

Introduction to ∞ -categories

Talk by Marco Robalo at DAGIT 2017

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Abstract

These are a copy of my notes on a talk given by Marco Robalo at Derived Algebraic Geometry in Toulouse (DAGIT) 2017: the content is purely his; the mistakes are all mine.

1 Motivation

1.1. Idea. An ∞ -category consists of

- objects;
- 1-morphisms between objects;
- n -morphisms between $(n - 1)$ -objects (for $n \geq 2$);
- composition laws for n -morphisms ($n \geq 1$) defined up to higher morphisms;
- associativity of compositions up to homotopy.

1.2. Proto-example. (Fundamental ∞ -groupoid) For a CW-complex X we have

- objects = points;
- 1-morphisms = homotopies;
- 2-morphisms = homotopies of homotopies;
- ... and so on.

1.3. Problem. No direct definition that is operational and simultaneously close to our intuition/desire (infinitely many axioms!).

1.4. Solution. Find a model category whose objects serve as models for ∞ -categories.

1.5. Modelling. Many classical examples:

- homotopy types can be modelled by topological spaces, simplicial sets, categories, etc.;
- homotopy theory of homotopy-commutative \mathbb{Q} -algebras can be modelled by dg-algebras;
- derived stacks can be modelled by simplicial presheaves.

1.6. Question. Why so many models?

1.7. Answer. Dwyer-Kan localisation: every model category has an associated ∞ -category that captures all the important information.

1.8. Question. If we have models then why care about ∞ -categories?

1.9. Answer. Many reasons:

- not all ∞ -categories have a model presentation;
- no ‘good enough’ definition of functors that relate different models (need an ∞ -functor between the associated ∞ -categories);
- models for diagrams are not always given by diagrams of models;
- proofs and statements become ‘simpler’.

2 Preliminary definitions

2.1. Category of simplices. Write Δ to be the **category of simplices**:

- $\text{ob}(\Delta) = \{[n]\}_{n \in \mathbb{N}}$ where $[n] = \{0 < 1 < \dots < n\}$ is the ordered set of natural numbers up to n ;
- $\text{Hom}_{\Delta}([m], [n])$ is the set of order-preserving maps from $[m]$ to $[n]$.

2.2. Simplicial notation. We use the following notation:

- $\text{sSet} = \text{Set}^{\Delta^{\text{op}}} = \text{Fun}(\Delta^{\text{op}}, \text{Set})$;
- $\Delta[n] = \text{Hom}_{\Delta}(-, [n]) \in \text{sSet}$;
- $S_n = \text{Hom}_{\text{sSet}}(\Delta[n], S)$ for $S \in \text{sSet}$;
- $\Lambda_n^i = \Delta[n] \setminus \{\text{interior and the face opposite the } j\text{-th vertex}\}$ is the **i -th horn** (for $n \geq 2$).

$$\Delta[2] = \begin{array}{ccc} & 1 & \\ \nearrow & \Downarrow & \searrow \\ 0 & \longrightarrow & 2 \end{array} \quad \Lambda_2^1 = \begin{array}{ccc} & 1 & \\ \nearrow & & \searrow \\ 0 & & 2 \end{array}$$

2.3. Nerve. The **nerve** $N(\mathcal{C})$ of a category \mathcal{C} is the simplicial set with

- n -simplices given by $(x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \dots \xrightarrow{f_n} x_n)$ in \mathcal{C} ;
- boundary maps given by composition (or forgetting the first/last object and morphism for the two edge cases);
- degeneracy maps given by inserting the identity.

2.4. Note. There is a set-bijection

$$\{\text{functors } \mathcal{C} \rightarrow \mathcal{D}\} \simeq \{\text{simplicial maps } N(\mathcal{C}) \rightarrow N(\mathcal{D})\}.$$

2.5. Lemma. There is an equivalence of categories $X \simeq N(\mathcal{C})$ if and only if all *inner* horns lift *uniquely*.

$$\begin{array}{ccc} \Lambda_n^i & \longrightarrow & X \\ \downarrow & \nearrow \exists! & \\ \Delta[n] & & \end{array} \quad \text{for all } i \in \{1, \dots, n-1\}$$

2.6. Composition. For example, “ Λ_2^1 gives composition”.

$$\begin{array}{ccccc} & & x_1 & & \\ f_1 \nearrow & & & \searrow f_2 & \\ x_0 & \xrightarrow{\exists! f_2 \circ f_1} & & & x_2 \end{array}$$

2.7. Associativity. As another example, “ Λ_3^1 gives associativity”:

$$x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} x_2 \xrightarrow{f_3} x_3 \text{ in } \mathcal{C}$$

corresponds to

$$\Lambda_2^1 \xrightarrow{(f_2, f_1)} N(\mathcal{C}) \quad \text{and} \quad \Lambda_2^1 \xrightarrow{(f_3, f_2)} N(\mathcal{C})$$

which generate compositions

$$\Delta[2] \xrightarrow{(f_2, f_1)} N(\mathcal{C}) \quad \text{and} \quad \Delta[2] \xrightarrow{(f_3, f_2)} N(\mathcal{C}).$$

So $(f_3 \circ f_2) \circ f_1 = f_3 \circ (f_2 \circ f_1)$ if and only if we can ‘fill the back face of the tetrahedron with vertices x_0, x_1, x_2 , and x_3 ’, i.e. if and only if we can extend $\Lambda_3^1 \rightarrow N(\mathcal{C})$ to $\Delta[3] \rightarrow N(\mathcal{C})$.

2.8. Exercise. What does the lifting of Λ_3^2 tell us?

2.9. Summary. The lifting property for inner horns ($0 < i < n$) gives composition and associativity laws; for outer horns ($i = 0, n$) it gives inverses.

$$\begin{array}{ccc} & x_0 & \\ \text{id}_{x_0} \nearrow & & \searrow \exists! \\ x_0 & \xrightarrow{f_1} & x_1 \end{array}$$

2.10. Kan complex. If $X \in \mathbf{sSet}$ is such that $X \simeq \mathbf{Sing}(T)$, where $\mathbf{Sing}(T)$ consists of singular simplices in a topological space T (i.e. continuous maps $|\Delta^n| \rightarrow T$) then we call it a **Kan complex**. Note that X is a Kan complex if and only if *all* horns lift, but *not necessarily* uniquely.

3 Quasi-categories

3.1. Quasi-category. A **quasi-category** is a simplicial set \mathcal{C} such that all *inner* horns lift, but *not necessarily* uniquely. This notion lies in between that of a Kan complex and that of the nerve of a category: we get compositions that aren’t unique, but their non-uniqueness is controlled by higher homotopy data. For example, consider

$$x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} x_2 \xrightarrow{\text{id}_{x_2}} x_2$$

which gives us two maps $u_1, u_2: \Delta[2] \rightarrow \mathcal{C}$. Similarly, for ‘associativity’ we can use the lifting of Λ_3^1 as before, after filling one face with the identity.

3.2. ∞ -category. We can use quasi-categories as a model for ∞ -categories: define the objects of a quasi-category \mathcal{C} to be the 0-simplices, and the n -morphisms to be the n -simplices.

3.3. ∞ -functor. Since functors are ‘maps that preserve commutative diagrams’, it makes sense to define an ∞ -functor to be a map of simplicial sets between quasi-categories, since these send n -simplices to n -simplices and preserve boundaries.

3.4. Homotopy category. Given an ∞ -category \mathcal{C} we define its **homotopy category** $\mathrm{h}\mathcal{C}$ to be the (1-)category with

- $\mathrm{ob}(\mathrm{h}\mathcal{C}) = \mathrm{ob}(\mathcal{C})$;
- $\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(x, y) = \mathrm{Hom}_{\mathcal{C}}(x, y) / \sim$, where $f \sim g$ if there exists a 2-morphism $u: \Delta[2] \rightarrow \mathcal{C}$ with boundary $(\mathrm{id}_y - g + f)$.

$$\begin{array}{ccc} & y & \\ f \nearrow & \Downarrow u & \searrow \mathrm{id}_y \\ x & \xrightarrow{g} & y \end{array}$$

Note that compositions are unique (and thus well defined) thanks to the lifting property, i.e. u_1, u_2 are identified in the homotopy category.

3.5. Subcategory. An ∞ -subcategory \mathcal{C}' of an ∞ -category \mathcal{C} is a sub-simplicial set obtained as a fibre product in sSet :

$$\begin{array}{ccc} \mathcal{C}' & \longrightarrow & \mathcal{C} \\ \downarrow & \lrcorner & \downarrow \\ N(\mathcal{D}) & \longrightarrow & N(\mathrm{h}\mathcal{C}) \end{array} \quad \text{where } \mathcal{D} \text{ is a subcategory of } \mathrm{h}\mathcal{C}.$$

3.6. Equivalence. A 1-morphism f in \mathcal{C} is called an **equivalence** if $[f]$ in $\mathrm{h}\mathcal{C}$ is an isomorphism.

3.7. ∞ -groupoid. An ∞ -category where *all* 1-morphisms are equivalences is called an ∞ -groupoid.

3.8. Proposition. An ∞ -category is an ∞ -groupoid if and only if it is a Kan complex.

3.9. Example. For T a topological space, $\mathrm{Sing}(T)$ is an ∞ -groupoid

4 Simplicial nerve and rectification

4.1. Mapping space. Let x, y be objects in an ∞ -category \mathcal{C} . We define the **mapping space** $\mathrm{Map}_{\mathcal{C}}(x, y)$ as the simplicial set obtained as the fibre product

$$\Delta[0] \times_{\mathcal{C}} \mathrm{Fun}(\Delta[1], \mathcal{C}) \times_{\mathcal{C}} \Delta[0]$$

where the maps are given by $x, y: \Delta[0] \rightarrow \mathcal{C}$ and $\mathrm{ev}_0, \mathrm{ev}_1: \mathrm{Fun}(\Delta[1], \mathcal{C}) \rightarrow \mathcal{C}$, where ev_n is the evaluation on n .

4.2. Note. Another notation used is $\mathrm{Hom}_{\mathcal{C}}^{\mathrm{LR}}(x, y)$, where we write the pullback as

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}^{\mathrm{LR}}(x, y) & \longrightarrow & \mathcal{C}^{\Delta[1]} \\ \downarrow & \lrcorner & \downarrow \\ \{x\} \times \{y\} & \longrightarrow & \mathcal{C} \times \mathcal{C} \end{array}$$

4.3. Proposition. $\mathrm{Map}_{\mathcal{C}}(x, y)$ is a Kan complex.

4.4. Note. $\pi_0 \mathrm{Map}_{\mathcal{C}}(x, y) \simeq \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(x, y)$.

4.5. Warning. These is *no* strict manifestation of composition:

$$\mathrm{Map}_{\mathcal{C}}(y, z) \times \mathrm{Map}_{\mathcal{C}}(x, y) \not\rightarrow \mathrm{Map}_{\mathcal{C}}(x, z).$$

4.6. Rectification. (Lurie) There exists $\widetilde{\mathrm{Map}}_{\mathcal{C}}(x, y) \in \mathrm{sSet}$ and *canonical* zig-zags of weak equivalences of simplicial sets

$$\widetilde{\mathrm{Map}}_{\mathcal{C}}(x, y) \xrightarrow{\sim} \{*\} \xleftarrow{\sim} \mathrm{Map}_{\mathcal{C}}(x, y)$$

such that we *do* get strict manifestations of composition maps. We write $\mathfrak{C}[\mathcal{C}]$ for the **rectified category**, which is simply \mathcal{C} but with Map replacing Map . Note that the rectified category is a true simplicial category.

4.7. Simplicial nerve. There exists a *non-trivial* extension of the nerve construction to simplicial categories that takes into account the simplicial structure, i.e. the **simplicial nerve** $N_{\Delta}(\mathcal{E})$ is a simplicial set when \mathcal{E} is a simplicial category.

4.8. Application. We can model a simplicial category \mathcal{E} by the simplicial set $N_{\Delta}(\mathcal{E})$.

4.9. Theorem. (Joyal-Lurie) There exists a model structure on sSet with

- cofibrant-fibrant objects = quasi-categories;
- weak equivalences = essentially surjective morphisms that induce a weak equivalence on mapping spaces.

4.10. Theorem. (Bergner) There exists a model structure on $\mathrm{Cat}^{\Delta^{\mathrm{op}}}$ with

- cofibrant-fibrant objects = simplicial categories enriched over Kan complexes;
- weak equivalences = essentially surjective morphisms that induce a weak equivalence on mapping spaces.

4.11. Theorem. (Lurie) The adjunction $(\mathfrak{C} \dashv N_{\Delta})$ forms a Quillen equivalence.

4.12. Example. Let \mathcal{M} be a simplicial model category and write $\mathcal{M}^{\mathrm{cf}}$ to mean the subcategory of cofibrant-fibrant objects. Then $\mathcal{M}^{\mathrm{cf}}$ is enriched over Kan complexes and so $N_{\Delta}(\mathcal{M}^{\mathrm{cf}})$ is a quasi-category.

4.13. Rectification of diagrams. Let \mathcal{D} be a category and \mathcal{M} a combinatorial simplicial model category. Then there is an equivalence of quasi-categories

$$\mathrm{Fun}\left(N(\mathcal{D}), N_{\Delta}(\mathcal{M}^{\mathrm{cf}})\right) \simeq N_{\Delta}\left((\mathcal{M}^{\mathcal{D}})^{\mathrm{cf}}\right)$$

where we endow $\mathcal{M}^{\mathcal{D}}$ with the projective model structure.

4.14. Explicit examples. The following three examples are prototypical.

I. The ∞ -**category \mathcal{S} of spaces**. This is $\mathcal{S} = N_{\Delta}(\mathrm{sSet}^{\mathrm{cf}})$ where

- sSet has the model structure used to study weak homotopy equivalences etc.;
- the cofibrant-fibrant objects are Kan complexes;
- $\mathrm{Map}_{\mathcal{S}}(x, y) \simeq \underline{\mathrm{Hom}}_{\Delta}(x, y)$.

II. The ∞ -**category Cat_{∞} of ∞ -categories**. This is $\mathrm{Cat}_{\infty} = N_{\Delta}(\mathrm{sSet})$ where sSet has the model structure as in Joyal-Lurie, but modified in some way so as to make it a simplicial model category.

III. The ∞ -**category $\mathrm{PreSh}(N(\mathcal{D}))$ of presheaves of spaces**. Here \mathcal{D} is an arbitrary category and sSet has the same model structure as in example I. Then

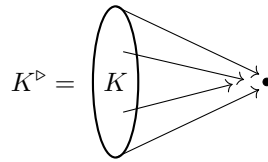
$$\mathrm{PreSh}(N(\mathcal{D})) = \mathrm{Fun}(N(\mathcal{D}^{\mathrm{op}}), \mathcal{S}) \simeq \mathrm{Fun}\left(N(\mathcal{D}^{\mathrm{op}}), N_{\Delta}(\mathrm{sSet}^{\mathrm{cf}})\right) \simeq N_{\Delta}\left((\mathrm{sSet}^{\mathcal{D}^{\mathrm{op}}})^{\mathrm{cf}}\right).$$

5 Homotopy colimits

5.1. Initial object. An **initial object** is some $\emptyset \in \mathcal{C}$ such that, for all $x \in \mathcal{C}$, the Kan complex $\mathrm{Map}_{\mathcal{C}}(\emptyset, x)$ is contractible.

5.2. Slogan. Universal objects are defined up to a contractible space of choices.

5.3. Cones. Given $K \in \mathrm{sSet}$ and an infinity functor $d: K \rightarrow \mathcal{C}$ we construct a new simplicial set K^{\triangleright} by formally adding an exterior vertex to K . Then a **cone under d** is a map of simplicial sets $\tilde{d}: K^{\triangleright} \rightarrow \mathcal{C}$ such that $\tilde{d}|_K = d$.



5.4. Proposition. Cones under d give a quasi-category $\mathcal{C}_{d/}$.

5.5. Colimits. A colimit is an initial object of $\mathcal{C}_{d/}$.

5.6. Proposition. Let $F: \mathcal{J} \rightarrow \mathcal{E}$ be a simplicial functor between Kan-enriched categories, and $c \in \mathcal{E}$ an object with a compatible family $\{\eta_j: F(j) \rightarrow c\}$. Then c is a **homotopy colimit of F** if and only if the induced map $N_{\Delta}(\mathcal{J} \rightarrow N_{\Delta}(\mathcal{E}))$ is a colimit diagram.

5.7. Example.

$$\begin{aligned} \operatorname{coeq}_{\mathcal{S}} \left(\begin{array}{ccc} & \operatorname{id} & \\ * & \rightrightarrows & * \\ & \operatorname{id} & \end{array} \right) &\simeq \operatorname{colim}_{\mathcal{S}} \left(\begin{array}{ccc} * & \coprod * & \rightarrow * \\ & \downarrow & \\ & * & \end{array} \right) \simeq \operatorname{hocolim} \left(\begin{array}{ccc} * & \coprod * & \rightarrow * \\ & \downarrow & \\ & * & \end{array} \right) \\ &\simeq \operatorname{colim}_{\mathbf{sSet}} \left(\begin{array}{ccc} * & \coprod * & \rightarrow \Delta[1] \\ & \downarrow & \\ & \Delta[1] & \end{array} \right) \simeq S^1 \in \mathbf{sSet} \end{aligned}$$

6 Localisation

7 Presheaves and ∞ -functors

8 Presentability

9 Symmetric monoidal ∞ -categories

10 Subtleties