De Rham cohomology

Let M be a smooth (often orientable, but not always) manifold of dimension n.

Tangent bundle

Definition: The *tangent space* T_pM at p is defined as

$$T_pM = \{\text{smooth paths } \alpha\colon (-1,1) \to M \mid \alpha(0) = p\}/\sim$$

where $\alpha \sim \beta$ iff $(\varphi \circ \alpha)'(0) = (\varphi \circ \beta)'(0)$ where φ is some chart on a neighbourhood of p.

Lemma: T_pM is a vector space of dimension n.

Definition: The *pushfoward* of a smooth map $f\colon M \to N$ of smooth manifolds is the map $f_*\colon T_pM \to T_{f(p)}N$ given by $f_*([\alpha]) = [f \circ \alpha]$.

Lemma: Given a chart (U, φ) around some point p we have a basis of T_pM consisting of elements $\frac{\partial}{\partial x_i}|_p = \varphi_*(e_i)$ where $\{e_i\}$ is the canonical basis of $\mathbb{R}^n$.

Definition: The *tangent bundle* TM is defined as $\coprod_{p \in M} T_pM$.

Lemma: The map $\pi\colon TM \to M$ given by $\pi(T_pM) = p$ gives TM the structure of a smooth vector bundle over M of rank n.

Cotangent bundle

Definition: The *cotangent space* T_p^*M at p is defined as the vector-space dual $(T_pM)^*$.

Definition: Given the basis $\left\{\frac{\partial}{\partial x_i}|_p\right\}$ of T_pM we have the dual basis $\{\mathrm{d}x^i|_p\}$ of T_p^*M consisting of *differentials*.

Definition: The *pullback* of a smooth map $f\colon M \to N$ of smooth manifolds is the map $f^*\colon T_{f(p)}^*N \to T_p^*M$ given by

$$f^*(\vartheta)\left(\sum_i \lambda_i \frac{\partial}{\partial x_i}|_p\right) = \vartheta\left(f_* \sum_i \lambda_i \frac{\partial}{\partial x_i}|_p\right).$$

Example: Let $f\colon M \to \mathbb{R}$ be smooth. Then $f_*\colon T_pM \to T_{f(p)}\mathbb{R} = \mathbb{R}$ and so $f_* \in T_p^*M$. Further, $f_* = \sum_i \frac{\partial \hat{f}}{\partial x_i} \mathrm{d}x^i|_p$ where $\hat{f}\colon \mathbb{R}^n \to \mathbb{R}$ is the local representation of f.

Definition: The *cotangent bundle* T^*M is defined as $\coprod_{p \in M} T_p^*M$.

Lemma: The cotangent bundle is a smooth vector bundle of M of rank n.

Vector fields and tensors

Definition/Lemma: A section of a smooth vector bundle on M is a *vector field* on M.

Definition: Given a vector space V with basis $\{v_1, \ldots, v_n\}$ we define the *space of alternating r-tensors* as the vector space $\bigwedge^r V$ with basis

$$\{v_{i_1}^* \wedge \ldots \wedge v_{i_r}^* \mid i_1 < \ldots < i_r\}$$

where the wedge product $\wedge$ satisfies $a \wedge b = -b \wedge a$ and $a \wedge a = 0$.

Note: We can give a better definition using general tensors, but we don't. Just note that an element of $\bigwedge^r V$ is a multilinear $V^r \to \mathbb{R}$.

Definition: Given a smooth $f\colon M \to N$ of smooth manifolds we define, for all r, the *pullback* $f^*\colon \bigwedge^r(T_{f(p)}N) \to \bigwedge^r(T_pM)$ given by

$$f^*(\vartheta)(X_1, \ldots, X_n) = \vartheta(f_*X_1, \ldots, f_*X_n)$$

where $X_i \in T_pM$.

Definition: The *(covariant) alternating r-tensor bundle* $\bigwedge^r TM$ is defined as $\coprod_{p\in M} \bigwedge^r(T_pM)$ and is a smooth vector bundle over M.

Differential forms

Definition: A *differential r-form* is a smooth section of $\bigwedge^r T^*M$. The space of all differential r-forms on M is denoted $\Gamma^r(M)$.

Note: A *differential* 1*-form* is a smooth section of the cotangent bundle, and a differential r-form is in particular a map $(T_pM)^r \to \mathbb{R}$.

Definition: Given a smooth section X of T^*M and $\omega \in \Gamma^r(M)$ we define the *contraction (with X)* as the differential $(r-1)$-form $i_X\omega$ given by

$$(i_X\omega)(X_1, \ldots, X_{r-1}) = \omega(X, X_1, \ldots, X_{r-1}).$$

Lemma: $i_X \circ i_X = 0$.

Lemma: Let $\sigma \in \Gamma^r(M)$ and $\omega \in \Gamma^s(M)$. Then

$$i_X(\sigma \wedge \omega) = (i_X\sigma) \wedge \omega + (-1)^r \sigma \wedge (i_X\omega).$$

Definition: Let $I = (i_1, \ldots, i_r)$. Then we write $\omega_I \mathrm{d}x^I$ to mean $\omega_{i_1\ldots i_r}\mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_r}$.

Lemma: For all r there exist unique linear maps $\mathrm{d}^r\colon \Gamma^r(M) \to \Gamma^{r+1}(M)$ such that, for all $f \in \Gamma^0(M)$, $\omega \in \Gamma^r(M)$, and $\sigma \in \Gamma^s(M)$,

- $\mathrm{d}^0 f = f_*$;
- $\mathrm{d}^{r+1} \circ \mathrm{d}^r = 0$;
- $\mathrm{d}^{r+s}(\omega \wedge \sigma) = \mathrm{d}^r \omega \wedge \sigma + (-1)^r \omega \wedge \mathrm{d}^s \sigma$.

Further, d is given by $\mathrm{d}(f \mathrm{d}x^I) = \mathrm{d}f \wedge \mathrm{d}x^I$.

Lemma: The differential d^r commutes with pullbacks, i.e. for smooth $f\colon M \to N$ and $\omega \in \Gamma^r(N)$ we have that $f^* \mathrm{d}^r \omega = \mathrm{d}^r f^* \omega$.

Integration over top forms

Definition: Given an atlas $(U_\alpha, \varphi_\alpha)_{\alpha \in I}$ on a smooth manifold M, we say that $\{\psi_\alpha\}_{\alpha \in I}$ is a *corresponding partition of unity* if (for all $\alpha \in I$) all of the following hold:

- $f_\alpha\colon M \to \mathbb{R}$ is smooth;
- $\operatorname{supp} f_\alpha \subseteq U_\alpha$;
- $\sum_{\alpha \in I} f_\alpha = 1$.

Definition: A differential form is said to be a *top form* if it lives in the highest-degree non-zero $\Gamma^k(M)$, i.e. $\Gamma^{\dim M}(M)$ when M is a smooth manifold.

Definition: Let $U \subset \mathbb{R}^n$ be open and $f \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n$ a compactly-supported top form on U. Then we define

$$\int_U f \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n = \int_U f \mathrm{d}x_1 \ldots \mathrm{d}x_n$$

where the right-hand side is Lebesgue integration.

Definition: Let M be a smooth n-dimensional orientable manifold and $\omega \in \Gamma^n(M)$ a compactly-supported top form. Let $(U_i, \varphi_i)_{i=1,\ldots,m}$ be the charts corresponding to the support of ω (by compactness) and let $\{\psi_i\}$ be a corresponding partition of unity. Then we define

$$\int_M \omega = \sum_{i=1}^m \int_{U_i} \psi_i \omega = \sum_{i=1}^m \int_{\varphi_i(U_i)} (\varphi_i^{-1})^* (\psi_i \omega).$$

Definition: Let M be an orientable smooth n-manifold. Then a top form ω is said to be an *orientation form* if it is non-vanishing and positively-oriented at each point of M, i.e. its value evaluated at a positively-oriented basis at any point is positive.

Lemma: Let M be an orientable smooth n-manifold. Then there exists an orientation form ω_0. Further,

$$\int_M \omega_0 > 0.$$

De Rham cohomology

Let M be a smooth orientable n-manifold.

Definition: The *de Rham complex* of M is the cochain complex $(\Gamma^\bullet(M), \mathrm{d}^\bullet)$. The *de Rham cohomology* of M is the cohomology of this complex, written $\mathrm{H}^\bullet_{\mathrm{dR}}(M)$.

Definition: A differential k-form ϑ is said to be *closed* if $\mathrm{d}^k\vartheta = 0$ (i.e. if it is a cycle) and *exact* if there exists some $(k-1)$-form ω such that $\vartheta = \mathrm{d}^{k-1}\omega$ (i.e. if it is a boundary).

Lemma: The pullback f^* of a smooth $f\colon M \to N$ induces a map $f^*\colon \mathrm{H}^\bullet_{\mathrm{dR}}(N) \to \mathrm{H}^\bullet_{\mathrm{dR}}(M)$.

Lemma: The de Rham cohomology of M is invariant under (smooth) homotopy, i.e. if $M \simeq N$ then $\mathrm{H}^\bullet_{\mathrm{dR}}(M) \cong \mathrm{H}^\bullet_{\mathrm{dR}}(N)$. In particular, if $f, g\colon M \to N$ are smooth and (smoothly) homotopic then $f^* = g^*$.

Lemma: Let M be a smooth connected compact orientable n-manifold. Then integration of top forms gives an isomorphism $\int_M\colon \mathrm{H}^n_{\mathrm{dR}}(M) \xrightarrow{\sim} \mathbb{R}$. If M is instead non-orientable then $\mathrm{H}^n_{\mathrm{dR}}(M) = 0$

Stokes' theorem

Let M be a smooth orientable n-manifold.

Definition: Let $\sigma\colon \Delta_p \to M$ be a smooth singular p-simplex and $\omega \in \Gamma^p(M)$ a closed p-form. Then we define

$$\int_\sigma \omega = \int_{|\Delta_p|} \sigma^*\omega$$

where the right-hand side is the integral of a top form over the submanifold (with corners) $|\Delta_p| \subset \mathbb{R}^p$. We can extend this linearly to any smooth p-chain $\sum_{i=1}^k \lambda_i\sigma_i$.

Lemma: Let c be a smooth p-chain in M and $\omega \in \Gamma^{p-1}(M)$. Then

$$\int_{\partial c} \omega = \int_c \mathrm{d}\omega$$

where ∂ is the simplicial boundary map.

Proof: We only need to prove the statement for smooth p-simplices, since the result then follows by linearity, so let σ be a smooth p-simplex. Now

$$\int_\sigma \mathrm{d}\omega = \int_{\Delta_p} \sigma^*\mathrm{d}\omega = \int_{\Delta_p} \mathrm{d}\sigma^*\omega = \int_{\partial\Delta_p} \sigma^*\omega$$

by commutativity of the differential and pullbacks, and by definition. Next, using the definition of the coboundary map ∂ we see that

$$\int_{\partial\Delta_p} \sigma^*\omega = \int_{\sum_{i=0}^{p}(-1)^i \Delta_p \circ F_{i,p}} \sigma^*\omega$$

where $F_{i,p} \colon \Delta_{p-1} \to \Delta_p$ is just $[0, 1, \ldots, \hat{i}, \ldots, p]$, i.e. the i-th face map. Then

$$\int_{\sum_{i=0}^{p}(-1)^i \Delta_p \circ F_{i,p}} \sigma^*\omega = \sum_{i=0}^{p}(-1)^i \int_{\Delta_p \circ F_{i,p}} \sigma^*\omega = \sum_{i=0}^{p}(-1)^i \int_{\Delta_{p-1}} F_{i,p}^*\sigma^*\omega.$$

Finally,

$$\sum_{i=0}^{p}(-1)^i \int_{\Delta_{p-1}} F_{i,p}^*\sigma^*\omega = \sum_{i=0}^{p}(-1)^i \int_{\Delta_{p-1}} (\sigma \circ F_{i,p})^*\omega = \sum_{i=0}^{p}(-1)^i \int_{\sigma \circ F_{i,p}} \omega = \int_{\partial\sigma} \omega.$$