Lecture_17

March 11, 2022

1 Equilibrium temperatures of solar system bodies.

The questio we wish to anwer today is this: Given a bodeies distance from a star, what is the surface temperature of an object?

1.1 Mercury

To answer this, we will use Mercury as an example, as it has no atmosphere (which makes our lives a bit simpler). It has an elliptical orbit, and the minimum distance from the Sun is $d_min = 4.6 \times 10^10$ m. We'll use this as a benchmark for our calculation.

If the surface is in equilibrium, then the energy absorbed by the surface will be the same as the energy radiated (much like in our discussion of black body spectra). We'll also approxmiate the surface of Mercury to be a perfect black body ($\epsilon = 1$).

Consider an area A on the surface of Mercury facing the Sun. The power absorbed by this area is

$$P_{\text{abs}} = \frac{L_\odot}{4\pi d_{\text{min}}^2} A$$

We've assumed here that there's very little curvature of the surface within area A in this approximation. The power emitted by this area is given by the Stefan Boltzmann law, and is

$$P_{\text{rad}} = \sigma A T_{\text{Mer}}^4$$

So,

$$P_{\text{abs}} = P_{\text{rad}}$$

$$\frac{L_\odot}{4\pi d_{\text{min}}^2} A = \sigma A T_{\text{Mer}}^4$$

Using $L_\odot = \sigma 4\pi R_\odot^2 T_\odot^4$, then we get

$$T_{\text{Mer}} = T_\odot \sqrt{\frac{R_\odot}{d_{\text{min}}}}$$

Plugging in for $T_\odot$ and $R_\odot$, we get $T_{\text{Mer}} = 714$ K. This temperature falls off at the edges of the side of Mercury, as the projected surface area visible to the Sun at the poles of Mercury are much smaller than at the equator.

So the temperature we've calculated is the maximum surface temperature. Now, let's try and calculate the average temperature on the side of Mercury facing the Sun.

$$P_{\text{abs}} = \frac{L_\odot}{4\pi d_{\text{min}}^2} \pi R_{\text{Mer}}^2$$

where that last term is now the projected surface are of Mercury as viewed from the Sun (so the surface area is that of a disc). The emitted power is given by

$$P_{\mathrm{rad}} = \sigma(2\pi R_{\mathrm{Mer}}^2)T_{\mathrm{Mer}}^4$$

where here, we've said that Mercury is only radiating over 1/2 a hemisphere (that is, the hemisphere being heated by the Sun!). So, now by balancing these again, we get

$$T_{\mathrm{Mer}} = \frac{1}{2^{1/4}} T_\odot \sqrt{\frac{R_\odot}{d_{\mathrm{min}}}}$$

where gives a temperature of 600 K. Interestingly, the radius of the object did not occur in that last equation. As such, the average temperature of a body depends solely on the solar radius, solar tempearture, and the distance to the object. This makes is very useful for estimating the temperatures which exoplanets at various distances from their host stars should be, without requiring any knowledge of the size or mass of the planet (assuming there is no atmosphere and the planey is a perfect black body!).

1.2 Earth

Let's very quickly consider the Earth

$$T_{\mathrm{Earth}} = \frac{1}{2^{1/4}} T_\odot \sqrt{\frac{R_\odot}{d_\oplus}}$$

which gives $T_{\mathrm{Earth}} = 333$ k or 60^{o} celsius. That maybe runs a bit high (especially since it's an average, not a maximum), so what assumptions do you think we'd need to properly adjust for this?

1.2.1 Reflection

Not all of the light incident on the Earth is absorbed - the ocean and our atmosphere reflect some incident radiation. The fraction reflected is given by the albedo, a, of a planet. For Earth, $a \sim 0.30$. We can account for this using:

$$P_{\mathrm{abs}} = (1-a)\frac{L_\odot}{4\pi d_{\mathrm{min}}^2}\pi R_\oplus^2$$

This gives

$$T_\oplus = \left(\frac{1-a}{2}\right)^{1/4} T_\odot \sqrt{\frac{R_\odot}{d_\oplus}}$$

which gives a temperature of $T_\oplus = 304.6$ k or 31.6^{o} C. So much better already!

1.2.2 Redistribution of energy

There are other effects, such as - The continous rotation of Earth. This means the surface exposure to radiation is constantly changing, which will help lower the temperature. - The atmosphere and oceans circulate, which spreads heat energy absorbed around a much larger fraction of the planet than we've used. This will also lower the temperature. - Internal energy of the planey due to cooling since formation helps increase the temperature.

Again, as a first order approxmiation, we could say that the first 2 effects mean that the energy absorbed by the Earth is in fact spread over the entire surface of the planet. This means

$$P_{\rm rad} = \sigma(4\pi R_\oplus^2)T_\oplus^4$$

This would then give a temperature of

$$T_\oplus = (1-a)^{1/4}\, T_\odot \sqrt{\frac{R_\odot}{2d_\oplus}}$$

This then gives a temperature of $T_\oplus$ = 256 k or -17$^{\rm o}$ C. Which is way too cool - so we've likely overcompensated.

1.2.3 Greenhouse Effect

Earth's atmosphere traps some absorbed energy near Earth's surface ("Greenhouse effect"). This raises the Earth's temperature to around 15$^{\rm o}$ C.

2 The Solar System

Since our discussion has brought us naturally to discussing the planets, we're now going to focus on the formation of the solar system for a bit.

2.1 Structure

The mass within the sollar system is distributed approximately as follows. First, the Sun has a mass of $M_\odot = 2 \times 10^{30}$ kg. Next, the planets have masses (relative to the Earth's mass, $M_\oplus$) of

	Mecury	Venus	Earth	Mars	Jupiter	Saturn	Uranus	Neptune
Mass($M_\oplus$)	0.055	0.815	1.0	0.107	318	95	14.5	17.1

Finally, for completeness, Pluto has a mass of 0.0022 $M_\oplus$.

In total, the planets have a mass $446M_\oplus = 0.0014M_\odot$, meaning the planets really only account for a rounding error of the solar systems total mass! They follow Keplerian orbits around the Sun.

2.2 Atmospheres

Now, in the previous section, we were discussing the temperatures planets may have due to heating by the Sun, and one of the critical elements which affected the calculation of the temperature was the inclusion of an atmosphere. So, let's consider what atmosphere's planets may have.

To escape from a planet with mass M and radius R, a gas particle of mass m must be moving upward with a velocity $\geq$ the escape velocity:

$$v_{\rm esc} = \sqrt{\frac{2GM}{R}}.$$

Now, if the gas in the atmosphere has a temperature T, then

$$\frac{3}{2}kT = \frac{1}{2}mv_{\text{rms}}^2$$

$$v_{\text{rms}} = \sqrt{\frac{3kT}{m}}$$

For appreciable number of gas particles to escape, then $v_{\text{esc}} \sim v_{\text{rms}}$. This gives the temperature at which a gas is likely to escape.

$$T_{\text{esc}} \sim \frac{2}{3}\frac{GMm}{kR}$$

So the temperature depends on $\frac{M}{R}$, as everything else is the same for all planets. In practice, there will be appreciable escape when $v_{\text{rms}} > 1/6v_{\text{esc}}$, since many particles in the gas have $v > v_{\text{rms}}$. So, let's revisit our Table from above, but for $\frac{M}{R}$ (with both given in Earth Units)

	Mecury	Venus	Earth	Mars	Jupiter	Saturn	Uranus	Neptune
$\frac{M}{R}$	0.14	0.86	1.0	0.19	28.4	10.1	3.6	4.5

So the escape temperatures are much higher for the gas giants, especially Jupiter and Neptune, while their actual temperature's are lower than the terrestrial planets (as they are further from the Sun). Because of this, they can retain ligher molecules such as H2 and He, which terrestiral planets lose very easily.