14/12/16
Masterclass, Angers

Hilbert Schemes

(I was late) ...
(wow)

~~we gloss over schemey details~~

i.e. assume reducedness etc.

?

- **Proposition**: if $p_1\ p_2\ p_3\ p_4$ are general points then $\operatorname{codim}(I_2(X), S_2) = 4$

- **Proposition**: if $p_1, \ldots, p_4$ are on a line then $\operatorname{codim}(I_2(X), S_2) = 3$

- **proof**: suppose the line is $y=0$
  the conic meets all four points and so contains the line $y=0$
  possible equations: $y(ax+by+cz)$
  so $\dim I_2 = 3$

- **Problem**: find a uniform degree which works for all X

- **question**: $X(t) = (0,0) \cup (t,0)$
  what is $\lim_{t\to 0} X(t)$?

  if we define it to be $(0,0)$ then we lose a point (so no hope of continuity)

  well $I(X(t)) = (y, (x-t)x)$
  so $I(X_0) = (y, x^2)$
  and $\dim k[x,y]/(y,x^2) = \dim k[x]/(x^2) = 2$
  $1, dx$

- **problem**: need to consider ideals and not only geometric objects
  i.e. schemes not varieties

- **Informally**: $\operatorname{Hilb}^p(\mathbb{A}^2) = \{ I \triangleleft k[x,y] \mid \dim k[x,y]/I = p \}$
  $\operatorname{Hilb}^p(\mathbb{A}^2) = \{ I \triangleleft k[x,y] \mid \dim (k[x,y,z]_d / I_d) = p \ \forall d \gg 0 \}$
  hom. and saturated $\longrightarrow$ for all polynomials $p \in k[x_0, \ldots, x_n]$
  $[x_0 p, \ldots, x_n p \in I \Rightarrow p \in I]$
  (here $x = x_0,\ y = x_1$)

- **Proposition**: $I \triangleleft S$ hom. sat. then $\exists P_I$ polynomial s.t. $\operatorname{codim}(I_d, S_d) = P_I(d)$ $\forall d \gg 0$
  and P_I is called the Hilbert polynomial of I

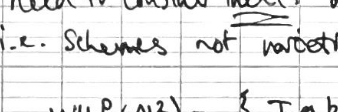
- **example**: $\mathbb{P}^2 \subset \mathbb{P}^4$, $P_{I(\mathbb{P}^2)} = \binom{d+2}{2}$
  (no of degree d monomials in three variables)

- **example**: $\mathbb{P}^1 \subset \mathbb{P}^n$ then $P_{I(\mathbb{P}^1)} = d+1$
  since $I(\mathbb{P}^1) = (x_2, \ldots, x_n)$ so
  $I_d(\mathbb{P}^1) = \ldots ?$ well
  $k[x_0, \ldots, x_n] / I_d(X) = \underbrace{(x_0^d, x_0^{d-1}x_1, \ldots, x_1^d)}_{d+1 \text{ elements}}$

- **Answer** (to something): $I_1 \sim I_2 \iff P_{I_1} = P_{I_2}$

- **New question**: what is / how do we construct $\operatorname{Hilb}^p(\mathbb{P}^n) = \{ I \subset S,\ P_I = p \}$

- **proposition**:



~~Suppose Z ∈ ... ℙ^n ...~~

suppose $\exists P$ polynomial $\to Z \subset C \setminus \{x\} \times \mathbb{P}^n$
and $Z \subset$ ~~$C \times \mathbb{P}^n \setminus \{p\}$~~ closed
s.t. $\forall q \in C$, $P_{I(Z_q)} = P$
If C smooth (near x) then
$\exists!\ \tilde{Z} \subset C \times \mathbb{P}^n$ s.t. $P_{I(\tilde{Z}_x)} = P$
and $\tilde{Z}$ extends Z
(and $\tilde{Z}$ is the schematic closure of Z)

* ~~uniformisation problem~~

  what are the possible Hilbert polynomials?

* $P = P_r(d) \binom{d+r}{r}$ is possible ($\mathbb{P}^r$)

* **example**: $3d+1 = \underbrace{(d+1)+d+(d-1)}_{\text{three lines}} \underbrace{+1}_{\text{point}}$

  (can realize this in $\mathbb{P}^2$)

  so this is a Hilbert polynomial

* **proposition**: if $r_1 \geq \ldots \geq r_k$ then

  $P = P_{r_1}(d) + P_{r_2}(d-1) + \ldots P_{r_k}(d-k+1)$

  is possible (and however: converse is true!)

  (by taking unions of $\mathbb{P}^{r_i}$)

  {there are lots of subtle issues here!

* **Definition**: no of terms in these sorts of decompositions is the Gotzmann number of P

* **example**: $d^2 - d - 4$

  $= \binom{d+2}{2} + \binom{d+1}{2} + \lambda d + \mu$

  with $\underline{\lambda < 0}$

  so NOT a Hilbert polynomial

* (improved) **Theorem**: $I \trianglelefteq S$, $P = P_I$

  $r =$ Gottmann numbers of P

  then $\operatorname{codim}(I_d, S_d) = P(d)$ for all $\underline{d \geq r}$

* ~~universal property: morphisms to $\mathrm{Hilb}(\mathbb{P}^n)$~~

* **Definition**: as a set,

  $\mathrm{Hilb}^P(\mathbb{P}^n) = \{ \underset{\text{hom. sat.}}{I \trianglelefteq S} \mid P_I = P \}$

* **Theorem**:

  $\mathrm{Hilb}^P(\mathbb{P}^n) \longrightarrow G(\overset{(\dim S_r)-P(r)}{}, S_r)$

  $I \longmapsto I_r$

  (r the Gotzmann number)

  is injective

  and $\mathrm{Im}(\varphi)$ can be defined by equations

  (in fact, determinants)

  (but also more recent, simpler, ones) much lower degree