# Grassmannians and Hilbert schemes

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August 15, 2018

**Abstract**

Notes taken at a masterclass given by Laurent Evain at Angers on the 13-15th December 2016. More likely than not that some mistakes have been introduced during typesetting and important points missed: please email `timhosgood@gmail.com` with any corrections.

## Contents



**0.0.1 Warning.** We are sometimes sloppy with defining notation, but hopefully in an understandable way. Usually if any notation isn't defined it has been carried over from a previous section.

# 1 Grassmannians

**1.0.1 Notation.** $\mathrm{k}$ is an algebraically-closed field; the natural numbers include zero; $\{*\}$ is an arbitrary singleton set; $\mathbb{P}^n$ is projective $n$-space over $\mathrm{k}$; $\mathbb{A}^n$ is affine $n$-space over $\mathrm{k}$; $\langle v_i \mid i \in I\rangle$ means the subspace generated by the vectors $\{v_i\}_{i\in I}$; $\{e_1,\ldots,e_n\}$ is the canonical basis of $k^n$; $V^*$ is the dual space of a vector space $V$, and the dual basis of $\{v_1,\ldots,v_n\}$ is written $\{v_1^*,\ldots,v_n^*\}$; $S_n$ is the symmetric group on $n$ elements; we write $S\subset T$ to mean $S\subseteq T$ (and write $S\subsetneq T$ if $S\subset T$ but $S\neq T$).

## 1.1 Set-theoretic introduction

**1.1.1 The idea.** Projective space can be thought of as the set of lines through the origin on a vector space under some equivalence relation. We can generalise this by looking at linear subspaces of some fixed arbitrary dimension.

**1.1.2 Definition.** For a $\mathrm{k}$-vector space $W$ and natural number $p \in \mathbb{N}$ we define the *$p$-Grassmannian associated to $W$* as the set of dimension-$p$ subspaces of $W$, i.e.

$$\mathrm{G}(p, W) = \{V \subset W \mid \dim V = p\}.$$

For $q \in \mathbb{N}$ we define $\mathrm{G}(p, q) = \mathrm{G}(p, \mathrm{k}^q)$.

**1.1.3 Trivial examples.** $\mathrm{G}(0, W) = \mathrm{G}(p, p) = \{*\}$.

**1.1.4 Less trivial examples.** $\mathrm{G}(1, q) = \mathrm{G}(q-1, q) \cong \mathbb{P}^{q-1}$. To see this, note that an element of e.g. $\mathrm{G}(q-1, q)$ is given uniquely, up to multiplication by a non-zero scalar $\lambda \in \mathrm{k}^\times$, by a linear form $L = a_0 x_0 + \ldots + a_{q-1} x_{q-1}$. This gives us the bijection $L \leftrightarrow [a_0 : \ldots : a_{q-1}] \in \mathbb{P}^{q-1}$.

**1.1.5 An explicit example.** An element of $\mathrm{G}(2, 4)$ is determined by a pair of linear forms, say $L = \sum_{i=0}^{3} a_i x_i$ and $M = \sum_{i=0}^{3} b_i x_i$. We can calculate the determinants $D_{ij}$ of the six $2 \times 2$-minors of the $2 \times 4$-matrix whose rows are $(a_j)$ and $(b_j)$ (e.g. $D_{01} = a_0 b_1 - a_1 b_0$). These are determined up to a constant, and are invariant under choosing a different pair of linear forms to represent the same element. It can be checked that $D = D_{01} D_{23} - D_{02} D_{13} + D_{03} D_{12}$ satisfies $D = 0$. In fact, the map $\varphi\colon\ \mathrm{G}(2, 4) \to \mathbb{P}^5$ given by $V \mapsto (D_{ij})$ has image given exactly by the vanishing of $D$, i.e.

$$\operatorname{Im} \varphi = \{[D_{01} : D_{02} : D_{03} : D_{12} : D_{13} : D_{23}] \in \mathbb{P}^5 \mid D = 0\}.$$

**1.1.6 A trick.** Let $V = \langle e_1, e_2 + e_3 \rangle \subset W = k^4$. Then $V$ is defined by the vanishing of $e_4$ and $e_2 - e_3$, i.e. by $e_4^*$ and $e_2^* - e_3^*$ in $W^*$. Taking the wedge product gives us $e_4^* \wedge (e_2^* - e_3^*) = -e_2^* \wedge e_4^* + e_3 \wedge e_4^*$ from which we can read off $D_{24} = -1$ and $D_{34} = 1$. This is explained in the next section.

## 1.2 Dual view

**1.2.1 The idea.** Look at the generators of a subspace $V$ rather than its equations.

**1.2.2 Explicit example.** With $V$ as in 1.1.6 we associate $V \leftrightarrow e_1 \wedge (e_2 + e_3)$.

**1.2.3 Speedy notation.** Write $e_I = e_{i_1} \wedge \ldots \wedge e_{i_r}$ where $I \subset \{1, \ldots, n\}$ and $i_1 < \ldots < i_r$. Given such an $I$, write $J = \{1, \ldots, n\} \setminus I$ and $\sigma_{IJ} = (I\,J) \in S_n$. Given $\lambda e_I$ for some $\lambda \in \mathrm{k}^\times$ we write $p_I = \lambda$.

**1.2.4 Another explicit example.** Look at $V \in \mathrm{G}(2, 3)$ defined by $\sum_{i=1}^{3} q_i e_i^*$ where $q_i = i$, so that $V$ has a basis $\{2e_1 - e_2, 3e_1 - e_3\}$. Then

$$(2e_1 - e_2) \wedge (3e_1 - e_3) = -2e_{1,3} + 3e_{1,2} + e_{2,3}.$$

Notice that, up to a sign, $p_I = q_J$ where $J = \{1, 2, 3\} \setminus I$.

**1.2.5 Proposition/definition.** Let $V \subset W \cong \mathrm{k}^q$ be of dimension $p$ with basis $\{b_1, \ldots, b_p\}$ and defined by equations $\{\varphi_1, \ldots, \varphi_{q-p}\}$. Write $\bigwedge_{i=1}^{p} b_i = \sum p_I e_I = p_V$ where the sum is taken over all $I = (i_1, \ldots, i_p)$ with $i_1 < \ldots < i_p$. Similarly, write $\bigwedge_{j=1}^{q-p} \varphi_j = \sum q_J e_J^* = q_V$, where $J = \{1, \ldots, q\} \setminus I$ is also ordered with $j_1 < \ldots < j_{q-p}$. Then the $p_I$ and $q_J$ are called *Plücker coordinates* and are related by $p_I = \operatorname{sgn} \sigma_{IJ} q_I$.

**1.2.6 Definition.** The *Plücker embedding* is the morphism $\mathrm{Pl}\colon\ \mathrm{G}(p, q) \to \mathbb{P}^N$, where $N = \binom{q}{p} - 1$, given by $V \mapsto (p_I(V))$, i.e. by taking the Plücker coordinates.

## 1.3 Nested spaces

**1.3.1 The idea.** We can see if a point lies on a hyperplane by taking the equations that define the hyperplane and evaluating them at the point, i.e. we take the dual of one subspace and evaluate it on the other.

**1.3.2 Nesting.** Two subspaces $U, V \subset W$ are said to be *nested* if either $U \subset V$ or $V \subset U$.

**1.3.3 Duality.** There is a canonical isomorphism $\mathbb{P}\left(\bigwedge^p W\right) \cong \mathbb{P}\left(\bigwedge^{q-p} W^*\right)$. Using the $p_I(V)$ as Plücker coordinates means we are working in the left-hand side; using the $q_J(V)$ means we are working in the right-hand side. We are free to switch between the two to describe $V$.

**1.3.4 Explicit example.** Let $U = \langle e_1, e_2 + ae_3 \rangle, V = \langle e_1, e_2 + be_4 \rangle \subset \mathrm{k}^4$ for some $a, b \in \mathrm{k}$. Then $U = V$ if and only if $a, b = 0$. Now we calculate the Plücker coordinates: $p_V = e_{1,2} + ae_{1,3}$ and $q_V = -be^*_{2,3} - e^*_{3,4}$. We 'evaluate' $q_V$ at $p_V$ by taking their product:

$$p_V \cdot q_V = be_1 \wedge e^*_3 - abe_1 \wedge e^*_2 + ae_1 \wedge e^*_4.$$

This product is calculated by using the rules that $e_i \cdot e^*_i = e^*_i(e_i) = 0$ and the linearity and anticommutativity of the wedge product (and is further explained below).

**1.3.5 The product.** Generally we define the above product on the level of tensors (since the exterior/wedge product is a quotient of the tensor product), i.e.

$$\varphi\colon (\wedge^m W \otimes \wedge^n W^*) \to (\wedge^{m-1} W \otimes \wedge^{n-1} W^*)$$

and $p_V \cdot q_V = \varphi(p_V \otimes q_V)$.

**1.3.6 Proposition.** Let $U, V \subset W$. Then $\varphi(p_V \otimes q_U) = 0$ if and only if $U$ and $V$ are nested.

**1.3.7 Corollary.** There exists some projective variety $Q \subset \mathbb{P}^N$ defined by quadrics (called the *Plücker relations*) such that $\mathrm{Im}\,(\mathrm{Pl}) \subset Q$.

*Proof.* $V \in \mathrm{G}(p, W) \iff V \subset V \iff \varphi(p_V \otimes q_V) = 0$. This last equation is of bi-degree $(1, 1)$ but, up to a sign, $p_V = q_V$, so it can be thought of as being degree-2 in $p_V$. $\square$

**1.3.8 Theorem.** $\mathrm{Im}\,(\mathrm{Pl}) = Q$.

## 1.4 Algebraic structure

**1.4.1 Idea.** We give the Grassmannian a natural algebraic structure by exhibiting it as an object with a universal property.

**1.4.2 More concrete idea.** Find an equivalence between morphisms $X \to \mathrm{G}(p, q)$ and bundles on $X$, noting that $\mathrm{G}(p, q)$ has a universal bundle.

**1.4.3 Definition.** A *vector bundle (of rank $r$)* on a vector space $X$ is an object/morphism pair $(\varphi\colon F \to X)$ such that

$$\forall p \in X \quad \exists U_p \underset{\text{open}}{\subset} X \quad \text{s.t.} \quad p \in U_p \text{ and } \varphi^{-1}(U_p) \cong U_p \times \mathrm{k}^r$$

and we write $F_p = \varphi^{-1}(p)$ to mean the *fibre over $p$*. **n.b.** the actual definition is slightly more involved, since we need to ensure that the fibres glue together nicely, but we refer the reader elsewhere for the full definition.

**1.4.4 Example.** Consider $\mathrm{k}^3 \setminus \{0\} \times \mathrm{k}^3$ with coordinates $(a_1, a_2, a_3, x_1, x_2, x_3)$. Define a bundle $F$ on $\mathrm{k}^3 \setminus \{0\}$ by setting $F_{(a_1,a_2,a_3)} = \{a_1x_1 + a_2x_2 + a_3x_3 = 0\}$. This is a rank-2 bundle.

**1.4.5 Definition.** Given a continuous map $f\colon X \to Y$ and a bundle $F \to Y$ we define a bundle $f^*F$ on $X$, called the *pullback $F$ under $f$*, by $(f^*F)_x = F_{f(x)}$

**1.4.6 Example.** Let $F$ be the bundle from 1.4.4 and define $f\colon \mathrm{k} \to \mathrm{k}^3 \setminus \{0\}$ by $t \mapsto (t^2, t^3, 1)$. Then $(f^*F)_t = \{t^2x_1 + t^3x_2 + x_3 = 0\}$.

**1.4.7 Definition.** Define a rank-$p$ bundle $U \to \mathrm{G}(p, W)$ by $U_V = V$. This is called the *universal bundle of the Grassmannian*.

**1.4.8 Theorem.** Specifying a morphism $X \to \mathrm{G}(p, W)$ is equivalent to specifying a rank-$p$ bundle $F \to X$ such that $F \subset X \times W$.

**1.4.9 Theorem.** Given a rank-$p$ bundle $F \to X$ there exists a unique $f\colon X \to \mathrm{G}(p, W)$ such that $f^*U \cong F$ (as bundles), and this $f$ is exactly the morphism given by $x \mapsto F_x$.

# 2 Hilbert schemes

**2.0.1 Notation.** Given a graded ring $R$ we write $R_d$ to mean the degree-$d$ part; given some ideal $I \triangleleft R$ we write $I_d$ to mean the degree-$d$ part; we write $R^{(i,j)} = \mathrm{k}[x_i, \ldots, x_j]$.

**2.0.2 Warning.** We ignore many scheme-theoretic details (i.e. assume reducedness) and don't always rigorously define all of our ideas (for the sake of time), especially commonly-known ones. From hereon in these notes are intended more for somebody with a reasonable (but by no means large) working knowledge of algebraic geometry.

## 2.1 Ideals

**2.1.1 Question.** How do we distinguish similar objects in projective space?

**2.1.2 Simple cases.** We can consider vector spaces up to dimension, and discrete sets of points up to cardinality, i.e. $\{p_1, \ldots, p_r\} \sim \{q_1, \ldots, q_s\} \iff r = s$.

**2.1.3 Idea.** We know that we can associate to each subset $X \subset \mathbb{P}^n$ some ideal $I(X) \triangleleft R^{(0,n)}$, and vice versa. But the ideal might contain more (scheme-theoretic) information than the subset. So we want to find some property that is shared by 'similar' ideals, like we have dimension for vector spaces and cardinality for discrete sets. This will be the *Hilbert polynomial*.

**2.1.4 Example.** Consider $X = \{0\} \subset \mathbb{A}^1$. Then $I(X) = (x) \triangleleft \mathrm{k}[x]$. Then 'set-theoretically' $(x) = (x^2) = X$, but 'scheme-theoretically' $I(X) = (x) \neq (x^2)$.

**2.1.5 Explicit example/definition.** Let $p = (a, b) \in \mathbb{A}^2$. Then we have the following ideals: the affine ideal $I_*(p) = (x - a, y - b) \triangleleft R^{(0,1)}$ and the projective ideal $I(p) = (x - az, y - bz) \triangleleft R^{(0,2)}$. Here

$$I_2(p) = \big(x(x-az), y(x-az), z(x-az), x(y-bz), y(y-bz), z(y-bz)\big)$$

and this is a subspace of $R_2^{(0,2)} = \mathrm{k}[x^2, y^2, z^2, xy, xz, yz]$. It can be checked that $\dim R_2^{(0,2)}/I_2(p) = 1$ and thus $\dim I_2(p) = 5$, i.e. $I_2(p) \in \mathrm{G}(5, R_2^{(0,2)})$.

**2.1.6 Proposition.** If $X = \{p_1, \ldots, p_4\} \subset \mathbb{P}^2$ consists of collinear points then $\operatorname{codim}(I_2(X), S_2) = 3$.

*Proof.* Suppose the line is $y = 0$. Then the conic meets all four points and so contains the line $y = 0$. Thus the possible equations of such a conic are of the form $y(ax + by + cz)$, thus $\dim I_2 = 3$. $\square$

**2.1.7 Proposition.** If $X = \{p_1, \ldots, p_4\} \subset \mathbb{P}^2$ consists of points in general position then $\operatorname{codim}(I_2(X), S_2) = 4$.

**2.1.8 Problem.** We want to be able to consider ideals as elements of a Grassmannian, but depending on the geometry of our subsets the ideals lie in different Grassmannians. Can we find a uniform degree which works for all $X$?

## 2.2 Limits

**2.2.1 Question.** Let $X(t) = \{(0,0), (t,0)\} \subset \mathbb{A}^2$. What is a 'good' definition of $\lim_{t\to0} X(t)$?

**2.2.2 Bad answer.** Naïvely, $\lim_{t\to0} X(t) = \{(0,0)\}$. This is not a 'continuous' notion, since we collapse two points to one. The ideal associated to this set is $(x, y)$.

**2.2.3 Better answer.** Work with ideals, so that we retain the differential information: $I(X(t)) = (y, (x-t)x) \xrightarrow{t\to0} (y, x^2)$. This has codimension $2$ since $\mathrm{k}[x,y]/(y, x^2)$ has $\{1, x\}$ as a $\mathrm{k}$-basis.

**2.2.4 Technical lemma.** Let $f \in \mathrm{k}[d]$ be a polynomial, $C \subset \mathbb{P}^2$ a curve, and $x \in C$ a point such that $C$ is smooth near $x$. Suppose we have some closed $Z \subset C \setminus \{x\} \times \mathbb{P}^n$ such that $h_{I(Z_q)} = f$ for all $q \in C$, where $Z_q$ is the fibre over $q$ of the bundle $C \times \mathbb{P}^n$. Then there exists a unique $Z' \subset C \times \mathbb{P}^n$ such that $h_{I(Z'_x)} = f$ and $Z'$ extends $Z$. In fact, $Z'$ is exactly the (scheme) closure of $Z$.

**2.2.5 Definition.** An ideal $I \lhd R^{(0,n)}$ is *saturated* if $\forall f \in R^{(0,n)}\ \big[x_0 f, \ldots, x_n f \in I \implies f \in I\big]$.

**2.2.6 Informal definition.** We make the following temporary definitions:

$$\mathrm{Hilb}^m(\mathbb{A}^2) = \{I \lhd k[x,y] \mid \mathrm{codim}(I, k[x,y]) = m\}$$

$$\mathrm{Hilb}^m(\mathbb{P}^2) = \{I \lhd k[x,y,z] \mid I \text{ homogeneous saturated, } \mathrm{codim}(I_d, k[x,y,z]_d) = m \text{ for all sufficiently large } d\}$$

## 2.3 Hilbert polynomials

**2.3.1 Proposition/definition.** If $I \lhd R^{(0,n)}$ is homogeneous and saturated then there exists a polynomial $h_I(d)$ such that $\mathrm{codim}(I_d, R_d^{(0,n)}) = h_I(d)$ for all sufficiently large $d$. This $h_I$ is called the *Hilbert polynomial*.

**2.3.2 Explicit example.** $\mathbb{P}^1 \subset \mathbb{P}^n$ has Hilbert polynomial $h_{I(\mathbb{P}^1)}(d) = d+1$ since $I(\mathbb{P}^1) = (x_2, \ldots, x_n)$ and so $k[x_0, \ldots, x_n]_d / I_d(\mathbb{P}^1)$ has $\{x_0^d, x_0^{d-1}x_1, \ldots, x_1^d\}$ as a basis. Note that that $h_{I(\mathbb{P}^1)}$ doesn't depend on $n$.

**2.3.3 Explicit example.** $\mathbb{P}^2 \subset \mathbb{P}^n$ has Hilbert polynomial $h_{I(\mathbb{P}^2)} = \binom{d+2}{d}$ by a similar argument.

**2.3.4 Answer.** Recalling 2.1.1 we say that $I(X) \sim I(Y) \iff H_{I(X)} = H_{I(Y)}$.

**2.3.5 Question.** How do we construct $\{I \lhd R^{(0,n)} \mid I \text{ hom. sat., } h_I = f\}$?

## 2.4 Uniformisation

**2.4.1 Question.** Which polynomials are Hilbert polynomials of some ideal?

**2.4.2 Partial answer.** $f_n(d) = \binom{d+n}{n}$ is the Hilbert polynomial of $\mathbb{P}^n$.

**2.4.3 Proposition.** If $r_1 \geqslant \ldots \geqslant r_k$ then

$$f_{r_1}(d) + f_{r_2}(d-1) + \ldots + f_{r_k}(d-k+1)$$

is the Hilbert polynomial of the unions of the $\mathbb{P}^{r_i}$ (though there are lots of subtle points here).

**2.4.4 Theorem.** Every Hilbert polynomial is of this form.

**2.4.5 Explicit example.** $3d+1 = (d+1) + d + (d-1) + 1$ and so is the Hilbert polynomial of the union of three lines and a point (and we can take this union in $\mathbb{P}^2$ it turns out).

**2.4.6 Explicit non-example.** $d^2 - d - 4$ has a negative $d$ coefficient, and so can never be realised as a Hilbert polynomial.

**2.4.7 Definition.** The number of terms in this sort of decomposition is called the *Gotzmann number*.

**2.4.8 Theorem.** Let $I \lhd R^{(0,n)}$ be saturated and homogeneous with Hilbert polynomial $h_I$ and Gotzmann number $r$. Then $\mathrm{codim}(I_d, S_d) = h_I(d)$ for all $d \geqslant n$. Further, this bound is optimal.

**2.4.9 Definition.** As a set, we define $\mathrm{Hilb}^f(\mathbb{P}^n) = \{I \lhd R^{(0,n)} \mid I \text{ hom. sat., } h_I = f\}$

**2.4.10 Theorem.** Let $S = R^{(0,n)}$. Then the map

$$\begin{aligned} \varphi \colon \mathrm{Hilb}^f(\mathbb{P}^n) &\to \mathrm{G}(\dim S_r - f(r), S_r) \\ I &\mapsto I_r \end{aligned}$$

is injective. Further, $\operatorname{Im}\varphi$ can be defined by algebraic equations.

# 3 Applications

## 3.1 Interpolation in degree one

**3.1.1 Theorem.** Given distinct $a_0, \ldots, a_d, b_0, \ldots, b_d \in \mathbb{P}^1$ there exists a unique $P \in \mathrm{k}[x]_d$ such that $P(a_i) = b_i$ for all $i$.

**3.1.2 Proof (idea).** Consider the map $\mathrm{k}[x]_d \to \mathrm{k}^{d+1}$ given by $P \mapsto (P(a_0), \ldots, P(a_d))$. Using the basis $\{1, x, x^2, \ldots, x^d\}$ we can write this map as a Vandermonde matrix, whose determinant is $\prod_{i \leqslant j}(a_i - a_j) \neq 0$.

**3.1.3 Proposition/definition.** Let $S = \mathrm{k}[x, y]$ and $p_1, \ldots, p_r \in \mathbb{P}^1$. Define

$$S_d(-p_1 \ldots - p_r) = \{f \in S_d \mid f(p_i) = 0 \ \forall i\}.$$

Then $\operatorname{codim}(S_d(-p_1 \ldots - p_r), S_d) = \min\{r, d+1\}$

## 3.2 Interpolation in degree two

**3.2.1 Question.** How many lines in $\mathbb{P}^2$ are there that go through three specified points?

**3.2.2 Answer.** Generally there are none: $L = ax + by + cz$ has three conditions imposed on it and so reduces to $0$. Here, by 'generally' we mean that the points are in general position.

**3.2.3 Proposition.** Let $p_1, \ldots, p_r \in \mathbb{P}^2$ be in general position. Then

$$\operatorname{codim}(S_d(-p_1 \ldots - p_r), S_d) = \min(r, \dim S_d).$$

**3.2.4 Example.** Let $d = 2$ and $r = 6$. Now $\dim S_2 = 6$, so we expect no conic to contain six given points. But what is $\dim S_2(-p_1 \ldots - p_6)$? As in 3.1.1 we can consider where the determinant of a matrix $M$ vanishes. There exists a Zariski open subset $U \subset (\mathbb{P}^2)^6$ such that $M$ has maximal rank, and so it suffices to find one position of points where $M$ is invertible. Say the $p_i$ are such that three are collinear on the line $L$. Then by Bézout we know that any conic $C$ going through these three collinear points must contain $L$, and so must be the union of $L$ and another line. But if the remaining three points aren't collinear then the conic cannot pass through all six points.

**3.2.5 Proof (of 3.2.3) (idea).** !!!
*See the (terrible) scans for the rest — I have yet to finish typing this up.*

## 3.3 Multiplicities

**3.3.1 Definition.**

**3.3.2 Dimension one.**

**3.3.3 Dimension two.**

**3.3.4 Example.**

**3.3.5 Example.**

## 3.4 Quadratic transformations

## 3.5 Hilbert's 14th problem