Introduction to ∞-categories

Talk by Marco Robalo at DAGIT 2017
Typed by Timothy Hosgood

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Abstract

These are a copy of my notes on a talk given by Marco Robalo at Derived Algebraic Geometry in Toulouse (DAGIT) 2017: the content is purely his; the mistakes are all mine.

1 Motivation

1.1. Idea. An ∞**-category** consists of

- objects;
- 1-morphisms between objects;
- n-morphisms between $(n-1)$-objects (for $n \geqslant 2$);
- composition laws for n-morphisms ($n \geqslant 1$) defined up to higher morphisms;
- associativity of compositions up to homotopy.

1.2. Proto-example. (Fundamental ∞-groupoid) For a CW-complex X we have

- objects = points;
- 1-morphisms = homotopies;
- 2-morphisms = homotopies of homotopies;
- ... and so on.

1.3. Problem. No direct definition that is operational and simultaneously close to our intuition/desire (infinitely many axioms!).

1.4. Solution. Find a model category whose objects serve as models for ∞-categories.

1.5. Modelling. Many classical examples:

- homotopy types can be modelled by topological spaces, simplicial sets, categories, etc.;
- homotopy theory of homotopy-commutative $\mathbb{Q}$-algebras can be modelled by dg-algebras;
- derived stacks can be modelled by simplicial presheaves.

1.6. Question. Why so many models?

1.7. Answer. Dwyer-Kan localisation: every model category has an associated ∞-category that captures all the important information.

1.8. Question. If we have models then why care about ∞-categories?

1.9. Answer. Many reasons:

- not all ∞-categories have a model presentation;
- no 'good enough' definition of functors that relate different models (need an ∞-functor between the associated ∞-categories);
- models for diagrams are not always given by diagrams of models;
- proofs and statements become 'simpler'.

2 Preliminary definitions

2.1. Category of simplices. Write Δ to be the **category of simplices**:

- $\mathrm{ob}(\Delta) = \{[n]\}_{n\in\mathbb{N}}$ where $[n] = \{0 < 1 < \ldots < n\}$ is the ordered set of natural numbers up to n;
- $\mathrm{Hom}_\Delta([m],[n])$ is the set of order-preserving maps from $[m]$ to $[n]$.

2.2. Simplicial notation. We use the following notation:

- $\mathsf{sSet} = \mathsf{Set}^{\Delta^{\mathrm{op}}} = \mathsf{Fun}(\Delta^{\mathrm{op}}, \mathsf{Set})$;
- $\Delta[n] = \mathrm{Hom}_\Delta(-,[n]) \in \mathsf{sSet}$;
- $S_n = \mathrm{Hom}_{\mathsf{sSet}}(\Delta[n], S)$ for $S \in \mathsf{sSet}$;
- $\Lambda_n^i = \Delta[n] \setminus \{\text{interior and the face opposite the } j\text{-th vertex}\}$ is the i**-th horn** (for $n \geqslant 2$).

$$\Delta[2] = \begin{tikzcd} & 1 \arrow[dr] \arrow[d, phantom, "\Downarrow"] & \\ 0 \arrow[ur] \arrow[rr] & {} & 2 \end{tikzcd} \qquad \Lambda_2^1 = \begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[ur] & & 2 \end{tikzcd}$$

2.3. Nerve. The **nerve** $N(\mathcal{C})$ of a category $\mathcal{C}$ is the simplicial set with

- n-simplices given by $(x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \ldots \xrightarrow{f_n} x_n)$ in $\mathcal{C}$;
- boundary maps given by composition (or forgetting the first/last object and morphism for the two edge cases);
- degeneracy maps given by inserting the identity.

2.4. Note. There is a set-bijection

$$\{\text{functors } \mathcal{C} \to \mathcal{D}\} \simeq \{\text{simplicial maps } N(\mathcal{C}) \to N(\mathcal{D})\}.$$

2.5. Lemma. There is an equivalence of categories $X \simeq N(\mathcal{C})$ if and only if all *inner* horns lift *uniquely*.

$$\begin{tikzcd} \Lambda^i_n \arrow[r] \arrow[d, hook] & X \\ \Delta[n] \arrow[ur, dashed, "\exists!"'] & \end{tikzcd} \qquad \text{for all } i \in \{1, \ldots, n-1\}$$

2.6. Composition. For example, "Λ^1_2 gives composition".

$$\begin{tikzcd} & x_1 \arrow[dr, "f_2"] & \\ x_0 \arrow[ur, "f_1"] \arrow[rr, dashed, "\exists! f_2 \circ f_1"] & & x_2 \end{tikzcd}$$

2.7. Associativity. As another example, "Λ^1_3 gives associativity":

$$x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} x_2 \xrightarrow{f_3} x_3 \text{ in } \mathcal{C}$$

corresponds to

$$\Lambda^1_2 \xrightarrow{(f_2,f_1)} N(\mathcal{C}) \quad \text{and} \quad \Lambda^1_2 \xrightarrow{(f_3,f_2)} N(\mathcal{C})$$

which generate compositions

$$\Delta[2] \xrightarrow{(f_2,f_1)} N(\mathcal{C}) \quad \text{and} \quad \Delta[2] \xrightarrow{(f_3,f_2)} N(\mathcal{C}).$$

So $(f_3 \circ f_2) \circ f_1 = f_3 \circ (f_2 \circ f_1)$ if and only if we can 'fill the back face of the tetrahedron with vertices x_0, x_1, x_2, and x_3', i.e. if and only if we can extend $\Lambda^1_3 \to N(\mathcal{C})$ to $\Delta[3] \to N(\mathcal{C})$.

2.8. Exercise. What does the lifting of Λ^2_3 tell us?

2.9. Summary. The lifting property for inner horns ($0 < i < n$) gives composition and associativity laws; for outer horns ($i = 0, n$) it gives inverses.

$$\begin{tikzcd} & x_0 \arrow[dr, dashed, "\exists!"] & \\ x_0 \arrow[ur, "\mathrm{id}_{X_0}"] \arrow[rr, "f_1"] & & x_1 \end{tikzcd}$$

2.10. Kan complex. If $X \in$ sSet is such that $X \simeq \mathrm{Sing}(T)$, where $\mathrm{Sing}(T)$ consists of singular simplices in a topological space T (i.e. continuous maps $|\Delta^n| \to T$) then we call it a **Kan complex**. Note that X is a Kan complex if and only if *all* horns lift, but *not necessarily* uniquely.

3 Quasi-categories

3.1. Quasi-category. A **quasi-category** is a simplicial set $\mathcal{C}$ such that all *inner* horns lift, but *not necessarily* uniquely. This notions lies in between that of a Kan complex and that of the nerve of a category: we get compositions that aren't unique, but their non-uniqueness is controlled by higher homotopy data. For example, consider

$$x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} x_2 \xrightarrow{\mathrm{id}_{x_2}} x_2$$

which gives us two maps $u_1, u_2 \colon \Delta[2] \to \mathcal{C}$. Similarly, for 'associativity' we can use the lifting of Λ^1_3 as before, after filling one face with the identity.

3.2. ∞-category. We can use quasi-categories as a model for ∞**-categories**: define the objects of a quasi-category $\mathcal{C}$ to be the 0-simplices, and the n-morphisms to be the n-simplices.

3.3. ∞-functor. Since functors are 'maps that preserve commutative diagrams', it makes sense to define an ∞**-functor** to be a map of simplicial sets between quasi-categories, since these send n-simplices to n-simplices and preserve boundaries.

3.4. Homotopy category. Given an ∞-category $\mathcal{C}$ we define its **homotopy category** $\mathrm{h}\mathcal{C}$ to be the (1-)category with

- $\mathrm{ob}(\mathrm{h}\mathcal{C}) = \mathrm{ob}(\mathcal{C})$;
- $\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(x, y) = \mathrm{Hom}_{\mathcal{C}}(x, y) \,/\, \sim$, where $f \sim g$ if there exists a 2-morphism $u\colon \Delta[2] \to \mathcal{C}$ with boundary $(\mathrm{id}_y - g + f)$.

$$\begin{tikzcd} & y \arrow[dr, "\mathrm{id}_y"] \arrow[d, phantom, "\Downarrow u"] & \\ x \arrow[ur, "f"] \arrow[rr, "g"'] & \, & y \end{tikzcd}$$

Note that compositions are unique (and thus well defined) thanks to the lifting property, i.e. u_1, u_2 are identified in the homotopy category.

3.5. Subcategory. An ∞**-subcategory** $\mathcal{C}'$ of an ∞-category $\mathcal{C}$ is a sub-simplicial set obtained as a fibre product in sSet:

$$\begin{tikzcd} \mathcal{C}' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{C} \arrow[d] \\ N(\mathcal{D}) \arrow[r] & N(\mathrm{h}\mathcal{C}) \end{tikzcd}$$

where $\mathcal{D}$ is a subcategory of $\mathrm{h}\mathcal{C}$.

3.6. Equivalence. A 1-morphism f in $\mathcal{C}$ is called an **equivalence** if $[f]$ in $\mathrm{h}\mathcal{C}$ is an isomorphism.

3.7. ∞-groupoid. An ∞-category where *all* 1-morphisms are equivalences is called an ∞**-groupoid**.

3.8. Proposition. An ∞-category is an ∞-groupoid if and only if it is a Kan complex.

3.9. Example. For T a topological space, $\mathrm{Sing}(T)$ is an ∞-groupoid

4 Simplicial nerve and rectification

4.1. Mapping space. Let x, y be objects in an ∞-category $\mathcal{C}$. We define the **mapping space** $\mathrm{Map}_{\mathcal{C}}(x, y)$ as the simplicial set obtained as the fibre product

$$\Delta[0] \times_{\mathcal{C}} \mathsf{Fun}(\Delta[1], \mathcal{C}) \times_{\mathcal{C}} \Delta[0]$$

where the maps are given by $x, y\colon \Delta[0] \to \mathcal{C}$ and $\mathrm{ev}_0, \mathrm{ev}_1\colon \mathsf{Fun}(\Delta[1], \mathcal{C}) \to \mathcal{C}$, where ev_n is the evaluation on n.

4.2. Note. Another notation used is $\mathrm{Hom}^{\mathrm{LR}}_{\mathcal{C}}(x,y)$, where we write the pullback as

$$\begin{tikzcd} \mathrm{Hom}^{\mathrm{LR}}_{\mathcal{C}}(x,y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{C}^{\Delta[1]} \arrow[d] \\ \{x\}\times\{y\} \arrow[r] & \mathcal{C}\times\mathcal{C} \end{tikzcd}$$

4.3. Proposition. $\mathrm{Map}_{\mathcal{C}}(x,y)$ is a Kan complex.

4.4. Note. $\pi_0\mathrm{Map}_{\mathcal{C}}(x,y) \simeq \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(x,y)$.

4.5. Warning. These is *no* strict manifestation of composition:

$$\mathrm{Map}_{\mathcal{C}}(y,z)\times\mathrm{Map}_{\mathcal{C}}(x,y) \not\to \mathrm{Map}_{\mathcal{C}}(x,z).$$

4.6. Rectification. (Lurie) There exists $\widetilde{\mathrm{Map}}_{\mathcal{C}}(x,y) \in$ sSet and *canonical* zig-zags of weak equivalences of simplicial sets

$$\widetilde{\mathrm{Map}}_{\mathcal{C}}(x,y) \xrightarrow{\sim} \{*\} \xleftarrow{\sim} \mathrm{Map}_{\mathcal{C}}(x,y)$$

such that we *do* get strict manifestations of composition maps. We write $\mathfrak{C}[\mathcal{C}]$ for the **rectified category**, which is simply $\mathcal{C}$ but with $\widetilde{\mathrm{Map}}$ replacing Map. Note that the rectified category is a true simplicial category.

4.7. Simplicial nerve. There exists a *non-trivial* extension of the nerve construction to simplicial categories that takes into account the simplicial structure, i.e. the **simplicial nerve** $N_\Delta(\mathcal{E})$ is a simplicial set when $\mathcal{E}$ is a simplicial category.

4.8. Application. We can model a simplicial category $\mathcal{E}$ by the simplicial set $N_\Delta(\mathcal{E})$.

4.9. Theorem. (Joyal-Lurie) There exists a model structure on sSet with

- cofibrant-fibrant objects = quasi-categories;
- weak equivalences = essentially surjective morphisms that induce a weak equivalence on mapping spaces.

4.10. Theorem. (Bergner) There exists a model structure on $\mathsf{Cat}^{\Delta^{\mathrm{op}}}$ with

- cofibrant-fibrant objects = simplicial categories enriched over Kan complexes;
- weak equivalences = essentially surjective morphisms that induce a weak equivalence on mapping spaces.

4.11. Theorem. (Lurie) The adjunction $(\mathfrak{C} \dashv N_\Delta)$ forms a Quillen equivalence.

4.12. Example. Let $\mathcal{M}$ be a simplicial model category and write $\mathcal{M}^{\mathrm{cf}}$ to mean the subcategory of cofibrant-fibrant objects. Then $\mathcal{M}^{\mathrm{cf}}$ is enriched over Kan complexes and so $N_\Delta(\mathcal{M}^{\mathrm{cf}})$ is a quasi-category.

4.13. Rectification of diagrams. Let $\mathcal{D}$ be a category and $\mathcal{M}$ a combinatorial simplicial model category. Then there is an equivalence of quasi-categories

$$\mathsf{Fun}\Big(N(\mathcal{D}), N_\Delta(\mathcal{M}^{\mathrm{cf}})\Big) \simeq N_\Delta\Big((\mathcal{M}^{\mathcal{D}})^{\mathrm{cf}}\Big)$$

where we endow $\mathcal{M}^{\mathcal{D}}$ with the projective model structure.

4.14. Explicit examples. The following three examples are prototypical.

I. The ∞**-category** $\mathcal{S}$ **of spaces**. This is $\mathcal{S} = N_\Delta(\mathsf{sSet}^{\mathrm{cf}})$ where

- sSet has the model structure used to study weak homotopy equivalences etc.;
- the cofibrant-fibrant objects are Kan complexes;
- $\mathrm{Map}_{\mathcal{S}}(x, y) \simeq \underline{\mathrm{Hom}}_\Delta(x, y)$.

II. The ∞**-category** Cat_∞ **of** ∞**-categories**. This is $\mathsf{Cat}_\infty = N_\Delta(\mathsf{sSet})$ where sSet has the model structure as in Joyal-Lurie, but modified in some way so as to make it a simplicial model category.

III. The ∞**-category** $\mathsf{PreSh}(N(\mathcal{D}))$ **of presheaves of spaces**. Here $\mathcal{D}$ is an arbitrary category and sSet has the same model structure as in example I. Then

$$\mathsf{PreSh}(N(\mathcal{D})) = \mathsf{Fun}(N(\mathcal{D}^{\mathrm{op}}), \mathcal{S}) \simeq \mathsf{Fun}\Big(N(\mathcal{D}^{\mathrm{op}}), N_\Delta(\mathsf{sSet}^{\mathrm{cf}})\Big) \simeq N_\Delta\Big((\mathsf{sSet}^{\mathcal{D}^{\mathrm{op}}})^{\mathrm{cf}}\Big).$$

5 Homotopy colimits

5.1. Initial object.

6 Localisation

7 Presheaves and ∞-functors

8 Presentability

9 Symmetric monoidal ∞-categories

10 Subtleties