Lecture_05

January 30, 2023

1 Tidal Forces

1.1 Mass Elements in a Line

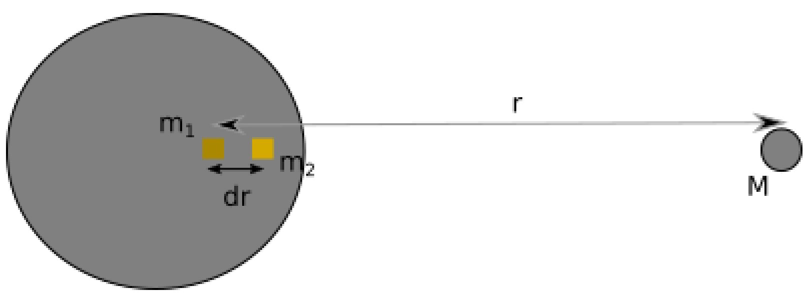
Consider Newton's 3rd law of Graviation:

$$F = G\frac{M_1 M_2}{r^2}$$

When initially deriving this equation, both M_1 and M_2 were treated as point masses, as they were both assumed to be spherically symmetric. The first question we want to ask is what happens if we do not treat them as point masses?

The forces across the body will differ, because of the $1/R^2$ dependance on the force. Consider two mass elements, m_1 and m_2, within the Earth. Because of Newton's Shell theorem, both elements see would see the Moon as a point mass, but they will both feel different forces, because of their different distances to the Moon. This differentical force is known as the tidal force.

Let us set up the problem as follows: m_1 lies a distance R from the centre of mass of a nearby body whose mass is M. m_2 is a second point mass, which lies along the line joining m_1 and M, but is slightly closer to M by a distance dr.



The force exerted on m_1 is:

$$F_{m1} = G\frac{m_1 M}{r^2}$$

The differential force between m_1 and m_2, dF_m, can be expressed as

$$dF_m = \left(\frac{dF_m}{dr}\right) dr = -2G\frac{mM}{r^3} dr$$

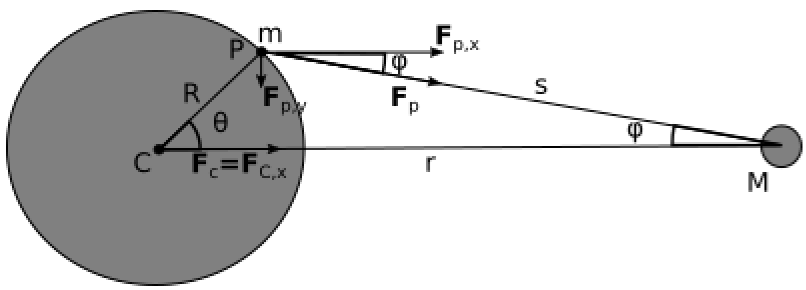
This is the "differentical force". It has a $1/r^3$ dependence, which is a signifcantly stronger dependance than $1/r^2$ dependance of the gravitational force. To get a feel for where this may be important, lets look at what the differential force on opposite sides of the Earth would be when we consider (1) the moon and (2) the Sun.

$$\frac{\Delta F_m}{\Delta F_\odot} = \frac{M_\oplus M_m}{d_m^3} R_\oplus \frac{d_\odot^3}{M_\oplus M_\odot R_\oplus} = \frac{M_m}{d_m^3} \frac{d_\odot^3}{M_\odot} = 2.14$$

So, the differential force on the Earth due to the moon is 2 times stronger than the force due to the Sun, which naturally explains why the Earth's tides are dominated by the lunar cycle.

1.2 Mass Elements across the sphere

Ok, let's now consider the difference in forces between points located at the centre of a body, and at its surface. Consider the following setup



For a test mass, m, located at the centre of the object on the left, then the graviational force felt due to the body on the right (M), is the normal gravitational force. Breaking it into its x and y components, we have

$$F_C = F_{C,x} = G\frac{Mm}{r^2}, \qquad F_{C,y} = 0$$

Now consider a test mass, m, located at point P on the surface of the sphere. This point lies a distance s from the large body on the right. In this case, the component forces are

$$F_{P,x} = G\frac{Mm}{s^2}\cos(\phi), \qquad F_{P,y} = -G\frac{Mm}{s^2}\sin(\phi) \tag{1}$$

The differential force is then given by

$$\mathbf{\Delta F} = \mathbf{F_P} - \mathbf{F_C} = GMm\left(\frac{\cos(\phi)}{s^2} - \frac{1}{r^2}\right)\hat{\mathbf{i}} - G\frac{Mm}{s^2}\sin(\phi)\hat{\mathbf{j}}$$

In essence, our work is now done, but there are a lot of variables in that last equation. Let's try and simplify some things. Recall the law of cosines:

$$s^2 = R^2 + r^2 - 2Rr\cos(\theta) = r^2\left(\frac{R^2}{r^2} + 1 - 2\frac{R}{r}\cos(\theta)\right)$$

Now, in most cases, $r >> R$. As such, $\frac{R^2}{r^2} << 1$, so can be neglected, giving

$$s^2 \approx r^2 \left(1 - 2\frac{R}{r}\cos(\theta)\right)$$

Substituting back in above, we then have

$$\mathbf{\Delta F} = GMm\left(\frac{\cos(\phi)}{r^2\left(1 - 2\frac{R}{r}\cos(\theta)\right)} - \frac{1}{r^2}\right)\widehat{\mathbf{i}} - G\frac{Mm}{r^2\left(1 - 2\frac{R}{r}\cos(\theta)\right)}\sin(\phi)\widehat{\mathbf{j}}$$

$$\mathbf{\Delta F} = \frac{GMm}{r^2}\left(\frac{\cos(\phi)}{\left(1 - 2\frac{R}{r}\cos(\theta)\right)} - 1\right)\widehat{\mathbf{i}} - \frac{GMm}{r^2}\frac{1}{\left(1 - 2\frac{R}{r}\cos(\theta)\right)}\sin(\phi)\widehat{\mathbf{j}}$$

Noting that $r >> R$ and that $(1+x)^{-1} \approx 1 - x$, we can use the substitution $\frac{1}{\left(1-2\frac{R}{r}\cos(\theta)\right)} \approx 1 + 2\frac{R}{r}\cos(\theta)$, we get the expression:

$$\mathbf{\Delta F} = \frac{GMm}{r^2}\left[\cos(\phi)\left(1 + 2\frac{R}{r}\cos(\theta)\right) - 1\right]\widehat{\mathbf{i}} - \frac{GMm}{r^2}\left(1 + 2\frac{R}{r}\cos(\theta)\right)\sin(\phi)\widehat{\mathbf{j}}$$

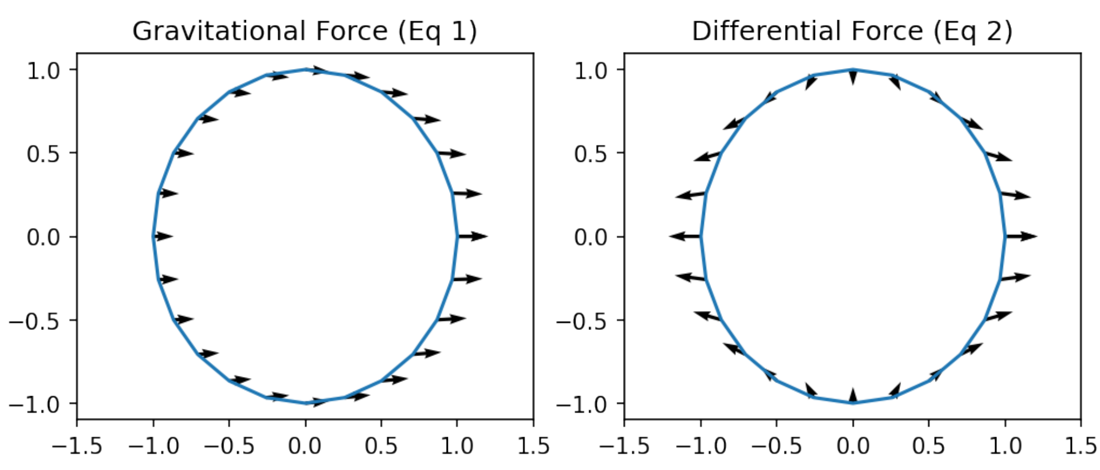
Finally, using the small angle approxmation gives $\cos(\phi) \approx 1$ and $\sin(\phi) \approx \frac{R\sin(\theta)}{r}$ we get

$$\mathbf{\Delta F} = \frac{GMm}{r^2}2\frac{R}{r}\cos(\theta)\widehat{\mathbf{i}} - \frac{GMm}{r^2}\left(1 + 2\frac{R\cos(\theta)}{r}\right)\frac{R\sin(\theta)}{r}\widehat{\mathbf{j}}$$

Excluding terms of order $\frac{R^2}{r^2} << 1$ again, we get

$$\mathbf{\Delta F} = \frac{GMmR}{r^3}2\cos(\theta)\widehat{\mathbf{i}} - \frac{GMmR}{r^3}\sin(\theta)\widehat{\mathbf{j}}$$

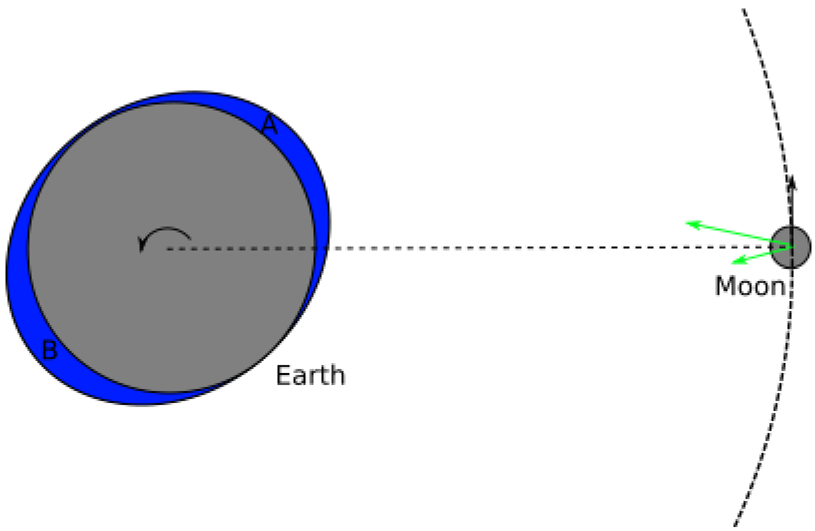
$$\mathbf{\Delta F} = \frac{GMmR}{r^3}\left(2\cos(\theta)\widehat{\mathbf{i}} - \sin(\theta)\widehat{\mathbf{j}}\right) \tag{2}$$



This is why we get two high tides per day. It is not the case that anything feels a force away from the moon on Earth, but just that the force on one side of the Earth is less than on the other.

1.3 Tidal Friction

So now let's consider the effects of rotation of the body. Consider the situation below, where tidal bulges have developed at points A and B, and because of friction between the Earth's surface and the ocean, the ocean's bulges are rotated along with the Earth's surface.



The forces acting on the moon because of bulges A and B are shown as green arrows.

- The force due to bulge A is dominant - meaning the moon is sped up in its orbit.
- The moons force on the dial bulges acts to slow down Earth's rotation -> This is tidal friction

The Earth's rotation is slowing, which means there is a loss of angular momentum. However, angular momentum within the Earth-Moon system is conserved. So, **the moon must be gaining angular momentum**.

1.3.1 Where does this angular momentum go?

Assume the moon (mass m) is moving in a circular orbit, of radius r. The angular momentum of the moon is then:

$$L = mrv$$

To find v, let's equate the gravitational force experienced by the moon with the centripetal force

$$\frac{GM_{\oplus}m}{r^2} = \frac{mv^2}{r} v = \left(\frac{GM_{\oplus}}{r}\right)^{1/2}$$

This means the angular momentum is given by

$$L = (GM_{\oplus})^{1/2} mr^{1/2}$$

The first three terms are constants, so $L \propto r^{1/2}$. So if L increases (which it must if the total L of the Earth-Moon system is conserved), then r increases. Which means the Moon moves further away!

Earth's rotation is slowing at a rate of about 0.0016 s/century (small but measureable). So the moon is drifting away!

Let's now also consider the effects of the tidal forces on the Moon by the Earth.

$$\Delta F \sim \frac{GMmR}{r^3}$$

Just to make sure we know what's what, M is the mass of the body causing the tidal force, R is the radius of the body feeling the tidal force, and r is the distance between them. If we then look at the ratio of the tidal forces felt by the Moon and the Earth, we get:

$$\frac{M_\oplus R_\mathrm{m}}{M_\mathrm{m} R_\oplus} \sim 22$$

Tidal friction is much stronger on the Moon versus on the Earth. This means the Moons rotation slows faster than the Earths, and is why the Moon is now tidal locked into a synchronous rotation (the same side always faces us).

Synchronous rotation is very common - seen in binary stars, and in Sun-Mercury system.