... and now for something completely different...

Set Theory

Actually, you will see that logic and set theory are very closely related.

Set Theory

- Set: Collection of objects (called elements)
- $a \in A$ "a is an element of A"
  "a is a member of A"
- $a \notin A$ "a is not an element of A"
- $A = \{a_1, a_2, \ldots, a_n\}$ "A contains $a_1, \ldots, a_n$"
- Order of elements is insignificant
- It does not matter how often the same element is listed (repetition doesn't count).

Set Equality

Sets A and B are equal if and only if they contain exactly the same elements.

Examples:

- $A = \{9, 2, 7, -3\}$, $B = \{7, 9, -3, 2\}$: $A = B$
- $A = \{dog, cat, horse\}$,
  $B = \{cat, horse, squirrel, dog\}$: $A \neq B$
- $A = \{dog, cat, horse\}$,
  $B = \{cat, horse, dog, dog\}$: $A = B$

Examples for Sets

"Standard" Sets:

- Natural numbers **N** = {0, 1, 2, 3, …}
- Integers **Z** = {…, -2, -1, 0, 1, 2, …}
- Positive Integers $\mathbf{Z}^+$ = {1, 2, 3, 4, …}
- Real Numbers **R** = {47.3, -12, π, …}
- Rational Numbers **Q** = {1.5, 2.6, -3.8, 15, …}

(correct definitions will follow)

Examples for Sets

- A = ∅ "empty set/null set"
- A = {z} Note: z∈A, but z ≠ {z}
- A = {{b, c}, {c, x, d}} set of sets
- A = {{x, y}} Note: {x, y} ∈A, but {x, y} ≠ {{x, y}}
- A = {x | P(x)} "set of all x such that P(x)"

  P(x) is the membership function of set A

  ∀x (P(x) → x∈A)
- A = {x | x∈ **N** ^ x > 7} = {8, 9, 10, …} "set builder notation"

Examples for Sets

We are now able to define the set of rational numbers Q:

$\mathbf{Q} = \{a/b \mid a \in \mathbf{Z} \wedge b \in \mathbf{Z}^+\}$, or

$\mathbf{Q} = \{a/b \mid a \in \mathbf{Z} \wedge b \in \mathbf{Z} \wedge b \neq 0\}$

And how about the set of real numbers R?

$\mathbf{R} = \{r \mid r \text{ is a real number}\}$

That is the best we can do. It can neither be defined by enumeration nor builder function.

Subsets

$A \subseteq B$ "A is a subset of B"

$A \subseteq B$ if and only if every element of A is also an element of B.

We can completely formalize this:

$A \subseteq B \Leftrightarrow \forall x\ (x \in A \rightarrow x \in B)$

Examples:

A = {3, 9}, B = {5, 9, 1, 3}, $A \subseteq B$? true

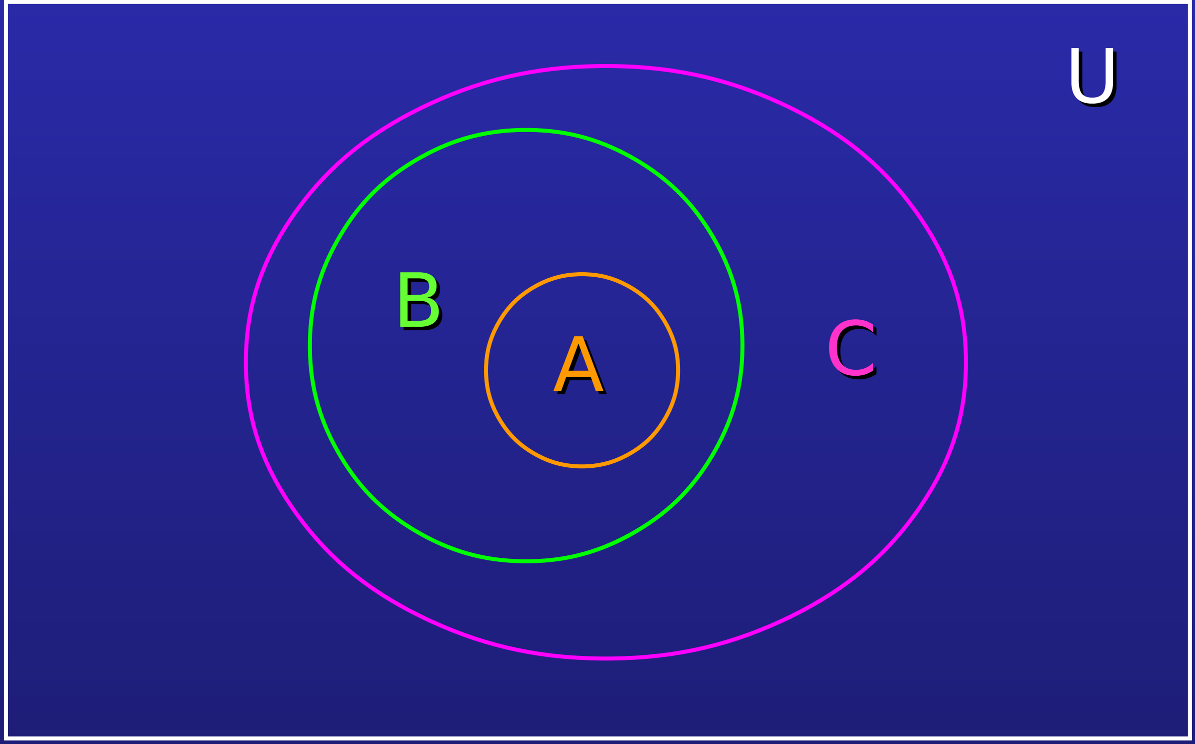
A = {3, 3, 3, 9}, B = {5, 9, 1, 3}, $A \subseteq B$? true

A = {1, 2, 3}, B = {2, 3, 4}, $A \subseteq B$? false

Subsets

Useful rules:

- $A = B \Leftrightarrow (A \subseteq B) \wedge (B \subseteq A)$
- $(A \subseteq B) \wedge (B \subseteq C) \Rightarrow A \subseteq C$ (see Venn Diagram)



Subsets

Useful rules:

- $\emptyset \subseteq A$ for any set A
  (but $\emptyset \in A$ may not hold for any set A)
- $A \subseteq A$ for any set A

Proper subsets:

$A \subset B$ "A is a proper subset of B"

$A \subset B \Leftrightarrow \forall x\ (x \in A \rightarrow x \in B) \wedge \exists x\ (x \in B \wedge x \notin A)$

or

$A \subset B \Leftrightarrow \forall x\ (x \in A \rightarrow x \in B) \wedge \neg \forall x\ (x \in B \rightarrow x \in A)$

Cardinality of Sets

If a set S contains n distinct elements, $n \in \mathbf{N}$, we call S a finite set with cardinality n.

Examples:

A = {Mercedes, BMW, Porsche}, |A| = 3

B = {1, {2, 3}, {4, 5}, 6} |B| = 4

C = ∅ |C| = 0

D = { $x \in \mathbf{N}$ | $x \leq 7000$ } |D| = 7001

E = { $x \in \mathbf{N}$ | $x \geq 7000$ } E is infinite!

The Power Set

P(A) "power set of A" (also written as 2^A)

$P(A) = \{B \mid B \subseteq A\}$ (contains all subsets of A)

Examples:

$A = \{x, y, z\}$

$P(A) = \{\varnothing, \{x\}, \{y\}, \{z\}, \{x, y\}, \{x, z\}, \{y, z\}, \{x, y, z\}\}$

$A = \varnothing$

$P(A) = \{\varnothing\}$

Note: $|A| = 0$, $|P(A)| = 1$

The Power Set

Cardinality of power sets: $| P(A) | = 2^{|A|}$

- Imagine each element in A has an "on/off" switch
- Each possible switch configuration in A corresponds to one subset of A, thus one element in P(A)

A	1	2	3	4	5	6	7	8
x	x	x	x	x	x	x	x	x
y	y	y	y	y	y	y	y	y
z	z	z	z	z	z	z	z	z

- For 3 elements in A, there are $2 \times 2 \times 2 = 8$ elements in P(A)

Cartesian Product

The ordered n-tuple $(a_1, a_2, a_3, \ldots, a_n)$ is an ordered collection of n objects.

Two ordered n-tuples $(a_1, a_2, a_3, \ldots, a_n)$ and $(b_1, b_2, b_3, \ldots, b_n)$ are equal if and only if they contain exactly the same elements in the same order, i.e. $a_i = b_i$ for $1 \leq i \leq n$.

The Cartesian product of two sets is defined as:

$A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$

Cartesian Product

Example:

A = {good, bad}, B = {student, prof}

$A \times B$ = { (good, student), (good, prof), (bad, student), (bad, prof) }

$B \times A$ = { (student, good), (prof, good), (student, bad), (prof, bad) }

Example: A = {x, y}, B = {a, b, c}

$A \times B$ = {(x, a), (x, b), (x, c), (y, a), (y, b), (y, c)}

Cartesian Product

Note that:

- $A \times \emptyset = \emptyset$
- $\emptyset \times A = \emptyset$
- For non-empty sets A and B: $A \neq B \Leftrightarrow A \times B \neq B \times A$
- $|A \times B| = |A| \cdot |B|$

The Cartesian product of two or more sets is defined as:

$A_1 \times A_2 \times \ldots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i \text{ for } 1 \leq i \leq n\}$

Set Operations

Union: $A \cup B = \{x \mid x \in A \vee x \in B\}$

Example: $A = \{a, b\}$, $B = \{b, c, d\}$

$A \cup B = \{a, b, c, d\}$

Intersection: $A \cap B = \{x \mid x \in A \wedge x \in B\}$

Example: $A = \{a, b\}$, $B = \{b, c, d\}$

$A \cap B = \{b\}$

Cardinality: $|A \cup B| = |A| + |B| - |A \cap B|$

Set Operations

Two sets are called disjoint if their intersection is empty, that is, they share no elements:

$A \cap B = \emptyset$

The difference between two sets A and B contains exactly those elements of A that are not in B:

$A-B = \{x \mid x \in A \wedge x \notin B\}$

Example: $A = \{a, b\}$, $B = \{b, c, d\}$, $A-B = \{a\}$

Cardinality: $|A-B| = |A| - |A \cap B|$

Set Operations

The complement of a set A contains exactly those elements under consideration that are not in A: denoted A^c (or $\overline{A}$ as in the text)

$A^c = U - A$

Example: $U = \mathbf{N}$, $B = \{250, 251, 252, \ldots\}$

$B^c = \{0, 1, 2, \ldots, 248, 249\}$

Logical Equivalence

Equivalence laws

- Identity laws, $P \wedge T \equiv P$,
- Domination laws, $P \wedge F \equiv F$,
- Idempotent laws, $P \wedge P \equiv P$,
- Double negation law, $\neg(\neg P) \equiv P$
- Commutative laws, $P \wedge Q \equiv Q \wedge P$,
- Associative laws, $P \wedge (Q \wedge R) \equiv (P \wedge Q) \wedge R$,
- Distributive laws, $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$,
- De Morgan's laws, $\neg(P \wedge Q) \equiv (\neg P) \vee (\neg Q)$
- Law with implication $P \rightarrow Q \equiv \neg P \vee Q$

Set Identity

Table 1 in Section 1.7 shows many useful equations

- Identity laws, $A \cup \emptyset = A$, $A \cap U = A$
- Domination laws, $A \cup U = U$, $A \cap \emptyset = \emptyset$
- Idempotent laws, $A \cup A = A$, $A \cap A = A$
- Complementation law, $(A^c)^c = A$
- Commutative laws, $A \cup B = B \cup A$, $A \cap B = B \cap A$
- Associative laws, $A \cup (B \cup C) = (A \cup B) \cup C$, ...
- Distributive laws, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$, ...
- De Morgan's laws, $(A \cup B)^c = A^c \cap B^c$, $(A \cap B)^c = A^c \cup B^c$
- Absorption laws, $A \cup (A \cap B) = A$, $A \cap (A \cup B) = A$
- Complement laws, $A \cup A^c = U$, $A \cap A^c = \emptyset$

Set Identity

How can we prove $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$?

Method I: logical equivalent

$$
\begin{aligned}
& x \in A \cup (B \cap C) \\
\Leftrightarrow\ & x \in A \vee x \in (B \cap C) \\
\Leftrightarrow\ & x \in A \vee (x \in B \wedge x \in C) \\
\Leftrightarrow\ & (x \in A \vee x \in B) \wedge (x \in A \vee x \in C) \quad \text{(distributive law)} \\
\Leftrightarrow\ & x \in (A \cup B) \wedge x \in (A \cup C) \\
\Leftrightarrow\ & x \in (A \cup B) \cap (A \cup C)
\end{aligned}
$$

Every logical expression can be transformed into an equivalent expression in set theory and vice versa.

Set Operations

Method II: Membership table

1 means “x is an element of this set”

0 means “x is not an element of this set”

A B C	$B \cap C$	$A \cup (B \cap C)$	$A \cup B$	$A \cup C$	$(A \cup B) \cap (A \cup C)$
0 0 0	0	0	0	0	0
0 0 1	0	0	0	1	0
0 1 0	0	0	1	0	0
0 1 1	1	1	1	1	1
1 0 0	0	1	1	1	1
1 0 1	0	1	1	1	1
1 1 0	0	1	1	1	1
1 1 1	1	1	1	1	1

... and the following mathematical appetizer is about...

Functions

Functions

A function f from a set A to a set B is an assignment of exactly one element of B to each element of A.

We write

$f(a) = b$

if b is the unique element of B assigned by the function f to the element a of A.

If f is a function from A to B, we write

$f: A \rightarrow B$

(note: Here, "$\rightarrow$" has nothing to do with if… then)

Functions

If $f:A \to B$, we say that A is the domain of f and B is the codomain of f.

If $f(a) = b$, we say that b is the image of a and a is the pre-image of b.

The range of $f:A \to B$ is the set of all images of all elements of A.

We say that $f:A \to B$ maps A to B.

Functions

Let us take a look at the function $f:P \rightarrow C$ with

P = {Linda, Max, Kathy, Peter}

C = {Boston, New York, Hong Kong, Moscow}

f(Linda) = Moscow

f(Max) = Boston

f(Kathy) = Hong Kong

f(Peter) = New York

Here, the range of f is C.

Functions

Let us re-specify f as follows:

f(Linda) = Moscow

f(Max) = Boston

f(Kathy) = Hong Kong

f(Peter) = Boston

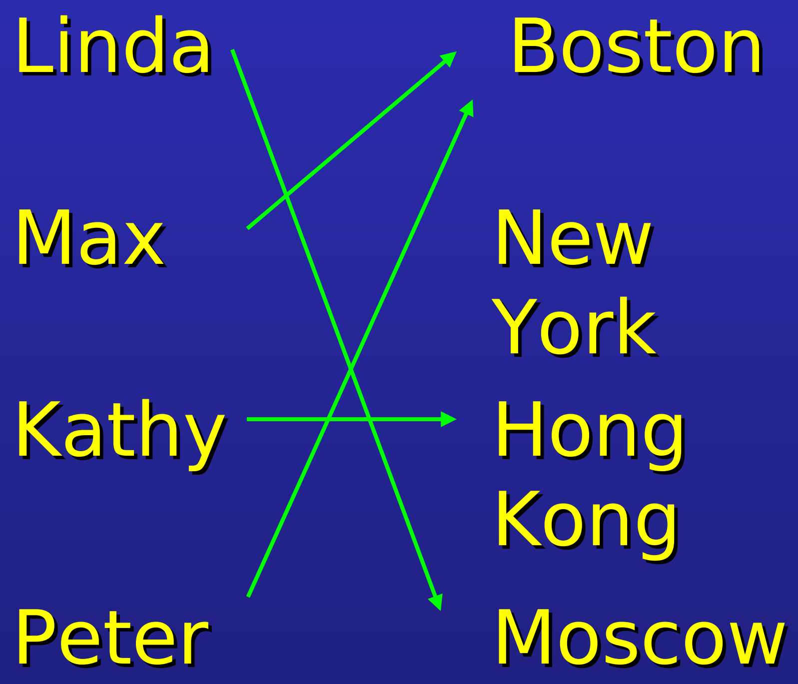
Is f still a function? yes

What is its range? {Moscow, Boston, Hong Kong}

Functions

Other ways to represent f:

x	f(x)
Linda	Moscow
Max	Boston
Kathy	Hong Kong
Peter	Boston



Functions

If the domain of our function f is large, it is convenient to specify f with a formula, e.g.:

$f: \mathbf{R} \to \mathbf{R}$

$f(x) = 2x$

This leads to:

$f(1) = 2$

$f(3) = 6$

$f(-3) = -6$

…

Functions

Let f_1 and f_2 be functions from A to **R**.

Then the sum and the product of f_1 and f_2 are also functions from A to **R** defined by:

$(f_1 + f_2)(x) = f_1(x) + f_2(x)$

$(f_1 f_2)(x) = f_1(x)\, f_2(x)$

Example:

$f_1(x) = 3x, \quad f_2(x) = x + 5$

$(f_1 + f_2)(x) = f_1(x) + f_2(x) = 3x + x + 5 = 4x + 5$

$(f_1 f_2)(x) = f_1(x)\, f_2(x) = 3x\,(x + 5) = 3x^2 + 15x$

Functions

We already know that the range of a function $f:A \to B$ is the set of all images of elements $a \in A$.

If we only regard a subset $S \subseteq A$, the set of all images of elements $s \in S$ is called the image of S.

We denote the image of S by f(S):

$f(S) = \{f(s) \mid s \in S\}$

Functions

Let us look at the following well-known function:

f(Linda) = Moscow

f(Max) = Boston

f(Kathy) = Hong Kong

f(Peter) = Boston

What is the image of S = {Linda, Max} ?

f(S) = {Moscow, Boston}

What is the image of S = {Max, Peter} ?

f(S) = {Boston}

Properties of Functions

A function $f:A \rightarrow B$ is said to be one-to-one (or injective), if and only if

$$\forall x, y \in A\ (f(x) = f(y) \rightarrow x = y)$$

In other words: f is one-to-one if and only if it does not map two distinct elements of A onto the same element of B.

Properties of Functions

And again…

f(Linda) = Moscow

f(Max) = Boston

f(Kathy) = Hong Kong

f(Peter) = Boston

Is f one-to-one?

No, Max and Peter are mapped onto the same element of the image.

g(Linda) = Moscow

g(Max) = Boston

g(Kathy) = Hong Kong

g(Peter) = New York

Is g one-to-one?

Yes, each element is assigned a unique element of the image.

Properties of Functions

How can we prove that a function f is one-to-one?

Whenever you want to prove something, first take a look at the relevant definition(s):

$\forall x, y \in A\ (f(x) = f(y) \rightarrow x = y)$

Example:

$f: \mathbf{R} \rightarrow \mathbf{R}$

$f(x) = x^2$

Disproof by counterexample:

$f(3) = f(-3)$, but $3 \neq -3$, so f is not one-to-one.

Properties of Functions

… and yet another example:

f:**R**→**R**

f(x) = 3x

One-to-one: $\forall x, y \in A\ (f(x) = f(y) \rightarrow x = y)$

To show: $f(x) \neq f(y)$ whenever $x \neq y$ (indirect proof)

$x \neq y$

$\Leftrightarrow 3x \neq 3y$

$\Leftrightarrow f(x) \neq f(y)$,

so if $x \neq y$, then $f(x) \neq f(y)$, that is, f is one-to-one.

Properties of Functions

A function $f:A \rightarrow B$ with $A,B \subseteq R$ is called strictly increasing, if

$\forall x,y \in A \ (x < y \rightarrow f(x) < f(y))$,

and strictly decreasing, if

$\forall x,y \in A \ (x < y \rightarrow f(x) > f(y))$.

Obviously, a function that is either strictly increasing or strictly decreasing is one-to-one.

Properties of Functions

A function $f:A\rightarrow B$ is called onto, or surjective, if and only if for every element $b\in B$ there is an element $a\in A$ with $f(a) = b$.

In other words, f is onto if and only if its range is its entire codomain.

A function $f: A\rightarrow B$ is a one-to-one correspondence, or a bijection, if and only if it is both one-to-one and onto.

Obviously, if f is a bijection and A and B are finite sets, then $|A| = |B|$.

Properties of Functions

Examples:

In the following examples, we use the arrow representation to illustrate functions f:A→B.

In each example, the complete sets A and B are shown.

Properties of Functions

Is f injective?

No.

Is f surjective?

No.

Is f bijective?

No.

Properties of Functions

Is f injective?

No.

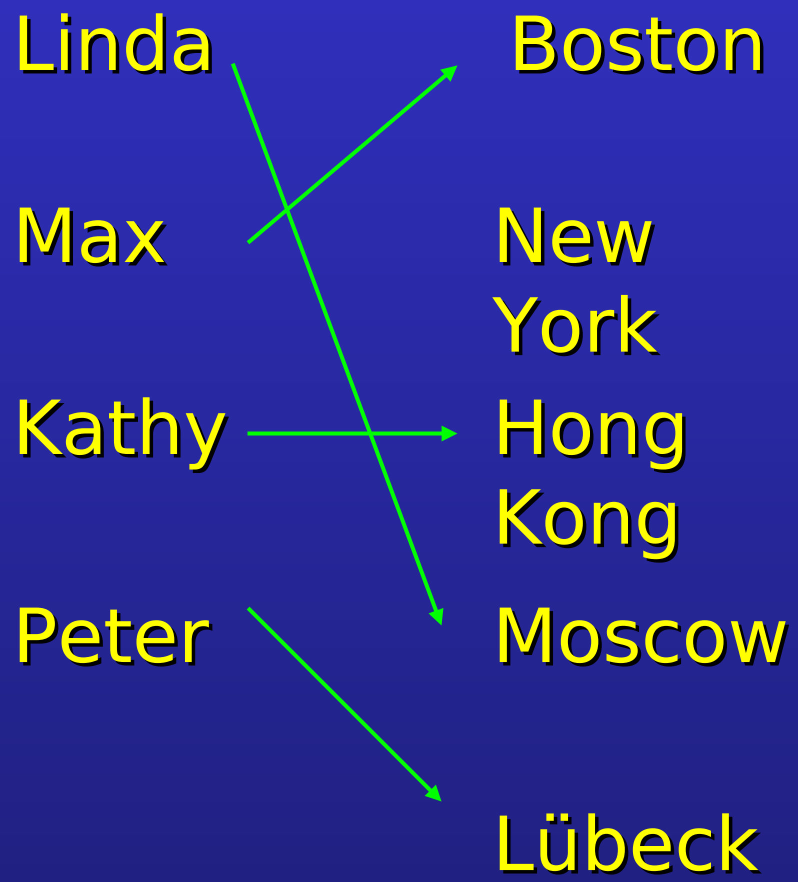
Is f surjective?

Yes.

Is f bijective?

No.

Properties of Functions



Is f injective?

Yes.

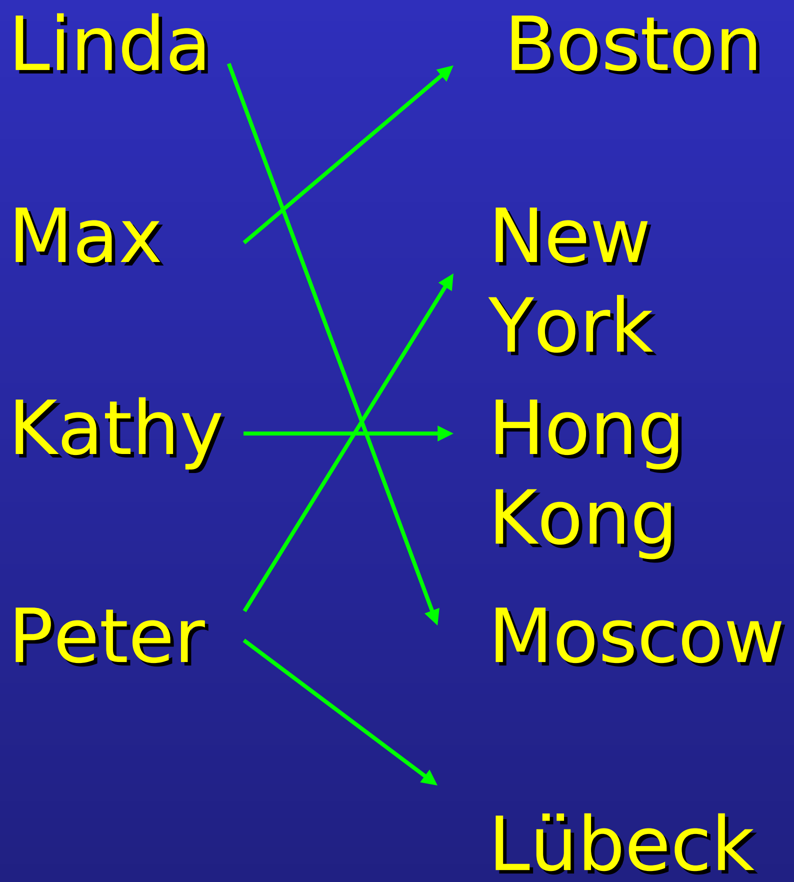
Is f surjective?

No.

Is f bijective?

No.

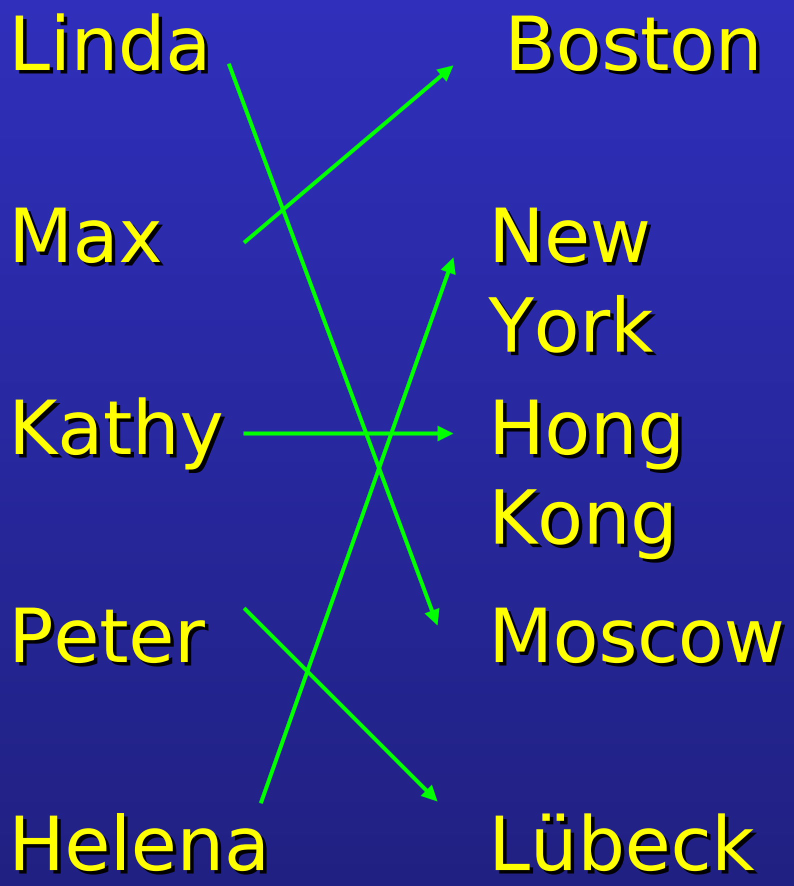
Properties of Functions



Is f injective?

No! f is not even a function!

Properties of Functions



Is f injective?

Yes.

Is f surjective?

Yes.

Is f bijective?

Yes.

Inversion

An interesting property of bijections is that they have an **inverse function**.

The **inverse function** of the bijection $f:A \rightarrow B$ is the function $f^{-1}:B \rightarrow A$ with

$f^{-1}(b) = a$ whenever $f(a) = b$.

Inversion

Example:

f(Linda) = Moscow

f(Max) = Boston

f(Kathy) = Hong Kong

f(Peter) = Lübeck

f(Helena) = New York

Clearly, f is bijective.

The inverse function f^{-1} is given by:

f^{-1}(Moscow) = Linda

f^{-1}(Boston) = Max

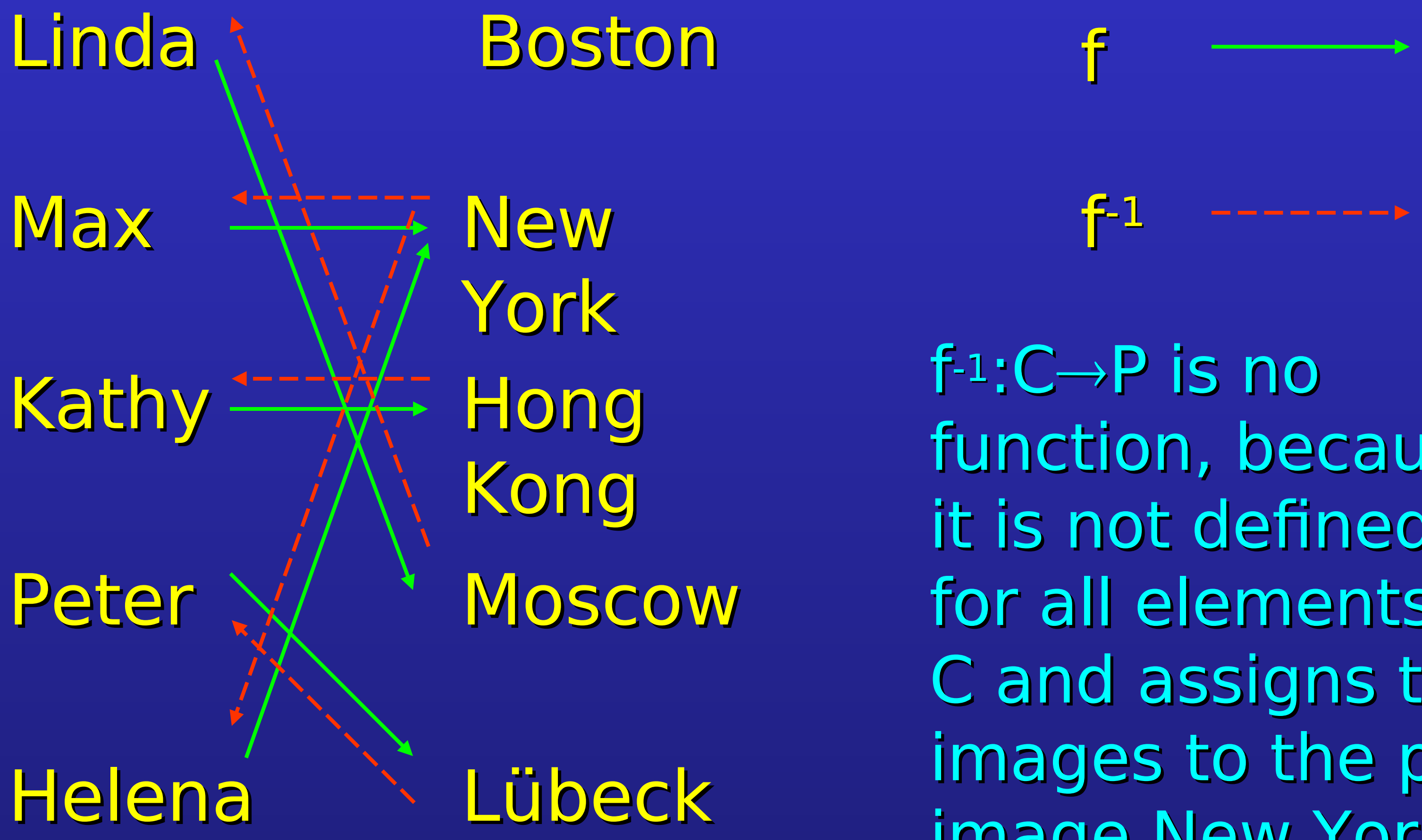
f^{-1}(Hong Kong) = Kathy

f^{-1}(Lübeck) = Peter

f^{-1}(New York) = Helena

Inversion is only possible for bijections

Inversion



$f^{-1}: C \rightarrow P$ is no function, because it is not defined for all elements of C and assigns two images to the pre-image New York.

Composition

The **composition** of two functions $g:A \rightarrow B$ and $f:B \rightarrow C$, denoted by $f \circ g$, is defined by

$(f \circ g)(a) = f(g(a))$

This means that

- **first**, function g is applied to element $a \in A$, mapping it onto an element of B,
- **then**, function f is applied to this element of B, mapping it onto an element of C.
- **Therefore**, the composite function maps from A to C.

Composition

Example:

$f(x) = 7x - 4$, $g(x) = 3x$,

$f:\mathbf{R}\to\mathbf{R}$, $g:\mathbf{R}\to\mathbf{R}$

$(f\circ g)(5) = f(g(5)) = f(15) = 105 - 4 = 101$

$(f\circ g)(x) = f(g(x)) = f(3x) = 21x - 4$

Composition

Composition of a function and its inverse:

$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = x$

The composition of a function and its inverse is the **identity function** $i(x) = x$.

Graphs

The **graph** of a function $f:A \rightarrow B$ is the set of ordered pairs $\{(a, b) \mid a \in A \text{ and } f(a) = b\}$.

The graph is a subset of $A \times B$ that can be used to visualize f in a two-dimensional coordinate system.

Floor and Ceiling Functions

The **floor** and **ceiling** functions map the real numbers onto the integers ($\mathbf{R} \rightarrow \mathbf{Z}$).

The **floor** function assigns to $r \in \mathbf{R}$ the largest $z \in \mathbf{Z}$ with $z \leq r$, denoted by $\lfloor r \rfloor$.

Examples: $\lfloor 2.3 \rfloor = 2$, $\lfloor 2 \rfloor = 2$, $\lfloor 0.5 \rfloor = 0$, $\lfloor -3.5 \rfloor = -4$

The **ceiling** function assigns to $r \in \mathbf{R}$ the smallest $z \in \mathbf{Z}$ with $z \geq r$, denoted by $\lceil r \rceil$.

Examples: $\lceil 2.3 \rceil = 3$, $\lceil 2 \rceil = 2$, $\lceil 0.5 \rceil = 1$, $\lceil -3.5 \rceil = -3$

Now, something about

BooleanAl gebra

(section 10.1)

Boolean Algebra

Boolean algebra provides the operations and the rules for working with the set **{0, 1}.**

These are the rules that underlie **electronic circuits**, and the methods we will discuss are fundamental to **VLSI design**.

We are going to focus on three operations:

- Boolean complementation,
- Boolean sum, and
- Boolean product

Boolean Operations

The **complement** is denoted by a bar (on the slides, we will use a minus sign). It is defined by

$-0 = 1$ and $-1 = 0$.

The **Boolean sum**, denoted by + or by OR, has the following values:

$1 + 1 = 1$, $1 + 0 = 1$, $0 + 1 = 1$, $0 + 0 = 0$

The **Boolean product**, denoted by $\cdot$ or by AND, has the following values:

$1 \cdot 1 = 1$, $1 \cdot 0 = 0$, $0 \cdot 1 = 0$, $0 \cdot 0 = 0$

Boolean Functions and Expressions

Definition: Let $B = \{0, 1\}$. The variable x is called a **Boolean variable** if it assumes values only from B.

A function from B^n, the set $\{(x_1, x_2, \ldots, x_n) \mid x_i \in B,$

$1 \leq i \leq n\}$, to B is called a **Boolean function of degree n**.

Boolean functions can be represented using expressions made up from the variables and Boolean operations.

Boolean Functions and Expressions

The **Boolean expressions** in the variables x_1, x_2, …, x_n are defined recursively as follows:

- 0, 1, x_1, x_2, …, x_n are Boolean expressions.
- If E_1 and E_2 are Boolean expressions, then $(-E_1)$, (E_1E_2), and $(E_1 + E_2)$ are Boolean expressions.

Each Boolean expression represents a Boolean function. The values of this function are obtained by substituting 0 and 1 for the variables in the expression.

Boolean Functions and Expressions

For example, we can create Boolean expression in the variables x, y, and z using the "building blocks"
0, 1, x, y, and z, and the construction rules:

Since x and y are Boolean expressions, so is xy.

Since z is a Boolean expression, so is (-z).

Since xy and (-z) are expressions, so is xy + (-z).

… and so on…

Boolean Functions and Expressions

Example: Give a Boolean expression for the Boolean function F(x, y) as defined by the following table:

x	y	F(x, y)
0	0	0
0	1	1
1	0	0
1	1	0

Possible solution: $F(x, y) = (-x) \cdot y$

Boolean Functions and Expressions

Another Example:

x	y	z	F(x, y, z)
0	0	0	1
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	0

Possible solution I:

F(x, y, z) = -(xz + y)

Possible solution II:

F(x, y, z) = (-(xz))(-y)

Boolean Functions and Expressions

There is a simple method for deriving a Boolean expression for a function that is defined by a table. This method is based on **minterms**.

Definition: A **literal** is a Boolean variable or its complement. A **minterm** of the Boolean variables $x_1, x_2, \ldots, x_n$ is a Boolean product $y_1 y_2 \ldots y_n$, where $y_i = x_i$ or $y_i = -x_i$.

Hence, a minterm is a product of n literals, with one literal for each variable.

Boolean Functions and Expressions

Consider F(x,y,z) again:

x	y	z	F(x, y, z)
0	0	0	1
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	0

F(x, y, z) = 1 if and only if:

x = y = z = 0 or

x = y = 0, z = 1 or

x = 1, y = z = 0

Therefore,

F(x, y, z) =
(-x)(-y)(-z) +
(-x)(-y)z +
x(-y)(-z)

Boolean Functions and Expressions

Definition: The Boolean functions F and G of n variables are **equal** if and only if $F(b_1, b_2, \ldots, b_n) = G(b_1, b_2, \ldots, b_n)$ whenever $b_1, b_2, \ldots, b_n$ belong to B.

Two different Boolean expressions that represent the same function are called **equivalent**.

For example, the Boolean expressions xy, $xy + 0$, and $xy \cdot 1$ are equivalent.

Boolean Functions and Expressions

The **complement** of the Boolean function F is the function –F, where $-F(b_1, b_2, \ldots, b_n) = -(F(b_1, b_2, \ldots, b_n))$.

Let F and G be Boolean functions of degree n. The **Boolean sum F+G** and **Boolean product FG** are then defined by

$(F + G)(b_1, b_2, \ldots, b_n) = F(b_1, b_2, \ldots, b_n) + G(b_1, b_2, \ldots, b_n)$

$(FG)(b_1, b_2, \ldots, b_n) = F(b_1, b_2, \ldots, b_n)\, G(b_1, b_2, \ldots, b_n)$

Boolean Functions and Expressions

Question: How many different Boolean functions of degree 1 are there?

Solution: There are four of them, F_1, F_2, F_3, and F_4:

x	F_1	F_2	F_3	F_4
0	0	0	1	1
1	0	1	0	1

Boolean Functions and Expressions

Question: How many different Boolean functions of degree 2 are there?

Solution: There are 16 of them, $F_1, F_2, \ldots, F_{16}$:

x	y	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	F_{11}	F_{12}	F_{13}	F_{14}	F_{15}	F_{16}
0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
0	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
1	0	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1

Boolean Functions and Expressions

Question: How many different Boolean functions of degree n are there?

Solution:

There are 2^n different n-tuples of 0s and 1s.

A Boolean function is an assignment of 0 or 1 to each of these 2^n different n-tuples.

Therefore, there are 2^{2^n} different Boolean functions.

Duality

There are useful identities of Boolean expressions that can help us to transform an expression A into an equivalent expression B (see Table 5 on page 705 in the textbook).

We can derive additional identities with the help of the **dual** of a Boolean expression.

The dual of a Boolean expression is obtained by interchanging Boolean sums and Boolean products and interchanging 0s and 1s.

Duality

Examples:

The dual of $x(y + z)$ is $x + yz$.

The dual of $-x \cdot 1 + (-y + z)$ is $(-x + 0)((-y)z)$.

The **dual of a Boolean function F** represented by a Boolean expression is the function represented by the dual of this expression.

This dual function, denoted by F^d, **does not depend** on the particular Boolean expression used to represent F.

Duality

Therefore, an identity between functions represented by Boolean expressions **remains valid** when the duals of both sides of the identity are taken.

We can use this fact, called the **duality principle**, to derive new identities.

For example, consider the absorption law $x(x + y) = x$.

By taking the duals of both sides of this identity, we obtain the equation $x + xy = x$, which is also an identity (and also called an absorption law).

Definition of a Boolean Algebra

All the properties of Boolean functions and expressions that we have discovered also apply to **other mathematical structures** such as propositions and sets and the operations defined on them.

If we can show that a particular structure is a Boolean algebra, then we know that all results established about Boolean algebras apply to this structure.

For this purpose, we need an **abstract definition** of a Boolean algebra.

Definition of a Boolean Algebra

Definition: A Boolean algebra is a set B with two binary operations $\vee$ and $\wedge$, elements 0 and 1, and a unary operation – such that the following properties hold for all x, y, and z in B:

$x \vee 0 = x$ and $x \wedge 1 = x$ (identity laws)

$x \vee (-x) = 1$ and $x \wedge (-x) = 0$ (domination laws)

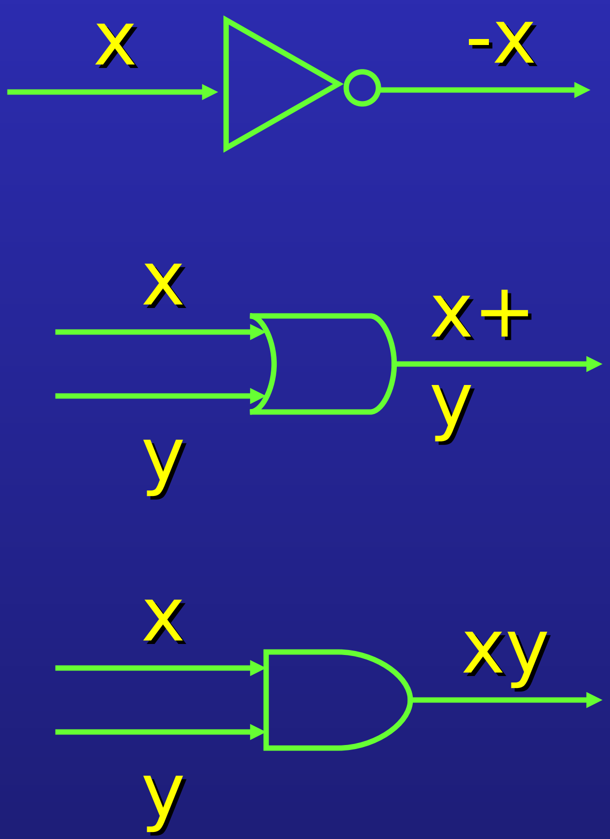
$(x \vee y) \vee z = x \vee (y \vee z)$ and
$(x \wedge y) \wedge z = x \wedge (y \wedge z)$ and (associative laws)

$x \vee y = y \vee x$ and $x \wedge y = y \wedge x$ (commutative laws)

$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ and
$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ (distributive laws)

Logic Gates

Electronic circuits consist of so-called gates.
There are three basic types of gates:



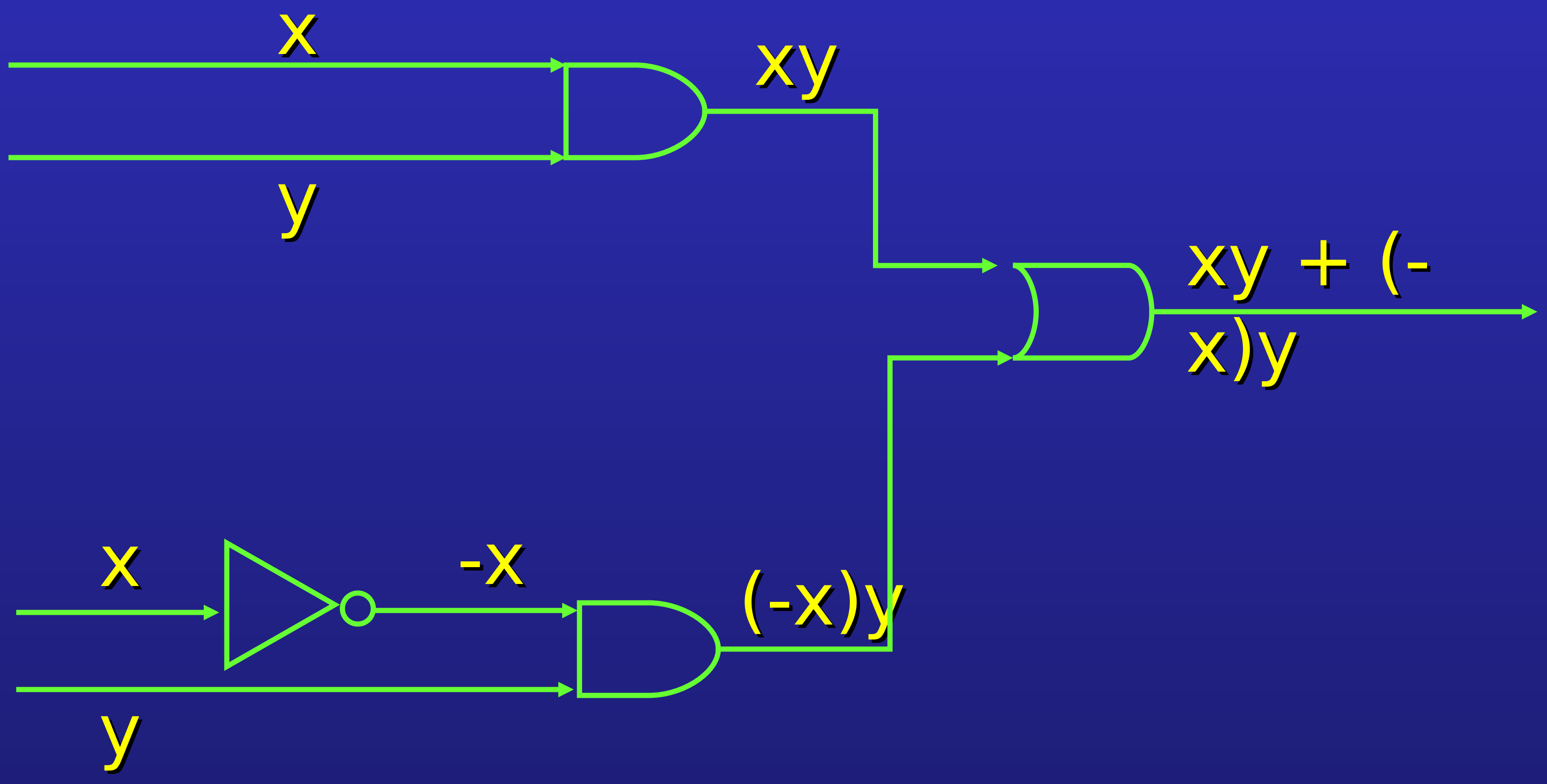
inverter

OR gate

AND gate

Logic Gates

Example: How can we build a circuit that computes the function xy + (-x)y ?



Logic, Sets, and Boolean Algebra

Logic	**Set**	**Boolean Algebra**
False	$\emptyset$	0
True	U	1
$A \wedge B$	$A \cap B$	$A \cdot B$
$A \vee B$	$A \cup B$	$A + B$
$\neg A$	A^c	$\overline{A}$

Compare the equivalence laws of them