Introduction to infinity categories

Talk by Marco Robalo at DAGIT 2017
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Abstract

These are a copy of my notes on a talk given by Marco Robalo at the seminar Derived Algebraic Geometry in Toulouse (DAGIT) 2017: the content is purely his; the mistakes are all mine.

1 Motivation

1.1 Idea. An ∞**-category** consists of

- objects;
- 1-morphisms between objects;
- n-morphisms between $(n-1)$-objects (for $n \geqslant 2$);
- composition laws for n-morphisms ($n \geqslant 1$) defined up to higher morphisms;
- associativity of compositions up to homotopy.

1.2 Proto-example. (Fundamental ∞-groupoid) For a CW-complex X we have

- objects = points;
- 1-morphisms = homotopies;
- 2-morphisms = homotopies of homotopies;
- ... and so on.

1.3 Problem. No direct definition that is operational and simultaneously close to our intuition/desire (infinitely many axioms!).

1.4 Solution. Find a model category whose objects serve as models for ∞-categories.

1.5 Modelling. Many classical examples:

- homotopy types can be modelled by topological spaces, simplicial sets, categories, etc.;
- homotopy theory of homotopy-commutative $\mathbb{Q}$-algebras can be modelled by dg-algebras;
- derived stacks can be modelled by simplicial presheaves.

1.6 Question. Why so many models?

1.7 Answer. Dwyer-Kan localisation: every model category has an associated ∞-category that captures all the important information.

1.8 Question. If we have models then why care about ∞-categories?

1.9 Answer. Many reasons:

- not all ∞-categories have a model presentation;
- no 'good enough' definition of functors that relate different models (need an ∞-functor between the associated ∞-categories);
- models for diagrams are not always given by diagrams of models;
- proofs and statements become 'simpler'.

2 Preliminary definitions

3 Quasi-categories

4 Simplicial nerve and rectification

5 Homotopy colimits

6 Localisation

7 Presheaves and ∞-functors

8 Presentability

9 Symmetric monoidal ∞-categories

10 Subtleties