

Lecture_11

February 27, 2024

1 The Solar System

In this lecture, we are going to discuss

- Equilibrium temperatures of planets.

The question we wish to answer today is this: Given the distance between a planet and a star, what is the surface temperature of the planet?

1.1 Mercury

To answer this, we will use Mercury as an example, as it has no atmosphere (which makes our lives a bit simpler). It has an elliptical orbit, and the minimum distance from the Sun is $d_{\min} = 4.6 \times 10^{10}$ m. We'll use this as a benchmark for our calculation.

If the surface is in equilibrium, then the energy absorbed by the surface will be the same as the energy radiated (much like in our discussion of black body spectra). We'll also approximate the surface of Mercury to be a perfect black body ($\epsilon = 1$).

Consider an area A on the surface of Mercury facing the Sun. The power absorbed by this area is

$$P_{\text{abs}} = \frac{L_{\odot}}{4\pi d_{\min}^2} A$$

We've assumed here that there's very little curvature of the surface within area A in this approximation. The power emitted by this area is given by the Stefan Boltzmann law, and is

$$P_{\text{rad}} = \sigma A T_{\text{Mer}}^4$$

So,

$$P_{\text{abs}} = P_{\text{rad}} \\ \frac{L_{\odot}}{4\pi d_{\min}^2} A = \sigma A T_{\text{Mer}}^4$$

Using $L_{\odot} = \sigma 4\pi R_{\odot}^2 T_{\odot}^4$, then we get

$$T_{\text{Mer}} = T_{\odot} \sqrt{\frac{R_{\odot}}{d_{\min}}}$$

Plugging in for T_{\odot} and R_{\odot} , we get $T_{\text{Mer}} = 714$ K. This temperature falls off at the edges of the side of Mercury, as the projected surface area visible to the Sun at the poles of Mercury are much smaller than at the equator.

So the temperature we've calculated is the maximum surface temperature that should occur on a planet that absorbs all incident radiation, which is also known as the **subsolar temperature**. Now, let's try and calculate the average temperature on the side of Mercury facing the Sun.

$$P_{\text{abs}} = \frac{L_{\odot}}{4\pi d_{\text{min}}^2} \pi R_{\text{Mer}}^2$$

where that last term is now the projected surface area of Mercury as viewed from the Sun (so the surface area is that of a disc). The emitted power is given by

$$P_{\text{rad}} = \sigma(2\pi R_{\text{Mer}}^2) T_{\text{Mer}}^4$$

where here, we've said that Mercury is only radiating over 1/2 a hemisphere (that is, the hemisphere being heated by the Sun!). So, now by balancing these again, we get

$$T_{\text{Mer}} = \frac{1}{2^{1/4}} T_{\odot} \sqrt{\frac{R_{\odot}}{d_{\text{min}}}}$$

where gives a temperature of 600 K. Interestingly, the radius of the object did not occur in that last equation. As such, the average temperature of a body depends solely on the solar radius, solar temperature, and the distance to the object. This makes is very useful for estimating the temperatures which exoplanets at various distances from their host stars should be, without requiring any knowledge of the size or mass of the planet (assuming there is no atmosphere and the planet is a perfect black body!).

1.2 Earth

Let's very quickly consider the Earth

$$T_{\text{Earth}} = \frac{1}{2^{1/4}} T_{\odot} \sqrt{\frac{R_{\odot}}{d_{\oplus}}}$$

which gives $T_{\text{Earth}} = 333 \text{ K}$ or $60^{\circ} \text{ Celsius}$. That maybe runs a bit high (especially since it's an average, not a maximum), so what assumptions do you think we'd need to properly adjust for this?

1.2.1 Reflection

Not all of the light incident on the Earth is absorbed - the ocean and our atmosphere reflect some incident radiation. The fraction reflected is given by the albedo, a , of a planet. For Earth, $a \sim 0.30$. We can account for this using:

$$P_{\text{abs}} = (1 - a) \frac{L_{\odot}}{4\pi d_{\text{min}}^2} \pi R_{\oplus}^2$$

This gives

$$T_{\oplus} = \left(\frac{1 - a}{2} \right)^{1/4} T_{\odot} \sqrt{\frac{R_{\odot}}{d_{\oplus}}}$$

which gives a temperature of $T_{\oplus} = 304.6 \text{ K}$ or 31.6° C . So much better already!

1.2.2 Redistribution of energy

There are other effects, such as

- The continuous rotation of Earth. This means the surface exposure to radiation is constantly changing, which will help lower the temperature.
- The atmosphere and oceans circulate, which spreads heat energy absorbed around a much larger fraction of the planet than we've used. This will also lower the temperature.
- Internal energy of the planet due to cooling since formation helps increase the temperature.

Again, as a first order approximation, we could say that the first 2 effects mean that the energy absorbed by the Earth is in fact spread over the entire surface of the planet. This means

$$P_{\text{rad}} = \sigma(4\pi R_{\oplus}^2)T_{\oplus}^4$$

This would then give a temperature of

$$T_{\oplus} = (1 - a)^{1/4} T_{\odot} \sqrt{\frac{R_{\odot}}{2d_{\oplus}}}$$

This then gives a temperature of $T_{\oplus} = 256 \text{ K}$ or -17° C . Which is way too cool - so we've likely overcompensated.

1.2.3 Greenhouse Effect

Earth's atmosphere traps some absorbed energy near Earth's surface ("Greenhouse effect"). This raises the Earth's temperature to around 15° C .

1.3 Atmospheres

One of the key differences between the equilibrium temperatures of the inner rocky planets and the outer gas giants is due to their different albedos. This is mainly down to the chemical composition of their surface. So why are the compositions between the inner planets and the outer planets so different?

To understand this, we first need to understand the concept of escape velocity. To do this, consider the question: what velocity is required such that a particle launched from a planet's surface will drop to a velocity of 0 at infinite distance? The solution is simply:

$$\frac{1}{2}mv_i^2 - \frac{GmM}{r_i} = \frac{1}{2}mv_f^2 - \frac{GmM}{r_f} \quad (1)$$

$$\frac{1}{2}mv_i^2 - \frac{GmM}{r_i} = 0 \quad (2)$$

$$v_i = \sqrt{\frac{2GM}{r_i}} \quad (3)$$

As such, a particle moving at velocity v_i when located a distance r_i from the centre of a body is destined to escape the gravitational potential of that object. So, now recall the Maxwell-Boltzmann velocity distribution, which has a very long tail at high velocities. It is these particles

that are likely to have a velocity higher than the escape velocity for a planet, and will leave, leading to a diminished atmosphere.

However, there is another effect here which counters particles escaping - collisions! In a dense atmosphere, a particle only need travel a short distance prior to interacting with another particle, changing its velocity. However, since the atmosphere will gradually get weaker the higher you go, at some point the density will become low enough that particles with a velocity $> v_{\text{esc}}$ will not undergo an interaction prior to escaping. This height is referred to as the **exobase**, and the atmosphere above this point is referred to as the **exosphere**.

Let's now figure out where the exobase lies. The height, z_{ex} is given by

$$\int_{z_{\text{ex}}}^{\infty} \sigma n(z) dz = \sigma N(z_{\text{ex}}) = 1$$

where $N(z_{\text{ex}})$ is the total column density above z_{ex} . As such, the exobase occurs approximately at the distance at which $N(z_{\text{ex}}) \sim \frac{1}{\sigma}$. For molecules like O_2 , $\sigma \sim 10^{-18} \text{ m}^{-2}$, and so the exobase occurs where $N(z_{\text{ex}}) \sim 10^{18} \text{ m}^{-2}$. For the Earth, this happens at roughly 500 km. At the exobase, the rms speed of the particles is

$$v_{\text{rms}} = \left(\frac{3k_{\text{B}}T}{m} \right)^{1/2}$$

For appreciable number of gas particles to escape, then $v_{\text{esc}} \sim v_{\text{rms}}$. This gives the temperature at which a gas is likely to escape.

$$T_{\text{esc}} \sim \frac{2}{3} \frac{GMm}{k_{\text{B}}R}$$

So the temperature depends on $\frac{M}{R}$, as everything else is the same for all planets. In practice, there will be appreciable escape when $v_{\text{rms}} > 1/6 v_{\text{esc}}$, since many particles in the gas have $v > v_{\text{rms}}$.

1.4 The Solar System

1.4.1 Mass distribution

Now that we know this, let's look at the structure of the Solar System. The mass within the solar system is distributed approximately as follows. First, the Sun has a mass of $M_{\odot} = 2 \times 10^{30} \text{ kg}$. Next, the planets have masses (relative to the Earth's mass, M_{\oplus}) of

	Mercury	Venus	Earth	Mars	Jupiter	Saturn	Uranus	Neptune
Mass(M_{\oplus})	0.055	0.815	1.0	0.107	318	95	14.5	17.1

Finally, for completeness, Pluto has a mass of $0.0022 M_{\oplus}$.

In total, the planets have a mass $446M_{\oplus} = 0.0014M_{\odot}$, meaning the planets really only account for a rounding error of the solar system's total mass! They follow Keplerian orbits around the Sun.

So, let's revisit our Table from above, but for $\frac{M}{R}$ (with both given in Earth Units)

	Mecury	Venus	Earth	Mars	Jupiter	Saturn	Uranus	Neptune
$\frac{M}{R}$	0.14	0.86	1.0	0.19	28.4	10.1	3.6	4.5

So the escape temperatures are much higher for the gas giants, especially Jupiter and Neptune, while their actual temperature's are lower than the terrestrial planets (as they are further from the Sun). Because of this, they can retain lighter molecules such as H₂ and He, which terrestrial planets lose very easily.

1.4.2 Angular Momentum distribution

There are two types of angular momentum associated with motion in the solar system:

- Rotational (Earth spinning around its axis), with a momentum of inertia of $I = \frac{2}{5}mr^2$, and $J_{\text{rot}} = I\omega_{\text{rot}}$
- Orbital (Earth moving around the Sun), with a momentum of inertia of $I = md^2$ if we treat the body as a point source, and $J_{\text{orb}} = I\omega_{\text{orb}}$

Let's look at the various bodies within the solar system, and how much of the solar systems angular momentum they contain.

The Sun There is a very small amount of orbital momentum as it orbits the solar system barycenter, and for this calculation we're going to neglect it. For the rotational angular momentum, J_{rot} , then $m = M_{\odot} = 2 \times 10^{30}$ kg, $r = R_{\odot} = 7 \times 10^8$ m, and has a rotational period of 26 days (2.25×10^6 s), which means

$$\omega_{\text{rot},\odot} = \frac{2\pi}{2.25 \times 10^6} = 2.80 \times 10^{-6} \text{ rad/s}$$

This gives a rotational angular momentum of the Sun of

$$J_{\text{rot},\odot} = \frac{2}{5}M_{\odot}R_{\odot}^2\omega_{\text{rot},\odot} = 1.07 \times 10^{42} \text{ J s}$$

The Sun's rotational angular momentum is 1.0767197674333849e+42 J s

The Earth So planets have both significant orbital and rotational angular momentum. To put the angular momentum of the Sun into context, let's look at the angular momentum of the Earth. Here, $m = M_{\oplus} = 6 \times 10^{24}$ kg, $r = R_{\oplus} = 6.4 \times 10^6$ m, $P_{\text{rot}} = 1$ day, $\omega_{\text{rot}} = 7.3 \times 10^{-5}$ rad/s. This means

$$J_{\text{rot},\oplus} = \frac{2}{5}M_{\oplus}R_{\oplus}^2\omega_{\text{rot},\oplus} = 7.06 \times 10^{33} \text{ J s}$$

which means the ratio of the Earth's rotational angular momentum to the Sun is

$$\frac{J_{\text{rot},\oplus}}{J_{\text{rot},\odot}} = \frac{7.06 \times 10^{33}}{1.07 \times 10^{42}} = 6.7 \times 10^{-9}$$

meaning the Earth's rotational angular momentum is negligible compared to that of the Sun.

The Earth's rotational angular momentum is 7.067092438108647e+33 J s

The orbital angular momentum of the Earth, with a $P_{\text{orb}} = 365.25$ days, and $d = 1$ AU. This gives an orbital angular momentum of

$$J_{\text{orb},\oplus} = M_{\oplus} d^2 \omega_{\text{orb},\oplus} = 2.66 \times 10^{40} \text{ J s}$$

The Earth's orbital angular momentum is 2.661084907751392e+40 J s

So the ratio of Earth's orbital to Earth's rotational angular momentum is:

$$\frac{J_{\text{orb},\oplus}}{J_{\text{rot},\oplus}} = 2.66 \times 10^{40} / 7.06 \times 10^{33} = 3.7 \times 10^6$$

meaning the rotational angular momentum of the Earth is negligible.

However, the ratio of the Earth's orbital angular momentum to the Sun's rotational angular momentum is

$$\frac{J_{\text{orb},\oplus}}{J_{\text{rot},\odot}} = 2.66 \times 10^{40} / 1.07 \times 10^{42} = 0.024$$