Problem Set 2

Michael

October 2, 2025

Question 1

Let T be the collection of all topological spaces. We define a relation $\sim$ on T by $X \sim Y$ if and only if X is homeomorphic to Y. Show that $\sim$ is an equivalence relation.

Proof. To show that $\sim$ is an equivalence relation, we must prove that it is reflexive, symmetric, and transitive.

Reflexivity: $(X \sim X)$

We must show that any topological space X is homeomorphic to itself. The identity map, $id_X : X \to X$, is a homeomorphism because it is a bijection, continuous, and its inverse (itself) is continuous. Thus, $X \sim X$.

Symmetry: (If $X \sim Y$, then $Y \sim X$)

We assume there exists a homeomorphism $f : X \to Y$. Its inverse, $f^{-1} : Y \to X$, is also a homeomorphism because it is a bijection, continuous (by definition of f), and its inverse, $(f^{-1})^{-1} = f$, is continuous. Thus, if $X \sim Y$, then $Y \sim X$.

Transitivity: (If $X \sim Y$ and $Y \sim Z$, then $X \sim Z$)

We assume there exist homeomorphisms $f : X \to Y$ and $g : Y \to Z$. The composite map $h = g \circ f : X \to Z$ is a homeomorphism. It is a bijection and continuous because compositions of bijections/continuous functions are bijections/continuous. Its inverse, $h^{-1} = f^{-1} \circ g^{-1}$, is also continuous as it is the composition of two continuous functions. Thus, the relation is transitive.

Since the relation $\sim$ is reflexive, symmetric, and transitive, it is an equivalence relation. □

Question 2

(a) Extend the basis of W to V

To extend the basis of W to V, we first take the basis of W and augment it with the basis vectors of V. We then iterate through this augmented list, removing any vector that is a linear combination of the preceding ones, until we have a set of n linearly independent vectors. This new basis for V is then transformed into an orthonormal basis, $\{u_1, \ldots, u_n\}$, using the Gram-Schmidt process. We ensure the first k vectors, $\{u_1, \ldots, u_k\}$, form an orthonormal basis for W.

(b) Compute the coset and find the basis for V/W

A general element $v \in V$ can be written as $v = v_W + v_{W^\perp}$, where $v_W = \sum_{i=1}^{k} c_i u_i$ is in W and $v_{W^\perp} = \sum_{i=k+1}^{n} c_i u_i$ is orthogonal to W. The coset $v + W$ is computed as:

$$v + W = (v_W + v_{W^\perp}) + W = v_{W^\perp} + (v_W + W) = v_{W^\perp} + W$$

The basis for the quotient space V/W is the set of cosets of the basis vectors that are orthogonal to W:

$$\text{Basis}(V/W) = \{u_{k+1} + W, u_{k+2} + W, \ldots, u_n + W\}$$

(c) Determine the formula for the dimension of V/W

The dimension of V/W is the number of vectors in its basis, which is $n - k$. Since $n = \dim(V)$ and $k = \dim(W)$, we have the formula:

$$\dim(V/W) = \dim(V) - \dim(W)$$

Question 3

(a) Find the matrix presentation M_k of ∂_k

The dimensions of the chain groups are $\dim(C_0) = 7$, $\dim(C_1) = 12$, and $\dim(C_2) = 3$. The two non-trivial boundary matrices are M_2 ($\partial_2 : C_2 \to C_1$) and M_1 ($\partial_1 : C_1 \to C_0$).

The Matrix M_2 for $\partial_2 : C_2 \to C_1$: This is a 12×3 matrix.

$$M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The Matrix M_1 for $\partial_1 : C_1 \to C_0$: This is a 7×12 matrix.

$$M_1 = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

The matrices M_3 and M_0 are zero maps.

(b) Put M_k in reduced row echelon form

RREF of M_2: Has **3 pivot columns** and **0 non-pivot columns**. **RREF of M_1:** Has **6 pivot columns** and **6 non-pivot columns**.

(c) Compute the Betti numbers

Using the formula $\beta_k = \dim(\mathrm{Ker}(\partial_k)) - \dim(\mathrm{Im}(\partial_{k+1}))$:

- $\boldsymbol{\beta_2} = \dim(\mathrm{Ker}(\partial_2)) - \dim(\mathrm{Im}(\partial_3)) = 0 - 0 = \mathbf{0}$.
- $\boldsymbol{\beta_1} = \dim(\mathrm{Ker}(\partial_1)) - \dim(\mathrm{Im}(\partial_2)) = 6 - 3 = \mathbf{3}$.
- $\boldsymbol{\beta_0} = \dim(\mathrm{Ker}(\partial_0)) - \dim(\mathrm{Im}(\partial_1)) = 7 - 6 = \mathbf{1}$.

(d) Compute the Euler Characteristic $\chi(K)$

We compute $\chi(K)$ in two ways:

1. **By counting simplices:**

$$\chi(K) = 7 - 12 + 3 = \mathbf{-2}$$

2. **Via Betti numbers (Euler-Poincaré):**

$$\chi(K) = \beta_0 - \beta_1 + \beta_2 = 1 - 3 + 0 = \mathbf{-2}$$

The results match, confirming our calculations.