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15/12/16
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Applications

- Interpolation in dimension one

- Theorem: given $a_1, \dots, a_d, b_1, \dots, b_d \in K$
  $\exists!\ P \in K[x]_d$ s.t. $P(a_i) = b_i$
  proof (well known) Lagrange
  (rewriting) consider $K[x]_d \to K^{d+1}$, $P \mapsto (P(a_0), \dots, P(a_d))$
  Then in the basis $\{1, x, x^2, \dots, x^d\}$
  this is
  $$\begin{pmatrix} 1 & a_0 & \cdots & a_0^d \\ \vdots & & & \vdots \\ 1 & a_d & \cdots & a_d^d \end{pmatrix}$$ (a Vandermonde)
  so $\det = \prod_{i<j} (a_i - a_j)$

Interpolation in dimension two

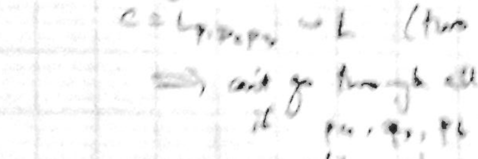
how many lines in $\mathbb{P}^2$ that go through three points? generally none (since $L = ax+by+cz$ has three conditions imposed upon it $\Rightarrow L = 0$)
unless they're not in general position!
↑ number of conditions imposed by r points depends on the geometry

- Proposition: Let $S = K[x,y]$, $p_i \in \mathbb{P}^2$
  define $S_d(-p_1 \dots -p_r) = \{f \in S_d \mid f(p_i) = 0\ \forall i\}$
  then if $r \le d+1$
  $\operatorname{codim}(S_d(-p_1 \dots -p_r), S_d) = r$

- Proposition: Let $p_1, \dots, p_r \in \mathbb{P}^2$ be in general position
  Then
  $\operatorname{codim}(S_d(-p_1 \dots -p_r), S_d) = \min(r, \dim S_d)$
- proof
  $\operatorname{codim}(S_d(-p_1 \dots -p_r), S_d(-p_1 \dots -p_{r-1})) = 1$

- e.g. $d = 2$, $r = 6$
  now $\dim(S_2) = 6$ so we expect no conic to contain all six points
  so what is $\dim S_2(-p_1 \dots -p_6)$?
  → we have a matrix M as before that depends on the p_i and we wish to compute $\det M$

$\exists\, U \subset (\mathbb{P}^2)^6$ (nonempty), open such that M has maximal rank on U; it suffices to find any position where this invertible i.e. $\det M \neq 0$



none $C \supset \{p_1, \dots, p_6\}$
⇓ (because)
$C \supset L$ $p_1 p_2 p_3$
⇓
$C = L_{p_4 p_5 p_6} \cup L$ (two lines)
⟹ can't go through all points if p_4, p_5, p_6 aren't collinear!

- Another point:
  consider $I(Z) = (x^3, x^2y, xy^2, y^3)$
  so set-theoretically $Z = \{(0,0)\}$
  (but scheme-theoretically it has sextuple conditions there)
  Lemma: $\exists$ parametrizations $p_1(t), \dots, p_6(t)$
  such that $\lim_{t \to 0} (p_1(t) \cup \dots \cup p_6(t)) = Z$
  (no $Z \in \mathrm{Hilb}^6$) (i.e. $p_i(t) \neq p_j$ for $t \neq 0$)
  clearly $S_2(-Z) = \{0\}$ (by definition of $I(Z)$)
  so $S_2(\dots) = \{0\}$
  apply "continuity"

| Same problem with multiplicities! — dimension are: $S_d(-p_1^{m_1} - \dots - p_r^{m_r})$
$\shortparallel$
$\{f \in S_d \mid f(p_i)=0,\ f'(p_i)=0, \dots, f^{(m_i-1)}(p_i)=0\}$

dimension two: here we have two derivative conditions plus one vanishing condition for a point of multiplicity two so multiplicity m gives $\frac{m(m+1)}{2}$ conditions

~~proposition:~~ and codim$(S_d(p_1^{m_1} \dots p_r^{m_r}), S_d)$
$= \sum m_i$ (when $\sum m_i$ is small) $d \geq 1$

HERE intuition fails! $\sum \frac{m_i(m_i+1)}{2}$ doesn't give the right answer! even in general position

e.g. conics C with $p_1, p_2 \in C$ of mult. 2 ... expect $C = 0$
But we have (always) the double line through p_1, p_2 so C is degenerate but not zero!

e.g. quartics containing the 5 points with mult 2 so 15 conditions, 3 per point and quartics have 15 coefficients so expect no quartic
But can always take a double conic through the five points

quadratic transformation: $\varphi: \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ (not defined everywhere, e.g. at $x=y=0$)
$[x:y:z] \longmapsto [yz:xz:xy]$

note: (general position ... general position)

now let $p(t) = [a(t):b(t):1] \xrightarrow{t \to 0} [0:0:1]$
what is lim $\varphi(p(t))$? well
$\varphi(p(t)) = [bt:at:abt^2] = [b:a:abt]$
$\longrightarrow [b:a:0]$

φ (mostly true)

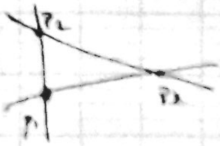
note: φ is involutive, i.e. $\varphi^2 = \mathrm{id}$
i.e. $\varphi = \varphi^{-1}$
(just calculate φ^2)

φ^{-1} maps line to point so φ "maps" point to line
i.e. involution or limit any lets us extend φ to all points (but it's no longer a function)

Thus if curve goes through $[0:0:1]$ with mult. m then under φ the curve goes through $\{z=0\}$ m times (and vice versa)

Now consider curve ~~...~~ with
$(d, m_1, \dots, m_r)$
↑ degree ↑ mult of intersection with p_1

where



and p_4, p_5, p_6 are elsewhere (in general position!)

so mult of φ(curve) through p_1 is $\underline{d - m_2 - m_3}$

thus $\varphi: (d, m_1, \dots, m_r)$
$\mapsto (2d - m_1 - m_2 - m_3, d - m_2 - m_3, d - m_1 - m_3, d - m_1 - m_2, m_4, \dots, m_r)$

- Properties: [illegible]
  - [illegible]

e.g. $(2,2,2,2)$ [illegible] $(0,0,0,-2)$ + $(0,0,0,0)$
multiple [illegible]

so going backwards
[illegible] started from $(2,2,2,2)$

e.g. $(4,2,2,2,2,2)$ [illegible] $(2,0,0,0,2,2)$ [illegible] $(0,0,0,-2,0,0)$
[illegible] could have specified [illegible] at [illegible] point
which corresponds (birationally) to a curve

negative [illegible] corresponds to [illegible] curve

- Definition: a pattern $(k, m_1, \ldots)$ [illegible]
  is Cremona unstable [illegible] if [illegible]
  [illegible] cannot produce negative
  multiplicities ≤ -2

- Conjecture: if $(d, m_1, \ldots, m_r)$
  is unstable then
  [illegible]

- Results: true for $r \leq 9$
  true for [illegible]
  true for [illegible]
  true for [illegible]

- Hilbert's 14th problem (specific case)
  $K[x_1, \ldots, x_n] \circlearrowleft G$
  So we have the G-invariants
  $K[x_1, \ldots, x_n]^G$
  Is this finitely generated as a K-algebra?

  e.g. $G = S_n$ then
  $K[x_1, \ldots, x_n]^{S_n} = K[\sigma_1, \ldots, \sigma_n]$
  where σ_1 and σ_n elementary symmetric polynomials

- Answer: no!
  (true if G is reductive)

- Counterexample
  Idea: prove that there is no curve of degree [illegible] in $\mathbb{P}^2$
  containing 16 points with multiplicities [illegible]