Computational Topology - Problem Set 1

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1. Determine which of the following is a proper topology for the integers $\mathbb{Z}$.

(a) Let $\mathbb{Z}_O$ denote the odd integers and $\mathbb{Z}_E$ denote the even integers. $\tau = \{\emptyset, \mathbb{Z}_O, \mathbb{Z}_E, \mathbb{Z}\}$

This set is a proper topology on $\mathbb{Z}$ because it satisfies the three axioms:

1. The empty set and the whole space are in τ: Both $\emptyset$ and $\mathbb{Z}$ are included in the set.
2. Closed under arbitrary unions: The union of any combination of the sets results in a set that is already in τ. For instance, $\mathbb{Z}_O \cup \mathbb{Z}_E = \mathbb{Z}$, which is in τ. All other unions are trivial (e.g., $\emptyset \cup \mathbb{Z}_O = \mathbb{Z}_O$).
3. Closed under finite intersections: The intersection of any finite number of sets is also in τ. The only non-trivial intersection is $\mathbb{Z}_O \cap \mathbb{Z}_E = \emptyset$, which is in τ.

(b) Let $k \in \mathbb{N}$ and $\mathbb{Z}_k$ denote the integers that whose absolute value is less than k (i.e. $|x| < k$). $\tau = \{\mathbb{Z}_k | k \in \mathbb{Z}\}$

This set is not a proper topology. It fails the first axiom because $\mathbb{Z}$ is not an element of τ (there is no integer k for which the set $\{x \in \mathbb{Z} : |x| < k\}$ is equal to the infinite set $\mathbb{Z}$). To make this a proper topology, we must add $\mathbb{Z}$ to the set:

$$\tau' = \tau \cup \{\mathbb{Z}\}$$

This corrected set is now a proper topology:

1. $\emptyset$ is in the set (since there is no x that satisfies $|x| < k$ for $k \leq 0$), and we have added $\mathbb{Z}$.
2. The union of any collection of sets $\mathbb{Z}_{k_i}$ is $\mathbb{Z}_{\sup(k_i)}$, which is in τ. If the set of indices is unbounded, the union is $\mathbb{Z}$, which is in τ'.

3. The intersection of any two sets $\mathbb{Z}_{k_1} \cap \mathbb{Z}_{k_2}$ is $\mathbb{Z}_{\min(k_1,k_2)}$, which is in τ.

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2. Let $(\mathbb{Z}, \tau)$ be a topological space. Suppose that $\{\{n, n+1\} | n \in \mathbb{Z}\} \subset \tau$. Prove that τ is the discrete topology.

To prove that τ is the discrete topology, we must show that any subset of $\mathbb{Z}$ is an open set. We will show that every singleton set $\{m\}$ for any $m \in \mathbb{Z}$ is in τ.

1. We are given that for any integer n, the set $\{n, n+1\}$ is in τ.
2. This implies that the set $\{n-1, n\}$ is also in τ (by simply choosing $n-1$ as our integer).
3. The axioms of a topology state that it must be closed under finite intersections. Let's take the intersection of two adjacent open sets:

$$\{n-1, n\} \cap \{n, n+1\} = \{n\}$$

4. Since both $\{n-1, n\}$ and $\{n, n+1\}$ are in τ, their intersection, the singleton set $\{n\}$, must also be in τ. This is true for any integer n.
5. Now, let A be any arbitrary subset of $\mathbb{Z}$. We can express A as the union of all its singleton elements:

$$A = \bigcup_{a \in A} \{a\}$$

6. Since every singleton set $\{a\}$ is in τ, and a topology is closed under arbitrary unions, the set A must be in τ.

Because any subset $A \subseteq \mathbb{Z}$ is in τ, τ is the power set of $\mathbb{Z}$, which is the definition of the discrete topology.

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3. Let $X = \{1, 2, 3, 4\}$. Show that (X, τ) and (X, Υ) are homeomorphic.

Given the topologies:

$$\tau = \{\emptyset, \{2\}, \{1, 2\}, \{3, 4\}, X\}$$

$$\Upsilon = \{\emptyset, \{1\}, \{2, 3\}, \{1, 4\}, X\}$$

To show that (X, τ) and (X, Υ) are homeomorphic, we need to find a function $f : X \to X$ that is a bijection, continuous, and has a continuous inverse. Consider the function $f : (X, \tau) \to (X, \Upsilon)$ such that:

- $f(1) = 4$
- $f(2) = 1$
- $f(3) = 2$
- $f(4) = 3$

This function is a bijection as it is a permutation of the elements of X.

1. Continuity of f: We must show that for every open set $U \in \Upsilon$, its preimage $f^{-1}(U)$ is open in τ.

- $f^{-1}(\emptyset) = \emptyset \in \tau$
- $f^{-1}(\{1\}) = \{2\} \in \tau$
- $f^{-1}(\{2,3\}) = \{3,4\} \in \tau$
- $f^{-1}(\{1,4\}) = \{1,2\} \in \tau$
- $f^{-1}(X) = X \in \tau$

Since the preimage of every open set in Υ is open in τ, f is continuous.

2. Continuity of f^{-1} (Open Map property): We must show that for every open set $V \in \tau$, its image $f(V)$ is open in Υ.

- $f(\emptyset) = \emptyset \in \Upsilon$
- $f(\{2\}) = \{1\} \in \Upsilon$
- $f(\{1,2\}) = \{4,1\} = \{1,4\} \in \Upsilon$
- $f(\{3,4\}) = \{2,3\} \in \Upsilon$
- $f(X) = X \in \Upsilon$

Since the image of every open set in τ is open in Υ, f^{-1} is continuous. Because f is a continuous bijection with a continuous inverse, it is a homeomorphism, and the spaces (X, τ) and (X, Υ) are homeomorphic.