Mathematics in the Modern World

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General Education

LAGUNA UNIVERSITY

Vision

Laguna University shall be a socially responsive educational institution of choice providing holistically developed individuals in the Asia-Pacific Region.

Mission

Laguna University is committed to produce academically prepared and technically skilled individuals who are socially and morally upright.

Course Code: **GE 3**

Course Description: This course deals with nature of mathematics, appreciation of its practical, intellectual, and aesthetic dimension, and application of mathematical tools in daily life. It also begins with an introduction to the nature of mathematics as an exploration of patterns (in nature and the environment) and as application of inductive and deductive reasoning. By exploring these topics, students are encouraged to go beyond the typical understanding of mathematics as merely a set of formula but as a source of aesthetics in patterns of nature, for example and a rich language in itself(and of science) governed by logic and reasoning.

The course then proceeds to survey ways in which mathematics provides a tool for understanding and dealing with various aspects of present-day living, such as managing personal finances, making social choices, appreciating geometric designs, understanding codes used in data transmission and security, and dividing limited resource fairly. These aspects will provide opportunities for actually doing mathematics in a broad range of exercises that bring out the various dimensions of mathematics as a way of knowing, and test the students' understanding and capacity.

Course Intended Learning Outcomes (CILO):

At the end of the course, students should be able to:

1. Use different types of reasoning to justify statements and arguments made about mathematics and mathematical concepts.
2. Discuss the language and symbols of mathematics.
3. Use a variety of statistical tools to process and manage numerical data
4. Analyze codes and coding schemes used for identification, privacy, and security purposes.
5. Use mathematics in other areas such as finance, voting, health and medicine, business, environment, arts and design, and recreation.
6. Appreciate the nature and uses of mathematics in everyday life.
7. Affirm honesty and integrity in the application of mathematics to various human endeavors.

Course Requirements:

- **Assessment Tasks - 60%**
- **Major Exams- 40%**

Periodic Grade 100%

Prelim Grade = 60% (Activity 1-4) + 40% (Prelim exam)

Midterm Grade = 30%(Prelim Grade) + 70 %[60% (Activity 5-7) + 40% (Midterm exam)]

Final Grade = Total CS + Final Exam x 70% + 30% of the Midterm

Final Grade = 30%(Midterm Grade) + 70 %[60% (Activity 8-10) + 40% (Final exam)]

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MODULE 1
THE NATURE OF MATHEMATICS

Introduction

This chapter introduces you to the various definitions and foundations of mathematics. The study describes mathematics in various aspects. Mathematics in our modern world is a system of knowing or understanding the term ‘Mathematics’ has been *construed and explained in various ways*.

Learning Outcomes

At the end of this module, students should be able to:

1. Recognize nature of mathematics in our world
2. Demonstrate understanding of the nature of mathematics in our world
3. Identify the foundations of mathematics and major contributions of the given scientists and their importance
4. Describing the nature of mathematics in our world, its patterns, sequences and behavior, how it is expressed, represented and used.
5. Identifying the Identify patterns in nature and their regularities

Lesson 1. The Importance of Mathematics in Various Aspects of Life

Mathematics in our modern world is a system of knowing or understanding the term ‘Mathematics’ has been construed and explained in various ways. *There are several definition of mathematics.*

Figure 1. Mathematics classes in a traditional classroom (right) and in an online setting (left)

Mathematics Has Its Own Language

Mathematics has its own language. It consists of mathematical terms, concepts, formula, theories and principles, numbers and operations, interrelations, combinations, and signs. Mathematical language is well defines, useful and clear. Mathematics is the last letter of the acronym STEM for science, technology, engineering and mathematics (Hess, 2013) program in the Senior High School.

Thus, at the tertiary level, mathematics in the modern world is a general education subject as specified by the Commission on Higher Education (CHED) Memorandum Order 20 Series of 2013. Mathematics is the most appropriate language for science.

Mathematics is the Science of Logical Reasoning

Mathematics is the science that deals with the logic of shape, quantity and arrangement (Hom, 2013). Logical reasoning is based on facts. Reasoning can be inductive or deductive. Inductive reasoning arises when +++a person uses the facts based on his observations and experiences. If a person performs a generalization based from self- evident truths, facts, postulates or actions. If the new facts coincides with previously established facts, it is logical or rational.

Mathematics is challenging to some, yet simple and affluent to others. Mathematics develops our reasoning, analytical thinking, infuse practicality and can be applied in our day –to-day activities (Hom, 2013).

Solving a mathematical problem is like solving our problems encountered or will be encountering in the future. Here, we gather data or facts, identify the problem, and the formula, use a system, process or procedure of solving based on the data gathered or facts. Last but not the least, to arrive at a conclusion or final value/solution (Hom, 2013).

Mathematics Is the Science of Quantity and Space

Mathematics as the science of quantity and space, deals with quantifiable facts, and relationships, problems and solutions involving space and form. This is due to our first-hand knowledge and various types of experiences. Any object or thing can be measured, divided, or multiplied. A line, weight, time can be measured. Weight is a quantity that can be measured, in grams, kilograms, pounds or ounces.

Meriam-Webster (n.d.) defines mathematics as the Science of numbers and their operations, the interrelations OF THUIS, combinations, generalizations, and abstractions and of space, measurement, and generalizations. But ultimately all types of explanations conclusively end at some kinds of relationship with number and space. Mathematics, is therefore, a part and product of the human or people's observations, facts gathered, measures, and conclusions made.

Mathematics Is A Means Of Generalization, To Draw Conclusion And Judgement, Inference Or Decisions, And Helps Predict The Behavior Of Nature And Phenomena In The World

Inferring is an inductive process. In this, the student of learner can construct new meaning when they recognize a pattern or relationship. It can be within what they already know or what they are knowledgeable at. The student or learner draws numerical inferences based on the given information and data. He or she draws numerical inferences on the basis of given information and data.

Mathematics is a conceptual language. Data or facts must be available so that generalizations can be made. Corwin.com (n.d.) defines "generalization" is a way to comprehend or form the relationship between two or more concepts.

For the student to understand the concepts, the teacher or professor summarize what we would like our students to take away after the unit or lesson. As Inferring is an inductive process, predicting is a deductive process. Once students have been able to infer a pattern, they test that pattern by making a prediction (Corwin.com, n.d.).

With repetition, the students learn how to infer and predict within the context of mathematics teaching or instruction. As to confirm the understanding of the student on the mathematical concepts, through a multi-sensory approach, they need to experience the patterns again and again in order to truly understand them.

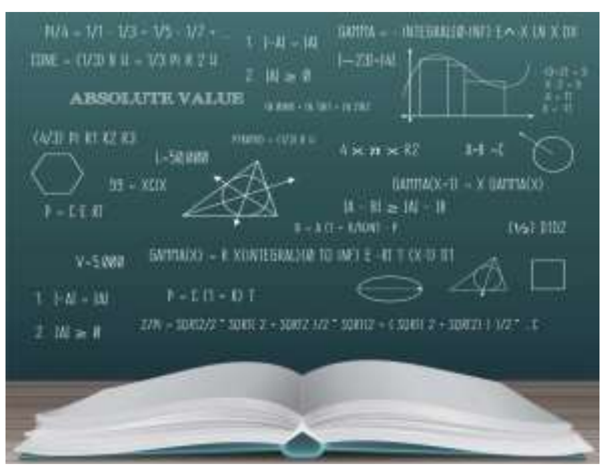


Figure 2. Mathematics and Education (Akshay, 2020)

Mathematics Is an Art

We see a great diversity of living things, from the microscopic to the gigantic (Study.com, n.d.). The human person conceives most simple to the complex; from bright colors to dull ones. One of the most intriguing things we see in nature is patterns. We tend to think of patterns as sequences or designs that are orderly and that repeat.

We see patterns in nature. It can be the most interesting that we can find. It helps organize patterns and regularities in the world. Mathematics shows the hidden patterns in nature that help us understand the world around us. It is a discipline that deals with data, measurements, and observations from science. In addition to those, are inference, deduction, and proofs. Included also are the mathematical models that are used in natural phenomena, the human behavior, and social systems. Again, in mathematics, it involves observation, conjectures and calculations.

There is also mathematics in art. Since Pythagoras, the most famous mathematician, discovered numerical reasons in musical harmony, the relationship between mathematics and art has been permanent. These aspects of mathematics make them a bridge between the humanities and the natural sciences, between the two cultures (Akshay, 2020).

Mathematics Is an Applied Science for the Expression of Other Sciences

Mathematics is not an invention (Bellis, 2020). All of the discoveries and laws of science are not considered inventions. The inventions of Thomas Edison, Albert Einstein, the Wright brothers, Alexander Graham Bell and Nicola Tesla, Karaoke Sing along System is a Filipino invention. Inventions like the cotton gin, the Internet, automobile, gadgets and appliances. All material things and processes are considered inventions. There are several different branches of mathematical science, which include algebra, geometry and calculus.

Mathematics is at the center of our culture and its history is often confused with that of philosophy. Just as the cosmological and evolution theories have exerted considerable influence on the conception that humans have of ourselves, the non-Euclidean geometries have allowed new ideas about the universe and theorems of mathematical logic have revealed the limitations of the deductive method. (Akshay, 2020)

Figure 3. Mathematics and Education (Akshay, 2020)

Mathematics is the science of logical reasoning

Mathematics is the science that deals with the logic of shape, quantity and arrangement (Hom, 2013). Logical reasoning is based on facts. Reasoning can be inductive or deductive. Inductive reasoning arises when a person uses the facts based on his observations and experiences. If a person performs a generalization based from self- evident truths, facts, postulates or actions. If the new facts coincides with previously established facts, it is logical or rational (Hom, 2013).

Mathematics is the method of progress of various subjects, in the world in the improvement or our civilization

Mathematics has its significant role in science and technology. Mathematical knowledge is applied in the study of science and in this different branches. For example physics, chemistry, biology, and other sciences. In addition, it is useful for different branches of science but also helps in its progress and organization. Mathematics has developed far beyond basic counting. This growth has been greatest in society's complex enough to sustain these Assessment Task and to provide leisure for contemplation and the opportunity to build on the achievements of earlier mathematicians.

Math is all around us. Its product are all around us. The mobile devices, architecture of our house, school or malls. Even money, investments, finance, engineering, and even sports. People will often think of mathematics as it is applied in sciences and engineering. The awareness in mathematics will bring about ways for people to solve problems, make decisions. In this COVID-A9 pandemic, the increasing no. of infected persons allow the government to find means of solving the problems, increase the testing centers and health facilities, protective equipment, laws and policies to overcome the increasing number of infected persons, young and old (Hom, 2013).

Mathematics Education and the Future of a Child or Student

Mathematics for learning is the same as learning in a sports. In a sports, a person's strength, tenacity, smart and healthy. Parents prepare you with your future that is to have the skills, good health the potential to earn money or income to support himself, his family and his other material dreams or needs. Mathematics will help the person earn income, invest and not to lose money. In particular, if you are the least bit familiar with statistics and calculations of interest, in a very easy way you will recognize the economic fraud and sellers of fog. With the help of science like mathematics, you will avoid a waste of money on various projects and tips that you believe can help you (Akshay, 2020). Mathematics is a ticket to the world and to solve global poverty.

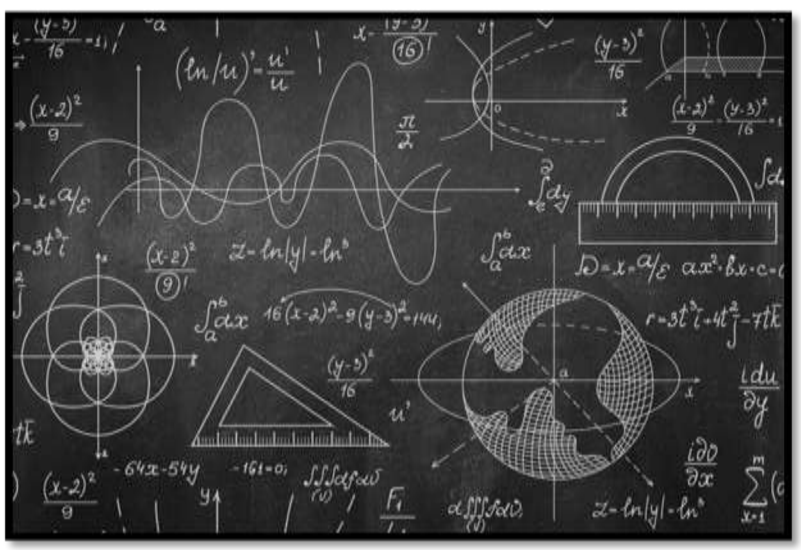
Mathematics and the Future

Mathematics will be more and widely used and well- represented in the future. It becomes a valuable and inevitable factor in every profession, industries and profession. A teacher or professor calculates time and plans to manage the class, computes the grades of the students, too. A policeman and or military personnel relies on the data. Politicians and Journalists analyze more and speaks less on issues, until they are sure of it. On COVID-19 pandemic, World Health Organization the Philippines Department of Education release the no. of positive cases, cured and death cases. Future mechanics and carpenters will use optimization electronics and analysis as much as they will use a hammer and a wrench (Akshay, 2020).

Lesson 2. The Historical Foundations of Mathematics

Since the beginning of recorded history, mathematic discovery has been at the forefront of every civilized society, and in use in even the most primitive of cultures. The needs of math arose based on the wants of society. The more complex a society, the more complex the mathematical needs. Primitive tribes needed little more than the ability to count, but also relied on math to calculate the position of the sun and the physics of hunting (Hom, 2013).

As an example, when civilization began to trade, a need to count was created. When humans traded goods, they needed a way to count the goods and to calculate the cost of those goods. The very first device for counting numbers was, of course, the human hand and fingers represented quantities. And to count beyond ten fingers, mankind used natural markers, rocks or shells. From that point, tools such as counting boards and the abacus were invented. (Bellies, 2020)



Source: https://www.rd.com/list/math-lessons-real-life/

Figure 4. Mathematics

There were several civilizations contributed to mathematics as we know it today. There are China, India, Egypt, Central America and Mesopotamia. The Sumerians were the first people to develop a counting system (Eduwoke, n.d.). These mathematicians developed arithmetic, which includes basic operations, multiplication, fractions and square roots.

The Sumerians' system passed through the Akkadian Empire to the Babylonians around 300 B.C. Six hundred years later, in America, the Mayans developed elaborate calendar systems and were skilled astronomers (LiveScience, 2013). About this time, the concept of zero was developed.

Mathematics as an organized science did not exist until the classical Greek period from 600 to 300 B.C. (Bellis, 2020). There were, however, prior civilizations in which the beginnings or rudiments of mathematics were formed.

The past 2000 years, bring forth a solid foundations of mathematics Euclid's Elements (*c.* 300 BCE), which presented a set of formal logical arguments based on a few basic terms and axioms, provided a systematic method of rational exploration that guided mathematicians, philosophers, and scientists well into the 19th century (Lambek & Redpath, 2017).

Sir Isaac Newton's notion of fluxions or derivatives) in 1665. Fluxion, in mathematics is the original term for derivative, in the calculus, raised by the Anglo-Irish empiricist George Berkeley (among others), did not call into question the basic foundations of mathematics (Britannica.com, 2017).

The discovery in the 19th century of consistent alternative geometries, however, precipitated a crisis, for it showed that Euclidean geometry, based on seemingly the most intuitively obvious axiomatic assumptions, did not correspond with reality as mathematicians had believed. This, together with the bold discoveries of the German mathematician Georg Cantor in set theory, made it clear that, to avoid further confusion and satisfactorily answer paradoxical results, a new and more rigorous foundation for mathematics was necessary. Mathematics has served as a model for rational inquiry in the West and is used extensively in the sciences.

The 20th-century quest to rebuild mathematics on a new basis independent of geometric intuitions. Early efforts included those of the logicist school of the British mathematicians Bertrand Russell and Alfred North Whitehead, the formalist school of the German mathematician David Hilbert, the intuitionist school of the Dutch mathematician

L.E.J. Brouwer, and the French set theory school of mathematicians collectively writing under the pseudonym of Nicolas Bourbaki. Some of the most promising current research is based on the development of category theory by the American mathematician Saunders Mac Lane and the Polish-born American mathematician Samuel Eilenberg following World War II.

Lesson 3. The ABCs of Mathematics

Below are the important developments in the history of mathematics, beginning from A to Z.

Abacus

The abacus was among the first tools in counting. It was invented around 1200 B.C. in China and was used in many ancient civilizations, including Persia and Egypt.

Figure 5. Abacus (Tchi, 2019)

Accounting

Accounting in the Renaissance era was used to monitor the lower socio-economic classes' property owners. The Italians of the Renaissance period (14th through 16th century) are widely recognized to be the fathers of modern accounting.

Figure 6. Renaissance Accounting (Spatial Perception & Concrete Experience, n.d.)

Algebra

The first treatise on algebra was written by Diophantus of Alexandria in the 3rd century B.C. Algebra comes from the Arabic word al-jabr, an ancient medical term meaning "the reunion of broken parts." Al-Khawarizmi is another early algebra scholar and was the first to teach the formal discipline.


DIOPHANTI
ALEXANDRINI
ARITHMETICORVM
LIBRI SEX.
LVTETIAE PARISIORVM,
fub Scuto Solari.
M. DC. XXI.

Figure 7. Algebra

(Arithmetica by Diophantus of Alexandria, n.d.)

Archimedes

Archimedes was a mathematician and inventor from ancient Greece. He was best known for his discovery of the relationship between the surface and volume of a sphere and its circumscribing cylinder for his formulation of a hydrostatic principle (Archimedes' principle) and for inventing the Archimedes screw (a device for raising water) (Bellies, 2020).

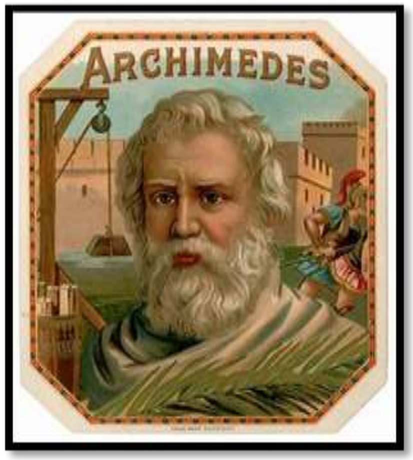


Figure 8. Archimedes

Gottfried Wilhelm Leibniz

Gottfried Wilhelm Leibniz (1646-1716) was a German philosopher, mathematician and logician who is probably most well known for having invented differential and integral calculus. He did this independently of Sir Isaac Newton. (Lim, n.d.)

Figure 9. Gottfried Wilhelm Leibniz

Graph

A graph is a pictorial representation of statistical data or of a functional relationship between variables. William Playfair (1759-1823) is generally viewed as the inventor of most graphical forms used to display data, including line plots, the bar chart, and the pie chart (Bellies, 2020).

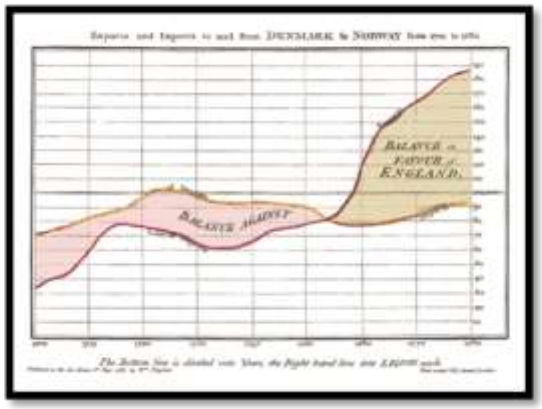


Figure 10. William Playfair Timeline Series

Math Symbol

In 1557, the "=" sign was first used by Robert Record. In 1631, came the “>" sign.

Pythagoreanism

Pythagoreanism is a school of philosophy and a religious brotherhood believed to have been founded by Pythagoras of Samos, who settled in Croton in southern Italy about 525 B.C. The group had a profound effect on the development of mathematics (Thesleff, 2013).

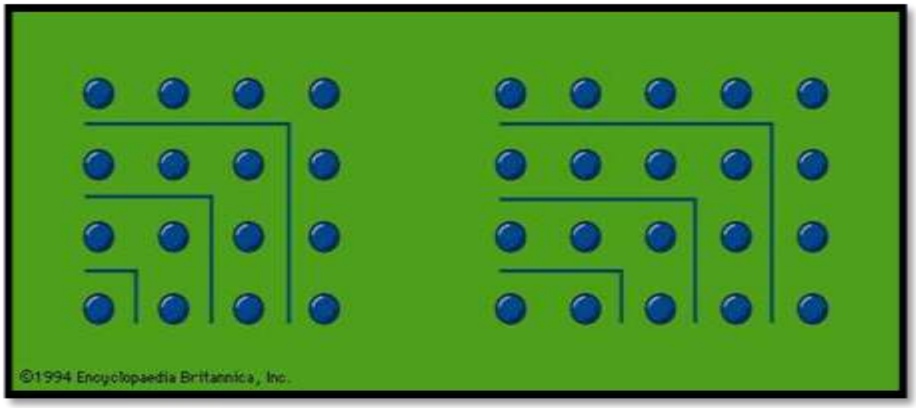


Figure 11. Gnomons of Pythagorean Number Theory (Encyclopedia Britannica, Inc.)

Protractor

The simple protractor is an ancient device. As an instrument used to construct and measure plane angles, the simple protractor looks like a semicircular disk marked with degrees, beginning with 0º to 180º. (How Maths Can Answer Questions We Haven't Thought of Yet, n.d.)

The first complex protractor was created for plotting the position of a boat on navigational charts. Called a three-arm protractor or station pointer, it was invented in 1801 by Joseph Huddart, a U.S. naval captain. The center arm is fixed, while the outer two are rotatable and capable of being set at any angle relative to the center one.

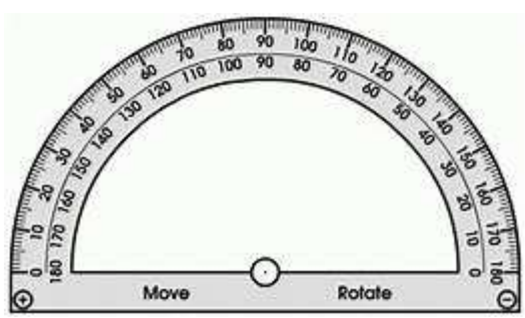


Figure 12. (Modern) Protractor

Slide Rulers

Circular and rectangular slide rules, an instrument used for mathematical calculations, were both invented by mathematician William Oughtred (Oughtred, 2020).

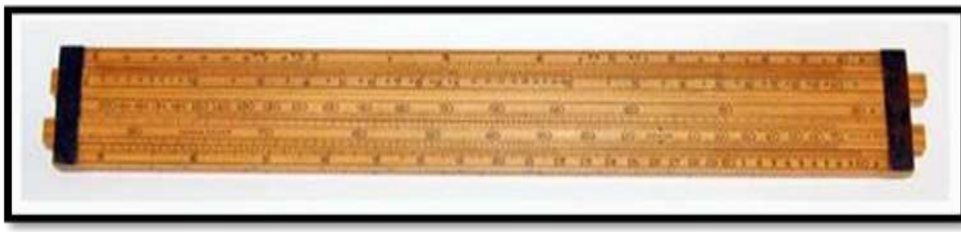


Figure 13. Slide Rule

Zero

Zero was invented by the Hindu mathematicians Aryabhata and Varamihara in India around or shortly after the year 520 A.D. (Bellies, 2020).

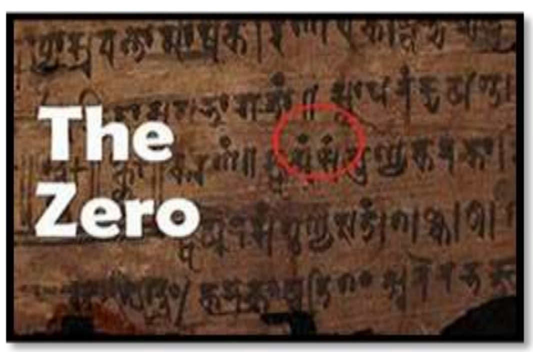


Figure 14. The (Ancient) Zero

Lesson 4. Patterns in Nature

Patterns are regular, repeated, or recurring forms of design. Pattern is a combination of elements or shapes repeated in a recurring and regular arrangement (Lamp, n.d.).

At a glance, the natural world may appear overwhelming in its diversity and complexity, there are regularities running through it, from the hexagons of a honeycomb to the spirals of a seashell and the branching veins of a leaf (Ball, 2016).

Fessenden (2016) described the curl of a chameleon's tail, the spiral of a pinecone's scales and the ripples created by wind moving grains of sand all have the power to catch the eye and intrigue the mind. According to her, when Charles Darwin first proposed the theory of evolution by natural selection in 1859. Darwin elaborated on the subject with his theory of evolution and his book, *On the Origin of Species*, published in 1859. His work with Darwin's finches and his ideas on survival of the fittest explained the mechanism of natural selection and how it could lead to a proliferation of many different kinds of organisms (Markgraf, 2019).

Figure 15. Chamelion's Tail (Fessenden, 2016)

It encouraged science enthusiasts to find reasons for the natural patterns seen in beasts of the land, birds of the air and creatures of the sea. The peacock's plumage, the spots of a shark must all serve some adaptive purpose, they eagerly surmised (Fessenden, 2016).

Ball (2016) described the Patterns in Nature explores not only the math and science but also the beauty and artistry behind nature's awe-inspiring designs. Unlike the patterns we create in technology, architecture, and art, natural patterns are formed spontaneously from the forces that act in the physical world. Spirals, stripes, branches, and fractals, say—recur in places that seem to have nothing in common, as when the markings of a zebra mimic the ripples in windblown sand are common patterns in nature.

That's because, as Patterns in Nature shows, at the most basic level these patterns can often be described using the same mathematical and physical principles: there is a surprising underlying unity in the kaleidoscope of the natural world (Ball, Patterns in Nature: Why the Natural World Looks the Way It Does).

Math Patterns in Nature

The following are mathematical patterns:

Fractal

A fractal is a detailed pattern that looks similar at any scale and repeats itself over time. A fractal's pattern gets more complex as you observe it at larger scales. This example of a fractal shows simple shapes multiplying over time, yet maintaining the same pattern. Examples of fractals in nature are snowflakes, trees branching, lightning, and ferns (Math Patterns in Nature, 2020).

Ferns grow naturally in the wood because there is a lot of tree cover and they are shade loving. A snowflake begins to form when an extremely cold water droplet freezes onto a pollen or dust particle in the sky. This creates an ice crystal. As the ice crystal falls to the ground, water vapor freezes onto the primary crystal, building new crystals – the six arms of the snowflake (NORA, 2016).

Figure 16. Ferns
(Features of the Wood: Fern Garden, 2013)

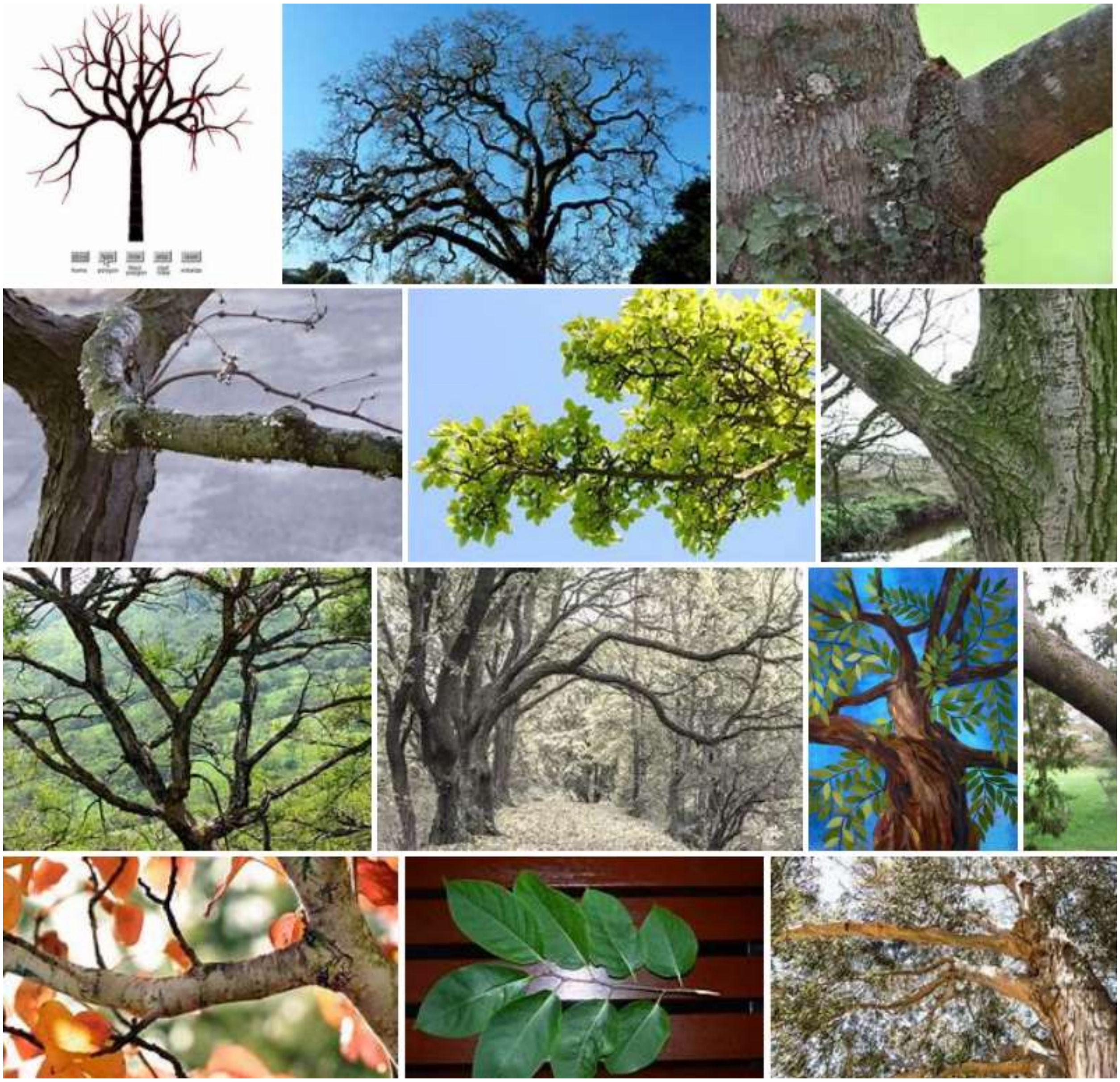


Figure 17. Trees Branching

(Google, n.d.)

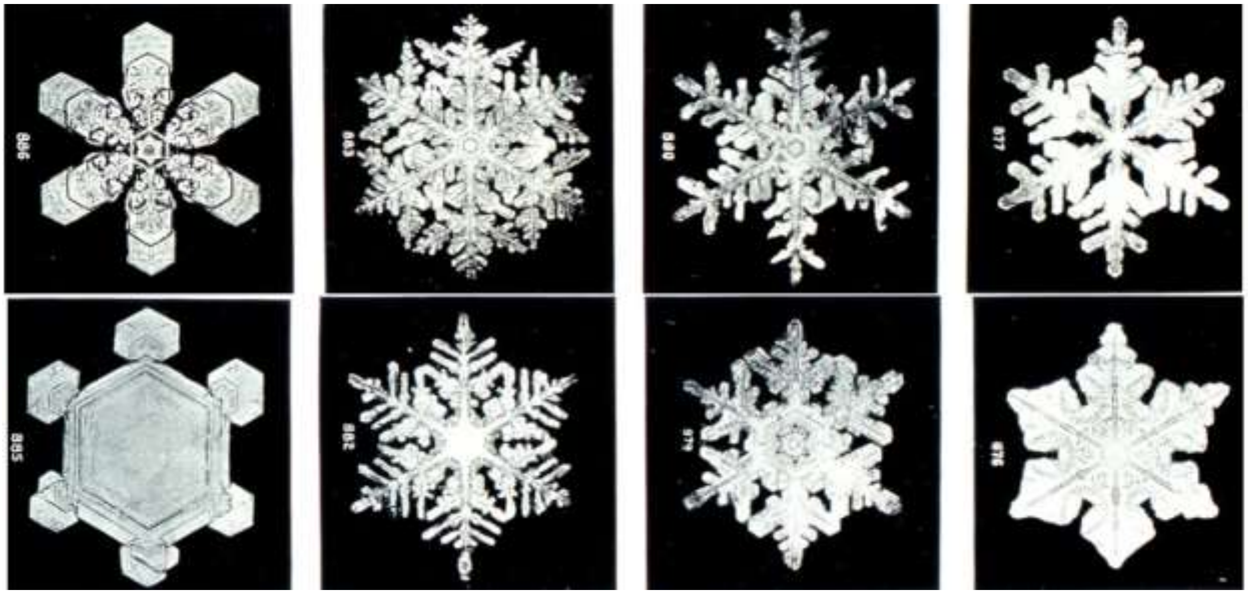


Figure 18. Snowflakes

(NORA, 2016)

The intricate shape of a single arm of the snowflake is determined by the atmospheric conditions experienced by entire ice crystal as it falls. A crystal might begin to grow arms in one manner, and then minutes or even seconds later, slight changes in the surrounding temperature or humidity causes the crystal to grow in another way. Although the six-sided shape is always maintained, the ice crystal (and its six arms) may branch off in new directions. Because each arm experiences the same atmospheric conditions, the arms look identical (How do snowflakes form? Get the science behind snow, 2016)

Figure 19. Spirals

(Woolfe, 2013)

Spirals

Spirals are a common shape found in nature, as well as in sacred architecture. In the natural world, we find spirals in the DNA double helix, sunflowers, the path of draining water, weather patterns (including hurricanes), vine tendrils, phyllotaxis (the arrangement of leaves on a plant stem), galaxies, the horns of various animals, mollusc shells, the nautilus shell, snail shells, whirlpools, ferns and algae. Look at a cross-section of red cabbage and you will see spirals. Look at your fingertip, where you would make a fingerprint, and you will see a spiral. Even the shape of your hair at the crown of your head…a spiral. Spirals seem to permeate many diverse natural formations: inorganic and organic, lifeless and alive, non-conscious and conscious (Woolfe, 2013).

Woolfe (2013) points out, supercomputer simulations have shown that the spiral arms of galaxies are not transient, random formations, but are "self-perpetuating, persistent, and surprisingly long lived." The spiral structure is not just a chance happening. Spiral galaxies are probably the most common kind of galaxy in the universe (our own Milky Way galaxy is a spiral galaxy).

Figure 20. The Mily Way

(Woolfe, 2013)

Figure 21. *Sunflower*

(Woolfe, 2013)

Plants may display logarithmic spirals, usually in the form of a Fibonacci spiral if based on the Fibonacci sequence. The Fibonacci spiral itself is an approximation of the golden spiral, which is based on the famous golden ratio (represented by the Greek letter phi). This approximation exists because the ratio between successive Fibonacci numbers is close to the golden ratio. It is argued that logarithmic spirals are so common in biological organisms because it is the most efficient way for something to grow. By maintaining the same shape through each successive turn of the spiral, the least amount of energy is used.

Each new cell in a plant is formed in turns, so how much of a turn should you have between new cells? If you don't have a turn at all, you get a straight line, but that is a very poor design since it involves many gaps. The best solution to this problem is the 'golden ratio' which creates a round shape with no gaps in it – the sunflower discovered this purely through the process of natural selection. The golden ratio works because it is an irrational number, meaning it cannot be turned into a fraction – in fact, it is as far as you can get from any fraction (Woolfe, 2013).

Figure 22. Snails

(Woolfe, 2013)

Figure 23. Lightning

(Smyth, 2020)

Lightning is an electrical discharge caused by imbalances between storm clouds and the ground, or within the clouds themselves. Most lightning occurs within the clouds (Smyth, 2020). "Sheet lightning" describes a distant bolt that lights up an entire cloud base. Other visible bolts may appear as bead, ribbon, or rocket lightning. In the figure, Lightning forks and rejoins itself over Table Mountain and Lion's Head in Cape Town, South Africa. Central Africa is the area of the world where lightning strikes most frequently (Smyth, 2020).

Figure 24. Various Patterns
(Hunter, 2020)

Varonoi

A Voronoi pattern provides clues to nature's tendency to favor efficiency: the nearest neighbor, shortest path, and tightest fit. Each cell in a Voronoi pattern has a seed point. Everything inside a cell is closer to it than to any other seed. The lines between cells are always halfway between neighboring seeds. In computer graphics, Voronoi patterns consist of lines connecting points that are the centers of circles which intersect with the model's triangle mesh to create a tube like structure replacing the original polygons (Marshallpeck, 2020).

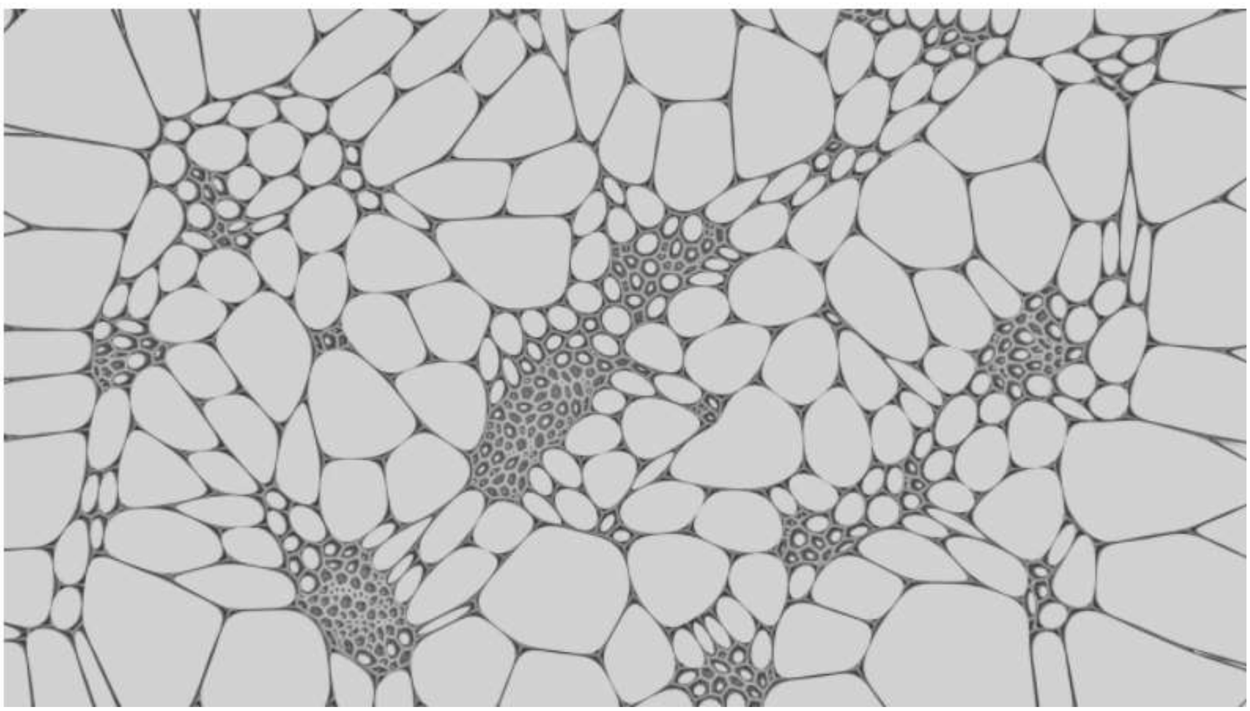


Figure 25. Varonoi
(Hunter, 2020)

Other examples of Voronoi patterns are the skin of a giraffe, corn on the cob, honeycombs, foam bubbles, the cells in a leaf, and a head of garlic. (Math Patterns in Nature, 2020)

Lesson 5. Numbers, Patterns and Sequences

Natural numbers

Natural numbers can mean either "counting numbers" {1, 2, 3, ...}, or "whole numbers" {0, 1, 2, 3, ...}, depending on the subject. In fact, a Real Number can be thought of as any point anywhere on the number line. Writing numbers down on a number line makes it easy to tell which numbers are greater or lesser. we can use the number line to help us add. we always move to the right to add. Any number that can be written as a fraction is called a Rational Number. So, if "p" and "q" are integers (remember we talked about integers), then p/q is a rational number. (Math Open Reference, 2020)

- Example: If p is 3 and q is 2, then: **p/q** = 3/2 = **1.5** is a rational number

The Number Line

- We can use the number line to help us subtract. We always move to the left to subtract.
- A number on the left is less than a number on the right.

Examples:

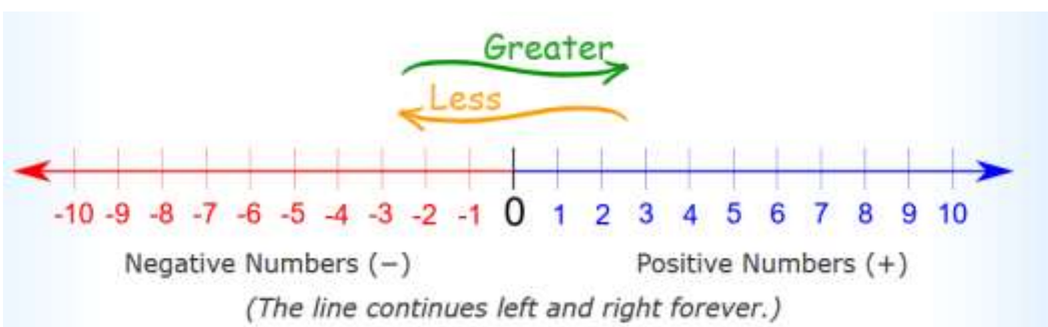


Figure 24. The Number Line (mathisfun.com)

5 is less than 8

−1 is less than 1

−8 is less than −5

- A number on the **right is greater** than a number on the left.

Examples:

8 is greater than 5

1 is greater than −1

−5 is greater than −8

Examples:

Maria borrowed $3 to pay for his lunch

Jose borrowed $5 to pay for her lunch

Rhian had enough money for lunch and has $3 left overPlace these people on the number line to find who is poorest and who is richest.

Owing money is negative

Having money is positive

- So Maria has "−3", Jose has "−5" and Rhian has "+3"

Now it is easy to see that:

- Maria is poorer than Jose (−5 is less than −3) and John is poorer than Rhian (−3 is less than 3) and Rhian is, of course, the richest!

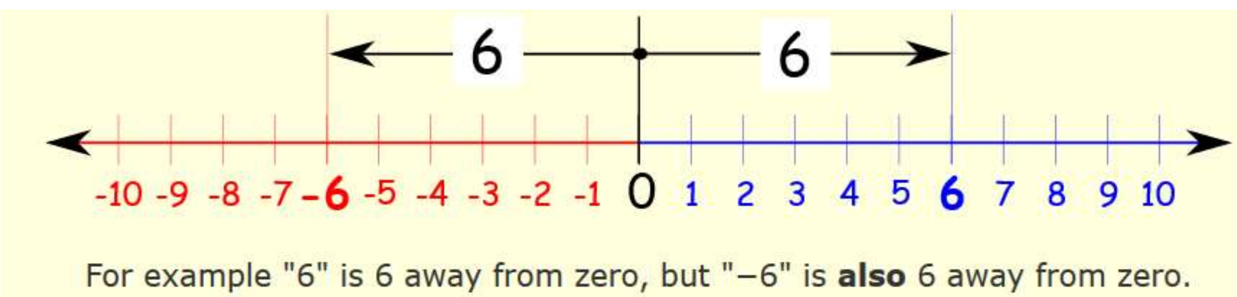


Figure 24. The Absolute Number (mathisfun.com)

- Whole Numbers are simply the numbers 0, 1, 2, 3, 4, 5, ... (and so on)

Examples: 0, 7, 212 and 1023 are all whole numbers

But, numbers like ½, 1.1 and 3.5 are not whole numbers.

- Counting Numbers are Whole Numbers, but without the zero. Because you can't "count" zero.
- So they are 1, 2, 3, 4, 5, ... (and so on).

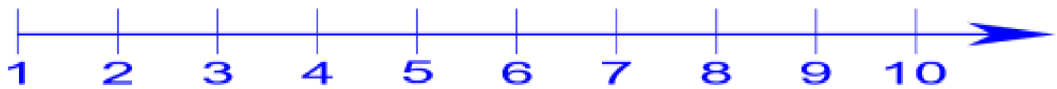


Figure 25. The Number Line (positive integers)

- Integers are like whole numbers, but they also include negative numbers ... but still no fractions allowed!
- So, integers can be negative {-1, -2,-3, -4, -5, ... }, positive {1, 2, 3, 4, 5, ... }, or zero {0}

We can put that all together like this:

Integers = { ..., -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, ... }

Examples: −16, −3, 0, 1 and 198 are all integers.

But numbers like ½, 1.1 and 3.5 are not integers.

The idea of zero, though natural to us now, was not natural to early humans ... if there is nothing to count, how can we count.

There are the numbers:

- Integers = { ..., -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, ... }
- Negative Integers = { ..., -5, -4, -3, -2, -1 }
- Positive Integers = { 1, 2, 3, 4, 5, ... }
- Non-Negative Integers = { 0, 1, 2, 3, 4, 5, ... } *(includes zero)*

- And when you only want positive integers, say "positive integers". It is not only accurate, it makes you sound intelligent. Like this (note: zero isn't positive or negative):

Name	Numbers	Examples
Whole Numbers	{ 0, 1, 2, 3, 4, 5, ... }	0, 27, 398, 2345
Counting Numbers	{ 1, 2, 3, 4, 5, ... }	1, 18, 27, 2061
Integers	{ ... -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, ... }	-15, 0, 27, 1102

Figure 25. Examples of Whole number, counting number and integers.

Rational Numbers:

{p/q : p and q are integers, q is not zero}

So half (½) is a rational number. And **2** is a rational number also, because we could write it as **2/1**. So, Rational Numbers include: all the **integers** and all **fractions**.

Even a number like 13.3168980325 is a Rational Number:

13.3168980325 = 133,168,980,325 / 10,000,000,000

So, the **square root of 2** (√2) is an irrational number. It is called irrational because it is not rational (can't be made using a simple ratio of integers). It isn't crazy or anything, just not rational. And we know there are many more irrational numbers. Pi (π) is a famous one.

So irrational numbers are useful. We need them to: *find the diagonal distance across some squares, to work out lots of calculations with circles (using π), and more, So we really should include them.*

And i (the square root of -1) times any Real Number is an Imaginary Number. So these are all Imaginary Numbers:

3i

-6i

0.05i

πi

Real Vs. Imaginary Numbers

Real number is a number whose square is non-negative. In mathematical notation, we denote the set of real numbers by the symbol R. Therefore for all x, if x ϵ R then x2 ≥ 0. In a more rigorous way, can introduce the set of real numbers as the unique, complete totally ordered field with the binary operation + and . along with the order relation <. This order relation follows the trichotomy law, which states that given two real numbers x and y, one and only one of these 3 holds; x >y, x <y or x =y.In fact they are often used together (Difference Between Real Numbers and Imaginary Numbers, 2011).

An imaginary number is a number whose square is negative. In other words, numbers like √(-1), √(-100) and √(-*e*) are imaginary numbers. All the imaginary numbers can be written in the form *a* **i** where **i** is the 'imaginary unit' √(-1) and *a* is a non-zero real number. (Observe that $\mathbf{i}^2 = -1$) (Difference Between Real Numbers and Imaginary Numbers, 2011).

Complex Numbers

If we put a Real Number and an Imaginary Number together we get a new type of number called a Complex Number and here are some examples:

- 3 + 2i
- 27.2 - 11.05i

Type of Number	Quick Description
Counting Numbers	{1, 2, 3, ...}
Whole Numbers	{0, 1, 2, 3, ...}
Integers	{..., -3, -2, -1, 0, 1, 2, 3, ...}
Rational Numbers	p/q : p and q are integers, q is not zero
Irrational Numbers	Not Rational
Real Numbers	Rationals and Irrationals
Imaginary Numbers	Squaring them gives a negative Real Number
Complex Numbers	Combinations of Real and Imaginary Numbers

Figure 26. Examples of Type of Number and Quick Description

Patterns in Numbers

The first thing you might notice about the sequence 1, 4, 9, 16, 25, 36, 49 is that the numbers all keep getting bigger and bigger. Is that a pattern? Sure. But is it an interesting pattern? In other words, can it be used to help figure out what the next number in the sequence has to be—since that's what we're really interested in here (Marshall, 2012).

What comes next?

Figure 27. Sample sequence (Smiley figure)

What would be the next face of the sequence?

- *There is an alternating face/s. Logically, the next face is the:* _____

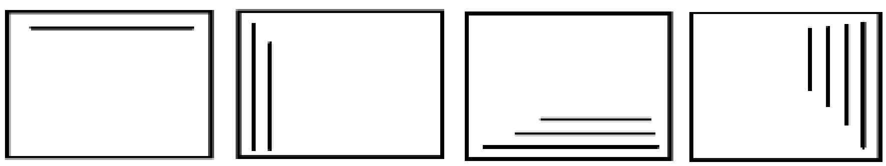


Figure 27. Boxes with lines in sequence

The lines seem to rotate in 90-degree interval

The length of the lines in the square is decreasing

Look into the no. of lines in the box

What comes next? ______________________________

Sequence

A sequence is an ordered list of numbers (or other elements like geometric objects), that often follow a specific pattern or function. Sequences can be both finite and infinite. It is a chain of numbers (or other objects) that usually follow a particular pattern. The individual elements in a sequence are called terms (Methigon, 2020).

The terms of a sequence are all its individual numbers or elements. It is a string of objects, like numbers, that follow a particular pattern. The individual elements in a sequence are called terms. An ordered list of numbers called terms that may have repeated values. The arrangement of the terms is set by a definite rule. Examples below,

3 6 9 12 15 18 21	*Add 3 to the previous number to get the next number.*
0 12 24 36 48 60 72	*Add 12 to the previous number to get the next number*
10 8 16 14 28 26 52	*Alternately subtract 2 and multiple by 2 to get the next number*
0 2 6 12 20 30 42	*Add increasing even numbers to get the next number*

Here are a few examples of sequences. Can you find their patterns and calculate the next two terms? Share your answers to the class:

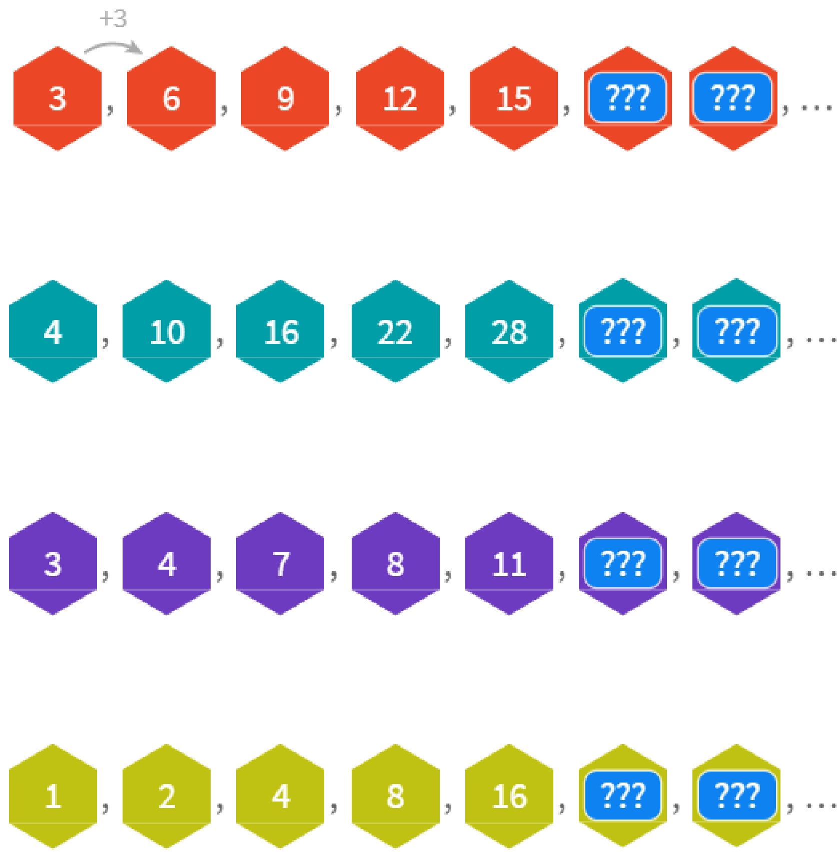


Figure 28. Numbers in Sequence (Methigon, 2020)

ASSESSMENT TASK 1-1

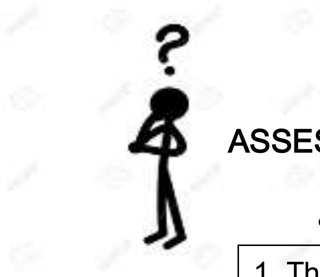
Instruction: Write/type your answer on a Word document (and print/upload) on letter-size bond paper/s.

1. Look for two (2) great mathematicians, and discuss their contributions in mathematics.

2. Georg Cantor's greatest contribution to mathematics was probably his Diagonal Method, which has been borrowed to prove, among other things, the impossibility of a solution to the halting problem. What did Cantor use this method to prove?

3. Form a group of five (5) students, presents your ABCs of mathematics/Video clip or in Word/app.

M –
A –
T –
H –
E –
M –
A –
T –
I -
C -
S –



ASSESSMENT TASK 1-2

- ESSAY. Reflect and Discuss on the following:

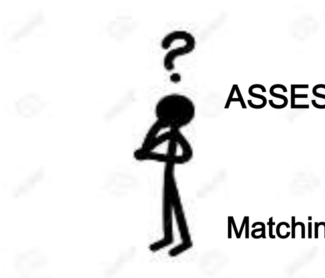
1. The Story of Mathematics: https://www.storyofmathematics.com/
2. Mathematics : https://www.storyofmathematics.com/
3. Mathematics in the Modern World : https://www.youtube.com/watch?v=mO_Y10QBD3Y&feature=youtu.be

4. Mathematics in the Modern World: https://www.youtube.com/watch?v=mO_Y10QBD3Y&feature=youtu.be
5. Mathematics in the Modern World https://www.youtube.com/watch?v=mO_Y10QBD3Y&feature=youtu.be
6. Math is the hidden secret to understanding the world https://www.youtube.com/watch?v=ZQElzjCsl9o&feature=youtu.behttps: //

ASSESSMENT TASK 1-3

Instruction: Read carefully then answer the following questions, Write your answer on the space provided.

	1.	It is the method of progress of various subjects, in the world in the improvement or our civilization
	2.	A mathematician and inventor from ancient Greece best known for his discovery of the relationship between the surface and volume of a sphere and its circumscribing cylinder
	3.	One of the first tools for counting invented around 1200 B.C
	4.	Circular and rectangular slide rules, an instrument used for mathematical calculations, were both invented by mathematician.
	5.	It was invented by the Hindu mathematicians Aryabhata and Varamihara in India around or shortly after the year 520 A.D.



ASSESSMENT TASK 1-4

Matching Type

Match Column A with the corresponding answer on Column B.

	A		B
1.	Calendar Systems	A.	*Mathematics*
2.	It is a school of philosophy and a religious brotherhood believed to have been founded by Pythagoras of Samos.	B.	*Mathematics has its own language:*
3.	In our modern world, it is a system of knowing or understanding.	C.	*Mayans*
4.	It consists of mathematical terms, mathematical concepts, formula, theories, principles, numbers and their operations, interrelations, combinations, and signs.	D.	*William Playfair*
5.	A graph is a pictorial representation of statistical data or of a functional relationship between variables.	E.	*Pythagoreanism*

ASSESSMENT TASK 1-5

1. Which one of the following is not a rational number?

   a. ½ b. 22/7 c. Π

2. Which one of these numbers is real?

   a. I_2 b. i^4 C. $5i^4$

Find the pattern and continue the following sequences. For the first four sequences also try to find a closed form expression.

3. 5, 8, 11, 14, 17, ___, ___, ___, …
4. 25, 21, 17, 13, 9, ___, ___, ___, …
5. 4, 6, 9, 13, 18, ___, ___, ___, ___, …
6. 1, 3, 9, 27, 81, ___, ___, ___, ___, …
7. 3, 6, 5, 10, 9, 18, ___, ___, ___, ___, …
8. 2, 4, 8, 10, 20, 22, 44, ___, ___, ___, ___, …
9. 1, 2, 4, 5, 8, 9, 13, ___, ___, ___, ___, …
10. 3, 3, 4, 8, 10, 30, 33, ___, ___, ___, ___, …

B. Look for the pattern and write the missing number.

1. 1, 4, 9, 16, 25, ____ , ____ , ____ , ____
2. 1, 3, 6, 8, 16, 18, 36, ____ , ____ , ____ , ____
3. 4, 13, 22, 31, ____ , ____ , ____ , ____
4. 64, 32, 16, 8, ____ , ____ , ____ , ____
5. 10, 6, 24, 20, 80, ____ , ____ , ____ , ____
6. -3, 6, -12, 24, ____ , ____ , ____ , ____
7. 8, 20, 36, 56, ____ , ____ , ____ , ____
8. 3, 12, 48, ____ , ____ , ____ , ____
9. 4, 20, 100, ____ , ____ , ____ , ____
10. $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{6}$, ____ , ____ , ____ , ____

C. Find the missing number in each the following patterns.

1.

11	=	1
11 x 11	=	121
11 x 11 x 11	=	1331
11 x11 x 11 x 11	=	____________________
11 x 11 x11 x 11 x 11	=	____________________
______________________	=	____________________

2.

21 x 9	=	189
321 x 9	=	2889
4321 x 9	=	38889
54321 x 9	=	488889
654321 x 9	=	____________________
7654321 x 9	=	____________________

3.

64	8	16
81	9	27
100	10	40
121	11	55
144	________	________
169	________	________

SUMMARY

Our world is becoming increasingly complex, and the intricate situations that arise with globalization, terrorism, climate change, and changing demographics require multifaceted approaches to understanding.

Mathematics reveals hidden patterns that help us understand the world around us. Now much more than arithmetic and geometry, mathematics today is a diverse discipline that deals with data, measurements, and observations from science; with inference, deduction, and proof; and with mathematical models of natural phenomena, of human behavior, and of social systems. Understanding the historical foundations of mathematics will help the people appreciate the importance mathematics.

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MODULE 2
MATHEMATICAL LANGUAGE AND SYMBOLS

Introduction

Mathematics is written in a symbolic language that is intended to express mathematical thoughts, notations, methods and facts, that is how to read and write mathematics as agreed upon by the mathematicians by which we call mathematical conventions. The language of mathematics has its own symbols and rules of grammar that are relatively different from those of English. This language consists of expressions or sentences written in symbols, just like the way mathematicians traditionally write them. This module will enlighten you on the different terminologies like PEMDAS, variables, universal and existential statements and other conventions in Mathematics.

Learning Outcomes

At the end of the lesson, the students should be able to:

1. Explain the nature of mathematics as a language.
2. Describe and exemplify the different symbols, syntax and rules of mathematics.
3. Write the mathematical language as expression or a sentence.
4. Insert mathematical symbols in Word, into equations or text by using the equation tools.
5. Describe and exemplify the mathematical conventions;
6. Perform operations on mathematical expressions correctly using PEMDAS;
7. Understand some of the language of variables that is a foundation for more mathematical thoughts;
8. Describe the different mathematical statements: Universal statements, Conditional statements and Existential statements.

Lesson 1: The Language, Symbols, Syntax and Rules of Mathematics

Mathematics as a language has symbols to define a formula or to represent a constant. It has syntax to make the mathematical expression or sentence well-formed to make the characters and symbols distinct and valid that do not violate the rules. Mathematics meets all of these requirements. The symbols, their meanings, syntax, and grammar are the same throughout the world (Helmenstine, 2019).

Helmenstine (2019) described Mathematics as the language of science. Galileo Galilei , an Italian astronomer and physicist is attributed with the quote, "*Mathematics is the language in which God has written the universe*." Syntax as defined in Oxford dictionary is the arrangement of words and phrases to create well-formed sentences in a language.

Mathematical symbols can label numbers (constants), variables, operations, functions, brackets, punctuation and grouping to help define the order of operations and other features of logical syntax. A mathematical concept is not defined by the symbol chosen to represent it, rather it is the convention (a custom or a way of acting and doing things that is widely accepted and followed) that dictates the meaning (Alida, n.d.).

The language of mathematics makes it easy to express the kinds of symbols, syntax and rules that mathematicians like to do and is characterized by the following:

Precise (able to make very fine distinctions), for example + means add, - means subtract, x means multiply and ÷ means divide.

Concise(able to say things briefly), for example the long English sentence Eight plus two equals ten can be shortened with the use of symbols 8 + 2 = 10.

Powerful (able to express complex thoughts with relative ease), for example the application of critical thinking and problem solving skill which entails the comprehension, analysis and reasoning to obtain the most accurate solution. (Daligdig, 2019)

Lesson 2. Writing Mathematical Language as an Expression or A Sentence

A mathematical expression is a finite combination of symbols that is well-formed according to rules that depend on the context or situation (Michael, n.d.). It is a correct arrangement of mathematical symbols used to represent a mathematical object of interest.

An expression does not state a complete thought hence there is no point to ask if an expression is true or false. In mathematics, an expression or mathematical expression is a finite combination of symbols that is well-formed according to rules that depend on the context. Mathematical symbols can designate numbers (constants), variables, operations, functions, punctuation, grouping, and other aspects of logical syntax (Michael, n.d.).

An expression is the mathematical analogue of an English noun; it is a correct arrangement of mathematical symbols used to represent a mathematical object of interest. (Expressions Versus Sentences, n.d.)

The most common types of expressions are numbers, sets and functions. Numbers have lots of different names, for example for the expressions:

5	2+3	10/2	(6-2)+1	1+1+1+1+1

The basic syntax for encoding mathematical formulas or expressions in the system enables you to quickly enter expressions using 2-D notation, whereby forgetting the use of parentheses “( )” is the common mistake. For example, 2/ (m+2) is different from 2/m+2. In the latter case the system will interpret it as (2/m) +2. Other examples of expressions are 7d -5, $6x^2+5x-9$ and $\frac{p-2}{p^2}$.

On the other hand, a mathematical sentence is the analogue of an English sentence (Expressions Versus Sentences, n.d.). It is a correct arrangement of mathematical symbols that states a complete thought.

Sentences have verbs and in the sentence $5 + 3 = 8$, "=" is the verb. A sentence can be (always) true, (always) false or sometimes true / sometimes false. $5 + 3 = 8$ is true but the sentence $5+3 = 7$ is false.

The sentence $y+2 = 2 + y$ is always true no matter what number is chosen for y. The sentence $m=4$ is true when $m=4$ and false otherwise.

Lesson 3: The Mathematical Conventions

A mathematical convention is a fact, name, notation or usage which is generally agreed upon by mathematicians just like the order of operations which is denoted by the acronym PEMDAS.

Almost all mathematical names and symbols are conventional. A name or notation that has been in use for quite some time is more likely to become a mathematical convention. However, some notational questions fail to develop conventional solutions, usually because two or more competing conventions achieve wide-spread usage. A convention is defined as "a place where people convene, or come together", hence the phrase "mathematical convention" is also used to denote a convention whose purpose is mathematical (Wikipedia, n.d.).

Lesson 4: PEMDAS

PEMDAS is just a set of rules that prioritize the order or sequence of operations starting from the most important to the least important. In simplifying any expression, all exponents should be simplified first, followed by multiplication and division from left to right and, finally, addition and subtraction from left to right (Wikipedia, n.d.). By convention, the following steps must be followed to perform the order of operations correctly:

(1) Simplify first everything inside the parentheses
(2) Simplify all exponential number in the expression
(3) Multiply and divide whichever **comes first, from left to right**
(4) Add and subtract whichever **comes first, from left to right**

Steps nos. 3 and 4 are critical points to observe or else, the answer will be wrong

Lesson 5: Speaking Mathematically

A variable is a symbol used to represent a random element of a set or an unspecified term of the theory, or a basic object of the theory, which is manipulated without referring to its possible intuitive interpretation. Aside from adding them to numbers, they are commonly used to represent vectors, matrices, functions and in writing universal, conditional and existential statements.

Doing algebraic computations with variables as if they were plain numbers allows one to solve a range of problems in a single computation, for example: $2m + 3m = 5m$,$7b^2 – 2b^2 = 5b^2$ and $2p + 3w = 2p + 3w$ (cannot add nor subtract if the variables used are different).

Any letter can be used as a variable either uppercase or lowercase. Your ability to use several variables in one equation will become important when more complex math applications are involved.

Furthermore, variables have other specific names such as:

- An **unknown** is a variable in an equation which has to be solved for.
- An indeterminate is a symbol, commonly called variable, that appears in a polynomial or a formal power series. Formally speaking, an indeterminate is not a variable, but a constant in the polynomial ring or the ring of formal power series. However, because of the strong relationship between polynomials or power series and the functions that they define, many authors consider indeterminate as a special kind of variables.
- A parameter is a quantity (usually a number) which is a part of the input of a problem, and remains constant during the whole solution of this Problem. For example, in mechanics the mass and the size of a solid body are *parameters* for the study of its movement. In computer science, *parameter* has a different meaning and denotes an argument of a function.
- Free variables and bound variables\A random variable is a kind of variable that is used in probability theory and its applications.

All these denominations of variables are of semantic nature and that the way of computing with them (syntax) is the same for all (Wikipedia, n.d.)

On the other hand, statement or proposition is a sentence that is true or false but not both. In Mathematics, the following are the types of statements:

Existential Statements

- State that something exists
- Usually describe ideas that are true for certain elements
- contain words like "there exists," "there is at least one," or "for some"
- There is at least one thing for which the property is true. (there exists)
- One element only is needed to be found that satisfies the statement in order to prove it.

Some examples of existential statements are -

- There exists a natural number *n*, such that n x n = 36
- There exists an integer *z*, such that z^2=25
- There is at least one number *n*, belonging to a set of Natural numbers, such that a x n = a

  A more concise and legible way of expressing the above statements are the following:

- $\exists$ n $\in \mathbb{N}$, $\ni$ n.n = 36
- $\exists$ n $\in \mathbb{Z}$, $\ni$ z^2 = 25
- $\exists$ n $\in \mathbb{N}$, $\ni$ a.n = a

  With the reference of the symbols and their meanings:

  - $\exists$ - there exists
  - $\in$ - belongs to
  - $\ni$ - such that
  - $\mathbb{N}$ - The set of natural numbers 1 to infinity
  - $\mathbb{Z}$ - The set of integers {-infinity .... -2, -1, 0, 1, 2,.... infinity}
  - $\forall$ - For all values of

Universal Statements

- statements that hold true for all elements of a set
- describe ideas that are valid for all elements within the context
- the presence of the keywords **"For all…", " Given any"** can lead us to a safe assumption that the statement is universal

Universal Conditional statements

- Both universal and conditional, e.g. **for all** animals a, **if** a is a dog, **then** a is a mammal.
- Quite often implied rather than specific in informal English; for example: The following statements are rewritten formally in the enclosed answers:
- Any student with a GPA of better than 3.5 must study a lot. ($\forall$ Students ***x***, if ***x***'s GPA is better than 3.5, x studies a lot.)
- If a polygon has 3 sides, it must be a triangle. ($\forall$ Polygons ***p***, if ***p*** has 3 sides, ***p*** is a triangle.)
- All real numbers are positive when squared. ($\forall$ Numbers n, if n is real, n^2 is positive.)
-

Conditional statements is anchored on the concept of a universal conditional statement. It is symbolically represented as: $\forall$ x, if **P(*x*)**, then **Q(*x*)** or $\forall$ x, $P(x) \rightarrow Q(x)$, which means, if one thing is true then some other thing is also true. (if-then).

According to Dr. Daniel Freeman in his lecture on discrete math, there are the so called universal existential statements and existential universal statements. He differentiated the two types of statements as follows:

- **Universal Existential Statements** are universal because the first part of the statement says that a certain property is true for all objects of a given type, and it is existential because its second part asserts the existence of something. For example: Every real number has an additive inverse.
- **Existential Universal Statements** assert that a certain object exists in the first part of the statement and says that the object satisfies a certain property for all things of a certain kind in the second part. For example: There is a positive integer that is less than or equal to every positive integer.

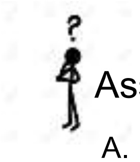
Supplemental Materials:

Mathematical Symbols with meanings: https://www.youtube.com/watch?v=BouqkWYIfzg&t=7s)

https://www.youtube.com/watch?v=1SBwQSOV9wk

https://www.youtube.com/watch?v=ZzeDWFhYv3E

https://youtu.be/izphl3Gu9_k



Assessment Task

A. Reflections: Write 5 importance of the language of mathematics in your life as a student. Use the 3 characteristics of the language mentioned in the content and discussion. [5]

B. Write the mathematical symbols of the following: [5]

1. Two times x ____________
2. Five plus three equals eight ____________
3. Nine is greater than four ____________
4. Negative eleven is less than negative nine ____________
5. Five divided p equals ten ____________

C. Identify numbers 1 to 5 in (B) if it is an expression or a sentence. Circle the correct answer. [5]

1. Sentence, expression
2. Sentence, expression
3. Sentence, expression
4. Sentence, expression
5. Sentence, expression

D. Perform the indicated operations using PEMDAS. Always remember, Honesty is the best policy. The use of calculator is not allowed. Write the final answer beside each expression. [15]

1) $21 \div 3 + (3 \times 9) \times 9 + 5$ ________________

2) $18 \div 6 \times (4 - 3) + 6$ ________________

3) $14 - 8 + 3 + 8 \times (24 \div 8)$ ________________

4) $4 \times 5 + (14 + 8) - 36 \div 9$ ________________

5) $(17 - 7) \times 6 + 2 + 56 - 8$ ________________

6) $(28 \div 4) + 3 + (10 - 8) \times 5$ ________________

7) $12 - 5 + 6 \times 3 + 20 \div 4$ ____________________

8) $36 \div 9 + 48 - 10 \div 2$ ____________________

9) $10 + 8 \times 90 \div 9 - 4$ ____________________

10) $8 \times 3 + 70 \div 7 - 7$ ____________________

11) $31b - 9b + 8b$ ____________________

12) $4a \times 8a^2 + 11a$ ____________________

13) $6m - 5m + 2n$ ____________________

14) $4p \times p^2 + 9p^3$ ____________________

15) $2cd \times 7cd^2 + 8cd$ ____________________

E. Identify the type of statement and underline the correct answer. [5]

1) For all even n numbers, n^2 is even.(existential, universal, conditional)
2) There exists one natural number n, such that n2 is less than 2. .(existential, universal, conditional)
3) If a polygon has exactly four sides, then it is a quadrilateral. .(existential, universal, conditional)
4) For some integer x, x is divisible by 5. .(existential, universal, conditional)
5) The square of a real number is nonnegative. .(existential, universal, conditional)

SUMMARY

Mathematicians accept the conventions so that the other mathematicians could understand what they write without constantly having to redefine basic terms. The fact that one evaluates multiplication before addition in the expression 3 + 7 x 6 is merely conventional. There is nothing significant about the order of operations, rather the mathematicians abide by conventions in order to allow other mathematicians to understand what they write without having to redefine basic terms repetitively. The letters PEMDAS and the words parenthesis, exponents, multiplication, division, addition, subtraction may not be very meaningful for someone trying to remember this order, hence we can remember it using the phrase:

Please **E**xcuse **M**y **D**ear **A**unt **S**ally and then it may be easier to remember the order of operations given in PEMDAS. Variables are unknown (or arbitrary) number(s) or objects represented by the notation: x, y, a, b, …. . a1, …, an etc. In Mathematics, an Existential statement says that something exists or is true for certain elements. On the other hand, a Universal statement is a statement which expresses the fact that all objects have a particular property and that a concept is true for a set of elements. On seeing some keywords like “there exists”, “for some…””For all…”, “Given any...” one may safely assume that the statement is either universal, conditional or existential. The existential and universal statements further our abilities to speak mathematically and with the use of symbolic notations, their meanings become clearer and specific.

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MODULE 3
PROBLEM SOLVING AND REASONING

Introduction

Problem solving goes hand in hand with reasoning. Mathematical reasoning is the critical skill that enables a student to make use of all other mathematical skills. With the development of mathematical reasoning, students recognize that mathematics makes sense and can be understood. They learn how to evaluate situations, select problem-solving strategies, draw logical conclusions, develop and describe solutions, and recognize how those solutions can be applied. Mathematical reasoners are able to reflect on solutions to problems and determine whether or not they make sense. They appreciate the pervasive use and power of reasoning as a part of mathematics (New Jersey Mathematics Curriculum Framework, 1996).

Problem solving is done every day by every person. It plays a vital role in the education of every learner. Problem solving is a process of applying Mathematics in an unfamiliar situation. Problem solving allows individuals to know more about the nature of mathematical activity, mathematical process and mathematical thinking. It includes examining the question to find the key ideas, choosing an appropriate strategy, doing the maths, finding the answer and then re-checking. Problem solving encompasses the set of cognitive procedures and thought processes that we apply to reach a goal when we must overcome obstacles to reach that goal. Reasoning encompasses the cognitive procedures we use to make inferences from knowledge and draw conclusions (UTSC, 2006). Logical reasoning is a useful tool in many areas, including solving math problems. Logical reasoning is the process of using rational, systemic steps, based on mathematical procedure, to arrive at a conclusion about a problem (Smith, 2017).

The future needs problem-solvers with reasoning skills. A better problem solver is the one who can find a resolution of which the path to the answer is not immediately known. But as education shifts its focus to the critical and creative angle of mathematics problems, we can’t lose sight of the abilities and skills that make this thinking possible: mathematical fluency. Mathematical fluency skills help students think faster and more clearly, giving them the energy, attention and focus to tackle complex problem-solving and reasoning questions (3P Learning, 2020).

Learning Outcomes

At the end of this module, students should be able to:

1. Apply inductive and deductive reasoning to solve problems;
2. Solve problems involving patterns and recreational problems following Polya’s strategy; and
3. Organize one’s methods and approaches for proving and solving problems.

Lesson 1. Inductive Reasoning (Cengage, 2018)

The type of reasoning that uses specific examples to reach a general conclusion of something is called *inductive reasoning*. The conclusion formed by using inductive reasoning is called a *conjecture*. A conjecture is an idea that may or may not be correct (Cengage, 2018).

Inductive reasoning is used in many ways. It is used to predict the next number in a list, to make a conjecture about an arithmetic procedure and to solve an application (Baltazar, et al., 2018).

Example:

1. Use inductive reasoning to predict the next number in each of the following list.
   a. 3, 6, 9, 12, 15, ___
      Solution: Since each succeeding number is 3 units larger than the preceding number, then the next number in the list is 3 larger than 15, which is 18.
   b. 1, 4, 9, 16, 25, ___
      Solution: Observe that $1 = 1^2$, $4 = 2^2$, $9 = 3^2$, $16 = 4^2$, and $25 = 5^2$, the numbers are all perfect squares. Then, it can be predicted that the next number will be 36 which is the square of 6.
   c. 1, 3, 6, 10, 15, ___
      Solution: 1 and 3 differ by 2 units, 3 and 6 differ by 3 units, 6 and 10 differ by 4 units, and 10 and 15 differ by 5 units. It appears that the difference between any two numbers is increasing starting from 2 units, then the next number in the list will be 6 units larger than 15 which gives an answer of 21.

2. Use inductive reasoning for a procedure given below to make a conjecture about the relationship between the size of the resulting number and the size of the original number.

Procedure: Pick a number. Multiply the number by 8. Add 6 to the product.
Divide the sum by 2. Subtract 3 from the quotient.

Solution: Suppose 2, 5, 7 are picked.

a. Pick a number		2	5	7
b. Multiply the number by 8	16	40	56	
c. Add 6 to the product		22	46	62
d. Divide the sum by 2		11	23	31
e. Subtract 3 from the quotient.		8	20	28

Notice that the resulting number is four times the original number. So if we picked 400, then the resulting number will be 1600.

3. Use inductive reasoning to solve an application: The following table shows some results obtained by Galileo Galilei (1564-1642) for pendulums of various lengths. (The length of 10 inches has been designated as 1 unit.) Answer the questions below the table.

Length of Pendulum (in units)	Period of Pendulum (in heartbeats)
1	1
4	2
9	3
16	4
25	5
36	6

a. If a pendulum has length of 49 units, what is its period?
b. If the length of the pendulum is quadrupled, what happens to its period?

Solution:

a. Observe that each pendulum has a period that is the square root of its length, $\sqrt{1}=1$, $\sqrt{4}=2$, $\sqrt{9}=3$, $\sqrt{16}=4$, $\sqrt{25}=5$, and $\sqrt{36}=6$. Then the conjecture that a pendulum with length of 49 units will have a period of 7 heartbeats ($\sqrt{49}=7$).
b. Notice the pendulum of length 4 units has a period that is twice that of a pendulum with length of 1 unit. A pendulum with length 16 has a period that is twice of a pendulum with length of 4 units. Therefore, quadrupling the length of a pendulum doubles it period.

Lesson 2. Deductive Reasoning (Baltazar et al., 2018)

1. **Deductive reasoning** is a process of reaching a conclusion by applying general principles, assumptions or procedures (Cengage, 2018). Reasoning deductively, known facts are used to make logical conclusions that must be true. A logic puzzle like Sudoku puzzle uses deductive reasoning (Baltazar, et al., 2018).

Example:

Use deductive reasoning to make a conjecture. Consider the following procedure: Pick a number. Multiply the number by 8. Add 6 to the product. Divide the sum by 2. Subtract 3 from the quotient.

Solution: Suppose *n* is picked as a number.

a. Pick a number	n
b. Multiply the number by 8	8n
c. Add 6 to the product	8n + 6
d. Divide the sum by 2	(8n + 6)/2 = 4n + 3
e. Subtract 3 from the quotient.	4n + 3 – 3 = **4n**

A number n is picked and the result is 4n after following the procedure. It produces a number that is four times the original number. So, if for example, number 10 is chosen, then the resulting number will be 4 times 10 which yields 40.

2. Solve a Logic Puzzle

Each of the four friends Donna, Sarah, Nikki, and Xhanelle, has a different pet (fish, cat, dog and snake). From the following clues, determine the pet of each.

1. Sarah is older than her friend who owns the cat and younger than her friend who owns the dog.
2. Nikki and her friend who owns the snake are both of the same age and they are the youngest members of their group.
3. Donna is older than her friend who owns the fish.

Solution:

From clue 1, Sarah does not own a cat nor a dog.

From clue 2, Nikki does not own a snake.

From clue 4, Donna does not own a fish.

	fish	cat	dog	Snake
Donna	x			
Sarah		x	x	
Nikki				x
Xhanelle				

From clue 1, Sarah is not the youngest, then she does not own a snake.

From clue 2, Nikki being the youngest does not own a dog.

There are now 3 x's in the pets in Sarah's row, therefore she owns a fish.

	fish	cat	dog	Snake
Donna	x			
Sarah	/	x	x	x
Nikki			x	x
Xhanelle				

From clue 3, Donna is older than Sarah, hence she owns the dog.

	Fish	cat	dog	Snake
Donna	x	x	/	x
Sarah	/	x	x	x
Nikki			x	x
Xhanelle				

There are 3 x's in the snake column except for Xhanelle, so she owns the snake.

	Fish	cat	dog	Snake
Donna	x	x	/	x
Sarah	/	x	x	x
Nikki			x	x
Xhanelle	x	x	x	/

Nikki is the only one left, therefore she owns the cat.

	Fish	Cat	dog	Snake
Donna	x	x	/	x
Sarah	/	x	x	x
Nikki	x	/	x	x
Xhanelle	x	X	x	/

To summarize, Donna owns a dog, Sarah owns a fish, Nikki owns a cat and Xhanelle owns a snake.

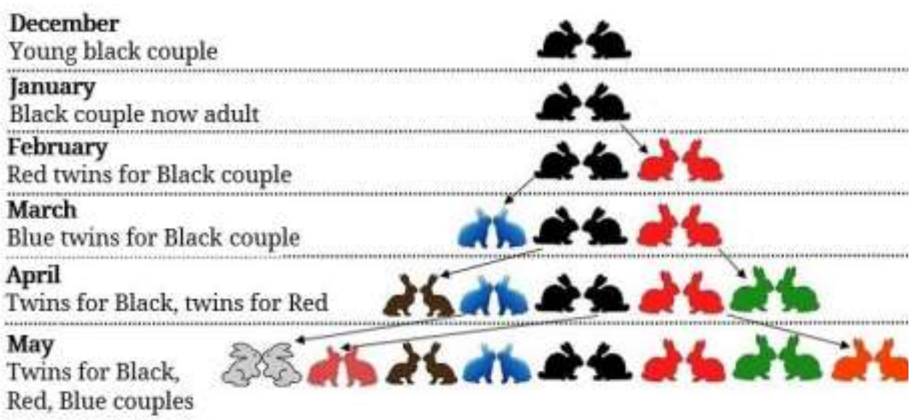
Supplementary Activity: Watch Can you solve "Einstein's Riddle"? - Dan Van der Vieren on https://www.youtube.com/watch?v=1rDVz_Fb6HQ

Lesson 3. Problem Solving with Patterns

In solving a problem, most students look for patterns for them to figure out what to do to arrive at an answer. They look for items or numbers that are repeated or a series of events that repeat (Baltazar et al., 2018).

FIBONACCI SEQUENCE

One of the famous mathematicians who spent so much time in solving problems using patterns is Leonardo of Pisa, also known as Fibonacci (Hartwell, 2016). In 1202, Fibonacci wrote the book *Liber Abaci* which contains his famous rabbit problem. Fibonacci's problem concerning the birth rate of rabbits paved the way to the discovery of a phenomenal sequence of numbers known as the *Fibonacci sequence* (Knott, 2015).



F_n (pair/s)	Month
1	1
1	2
2	3
3	4
5	5
8	6

Source: http://oldeuropeanculture.blogspot.com/2018/02/fibonacci_24.html

Figure 1: Fibonacci's Rabbit Problem

Statement of Fibonacci's rabbit problem: At the beginning of a month, Fibonacci started with a pair of newborn rabbits. After a month the rabbits produced no offspring; however, every month thereafter, the pair of rabbits produces another pair of rabbits. The offspring reproduce in exactly the same manner. If none of the rabbits dies, how many pairs of rabbits will there be at the start of each succeeding month (Thomas, 2015)?

Fibonacci discovered that a Fibonacci number can be found by adding its previous two Fibonacci numbers. The first six terms of the Fibonacci sequence are 1, 1, 2, 3, 5 , 8. F_n will be used to denote the *n*th term of the Fibonacci sequence (Protonotarios, 2020).

Fibonacci Numbers

Definition: $F_1 = 1$, $F_2 = 1$, $F_n = F_{n-1} + F_{n-2}$, for $n \geq 3$.

Example: Use the definition of Fibonacci numbers to find the seventh, eighth and tenth Fibonacci numbers.

Solution: $F_7 = F_6 + F_5 = 8 + 5 = 13$

$F_8 = F_7 + F_6 = (F_6 + F_5) + F_6 = (8 + 5) + 8 = 21$

$F_{10} = F_9 + F_8 = (F_8 + F_7) + F_8 = (21 + 13) + 21 = 55$

BINET'S FORMULA

It is easy to find the *n*th Fibonacci number if the two of previous numbers are known. *Jacques Binet* in 1543 was able to find a formula for the *n*th Fibonacci number (Knott, 2016).

Binet's Formula

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

Example: Use Binet's formula and a calculator to find the 20th Fibonacci number.

Solution:

$$F_{20} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{20} - \left(\frac{1-\sqrt{5}}{2}\right)^{20}\right] = 6765$$

PASCAL'S TRIANGLE

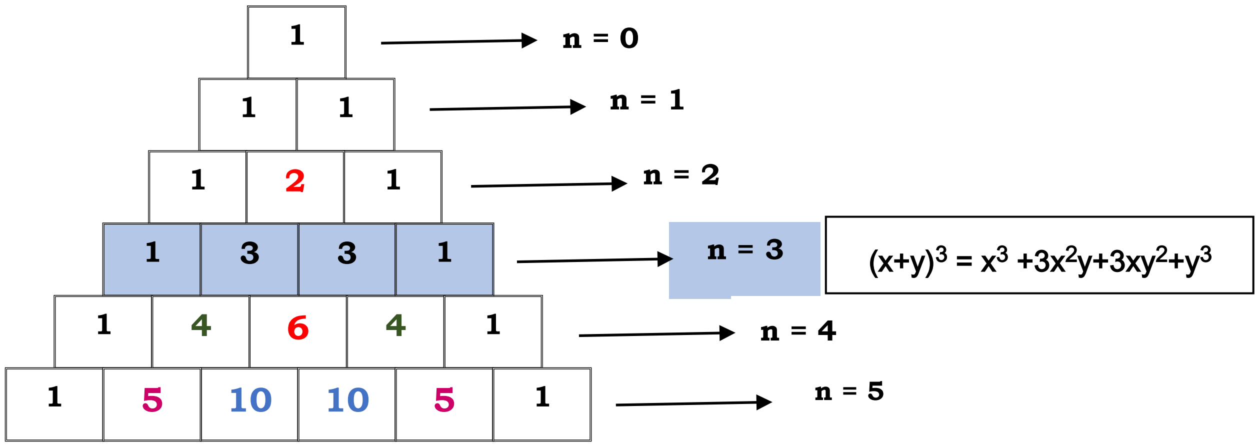
Another famous mathematician who loves patterns is Blaise Pascal (1623-1662). ***Pascal's triangle***, in algebra, a triangular arrangement of numbers that gives the coefficients in the expansion of any binomial expression, such as $(x + y)^n$, where n represents the exponent. It is named for the 17th-century French mathematician Blaise Pascal, but it is far older (Chegg Inc., 2020).

Chinese mathematician Jia Xian devised a triangular representation for the coefficients in the 11th century (Gregersen, 2019). His triangle was further studied and popularized by Chinese mathematician Yang Hui in the 13th century, for which reason in China it is often called the Yanghui triangle. It was included as an illustration in Chinese mathematician Zhu Shijie's *Siyuan yujian* (1303; "Precious Mirror of Four Elements"), where it was already called the "Old Method." The remarkable pattern of coefficients was also studied in the 11th century by Persian poet and astronomer Omar Khayyam (Hosch, 2009).

Here's how to construct the Pascal's Triangle: The arrangement is similar to stacking of shoe boxes. The top number, left and right sides are all 1.

1						$\rightarrow$	1
1	1					$\rightarrow$	1, 1
1	2	1				$\rightarrow$	1+1=2
1	3	3	1			$\rightarrow$	1+2=3, 2+1=3
1	4	6	4	1		$\rightarrow$	1+3=4, 3+3=6, 3+1=4
1	5	10	10	5	1	$\rightarrow$	1+4=5, 4+6=10, 6+4=10, 4+1=5

In algebra, expanding $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$ is just a simple special product process that can also be solved using the Pascal's triangle. But then expanding $(x+y)^6$ will require long solution. Let us first examine the expansion of $(x+y)^3$.



Observe that:

1. The exponent is 3 (n = 3).
2. There are 4 boxes which contain the numbers 1, 3, 3, 1 that represents the numerical coefficient of each term in the expansion.

   1st term = 1

   2nd term =3

   3rd term =3

   4th term =1

   $$\boxed{(x+y)^3 = \underline{1}x^3 + \underline{3}x^2y + \underline{3}xy^2 + \underline{1}y^3}$$

3. Since n = 3, the highest exponent for the 1st and 4th terms is 3. Notice that the exponent of x is in descending order (starting from the first term, ending at the 3rd term) and the exponent of y is in ascending order (starting from the 2nd term, ending at the last term).

   $(x+y)^3 = x^3 + 3x^2y + 3x^1y^2 + y^3$ $(x+y)^3 = x^3 + 3x^2y^1 + 3xy^2 + y^3$

NOTE: If the exponent of the binomial is n, then the number of terms is (n+1). (i.e. n = 6, there are 6+1=7 terms in the expansion of binomials.

Supplementary Activity

Read Pascal's triangle on https://www.mathsisfun.com/pascals-triangle.html to discover more hidden patterns of numbers.

Lesson 4. Polya's Problem Solving Strategy

George Polya had an important influence on problem solving in mathematics education. *How to Solve It* (1945) is one of his best known books published. He outlines a strategy for solving problems from virtually any discipline. He stated that good problem solvers tend to forget the details and tend to focus on the structure of the problem, while poor problem solvers focus on the opposite (Polya, 1957). He designed the following:

4-Step Process (EDCON, 1994)

1. Understand the problem. (SEE)
   Read and understand the problem. Identify what is the given information, known data or values and what is the unknown and to be solved as required by the problem.
   a. Can you restate the problem in your own words?
   b. Can you determine what is known about these types of problems?
   c. Is there missing information that if known would allow you to solve the problem?
   d. Is there extraneous information that is not needed to solve the problem?
   e. What is the goal?
2. Devise a plan. (PLAN)
   Think of a way to solve the problem by setting up an equation, drawing a diagram, and making a chart that will help you find the unknown and the solution. To start devising a plan, try doing the following:
   a. Make a list of the known information.
   b. Make a list of information that is needed.
   c. Draw a diagram.
   d. Make an organized list that shows all the possibilities.
   e. Make a table or a chart.
   f. Work backwards.
   g. Try to solve similar but simpler problem.
   h. Write an equation, as possible define what each variable represents.
   i. Perform an experiment.
   j. Guess at a solution and then check the result.

3. Carry out the plan. (DO)

   Solve the equation you have set up and observe analytical rules and procedures until you arrive at an answer.

   a. Work carefully.
   b. Keep an accurate and neat record of all your attempts.
   c. Realize that some of your initial plans will not work and that you will have to devise another plan and modify your existing plan.

4. Look back. (CHECK)

   In order to validate the obtained value, you need to verify and check if the answer makes sense or correct based on the situation posed in the problem. Label your final correct answer.

   a. Ensure that the solution is consistent with the facts of the problem.
   b. Interpret the solution within the context of the problem.
   c. Ask yourself whether there are generalizations of the solution that you could apply to similar problems.

Assessment Task 1-1

Inductive Reasoning

I. Predict the next number/s in each list.

1. 27, 24, 21, 18, _____, _____
2. 6, 1, 9, 5, 12, 9, 15, _____, _____
3. 1000, 200, 40, _____

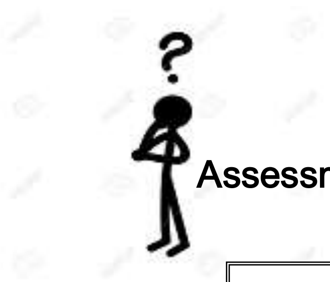
II. Follow the given procedure. Make a conjecture about the relationship between the size of the resulting number and the size of the original number.

1. Take your age.
2. Multiply it by 2
3. Add 5
4. Multiply the sum by 50
5. Subtract 365
6. Add 115

III. Solve an application. A tsunami is a sea wave produced by an underwater earthquake. The height of the tsunami as it approaches land depends on the velocity of the tsunami. Use the table below to answer each of the following questions.

Velocity of tsunami (feet/second)	Height of tsunami (feet)
6	4
9	9
12	16
15	25
18	36
21	49
24	64

1. What happens to the height of the tsunami when its velocity is doubled?



Assessment Task 1-2

Deductive Reasoning

I. Establish a conjecture following the given procedure.

1. Pick any counting number
2. Multiply the number by 6
3. Add 8 to the product
4. Divide the sum by 2
5. Subtract 4 from the quotient

II. Solve a Logic Puzzle: Each of four siblings (Edmund, Genalyn, Madelyn, and Sonia) bought four different cars. One chooses a Honda car, another Mitsubishi car, another Toyota car, and the other Suzuki car. From the following clues, determine which sibling bought which car.

1. Edmund, living alone, stays next door to his sister who bought the Honda car and very far from his sister who bought the Suzuki car.
2. Genalyn, living alone also, is younger than the one who bought the Mitsubishi car and older than her sibling who bought the Toyota car.
3. Madelyn did not like Toyota and Suzuki cars. But she and her sibling, who bought the Toyota car live in the same house.

Assessment Task 1-3

Problem Solving with Patterns

I. Find the 11th Fibonacci number using the Fibonacci numbers conjecture.

II. Use Binet's formula and a calculator to find the 150th Fibonacci number.

III. Expand $(x+y)^8$ using the Pascal's triangle.

TRUE or FALSE. Write T if the statement if equation is True, otherwise write F.

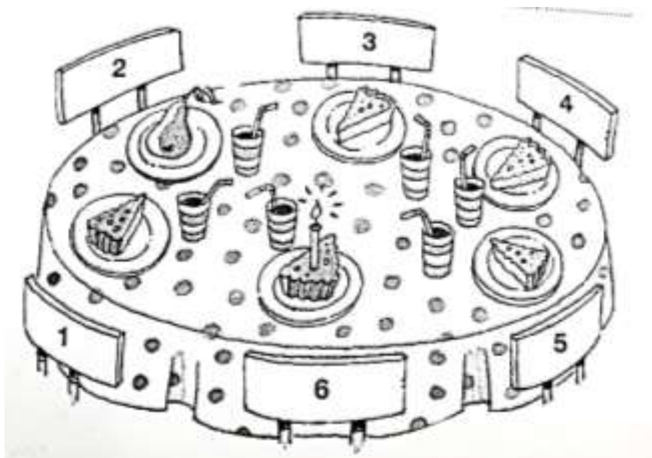
	1. 19 is a prime number, thus F_{19} is a prime number?
	2. F_{19} is 144 the index no. is equal to its digit sum?
	3. Every 3rd Fibonacci is a multiple of 2?
	4. The digit sum of 89 is 17, therefore F_{17} = 89?
	5. F_6 = 8 is the first Fibonacci number with 2 as a factor?
	6. 7 is a factor of 8? 7. F_3 and F_4 are prime factors of F_{12}?
	8. Every 4th Fibonacci numbers is a multiple of 3?
	9. Every 4th Fibonacci numbers is an odd number? 10. F_{12} = 144 the index number 12 is a factor of 144, 15 is a factor of F_{15}?

Assessment Task 1-4

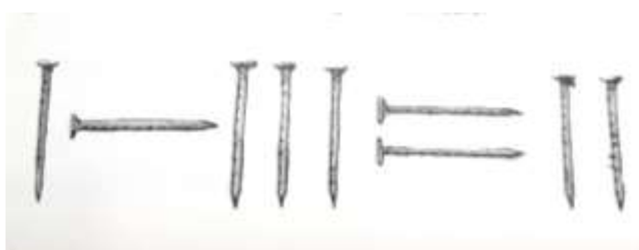
Use Polya's Problem Solving Strategy to solve the following riddles.

1. Mark is six today. He has invited five friends to his birthday party. At 4 o'clock they will sit down to tea. Can you decide who will sit where? (Brown, 2001)

a. Isabel sits next to Mark.	Chair 1: __________
b. Julie sits between two boys.	Chair 2: __________
c. Willian sits facing Mark.	Chair 3: __________
d. Marina does not eat cake.	Chair 4: __________
e. Isabel is not facing Marina.	Chair 5: __________
f. Thomas sits beside Mark.	Chair 6: __________



2. There is a mistake in this sum. 1 minus 3 is not equal to 2. Move one nail to make the sum right (Brown, 2001).



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SUMMARY

The type of reasoning that uses specific examples to reach a general conclusion of something is called *inductive reasoning*. Inductive reasoning is used in many ways. It is used to predict the next number in a list, to make a conjecture about an arithmetic procedure and to solve an application. Deductive reasoning is a process of reaching a conclusion by applying general principles, assumptions or procedures.

One of the famous mathematicians who spent so much time in solving problems using patterns is Leonardo of Pisa, also known as Fibonacci. Fibonacci's problem concerning the birth rate of rabbits paved the way to the discovery of a phenomenal sequence of numbers known as the *Fibonacci sequence*.

Jacques Binet in 1543 was able to find a formula for the nth Fibonacci number called Binet's formula. **Pascal's triangle**, in algebra, a triangular arrangement of numbers that gives the coefficients in the expansion of any binomial expression, such as $(x + y)^n$, where n represents the exponent.

George Polya had an important influence on problem solving in mathematics education. *How to Solve It* (1945) is one of his best known books published. He outlines a strategy for solving problems from virtually any discipline. Polya's problem solving strategy are: understand the problem, devise a plan, carry out the plan; and look back.

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MODULE 4
THE LANGUAGE OF SETS AND MATHEMATICAL LOGIC

Introduction

There is a natural relationship between sets and logic. Mathematics is composed of sets that are logically grouped (Logic and Sets, n.d.). Years of studying and centuries of continued research established the connection between them. From the logical point of view, things are arranged by sets. A house, for example, is subdivided into different rooms of different purposes. Things around us are grouped into sets according to their function and purpose. God created humans to think logically so as to govern earth systematically.

Learning Outcomes

At the end of this module, students should be able to:

1. Define sets;
2. Differentiate the types of set;
3. Write sets in roster and rule method;
4. Perform set operations;
5. Illustrate and symbolize propositions;
6. Perform different types of operations on propositions; and
7. Construct truth tables of propositions.

Lesson 1. Set Notation

Use of the word *set* as formal mathematical term was introduced in 1879 by Georg Cantor (1845-1918). A **set** is a collection of **elements** (or objects) and can be considered as an element itself (New Castle University, 2018). It contains no duplicate. It is used to group and describe elements which share a similar property. It is described using braces { } and is denoted using capital letters (Baltazar et al., 2018).

The objects used to form a set are called its *element* or its members. The elements of a set are written inside a pair of curly braces and are represented by commas and are written in small letters (www.math-only-math.com/, 2020).

Here are some of the symbols used for set discussion (Baltazar et al., 2018):

Σ	the sum of
$\exists$	there exists
$\forall$	for every (for any)
$\in$	element of (or member of)
$\notin$	not an element of (or not a member of)
$\subseteq$	subset of
$\Rightarrow$	if ..., then
$\Leftrightarrow$	if and only if
$\mathbb{R}$	set of real numbers
$\mathbb{N}$	set of natural numbers
$\mathbb{Z}$	set of integers
$\mathbb{Q}$	set or rational numbers
∞	infinity

The following are examples of sets:

1. The numbers 1, 3, 7 and 10
2. A={x|x>0}
3. The set of natural numbers N = {1, 2, 3, ...}
4. The integer numbers $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$
5. C = {x/x is a natural number and x < 8}

NOTE:

1. *The three dots in enumerating the elements of the set are called ellipsis which indicate a continuing pattern(infinite set) and is read "and so forth".*
2. *A={x|x>0} is read as "A is the set of all x such that (|) x is greater that 0.*

Lesson 2. Types of Set

Null set. A null set is any set that does not contain any element. It is also called empty or void set. The symbol used to represent an empty set is { } or ∅. "The set of natural numbers between 9 and 10" is a null set since there is no natural number between 9 and 10 (www.toppr.com, n.d.).

Singleton set. A singleton set is a set containing only one element. Unit set is another name for this type of set. "The set of natural numbers between 8 and 10" is a singleton set since 9 is the only natural number between 8 and 10 (Warier, n.d.).

Universal Set. A universal set contains all the elements of a problem under consideration. The set of Real Numbers is a universal set for ALL natural, whole, odd, even, rational and irrational numbers. It is denoted by the symbol U.

Finite and infinite sets. Any set which is empty or contains a definite and countable number of elements is called a finite set. Sets defined otherwise, for uncountable or indefinite numbers of elements are referred to as infinite sets (BYJU'S Web Site, 2020).

Examples:

Finite set: A = {a, e, i, o, u}

Infinite Set: B = {1, 2, 3, . . .}

Equal and unequal sets. Two sets X and Y are said to be equal if they have exactly the same elements irrespective of the order of appearance in the set. Equal sets are represented as X = Y. Otherwise, the sets are referred to as unequal sets, which are represented as X ≠ Y.

Examples:

1. If X = {a, e, i, o, u} and H = {o, u, i, a, e} then both of these sets are equal.
2. If C = {1, 3, 5, 7} and D = {1, 3, 5, 9} then both of these sets are unequal.
3. If A = {b, o, y} and B = {b, o, b, y, y} then also A = B because both contain same elements.

Equivalent sets. Equivalent sets are those which have an equal number of elements irrespective of what the elements are.

Example: A = {1, 2, 3, 4, 5} and B = {x : x is a vowel letter} are equivalent sets because both these sets have 5 elements each.

Power set. The collection of ALL the subsets of a given set is called a power set of that set under consideration.

Example: Given A = {a, b} then Power set of A [P (A)] = φ, {a}, {b} and {a, b}.
If n (A) = m then generally, n[P (A)] = 2^m

Lesson 3. Defining a Set

There are two ways in defining a set. These are known as the roster method and the rule method (www.toppr.com, n.d.).

Roster Method (or Listing method). This method involves writing the members of a set as a list, separated by commas and enclosed within curly braces. For example, the four seasons are a set and could be written as {Summer, Autumn, Spring, Winter}.

***Note**: The order of the elements in the list doesn't matter.*

For example, it could also be written as {Spring, Autumn, Summer, Winter} or {Winter, Autumn, Spring, Summer}.

Rule Method (or Set builder notation). This method involves writing down the properties that are shared by every member of the group. This rule is written inside a pair of curly braces and can be written either as a statement or expressed symbolically or written using a combination of statements and symbols (Chegg Inc , 2020).

For example, a set A whose elements included all the positive integers could be defined symbolically as: A={x|x>0} or in statement form as: A={x|x is a positive number}

Note: 1. The rule method is often preferred when defining larger sets where it would be difficult or time consuming to list all of the elements in a set.
2. The rule method is stated singularly.

Example: Write the given sets in roster and rule method.

1. A is the set of all even natural numbers less than or equal to 10
2. B is the set of all integers less than -3.

Solution:

1. Roster method: A = {2, 4, 6, 8, 10}
   Rule method: A = {x/x is an even natural number less than or equal to 10}
   or A = {x is an even natural number/x $\leq$ 10}

2. Roster method: B = {-4, -5, -6, . . .}
   Rule method: B = {x/x is an integer less than -3} or B = {x is an integer/x < -3}

Lesson 4. Basic Set Operations (Daligdig, 2019)

Basic operations in mathematics involve addition and subtraction as ways of combining numbers. Other ways of combining numbers are the operations on sets like union, intersection, and complementation.

Union. The union of two sets A and B, denoted by A ∪ B, is the set whose elements are either members of A or B, or both (Math Goodies, 2020).

For example:

1. If A = {1, 2, 3} and B = {4, 5} then A ∪ B = {1, 2, 3, 4, 5}.
2. If C = {3, 6, 9} and D = {6, 9, 10, 11}, then C ∪ D = {3, 6, 9, 10, 11}
3. If E = 5, 6, 7, 8}, F = {6, 7, 8, 9, 10}, G = {5, 6}

Find:	Answers:
a. E ∪ F	= {5, 6, 7, 8, 9, 10}
b. F ∪ G	= {5, 6, 7, 8, 9, 10}
c. E ∪ (F ∪ G)	= {5, 6, 7, 8, 9, 10}
d. (G ∪ F) ∪ E	= {5, 6, 7, 8, 9, 10}
e. E ∪ E	= {5, 6, 7, 8}

Intersection. The intersection of two sets A and B, denoted by A ∩ B, whose elements are common to both A and B.

For example:

1. If A = {4, 5, 6} and B = {5, 6, 7, 8}, then A ∩ B = {5, 6}
2. If C = {m, o, p, q} and D = {m, p}, then C ∩ D = {m, p}
3. If P = {a, b, c}, Q = {c, b, d, e}, and R = {c, f, g}

Find:	Answers:
a. P ∩ Q	= {B, C}
b. Q ∩ R	= {c}
c. (P ∩ Q) ∩ R	= {c}

Complement of a Set. It is not possible to talk about complement of a set without mentioning universal set.

For example:

1. The universal set U is given by the following:

   U = {2, 4, 6, 8, 10}, and two other sets

   A = {6, 8, 10} and B = {2, 4, 6}

The complement of A, denoted by A', is the set whose elements are in the universal set but not in set A. Thus A' = {2, 4} and B' = {8, 10}. To check the answer, the union of A and A' is equal to the universal set ($A \cup A' = U$).

Thus $A \cup A'$ = {6, 8, 10} $\cup$ {2, 4} = {2, 4, 6, 8, 10} and

$B \cup B'$ = {2, 4, 6} $\cup$ {8, 10} = {2, 4, 6, 8, 10}.

2. U = { r, s, t, u, v}, P = {r, s, t}, Q = {u, v}, S = {t, v}, T = {r, s, t, u, v}

Find:	Answers:
a. P'	= {u, v}
b. Q'	= {r, s, t}
c. S'	= {r, s, u}
d. T'	= $\varnothing$

3. U = {1, 3, 5, 7, 9}, X = {1, 3, 5}, Y = {3, 7}, Z = {1, 3, 5, 7}

   Find:

   a. $(X \cap Y)'$
   b. $(Y \cap Z)'$
   c. $(X \cup Y)'$

   Solution:

   a. $(X \cap Y)'$ = {3}' = {1, 5, 7, 9}

      Find the intersection of X and Y, then get its complement.

   b. $(Y \cap Z)'$ = {3, 7}' = {1, 5, 9}

      Get the intersection of Y and Z, then find its complement.

   c. $(X \cup Y)'$ = {1, 3, 5, 7} = {9}

      Start by getting the union of X and Y, then find the complement of the result.

Lesson 5. Basic Logical Connectives and Truth Tables (CHED, 2016)

There is conflict in all relationships. Conflict specifically means disagreements and arguments (https://www.loveisrespect.org/, 2017). To avoid uncertainty in the validity of mathematical statements, a powerful language of logic is employed in asserting truths of statements. The use of logic illustrates the importance of precision and conciseness in communicating mathematics.

Logic, as defined by Kueker (n.d.), is the analysis of methods of reasoning. It is interested in the form rather than the content of the argument. Mathematical logic is the study of reasoning as used in mathematics. Mathematical reasoning consists of drawing correct conclusions from given hypotheses. A statement being a logical consequence of some other statements is a basic concept of reasoning. The use of "therefore" (in ordinary mathematical English) customarily indicates that the following statement is a consequence of what comes before (Baltazar et al., 2018).

STATEMENTS (or verbal assertions). The fundamental property of a statement is that it is either true or false, but not both. The truthfulness or falsity of a statement is called *truth value*.

PROPOSITION. A proposition is a statement or a declarative sentence which is either true (T) or false (F) without additional information. Propositions are usually denoted by small letters (*p, q, r* -with or without subscripts). An example is a claim by Netflix viewers that the best comedy movie of the year is "Lady Bird". This statement is a proposition that is T or F but not both.

p: Everyone should study logic.

maybe read

p is the proposition "Everyone should study logic."

Propositions in English and Symbolic Form

Proposition in symbolic form	Proposition in English
$2 + 5 = 7$	The sum of two and five is seven.
$2 \in \mathbb{N}$	Two is a natural number.
$-100 \in \mathbb{Z}$	Negative 100 is an integer.
$\pi \notin \mathbb{Q}$	The constant π is an irrational number.
$\pi \in \mathbb{R}$	The constant π is a real number.
$\sqrt{2} < 2$	The square root of 2 is less than 2.
$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$	The set of all natural numbers is a subset of the set of all integers, the set of all integers is a subset of the set of all rational numbers, and the set of all rational numbers is a subset of the set of all real numbers. (Answers.com, 2020)

Example:

1. Each of the following statements is a proposition. Which are true and which are false? If the answer is false, state why.
   a. 9 is a prime number.
   b. $5 + 3 = 8$
   c. $x^2 + y^2 \geq 0$
   d. $10 < -3$

Answers:

a. False. Prime numbers have no other factors than 1 and itself, 9 can be expressed as $3 \cdot 3$.
b. True
c. True
d. False. A negative number is always less than a positive number.

2. Determine whether each of the following is a proposition or not. If a proposition, give its truth value.
   a. Batanes is an island in the Philippines.

b. Find a number which divides your age.

c. My seatmate will get a perfect score in the Logic exam.

d. Welcome to the Philippines!

e. 3 + 2 = 5

f. What is the domain of the function?

g. $f(x) = \frac{\sqrt{x}}{x+1}$ is a rational function.

Answers:

a. "Batanes is an island in the Philippines." is a declarative sentence. Hence, the statement is a true proposition.
b. This is not a proposition since this is an imperative sentence.
c. The statement is a proposition (declarative sentence). Although the truth value will only be known after the logic exam, it can only be either true or false but not both.

Remark: For a declarative sentence to be a proposition, it is not necessary that its true value is immediately known.

d. It is not a proposition since the statement is an exclamatory sentence.
e. The statement is a true mathematical sentence which is a declarative sentence. Therefore, it is a true proposition.
f. It is not a proposition, it is an interrogative sentence.
g. The given is a declarative sentence. It is a false proposition. The function *f* is irrational since the numerator of a function is not a polynomial.

Remark: False mathematical sentences are still propositions.

Truth Table. A truth table is a table that shows the truth value of a compound statement for all possible truth values of its simple statements. Given a proposition, its truth table shows all its possible truth values. It is useful in displaying all the possible truth value combinations of two or more propositions (Gregersen, 2019).

Construct a truth table for: a) a proposition *p*;

b) two propositions *p* and *q*; and

c) three propositions *p*, *q* and *r*.

a. Since a proposition has two possible truth values ($2^1 = 2$), a proposition p would have the following truth table.

p
T
F

b. The possible truth value combinations of the propositions *p* and *q* will have 4 rows ($2^2 = 4$). The propositions will have the following truth table.

p	*q*
T	T
T	F
F	T
F	F

c. The truth table for the propositions p, q and r have 8 rows ($2^3 = 8$)

p	*q*	*r*
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

NOTE: A truth table involving *n* propositions has 2^n rows.

LOGICAL CONNECTIVES

Logical Connectives. A Logical Connective is a symbol which is used to connect two or more propositional or predicate logics in such a manner that resultant logic depends only on the input logics and the meaning of the connective used (Parahar, 2019).

Generally, there are five connectives which are:

- NEGATION – not (¬ or ~)
- CONJUNCTION – and (∧)
- DISJUNCTION – or (∨)
- CONDITIONAL – if-then (⟹)
- BICONDITIONAL – If and only if (⟺).

Implication

***NEGATION*. Given any statement *p*, another statement, called the negation of *p*, can be formed by writing "It is false that . . ." before *p* or, if possible, by inserting in *p* the word NOT.** The negation of a proposition *p* (written as $\neg p$ or $\sim p$) is false when *p* is true and is true when *p* is false (Parahar, 2019). In other words, the truth value of the negation of a statement is always the opposite of the truth value of the original statement (Lipschutz, 1981).

The truth table is as follows:

p	$\sim p$
T	F
F	T

Example:

1. Consider the following statements:

    (a) Manila is in the Philippines.

    (b) It is false that Manila is in the Philippines.

    (c) Manila is not in the Philippines.

    Then (2) and (3) are each the negation of (1).

    Notice that (1) is true and (2) and (3), its negations, are false.

2. Consider the following statements:

    (a) 2 + 2 = 5

    (b) It is false that 2 + 2 = 5.

    (c) $2 + 2 \neq 5$

    Then (2) and (3) are each the negation of (1).

    Notice that (1) is false and (2) and (3), its negations, are true.

3. State the negation of the following propositions.

    a. n_1: $p(x) = \frac{x-1}{x+2}$ is a polynomial function.

    b. n_2: 2 is an odd number.

    c. n_3: The *tinikling* is the most difficult dance.

    d. n_4: Everyone in Visayas speaks Cebuano.

Solution: The negation of the given propositions are given below.

a. $\sim n_1$: "It is not true that $p(x) = \frac{x-1}{x+2}$ is a polynomial function." or

$\sim n_1$: "$p(x) = \frac{x-1}{x+2}$ is not a polynomial function."

b. $\sim n_2$: "It is not true that 2 is an odd number." or

$\sim n_2$: "2 is an even number."

c. $\sim n_3$: The *tinikling* is not the most difficult dance.

d. $\sim n_4$: Not everyone in Visayas speaks Cebuano.

***CONJUNCTION*. Any two statements can be combined by the word** AND to form a composite statement called the conjunction of the original statements. The AND operation of two propositions *p* and *q* (written as $p \wedge q$) is true if both the propositional variable *p* and *q* is true (Parahar, 2019). In other words, the conjunction of two statements is true only if each component is true (Lipschutz, 1981). The propositions *p* and *q* are called *conjuncts*.

The truth table is as follows:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

NOTE: Conjunctions do not always use the word "and". Words like "but", "even though", "yet" and "while" are also used to flag conjunctions.

Example:

1. Let p: It is raining and q: The sun is shining, then $p \wedge q$ denotes the statement "It is raining and the sun is shining."

2. The symbol $\wedge$ can be used to define the intersection of two sets, specifically,

$$A \cap B = \{x \mid x \in A \wedge x \in B\}$$

3. Consider the following four statements:

(a) Santa Cruz is in Laguna and 2 + 2 = 5.	False
(b) Santa Cruz is in Quezon and 2 + 2 = 4.	False
(c) Santa Cruz is in Quezon and 2 + 2 = 5.	False
(d) Santa Cruz is in Laguna and 2 + 2 = 4.	True

Only (d) is true. Each of the other statements is false since at least one of its substatements is false.

4. Let "p: Angels exist." and "q: $\pi > 3$". Express the following conjunctions as English sentences or in symbols, as the case may be.

   a. $p \wedge q$

   b. $p \wedge (\sim q)$

   c. Angels do not exist and $\pi \leq 3$.

   d. While angels do not exist, $\pi > 3$.

Solution:

   a. $p \wedge q$: Angels exist and $\pi > 3$.

   b. $p \wedge (\sim q)$: Angels exist and $\pi \leq 3$. or

$p \wedge (\sim q)$: Angels exist, yet $\pi \leq 3$.

c. $\sim p \wedge \sim q$

d. $(\sim p) \wedge q$

DISJUNCTION. **Any two statements can be combined by the work** OR (in the sense of "and/or") to form a new statement which is called the disjunction of the original two statements. The OR operation of two propositions *p* and *q* (written as $p \vee q$) is true if at least any of the propositional variable *p* or *q* is true (Parahar, 2019). In other words, the disjunction of two statements is false only if each component is false. The propositions *p* and *q* are called *disjuncts* (Lipschutz, 1981).

The truth table is as follows:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Example:

1. Let *p*: He studied French at the university and *q*: He lived in France, then $p \vee q$ is the statement "He studied French at the university or he lived in France."
2. The symbol $\vee$ can be used to define the union of two sets, specifically,

   $A \cup B = \{x \mid x \in A \vee x \in B\}$

3. Consider the following four statements:

   (a) Bubukal is in Santa Cruz and 2 + 2 = 5. True

   (b) Bubukal is in Pila and 2 + 2 = 4. True

   (c) Bubukal is in Pila and 2 + 2 = 5. True

   (d) Bubukal is in Santa Cruz and 2 + 2 = 4. False

   Only (d) is false. Each of the other statements is true since at least one of its components is true.

4. Let "*p*: Pat has a date with Camille.", "*q*: Aaron is sleeping." and "*r*: Kianna is eating." Express the following conjunctions as English sentences or in symbols, as the case may be.

   a. $p \vee q$

   b. $q \vee (\sim r)$

   c. $p \vee (q \vee r)$

   d. Either Pat has a date with Camille or Aaron is sleeping, or Kianna is eating.

   e. Either Pat has a date with Camille and Aaron is sleeping, or Kianna is eating.

   f. Either Pat has a date with Camille, or Aaron is sleeping and Kianna is eating.

   g. Either Pat has a date with Camille and Aaron is sleeping, or Pat has a date with Camille and Kianna is eating.

Solution:

a. $p \vee q$: Pat has a date with Camille or Aaron is sleeping.

b. $q \vee (\sim r)$: Either Aaron is sleeping or Kianna is not eating.

c. $p \vee (q \vee r)$: Either Pat has a date with Camille, Aaron is sleeping or Kianna is eating.

d. $(p \vee q) \vee r$

e. $(p \wedge q) \vee r$

f. $p \vee (q \wedge r)$

g. $(p \wedge q) \vee (p \wedge r)$

***CONDITIONAL*. Many** statements, especially in mathematics, are of the form "If *p* then *q*". Such statements are called <u>conditional</u> statements. An implication $p \implies q$ is the proposition "if *p*, then *q*" . It is false if *p* is true and *q* is false. The rest cases are true (Parahar, 2019). In other words, a true statement cannot imply a false statement (Lipschutz, 1981).

The conditional $p \implies q$ can also be read:

a.) *p* implies *q*c.) *p* is sufficient for *q* e.) *q* if *p*

b.) *p* only if *q* d.) *q* is necessary for *p* f.) *q* is implied by *p*

The truth table is as follows:

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Example:

1. **Consider the following statements:**

   (a) Bubukal is in Santa Cruz and 2 + 2 = 5. False

   (b) Bubukal is in Pila and 2 + 2 = 4. True

   (c) Bubukal is in Pila and 2 + 2 = 5. True

   (d) Bubukal is in Santa Cruz and 2 + 2 = 4. True

Only (a) is a false statement; the others are true.

2. **Suppose Lyra is a Grade 11 student, analyze the truth values the following conditionals.**

   p_1: If Lyra is in Grade 11, then she is a senior high school student.

   p_2: If Lyra is in Grade 11, then she is working as a lawyer.

   p_3: If Lyra has a degree in Education, then she believes in true love.

   Solution:

   p_1: "Lyra is in Grade 11" and "Lyra is a senior high school student" are both true, then p_1 is true.

p_2: "Lyra is in Grade 11" is true and "Lyra is working as a lawyer" is false because a Grade 11 student is not yet qualified to be a lawyer, then p_2 is false.

p_3: "Lyra has a degree in Education" is false and the truth value of "Lyra believes in true love" is not known, then p_3 is true regardless of the truth value of the conclusion.

3. Determine the truth value of the following propositions.

   a.) If 2 > 0, then there are more than 100 million Filipinos.

   b.) If 2 > 0, then there are only 5 languages spoken in the Philippines.

   c.) If 2 < 0, then it is more fun in the Philippines.

Solution: The number 2 is a positive number, so the proposition "2 > 0" is true while "2 < 0" is false.

a.) "2 > 0" is true and "There are more than 100 million Filipinos" is also true, then the conditional is true.

b.) "2 > 0" is true and "There are only 5 languages spoken in the Philippines" is not true, therefore the conditional is false.

c.) "2 < 0" is false, then the conditional is true whether it is more fun in the Philippines or not.

BICONDITIONAL. Another common statement is of the form "*p* if and only if *q*", *or simply,* "*p* iff *q*". Such statements are called biconditional statements. $p \Leftrightarrow q$ is biconditional logical connective which is true when p and q are same, i.e. both are false or both are true (Parahar, 2019). In other words, if *p* and *q* have the same truth value, then $p \Leftrightarrow q$ is true; if *p* and *q* have opposite truth values, then $p \Leftrightarrow q$ is false (Lipschutz, 1981).

The biconditional $p \Leftrightarrow q$ can also be read: *p* is necessary and sufficient for *q*.

The truth table is as follows:

p	q	$p \Leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

The statement $p \Leftrightarrow q$ is equivalent to the statement $p \Leftrightarrow q = (p \Rightarrow q) \wedge (q \Rightarrow p)$.

Example:

1. Consider the following statements:

   (a) Bubukal is in Santa Cruz and 2 + 2 = 5. False

   (b) Bubukal is in Pila and 2 + 2 = 4. False

   (c) Bubukal is in Pila and 2 + 2 = 5. True

   (d) Bubukal is in Santa Cruz and 2 + 2 = 4. True

(c) and (d) are true, and (a) and (b) are false.

2. **Suppose Lyra is a Grade 11 student, analyze the truth values the following biconditionals.**

   p_1: Lyra is in Grade 11 if and only if she is a senior high school student.

   p_2: Lyra is in Grade 11 if and only if she is working as a lawyer.

   p_3: Lyra has a degree in Education if and only if she believes in true love.

   Solution:

   p_1: The components of p_1 are true, hence the biconditional is true.

   p_2: "Lyra is in Grade 11" is true and "Lyra is working as a lawyer" is false, then p_2 is false.

p_3: The truth value of the biconditional depends on whether Lyra believes in true love or not.

$$\left.\begin{array}{l}\text{“Lyra has a degree in Education” is false.}\\ \text{“Lyra believes in true love” is true.}\end{array}\right\}\ \text{False}$$

$$\left.\begin{array}{l}\text{“Lyra has a degree in Education” is false.}\\ \text{“Lyra does not believes in true love” is false.}\end{array}\right\}\ \text{True}$$

Lesson 6. Converse, Inverse and Contrapositive (Baltazar, et al., 2018)

Suppose p and q are propositions.

			Converse	Inverse	Contrapositive
p	q	$p \Rightarrow q$	$q \Rightarrow p$	$\sim p \Rightarrow \sim q$	$\sim q \Rightarrow \sim p$
T	T	T	T	T	T
T	F	F	T	T	F
F	T	T	F	F	T
F	F	T	T	T	T

Example:

1. Give the converse, inverse and contrapositive of the following implications.
   a. If this movie is interesting, then I am watching it.
   b. If p is a prime number, then it is odd.

   Answers:

   a. Inverse: If this movie is not interesting, then I am not watching it.
      Converse: If I am watching this movie, then it is interesting.
      Contrapositive: If I am not watching this movie, then it is not interesting.

   b. Inverse: If p is not a prime number, then it is not odd.
      Converse: If p is an odd number, then it is.
      Contrapositive: If p is not an odd number, then it is not prime.

2. Identify the three simple propositions in the statement "A function f has an inverse if and only if f is one-to-one and onto." and label them p, q and r.

   p: A function f has an inverse.
   q: f is one-to-one.
   r: f is onto.

3. Express symbolically the statement in number 2 using the logical operators.

   Answer: $p \Leftrightarrow (q \wedge r)$ or $[p \Rightarrow (q \wedge r)] \wedge [(q \wedge r) \Rightarrow p]$

Lesson 7. Quantifiers (Lipschutz, 1981)

In mathematics, the phrases 'there exists' and 'for all' play a huge role in logic and logic statements (Alida, 2020). Quantifiers are used to describe the variable(s) in a statement. These are words, expressions, or phrases that indicate the number of elements that a statement pertains to. In mathematical logic, there are two quantifiers: 'there exists' and 'for all' (Alida, 2020).

Universal quantifier. The universal quantifier usually written in the English language as "for all" or "for every". It is denoted by the symbol $\forall$.

Existential quantifier. The existential quantifier expressed in words as "there exists" or "for some". This quantifier is denoted by $\exists$.

Compound Quantifiers. When one quantity is involved in one statement, it is common to encounter more than one quantifier for that statement. Special care must be taken in the order in which these quantifiers appear.

Order does not matter when using the same quantifier. For example, the statement $\forall x, \forall y, P(x, y)$ is the same as $\forall y, \forall x, P(x, y)$. Similarly, for the quantified sentence $\exists x, \exists y, P(x, y)$ is equivalent to $\exists y, \exists x, P(x, y)$. For mixed quantifiers, order definitely is important. The statement $\forall x, \exists y, P(x, y)$ is never always equivalent to the sentence $\exists y, \forall x, P(x, y)$.

Example: Write as an English sentence and determine if it is true or false.

a. $\forall x \in \mathbb{R}, \exists y \in \mathbb{R},\ x + y = 10$
b. $\forall x \in \mathbb{Z}^{+}, \exists y \in \mathbb{R},\ y^2 = x$

Solution:

a. For every real number *x*, there exists a real number *y* such that the sum of *x* and *y* is equal to 10. TRUE
b. For every positive integer *x*, there exists a real number *y* such that the square of *y* is equal to *x*. TRUE

Lesson 8. Constructing Truth Tables (CHED, 2016)

When you're constructing a truth table, consider all possible assignments of True (T) and False (F) to the component statements (Ikenaga, 2020). For example, suppose the component statements are *p*, *q*, and *r*. Each of these statements can be either true or false, so there are $2^3 = 8$ possibilities. When listing the possibilities, assign truth values to the component statements in a systematic way to avoid duplication or omission.

Example:

1. Let *p* and *q* be propositions. Construct the truth table of the compound proposition$(p \Rightarrow q) \wedge (q \Rightarrow p)$.

Solution:

Step 1: Since there are two propositions *p* and *q* involved, the truth table should have four rows which consist of all possible truth values combination of *p* and *q*.

p	q	$p \Rightarrow q$	$q \Rightarrow p$	$(p \Rightarrow q) \wedge (q \Rightarrow p)$
T	T			
T	F			
F	T			
F	F			

Step 2: The given proposition is a conjunction of the conditionals $(p \Rightarrow q)$ *and* $(q \Rightarrow p)$ as the conjuncts. In the next two columns encode the truth values of these conditonals.

p	q	$p \Rightarrow q$	$q \Rightarrow p$	$(p \Rightarrow q) \wedge (q \Rightarrow p)$
T	T	T	T	
T	F	F	T	
F	T	T	F	
F	F	T	T	

Step 3: In the final column, encode the truth values of the conjunction $(p \Rightarrow q) \wedge (q \Rightarrow p)$ using the third and fourth columns.

p	q	$p \Rightarrow q$	$q \Rightarrow p$	$(p \Rightarrow q) \wedge (q \Rightarrow p)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

2. Construct the truth table of the compound proposition

   $s: [(p \Rightarrow r) \wedge (q \Rightarrow r)] \Rightarrow [(p \vee q) \Rightarrow r]$

   Solution: Since there are 3 propositions, there should be 8 rows.

Step 1:

p	q	r	$p \Rightarrow r$	$q \Rightarrow r$	$(p \Rightarrow r) \wedge (q \Rightarrow r)$	$p \vee q$	$(p \vee q) \Rightarrow r$	s
T	T	T						
T	T	F						
T	F	T						
T	F	F						
F	T	T						
F	T	F						
F	F	T						
F	F	F						

Step 2:

p	q	r	$p \Rightarrow r$	$q \Rightarrow r$	$(p \Rightarrow r) \wedge (q \Rightarrow r)$	$p \vee q$	$(p \vee q) \Rightarrow r$	s
T	T	T	T	T				
T	T	F	F	F				
T	F	T	T	T				
T	F	F	F	T				
F	T	T	T	T				
F	T	F	T	F				
F	F	T	T	T				
F	F	F	T	T				

Step 3:

p	q	r	$p \Rightarrow r$	$q \Rightarrow r$	$(p \Rightarrow r) \wedge (q \Rightarrow r)$	$p \vee q$	$(p \vee q) \Rightarrow r$	s
T	T	T	T	T	T			
T	T	F	F	F	F			
T	F	T	T	T	T			
T	F	F	F	T	F			
F	T	T	T	T	T			
F	T	F	T	F	F			
F	F	T	T	T	T			
F	F	F	T	T	T			

Step 4:

p	q	r	$p \Rightarrow r$	$q \Rightarrow r$	$(p \Rightarrow r) \wedge (q \Rightarrow r)$	$p \vee q$	$(p \vee q) \Rightarrow r$	s
T	T	T	T	T	T	T		
T	T	F	F	F	F	T		
T	F	T	T	T	T	T		
T	F	F	F	T	F	T		
F	T	T	T	T	T	T		
F	T	F	T	F	F	T		
F	F	T	T	T	T	F		
F	F	F	T	T	T	F		

Step 5:

p	q	r	$p \Rightarrow r$	$q \Rightarrow r$	$(p \Rightarrow r) \wedge (q \Rightarrow r)$	$p \vee q$	$(p \vee q) \Rightarrow r$	s
T	T	T	T	T	T	T	T	
T	T	F	F	F	F	T	F	
T	F	T	T	T	T	T	T	
T	F	F	F	T	F	T	F	
F	T	T	T	T	T	T	T	
F	T	F	T	F	F	T	F	
F	F	T	T	T	T	F	T	
F	F	F	T	T	T	F	T	

Step 6:

p	q	r	$p \Rightarrow r$	$q \Rightarrow r$	$(p \Rightarrow r) \wedge (q \Rightarrow r)$	$p \vee q$	$(p \vee q) \Rightarrow r$	s
T	T	T	T	T	T	T	T	T
T	T	F	F	F	F	T	F	T
T	F	T	T	T	T	T	T	T
T	F	F	F	T	F	T	F	T
F	T	T	T	T	T	T	T	T
F	T	F	T	F	F	T	F	T
F	F	T	T	T	T	F	T	T
F	F	F	T	T	T	F	T	T

The last column of the truth table consists entirely of T. This means that the proposition $s: [(p \Rightarrow r) \wedge (q \Rightarrow r)] \Rightarrow [(p \vee q) \Rightarrow r]$ is always true for all possible combinations of the truth values of p, q and r. Such propositions are called ***tautologies***.

Definition: A proposition that is always true is called ***tautology***, while a proposition that is always false is called a ***contradiction***. A tautology is denoted by τ and a contradiction by ϕ (Ikenaga, 2019).

---

3. Using truth tables, show the following:
   i. $p \vee \tau$ is a tautology
   ii. $p \wedge \phi$ is a contradiction

Solution:

i. The second column consists of T since τ is a proposition which is always true. The proposition $p \vee \tau$ is true for any truth value of p; hence, it is a tautology.

p	τ	$p \vee \tau$
T	T	T
F	T	T

ii. The second column consists of F since ϕ is a proposition which is always false. The proposition $p \wedge \phi$ is false for any truth value of p; hence, it is a contradiction.

p	ϕ	$p \wedge \phi$
T	T	T
F	T	T

Assessment Task 1-1

I. Write the given sets in roster and rule method.

1. A is the set of counting numbers greater than -2 and less than 8.

   Roster method:

   Rule method:

2. B is the set of integers less than 0

   Roster method:

   Rule method:

II. Write the given sets in rule method.

1. $A = \{1, 3, 5, 7\}$

2. $B = \{-4, -5, -6, \ldots\}$

III. Express the given set in roster method.

1. $A = \{x \mid x \textit{ is a positive integer less than } 2\}$

2. $B = \{x \mid x \geq -2\}$

Assessment Task 1-2

Perform the indicated set operations.

Given: $U = \{a, b, c, d, e\}$

$A = \{a, c, e\}$

$B = \{a, d\}$

$C = \{a, b, c, d, e\}$

$D = \{\}$

1. $A' \cap B'$

2. $C' \cup B'$

3. $D' \cap A$

4. $(A \cup D)'$

5. $(B' \cup C') \cap A$

Assessment Task 1-3

I. Write T if the proposition is true, otherwise write F.

_____1. $\sqrt{2}$ is an irrational number.

_____2. $7 - 2 = 4$

_____3. 2 is the only prime number that is even.

_____4. The constant π is a real number.

_____5. $f(x) = \frac{1}{x}$ is a radical function.

II. Determine whether each of the following statement is a proposition or not. Write P if proposition, NP if not.

_____1. Mabuhay!

_____2. Jose Rizal is our national hero.

_____3. $\sqrt{3}$ is an integer.

_____4. Greet your audience.

_____5. If an integer is even, then its square is also even.

III. Express the following conjunctions as English sentences or in symbols, as the case may be.

1. 5 is an integer

2. $8 \in N$

3. $6 < 12$

4. The constant e is an irrational number.

5. $10 = \sqrt{100}$

Assessment Task 1-4

I. Let p be “It is cold” and let q be “It is raining”. Give a simple verbal sentence which describes each of the following statements.

1. $p \vee {\sim}q$
2. ${\sim}p \Rightarrow {\sim}q$
3. $p \Leftrightarrow {\sim}q$
4. ${\sim}{\sim}q$
5. $(p \wedge {\sim}q) \Rightarrow p$

II. Let Let p be “He is tall” and let q be “He is handsome”. Write each of the following statements in symbolic form using p and q.

1. He is tall and handsome.
2. It is false that he is short or handsome.
3. He is neither tall nor handsome.
4. He is tall, or he is short and handsome.
5. It is not true that he is short or not handsome.

Assessment Task 1-5

I. Give the converse, inverse and contrapositive of the following implications.

1. If π is an irrational number then it is a number that goes on forever.

   Inverse:

   Converse:

   Contrapositive:

2. If i is a complex number then it is not a real number.

   Inverse:

   Converse:

   Contrapositive:

II. Write the following quantifiers as an English sentence.

1. There exists an integer x, such that $5 - x = 2$.

2. There exists an integer x such that $3x - 2 = 0$.

3. The square of every real number is greater than 0.

Assessment Task 1-6

I. Determine the truth value of each of the following composite statements.

1. If $3 + 2 = 7$, the $4 + 4 = 8$.

2. It is not true that $2 + 2 = 5$ if and only if $4 + 4 = 10$.

3. It is not true that $1 + 1 = 3$ or $2 + 1 = 3$.

II. Construct the truth table of the following compound propositions.

1. $(p \wedge q) \wedge \sim(p \vee q)$

2. $\sim(p \wedge q) \vee \sim(q \Leftrightarrow p)$

SUMMARY

- Use of the word *set* as formal mathematical term was introduced in 1879 by Georg Cantor (1845-1918).
- A **set** is a collection of **elements** (or objects) and can be considered as an element itself.
- The types of set are null, singleton, universal, finite, infinite, equal, equivalent and power set.
- The two ways of writing a set are roster and rule method.
- Set operation is composed of union, intersection and complement of set.
- Logic, as defined by David W. Kueker, is the analysis of methods of reasoning. It is interested in the form rather than the content of the argument.
- The fundamental property of a statement is that it is either true or false, but not both.
- A proposition is a statement or a declarative sentence which is either true (T) or false (F) without additional information. Propositions are usually denoted by small letters (*p, q, r* - with or without subscripts).
- A truth table is a table that shows the truth value of a compound statement for all possible truth values of its simple statements.
- A Logical Connective is a symbol which is used to connect two or more propositional or predicate logics in such a manner that resultant logic depends only on the input logics and the meaning of the connective used.
- The five logical connectives are negation, conjunction, disjunction, conditional and biconditional.
- In mathematical logic, there are two quantifiers: universal and existential.
- When you're constructing a truth table, consider all possible assignments of True (T) and False (F) to the component statements.
- When listing the possibilities, assign truth values to the component statements in a systematic way to avoid duplication or omission.

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