Lecture_10

February 22, 2023

1 Radiation Pressure

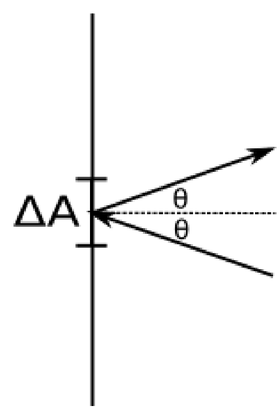
So now we are going to look at a concept which leans on the same analysis and tricks as used when dealing with black body spectra. In our analysis, we found that the energy hitting the wall of a container filled with black body radiation, coming from a volume dV, was

$$dE_{\mathrm{abs}} = u(T) r^2 \sin\theta dr d\theta d\phi \frac{\Delta A \cos(\theta)}{4\pi r^2}$$

$$dE_{\mathrm{abs}} = u(T) \frac{\Delta A}{4\pi} \cos\theta \sin\theta dr d\theta d\phi$$

1.1 Isotropic Radiation

Now, suppose the photons are reflected from the wall, as in the below figure.



The change in momentum perpendicular to the wall for photons incident at a angle θ is

$$dp_\perp = \frac{dE}{c}\cos\theta - \left(-\frac{dE}{c}\cos\theta\right)$$

$$dp_\perp = \frac{2dE}{c}\cos\theta$$

$$dp_\perp = \frac{u(T)\Delta A}{2\pi c}\cos^2\theta \sin\theta dr d\theta d\phi$$

If we want the total change from all photons, then we need to integrate over the hemisphere (as done in the previous lecture)

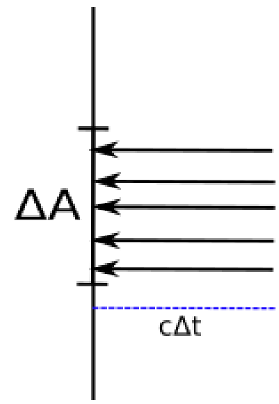
$$\Delta p_\perp = \frac{u(T)\Delta A}{2\pi c}\int_0^{c\Delta t} dr \int_0^{\pi/2} \cos^2\theta \sin\theta d\theta \int_0^{2\pi} d\phi$$

The integral over θ becomes $-\frac{1}{3}cos^3\theta$, which when evaluated at the limits gives $\frac{1}{3}$. So the total expression becomes

$$\Delta p_{\perp} = \frac{1}{3}u\Delta A\Delta t$$

$$\frac{\Delta p_{\perp}/\Delta t}{\Delta A} = \frac{F_{\perp}}{\Delta A} = P_{\text{rad}} = \frac{1}{3}u$$

This is the pressure exerted due to isotropic radiation. ## Directed Radiation If the radiation is not isotropic, but is directed, things need a bit of rearranging.



The energy contained within the volume of photons that will hit ΔA in Δt is:

$$E = u\Delta A\ c\ \Delta t = I\Delta A\Delta t$$

where $I = uc$ is the intensity.

$$\Delta p_{\perp} = \frac{E}{c} - (-\frac{E}{c}) = \frac{2E}{c} = \frac{2}{c}u\Delta A\ c\ \Delta t$$

$$\Delta p_{\perp} = 2u\Delta A\ \Delta t$$

$$\frac{\Delta p_{\perp}/\Delta t}{\Delta A} = \frac{F_{\perp}}{\Delta A} = P_{\text{rad}} = 2u = 2\frac{I}{c}$$

If the directed radiation is not reflected, but instead absorbed, then we get that

$$\Delta p_{\perp} = 0 - (-\frac{E}{c}) = \frac{E}{c}\ , P_{\text{rad}} = u = \frac{I}{c}$$

1.2 Example

A comet releases dust, which is pushed radially away from the Sun by the Sun's radiation pressure. Dust particles have a radius R and a density of $\rho = 3.5 \times 10^3$ kg m^{-3}. These dust particles fully absorb sunlight hitting them. For what R does the graviational force due to the Sun balance the outward force due to radiation pressure? We're going to first treat the Sun as a point source of radiation. We are also going to say that F_{rad} is outwards and F_{grav} is inwards.

$$F_{\text{rad}} = P_{\text{rad}}A = \frac{I}{c}\pi R^2$$

$$F_{\text{rad}} = \frac{1}{c}\frac{L_\odot}{4\pi d^2}\pi R^2$$

$$F_{\text{rad}} = \frac{1}{c}\frac{\sigma 4\pi R_\odot^2 T_\odot^4}{4\pi d^2}\pi R^2$$

$$F_{\text{rad}} = \frac{1}{c}\frac{\sigma R_\odot^2 T_\odot^4}{d^2}\pi R^2$$

The graviational force is easier to compute:

$$F_{\text{grav}} = \frac{GM_\odot \rho \frac{4}{3}\pi R^3}{d^2}$$

So now if we balance these forces, we get

$$F_{\text{rad}} = F_{\text{grav}}$$

$$\frac{1}{c}\frac{\sigma R_\odot^2 T_\odot^4}{d^2}\pi R^2 = \frac{GM_\odot \rho \frac{4}{3}\pi R^3}{d^2}$$

$$R = \frac{3}{4}\frac{\sigma R_\odot^2 T_\odot^4}{cG\rho M_\odot} = 5800\text{A}\left(\frac{1000\text{kgm}^{-3}}{\rho}\right) = 0.166\mu\text{m}$$

Dust particles with this radius feel no net force - which means they feel no acceleration. As such, they travel in a straight line at constant velocity. Particles with a radius bigger than this have $F_{\text{grav}} > F_{\text{rad}}$, and so curve towards the Sun, while paricle with R less than this value $F_{\text{grav}} < F_{\text{rad}}$, meaning the path curves away from the Sun.

The collection of all of these paths forms the dust tail of a comet, see the course attachement for a diagram of this effect.

2 Equilibrium temperatures of solar system bodies.

The question we wish to anwer today is this: Given the distance between a planet and a star, what is the surface temperature of the planet?

2.1 Mercury

To answer this, we will use Mercury as an example, as it has no atmosphere (which makes our lives a bit simpler). It has an elliptical orbit, and the minimum distance from the Sun is $d_{\text{min}} = 4.6 \times 10^{10}$ m. We'll use this as a benchmark for our calculation.

If the surface is in equilibrium, then the energy absorbed by the surface will be the same as the energy radiated (much like in our discussion of black body spectra). We'll also approxmiate the surface of Mercury to be a perfect black body ($\epsilon = 1$).

Consider an area A on the surface of Mercury facing the Sun. The power absorbed by this area is

$$P_{\text{abs}} = \frac{L_\odot}{4\pi d_{\text{min}}^2}A$$

We've assumed here that there's very little curvature of the surface within area A in this approximation. The power emitted by this area is given by the Stefan Boltzmann law, and is

$$P_{\text{rad}} = \sigma A T_{\text{Mer}}^4$$

So,

$$P_{\text{abs}} = P_{\text{rad}}$$

$$\frac{L_\odot}{4\pi d_{\text{min}}^2} A = \sigma A T_{\text{Mer}}^4$$

Using $L_\odot = \sigma 4\pi R_\odot^2 T_\odot^4$, then we get

$$T_{\text{Mer}} = T_\odot \sqrt{\frac{R_\odot}{d_{\text{min}}}}$$

Plugging in for $T_\odot$ and $R_\odot$, we get $T_{\text{Mer}} = 714$ K. This temperature falls off at the edges of the side of Mercury, as the projected surface area visible to the Sun at the poles of Mercury are much smaller than at the equator.

So the temperature we've calculated is the maximum surface temperature. Now, let's try and calculate the average temperature on the side of Mercury facing the Sun.

$$P_{\text{abs}} = \frac{L_\odot}{4\pi d_{\text{min}}^2} \pi R_{\text{Mer}}^2$$

where that last term is now the projected surface are of Mercury as viewed from the Sun (so the surface area is that of a disc). The emitted power is given by

$$P_{\text{rad}} = \sigma (2\pi R_{\text{Mer}}^2) T_{\text{Mer}}^4$$

where here, we've said that Mercury is only radiating over 1/2 a hemisphere (that is, the hemisphere being heated by the Sun!). So, now by balancing these again, we get

$$T_{\text{Mer}} = \frac{1}{2^{1/4}} T_\odot \sqrt{\frac{R_\odot}{d_{\text{min}}}}$$

where gives a temperature of 600 K. Interestingly, the radius of the object did not occur in that last equation. As such, the average temperature of a body depends solely on the solar radius, solar tempearture, and the distance to the object. This makes is very useful for estimating the temperatures which exoplanets at various distances from their host stars should be, without requiring any knowledge of the size or mass of the planet (assuming there is no atmosphere and the planey is a perfect black body!).