Lecture_01

January 16, 2023

1 Introduction

Astrophysics is concerned with answering fundamental questions about the nature of the Universe. Answering these questions requires an understanding of classical mechanics (orbits), quantum mechanics (spectra)), thermodynamics (blackbodies), special relativity (distances to Galaxies), general relativity (cosmology/black holes), electromagnetism (polarisation/stellar structure), plasma physics (shocks/the Sun), condensed matter physics (neutron stars). As such, an astrophysicist needs to have a very wide knowledge base across all of the sub-disciplines of physics (while typically having a very deep knowledge on one field in particular).

Astrophysicists must also have a good background in statistics and data analysis, since a lot of modern astrophysics requires analyse of observations or simulations. In particular, Bayesian statistics, where one implicitly states what their prior beliefs are, is very commong in modern research.

Currently (and this is of course my own biased view), there are several fundamental questions which astrophysics is focused on answering:

- Compact object physics: How many black holes are there in the Universe, and what is the most general description of these objects? What is the equation of state of nuclear matter?
- Cosmology: What is the fate of our Universe? How is this related to dark matter and energy in the Universe, and are the “constants” which dictate the Universe’s future changing with time?
- Stellar physics: How do stars form, and how quickly do they evolve through their various life stages?
- Exoplanets: How many habitable planets are out there? What do we mean by habitable planets? Can we detect signatures of life in the atmospheres of these planets?

Given that this is a 24 lecture course over 12 weeks, we are not going to be able to answer these questions, but we will be developing the foundations required to consider these questions We will focus on overviews of 6 rather general topics:

- Orbital Mechanics.
- Blackbody radiation, stars, and their spectra.
- Planets and exoplanets.
- Galaxies.
- Cosmology.
- Special Relativity.

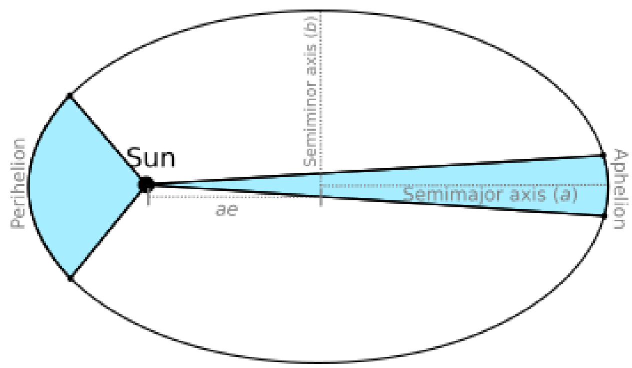
Each of these topics is going to be covered in further detail in 3rd (PY3109) and 4th year (PY4110, PY4111, PY4112, PY4126). To start with, we’re going to look at orbital mechanics.

2 Kepler's Law's of Planetary motion

Johannes Kepler used observations of Mars to arrive at the following three laws of planetary motion:

1. A planets orbit follows an ellipse, with the Sun located at one focus of the ellipse.
2. A line joining the Sun and a planet sweeps out an equal area in equal time intervals.
3. The square of the orbital period of a planet is proportional to the cubes of the semimajor axis of their orbits.

The below figure helps to demonstrate the 1st two laws, mainly that the areas shaded in blue, which are the areas swept out by a planet over the same interval but at the different parts of the orbit, are the same.



The third law doesn't have an easy visulations, but is defined as

$$\frac{P^2}{a^3} = \frac{4\pi^2}{G(M_1 + M_2)} \approx \frac{4\pi^2}{GM_\odot}$$

when we are talking about solar system objects (as the mass of the Sun is so much larger than any other objects in the solar system).

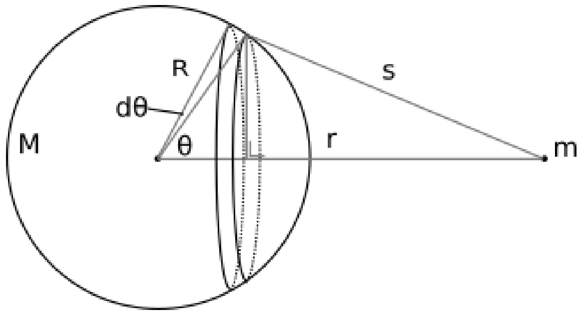
Historically, Kepler proposed these 3 laws from observations, and then Newton developed his theory of gravity to match the laws. However, this direction of doing things is pretty tricky, so to see where the three laws come from, we'll work in the opposite direction: let's assume Newton's theory of gravity, and we'll try and derive Kepler's 3 laws.

3 Newton's Shell Theorem

To determine Kepler's laws, we are going to assume that Newton's Law of gravitation is true. That is,

$$\mathbf{F} = -\frac{Gm_1m_2}{r^2}\hat{r}$$

for two point masses. The first thing we must do is build on this, and ask: how does gravity work when one object is not a point mass? Consider the setup below, where we are going to have a spherical shell of mass M and radius R, and a point massf of m. The centre of the masses are separated by a distance r.



We are going to break the problem down into 3 steps: * The potential energy due to the shell of mass M. * The potential energy due to a thin ring of mass $\mathrm{d}M$ on that shell. * The potential energy to a mass element of mass m_i within that thin ring.

Starting with that last point, the potential energy between our point mass and a mass element, m_i, on the ring is

$$U_i = -\frac{Gmm_i}{s}$$

The potential due to the entire ring is then simply

$$\mathrm{d}U = \sum_i -\frac{Gmm_i}{s} = -\frac{Gm\sum_i m_i}{s} = -\frac{Gm\mathrm{d}m}{s}.$$

Now, in order to consider the whole shell, we need a substitution for $\mathrm{d}m$. If we assume that the mass of the shell is uniformly distributed, then we have that

$$\mathrm{d}m = \frac{M}{A}\mathrm{d}A$$

where $\mathrm{d}A$ is the surface area of the ring. This area is given by

$$\mathrm{d}A = 2\pi R^2 \sin\theta \mathrm{d}\theta$$

Thus,

$$\frac{\mathrm{d}m}{M} = M\frac{\mathrm{d}A}{A} = \frac{2\pi R^2 \sin\theta \mathrm{d}\theta}{4\pi R^2} = \frac{M}{2}\sin\theta \mathrm{d}\theta$$

giving

$$\mathrm{d}U = -\frac{GMm}{2s}\sin\theta \mathrm{d}\theta.$$

We can't just integrate this to get U, as s is not independant of θ. Thankfully, we can use Pythagoras' theorem, which gives

$$s^2 = r^2 + R^2 - 2rR\cos\theta$$

which gives

$$2s\mathrm{d}s = 2rR\sin\theta \mathrm{d}\theta \tag{1}$$

$$\frac{s}{rR}\mathrm{d}s = \sin\theta \mathrm{d}\theta. \tag{2}$$

The potential thus becomes

$$\mathrm{d}U = -\frac{GMm}{2rR}.$$

Now, there are two ways we can integrate this, for two different setups (see below). Integrating from $r - R$ to $r + R$ (which are the maximum and minimum values for s for integrating over the sphere) gives

$$U = -\frac{GMm}{2rR}\int_{r-R}^{r+R} \mathrm{d}s = -\frac{GMm}{2rR}\left[R + r - r + R\right] = -\frac{GMm}{2r}$$

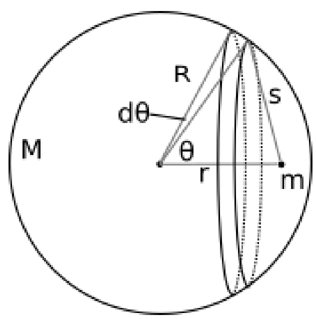
The gravitational force is thus given by

$$F = -\frac{\mathrm{d}U}{\mathrm{d}r} = -\frac{GMm}{r^2}$$

The meaning of this equation is quite important. It shows that the point mass sees the shell of mass M as if it were all residing at the centre of shell. Now, we can generalise this to sphere of uniform mass density by just saying that a sphere is nothing more than an infinite set of concentric shells. As such, we can treat spheres of uniform density (stars/planets/moons) as if they are point masses.

3.0.1 Special Case ($r < R$)

There's another interesting side effect to all of this. Consider the setup below.



In this setup, the point mass lies within the shell. The initial setups for calculating U are the same:

$$\mathrm{d}U = -\frac{GMm}{2rR}.$$

However, when we now integrate, we integrate from $R - r$ to $R + r$. This gives

$$U = -\frac{GMm}{2rR}\int_{R-r}^{R+r} \mathrm{d}s = -\frac{GMm}{2rR}\left[r + R - R + r\right] = -\frac{GMm}{2R}$$

The force is then

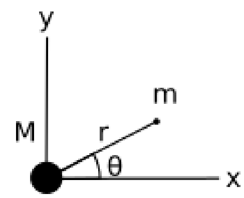
$$F = -\frac{\mathrm{d}U}{\mathrm{d}r} = 0$$

That is, the point mass doesn't feel any gravitational force when inside of the shell! Why is this is important? Imagine you were located half way between here and the Earth's core, and you wanted to know what the gravitational forces you feel are. Newton's shell theorem tells you that all of the mass closer to the core than you acts as if it were at the core, while all of the mass above you dones't exert any force at all.

4 Orbital Mechanics (Kepler's Laws)

4.1 Angular momentum & central forces (recap of PY2101)

So, now that we know that large bodies can be treated as if they were point masses, what can we derive from it? First let's consider two masses, M and m, located at the positions below.



While the above has the x and y axes labelled, it's far more convenient to work in polar coordinates. Given that

$$\hat{\mathbf{r}} = \cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}} \tag{3}$$
$$\hat{\boldsymbol{\theta}} = -\sin\theta\hat{\mathbf{i}} + \cos\theta\hat{\mathbf{j}} \tag{4}$$

then we know

$$\hat{\mathbf{r}} \cdot \hat{\boldsymbol{\theta}} = 0 \tag{5}$$
$$\hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}} = \hat{\mathbf{k}} \tag{6}$$
$$\tag{7}$$

From these definitions, we can derive the following time derivatives:

$$\frac{\mathrm{d}\hat{\mathbf{r}}}{\mathrm{d}t} = \hat{\boldsymbol{\theta}}\frac{\mathrm{d}\theta}{\mathrm{d}t} \tag{8}$$
$$\frac{\mathrm{d}\hat{\boldsymbol{\theta}}}{\mathrm{d}t} = -\hat{\mathbf{r}}\frac{\mathrm{d}\theta}{\mathrm{d}t} \tag{9}$$

The whole point of the above is that the velocity of the planet can then be written as

$$\mathbf{v} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \frac{\mathrm{d}r}{\mathrm{d}t}\hat{\mathbf{r}} + r\frac{\mathrm{d}\hat{\mathbf{r}}}{\mathrm{d}t} \tag{10}$$
$$= v_r\hat{\mathbf{r}} + v_t\hat{\boldsymbol{\theta}} \tag{11}$$
$$\tag{12}$$

The angular momentum of the planet is then

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}$$

which, when you take the derivative with respect to time, gives

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \times \mathbf{p} + \mathbf{r} \times \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}$$

using $\mathbf{p} = m\mathbf{v}$ and $\mathbf{F} = m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}$, we thus get

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = m(\mathbf{v} \times \mathbf{v}) + \mathbf{r} \times \mathbf{F}$$

The first term has to be 0 (a vector crossed with itself is 0) while for that last term must be zero as $\mathbf{F}$ is parallel to $\mathbf{r}$. Thus, for any central force

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = 0$$

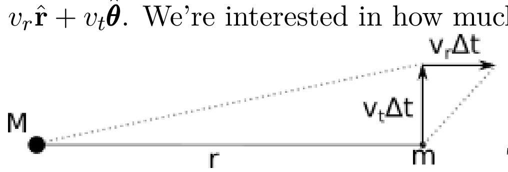
4.2 Kepler's Second Law

So, if angular momentum is convered under gravity, what does that tell us about the orbit of (for example) a planet around a star? The angular momentum of the planet can be written as

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} \tag{13}$$

$$\mathbf{L} = mrv_t\hat{\mathbf{k}} \tag{14}$$

Now imagine a setup like that shown below, where we have a planet moving with a velocity $\mathbf{v} = v_r\hat{\mathbf{r}} + v_t\hat{\boldsymbol{\theta}}$. We're interested in how much area is swept out by the orbit of the planet in a time Δt.



This is given by

$$\Delta A \approx \frac{1}{2}rv_t\Delta t + \frac{1}{2}rv_r\Delta t$$

In the limit where $r >> v_r\Delta t$ (which is true for planets when considering very short timescales), we end up with

$$\Delta A \approx \frac{1}{2}rv_t\Delta t$$

In the limit $t \to 0$ we then get

$$\lim_{t\to 0}\frac{\Delta A}{\Delta t} = \frac{\mathrm{d}A}{\mathrm{d}t} = \frac{1}{2}rv_t = \frac{L}{m}$$

The important thing about this last term is that both L and m are constants, which means

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \text{constant.}$$

This is Kepler's Second Law, and to fully understand it we need to determine Kepler's 1st Law.