Name: Minh Khai Tran
zID: 5168080

Question 5:

a) $f(n) = (\log_2(n))^2$ and $g(n) = \log_2\left(n^{\log_2(n)}\right)^2$

Using $\log(a^b) = b\log a$ we obtain:

$$g(n) = 2\log_2\left(n^{\log_2(n)}\right) = 2\log_2(n)\log_2(n) = 2\ (\log_2(n))^2$$
$$g(n) = \Theta((\log_2(n))^2$$

Therefore, $f(n) = \Theta(g(n))$

b) $f(n) = n^{10}$ and $g(n) = 2^{\sqrt[10]{n}} = 2^{n^{\frac{1}{10}}}$

We want to show that $f(n) = O(g(n))$, which means that we have to prove that:

$n^{10} < c2^{n^{\frac{1}{10}}}$ for some positive c and all sufficiently large n.

But, since the log function is monotonically increasing, this will hold just in case:

$$\log_2 n^{10} < \log_2 c + \log_2 2^{n^{\frac{1}{10}}}$$
$$10\log_2 n < \log_2 c + n^{\frac{1}{10}}$$

$\frac{(10\log_2 n)}{(\log_2 c + n^{\frac{1}{10}})} < 1$ for sufficiently large n. (1)

To this end we use L'Hôpital's to compute the limit

$$\lim_{n\to\infty}\frac{(10\log_2 n)}{\left(\log_2 c + n^{\frac{1}{10}}\right)} = \lim_{n\to\infty}\frac{(10\log_2 n)'}{\left(\log_2 c + n^{\frac{1}{10}}\right)'} = \lim_{n\to\infty}\frac{10\frac{1}{\ln(2)\,n}}{\frac{1}{10}n^{-\frac{9}{10}}} = \lim_{n\to\infty}\frac{100}{\ln(2)\,n^{\frac{1}{10}}} = 0.\ (2)$$

From (1) and (2) we got:

$$0 < 1$$

So (1) is always true with sufficiently large n.

Therefore, $f(n) = O(g(n))$.

c) $f(n) = n^{1+(-1)^n}$ and $g(n) = n$.

Assume $f(n) = O(g(n))$, so we need to prove that:

$n^{1+(-1)^n} < cn$ for some positive c and all sufficiently large n.

But, since the log function is monotonically increasing, this will hold just in case:

$$\log\left(n^{1+(-1)^n}\right) < \log(c) + \log(n)$$
$$(1 + (-1)^n)\log(n) < \log(c) + \log(n)$$
$$(-1)^n\log(n) < \log(c)$$

Just note that $(-1)^n > 0$ when n is even and $(-1)^n < 0$ when n is odd.

Thus, for any fixed constant $c > 0$ and assume $\log(c) > 0$.

For all n is a sufficiently even large integer, we have:

$$(-1)^n\log(n) > \log(c)$$

For all n is a sufficiently odd large integer, we have:

$$(-1)^n\log(n) < 0 < \log(c)$$

Thus, neither $f(n) = O(g(n))$ nor $f(n) = \Omega(g(n))$.