the reasoning above that there is a 18th class: The Poincaré–Betti series of modules define by matrices of linear forms of the special type encountered over the special Koszul algebras R_n (or even the linear “diagonal” parts of such series). In Section 4, we study modules defined by “all linear forms” with a similar result.

Remark 1.6. We could also redo the whole theory by replacing in the Jacobsson construction the free graded Lie algebras, by free graded Lie algebras with different signs. This leads to similar results for quotients of exterior algebras instead of quotients of polynomial rings.

2. Good and bad Koszul algebras

Definition 2.1. We say that a local ring (R, m) is good if for all finitely generated R-modules M the series $P_R^M(z)$ are all rational functions which can be put on a common denominator (independent of M) and that (R, m) is bad otherwise. We make similar definitions for homogeneous quotients of polynomial rings (e.g. quadratic algebras).

Example 2.2. If R is a Golod ring then it is good (cf. references in Section 5.2 of Avramov [3]). In particular the Koszul algebra

$$R = \frac{k[x, y, z, u]}{(x^2, xy, xz, xu)}$$

is good.

Example 2.3. If R is a complete intersection then it is good, and more generally any S which comes from a complete intersection R by one Golod map: $R \longrightarrow S$ is good (this follows from [7] (Theorem 1(d) formula (8) combined with Theorem 2(a) of Backelin and Roos [7]) (also first proved by Levin).

Here are more results and examples:

Theorem 2.4. (a) *All local rings of embedding dimension* $\leqslant 3$ *are good*.

(b) *Among the* 83 *quadratic algebras of embedding dimension* 4 *given in* [34] *the following* 3 *algebras of embedding dimension* 4 *are not good (i.e. bad)*:

(A) $R = K[x, y, z, u]/(x^2, xy, y^2, z^2, zu, u^2)$ $(case\ 54)$,

(B) $R = K[x, y, z, u]/(x^2, xy, z^2, zu, u^2)$ $(case\ 41)$,

(C) $R = K[x, y, z, u]/(x^2, xy, zu, u^2)$ $(case\ 21)$.

Speculation: The 83 quadratic algebras of embedding dimension 4 given in [34] probably describe all possibilities when it comes to classifying the Hilbert series of the algebras and the Hilbert series of their Koszul duals.

Proof of Theorem 2.4. In [4] it was proved that all local rings of embedding dimension $\leqslant 3$ could be obtained by a Golod map from a complete intersection. This proves (a) if we use the result mentioned in Example 2.3 above. The proof of (b) will be given below: all the 80 cases not equal to (51),(41) and (21) are Golod from complete intersection. The cases (A), (B) and (C) will be treated in Sections 5 and 6 below. □

Here is a result which can be used for distinguishing between bad and good rings of higher embedding dimension:

Proposition 2.5. *Let* $R \longrightarrow \bar{R}$ *be a large map in the sense of Levin* [18] *between local rings. If R is good, then so is* $\bar{R}$.

Remark 2.6. Any map $R \longrightarrow \bar{R}$ which is a retract is a large map. For more examples wee refer to [18].

Proof of Proposition 2.5. It is proved in [18] that if $R \longrightarrow \bar{R}$ is large, then for any $\bar{R}$-module M the following formula holds true:

$$P_R^M(z) = P_R^{\bar{R}}(z) P_{\bar{R}}^M(z), \tag{2.1}$$

where we consider M as an R-module by means of the map $R \longrightarrow \bar{R}$. This gives the result. □

3. Ext-algebras and modules over them

In this section, we develop some formulas that are special cases of some results from [28]. We begin by studying $P_R^M(z)$ for modules over a local ring (R, m) such that $m^3 = 0$. First we observe that if N is a minimal first syzygy of M, then $m^2 N = 0$. We can therefore assume from the beginning that $m^2 M = 0$. We note that the natural inclusion map $mM \longrightarrow M$ gives rise to a map

$$\mathrm{Ext}_R^*(M, k) \longrightarrow \mathrm{Ext}_R^*(mM, k) \tag{3.1}$$

of left $\mathrm{Ext}_R^*(k, k)$-modules (we use the Yoneda multiplication). Let the image of (3.1) be denoted by V (it is a left $\mathrm{Ext}_R^*(k, k)$-module). Furthermore, recall that for any graded module $U = \oplus U^i$ we denote by $s^{-1}U$ the module given by: $s^{-1}U^i = U^{i-1}$. Now, the long exact sequence of Ext associated to the short exact sequence

$$0 \longrightarrow mM \longrightarrow M \longrightarrow M/mM \longrightarrow 0 \tag{3.2}$$

essentially gives the following result:

Theorem 3.1. *Let (R, m) be any local ring with $m^3 = 0$. Let M be any R-module with $m^2 M = 0$. Then we have an exact sequence of $\mathrm{Ext}_R^*(k, k)$-modules*:

$$\begin{aligned} 0 &\longrightarrow s^{-1}V \longrightarrow s^{-1}\mathrm{Ext}_R^*(mM, k) \longrightarrow \mathrm{Ext}_R^*(M/mM, k) \\ &\longrightarrow \mathrm{Ext}_R^*(M, k) \longrightarrow V \longrightarrow 0, \end{aligned} \tag{3.3}$$

where the $\mathrm{Ext}_R^*(k, k)$-modules $s^{-1}\,\mathrm{Ext}_R^*(mM, k)$ and $\mathrm{Ext}_R^*(M/mM, k)$ are free of rank $|mM|$ and $|M/mM|$, respectively, and the map between them is given by a matrix of linear forms. Furthermore

$$P_R^M(z) = (|M/mM| - z\,|mM|)P_R(z) + (1 + z)V(z), \tag{3.4}$$

where $V(z)$ is the generating series of the graded vector space V. Furthermore, if R is a Koszul algebra then V is a free $\mathrm{Ext}_R^*(k, k)$-module so that $V(z) = B(z)P_R(z)$ where $B(z)$ is the generating function of a graded set B of basis elements of the $\mathrm{Ext}_R^*(k, k)$-module V.

Proof. The image of (3.1) is denoted by V so the exact sequence (3.3) *does* indeed follow from the long exact sequence of Ext_R obtained by applying the functor $\mathrm{Ext}_R^*(\cdot, k)$ to (3.2).

Now (3.4) follows by observing that the alternating sum of the Hilbert series of the modules in (3.3) is zero. Furthermore, if R is also a Koszul algebra, then lgldim $\mathrm{Ext}_R^*(k,k) \leqslant 2$ so that $s^{-1}V$ and therefore also V must be a free $\mathrm{Ext}_R^*(k,k)$-module. If B is a graded vector space corresponding to a $\mathrm{Ext}_R^*(k,k)$-basis of V then the Hilbert series $V(z)$ of V is equal to $B(z)P_R(z)$ and the formula (3.4) takes the form

$$P_R^M(z) = \frac{|M/mM| - z|mM| + B(z)(1+z)}{R(-z)}, \tag{3.5}$$

since $P_R(z) = R^!(z) = 1/R(-z)$ for Koszul algebras. $\square$

Corollary 3.2. *If the Koszul dual $R^!$ of the Koszul algebra R is left graded coherent [29,30](i.e. all kernels of maps between finitely generated graded free left $R^!$-modules are finitely generated), then all $P_R^M(z)$ are rational functions with a common denominator, so that R is a good Koszul algebra.*

Indeed in this case $B(z)$ is a polynomial and everything follows from formula (3.5).

Remark 3.3. Note that if $\tilde{R}$ is a complete intersection then $\mathrm{Ext}_{\tilde{R}}^*(k,k)$ is noetherian (to the left and right) and if $\tilde{R} \longrightarrow R$ is a Golod map (onto) of local rings, then $\mathrm{Ext}_R^*(k,k)$ is a graded coherent ring (to the left and right)[29,30].

Remark 3.4. Examples of Koszul algebras to which Theorem 3.1 applies are the R_n (for varying n) encountered earlier and it follows that the set of series $V(z)$ for these algebras are rationally related to the 18 classes of series mentioned in Remark 1.5 above.

4. General results about modules over the algebras R_n: generalizations

We first need a general theorem about "linear modules" over the rings R_n. For special modules M coming from a quadratic algebra N this was already done in Section 1.

Theorem 4.1. *Let*

$$R = R_n = \frac{k[x_1, x_2, \ldots, x_n]}{(x_1, x_2, \ldots, x_n)^2} \otimes \frac{k[X_1, X_2, \ldots, X_n]}{(X_1, X_2, \ldots, X_n)^2}$$

and let M be the R_n-module defined by a $s \times t$ matrix of linear forms in the $2n$ variables

$$x_1, x_2, \ldots, x_n, X_1, X_2, \ldots, X_n$$

(s *rows and* t *columns*):

$$s^{-1}R^t \longrightarrow R^s \longrightarrow M \longrightarrow 0.$$

Then M is graded, generated as an R_n-module by $\leqslant s$ generators of degree 0 *and the graded R-resolution of M consists of a linear and sublinear part that determine each other* (*the precise result is formula* (4.6) *below*).

Proof. The Hilbert series $M(z)$ of M is of the form $a + bz + cz^2$, so that the Hilbert series of the trivial extension $\mathscr{R} = R \propto s^{-1}M$ (here $s^{-1}M$ is the graded module defined by $s^{-1}M_i = M_{i-1}$) is

$$\mathscr{R}(z) = 1 + (2n + a)z + (n^2 + b)z^2 + cz^3.$$

Furthermore, the two-variable version of the theorem of Gulliksen [13] shows that the PB-series of $\mathscr{R}$ is given by the formula

$$P_{\mathscr{R}}(x, y) = \frac{P_R(x, y)}{1 - xyP_R^M(x, y)}. \tag{4.1}$$

In general even if M is not graded, the Ext-algebra of $\mathscr{R} = R \propto M$ sits in the middle of a Hopf algebra extension:

$$k \longrightarrow T(s^{-1}\operatorname{Ext}_R^*(M, k)) \longrightarrow \operatorname{Ext}_{\mathscr{R}}^*(k, k) \longrightarrow \operatorname{Ext}_{R_n}^*(k, k) \longrightarrow k, \tag{4.2}$$

where T is the tensor algebra (thus has global dimension 1) and $\operatorname{Ext}_{R_n}^*(k, k)$ has global dimension 2. Therefore, the middle Hopf algebra has global homological $\leqslant 3$ and we start therefore with some general results about a local ring S such that $\operatorname{Ext}_S^*(k, k)$ has global dimension 3.

The following result (due to Clas Löfwall) is mentioned and proved rather implicitly in the proof of Lemma 2.4 of Jacobsson [14]: Let (S, m) be a graded local ring (or a graded algebra) such that the Yoneda Ext-algebra $\operatorname{Ext}_S^*(k, k)$ has global dimension 3. Then (S, m) satisfies the condition L_3 [33] so that in particular

$$P_S(x, y)^{-1} = (1 + 1/x)/S^!(xy) - S(-xy)/x. \tag{4.3}$$

For the proof (Löfwall) see [14]. Now apply the formula (4.1) to $S = \mathscr{R} = R_n \propto s^{-1}M$. First $S^!(t) = P_S(t/y, y)|_{y=0}$ so that (4.1) implies that

$$S^!(t) = R^!(t)/(1 - t(P_R^M(t/y, y)|_{y=0})) \tag{4.4}$$

and

$$P_R^M(t/y, y)|_{y=0} = \sum_{i \geqslant 0} |\operatorname{Tor}_{i,i}^R(M, k)|t^i, \tag{4.5}$$

which will be denoted by $L_R^M(t)$ since it is the generating series for the "linear part" of the resolution of M. Secondly $S(t) = R(t) + tM(t)$, so if we put these two results in the formula (4.3) we obtain the formula:

$$P_R^M(x, y) = L_R^M(xy) + \frac{1}{x}\left(L_R^M(xy) - \frac{M(-xy)}{R(-xy)}\right), \tag{4.6}$$

where we have used that R is Koszul, so that $R^!(t) = 1/R(-t)$. $\square$

Remark 4.2. Note that the fact that R is Koszul with $R^!$ of global dimension 2 (then $m^3 = 0$) is the only thing that is used in the proof of Theorem 4.1. But Theorem 4.1 inspires one to study the case of R_n and modules defined by *generic* matrices of linear forms in more

Table 1
M is the cokernel of a generic map of linear forms $R^m \longrightarrow R^s$

s	The Hilbert series $M(t)$ of the graded R-module M					
	$m=1$	$m=2$	$m=3$	$m=4$	$m=5$	$m=6$
1	$1+3t+t^2$	$1+2t$	$1+t$	1	1	1
2	$2+7t+4t^2$	$2+6t$	$2+5t$	$2+4t$	$2+3t$	$2+2t$
3	$3+11t+8t^2$	$3+10t+4t^2$	$3+9t$	$3+8t$	$3+7t$	$3+6t$
4	$4+15t+12t^2$	$4+14t+8t^2$	$4+13t+4t^2$	$4+12t$	$4+11t$	$4+10t$
5	$5+19t+16t^2$	$5+18t+12t^2$	$5+17t+8t^2$	$5+16t+4t^2$	$5+15t$	$5+14t$
6	$6+23t+20t^2$	$6+22t+16t^2$	$6+21t+12t^2$	$6+20t+8t^2$	$6+19t+4t^2$	$6+18t$

Table 2
M is the cokernel of a generic map of linear forms $R^m \longrightarrow R^s$

s	Generating series of the linear resolution of the graded R-module M					
	$m=1$	$m=2$	$m=3$	$m=4$	$m=5$	$m=6$
1	$\frac{1}{1-t}$	K	K	K	K	K
2	$2+t$	$2+2t$	$2+3t+4t^2+4t^3$	K	K	K
3	$3+t$	$3+2t$	$3+3t$	$3+4t+4t^2$	$3+5t+8t^2+12t^3+16t^4+16t^5$	K
4	$4+t$	$4+2t$	$4+3t$	$4+4t$	$4+5t+t^2$	$4+6t+8t^2+8t^3$
5	$5+t$	$5+2t$	$5+3t$	$5+4t$	$5+5t$	$5+6t+4t^2$
6	$6+t$	$6+2t$	$6+3t$	$6+4t$	$6+5t$	$6+6t$

K means that we have a Koszul module, so that the series is $M(-t)/R(-t)$.

detail. Here, are some sample results obtained by calculating with Macaulay 2 [12] over $\mathbf{Q}$. In view of formula (4.6) it is sufficient to determine the Hilbert series of the corresponding module M (Table 1) and the linear resolution series $L_R^M(t)$ (Table 2).

Thus, we cannot hope for strange resolutions if we study matrices defined by generic linear forms over Koszul algebras. Indeed we have to consider more degenerate cases. This has been done in [32,35], and we here just summarize and slightly extend the results given there in the next section (in an earlier version of the present paper these results were presented without proofs):

5. Resolutions of modules in embedding dimension 4

Let

$$R = \frac{k[x,y,z,u]}{(x^2, xy, y^2, z^2, zu, u^2)}$$

and let M_α be the R-module which is the cokernel of the map $s^{-1}R^3 \longrightarrow R^2$ defined by following 2×3 matrix of linear terms:

$$\begin{pmatrix} y & x+\alpha u & u \\ z & -u & x+(\alpha-2)u \end{pmatrix}, \tag{5.1}$$

where α is an integer $\geqslant 2$. Then I claim that the Betti series for the linear part of the resolution of M_α is

$$L_R^{M_\alpha}(t) = \left(2 - t + \frac{t^{\alpha+1}}{1-t^\alpha}\right)\frac{1}{(1-t)^2}.$$

In [35, Theorem 4.1] we proved this (for the case when $k[x, y, z, u]$ is replaced by the exterior algebra in 4 variables of degree 1—but the same proof works for our present R. Since the Hilbert series of M_α is $2 + 5t$ the formula (4.6) of the preceding section where we put $x = z$ and $y = 1$ gives

$$P_R^{M_\alpha}(z) = P_R^{M_\alpha,k}(z) = \frac{2-9z+13z^2-4z^3}{(1-z)^2(1-2z)^2} + \frac{(1+z)z^\alpha}{(1-z)^2(1-z^\alpha)}. \tag{5.2}$$

In particular, if α goes to ∞ through the primes then the irreducible factors in the denominator of (5.2) will get a higher and higher degree. Thus the ring R is *not* a good Koszul algebra. Furthermore, we proved in [31] that if N is the R-module $R/(x-u, y-z)$ then the series $P_R^{N,N}(z)$ is an irrational function. It is quite possible that if R is a good Koszul algebra then $P_R^{N,M}(z)$ is a rational function for all R-modules N and M of finite length. If this were proved then the result just mentioned would prove in an alternative way that R is a bad Koszul algebra. Note that here N is an R-module of length 3 and $P_R^{N,N}(z)$ can be interpreted as the generating series of the *Hochschild homology* of the k-algebra $k[x, y]/(x^2, xy, y^2)$ We will now show how to deduce results that are analogous to (5.2) for the two other cases of bad Koszul algebras mentioned in Section 2 above: $R' = k[x, y, z, u]/(x^2, xy, z^2, zu, u^2)$ and $R'' = k[x, y, z, u]/(x^2, xy, zu, u^2)$. In these two cases we study the modules M'_α and M''_α defined by the matrices (5.1) over R' and R'', respectively. We start with M'_α. There is a natural ring map

$$R' \propto M'_\alpha \longrightarrow R \propto M_\alpha. \tag{5.3}$$

If we denote the R'-generators of M'_α by v and w and also (sic!) the R-generators of M'_α by v and w the ring to the left-hand side of (5.3) can be identified with

$$k[x, y, z, u, v, w]/(x^2, xy, z^2, zu, u^2, v^2, vw, w^2, yv + zw,$$
$$(x + \alpha u)v - uw, uv + (x + (\alpha - 2)u)w).$$

If we denote this ring by S then the map (5.3) is the natural map $S \longrightarrow S/(y^2)$. I claim that this map is a Golod map [16,17]. For this we cite Backelin [5], pp. 12–13:

"...some modifications of [17, Theorem 2.3] may be applied. In fact Levin states that if B is a local ring with the maximal ideal m, if $x \in m$ and if a is an ideal such that $xa \subset m^2$ then $B \to B/xa$ is Golod provided $(0 : x) = 0$. In his proof he, however only uses the weaker condition $(0 : x) \cap a = 0$."

We now apply this to the ring S the element y and the ideal (y). We have $(0 : y) = (x, zv)$ in S and from this follows that $(0 : y) \cap (y) = 0$ in S and it follows indeed that $S \longrightarrow S/(y^2)$ *is* a Golod map. Therefore

$$P_{R\propto M_\alpha}(x, y) = \frac{P_{R'\propto M'_\alpha}(x, y)}{1 - x(P_S^{S/(y^2)}(x, y) - 1)}. \tag{5.4}$$

Since we know the left-hand side of (5.4) all we have to do to get $P_{R'\propto M'_\alpha}(x, y)$ is to calculate the denominator of the right-hand side of (5.4) i.e. the minimal S-resolution of the S module S/y^2. But in S we also have $(0 : y^2) = (x, zv)$. Furthermore, $(x) \cap (zv) = 0$ so that $(x, zv) = (x) \oplus (zv)$ and therefore it is sufficient to calculate the S-resolutions of (x) and (zv). Now $(zv) = s^{-2}k$ and $(0 : x) = (y, x, zv) = (y) \oplus ((x) \oplus (zv))$ so we can continue recursively. This gives the following formula for the relative series:

$$P_S^{S/(y^2)}(x, y) - 1 = \frac{xy^2 + x^2y^4 P_S(x, y)}{1 - xy - x^2y^2}. \tag{5.5}$$

Combining this with (5.4) we get

$$\frac{1}{P_{S/y^2}(x, y)} = \frac{1}{P_S(x, y)}\left(1 - \frac{x^2y^2}{1 - xy - x^2y^2}\right) - \frac{x^3y^4}{1 - xy - x^2y^2}. \tag{5.6}$$

We will now prove that our ring S satisfies L_3. The most elegant way of doing this is to prove a more general result. First, note that if T is any graded algebra with generators in degree 1 and homogeneous relations of degree $\geqslant 2$ then the summation in the double series

$$P_T(x, y) = \sum_{i,j} |\mathrm{Tor}^T_{i,j}(k, k)| x^i y^j$$

is in fact only extended over $j \geqslant i$ since $\mathrm{Tor}^T_{i,j}(k, k) = 0$ for $j < i$. Therefore the formal power series

$$\frac{1}{P_T(x/y, y)} = \sum_{i\geqslant 0} C_i(x) y^i, \tag{5.7}$$

where the formal power series $C_i(x)$ are uniquely defined.

Proposition 5.1. *Let T be as above and $C_i(x)$ be defined by* (5.7). *The following conditions are equivalent*

(a) $C_i(x) = 0$ *for* $i \geqslant 2$
(b) $1/P_T(x, y) = (1 + 1/x)/T^!(xy) - T(-xy)/x$, *where $T(t)$ is the Hilbert series of T and $T^!(t)$ is the Hilbert series of the Koszul dual of T.*

Proof. If (a) is true, then

$$\frac{1}{P_T(x/y, y)} = C_0(x) + yC_1(x). \tag{5.8}$$

Now put first $y = 0$ in (5.8). We get

$$1/C_0(x) = \sum_{i \geqslant 0} |\mathrm{Tor}^T_{i,i}(k,k)|x^i = T^!(x). \tag{5.9}$$

Next put $y = -x$ in (5.8). This gives

$$\frac{1}{P_T(-1,-x)} = C_0(x) - xC_1(x). \tag{5.10}$$

But $1/P_T(-1, x)$ is equal to the Hilbert series $T(x)$ of the ring T as is easily seen by taking the alternating sum of the Hilbert series of a minimal T-resolution of k. Thus $T(-x) = C_0(x) - xC_1(x)$ and since $C_0(x) = 1/T^!(x)$ we get $C_1(x) = (1/T^!(x) - T(-x))/x$ and substituting these values in (5.8) and replacing x by xy we get the formula in (b). It is evident that (b) implies (a) and Proposition 5.1 is proved. □

Remark 5.2. We do not know if the right-hand side of (5.7) is always a polynomial in y, but this is at least the case if T is a Golod ring. There are variants of Proposition 5.1 (conditions L_4, etc.), where we assume that the right-hand side of (5.5) is of the form $C_0(x) + y^2C_2(x)$, etc.

Now, it is easy to finish the proof that our previous S satisfies condition (b) of Proposition 5.1: replace x by x/y in the formula (5.6). We obtain

$$\frac{1}{P_{S/y^2}(x/y, y)} = \frac{1}{P_S(x/y, y)}\left(1 - \frac{x^2}{1-x-x^2}\right) - \frac{x^3y}{1-x-x^2} \tag{5.11}$$

and since we know that $1/P_{S/y^2}(x/y, y) = C_0(x) + yC_1(x)$ we obtain from (5.11) that

$$P_S(x/y, y) = D_0(x) + yD_1(x)$$

for suitable $D_0(x)$ and $D_1(x)$ so that (a) of Proposition 5.1 is fulfilled for S and therefore (b) is so too. Furthermore $P_S(x, y)$ can be calculated from $P_{S/y^2}(x, y)$. Now, formula (4.6) can be applied not only to R and M_α but also to R' and M'_α and even to R'' and M''_α when we analyze the natural map

$$R'' \propto M''_\alpha \longrightarrow R' \propto M'_\alpha$$

in exactly the same way as we did above for (5.3) and we get that

$$L_R^{M_\alpha}(t) = L_{R'}^{M'_\alpha}(t) = L_{R''}^{M''_\alpha}(t) = \left(2 - t + \frac{t^{\alpha+1}}{1-t^\alpha}\right)\frac{1}{(1-t)^2}$$

is the same in all three cases (α integer $\geqslant 2$) but $M_\alpha(t) = 2 + 5t$, $R(t) = (1 + 2t)^2$, $M'_\alpha(t) = (2 + 3t - 3t^2)/(1 - t)$, $R'(t) = (1 + 2t)(1 + t + t^2)/(1 - t)$ and $M''_\alpha(t) = (2 + t - 4t^2 + 2t^3)/(1 - t)^2$, $R''(t) = (1 + t - t^2)^2((1 - t)^2$. Then we get as we said (5.2) by putting $x = z$, $y = 1$ in (4.6) for our R and M_α and the formula in the introduction by putting $x = z$, $y = 1$ in (4.6) for our R'' and M''_α.

To prove that there are pairs of modules of finite length over R' and R'' whose $\mathrm{Tor}(., -)$ series is irrational, we have to work much harder and prove more about Hochschild and cyclic homology. This will be done in the next section, where we will calculate explicitly the cyclic and Hochschild homology of $k[x,y]/(x^2,xy)$ i.e. in particular $\mathrm{Tor}_*^{R''}(M,M)$, where $R''=k[x,y,z,u]/(x^2,xy,zu,u^2)$ and $M=R''/(x-u,y-z)$. In this case M is not of finite length and we have to develop extra theory to handle M/z^2.

6. Applications of some explicit computations of Hochschild and cyclic homology

The main aim of this section is to prove the assertion about pairs of modules of finite length with irrational Tor series over the three Koszul algebras of embedding dimension 4 mentioned in Section 5. In doing so, we will first prove some results about Hochschild and cyclic homology and $\mathrm{Tor}(M,N)$ that are of independent interest.

Let us recall a few facts about Hochschild homology (for more details about results about Hochschild and cyclic homology used below we refer to the book by Loday [19]). We will only consider graded algebras R over a field k of characteristic 0. Let us recall that the ith Hochschild homology group of R, usually denoted by $\mathrm{HH}_i(R)$ is defined by

$$\mathrm{HH}_i(R)=\mathrm{Tor}_i^{R\otimes_k R^{\mathrm{op}}}(R,R),$$

where R^{op} is the ring with the opposite multiplication and where R is considered as an R bimodule in the natural way (thus an $R\otimes_k R^{\mathrm{op}}$-module).

It was proved by Connes that we have a natural exact sequence (for $i\geqslant 1$; for $i=1$ we should put $\mathrm{HC}_{i-1}(R)$ to the left in this sequence):

$$0\longrightarrow \overline{\mathrm{HC}}_{i-1}(R)\longrightarrow \mathrm{HH}_i(R)\longrightarrow \overline{\mathrm{HC}}_i(R)\longrightarrow 0,$$

where $\overline{\mathrm{HC}}_i(R)$ is the so-called reduced cyclic homology of R. Clearly, the Hochschild homology is bigraded and the cyclic homology is so too and this decomposition is compatible with the bigrading.

Recall that if

$$R=k[x_1,\ldots,x_n]/(x_1,\ldots,x_n)^2$$

then since R is commutative we have that

$$R\otimes_k R^{\mathrm{op}}=k[x_1,\ldots,x_n]/(x_1,\ldots,x_n)^2\otimes_k k[y_1,\ldots,y_n]/(y_1,\ldots,y_n)^2$$

is the algebra that we called earlier R_n. Furthermore

$$\mathrm{HH}_i(R)=\mathrm{Tor}_i^{R_n}(R_n/(x_1-y_1,\ldots,x_n-y_n),R_n/(x_1-y_1,\ldots,x_n-y_n)).$$

We calculated in [31, p. 181, Corollary 1, (i) and (iv)] the corresponding generating function for the Hochschild homology, which gives in particular the following explicit formula for the generating series in two variables of $\overline{\mathrm{HC}}_{*,*}(R)$:

$$\overline{\mathrm{HC}}_{*,*}(R)(x,y)=-\frac{1}{x}\sum_{t\geqslant 1}\frac{\phi(t)}{t}\,\log(1+n(-xy)^t), \tag{6.1}$$

where $\phi(t)$ is the Euler function and we proved in [31] that this series represents an irrational function for $n\geqslant 2$. In fact the proof of irrationality of Roos [31] can be generalized to a more general theorem that covers many more cases of irrationality. We will present this result as Theorem 7.1 in the next section, and here we just make a few preparations for using it. First, note that (6.1) shows that for the reduced cyclic homology of our special R only the terms of bidegree $(q-1, q)$ namely $\overline{\mathrm{HC}}_{q-1,q}(R)$ are possibly non-zero. Next, there is a general theorem due to Feigin and Tsygan [10] saying that for a general Koszul algebra A in characteristic 0 the graded reduced cyclic homology of A and its Koszul dual $A^!$ are related by the formula

$$\overline{\mathrm{HC}}_{n,q}(A) \simeq \overline{\mathrm{HC}}_{q-n-1,q}(A^!)^*, \tag{6.2}$$

where * means vector space dual (cf. [21] for the gradings). When

$$A = R = k[x_1, \ldots, x_n]/(x_1, \ldots, x_n)^2$$

our $\overline{\mathrm{HC}}_{q-1,q}(A)$ correspond to the dual $\overline{\mathrm{HC}}_{0,q}(A^!)$, and $A^!$ is the free algebra on n generators, i.e. $A^! = T(V)$, where V is a graded vector space of dimension n concentrated in degree 1 and therefore (6.1) gives for $x = 1$ and $y = z$ the generating series of $\overline{\mathrm{HC}}_{0,*}(T(V))$ in that case (note that $\overline{\mathrm{HC}}_{i,*}(T(V)) = 0$ for $i > 0$. Now, for *any* free algebra $T(V)$ over *any* positively graded vector space V we have $\overline{\mathrm{HC}}_{i,*}(T(V)) = 0$ for $i > 0$ and

$$\overline{\mathrm{HC}}_{0,*}(T(V))(z) = -\sum_{t=1}^{\infty} \frac{\phi(t)}{t} \log(1 - V((-1)^{t-1}z^t)), \tag{6.3}$$

where $V(z)$ is the generating series for the graded vector space V. This last formula is due to Clas Löfwall [21] (it is also cited in [25–27]). In the next Section 7, we will prove a general Theorem 7.1 giving conditions on $V(z)$ for (6.3) to be irrational. The conditions are that $V(z)$ is rational and that the coefficients of the Taylor development of $V'(z)/(1-V(z))$ satisfy certain inequalities (they are all automatically $\geqslant 0$). For $V(z) = nz$ these coefficients c_i in

$$\frac{V'(z)}{1-V(z)} = \sum_{i \geqslant 0} c_i z^i$$

are $c_i = n^{i+1}$ and they clearly satisfy the condition (A) of Theorem 7.1 if $n > 1$. Thus we have a "new" proof of irrationality of (6.1) for $n > 1$. We now treat another case, $R = k[x, y]/(x^2, xy)$ which is the "simplest" algebra that is not a complete intersection. Here

$$R^! = k\langle X, Y\rangle/(Y^2) = k\langle X\rangle \amalg \frac{k\langle Y\rangle}{(Y^2)} \tag{6.4}$$

Now, use the general formula for the cyclic homology of a coproduct [9] which in the special case at hand can be written (cf. also [21, Proposition 4]):

Proposition 6.1. *Let A and B be graded algebras and $A \amalg B$ their coproduct* (*over k*). *Then*

$$\overline{\mathrm{HC}}_n(A \amalg B) = \overline{\mathrm{HC}}_n(A) + \overline{\mathrm{HC}}_n(B)$$

for $n > 0$ and

$$\overline{\mathrm{HC}}_0(A \amalg B) = \overline{\mathrm{HC}}_0(A) + \overline{\mathrm{HC}}_0(B) + \overline{\mathrm{HC}}_0(T(A^+ \otimes B^+)), \tag{6.5}$$

where A^+ and B^+ are the positively graded parts of A and B and $T(A^+ \otimes B^+)$ is the tensor algebra of the tensor product (*over k*) *of the graded vector spaces A^+ and B^+.*

Now use Proposition 6.1 for the two different parts of $R^!$ in (6.4): $A = k\langle X\rangle$ *and* $B = (k\langle Y\rangle/(Y^2))$. The bigraded generating series for $\overline{\mathrm{HC}}_{*,*}(A)$ and $\overline{\mathrm{HC}}_{*,*}(B)$ are in this case rational (and well-known), and the "culprit" which is responsible for $\overline{\mathrm{HC}}_{0,*}(R^!)$ having an irrational series is the part $\overline{\mathrm{HC}}_0(T(A^+ \otimes B^+))$ of (6.5). In this case $V = A^+ \otimes B^+$ so that $V(z) = z^2/(1-z)$. In order to apply the Theorem 7.1, we have now to check the coefficients in the Taylor development of

$$\frac{V'(z)}{1-V(z)} = \frac{2z}{1-z-z^2} + \frac{z^2}{(1-z)(1-z-z^2)} = \sum_{i\geqslant 0} c_i z^i \tag{6.6}$$

and from (6.6) it is easy to see that the condition B of Theorem 7.1 is satisfied (for $n_0 = 1$). Thus the generating series for the Hochschild homology of $k[x, y]/(x^2, xy)$, i.e. the series for $S = k[x, y, z, u]/(x^2, xy, zu, u^2)$ and the S-module $M = S/(x-u, y-z)$:

$$\sum_{i\geqslant 0, j\geqslant 0} \dim_k(\mathrm{Tor}^S_{i,j}(M, M))x^i y^j \tag{6.7}$$

is an irrational function. However, M is not of finite length and we now want to reduce M correspondingly. Let us study $N = M/z^2$. I claim first that the map

$$\mathrm{Tor}^S_*(M, M) \longrightarrow \mathrm{Tor}^S_*(M, N) \tag{6.8}$$

is a monomorphism for $*\geqslant 2$. This follows directly from the commutative diagram where $\bar{S} = S/(y^2, z^2)$

$$\begin{array}{ccc} \mathrm{Tor}^S_*(M, M) & \longrightarrow & \mathrm{Tor}^S_*(M, N) \\ \downarrow & & \downarrow \\ \mathrm{Tor}^{\bar{S}}_*(N, N) & = & \mathrm{Tor}^{\bar{S}}_*(N, N), \end{array}$$

provided we can prove that the left vertical arrow is a monomorphism for $*\geqslant 2$. But this left vertical map is exactly the natural map between Hochschild homology groups (now $A = k[x, y]/(x^2, xy)$ and $\bar{A} = k[x, y]/(x^2, xy, y^2)$), so that $S = A \otimes A^{\mathrm{op}}$ and $\bar{S} = \bar{A} \otimes \bar{A}^{\mathrm{op}}$)

$$\mathrm{HH}_*(A) \longrightarrow \mathrm{HH}_*(\bar{A}).$$

But this map sits (for $n\geqslant 2$) as a middle vertical map in a diagram of two Connes exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \overline{\mathrm{HC}}_{n-1,q}(A) & \longrightarrow & \mathrm{HH}_{n,q}(A) & \longrightarrow & \overline{\mathrm{HC}}_{n,q}(A) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \overline{\mathrm{HC}}_{n-1,q}(\bar{A}) & \longrightarrow & \mathrm{HH}_{n,q}(\bar{A}) & \longrightarrow & \overline{\mathrm{HC}}_{n,q}(\bar{A}) & \longrightarrow & 0. \end{array}$$

Thus it is sufficient to prove that

$$\overline{\mathrm{HC}}_{n,q}(A) \longrightarrow \overline{\mathrm{HC}}_{n,q}(\bar{A})$$

is a monomorphism for $n \geqslant 1$ or (using the Feigin–Tsygan duality) that

$$\overline{\mathrm{HC}}_{q-n-1,q}(\bar{A}^!) \longrightarrow \overline{\mathrm{HC}}_{q-n-1,q}(A^!) \tag{6.9}$$

is an epimorphism for $n \geqslant 1$. But the only possible non-zero groups here are for $q = n + 1$ and $q = n + 2$ and the last case gives only a possible non-zero group for $A^!$. But if $q = n + 2$ and $n \geqslant 1$, then

$$\overline{\mathrm{HC}}_{1,n+2}(A^!) = \overline{\mathrm{HC}}_{1,n+2}(k\langle X\rangle) \oplus \overline{\mathrm{HC}}_{1,n+2}(k\langle Y\rangle(Y^2))$$

and the groups on the right-hand side are zero. Finally, we have to check that (6.9) is an epimorphism for $q = n + 1$, i.e. that

$$\overline{\mathrm{HC}}_{0,*}(k\langle X, Y\rangle) \longrightarrow \overline{\mathrm{HC}}_{0,*}(k\langle X, Y\rangle/(Y^2))$$

is an epimorphism. But this is part of Proposition 8 in [21] and also follows by direct inspection. Thus, we have now finished the proof that (6.8) is a monomorphism for $* \geqslant 2$. Consider now the exact sequence of S-modules (L is defined by this sequence and we recall that $N = M/z^2$):

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0. \tag{6.10}$$

Since (6.8) is a monomorphism for $* \geqslant 2$ it follows from the long exact sequence obtained when applying $\mathrm{Tor}^S_{*,*}(M, .)$ to (6.10) that if we can prove that the series for $\mathrm{Tor}^S_{*,*}(M, L)$ is rational then we get that the double series of $\mathrm{Tor}^S_{*,*}(M, N)$ is irrational.

Here is a very useful result which will be used to prove this (and other results) later:

Proposition 6.2. *Let A be an algebra (commutative) and let B^{op} be a quotient of A^{op}. Consider the map*

$$R = A \otimes_k A^{\mathrm{op}} \longrightarrow A \otimes_k B^{\mathrm{op}} = S$$

and consider A as an R-module by means of $A \otimes A^{op} \longrightarrow A$ and B as a quotient of that R-module. Then $\mathrm{Tor}^R_*(S, A) = 0$ *for* $* > 0$, *and* $\mathrm{Tor}^R_0(S, A) = B$.

Proof. This result is a special case of the standard isomorphism

$$\mathrm{Tor}^{A \otimes A^{\mathrm{op}}}(M \otimes_k N, A) \cong \mathrm{Tor}^A(M, N) \tag{6.11}$$

which holds over any k-algebra A for all right A-modules M and all left A-modules N : see for example Exercise 3 in Section 3 of Chapter VII of MacLane's book *Homology* (I owe this remark to the referee.) □

Corollary 6.3. *Under the hypothesis of the Proposition* 6.2, *the natural map*

$$\mathrm{Tor}^R_*(A, B) \longrightarrow \mathrm{Tor}^S_*(B, B)$$

is an isomorphism.

Proof of the Corollary 6.3. According to the Proposition 6.2 the natural change of rings spectral sequence:

$$E^2_{p,q} = \mathrm{Tor}^S_p(\mathrm{Tor}^R_q(S, A), B) => \mathrm{Tor}^R_n(A, B) \tag{6.12}$$

degenerates into the isomorphism $\mathrm{Tor}^R_n(A, B) \simeq \mathrm{Tor}^S_n(\mathrm{Tor}^R_0(S, A), B)$ and $\mathrm{Tor}^R_0(S, A) = B$. □

We will make three applications of this result:

Corollary 6.4. *Let $A = k[x, y]/(x^2, xy)$ so that $A^{\mathrm{op}} = k[z, u]/(zu, u^2)$ and let $B^{\mathrm{op}} = A^{\mathrm{op}}/(u)$ so that $R = A \otimes_k A^{\mathrm{op}} = k[x, y, z, u]/(x^2, xy, zu, u^2)$ and $S = A \otimes_k B^{\mathrm{op}} = R/(u)$. Then in Corollary 6.3, $A = R/(x - u, y - z)$ is the module we called M earlier and $B = R/(z, u, y - z)$, so the natural map*

$$\mathrm{Tor}^R_{*,*}(M, B) \longrightarrow \mathrm{Tor}^S_{*,*}(B, B) \tag{6.13}$$

is an isomorphism. But one sees easily that the L in the short exact sequence (6.10) is isomorphic to $s^{-2}B$ and (6.13) shows that the double series of $\mathrm{Tor}^R_{*,*}(M, L)$ is rational since the double series of $\mathrm{Tor}^S_{*,*}(B, B)$, i.e. the double series of $\mathrm{Tor}^{k[x,y,z]/(x^2,xy)}_{*,*}(B, B)$ is so.

Thus we have now finished the proof that the double series of $\mathrm{Tor}^R_{*,*}(M, N)$ is irrational. We now prove that $\mathrm{Tor}^R_{*,*}(M, N)$ is isomorphic to $\mathrm{Tor}^{R'}_{*,*}(N, N)$ where $R' = k[x, y, z, u]/(x^2, xy, z^2, zu, u^2)$ which solves the irrationality problem for ring R' too (N is of finite length over that ring):

Corollary 6.5. *Let $A = k[x, y]/(x^2, xy)$ so that $A^{\mathrm{op}} = k[z, u]/(zu, u^2)$ and let $B^{\mathrm{op}} = A^{\mathrm{op}}/(z^2)$ so that $R = A \otimes_k A^{\mathrm{op}} = k[x, y, z, u]/(x^2, xy, zu, u^2)$ and $S = A \otimes_k B^{\mathrm{op}} = R/(z^2)$. Then in Corollary 6.3, $A = R/(x - u, y - z)$ is the module we called M earlier and $B = R/(x, u, y - z, z^2) = M/z^2 = N$, and $S = R'$ and the natural map*

$$\mathrm{Tor}^R_{*,*}(M, N) \longrightarrow \mathrm{Tor}^{R'}_{*,*}(N, N) \tag{6.14}$$

is an isomorphism.

Thus the case R' is solved.

Finally, we have to prove that $\mathrm{Tor}^R_{*,*}(N, N)$ has an irrational series. Apply $\mathrm{Tor}^R_*(., N)$ to the exact sequence

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0.$$

We get an exact sequence

$$0 \longrightarrow \mathrm{Tor}^R_{*,*}(M, N) \longrightarrow \mathrm{Tor}^R_{*,*}(N, N) \longrightarrow \mathrm{Tor}^R_{*-1,*}(L, N) \longrightarrow 0, \tag{6.15}$$

where we have a monomorphism to the left since the isomorphism

$$\mathrm{Tor}^R_*(M, N) \simeq \mathrm{Tor}^{R'}_*(N, N)$$

factors in a natural way as a composition

$$\mathrm{Tor}_*^R(M, N) \longrightarrow \mathrm{Tor}_*^R(N, N) \longrightarrow \mathrm{Tor}_*^{R'}(N, N) \tag{6.16}$$

so that the left arrow in (6.16) is a monomorphism. Since, we know that $\mathrm{Tor}_{*,*}^R(M, N)$ has an irrational series we just have to prove that the series for $\mathrm{Tor}_{*-1,*}^R(L, N)$ in (6.15) is rational. But this is not difficult; indeed multiplication by z^2 on L is a monomorphism and applying $\mathrm{Tor}_*^R(N, .)$ on the corresponding exact sequence

$$0 \longrightarrow L \longrightarrow L \longrightarrow L/z^2 \longrightarrow 0$$

gives a long exact sequence which reduces to short exact sequences since $z^2 N = 0$. Now $\mathrm{Tor}_{*,*}^R(N, L/z^2)$ has a rational series.

7. The main result about irrational series

Here, we will use in an essential way the Skolem–Mahler–Lech theorem. Recall that this theorem says that the set of zeros of a linear recurrence sequence over a field of characteristic zero is the union of a finite set together with a finite set of arithmetical progressions. This was proved for the case of the rational field by Skolem, was generalized to any algebraic number field by Mahler and to any field of characteristic 0 by Lech ([15] also gave a counter-example in characteristic p). Later Mahler gave a different proof in the general case. For the references, cf. [8] and Remark 7.3 infra. The clue to all our proofs of irrationality of the Hochshild (and cyclic) homology series is the following theorem which uses the Skolem–Mahler–Lech theorem. The theorem can be generalized, but it covers most cases that we need.

Theorem 7.1 (*IRR*). *Let V be a strictly positively graded vector space (finite dimensional in each degree) over a field of characteristic* 0 *and let its generating series $V(z)$ be the expansion of a rational function* (*thus* $V(0) = 0$), *and assume that in the power series expansion*:

$$\frac{V'(z)}{1 - V(z)} = \sum_{i \geqslant 0} c_i z^i$$

(*where the c_i:s are automatically*) $\geqslant 0$) *we have one of two cases*:

Case A: $c_0 > 0$ *and* $c_i > c_0$ *for all even* $i > 0$.

Case B: $c_0 = 0$ *and* $c_i > 0$ *for all even i that are* $\geqslant i_0$, *for some* $i_0 > 0$.

Then the formal power series:

$$F(z) = -\sum_{n=1}^{\infty} \frac{\phi(n)}{n} \log(1 - V((-1)^{n-1} z^n)) \tag{7.1}$$

is an irrational function (here $\phi(n)$ is the Euler ϕ- function).

Proof. If $F(z)$ were rational $F'(z)$ would be so too. But

$$F'(z) = \sum_{n=1}^{\infty} \phi(n)(-1)^{n-1} z^{n-1} \frac{V'((-1)^{n-1}z^n)}{1 - V((-1)^{n-1}z^n)} \tag{7.2}$$

so that if we write

$$\frac{V'(z)}{1 - V(z)} = \sum_{i=0}^{\infty} c_i z^i, \tag{7.3}$$

we see that z times (7.2) can be written

$$zF'(z) = \sum_{n=1}^{\infty} \phi(n) \sum_{i=0}^{\infty} c_i((-1)^{n-1}z^n)^{i+1}. \tag{7.4}$$

Now put $G(z) = zF'(z)$. If $G(z)$ were rational then $(G(z) - G(-z))/2$ would be so too. But

$$\begin{aligned} \frac{G(z) - G(-z)}{2} &= \sum_{n \geqslant 1, n \text{odd}} \phi(n) \sum_{i \geqslant 0, i \text{even}} c_i z^{n(i+1)} \\ &= \sum_{N \geqslant 1, N \text{odd}} \left(\sum_{n|N} \phi(n) c_{\frac{N}{n}-1} \right) z^N. \end{aligned} \tag{7.5}$$

In order to analyze this we need a different reasoning in the two cases:

Case A: $c_0 > 0$ and all $c_i > c_0$ for $i > 0$ and i even.

Here the coefficient of z^N is (N is odd !)

$$\sum_{n|N} \phi(n) c_{\frac{N}{n}-1} = \phi(1) c_{N-1} + \sum_{n|N, n \neq 1} \phi(n) c_{\frac{N}{n}-1} \geqslant c_{N-1} + (N-1)c_0, \tag{7.6}$$

where we have used that $\sum_{n|N} \phi(n) = N$, that $\phi(1) = 1$ and that $c_i > c_0$ for i even and $i > 0$. Note also that we have equality in (7.6) if and only if N is an odd prime. Thus it follows that the *rational* function

$$\frac{G(z) - G(-z)}{2} - \sum_{N \geqslant 1, N \text{odd}} (c_{N-1} + (N-1)c_0) z^N$$

would have a Taylor expansion, where the coefficients are $\geqslant 0$ and are zero if and only if the indices are odd primes or even numbers. But, this contradicts the Skolem–Mahler–Lech theorem.

Case B: $c_0 = 0$ but from a certain even integer n_0 on, the c_i, where $i \geqslant n_0$ and i even are all strictly positive.

Now, since $c_0 = 0$, we have that the coefficient of z^N (N odd) in (7.5) is

$$\sum_{n|N} \phi(n) c_{\frac{N}{n}-1} = \phi(1) c_{N-1} + \sum_{n|N, n \neq 1, n \neq N} \phi(n) c_{\frac{N}{n}-1}$$

and here

$$\sum_{n|N, n\neq 1, n\neq N} \phi(n) c_{\frac{N}{n}-1} \tag{7.7}$$

is $\geqslant 0$ and is 0 (the empty sum) if N is a prime. But I claim that if N is not a prime, N odd and $N > n_0^2$ then this sum (7.7) is strictly positive. Indeed, if N is not a prime, then it can be written as a product $N = n_1 n_2$, where $n_1 > 1$ and $n_2 > 1$. Now at least one of these factors must be $> n_0$, for if both were $\leqslant n_0$, then we would have $N \leqslant n_0^2$, which contradicts our hypothesis. Thus if (say) $n_2 > n_0$, then $\phi(n_1)c_{n_2-1}$ occurs in the sum (7.7) and $n_2 - 1 \geqslant n_0$, so that c_{n_2-1} is strictly positive and therefore (7.7) is so too. Now we get that the *rational* function

$$\frac{G(z) - G(-z)}{2} - \sum_{N \geqslant 1, N \text{odd}} c_{N-1} z^N$$

has a Taylor expansion where the coefficients with index $i \geqslant n_0^2$ are $\geqslant 0$ and are zero if and only if i is even or an odd prime and again we obtain that rationality contradicts the Skolem–Mahler–Lech theorem and Theorem 7.1 (IRR) is proved. □

Remark 7.2. Some conditions like A or B are necessary, for if say $V(z) = z^t$ for a fixed $t \geqslant 1$ then the series in Theorem 7.1 is a rational function and A is violated if $t = 1$ and B is violated if $t \geqslant 2$. But for say $V(z) = 2z^t$ and $t \geqslant 1$ we still have irrationality, but none of the conditions A or B are satisfied for $t \geqslant 3$. But it we modify B to say that we want B after we have excluded the index from a finite number of arithmetical sequences then the irrationality proof still holds true for $V(z) = 2z^t$ and $t \geqslant 3$ (take away the finite set of arithmetic sequences $a + kt$ where $k \geqslant 1$ and $a = 0, \ldots, t-2$).

Remark 7.3. Theorem 7.1 generalizes a result that we proved in [31] and which was inspired by a result of Serre [36].

8. Some further explicit computations of Hochschild and cyclic homology

In this section, we do some calculations that indicate that all commutative Koszul algebras of embedding dimension $\leqslant 3$ which are not complete intersections have irrational Hochschild homology series (and also irrational cyclic homology series). We assume (to simplify) that the base field k is of characteristic 0 and is algebraically closed.

The embedding dimension 1 is trivial and it is well-known (cf. e.g. [6]) that the isomorphy classes of embedding dimension 2 algebras with quadratic relations that are not complete intersections are represented by $R_1 = k[x, y]/(x^2, xy, y^2)$ and $R_2 = k[x, y]/(x^2, xy)$. In the first case, we proved in [31] (and reproved in Section 6) that the Hochschild homology is irrational (even for any local (R, m) with $m^2 = 0$ and embedding dimension $\geqslant 2$). The second case was treated in Section 6 above. Now we pass to the embedding dimension 3 case. We will now deduce with the help of the paper [6] by Backelin and Fröberg (cf. [6, pp. 496–497], where also references to a paper by Emsalem and Iarrobino are given) that there are 23 different *isomorphy classes* of commutative Koszul algebras, which are not complete intersections $R=k[x,y,z]/(f_1,f_2,\ldots,f_t)$, where the f_i:s are quadratic forms. First it follows from [6] (Table 1 on p. 498) that there are 11 cases of such Koszul algebras when it comes to classification by Hilbert series or (equivalently) Poincaré–Betti series, namely (cf. [6]) 2b,3c,3d,3e,4a,4b,4c,4d,5a,5b,6a (here the first number in e.g. 4a means that it is a case where $t=4$, i.e. that we divide by 4 quadratic forms in (x,y,z)). However, we will see that these 11 cases split up into 23 isomorphy classes and *that the Hochschild* (*co*) *homology series is able to distinguish all these* 23 *isomorphy classes*! Furthermore, in probably all of these 23 cases the Hochschild homology series are irrational functions (but "close" to rational ones in the sense that only the first few terms in the series determine the whole series—this answers a question that was posed to me by Ralf Fröberg). We will treat 10 cases below with the help of our irrationality theorem in Section 7 and another five cases below—one of these cases is a result by Parhizgar—and we hope to return to the 8 cases left later.

We now turn to more details. We will label the 23 different cases A, B, C, ..., U, V, W. Thus in what follows we will study $R_? = k[x,y,z]/(I_?)$ where ? is one of $A,B,C,\ldots,U,V,W$ and where the different ideals $I_?$ are given as follows (the different isomorphy classes A and B (two quadratic forms in three variables) were of course known long ago):

Case A: $I_A=(x^2,xy)$.
Case B: $I_B=(xy,xz)$.
Case C: $I_C=(x^2,xy,z^2)$.
Case D: $I_D=(x^2,xy,(y+z)z)$.
Case E: $I_E=(xy,xz,x^2+y^2)$.
Case F: $I_F=(xy,xz,x^2+yz)$.
Case G: $I_G=(x^2,xy,y^2)$.
Case H: $I_H=(x^2,xy,yz)$.
Case I: $I_I=(xy,xz,yz)$.
Case J: $I_J=(x^2,xy,xz+y^2)$.
Case K: $I_K=(x^2,xy,xz)$.
Case L: $I_L=(x^2,xy,y^2,z^2)$.
Case M: $I_M=(x^2,xz,y^2,yz)$.
Case N: $I_N=(x^2,xy,xz,y^2)$.
Case O: $I_O=(xy,xz,yz,z^2)$.
Case P: $I_P=(y^2,xz,yz,xy-z^2)$.
Case Q: $I_Q=(x^2,z^2,y^2-xz,yz)$.
Case R: $I_R=(x^2,y^2,z^2,xz-yz)$.
Case S: $I_S=(x^2,y^2,xz-z^2,yz-z^2)$.
Case T: $I_T=(x^2,xy,xz,y^2,z^2)$.
Case U: $I_U=(x^2,xy,xz,y^2,yz)$.
Case V: $I_V=(x^2-y^2,y^2-z^2,xy,xz,yz)$.
Case W: $I_W=(x^2,xy,xz,y^2,yz,z^2)$.

Remark 8.1. The *isomorphy* classes C,D,E,F,G,H,I,J,K (three relations) were already mentioned in [6] and there they reduce to 3 cases when they study the Hilbert series. However, the Hochschild homology series are different in all these 9 cases. Indeed, the case 3c (Hilbert series $(1+2t-t^3)/(1-t)$) of Backelin and Fröberg [6] splits into the isomorphy classes C, D, E, F, the Case 3d (Hilbert series $(1+2t)/(1-t)$) of Backelin and Fröberg [6] splits into the isomorphy classes G, H, I, J and the Case 3e (Hilbert series $(1+t-2t^2+t^3)/(1-t)^2$) corresponds to case K here. All the four cases of 3c have *different* Hochschild and cyclic homology series, already for the first homology, indeed the generating series for the first reduced cyclic homology $\overline{\mathrm{HC}}_{1,*}(R_?)$ are for $? = C, D, E$ and F equal to

$$\frac{3t^2-t^3-t^4}{1-t}, \quad \frac{3t^2-2t^3-t^4}{1-t}, \quad \frac{3t^2-2t^3}{1-t} \quad \text{and} \quad \frac{3t^2-3t^3}{1-t}=3t^2,$$

respectively. The full series will be determined below for the case C. In the same way in the Case 3d the generating series for the first reduced cyclic homology $\overline{\mathrm{HC}}_{1,*}(R_?)$ for $? = G, H, I$ and J are equal to

$$\frac{3t^2}{1-t}, \quad \frac{3t^2-2t^3}{1-t}, \quad \frac{3t^2-3t^3}{1-t}=3t^2 \quad \text{and} \quad \frac{3t^2-t^3}{1-t},$$

respectively. The full series will be determined below for the cases G,H,I.

Remark 8.2. The cases L,M,N,O,P,Q,R,S with 4 relations correspond by duality to the classification of 2 "Hopf" quadratic forms in 3 non-commuting variables X, Y, Z. Here we have 8 cases (I thank Jörgen Backelin for help here):

X^2, Y^2, corresponding to (xy, xz, yz, z^2), i.e. case O,

$X^2, [X, Y]$, corresponding to (xz, yz, y^2, z^2), i.e. case N,

$X^2, [Y, Z]$, corresponding to (xy, xz, y^2, z^2), i.e. case M,

$X^2, [X, Y]+Z^2$, corresponding to $(xz, yz, y^2, xy-z^2)$, i.e. case P,

$[X, Y], [X, Z]$, corresponding to(x^2, y^2, z^2, yz), i.e. case L,

$[X, Y], [X, Z]+Y^2$, corresponding to (x^2, z^2, yz, y^2-xz), i.e. case Q,

$[X, Y], [X, Z]+[Y, Z]$, corresponding to $(x^2, y^2, z^2, xz-yz)$, i.e. case R,

$[X, Y], [X, Z]+[Y, Z]+Z^2$, corresponding to $(x^2, y^2, xz-z^2, yz-z^2)$, i.e. case S,

This should be compared to the classification of two quadratic forms in two commuting variables where we also have eight cases (cf. e.g. [37, p. 352]) but where the six cases (x^2, y^2), (x^2, yz), $(x^2, xy+z^2)$, $(xy, xz+y^2)$, $(xy, xz+yz)$, $(xy, xz+yz+z^2)$ give rise to complete intersections and can be ignored from our point of view. This is not so in the non-commutative Hopf case, thus we do have eight cases here, and whereas Backelin–Fröberg have only 3 cases 4a, 4b and 4c when it comes to classification with Hilbert series of the ring. Here are more details:

The case 4a has Hilbert series $1 + 3t + 2t^2$ and the cases L,Q,R,S have that Hilbert series too. However the first Hochschild *cohomology* of the corresponding ring are artinian modules and have Hilbert polynomials equal to

$$1 + 6t, \quad 2 + 6t, \quad 3 + 6t \quad \text{and} \quad 5 + 6t,$$

in cases S, R, Q and L, respectively. The Hochschild homology series will be determined in case L below. It is different from the corresponding series of Q,R and S (which probably are equal).

The case 4b has Hilbert series $(1 + 2t - t^2 - t^3)/(1 - t)$ and the cases M and P have that Hilbert series too. The whole Hochschild homology for the case M will be determined below (the case P seems to have the same Hochschild homology). But their first Hochschild *cohomology* series is different: it is $3 + 3t + t/(1 - t)$ in case M and $3 + 3t + 1/(1 - t)$ in case P.

The case 4c has Hilbert series $(1 + 2t - t^2)/(1 - t)$ and the cases O and N have that Hilbert series too. Their complete Hochschild homology will be termined below.

Remark 8.3. The cases T and V are non-isomorphic but have the same Hilbert series $1 + 3t + t^2$. They can, however, be distinguished already by their Koszul homology, since one of them is Gorenstein (case V) and the other one is not. All the three cases T,U,V, with five relations correspond dually to the classification of one quadratic "Hopf" form f in the three non-commuting dual variables X, Y, Z. Here, the cases $f = X^2$ and $f = X^2 + Y^2$ give dual rings isomorphic to R_U and R_T, respectively, and the case V corresponds to the dual ring for $f = X^2 + Y^2 + Z^2$.

We classify the cases according to the methods of proof:

(1) The cases B,H,I,K,M,N,O,T,U,W can all be treated using the Theorem 7.1 of Section 7. We will use the same method as we did when we solved the case $k[x, y]/(x^2, xy)$ in Section 6. Let us recall that if a graded algebra C is of the form $A \coprod B$ then

$$\overline{\mathrm{HC}}_{n,*}(C) = \overline{\mathrm{HC}}_{n,*}(A) + \overline{\mathrm{HC}}_{n,*}(B)$$

for $n > 0$ and

$$\overline{\mathrm{HC}}_{0,*}(C) = \overline{\mathrm{HC}}_{0,*}(A) + \overline{\mathrm{HC}}_{0,*}(B) + \overline{\mathrm{HC}}_{0,*}(T(A^+ \otimes B^+)), \tag{8.1}$$

and furthermore if D is a Koszul algebra, then according to the result by Feigin and Tsygan cited earlier

$$\overline{\mathrm{HC}}_{n,q}(C) \simeq \overline{\mathrm{HC}}_{q-n-1,q}(C^!)^*, \tag{8.2}$$

where $C^!$ is the Koszul dual and * means vector space dual (cf. [21] for the gradings).

We will see that in the 10 cases when ? is one of $B, H, I, K, M, N, O, T, U, W$ we have that $C = R_?^!$ can be decomposed into a direct sum $A \coprod B$ where A and B have rational cyclic and Hochschild homology series (U and W are exceptional, cf. below), but where the "culprit", causing the cyclic and Hochschild homology series to be irrational is $\overline{\mathrm{HC}}_{0,*}(T(A^+ \otimes B^+))$, i.e. the reduced cyclic homology series of $T(V)$, where the graded vector space V is given by $V = A^+ \otimes_k B^+$. We then have to prove that Theorem 7.1 can be used in all the 10 cases B,H,I,K,M,N,O,T,U,W:

Case B: The Koszul dual of R_B is

$$R_B^! = \frac{k\langle X, Y, Z\rangle}{(X^2, Y^2, Z^2, [X, Z])} = \frac{k\langle Y\rangle}{(y^2)} \coprod \frac{k\langle X, Z\rangle}{(X^2, Z^2, [X, Z])} = A \coprod B.$$

Here $A = (k\langle Y\rangle/(y^2))$ and $B = (k\langle X, Z\rangle)/((X^2, Z^2, [X, Z]))$ which are both Koszul duals of regular ring $k[y]$ and $k[x, z]$ which are as nice as you could wish. But $V(z) = (A^+ \otimes_k B^+)(z) = 2z^2 + z^3$ and

$$\frac{V'(z)}{1 - V(z)} = \frac{4z + 3z^2}{1 - 2z^2 - z^3} = \frac{1 + 2z}{1 - z - z^2} - \frac{1}{1 + z} = \sum_{i \geqslant 0} c_i z^i \tag{8.3}$$

and from this it is clear that the c_i satisfy the condition B of Theorem 7.1.

Case H: In this case the Koszul dual of R_H is

$$A \coprod B = \frac{k\langle Y\rangle}{(Y^2)} \coprod \frac{k\langle X, Z\rangle}{(Z^2, [X, Z])}$$

and $A^!$ is regular and $B^!$ is a complete intersection thus innocent from the rational series point of view. But $V(z) = (A^+ \otimes_k B^+)(z) = 2z^2/(1 - z)$ so that

$$\frac{V'(z)}{1 - V(z)} = \frac{4z - 2z^2}{1 - 2z - z^2 + 2z^3} = \frac{2}{1 - 2z} - \frac{2}{1 - z^2} = \sum_{i \geqslant 0} c_i z^i \tag{8.4}$$

and from this it is again clear that the c_i satisfy the condition B of Theorem 7.1.

Case I: Here

$$R_I^! = \frac{k\langle X\rangle}{(X^2)} \coprod \frac{k\langle Y, Z\rangle}{(Y^2, Z^2)} = A \coprod B,$$

where it is again clear that $A^!$ is regular and $B^!$ is a complete intersection. Furthermore, the $V(z)$ is the same as in case H and thus we have an irrational series here.

Case K: Now

$$R_K^! = k\langle X\rangle \coprod \frac{k\langle Y, Z\rangle}{(Y^2, [Y, Z], Z^2)} = A \coprod B,$$

where $A^!$ and $B^!$ are both regular and $V(z) = (2z^2 + z^3)/1 - z$ so that

$$\frac{V'(z)}{1 - V(z)} = \frac{1 + 4z + 3z^2}{1 - z - 2z^2 - z^3} - \frac{1}{1 - z}$$

and the irrationality follows again from case B of Theorem 7.1.

Case M: Now

$$R_M^! = \frac{k\langle X\rangle}{(X^2)} \coprod \frac{k\langle Y, Z\rangle}{([Y, Z])} = A \coprod B$$

and $A^!$ and $B^!$ are both regular and $V(z) = 2z^2 - z^3)/(1-z)^2$ so that

$$\frac{V'(z)}{1-V(z)} = \frac{2+2z-3z^2}{1-2z-z^2+z^3} - \frac{2}{1-z}$$

and we can again apply part B of Theorem 7.1.

Case N: Now

$$R_N^! = \frac{k\langle X,Y\rangle}{(X^2,[X,Y])} \coprod k\langle Z\rangle = A \coprod B$$

and $A^!$ and $B^!$ are complete intersections and $V(z) = 2z^2/(1-z)^2$ so that

$$\frac{V'(z)}{1-V(z)} = \frac{4z}{(1-z)(2-2z-z^2)},$$

to which Theorem 7.1 applies.

Case O: Now

$$R_O^! = \frac{k\langle X,Y\rangle}{(X^2,Y^2)} \coprod k\langle Z\rangle = A \coprod B$$

and the case is very similar to case N, with the same $V(z)$ (but cases N and O can be separated by the Hochschild *co*homology series, cf. remarks above.)

Case T: Here

$$R_T^! = k\langle X\rangle \coprod \frac{k\langle Y,Z\rangle}{([Y,Z])} = A \coprod B$$

and $A^!$ and $B^!$ are complete intersections and $V(z) = (2z^2 - z^3)/(1-z)^3$ so that

$$\frac{V'(z)}{1-V(z)} = \frac{3-2z}{1-3z+z^2} - \frac{3}{1-z}$$

and one checks that part B of Theorem 7.1 can again be applied.

Case U: Here

$$R_U^! = k\langle X,Y\rangle \coprod \frac{k\langle Z\rangle}{(Z^2)} = A \coprod B$$

and here $B^!$ is regular, *but* A is the free associative algebra on two generators and thus has irrational cyclic homology. But, here everything can be done very explicitly and we have $V(z) = 2z^2/(1-2z)$ and irrationality follows again.

Case W: Here $R_W^! = k\langle X,Y,Z\rangle$ and we treated this case already in [31].

Thus, we have finished the discussion of the 10 cases B,H,I,K,M,N,O,T,U,W.

We will now treat the 4 cases A,C,G,L. The corresponding $R^!$ are all of the form $A \otimes_k B$, where A has irrational cyclic homology and where B has rational cyclic homology. We can now use the Künneth Theorem 4.3.11 on p. 131 of Loday [19] to obtain irrationality (note that in our case the S of Loday [19] is zero).

Among the remaining 9 cases, already V has been treated is Parhizgar [25,26] (and Löfwall): Here

$$R_V^! = \frac{k\langle X, Y, Z\rangle}{(X^2 + Y^2 + Z^2)} = k\langle X, Y\rangle \coprod_{k[S]} k\langle Z\rangle,$$

where $k[S]$ is the free algebra on one generator S in degree 2 and where $k[S]$ is embedded in $k\langle X, Y\rangle$ and $k\langle Z\rangle$ by means of $S \to X^2 + Y^2$ and $S \to -Z^2$, respectively. Thus, we have a coproduct over a non-trivial algebra (in this case $k[S]$) and not the trivial one k but they were able to handle this case. If there were a generalization of Conjecture 1 of [26, p. 3662] (inspired by Chapter 3 of Feigin and Tsygan [9]) to a result that more generally expresses $\overline{\mathrm{HC}}(A \amalg_C B)$ in terms of $\overline{\mathrm{HC}(\mathrm{A})}$, $\overline{\mathrm{HC}(\mathrm{B})}$ and $\overline{\mathrm{HC}(\mathrm{C})}$ and a *Nil*-functor to the case when C is a (say) sub Hopf algebra of A and B and if furthermore there were some kind of Künneth formula for semi-tensor products of Hopf algebras then we would be able to treat the remaining 8 cases too. We hope to return to these cases and also to the two unique non-Koszul cases mentioned in [6], p. 496, in a later publication.

9. Open problems and conjectures

Problem 9.1. Let R be a quadratic algebra which is not a Koszul algebra (over a field of characteristic 0, say), and let $R^!$ be the Koszul dual of R. Find a generalization of the isomorphism between the reduced cyclic homology of R and $R^!$

$$\overline{\mathrm{HC}}_{n,q}(R) \simeq \overline{\mathrm{HC}}_{q-n-1,q}(R^!)^*, \tag{9.1}$$

proved by Feigin and Tsygan for the case of Koszul algebras. There probably is a very nice generalization if R is commutative and satisfies the condition L_3 [33] which involves the cyclic homology of a free algebra $T(V)$ which is trivial if R is Koszul [20].

Problem 9.2. Let R be an algebra, M an R-bimodule and $R \propto M$ the trivial extension of R by means of M. Calculate the Hochschild homology (and cyclic homology) of $R \propto M$ in terms of invariants of R and M, in particular try to get explicit information about the invariants $\mathrm{Nil}^+(R, M)$ of p. 100 of Feigin and Tsygan [9].

Problem 9.3. Try to prove the Conjecture 1 in [26] in the more general setting mentioned at the end of Section 8.

Finally, there is the Question already mentioned in Section 5,c) of Roos [35].

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