Statistics Bonus Lecture

There are normally two types of variables we are interested in - discrete and continuous variables.

- Discrete: For example, the number of particles within a box - it must be a whole, real number, and is finite.
- Continuous: For example, length or time.

Discrete probability distributions

Let x be some discrete random variable - that is, it can only take a finite number of values. We'll denote these values as x_i, and their probability of occurring will be P_i. First, we will require that

$$\sum_i P_i = 1.$$

The mean value of our variable is thus

$$\langle x \rangle = \sum_i x_i P_i.$$

We can also define

$$\left\langle x^2 \right\rangle = \sum_i x_i^2 P_i.$$

Continuous probability distributions

Let x be some continuous random variable - that is, it can only take a finite number of values. In this case, individual values of x do not have a well defined probability, so instead we talk about probability intervals, saying that $P(x)\mathrm{d}(x)$ is the probability of having a value between x and $x + \mathrm{d}x$. Thus, we now require

$$\int P(x)\mathrm{d}x = 1.$$

The mean and mean squared are then defined as

$$\langle x \rangle = \int x P(x)\mathrm{d}x.$$
$$\left\langle x^2 \right\rangle = \int x^2 P(x)\mathrm{d}x.$$

A very common example of a continuous probability distribution is the Gaussian, which has the general form

$$P(x) = Ce^{-\frac{x^2}{2a^2}}.$$

This distribution has a maximum at $x = 0$.Both C and a can be evaluated using the above requirements, and give

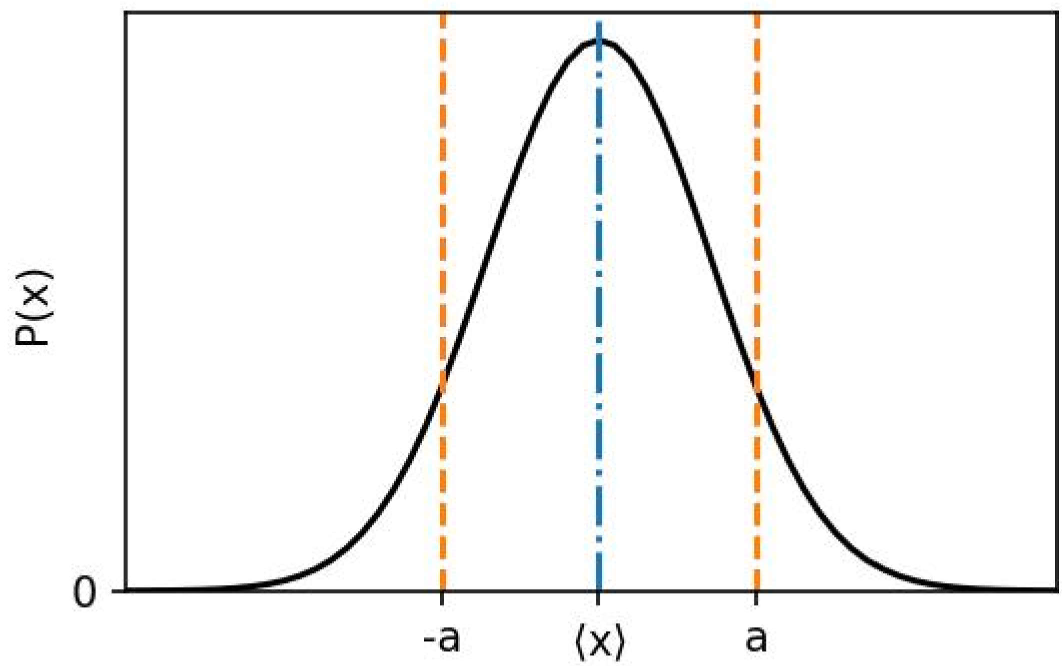
$$C = \frac{1}{\sqrt{2\pi a^2}}$$

and

$$a^2 = \langle x^2 \rangle.$$

An example of a Gaussian is plotted below - and note that a more general form of the Gaussian is

$$P(x) = \frac{1}{\sqrt{2\pi a^2}} e^{-\frac{(x-\langle x \rangle)^2}{2a^2}}.$$



Variance

So imagine we now have some set of values. We know how to calculate the average (whether they're continuous or discrete) - the next useful piece of information is the spread of values. Let's define the deviation as

$$x - \langle x \rangle$$

which tells us how far or below the average a given value lies. However, the average of this value for a distribution, $\langle x - \langle x \rangle \rangle$ should be 0! (Think about this to convince yourself it's true). So, instead, we use a quantity called the variance, σ, which is

$$\sigma^2 = \left\langle (x - \langle x \rangle)^2 \right\rangle$$
$$\sigma^2 = \left\langle x^2 \right\rangle - \langle x \rangle^2$$

Now think about the Gaussian again - using the above definition, we can rewrite the general expression as

$$P(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \mathrm{e}^{-\frac{(x-\langle x \rangle)^2}{2\sigma^2}}.$$

Now we can better understand the meaning of some of the terms - the variance is the standard deviation of a Gaussian distribution. Thus, when physicists discuss 1σ, 2σ, 3σ, it means we're taking about the probability of a result and how far from the central value of a Gaussian distribution the result lies.

Binomial distribution

Let's now look at a very specific case of a discrete probability distribution: the binomial distribution. We'll start by considering a Bernoulli Trial, which is an experiment where we can only have 2 outcomes: success or fail. A good example would be tossing a coin, and labelling heads as a success, and tails as a fail. Or (relating this to PS 1), if a particle can be described as being on the left side of a room (success) or right side (fail).

The important thing is we will say the probability of a success occurring is p, meaning the probability of a failure is $1 - p$.

Let's start by saying our variable is x, which can take a value of 1 for success and 0 for a failure. Using our above definitions, we can write down

$$\langle x \rangle = 0 \times (1-p) + 1 \times p = p$$
$$\left\langle x^2 \right\rangle = 0^2 \times (1-p) + 1^2 \times p = p$$
$$\sigma = \sqrt{\left\langle x^2 \right\rangle - \langle x \rangle^2} = \sqrt{p - p^2} = \sqrt{p(1-p)}$$

The binomial distribution, $P(n, k)$, is the probability of having k successes from n independent trails. First, the probability that we have k successes and $n - k$ failures is $p^k(1-p)^{n-k}$. The number of ways we can arrange our k successes (remembering our discussion of marbles in the first lecture) is the statistical weight, giving us

$$P(n, k) = \Omega\, p^k (1-p)^{n-k}.$$

This is the binomial distribution.