Lecture_21

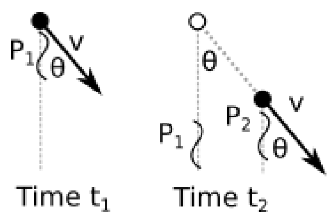
April 3, 2023

1 Relativistic Doppler Shift

In this lecture, we're going to discuss how the concepts of special relativity affect the Doppler shift, which was introduced previously in this course. We'll start by considering general time intervals, then reconsider this in the contact of the frequency and wavelength of light. Finally, we'll quickly show that what we've derived reduces to the expected clssical Doppler shift for low velocities.

1.1 The Doppler Factor

Consider a source of light moving toward an observer with speed v at an angle θ to our line of sight:



where p_1 is a photon emitted at time t_1 in the direction of Earth, and p_2 is a photon emitted at time t_2 towards Earth. What is the time interval between the arrival of the two photons at the observer?

Going back to the Lorentz transformation for time:

$$t_2 - t_1 = \gamma \left[(t_2' - t_1') + \frac{u(x_2' - x_1')}{c^2} \right]$$

Here, the positive x direction is away from the observer, and the relevant velocity component that we need is

$$u = -v \cos\theta$$

where the minus sign arrises are the source is moving in the negative x direction. Also, the distance travelled by p_1 in the time interval $\Delta t'$ in the primed frame is simply

$$x_2' - x_1' = \Delta x' = c\Delta t'.$$

So, combining all of these together, the above expression becomes

$$\Delta t_{\text{obs}} = \gamma \left(\Delta t' - \frac{v \cos\theta \Delta t'}{c} \right)$$

Now we can simplify this a bit more to get

$$\Delta t_{\text{obs}} = \gamma \Delta t' \left(1 - \frac{v\cos\theta}{c}\right) \equiv \frac{1}{D}\Delta t'$$

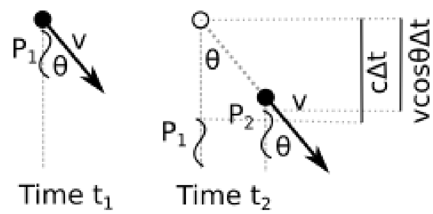
where

$$D = \frac{1}{\gamma\left(1 - \frac{v}{c}\cos\theta\right)}$$

and $\theta = 0$ corresponds to motion directly towards the observer. D is known as the *Doppler Factor*. It plays a role analogous to the Lorentz factor, but allows for motion at some angle to the line of sight.

When $\theta = 90^{\text{o}}$, then $D = \frac{1}{\gamma}$ and $\Delta t = \gamma \Delta t'$

We can gain insight into the expression for Δt by considering the above plot with a few additions



where Δt is the time between emission of the photons. The distance between photons p_1 and p_2 in this setup is:

$$\Delta x = c\Delta t - v\cos\theta \Delta t$$

And so the time mesaured between the arrival of the photons is

$$\Delta t_{\text{obs}} = \frac{\Delta x}{c} = \Delta t\left(1 - \frac{v}{c}\cos\theta\right)$$

Now, the emission of the photons happens at the same place in the rest (primed) frame of the emitting object - which means that $\Delta x' = 0$, so $\Delta = \gamma \Delta t'$, and so we arrive at

$$\Delta t_{\text{obs}} = \gamma \Delta t' \left(1 - \frac{v}{c}\cos\theta\right)$$

as we obtained previously. We see that the origin of the factor $\left(1 - \frac{v}{c}\cos\theta\right)$ is the motion of the source towards/aways from the observer.

1.2 Time interval durations

At this stage, it's worth asking ourselves the question: Is the observed time interval shorter or longer than the time interval in the rest frame of the source ($\Delta t' = \Delta t_{\text{rest}}$)? Starting with:

$$\Delta t_{\text{obs}} = \gamma \Delta t_{\text{rest}} \left(1 - \frac{v}{c}\cos\theta\right)$$

it's not clear there is a simple answer, because of the cos term. Let's break down what we know and see if we can get some insight

1.2.1 Source moving away from us $(90 < \theta \leq 180)$

- $\gamma > 1$ always.
- $\left(1 - \frac{v}{c}\cos\theta\right) > 1$ when $90 < \theta \leq 180$

Thus the two factorsact in the same sense when $90 < \theta \leq 180$ (that is, they are both > 1), so when the source is receding we have that

$$\Delta t_{\text{obs}} > \Delta t_{\text{rest}}.$$

This means the time interval in the rest frame is shorter than observed.

1.2.2 Source moving nearly directly towards us $(\theta << 90\ )$

Now, when $\theta << 90$ such that $\cos\theta$ is very large, then

$$\gamma\left(1 - \frac{v}{c}\cos\theta\right) < 1$$

the observed time interval will be shorter than the rest frame interval

$$\Delta t_{\text{obs}} < \Delta t_{\text{rest}}.$$

This can be very useful when you observe an event and know the sources motion relative to you, as it can allow you to figure out how long it took in the rest frame of the source. This becomes important for objects which may be moving towards us at very hight velocities, as events which seem to take seconds in our frame (and hence are hard to physically explain) may take much longer in the rest frame of the source.

1.3 Moving to frequency and wavelength

Suppose now that Δt_{obs} corresponds to the time between the arrival of two subsequent light wave crests. Then

$$\Delta t_{\text{obs}} = \frac{1}{\nu},\ \Delta t' = \frac{1}{\nu'} \tag{1}$$

$$\frac{1}{\nu} = \frac{\gamma}{\nu'}\left(1 - \frac{v}{c}\cos\theta\right) \tag{2}$$

$$\nu = \frac{\nu'}{\gamma\left(1 - \frac{v}{c}\cos\theta\right)} \tag{3}$$

$$\tag{4}$$

which is the relativistic Doppler shift. When the light source moves directly towards ($\theta = 0$) or away ($\theta = 180$) from the observer, then:

$$v_r \equiv -v\cos\theta = \mp v \tag{5}$$

$$v_r^2 = v^2 \tag{6}$$

where v_r is the radial component of the velocity, and the minus arises as we have said the source is moving towards us with a velocity v, but when we measure radial velocities from Earth, we typically measure a positive radial velocity as an object moving away from us, and a negative radial velocity as an object moving towards us. We then get

$$\gamma\left(1+\frac{v_r}{c}\right) = \frac{\left(1+\frac{v_r}{c}\right)}{\sqrt{1-\frac{v_r^2}{c^2}}} \tag{7}$$

$$= \frac{\left(1+\frac{v_r}{c}\right)}{\sqrt{\left(1+\frac{v_r}{c}\right)\left(1-\frac{v_r}{c}\right)}} \tag{8}$$

$$= \sqrt{\frac{\left(1+\frac{v_r}{c}\right)}{\left(1-\frac{v_r}{c}\right)}} \tag{9}$$

$$\tag{10}$$

and thus

$$\nu = \nu'\sqrt{\frac{\left(1-\frac{v_r}{c}\right)}{\left(1+\frac{v_r}{c}\right)}}$$

where here v_r is positive when source moving away from us, and negative when moving towards us. We can also write this in terms of wavelength using $\lambda\nu = c$:

$$\lambda = \lambda'\sqrt{\frac{\left(1+\frac{v_r}{c}\right)}{\left(1-\frac{v_r}{c}\right)}}$$

- Motion towards the observer: higher ν, shorter λ. - Motion away the observer: lower ν, longer λ.

1.4 Z, redshift

The fractional shift in wavelength is called the redshift:

$$Z = \frac{\Delta\lambda}{\lambda'} = \frac{\lambda-\lambda'}{\lambda'}$$

where λ' is the wavelength in the rest frame. With the above expression for λ, we get

$$Z = \sqrt{\frac{\left(1+\frac{v_r}{c}\right)}{\left(1-\frac{v_r}{c}\right)}} - 1$$

`<b>Example</b>`

The emission lines in the spectrum of the quasar 3C273 display a redshift of $Z = 0.158$ At what wavelength is the Hβ line observed in the spectrum of this quasar?

The rest wavelength of Hβ is 4861Å. So,

$$Z = \frac{\lambda-\lambda'}{\lambda'}$$

Solving for λ gives:

$$\lambda = \lambda'(1+Z) \tag{11}$$

$$= 5629 \tag{12}$$

The speed with which the quasar is moving from the Earth (due to the expansion of the Universe) is given by:

$$(Z+1)^2 = \frac{1+\frac{v_r}{c}}{1-\frac{v_r}{c}} \tag{13}$$

$$\frac{v_r}{c} = \frac{(Z+1)^2-1}{(Z+1)^2+1} \tag{14}$$

$$= 0.1456 \tag{15}$$

1.5 Sanity check

When $v_r/c << 1$, then we get (by using $(1+x)^n \sim 1+nx$)

$$\left(1+\frac{v_r}{c}\right)^{1/2}\left(1-\frac{v_r}{c}\right)^{-1/2} \sim \left(1+\frac{v_r}{2c}\right)\left(1+\frac{v_r}{2c}\right) \tag{16}$$

$$\sim 1+\frac{v_r}{c}+\frac{v_r^2}{4c^2} \tag{17}$$

$$\sim 1+\frac{v_r}{c} \tag{18}$$

Subbing this into

$$\lambda = \lambda' \sqrt{\frac{\left(1+\frac{v_r}{c}\right)}{\left(1-\frac{v_r}{c}\right)}}$$

we get

$$\lambda = \lambda' \left(1+\frac{v_r}{c}\right)$$

which is the non relativistic Doppler shift (phew!).