1. Let X and Y be topological spaces and let $\mathcal{T}$ be the set of all topological spaces. Recall, we say X and Y are homeomorphic if there exist a homeomorphism $f : X \longrightarrow Y$. Define the relation $\sim$ on $\mathcal{T}$ by, $X \sim Y$ if and only if X is homeomorphic to Y. Show that $\sim$ is and equivalence relation on $\mathcal{T}$.

2. In this question, we will explore how dimensionality of a vector space and quotient spaces are related.
   (a) Consider a vector space V such that the basis for V is $\{v_1, v_2, \ldots, v_n\}$. Now, let $\{w_1, w_2, \ldots, w_k\}$ be a basis for the subspace W such that $k \leq n$ and $w_i \in V$. Using the Gram-Schmidt process to extend the basis of W to V. *Hint: Add additional vectors from V to the set $\{w_1, w_2, \ldots, w_k\}$ then do the Gram-Schmidt process. The Wikipedia page on Gram-Schmidt is a great resource to understand this process.*
   (b) Write out what an element $v \in V$ using the basis formed in part a, then compute the corresponding coset. Finally, determine the basis for V/W. *Hint: Note that if $v \in W$ then $vW = W$*
   (c) Using part a and b, determine the formula for the dimension of V/W.

3. Recall the boundary map (i.e. a linear map) $\partial_k : C_k \longrightarrow C_{k-1}$ defined by:

$$\partial_k(\{v_{\alpha_1}, \ldots, v_{\alpha_{k-1}}\}) \mapsto \sum_{i=1}^{k-1} \{v_{\alpha_1}, \ldots, v_{\alpha_{i-1}}, v_{\alpha_{i+1}}, \ldots, v_{\alpha_{k-1}}\}$$

   In this problem we will compute the Betti numbers, β_k for the simplicial complex K in Fig 1
   (a) Recall from linear algebra that we can represent linear maps as matrices. (Here's a great video if you need a refresher: 3Blue1Brown - Linear transformations and matrices) Using the the simplicial complex K in Fig 1, Find the matrix presentation M_k of ∂_k for all $k \geq 0$. *Note: Your matrix should only have 1's and 0's since we are working in mod 2.*
   (b) Put M_k in reduced row echelon form for all $k \geq 0$ using Gaussian elimination.
   (c) Note that the number of zero columns in the reduced row echelon form is the dimension of the null space (i.e. the kernel of ∂_k); and the number of remaining columns is the dimension of the column space (i.e. the image of ∂_k). Using this information and formula derived in Question 2, Compute the Betti numbers for the simplicial complex K in Fig 1.
   (d) We gave two ways to compute the Euler Characteristic. Compute $\chi(K)$ both ways and show they are equal.

4. (Python Question)
   (a) Modify your python code from Problem Set 1 to compute the Betti numbers for a randomly generated complex.
   (b) Generate 100 complexes from a list of 10 vertices and record each set of Betti Numbers. Find the average for each k-th Betti number from part c and plot the results in histograms. Discuss your findings.

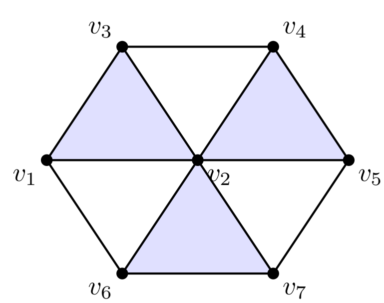

Figure 1: A simplicial complex formed from seven 0-simplices, twelve 1-simplices, and three 2-simplices.