Lecture_09

February 19, 2024

1 Interaction of radiation and matter - Black-body radiation

In this lecture, we are going to discuss

- Black body radiation
- Wien's displacement law
- The two frequency limits of black body radiation
  - Rayleigh Jeans limit
  - Wien limit
- Stefan-Boltzmann law

1.1 The continuum of spectra: Black-body radiation

We now know that if we were to look at the light coming from an object within the Universe, we'd expect to see features of particular widths at wavelengths associated with transitions between states of different atoms. However, what the continuum behaviour? That is, what function should describe the light outside of these wavlengths? To understand this, we need to consider black-body radiation.

First, a black-body is a physical body that absorbs all electromagnetic radiation incident on it. "A perfect absorber/emitter of radiation" - which means the efficiency of absorption/emissions is 1. The distribution of the energy density of the radiation with frequency/wavlenegth is determined by the objects temperature T, according to the Planck blackbody radiation law (see the website for the full derivation, combines information from PY2102 and PY2104).

$$u(\omega, T)d\omega = \frac{\hbar\omega^3 d\omega}{\pi^2 c^3 \left(\exp\left[\frac{\hbar\omega}{kT}\right] - 1\right)}$$

This gives emitted energy density as a function of ω. This function then describes the expected spectrum of an object outside of absorption/emission features. There are a couple of interesting features we can immediately work out. First, the peak of this function occurs at

$$\frac{du}{d\omega} = 0$$

which has the solution

$$(3 - \beta\hbar\omega)\exp(\beta\hbar\omega) = 3$$

where $\beta = 1/kT$. This is a transcendental equation, and must be either solved graphically or numerically. The solution is

$$\frac{\hbar\omega_{\rm peak}}{kT} = \frac{h\nu_{\rm peak}}{kT} = 2.822$$

which is the Wien displacement law. This relates the frequency at which a blackbody peaks with the temperature of the object. This peak moves towards higher frequencies for objects with higher temperatures. It can be used to inform the observing strategy for a particular object - cold objets peak at low frequencies (so need Radio/IR observations), while hot objects peak towards the UV/X-ray.

Also, note that the blackbody curve for a higher T lies completely above the blackbody curve for a lower T. Consider

$$\frac{u(\omega, T_1)}{u(\omega, T_2)} = \frac{\left(\exp\left[\frac{\hbar\omega}{kT_2}\right] - 1\right)}{\left(\exp\left[\frac{\hbar\omega}{kT_1}\right] - 1\right)}$$

which is always > 1 if $T_1 > T_2$.

Now, let's look at the equation in a bit more detail

1.1.1 Low-frequency limit for Black-body radiation

Consider first the condition $\frac{\hbar\omega}{kT} << 1$, which means that $\exp\left[\frac{\hbar\omega}{kT}\right] \sim 1 + \frac{\hbar\omega}{kT}$. This means that

$$u(\omega, T)d\omega = \frac{\hbar\omega^3 d\omega}{\pi^2 c^3 \left(\frac{\hbar\omega}{kT}\right)}$$

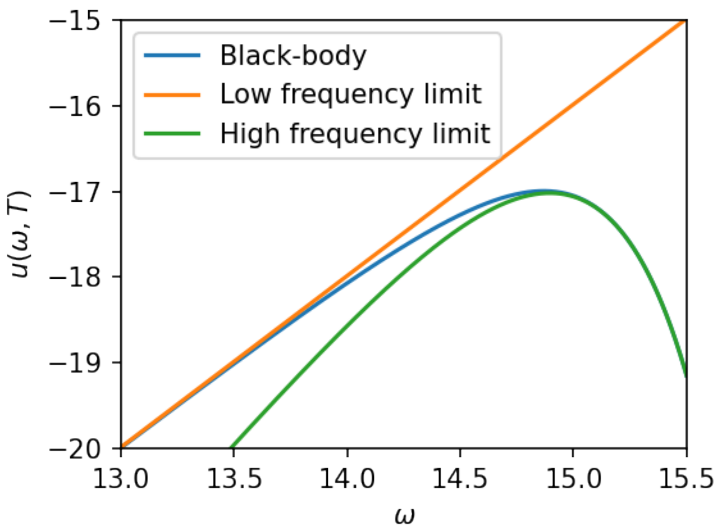
$$u(\omega, T)d\omega = \frac{kT\omega^2 d\omega}{\pi^2 c^3}$$

This is known as the Rayleigh Jeans (classical) limit. When first derived, it was quickly shown that it couldn't be fully correct. To understand why, consider the limit $\omega \to \infty$, then $u(\omega, T) \to \infty$. This is known as the "ultraviolet catastrophe", as it sugested that objects emitted infinite energy!

1.1.2 High-frequency limit for Black-body radiation

Consider now the condition $\frac{\hbar\omega}{kT} >> 1$, which means that $\exp\left[\frac{\hbar\omega}{kT}\right] >> 1$. This means that

$$u(\omega, T)d\omega = \frac{\hbar\omega^3}{\pi^2 c^3} \exp\left[-\frac{\hbar\omega}{kT}\right] d\omega$$



1.2 Total Energy density of Black-body radiation

Ok, so we know the shape of the spectrum now, and the two limits of it. Next, we want to ask the question - what is the total energy density of black body radiation? To do this, we must integrate over ω.

$$u(T) = \int_0^\infty u(\omega, T) d\omega$$

$$u(T) = \int_0^\infty \frac{\hbar \omega^3 d\omega}{\pi^2 c^3 \left(\exp\left[\frac{\hbar\omega}{kT}\right] - 1\right)}$$

Can tackle this using substitution of variables. Let $x = \frac{\hbar\omega}{kT}$, and $dx = \frac{\hbar d\omega}{kT}$. We then get

$$u(T) = \frac{\hbar}{\pi^2 c^3} \left[\frac{kT}{\hbar}\right]^4 \int_0^\infty \frac{x^3 dx}{e^x - 1}$$

That last term is a definite integral (can be looked up or thrown into a integral solver), and is $\frac{\pi^4}{15}$. So, we get

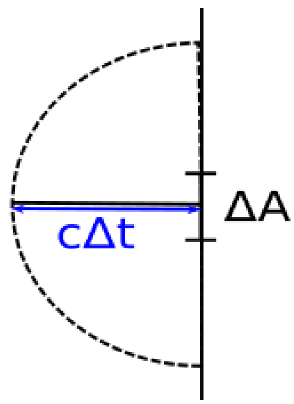
$$u(T) = \frac{\pi^2 k^4}{15\hbar^3 c^3} T^4 = aT^4$$

This gives you the energy density per unit area per unit time for blackbody radiation.

1.3 Emitted energy from the surface of a black-body container

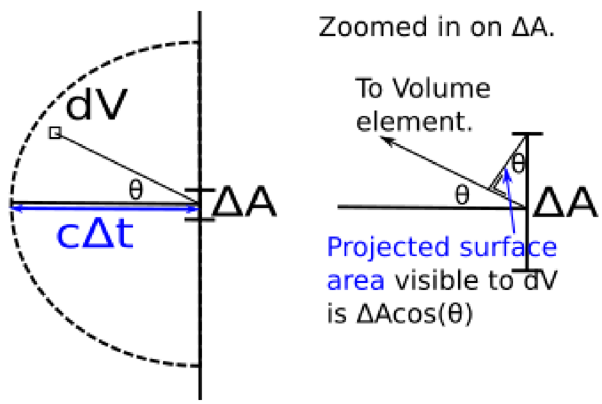
Now we want to ask the question "what is the energy emitted by a surface which has been heated by black body radiation". First, consider a cavity. Inside is full of black body radiation. Now, consider a small area ΔA on the wall of this cavity filled. If the cavity is in equilibrium with the black body radiation, then the energy absorbed by ΔA should be the same as the energy emitted by ΔA. So, in order to find out how much energy that area is emitting, we are first going to calculate how much energy is absorbed by the surface, and assume they are equal.

First, let's find the energy absorbed. This is given by the number and energy of the photons hitting ΔA. The total volume with photons capable of hitting ΔA in time Δt is a half sphere of radius $c\Delta t$, as shown below.



Now, consider a volume element in the half-sphere. If we assume that the photons are travelling isotropically (all directions). As such, the fraction that will hit ΔA is:

$$fraction = \frac{\Delta A \cos(\theta)}{4\pi r^2}$$



The total energy in the volume element is:

$$dE = u(T)dV$$

$$dE = u(T)r^2 \sin(\theta) dr d\theta d\phi$$

Therefore

$$dE_{\text{abs}} = u(T)r^2 \sin\theta dr d\theta d\phi \frac{\Delta A \cos(\theta)}{4\pi r^2}$$

The total energy absorbed by ΔA in time Δt is then given by integrating over the volume:

$$E_{\text{abs}} = \int_0^{c\Delta t} \int_0^{\pi/2} \int_0^{2\pi} \frac{u(T)\Delta A}{4\pi} \cos\theta \sin\theta dr d\theta d\phi$$

$$E_{\text{abs}} = \frac{u(T)\Delta A}{4\pi} \int_0^{c\Delta t} dr \int_0^{\pi/2} \cos\theta \sin\theta d\theta \int_0^{2\pi} d\phi$$

The first integral is just $c\Delta t$, the last is 2π, and the middle one can be done using the substitution $x = \sin\theta$. Doing this gives

$$E_{\text{abs}} = \frac{1}{4} c\, u(T)\Delta A \Delta t = E_{\text{emit}}$$

This is the energy absorbed/emitted by an area ΔA in time Δt. So now, the energy emitted per unit time per unit area by a black body surface is

$$\frac{L}{A} = \frac{c}{4} u(T) = \frac{1}{4} c\, a\, T^4 = \sigma T^4$$

where σ is the Stefan Boltzmann constant. Accounting for efficiency:

$$\frac{L}{A} = \epsilon \sigma T^4$$

where $0 < \epsilon < 1$, 1 for a perfect Black body.