Introduction to infinity categories

Talk by Marco Robalo at DAGIT 2017
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Monday 12th June, 2017

Abstract

These are a copy of my notes on a talk given by Marco Robalo at the seminar Derived Algebraic Geometry in Toulouse (DAGIT) 2017: the content is purely his; the mistakes are all mine.

1 Motivation

1.1. Idea. An **∞-category** consists of

- objects;
- 1-morphisms between objects;
- n-morphisms between $(n-1)$-objects (for $n \geqslant 2$);
- composition laws for n-morphisms ($n \geqslant 1$) defined up to higher morphisms;
- associativity of compositions up to homotopy.

1.2. Proto-example. (Fundamental ∞-groupoid) For a CW-complex X we have

- objects = points;
- 1-morphisms = homotopies;
- 2-morphisms = homotopies of homotopies;
- ... and so on.

1.3. Problem. No direct definition that is operational and simultaneously close to our intuition/desire (infinitely many axioms!).

1.4. Solution. Find a model category whose objects serve as models for ∞-categories.

1.5. Modelling. Many classical examples:

- homotopy types can be modelled by topological spaces, simplicial sets, categories, etc.;
- homotopy theory of homotopy-commutative $\mathbb{Q}$-algebras can be modelled by dg-algebras;
- derived stacks can be modelled by simplicial presheaves.

1.6. Question. Why so many models?

1.7. Answer. Dwyer-Kan localisation: every model category has an associated ∞-category that captures all the important information.

1.8. Question. If we have models then why care about ∞-categories?

1.9. Answer. Many reasons:

- not all ∞-categories have a model presentation;
- no 'good enough' definition of functors that relate different models (need an ∞-functor between the associated ∞-categories);
- models for diagrams are not always given by diagrams of models;
- proofs and statements become 'simpler'.

2 Preliminary definitions

2.1. Category of simplices. Write Δ to be the **category of simplices**:

- $\mathrm{ob}\Delta = \{[n]\}_{n\in\mathbb{N}}$ where $[n] = \{0 < 1 < \ldots < n\}$ is the ordered set of natural numbers up to n;
- $\mathrm{Hom}_\Delta([m],[n])$ is the set of order-preserving maps from $[m]$ to $[n]$.

2.2. Simplicial notation. We use the following notation:

- $\mathsf{sSet} = \mathsf{Set}^{\Delta^{\mathrm{op}}} = \mathsf{Fun}(\Delta^{\mathrm{op}}, \mathsf{Set})$;
- $\Delta[n] = \mathrm{Hom}_\Delta(-,[n]) \in \mathsf{sSet}$;
- $S_n = \mathrm{Hom}_{\mathsf{sSet}}(\Delta[n], S)$ for $S \in \mathsf{sSet}$;
- $\Lambda_n^i = \Delta[n] \setminus \{\text{interior and the face opposite the } j\text{-th vertex}\}$ is the i**-th horn**.

$$\Delta[2] = \begin{tikzcd} & 1 \arrow[dr] \arrow[d, phantom, "\Downarrow"] & \\ 0 \arrow[ur] \arrow[rr] & & 2 \end{tikzcd} \qquad \Lambda_2^1 = \begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[ur] & & 2 \end{tikzcd}$$

2.3. Nerve. The **nerve** $N(\mathcal{C})$ of a category $\mathcal{C}$ is the simplicial set with

- n-simplices given by $(X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \ldots \xrightarrow{f_n} X_n)$ in $\mathcal{C}$;
- boundary maps given by composition (or forgetting the first/last object and morphism for the two edge cases);
- degeneracy maps given by inserting the identity.

2.4. Note. There is a set-bijection

$$\{\text{functors } \mathcal{C} \to \mathcal{D}\} \simeq \{\text{simplicial maps } N(\mathcal{C}) \to N(\mathcal{D})\}.$$

2.5. Lemma. There is an equivalence of categories $X \simeq N(\mathcal{C})$ if and only if all *inner* horns lift *uniquely*.

$$\begin{tikzcd} \Lambda^i_n \arrow[r] \arrow[d, hook] & X \\ \Delta[n] \arrow[ur, dashed, "\exists!"'] & \end{tikzcd} \qquad \text{for all } i \in \{1, \ldots, n-1\}$$

2.6. Composition. For example, "Λ^1_2 gives composition".

$$\begin{tikzcd} & X_1 \arrow[dr, "f_2"] & \\ X_0 \arrow[ur, "f_1"] \arrow[rr, dashed, "\exists! f_2 \circ f_1"] & & X_2 \end{tikzcd}$$

2.7. Associativity. As another example, "Λ^1_3 gives associativity". $X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \xrightarrow{f_3} X_3$ in $\mathcal{C}$ corresponds to

$$\Lambda^1_2 \xrightarrow{(f_2, f_1)} N(\mathcal{C}) \quad \text{and} \quad \Lambda^1_2 \xrightarrow{(f_3, f_2)} N(\mathcal{C})$$

which generate compositions

$$\Delta[2] \xrightarrow{(f_2, f_1)} N(\mathcal{C}) \quad \text{and} \quad \Delta[2] \xrightarrow{(f_3, f_2)} N(\mathcal{C}).$$

So $(f_3 \circ f_2) \circ f_1 = f_3 \circ (f_2 \circ f_1)$ if and only if we can 'fill the back face of the tetrahedron with vertices X_0, X_1, X_2, and X_3', i.e. if and only if we can extend $\Lambda^1_3 \to N(\mathcal{C})$ to $\Delta[3] \to N(\mathcal{C})$.

2.8. Summary. The lifting property for inner horns ($0 < i < n$) gives composition and associativity laws; for outer horns ($i = 0, n$) it gives inverses.

$$\begin{tikzcd} & X_0 \arrow[dr, dashed, "\exists!"] & \\ X_0 \arrow[ur, "\mathrm{id}_{X_0}"] \arrow[rr, "f_1"] & & X_1 \end{tikzcd}$$

2.9. Kan complex. If XsSet is such that $X \simeq \mathrm{Sing}(T)$, where $\mathrm{Sing}(T)$ consists of singular simplices in a topological space T (i.e. continuous maps $|\Delta^n| \to T$) then we call it a **Kan complex**. Note that X is a Kan complex if and only if *all* horns lift, but *not necessarily* uniquely.