13/12/16

masterclass, Angers

Grassmannians

- $G(p,W) := \{V \subset W \mid \dim V = p\}$ (W: vector space)
- $G(p,q) := G(p, k^q)$
- e.g. $G(0,W) = \{*\}$, $G(p,p) = \{*\}$
- e.g. $G(1,q) \cong \mathbb{P}^{q-1}$
- e.g. $G(q-1,q) \cong \mathbb{P}^{q-1}$ (since elements of $G(q-1,q)$ are given by linear forms, and linear forms are in bijection with $\mathbb{P}^n$) $\lambda(a_0x_0 + \ldots + a_{q-1}x_{q-1})$

- e.g. $G(2,4)$
elements determined by $a_0x_0 + \ldots + a_3x_3 = 0$, $b_0x_0 + \ldots + b_3x_3 = 0$
calculate the six determinants e.g. $D_{01} = a_0b_1 - a_1b_0$
check that these are invariant and defined (up to a constant)
and $D_{01}D_{23} - D_{02}D_{13} + D_{03}D_{12} = 0$
then can show that $\varphi: G(2,4) \to \mathbb{P}^5$, $V \mapsto (D_{ij})$ has $\operatorname{Im}\varphi = V(D_{01}D_{23} - D_{02}D_{13} + D_{03}D_{12})$

- e.g. $V = \langle e_1, e_2+e_3\rangle \subset W = \langle e_1,e_2,e_3,e_4\rangle \cong k^4$
so $x_4 = 0$, $x_2 - x_3 = 0$ i.e. V defined by $e_4^*, e_2^* - e_3^* \in W^*$
now cheat/trick: $e_4^* \wedge (e_2^* - e_3^*) = \underbrace{-e_2^* \wedge e_4^*}_{D_{24}=-1} + \underbrace{e_3^* \wedge e_4^*}_{D_{34}=1}$ ← note that indexes are $1,\ldots,4$ not $0,\ldots,3$! just notational faffy problems

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- Dual view: generators of V rather than equations (use same $V \subset W$ as above)
i.e. $V \longleftrightarrow e_1 \wedge (e_2+e_3)$
- e.g. $G(2,3)$: look at V defined by $e_1^* + 2e_2^* + 3e_3^*$ (write q_i = coeff of e_i^*)
so V has basis $(2e_1 - e_2, 3e_1 - e_3)$
then take the wedge: $(2e_1-e_2)\wedge(3e_1-e_3) = \underbrace{-2e_1\wedge e_3}_{p_{13}=-2=-q_2} + \underbrace{3e_1\wedge e_2}_{p_{12}=3=q_3} + \underbrace{e_2\wedge e_3}_{p_{23}=1=q_1}$
- let $I \subset \{1,\ldots,n\}$, $J = I^c$, order increasingly
then define $\varepsilon_I := \varepsilon_J := \operatorname{sgn}\sigma_{(I,J)}$
where $\sigma_{(I,J)}: (1,\ldots,n) \mapsto (I,J)$
- proposition: $V \subset K^q$ with basis $\{b_1,\ldots,b_p\}$ of V and equations $\{\varphi_1,\ldots,\varphi_{q-p}\}$ defining V
write $\bigwedge_{i=1}^p b_i = \sum p_I e_I$ (e.g. $e_{(2,4)} = e_2 \wedge e_4$)
$\bigwedge_{j=1}^{q-p} \varphi_j = \sum q_J e_J^*$ then if $I = J^c$ then $p_I = \varepsilon_I q_J$
and p_I (or q_J) are called the Plücker coordinates
- Plücker embedding: $Pl: G(p,q) \to \mathbb{P}^{\binom{q}{p}-1}$, $V \mapsto (p_I(V))$

- when is a space inside another one? inspiration: e.g. is a point in a hyperplane? well we think of the hyperplane as in the dual space (defined by equations) and evaluate the hyperplane on the point

given $V \in G(p,W)$ (with $\dim W = q$)
we have $V \to V_1 \in \mathbb{P}(\Lambda^p W)$ coordinates p_I
$V \to V_2 \in \mathbb{P}(\Lambda^{q-p} W^*)$ " " q_J
(n.b. canonical iso $\mathbb{P}(\Lambda^p W) \cong \mathbb{P}(\Lambda^{q-p} W^*)$)

- e.g. $V = \langle e_1, e_2 + a e_3\rangle$
$U = \langle e_1, e_2 + b e_4\rangle$
so $V = U \iff a=b=0$

so $p_V = e_1\wedge e_2 + a e_1 \wedge e_3$
$q_U = e_3^* \wedge (be_2^* - e_4^*)$
$= -be_2^* \wedge e_3^* - e_3^* \wedge e_4^*$

now look at $p_V \cdot q_U$ ← this needs defining! $\boxed{!}$

REALLY just 'contraction' $(e_i \wedge e_j)\cdot(e_k^* \wedge e_l^*)$ has 4 terms

generalised Kronecker delta (e.g. $(e_i^*)(e_j) = \delta_{ij}$)

In this example $p_u \cdot q_v = b\, e_1 \wedge e_3^* - ab\, e_1 \wedge e_2^* + a\, e_1 \wedge e_4^*$

- **Proposition**: (generally) (i.e. for any u, v)
  $\varphi(p_v \otimes q_u) = 0 \iff u \subset v$ or $v \subset u$
  where $\varphi: \Lambda^{p} W \otimes \Lambda^{n-p} W^* \to \Lambda^{n-1} W \otimes \Lambda^{n-1} W^*$ is exactly this product talked about above

- **Corollary**: $\exists Q$ defined by quadrics (Plücker relations) s.t. $\mathrm{Im}(Pl) \subset Q$

proof: $V \in G(p, W)$ has $V \to v_1$ p_v, $\to v_2$ q_v
then $V \subset V \iff \varphi(p_v \otimes q_v) = 0$
↳ bidegree $(1,1)$ but (up to sign) p_v "=" q_v
so degree 2 in p_v

$\boxed{\text{Theorem: } \mathrm{Im}(Pl) = Q}$

- what is the natural algebraic structure on $G(p,q)$?

Idea: give it some universal property
more concrete idea: make morphisms to $Gr(p,q)$ equivalent to bundles on the source space

- **Vector bundle**: (of rank k on X)
  a morphism/object pair $F \xrightarrow{\varphi} X$ s.t.
  $\left[\forall p \in X\ \exists U_p \in Op(X) \text{ s.t. } p \in U_p \text{ and } \varphi^{-1}(U) \cong U \times k^k\right]$

- e.g. $F \to k^3 \setminus \{0\}$ by $F = \mathbb{V}(a_1x_1 + a_2x_2 + a_3x_3) \subset k^3\setminus\{0\} \times k^3$ $\;(a_i)\ (x_i)$
  is a rank-2 bundle on $X = k^3 \setminus \{0\}$

- **pullback**: $\varphi: Y \to X$, $F \to X$ bundle
  then $(\varphi^* F)_y = F_x$ (where F_p is the fibre over p)

e.g. (F as above) and
$\varphi: k \to k^3 \setminus \{0\}$
$t \mapsto (t^2, t^3, 1)$
so $\varphi^* F$ is given by
$\mathbb{V}(t^2 x_1 + t^3 x_2 + x_3) \subset k \times k^3$ $\;(t)\ (x_i)$

- grassmannians again: define $F \to Gr(p, W)$
  by F_x being the vector space parametrised by $x \in Gr(p,W)$.
  This is called the universal bundle of the Grassmannian (rank p)

n.b. $\varphi: X \to G(p, W)$ with $F \to G(p,W)$ (bundle) has $(\varphi^* F) \subset X \times W$

- **Theorem**: universal property of the Grassmannian
  the following data are equivalent:
  - a morphism $X \to Gr(p, W)$
  - a rank p bundle $F \to X$ with $F \subset X \times W$ (← this is vital)

- **Theorem**: given $F \to X$ (bundle)
  $\exists!\ \varphi: X \to G(p, W)$ s.t. $\varphi^* U_{Gr(p,W)} \cong F$ (as bundles)
  (where $U_{G(p,W)}$ is the universal bundle)

n.b. $\varphi: x \mapsto F_x$ is exactly this φ