Schemes: a cheat sheet

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April 27, 2018

Abstract

This is a cheat sheet to elementary scheme theory, following Ravi Vakil's wonderful notes/book *The Rising Sea*. It it a constant work-in-progress, but unfortunately not at the top of my to-do list at the moment. Hopefully one day I'll be able to come back and finish this off.

Contents

1 Conventions

- Any references of the form (1.1.A) are to Vakil's *The Rising Sea*
- All rings have an identity with $1 \neq 0$ and are commutative unless otherwise specified.
- Inclusions of sets (groups, rings, etc.) are *not* assumed to be strict: $U \subset V$ means $U \subseteq V$; if $U \neq V$ then we write $U \subsetneq V$.
- We write $\{*\}$ to mean a set with one element.
- Maps between topological spaces are assumed to be continuous unless otherwise specified.
- In an abelian category, given some morphism $\varphi\colon x \to y$, we write $\operatorname{Ker}\varphi$ to mean the object in the kernel pair, and $\ker\varphi$ to mean the morphism, i.e. $\ker\varphi\colon \operatorname{Ker}\varphi \to x$; we do similarly for the cokernel and image.
- We write $\mathcal{C}/X$ to be the slice category, i.e. its objects are pairs (Y, f) where $Y \in \mathcal{C}$ and $f\colon Y \to X$.
- Given some topological space X we write $\mathrm{Op}(X)$ to be the category of open sets of X, whose morphisms are given by inclusions.
- We sometimes write 'iff' to mean 'if and only if'.

2 Sheaves

2.1 Definitions

Here we write X to mean an arbitrary topological space, $\mathcal{F}$ an arbitrary sheaf on X, p an arbitrary point in X, U an arbitrary open set in X, etc.

2.1.1 Preliminary definitions

- *Presheaf*: a contravariant functor $\mathcal{F}\colon \mathrm{Op}(X) \to \mathcal{C}$ (where $\mathcal{C}$ is Set, Ab, CRing, or some similar category).
- *Section*: an element of $\mathcal{F}(U)$ for some $U \in \mathrm{Op}(X)$ (and if $U = X$ then the element is called a *global section*); **n.b.** depending on the context a section can also mean the right inverse of a continuous map, i.e. a section of $f\colon X \to Y$ is $\mu\colon Y \to X$ such that $f \circ \mu = \mathrm{id}_Y$ (and a left inverse is called a *retraction*).
- *Sheaf*: a presheaf $\mathcal{F}$ satisfying the *identity* and *gluability* axioms for all open sets $U \in \mathrm{Op}(X)$ and covers $\{U_i\}_{i \in I}$ of U:
  - *Identity/locality*: if $f_1, f_2 \in \mathcal{F}(U)$ agree in all $\mathcal{F}(U_i)$ then $f_1 = f_2$;

– *Gluability*: if $f_i \in \mathcal{F}(U_i)$ agree in all $\mathcal{F}(U_i \cap U_j)$ then there exists $f \in \mathcal{F}(U)$ such that f restricts to f_i in $\mathcal{F}(U_i)$.

Equivalently, we can ask that $\mathcal{F}$ satisfies the following exactness requirement:

$$\mathcal{F}(U) = \mathrm{eq}\left(\prod \mathcal{F}(U_i) \rightrightarrows \prod \mathcal{F}(U_i \cap U_j)\right).$$

We write $\mathcal{C}_X$ to mean the *category of sheaves in $\mathcal{C}$ on X*, where $\mathcal{C}$ is one of Set, Ab, CRing, etc.

- *Separated presheaf*: a presheaf satisfying only the identity axiom.

- *Stalk*: given a point $p \in X$ we define the stalk of $\mathcal{F}$ at p by

$$\begin{aligned}\mathcal{F}_p &= \varinjlim_{U \ni p} \mathcal{F}(U)\\ &= \big\{(f, U) \mid p \in U, f \in \mathcal{F}(U)\big\} / \sim\end{aligned}$$

where $(f, U) \sim (g, V)$ if there exists $W \subset U \cap V$ such that the restriction of f and g to W agree.

- *Germ*: an element of a stalk; given a particular $f \in \mathcal{F}(U)$ with $p \in U$ we say *the germ of f at p* to mean the image f_p of f in $\mathcal{F}_p$.

- *Compatible germs*: an element

$$(s_p)_{p \in U} \in \prod_{p \in U} \mathcal{F}_p$$

consists of *compatible germs* if there exists an open cover $\{U_i\}$ of U and sections $f_i \in \mathcal{F}(U_i)$ such that for all $p \in U_i$ the germ of f_i at p is s_p. (**n.b.** the notation can be confusing: $(s_p)_{p \in U}$ *doesn't* mean we take one section s and consider its germ at all points (otherwise this definition would be redundant!) but instead the section *varies* with the point at which we take the germ, so perhaps something more cumbersome like $(s_p^{(p)})_{p \in U}$ would be a better notation); see (2.4.C) in Section 2.2 and all of Section 2.3 for more information).

- *Sheafification*: a morphism $a \colon \mathcal{F} \to \mathcal{F}^a$ of presheaves such that $\mathcal{F}^a$ is a sheaf and, for any sheaf $\mathcal{G}$ and presheaf morphism $g \colon \mathcal{F} \to \mathcal{G}$ there *exists* a *unique* sheaf morphism $f \colon \mathcal{F}^a \to \mathcal{G}$ such that the following diagram commutes:

$$\begin{tikzcd} \mathcal{F} \arrow[r, "a"] \arrow[dr, "g"'] & \mathcal{F}^a \arrow[d, "f"] \\ & \mathcal{G} \end{tikzcd}$$

We can construct this explicitly by using compatible sections:

$$\mathcal{F}^a(U) = \left\{(f_p)_{p \in U} \in \prod\nolimits_{p \in U} \mathcal{F}_p \mid \forall p \in U\ \exists V_p \in \mathrm{Op}(U),\ s \in \mathcal{F}(V_p) \text{ s.t. } p \in V_p \text{ and } s_q = f_q\, \forall q \in V\right\}$$

- *Support of a sheaf*: when $\mathcal{F} \in \mathcal{C}_X$ we define

$$\operatorname{Supp} \mathcal{F} = \{p \in X \mid \mathcal{F}_p \neq t_{\mathcal{C}}\}$$

where $t_{\mathcal{C}}$ is the terminal object in $\mathcal{C}$ (so $t_{\mathsf{Set}} = \{*\}$, $t_{\mathsf{Ab}} = t_{\mathsf{CRing}} = 0$).

- *Support of a section*: when $\mathcal{F} \in \mathcal{C}_X$ and s is a global section we define

$$\operatorname{Supp}(s) = \{p \in X \mid s_p \neq t_{\mathcal{C}} \in \mathcal{F}_p\}$$

where $t_{\mathcal{C}}$ is the terminal object in $\mathcal{C}$.

2.1.2 Ringed spaces

- *Ringed space*: a pair $(X, \mathcal{O}_X)$ where $\mathcal{O}_X$ is a sheaf of rings on X, called the *structure sheaf*.
- *Functions on a ringed space*: the elements $f \in \mathcal{O}_X(U)$ for an open $U \subset X$ are called *functions* on U.
- *Locally ringed space*: a ringed space where every stalk is a local ring (i.e. has a unique maximal ideal).
- *Sheaf of $\mathcal{O}_X$-modules*: a sheaf $\mathcal{F}$ of abelian groups on X such that each $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module in a functorial way (**n.b.** this also makes each stalk $\mathcal{F}_p$ an $\mathcal{O}_{X,p}$-module).
- *Sheaf Hom*: the sheaf $\underline{\mathrm{Hom}}_{\mathcal{C}_X}(\mathcal{F}, \mathcal{G})$ defined by

$$\underline{\mathrm{Hom}}_{\mathcal{C}_X}(\mathcal{F}, \mathcal{G})(U) = \mathrm{Hom}_{\mathcal{C}_X}(\mathcal{F}|_U, \mathcal{G}|_U)$$

  where $\mathcal{C}$ is one of Set, Ab, CRing, etc. (**n.b.** this is **not** the same as $\mathrm{Hom}_{\mathsf{Set}}(\mathcal{F}(U), \mathcal{G}(U))$: the former is the collection of natural transformations, i.e. sheaf morphisms; the latter is the collection of set morphisms).
- *Dual of an $\mathcal{O}_X$-module*: $\mathcal{F}^\vee = \underline{\mathrm{Hom}}_{\mathcal{O}_X\text{-}\mathsf{Mod}}(\mathcal{F}, \mathcal{O}_X)$.

2.1.3 Abelian category structure

- *Subsheaves and quotient sheaves*: a sheaf $\mathcal{F}$ is a *subsheaf* (resp. *quotient sheaf*) of another sheaf $\mathcal{G}$ if there exists a monomorphism $\mathcal{F} \to \mathcal{G}$ (resp. epimorphism $\mathcal{G} \to \mathcal{F}$).
- *Presheaf (co)kernel*: given a morphism of *presheaves* $\varphi\colon \mathcal{F} \to \mathcal{G}$ we define presheaves by

$$(\mathrm{Ker}_{\text{pre}}\,\varphi)(U) = \mathrm{Ker}\,\varphi(U);$$
$$(\mathrm{Coker}_{\text{pre}}\,\varphi)(U) = \mathrm{Coker}\,\varphi(U).$$

  These satisfy the universal properties of (co)kernels in the category of presheaves on X.
- *Sheaf (co)kernel*: the presheaf kernel is already a sheaf: $\mathrm{Ker}\,\varphi = \mathrm{Ker}_{\text{pre}}\,\varphi$; the presheaf cokernel needs to be sheafified: $\mathrm{Coker}\,\varphi = (\mathrm{Coker}_{\text{pre}}\,\varphi)^a$.
- *Sheaf image*: $\mathrm{Im}\,\varphi = (\mathrm{Im}_{\text{pre}}\varphi)^a$.

2.1.4 Direct and inverse images

Let $\pi\colon X \to Y$ be a continuous map of topological spaces, $\mathcal{F}$ a sheaf of abelian groups on X, and $\mathcal{G}$ a sheaf of abelian groups on Y.

- *Direct image/pushforward*: define a sheaf on Y by

$$\pi_*\mathcal{F}(V) = \mathcal{F}(\pi^{-1}(V)).$$

  So $\pi_*\colon \mathsf{Ab}_X \to \mathsf{Ab}_Y$.
- *Inverse image:* define a **presheaf** on X by

$$\pi^{-1}_{\text{pre}}\mathcal{G}(U) = \varinjlim_{V \supset \pi(U)} \mathcal{G}(V)$$

and then sheafify to get the inverse image sheaf:

$$\pi^{-1}\mathcal{G} = (\pi_{\text{pre}}^{-1}\mathcal{G})^a.$$

So $\pi^{-1}\colon \mathsf{Ab}_Y \to \mathsf{Ab}_X$.

- *Extension by zero*: let $i\colon U \to Y$ be the inclusion of an **open** set and define the **presheaf**

$$(i_!^{\text{pre}}\mathcal{F})(W) = \begin{cases} \mathcal{F}(W) & \text{if } W \subset U; \\ 0 & \text{otherwise,} \end{cases}$$

and then sheafify to get the extension by zero:

$$i_!\mathcal{F} = (i_!^{\text{pre}}\mathcal{F})^a.$$

So $i_!\colon \mathcal{O}_U\text{-}\mathsf{Mod} \to \mathcal{O}_Y\text{-}\mathsf{Mod}$.

2.1.5 Common examples

Let $S \in \mathsf{Set}$ be an arbitrary set.

- *Skyscraper sheaf*:

$$i_{p,*}S(U) = \begin{cases} S & \text{if } p \in U; \\ \{*\} & \text{if } p \notin U. \end{cases}$$

- *Constant presheaf*: $\underline{S}_{\text{pre}}(U) = S$.
- *Constant sheaf*: $\underline{S}(U) = \{\text{locally constant } f\colon U \to S\}$ (**n.b.** this is the sheafification of the constant presheaf)

2.2 Facts

2.2.1 Presheaves

Motto: ***presheaves*** *work well on the level of* ***sections***

(2.3.H) A sequence of **presheaves** $0 \to \mathcal{F}_1 \to \ldots \to \mathcal{F}_n \to 0$ is exact if and only if it is exact on *all sections*, i.e. if and only if $0 \to \mathcal{F}_1(U) \to \ldots \to \mathcal{F}_n(U) \to 0$ is exact for all $U \in \mathrm{Op}(X)$.

2.2.2 Sheafification

(2.4.L) Sheafification is left-adjoint to the forgetful functor from sheaves to presheaves.

(2.4.8) Sheafification is equivalent to taking the sheaf of sections of the espace étalé of a presheaf (see Section 2.3).

2.2.3 Direct and inverse images

(2.6.B) $(\pi^{-1} \dashv \pi_*)$

(2.5.G), (2.6.E) Direct images are **left-exact** functors; inverse images and extensions by zero are **exact** functors.

(2.6.3) If $i\colon \{p\} \to Y$ is the inclusion of a point and $\mathcal{G}$ is a sheaf of abelian groups on Y then $i^{-1}\mathcal{G}$ is the stalk $\mathcal{G}_p$. Thus 'stalks are left adjoint to skyscrapers'.

(2.6.D) If $i\colon U \to Y$ is the inclusion of an **open** set and $\mathcal{G}$ is a sheaf of abelian groups on Y then $i^{-1}\mathcal{G}$ is the restriction $\mathcal{G}|_U$.

(2.6.F) If $j\colon Z \to Y$ is the inclusion of a **closed** set and $\mathcal{F}$ is a sheaf of sets on Z then

$$(j_*\mathcal{F})_p = \begin{cases} \mathcal{F}_p & \text{if } p \in Z; \\ \{*\} & \text{if } p \notin Z. \end{cases}$$

If $\mathcal{G}$ is a sheaf on Y **such that** $\operatorname{Supp}\mathcal{G} \subset Z$ then $\mathcal{G} \xrightarrow{\sim} i_*i^{-1}\mathcal{G}$. Thus 'a sheaf supported on a closed subset can be considered a sheaf on that subset.'

(2.6.G) If $i\colon U \to Y$ is the inclusion of an **open** set and $\mathcal{F}$ is an $\mathcal{O}_U$-module, then

$$(i_!\mathcal{F})_p = \begin{cases} \mathcal{F}_p & \text{if } p \in U; \\ 0 & \text{if } p \notin U. \end{cases}$$

(2.6.G) If $i\colon U \to Y$ is the inclusion of an **open** set, $j\colon Z \to Y$ is the inclusion of its **complement**, and $\mathcal{G}$ is an $\mathcal{O}_Y$-module, then the following sequence is short exact:

$$0 \to i_!i^{-1}\mathcal{G} \to \mathcal{G} \to j_*j^{-1}\mathcal{G} \to 0.$$

Further, $(i_! \dashv i^{-1})$.

2.2.4 Sheaves

Motto: ***sheaves*** *work well on the level of* ***stalks***

(2.2.C) $\mathcal{F}(\cup_i U_i) = \varprojlim \mathcal{F}(U_i)$.

(2.3.D) $\underline{\operatorname{Hom}}_{\mathsf{Set}_X}\big(\underline{\{p\}}, \mathcal{F}\big) \cong \mathcal{F}$;
$\underline{\operatorname{Hom}}_{\mathsf{Ab}_X}\big(\underline{\mathbb{Z}}, \mathcal{F}\big) \cong \mathcal{F}$;
$\underline{\operatorname{Hom}}_{\mathcal{O}_X\text{-}\mathsf{Mod}}\big(\mathcal{O}_X, \mathcal{F}\big) \cong \mathcal{F}$.

(2.4.B) The support $\operatorname{Supp}(s)$ of a section is a closed subset of X.

(2.4.A), (2.4.C) Sections of a **sheaf** (or even a separated presheaf) are determined by germs, i.e. the natural map

$$\mathcal{F}(U) \longrightarrow \prod_{p \in U} \mathcal{F}_p$$

is injective. Further, the image of this map is exactly the set of compatible sections.

(2.4.D) Morphisms of **sheaves** are determined by stalks, i.e. if $\varphi, \psi\colon \mathcal{F} \to \mathcal{G}$ and $\varphi_p = \psi_p$ for all $p \in X$ then $\varphi = \psi$.

(2.4.E) Isomorphisms of **sheaves** are determined by stalks, i.e. if $\varphi\colon \mathcal{F} \to \mathcal{G}$ induces isomorphisms $\varphi_p\colon \mathcal{F}_p \xrightarrow{\sim} \mathcal{G}_p$ for all $p \in X$ then $\mathcal{F} \cong \mathcal{G}$ (**n.b.** this does **not** say that if $\mathcal{F}_p \cong \mathcal{G}_p$ for all p then $\mathcal{F} \cong \mathcal{G}$).

(2.4.N), (2.4.O) For a morphism $\varphi\colon \mathcal{F} \to \mathcal{G}$ of **sheaves** the following are equivalent:

- φ is a monomorphism in the category of sheaves;
- φ is injective on the level of stalks;
- φ is injective on the level of sections.

The corresponding statements are also equivalent

- φ is an epimorphism in the category of sheaves;
- φ is surjective on the level of stalks.

(**n.b.** there is no third statement for epimorphisms; see (2.5.F) for more).

(2.5.A) Stalks and (co)kernels of **sheaves** commute, i.e. $\mathrm{Ker}(\mathcal{F} \to \mathcal{G})_p \cong \mathrm{Ker}(\mathcal{F}_p \to \mathcal{G}_p)$.

(2.5.D) A sequence of **sheaves** $0 \to \mathcal{F}_1 \to \ldots \to \mathcal{F}_n \to 0$ is exact if and only if it is exact on *all stalks*, i.e. if and only if $0 \to (\mathcal{F}_1)_p \to \ldots \to (\mathcal{F}_n)_p \to 0$ is exact for all $p \in X$.

(2.5.F), (2.5.G), (2.5.H) The following functors are all left exact:

- ($\mathsf{Ab}_X \to \mathsf{Ab}$) taking sections over U for any $U \in \mathrm{Op}(X)$;
- ($\mathsf{Ab}_X \to \mathsf{Ab}_Y$) the direct image π_* for any continuous $\pi\colon X \to Y$ (see Section 2.2.3);
- ($\mathsf{Ab}_X \to \mathsf{Ab}_X$) the covariant $\underline{\mathrm{Hom}}(\mathcal{F}, -)$ and the contravariant $\underline{\mathrm{Hom}}(-, \mathcal{F})$.

(2.5.2), (2.5.I) Ab_X and $\mathcal{O}_X$-Mod (where $(X, \mathcal{O}_X)$ is a ringed space) are abelian categories.

2.3 Étalé spaces and sheaves

n.b. there is a difference between *étalé* (from the verb *étaler*, roughly meaning 'spread out') and *étale* (a reasonably obscure nautical term: «l'étale» is the moment in between changing tides where the water is 'slack'). There is often confusion between the two terms, and it is common for them to be mixed up. The correct notions are of an « espace étalé » (i.e. an *étalé space*) but a « morphisme étale » (i.e. an *étale morphism*). We refrain from using the phrase *étalé morphism* to describe the map in an étalé space and instead just use the phrase 'local homeomorphism' (even though an étale morphism between two topological spaces is often defined to be exactly a local homeomorphism).

2.3.1 Étalé spaces and sheaves

$$
\begin{tikzcd}
\mathsf{Top}/X \arrow[drr, dashed, "\Gamma" near start] & & \mathsf{Set}_X^{\mathrm{pre}} \arrow[dll, dashed, "\Lambda"' near start] \\
\text{Étalé}\,(X) \arrow[u, hook, dashed] \arrow[rr, bend left, "\Gamma"] \arrow[rr, phantom, "\perp"] & & \mathsf{Set}_X \arrow[u, hook, dashed] \arrow[ll, bend left, "\Lambda"] \\
\mathsf{Cover}(X) \arrow[u, hook, dashed] \arrow[rr, bend left, "\Gamma"] & & \mathsf{Set}_X^{\mathrm{l.c.}} \arrow[u, hook, dashed] \arrow[ll, bend left, "\Lambda"]
\end{tikzcd}
$$

Figure 2.3.1.1: The adjunctions given by Γ and Λ

- *Local homeomorphism*: a (continuous) map $p\colon E \to X$ such that for all $e \in E$ there exists some open neighbourhood $U_e \subset E$ of e such that $p|_{U_e}\colon U_e \to p(U_e)$ is a homeomorphism (**n.b.** this is weaker than the notion of a *covering space* – see Section 2.3.3).
- *Étalé space over* X: a pair (E, p) where E is a topological space and $p\colon E \to X$ is a local homeomorphism. We write the full subcategory of Top/X consisting of étalé spaces over X as $\text{Étalé}\,(X)$.

- *Associated presheaf*: define the functor $\Gamma\colon \mathsf{Top}/X \to \mathsf{Set}_X^{\mathrm{pre}}$ by sending a slice to the sheaf of sections, i.e.
$$\Gamma\left(E \xrightarrow{p} X\right)(U) = \{\sigma\colon U \to E \mid p \circ \sigma = id_U\}.$$
It turns out that the image under Γ of *any* slice $(E \xrightarrow{p} X) \in \mathsf{Top}/X$ is in fact a sheaf.

- *Associated topological space*: define the functor $\Lambda\colon \mathsf{Set}_X^{\mathrm{pre}} \to \mathsf{Top}/X$ by sending a presheaf to the space consisting of the disjoint union (the coproduct in Set) of all the stalks, i.e.
$$\Lambda(\mathcal{F}) = \coprod_{x \in X} \mathcal{F}_x$$
with the topology *generated* form the basis of the germs of sections on open sets of X, i.e.
$$\mathrm{Op}\left(\Lambda(\mathcal{F})\right) = \{V_{U,f} \mid U \in \mathrm{Op}\, X, f \in \mathcal{F}(U)\}$$
where $V_{U,f} = (f_x)_{x \in U}$. The local homeomorphism $p\colon \Lambda(\mathcal{F}) \to X$ is just projection $p\colon f_x \mapsto x$ and thus $p^{-1}(x) = \mathcal{F}_x$ (**n.b.** hence we call $p^{-1}(x)$ the *stalk* at x, and not the fibre!).
It turns out that the image under Λ of *any* presheaf $\mathcal{F} \in \mathsf{Set}_X^{\mathrm{pre}}$ is in fact an étalé space.

- *Equivalence of sheaves and étalé spaces*: the functors Γ and Λ give an adjunction $(\Gamma \dashv \Lambda)$ between Top/X and $\mathsf{Set}_X^{\mathrm{pre}}$. Further, this adjunction descends to an **equivalence** between $\text{Étalé}\,(X)$ and Set_X (see Fig. 2.3.1.1).

2.3.2 Sheafification, inverse image, and compatible germs

Using the associated étalé space of a presheaf certain constructions become simpler to understand:

- Sheafification is the composition $\Gamma\Lambda$

- The étalé space of an inverse image is the pullback of the étalé space, i.e. if $\mathcal{F} \in \mathsf{Set}_X$, $\mathcal{G} \in \mathsf{Set}_Y$, and $\pi\colon X \to Y$ then
$$\Lambda(\pi^{-1}\mathcal{G}) \cong (X \times_Y \Lambda(\mathcal{G}) \to X)$$
or, equivalently, the following diagram commutes:

$$\begin{tikzcd}
\mathsf{Set}_X \arrow[d, "\Lambda"] & \mathsf{Set}_Y \arrow[l, "\pi^{-1}"'] \arrow[d, "\Lambda"] \\
\text{Étalé}\,(X) & \text{Étalé}\,(Y) \arrow[l, "X \times_Y \Lambda(-)"']
\end{tikzcd}$$

2.3.3 Covering spaces and locally-constant sheaves

- *Covering space over X*: a pair (C, π) where C is a topological space and $\pi\colon C \to X$ is a (continuous) surjective map such that for all $x \in X$ there exists some open neighbourhood $U_x \subset X$ such that $\pi^{-1}(U_x) = \bigsqcup_{i \in I} V_x^{(i)}$ is a union of disjoint open subsets $V_x^{(i)} \subset C$ We write the full subcategory of Top/X consisting of covering spaces over X as $\mathsf{Cover}(X)$.

- *Locally-constant sheaf*: a sheaf $\mathcal{F}$ on X such that there exists an open cover $X = \bigcup_{i \in I} U_i$ of X with $\mathcal{F}|_{U_i} \cong \underline{S^{(i)}}$ for some non-empty set $S^{(i)}$, i.e. 'locally, the sheaf is a constant sheaf'.

- *Equivalence of locally-constant sheaves and covering spaces*: the functors Γ and Λ descend to an **equivalence** between $\mathsf{Cover}(X)$ and $\mathsf{Set}_X^{\mathrm{l.c.}}$ (see Fig. 2.3.1.1).

3 Affine schemes

3.1 Definitions

3.1.1 Preliminary definitions

- *Spectrum of a ring*: the set $\operatorname{Spec} A = \{[\mathfrak{p}] \mid \mathfrak{p} \triangleleft_{\text{prime}} A\}$ for some commutative ring A (**n.b.** we often just write $\mathfrak{p}$ (and $\mathfrak{q}$) to mean a **prime** ideal, and similarly write $\mathfrak{m}$ to mean a **maximal** ideal).

- $\mathbb{V}$ *and* $\mathbb{I}$: for $I \triangleleft A$ define
$$\mathbb{V}(I) = \{[\mathfrak{p}] \in \operatorname{Spec} A \mid I \subset \mathfrak{p}\} \subset \operatorname{Spec} A;$$
for $S \subset \operatorname{Spec} A$ define
$$\mathbb{I}(S) = \bigcap_{[\mathfrak{p}] \in S} \mathfrak{p} \triangleleft A.$$

- *Zariski topology*: the closed sets of $\operatorname{Spec} A$ are exactly those of the form $\mathbb{V}(I)$ for some $I \triangleleft A$.

- *Distinguished open set*: given $f \in A$ define
$$\mathrm{D}(f) = \{[\mathfrak{p}] \in \operatorname{Spec} A \mid f \notin \mathfrak{p}\}.$$

- *Elements as functions*: We think of elements $f \in A$ as functions on $\operatorname{Spec} A$ by defining $f([\mathfrak{p}])$ as the image of f in the residue field $\kappa([\mathfrak{p}])$ at $[\mathfrak{p}]$, i.e.
$$f([\mathfrak{p}]) = f \pmod{\mathfrak{p}} \in K(A/\mathfrak{p})$$
where $K(R)$ is the *field of fractions* of a ring. We freely use the fact that localising and taking a quotient 'commute', i.e.
$$\kappa([\mathfrak{p}]) = A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p} = K(A/\mathfrak{p}).$$

- *Structure sheaf*: a sheaf defined on the basis of distinguished opens $\mathrm{D}(f)$ by
$$\mathcal{O}_{\operatorname{Spec} A}\big(\mathrm{D}(f)\big) = S^{-1}A$$
where $S = \{g \in A \mid \mathbb{V}(g) \subset \mathbb{V}(f)\} = \{g \in A \mid \mathrm{D}(f) \subset \mathrm{D}(g)\}$.

- *Affine scheme*: the set $\operatorname{Spec} A = \{\mathfrak{p} \triangleleft A\}$ with the *Zariski topology* and *structure sheaf* $\mathcal{O}_{\operatorname{Spec} A}$.

- *Affine n-space*: $\mathbb{A}^n_A = \operatorname{Spec} A[x_1, \ldots, x_n]$

- *Morphisms*: a ring homomorphism $\varphi\colon A \to B$ induces a morphism $\operatorname{Spec} B \to \operatorname{Spec} A$ since $\varphi^{-1}(\mathfrak{q}) = \mathfrak{p}$, i.e. the preimage of a prime ideal is prime.

- *Specilisation and generisation*: if $x, y \in X$ are points in a topological space such that $x \in \overline{\{y\}}$ then x is a specilisation of y, and y is a generisation of x.

- *Generic point of a closed subset $K \subset X$*: a point $p \in X$ such that $\overline{\{p\}} = K$.

- *Sheaf from an A-module*: for some A-module M we define the sheaf $\widetilde{M}$ on the distinguished base by
$$\widetilde{M}\big(\mathrm{D}(f)\big) = S^{-1}M = M \otimes_A \mathcal{O}_{\operatorname{Spec} A}\big(\mathrm{D}(f)\big)$$
where $S = \{g \in A \mid \mathbb{V}(g) \subset \mathbb{V}(f)\} = \{g \in A \mid \mathrm{D}(f) \subset \mathrm{D}(g)\}$.

3.1.2 Topological properties

- *Connected*: X cannot be written as a non-trivial disjoint union of two open sets.
- *Irreducible*: X is non-empty and cannot be written as a non-trivial union of two closed subsets.
- *Quasicompact*: for any open cover $X = \bigcup_{i\in I} U_i$ there exists a finite subcover, i.e. $J \subset I$ with $|J| < \infty$ such that $X = \bigcup_{i\in J} U_i$.
- *Noetherian*: X satisfies the descending chain condition for closed subsets, i.e. any sequence $Z_1 \supset Z_2 \supset \ldots$ of closed subsets eventually stabilises ($Z_r = Z_{r+1} = \ldots$).

3.2 Facts

Here A is a commutative ring, I and J are arbitrary ideals, S is a multiplicative set, $\mathfrak{p}$ is always a prime ideal, etc. We are sometimes sloppy with notation just for sake of time (an awful excuse). We write $\mathfrak{N}$ to mean the nilradical ideal of A.

3.2.1 Algebra

(3.4.E) $\sqrt{\cap_{i=1}^n I_i} = \cap_{i=1}^n \sqrt{I_i}$

(3.4.F) $\sqrt{I} = \cap_{I\subset\mathfrak{p}} \mathfrak{p}$

- $\cap_{\mathfrak{p}\in A} \mathfrak{p} = \mathfrak{N}$
- If M is an A-module and $S \subset A$ is a multiplicative set then $S^{-1}M = M \otimes_A S^{-1}A$.

3.2.2 Quotients and localisations

(3.2.J), (3.2.K), (3.4.I) There are inclusion-preserving bijections

$$\{\mathfrak{p} \lhd A/I\} \leftrightarrow \{\overline{\mathfrak{p}} \lhd A \mid I \subset \overline{\mathfrak{p}}\};$$
$$\{\mathfrak{p} \lhd S^{-1}A\} \leftrightarrow \{\overline{\mathfrak{p}} \lhd A \mid \overline{\mathfrak{p}} \cap S = \varnothing\}.$$

These give embeddings

$$\operatorname{Spec} A/I \hookrightarrow \operatorname{Spec} A \text{ as a \textit{closed} subset;}$$
$$\operatorname{Spec} A_f \hookrightarrow \operatorname{Spec} A \text{ as an \textit{open} subset;}$$
$$\operatorname{Spec} S^{-1}A \hookrightarrow \operatorname{Spec} A \text{ as an arbitrary subset for arbitrary } S.$$

§3.2.7 $\operatorname{Spec} A/I \subset \operatorname{Spec} A$ 'looks like' $\mathbb{V}(I) \subset \operatorname{Spec} A$.

§3.2.8 $\operatorname{Spec} A_f = \operatorname{Spec}\left(\{1, f, f^2, \ldots\}^{-1}A\right)$ 'looks like' the set of points where f doesn't vanish, i.e.

$$\mathrm{D}(f) = \operatorname{Spec} A_f = \operatorname{Spec} A \setminus \mathbb{V}(f).$$

§3.2.8 $\operatorname{Spec} A_{\mathfrak{p}} = \operatorname{Spec}\left((A \setminus \mathfrak{p})^{-1}A\right)$ 'looks like' a shred of $\operatorname{Spec} A$ near the subspace corresponding to $[\mathfrak{p}]$ (i.e. $\mathbb{V}(\mathfrak{p})$).

3.2.3 Nilpotents

(3.4.5) If $I \subset \mathfrak{N}$ is a radical of nilpotents then $\operatorname{Spec} A/I \to \operatorname{Spec} A$ is a homeomorphism.

3.2.4 Distinguished opens

(3.5.A) $\operatorname{Spec} A \setminus \mathbb{V}(I) = \bigcup_{f \in I} \mathrm{D}(f)$; in particular the distinguished open sets for a basis for the Zariski topology.

(3.5.D) $\mathrm{D}(f) \cap \mathrm{D}(g) = \mathrm{D}(fg)$.

(3.5.E) $\mathrm{D}(f) \subset \mathrm{D}(g)$ iff $f \in \sqrt{(g)}$ iff $g^{-1} \in A_f$.

(3.5.B) $\bigcup_{i \in I} \mathrm{D}(f_i) = \operatorname{Spec} A$ iff $(f_i)_{i \in I} = A$ iff there exists $J \subset I$ with $|J| < \infty$ such that $A = \sum_{i \in J} a_i f_i$ for some $a_i \in A$.

(3.5.F) $\mathrm{D}(f) = \varnothing$ iff $f \in \mathfrak{N}$.

3.2.5 $\mathbb{V}$ and $\mathbb{I}$

(3.6.L) $[\mathfrak{q}]$ is a specilisation of $[\mathfrak{p}]$ iff $\mathfrak{p} \subset \mathfrak{q}$, thus in particular $\mathbb{V}(\mathfrak{p}) = \overline{\{\mathfrak{p}\}}$.

- $\mathbb{V}(I) \cup \mathbb{V}(J) = \mathbb{V}(IJ)$.
- $\mathbb{V}(I) \cap \mathbb{V}(J) = \mathbb{V}(I + J)$.

(3.6.I), (3.7.1), (3.7.E) $\mathbb{V}$ and $\mathbb{I}$ give an inclusion-reversing bijection

$$\begin{aligned}
\{\text{radical ideals of } A\} &\leftrightarrow \{\text{closed subsets of } \operatorname{Spec} A\}; \\
\{\text{prime ideals of } A\} &\leftrightarrow \{\text{irreducible closed subsets of } \operatorname{Spec} A\}; \\
\{\text{maximal ideals of } A\} &\leftrightarrow \{\text{closed points of } \operatorname{Spec} A\}.
\end{aligned}$$

(3.7.E) Every irreducible closed subset of $\operatorname{Spec} A$ has exactly one generic point.

- The maximal ideals of $k[x_1, \ldots, x_n]$ when k is algebraically closed are exactly $(x_1 - a_1, \ldots, x_n - a_n)$ for $(a_1, \ldots, a_n) \in A^n$.

3.2.6 Structure sheaf

(4.1.A) The natural map $A_f \to \mathcal{O}_{\operatorname{Spec} A}\big(\mathrm{D}(f)\big)$ is an isomorphism.

(4.3.2) The global sections of the structure sheaf of an affine scheme give the associated ring, i.e. if $(X, \mathcal{O}_X) \cong (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ as ringed spaces (see Section 4.1.1) then

$$\Gamma(X, \mathcal{O}_X) = \Gamma\big(\mathrm{D}(1), \mathcal{O}_{\operatorname{Spec} A}\big) = A.$$

(4.3.F) The stalk $\mathcal{O}_{X,p}$ of the structure sheaf $\mathcal{O}_X$ at a point $p = [\mathfrak{p}]$ is the local ring $A_{\mathfrak{p}}$. In particular, affine schemes (and thus schemes) are locally ringed spaces.

3.2.7 Topological properties

(3.6.3) $\operatorname{Spec} A$ is **not** connected if and only if $A \cong A_1 \times A_2$ for some non-zero rings A_1, A_2.

(3.6.B) If $Z \subset X$ is irreducible (with the subspace topology) then so too is $\overline{Z}$.

(3.6.C) If A is an integral domain then $\operatorname{Spec} A$ is irreducible.

(3.6.D) An irreducible topological space is connected (but the converse is not necessarily true).

(3.6.G) $\operatorname{Spec} A$ is quasicompact for all A (**n.b.** but in general it can still have non-quasicompact open subsets, unless $\operatorname{Spec} A$ is Noetherian).

(3.6.H) Every closed subset of a quasicompact space is quasicompact.

(3.6.I) Closed points of $\operatorname{Spec} A$ (i.e. points p such that $\{p\}$ is closed) correspond to maximal ideals (see Section 3.2.5).

(3.6.J) If k is a field and A is a finitely-generated k-algebra then the closed points of $\operatorname{Spec} A$ are dense.

(3.6.O) Connected components of a topological space are closed (if the space is Noetherian then they are open too).

(3.6.Q) Every connected component C of a topological space X is the union of irreducible components of X.

(3.6.Q) If $S \subset X$ is both open and closed then it is the union of some of the connected components of X (if X is Noetherian then the converse holds too).

(3.6.T) If A is Noetherian then $\operatorname{Spec} A$ is Noetherian.

4 Schemes

4.1 Definitions

4.1.1 Preliminary definitions

- *Isomorphism of ringed spaces*: $\pi\colon (X, \mathcal{O}_X) \xrightarrow{\sim} (Y, \mathcal{O}_Y)$ is the data of a homeomorphism $\pi : X \to Y$ and an isomorphism of sheaves $\mathcal{O}_Y \xrightarrow{\sim} \pi_*\mathcal{O}_X$ (or equivalently $\pi^{-1}\mathcal{O}_Y \xrightarrow{\sim} \mathcal{O}_X$).
- *Affine scheme*: **redefined** as any ringed space isomorphic to $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ for some A.
- *Scheme*: a ringed space $(X, \mathcal{O}_X)$ such that every point $x \in X$ has an open neighbourhood U with $(U, \mathcal{O}_X|_U)$ isomorphic to an affine scheme (**n.b.** although we don't specify a topology on X, the fact that we require it to consist locally of affine schemes means that we still call its topology the Zariski topology – see (4.3.D)).
- *Isomorphism of schemes*: this is just an isomorphism of ringed spaces.
- *Open subscheme*: the ringed space $(U, \mathcal{O}_X|_U)$ for any open subset $U \subset X$ of a scheme.
- *Closed subscheme*: the ringed space $(Z, i^{-1}(\mathcal{O}_X/\mathcal{I}))$ for any closed subset $i\colon Z \hookrightarrow X$ of a scheme and $\mathcal{I} \subset \mathcal{O}_X$ an ideal sheaf.

4.1.2 Projective schemes

Unless otherwise stated, $S_\bullet$ is $\mathbb{Z}_{\geqslant 0}$-graded.

- *Graded ring over A*: a graded ring $S_\bullet$ with $S_0 = A$; the irrelevant ideal is $S_+ = \bigoplus_{d\geqslant 1} S_d$; $\S_\bullet$ is *finitely generated over A* if S_+ is a finitely-generated ideal; $S_\bullet$ is *generated in degree one* if it is generated by S_1 as an A-algebra.
- Proj *construction*: we define the scheme $\operatorname{Proj} S_\bullet$ as follows.
  - As a set,
$$\operatorname{Proj} S_\bullet = \big\{[\mathfrak{p}] \lhd S_\bullet \mid \mathfrak{p} \text{ is homogeneous}, S_+ \not\subseteq \mathfrak{p}\big\};$$

- As a topological space (with the *Zariski* topology), the closed sets are the $\mathbb{V}(I)$ for **homogeneous** ideals $I \lhd S_\bullet$ where
$$\mathbb{V}(I) = \{[\mathfrak{p}] \in \operatorname{Proj} S_\bullet \mid I \subset \mathfrak{p}\}$$
and we define $\mathrm{D}(f) = \operatorname{Proj} S_\bullet \setminus \mathbb{V}(f)$ as before;
- As a scheme, the structure sheaf $\mathcal{O}_{\operatorname{Proj} S_\bullet}$ is defined by gluing together sheaves on the open subschemes $\mathrm{D}(f)$, where we define
$$\mathcal{O}_{\operatorname{Proj} S_\bullet}|_{\mathrm{D}(f)} = \mathcal{O}_{\operatorname{Spec}((S_\bullet)_f)_0}$$
where $((S_\bullet)_f)_0$ is the degree-zero part of the localisation of $S_\bullet$ at $\{1, f, f^2, \ldots\}$.

- *Projective n-space*: $\mathbb{P}^n_A = \operatorname{Proj} A[x_0, \ldots, x_n]$.
- *The sheaf associated to a graded module*: (recall the definition of $\widetilde{M}$ from Section 3.1.1). Let M be a graded module over $S_\bullet$ and define the sheaf $\widetilde{M}$ by
$$\widetilde{M}\big(\mathrm{D}(f)\big) = \widetilde{(M_f)_0}.$$
- *Serre twisting sheaf*: (**n.b.** this comes much later in Vakil: §14, §15). When $S_\bullet$ is **generated in degree one**, for $n \in \mathbb{Z}_{\geqslant 0}$ define $M(n) = S_\bullet[n]$ i.e. $M_d = S_{d+n}$. Then write
$$\mathcal{O}(n) = \widetilde{M(n)} = \widetilde{S_\bullet[n]}.$$
For $-n \in \mathbb{Z}_{\geqslant 0}$ (i.e. $n < 0$) define
$$\mathcal{O}(-n) = \mathcal{O}(n)^\vee.$$

4.2 Facts

4.2.1 Schemes and ringed spaces

(4.3.B) If $f \in A$ then $\mathrm{D}(f) \leftrightarrow \operatorname{Spec} A_f$ induces a natural isomorphism of ringed spaces
$$\big(\mathrm{D}(f), \mathcal{O}_{\operatorname{Spec} A}|_{\mathrm{D}(f)}\big) \cong (\operatorname{Spec} A_f, \mathcal{O}_{\operatorname{Spec} A_f})$$
i.e. they are isomorphic as affine schemes.

(4.3.D) The affine open subsets of a scheme form a basis for the (Zariski) topology on X.

(4.3.G) If f is a function on a ringed space X (i.e. $f \in \Gamma(X, \mathcal{O}_X)$) then
$$\{x \in X \mid f_x \text{ is invertible}\} \text{ is open in } X.$$
If X is **locally** ringed then
$$\{x \in X \mid f(x) = 0\} \text{ is closed in } X.$$
Thus if f vanishes nowhere then it is invertible.

4.2.2 Algebra

4.5.D If $S_\bullet$ is a graded ring over A then it is finitely-generated (as a graded ring over A) iff it is finitely-generated as a graded A-algebra (i.e. generated over $A = S_0$ by a finite number of homogeneous elements of positive degree).