Lecture_16

March 13, 2023

1 Stellar Photospheres

In the previous lecture, we discussed how observational quantites (magnitude, distance, colour) can be translated into physical quantites (masses, radii, temperatures). However, in doing this, we never discussed what part of a star we are actually viewing when we take a spectrum, or take some photometry. It's important to figure this out, as the The goal of the first part of this lecture is to shed light (pun intended!) on this topic.

First, let's define what the photosphere is. Recall from our discussion about pressure broadening that the mean free path between particle collisions is given by

$$l = \frac{1}{n\sigma}$$

When discussion photons, a similar expression exists, where σ represents the cross sectional area for photons undergoing absorption or scattering. The **optical depth** τ is defined as 1 over this value:

$$\tau = \frac{1}{l} = \sigma \int_0^x n(x')dx' = \sigma n x \text{ (if n constant)}$$

where we have let the number density vary as a function of distance x. This quantity represents how much light is attunated when passing through a medium. For $\tau \ll 1$, then the light is not attunated by the medium, and the medium is referred to as optically thin. If $\tau \gg 1$, then the medium is optically thick.

When applying this to stars, it means that the inner parts of the star are optically thick, meaning we don't see the light coming from these regions. As you progress outwards from the Suns core, eventually either n or x become small enough that the optical depth drops below 1, in which case the medium becomes optically thin. It is from this region that light is emitted, and which we then observe with our telescopes. This region is called the photosphere. Now, let's use this information to understand both how large the photosphere is, and how this decides a stars luminosity class.

1.1 Scale Height of the photosphere

Imagine we have a spherical star that is in hydrostatic equilibrium, where the inward force of gravity is balanced by the outward force due to the internal pressure of the object. The pressure gradient for such an object is given by

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{GM(r)\rho}{r^2}$$

where $M(r)$ is the mass enclosed at a distance r, and ρ is the local mass density. If we assume the star behaves like an ideal gas, then we know that

$$P = nk_{\mathrm{B}}T = \frac{\rho k_{\mathrm{B}}T}{\mu m_{\mathrm{p}}}$$

where μ is the mean molecular mass of the gas, and depends on the mixture of elements and the degree of ionisation of the gas. First, let's use the standard definitions for elements that astronomers use: we care about hydrogen, helium, and metals (everything heavier than helium). The total mass density can then be written as

$$\rho = \rho_{\mathrm{H}} + \rho_{\mathrm{He}} + \rho_{\mathrm{metal}}$$

Normally, we discuss mass fractions: that is

$$X \equiv \frac{\rho_{\mathrm{H}}}{\rho} \tag{1}$$

$$Y \equiv \frac{\rho_{\mathrm{He}}}{\rho} \tag{2}$$

$$Z \equiv \frac{\rho_{\mathrm{metal}}}{\rho} = 1 - X - Y \tag{3}$$

$$\tag{4}$$

1.1.1 Pure Hydrogen

Imagine a gas composed of pure hydrogen. If completely neutral, then the number density of this gas is $n = \rho/m_p$ (neglecting the electron mass), while the mean molecular mass would be $\mu = 1$. If the gas is completely ionised, then $n = 2\rho/m_p$, as we have now separated out the electrons from the protons, doubling the number of particles. The mean molcular mass then becomes $\mu = 1/2$.

1.1.2 Pure Helium

For a pure helium gas which is completely neutrl, then $n = \rho/(4m_p)$ and $\mu = 4$. If the gas is completely ionised, we triple the number of particles, as two electrons are freed from each atom, meaning $n = 3\rho/(4m_p)$ and $\mu = 4/3$.

1.1.3 A gas of metals

For a gas made purely of metals, we will write A as the average number of nucleons per atom. The number density is $n = \rho/(Am_p)$ and the mean molecular mass is $\mu = 4$ when neutral. If the number of protons and neutrons in each nucleas is roughly equal, then $\sim A/2$ electrons will be freed when the gas is fully ionised, meaning the number density is $n \approx \rho/(2m_p)$ and the mean molecular mass is $\mu \approx 2$, if $A/2 \gg 1$.

So, if we then want to estimate the number density for a fully ionized gas consisting of hydrogen, helium, and metals, then we can use

$$n \approx X\left(\frac{2\rho}{m_p}\right) + Y\left(\frac{3\rho}{4m_p}\right) + X\left(\frac{\rho}{2m_p}\right) \tag{5}$$

$$n \approx \left(2X + \frac{3}{4}Y + \frac{1}{2}Z\right)\frac{\rho}{m_p} \tag{6}$$

and the mean molecular weight would be

$$\mu = \left(2X + \frac{3}{4}Y + \frac{1}{2}Z\right)^{-1}$$

For the Sun, where $X = 0.734, Y = 0.250, Z = 0.016$, then

$$\mu_\odot = 0.60$$

As a reference, let's consider what the mean molecular weight would be for a completely neutral gas:

$$\mu = \left(X + \frac{Y}{4} + \frac{Z}{A}\right)^{-1}$$

which would give

$$\mu_{\text{neutral}} = 1.25$$

where we've neglected the metal contribution, since they account for so little of the Sun's composition. The key take away here is this: the mean molecular weight is of order 1 for the Sun, regardless of how ionized it is.

So, if a star is in hydrostatic equilibrium, then we have that

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{GM(r)\rho}{r^2} = -g\rho$$

Recalling the ideal gas law from earlier, we can rewrite this as

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{g\mu m_p}{k_\mathrm{B}T}P$$

If we then assume that, within the photosphere, that g, μ, and T are all constant, then this equation can be integrated to

$$P \propto \exp\left(-\frac{g\mu m_p}{k_\mathrm{B}T}r\right) = \exp\left(-\frac{r}{H}\right)$$

where we've now defined the quantity of the scale height H to be

$$H = \frac{k_\mathrm{B}T}{g\mu m_p}$$

Using solar values for all of these ($T = 5800$, $\mu = 0.6$, $g = \frac{GM_\odot}{R_\odot^2} = 274\, m\, s^{-2}$), we get that

$$H \approx 300\,\text{km}$$

This is height over which the photosphere decays, as the pressure tends towards 0. When compared to the radius of the Sun (700,000km), then we find the scale height is a small fraction of this, which means the approximateon of constant g, μ, and T is reasonable.

1.2 Luminosity classes and pressure relation

In the precious lecture, we stated that the pressure is related to the radius, which we have now shown. The next step is to compare the pressures of different size stars, to justify the ordering of the luminosity classes as given previously.

First, let's rewrite the optical depth to be

$$\mathrm{d}\tau = -n(r)\sigma(r)\mathrm{d}r$$

where n is the number density and σ the average cross section for interactions. This is often written as

$$\mathrm{d}\tau = -\rho(r)\kappa(r)\mathrm{d}r.$$

where ρ is the mass density and κ is called the opacity, and is given by

$$\kappa = \frac{n\sigma}{\rho}$$

For the Sun, the opacity is $\kappa \sim 3\,\mathrm{m}^2\,\mathrm{kg}^{-1}$. Since

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -g\rho$$

we can say that

$$\frac{\mathrm{d}P}{\mathrm{d}\tau} = \frac{\mathrm{d}P}{\mathrm{d}r}\frac{\mathrm{d}r}{\mathrm{d}\tau} = \frac{g}{\kappa}$$

If this ratio is constant in a stars atmosphere, then we can say that

$$P \approx \frac{g}{\kappa}\tau$$

If we now just cosider the photosphere, where $\tau \sim 1$, then we have that

$$P_{\mathrm{Phot}} \approx \frac{g_{\mathrm{Phot}}}{\kappa_{\mathrm{Phot}}}$$

Since stars have similar compositions, it's reasonable to assume that κ_{Phot} is similar for all of them. This means that the pressure in any stars photosphere depends on g, when means when comparing two stars, the ratio of the pressures in their photospheres is approximately

$$\frac{P_1}{P_2} = \frac{M_1}{M_2}\left(R_2 R_1\right)^2$$

As an example, let's consider the stars Betelgeuse and Proxima Cen. Betelgeuse his a M2 I star, and Proxima Cen is a M5 V star - so they both have a similar temperature, but their luminosities are very different. If we use the known masses and radii for these stars, we find that

$$\frac{P_{\mathrm{Betel}}}{P_{\mathrm{Prox}}} \sim 10^{-6}$$

which goes to show that Betelgeuse, which has a significantly larger radius than Proxima Cen, has a much lower pressure in its photosphere. This in turn means the amount of pressure broadening in its absorption features is very small compared to Proxima Cen, which is why its luminosity class is I while Proxima Cen's is class V.