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Question 2:

- Let replace $y = x^{100}$ in the polynomial $P(x)$. We get:
$$P_y(y) = A_0 + A_1 y + A_2 y^2 \text{ where } A_0, A_1, A_2 \text{ can be arbitrarily large numbers.}$$
- If we can manage to compute somehow the product polynomial
$$P_y(y)^2 = P_y(y) * P_y(y) = C_0 + C_1 y + C_2 y^2 + C_3 y^3 + C_4 y^4$$
with only 5 multiplications, we can then obtain the squared $P(x)$
- Since the degree of $P(y)^2$ is 4 we need 5 values to uniquely determine $P_y(y)^2$
- We choose the smallest possible 5 integer values $-2,\ -1,\ 0, 1, 2$.
- Thus, we compute:
$$P_y(-2) = A_0 + A_1(-2) + A_2(-2)^2 = A_0 - 2A_1 + 4A_2$$
$$P_y(-1) = A_0 + A_1(-1) + A_2(-1)^2 = A_0 - A_1 + A_2$$
$$P_y(0) = A_0$$
$$P_y(1) = A_0 + A_1 1 + A_2(1)^2 = A_0 + A_1 + A_2$$
$$P_y(2) = A_0 + A_1(2) + A_2(2)^2 = A_0 + 2A_1 + 4A_2$$
As we see, these evaluations involve only additions because $2A = A + A$, $4A = A + A + A + A$
- We can now obtain $P_y(-2)^2, P_y(-1)^2, P_y(0)^2, P_y(1)^2, P_y(2)^2$ with only 5 multiplications of large numbers:
$$P_y(-2)^2 = P_y(-2) * P_y(-2) = (A_0 - 2A_1 + 4A_2) * (A_0 - 2A_1 + 4A_2)$$
$$P_y(-1)^2 = P_y(-1) * P_y(-1) = (A_0 - A_1 + A_2) * (A_0 - A_1 + A_2)$$
$$P_y(0)^2 = P_y(0) * P_y(0) = A_0 * A_0$$
$$P_y(1)^2 = P_y(1) * P_y(1) = (A_0 - A_1 + A_2) * (A_0 - A_1 + A_2)$$
$$P_y(2)^2 = P_y(2) * P_y(2) = (A_0 + 2A_1 + 4A_2) * (A_0 + 2A_1 + 4A_2)$$
- Thus, the coefficient of $P_y(y)^2$: C_0, C_1, C_2, C_3, C_4 we can obtain those by:
$$C_0 + C_1(-2) + C_2(-2)^2 + C_3(-2)^3 + C_4(-2)^4 = P_y(-2)^2$$
$$C_0 + C_1(-1) + C_2(-1)^2 + C_3(-1)^3 + C_4(-1)^4 = P_y(-1)^2$$
$$C_0 + C_1(0) + C_2(0)^2 + C_3(0)^3 + C_4(0)^4 = P_y(0)^2$$
$$C_0 + C_1(1) + C_2(1)^2 + C_3(1)^3 + C_4(1)^4 = P_y(1)^2$$
$$C_0 + C_1(2) + C_2(2)^2 + C_3(2)^3 + C_4(2)^4 = P_y(2)^2$$
Simplifying the left side, we obtain
$$C_0 - 2C_1 + 4C_2 - 8C_3 + 16C_4 = P_y(-2)^2$$
$$C_0 - C_1 + C_2 - C_3 + C_4 = P_y(-1)^2$$
$$C_0 = P_y(0)^2$$
$$C_0 + C_1 + C_2 + C_3 + C_4 = P_y(1)^2$$
$$C_0 + 2C_1 + 4C_2 + 8C_3 + 16C_4 = P_y(2)^2$$
Solve this system of linear equations for C_0, C_1, C_2, C_3, C_4 we obtain:
$$C_0 = P_Y(0)^2$$
$$C_1 = \frac{P_y(-2)^2}{12} - \frac{2P_y(-1)^2}{3} + \frac{2P_y(1)^2}{3} - \frac{P_y(2)^2}{12}$$
$$C_2 = -\frac{P_y(-2)^2}{24} + \frac{2P_y(-1)^2}{3} - \frac{5P_y(0)^2}{4} + \frac{2P_y(1)^2}{3} - \frac{P_y(2)^2}{24}$$
$$C_3 = -\frac{P_y(-2)^2}{12} + \frac{P_y(-1)^2}{6} - \frac{P_y(1)^2}{6} + \frac{P_y(2)^2}{12}$$
$$C_4 = \frac{P_y(-2)^2}{24} - \frac{P_y(-1)^2}{6} + \frac{P_y(0)^2}{4} - \frac{P_y(1)^2}{6} + \frac{P_y(2)^2}{24}$$
- Note that these expressions do not involve any multiplications of TWO large numbers and thus can be done in linear time.
- At this stage, we obtain the coefficients for the polynomial $P_y(y)^2$ by only 5 multiplications of TWO large numbers.
- After we obtain the coefficients for the polynomial $P_y(y)^2$, we replace $y = x^{100}$ back to the $P_y(y)^2$ to get $P(x)^2$.
$$P(x)^2 = C_0 + C_1 x^{100} + C_2 x^{200} + C_3 x^{300} + C_4 x^{400}$$