Lecture_08_Supp

February 16, 2023

1 Black-body radiation

1.1 The partition function

Recall that at the end of the thermal physics course, we began to consider the partition function of an ideal quantum gas. As a recap, consider a gas of N particles, which are non-interacting. For a perfect quantal gas, the energy of the system will be given by

$$U = \sum_r n_r \epsilon_r$$

where this is a sum over all available states, and n_r is the occupation number of each state. The occupation numbers are given by

$$n_r = 0, 1 \qquad \text{all r Fermi} - \text{Dirac} \tag{1}$$
$$n_r = 0, 1, 2, \dots \qquad \text{all r Bose} - \text{Einstein} \tag{2}$$

So, the partition function is then

$$Z(T, V, N) = \sum_{n_1 n_2, \dots, n_r} e^{-\beta \sum_r n_r \epsilon_r} \tag{3}$$
$$Z(T, V, N) = \sum_{n_1 n_2, \dots, n_r} e^{-\beta(n_1\epsilon_1 + n_2\epsilon_2 + \dots)} \tag{4}$$
$$Z(T, V, N) = \sum_{n_1=0}^{\infty} e^{-\beta(n_1\epsilon_1)} \sum_{n_2=0}^{\infty} e^{-\beta(n_2\epsilon_2)} \dots \tag{5}$$
$$Z(T, V, N) = \prod_i \sum_{n_i=0}^{\infty} e^{-\beta(n_i\epsilon_i)} \tag{6}$$

That is, the partition function is the product over all states (i) of the sum of all occupation numbers(n_i) of each state. If we differentiate $\ln Z$ with respect to the energy level ϵ_i while leaving the temperature and other energy levels constant, we obtain the mean occupation number $\bar{n}_i$ of the state i. That is

$$\bar{n}_i = \frac{1}{\beta}\left(\frac{\partial \ln Z}{\partial \epsilon_i}\right)_{T, \epsilon_r (r != i)}$$

From this, we can calculate all desired quantities of the system at hand.

Borrowing a few more results from quantum mechanics, we will start with the following information.

- Photons have a spin of 1, meaning they act like bosons.
- Photons do not interact with each other, meaning a gas of photons acts like a perfect quantal gas

So if we now have a sealed container filled with photons, where the walls are constantly absorbing and emitting photons, what does the flux density within that container look like? The partition function is given by

$$Z = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \dots e^{-\beta(n_1\epsilon_1 + n_2\epsilon_2 + \dots)} \tag{7}$$

$$Z = \prod_i \sum_{n_i=0}^{\infty} e^{-\beta(n_i\epsilon_i)} \tag{8}$$

For bosons, where $n_i = 0, 1, 2, 3, \dots$, then this becomes

$$Z = \prod_i \left(1 + e^{-\beta\epsilon_i} + e^{-2\beta\epsilon_i} + \dots\right)$$

which, using the geometric series $1 + x + x^2 + \dots = 1/(1-x)$ for $x < 1$, gives

$$Z = \prod_i \frac{1}{1 - e^{-\beta\epsilon_i}}$$

Thus, we get

$$\ln(Z) = \sum_i \ln(1 - e^{-\beta\epsilon_i})$$

The average occupation number of each state is thus given by

$$\bar{n}_i = \frac{1}{\beta}\left(\frac{\partial \ln Z}{\partial \epsilon_i}\right) = \frac{1}{e^{\beta\epsilon_i} - 1}$$

The next quantity that we need is the density of states. Again, this comes from PY2104 (Lecture 12), and is given by

$$g(k)\mathrm{d}k = \frac{1}{8}\frac{4\pi k^2 \mathrm{d}k}{(\pi/L)^3}2 = \frac{Vk^2\mathrm{d}k}{\pi^2}$$

where $\mathbf{k} = \frac{\mathbf{p}}{\hbar}$ is the wavevector. There is an extra factor of 2 in this term relative to what we dealt with in thermodynamics, and that is because electromagnetic waves can have 2 polarisations. Rewriting this in terms of momentum gives

$$g(p)\mathrm{d}p = \frac{Vp^2\mathrm{d}p}{\pi^2\hbar^3}$$

Now, we know that the energy of a photin is $E = \hbar\omega$. As such, it is more convenient to rewrite the above expression in terms of ω. The relation between p and ω is just $p = \hbar\omega/c$, and so

$$g(\omega)\mathrm{d}\omega = \frac{V\omega^2\mathrm{d}\omega}{\pi^2 c^3}$$

So, the mean number of photons between a frequency of ω and $\omega + \mathrm{d}\omega$ is then given by

$$\mathrm{d}N = \bar{n}g(\omega)\mathrm{d}\omega.$$

Substituting in the above gives

$$\mathrm{d}N = \frac{V}{\pi^2 c^3} \frac{\omega^2 \mathrm{d}\omega}{e^{\beta\hbar\omega} - 1}$$

The energy contained by these photons is thus

$$U = \hbar\omega \mathrm{d}N = \frac{V\hbar}{\pi^2 c^3} \frac{\omega^3 \mathrm{d}\omega}{e^{\beta\hbar\omega} - 1}$$

The energy density ($u = U/V$) is then

$$u(\omega, T)\mathrm{d}\omega = \frac{\hbar}{\pi^2 c^3} \frac{\omega^3 \mathrm{d}\omega}{e^{\beta\hbar\omega} - 1}$$