Grammars

Rule in form $\alpha \to \beta$, α, β are some words

Example:

$$\left[\begin{array}{l} S \to aSb \\ aSb \to aaSbb \\ S \to \varepsilon \end{array}\right.$$

$$\Sigma = \{a, b\}$$
$$N = \{S\}$$

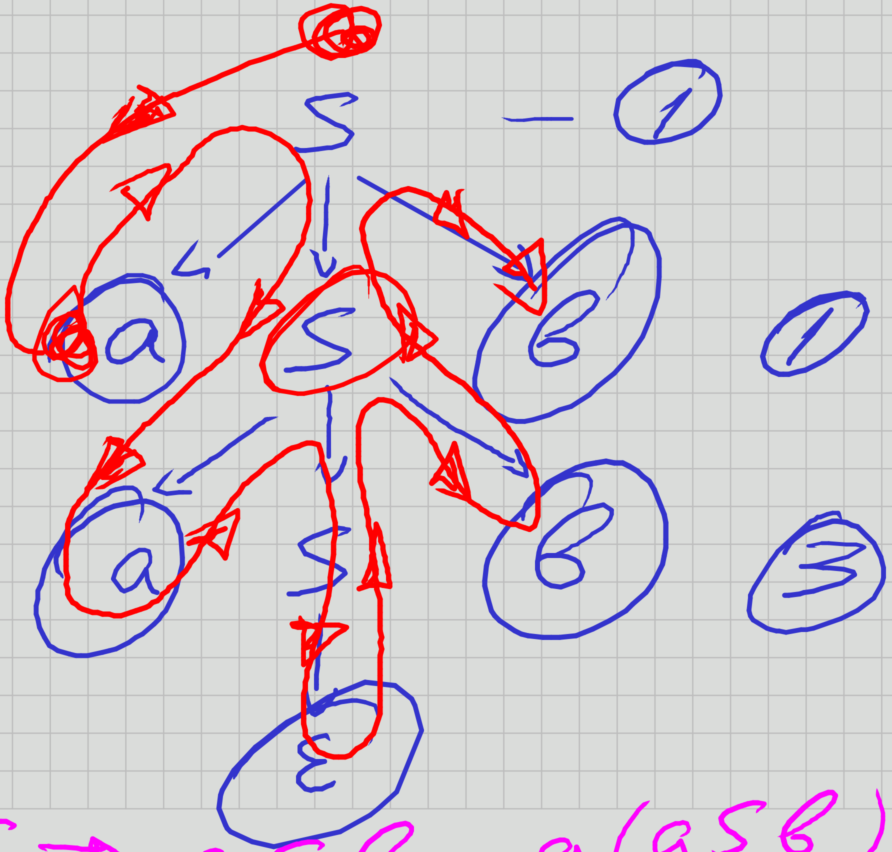
$$aabb \in L(G)$$

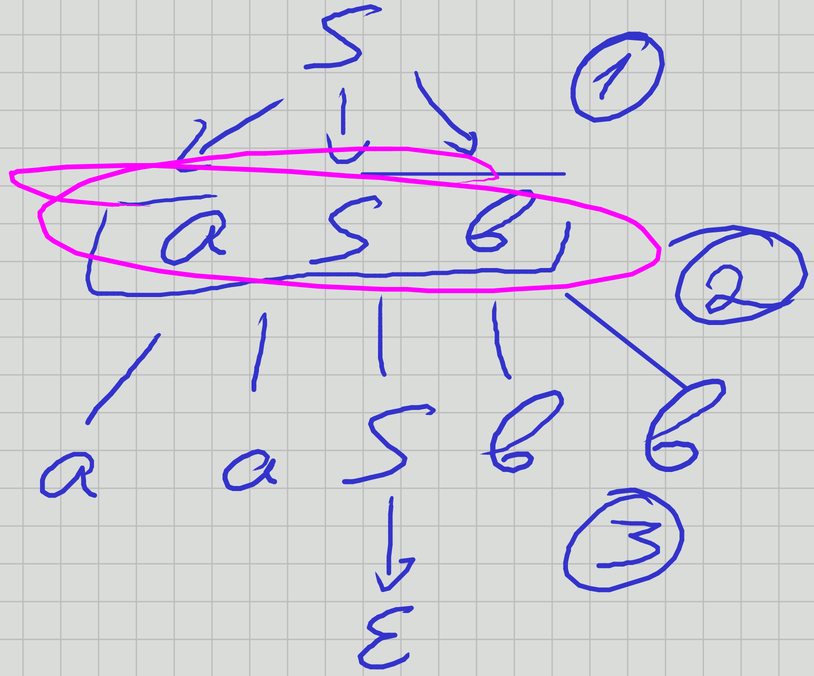
1. Σ alphabet of Terminal symbols
2. N alphabet of Nonterminal symbols
3. $S \in N$ axiom
4. $P \subseteq \{(\alpha, \beta) : \alpha, \beta \in (N \cup \Sigma)^*\}$ rules

Дерево вывода для слова из грамматики
Derivation (?) tree

① $S \to aSb$
② $aSb \to aaSbb$
③ $S \to \varepsilon$

$w = aabb$





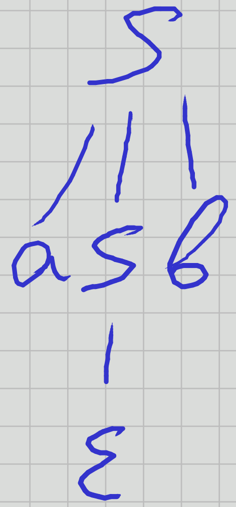
$S \to a\underline{S}b \quad a(a\underset{\varepsilon}{S}b)b \quad aa\varepsilon bb \quad aabb$

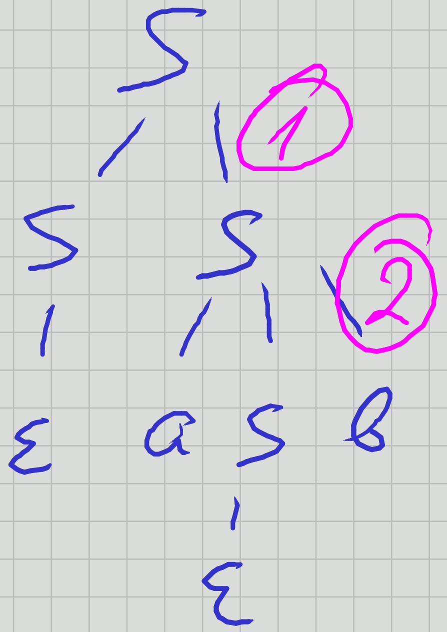
① $S \rightarrow SS$

② $S \rightarrow aSb$

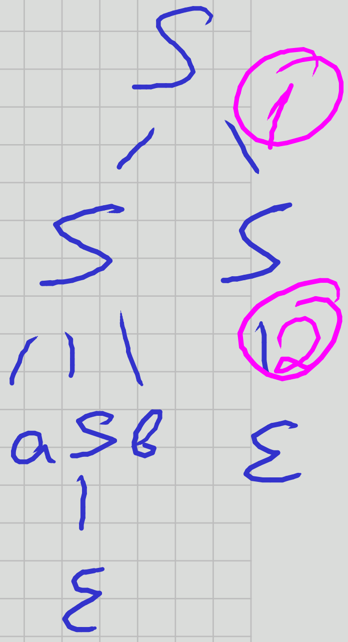
③ $S \rightarrow \varepsilon$

$w = ab$





$\varepsilon a \varepsilon b$



$a \varepsilon b \varepsilon$

Type 1 Grammars

$S \to \Sigma$

1. $\boxed{\text{S -> e}}$ => S shouldn't be on the right side of rules

2. for other rules

$|\beta| \geq |\alpha|$

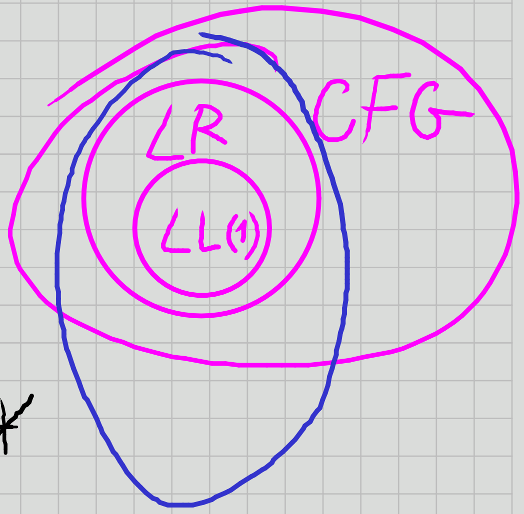
$\boxed{\alpha} \to \beta$

Context dependant Languages

Type 2 Grammars

$A \to \beta\ ,\ A \in N,\ \beta \in (N \cup \Sigma)^*$

Context free languages



Fortran, Lisp

C, Java

Type 3 Grammars

All rules :

$$A \to wB \quad \text{or} \quad A \to w$$

$$A, B \in N \qquad w \in \Sigma^*$$

$S \to Sa \mid Sbb$

$S \to Ta$

$T \to bb$

left linear

Right-linear Grammars

$$L(G_1) = L(G_2)$$

Regular Languages

$S \to aT$

$T \to Sb$

$S \to \varepsilon$

$S \to aSb$

$$\boxed{S \Rightarrow a\underline{T} \Rightarrow a\underline{S}b} \Rightarrow aaTb \Rightarrow$$

$$\Rightarrow aa\underline{S}bb \Rightarrow aabb$$

$$S \overset{*}{\Rightarrow} aSb \overset{*}{\Rightarrow} aaSbb$$

$\varepsilon \in L(G)$ $\qquad$ $\varepsilon, ab, aabb$

$\underline{S} \Rightarrow \varepsilon$

$aa\underline{S}bb \Rightarrow aa(a\underline{T})bb \Rightarrow aaa(\overset{\nearrow \varepsilon}{S}b)bb \Rightarrow$

$\Rightarrow aaabbb$

$aaabbb \in L(G) = \{a^n b^n\} \overset{?}{\in} REG$

---

example: Grammar for language of correct scope stuctures

$S \rightarrow SS \mid (S) \mid \varepsilon$ $\qquad$ $S \rightarrow aSb$

$(())$ $\qquad$ $((()())())$

$S \rightarrow (S)$

$S \rightarrow [S]$

$S \rightarrow \{S\}$

$$L(G) = \{ w \mid S \overset{*}{\Rightarrow} w \}$$

equivalent
models
of
computation

Type 0: рекурсивно перечислимые
recursively enumerable languages
Turing Machine accepts words from this language

слова принимаются, но не распознаются

Type 1: Context dependant

Turing Machine accepts words from this language
but we are not allowed to use more memory cells
that was used to write down input word

Type 2: Context Free (CF)
NFA + Stack = PDA

Type 3: Regular
NFA

Однозначные и неоднозначные грамматики

unambiguous and ambiguous grammars

deliberately ambiguous grammars

Grammar is ambiguous if there exists a word (in the language of grammar) that has more that one derivation path.

$$S \to (S)S \mid \varepsilon$$

example $S \to SS \mid (S) \mid \varepsilon$

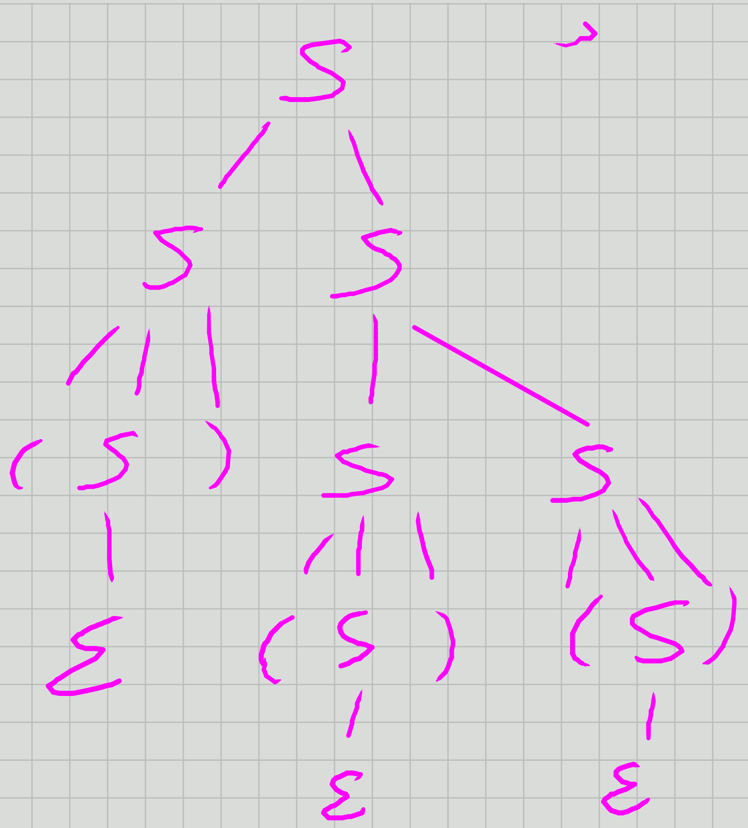
1) $S \Rightarrow \underline{S}S \Rightarrow (S)\underline{S} \Rightarrow (S)SS \Rightarrow \ldots \Rightarrow$

$\Rightarrow ()()()$

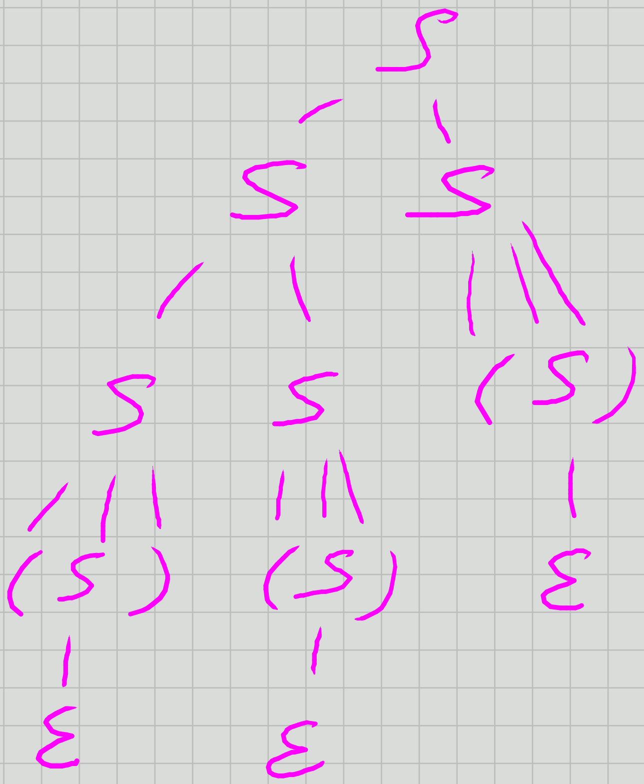
2) $S \Rightarrow S\underline{S} \Rightarrow S(S) \Rightarrow SS(S) \Rightarrow \ldots \Rightarrow$

$\Rightarrow ()()()$

Left Derivation Tree ?



Right Derivation Tree ?



example

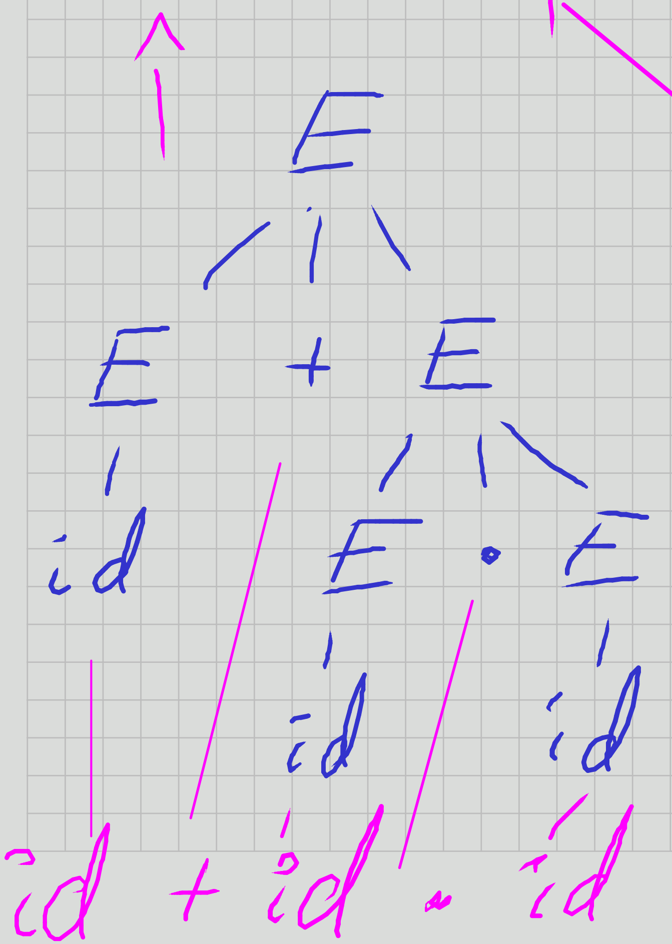
$E \to E + E \mid E \cdot E \mid id$

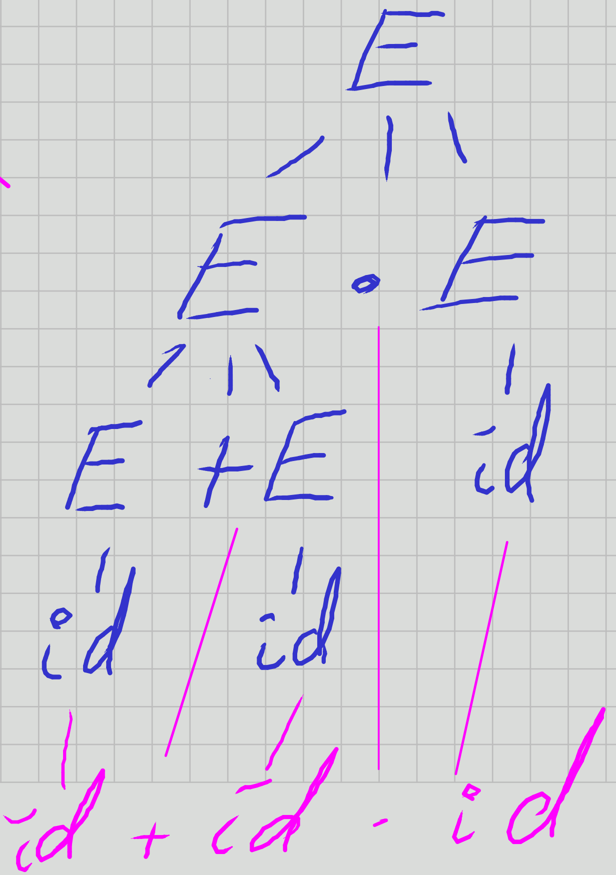
$\underline{id + id \cdot id}$

$x + y \cdot z$

0x1B [illegible]

&x
&y
&z





$2+2 \cdot 2$

$2 \cdot 2 = 4$
$4+2=6$

$2+2=4$
$4 \cdot 2=8$

Uebungsblatt 5-3

Gegeben sei die KFG $G = (N, \Sigma, P, S)$ mit $N = \{S, T\}$, $\Sigma = \{a, b, c, d\}$ und $P = \{S \to aSb | T, T \to cTd | \varepsilon\}$.

(i) Geben Sie die Ableitung für *aacdbb* an.

(ii) Geben Sie den Ableitungsbaum für *ccdd* an.

(iii) Beschreiben Sie die von G erzeugte Sprache formal.

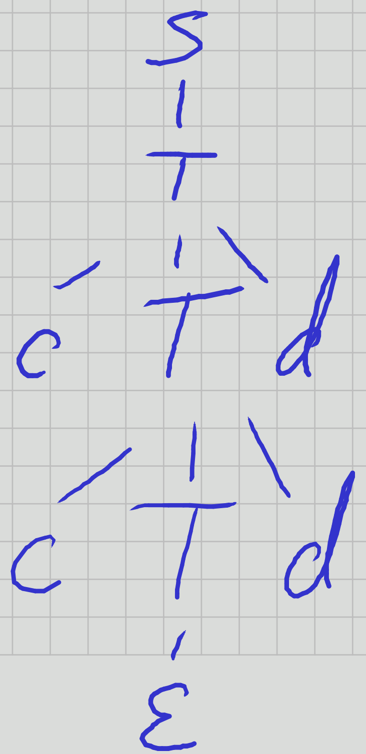
1 $S \to aSb$
2 $S \to T$
3 $T \to cTd$
4 $T \to \varepsilon$

I $S \overset{1}{\Longrightarrow} aSb \overset{1}{\Longrightarrow} aaSbb \overset{2}{\Longrightarrow}$

$\overset{2}{\Longrightarrow} aaTbb \overset{3}{\Longrightarrow} aacTdbb \overset{4}{\Longrightarrow}$

$\overset{4}{\Longrightarrow} aacdbb$

II



III

$S \to aSb$

$S \overset{*}{\Longrightarrow} a^n S b^n$

$T \overset{*}{\Longrightarrow} c^m T d^m$

$S \overset{*}{\Longrightarrow} a^n c^m T d^m b^n$

$L = \{a^n c^m d^m b^n$
$n, m \geq 0\}$

Uebungsblatt 5-4

Aufgabe 4:
Gegeben sei die KFG $G = (N, \Sigma, P, S)$ mit $N = \{S, T, U\}$, $\Sigma = \{0, 1\}$ und $P = \{S \to TU, T \to 0T1|01, U \to 1U0|\varepsilon\}$.

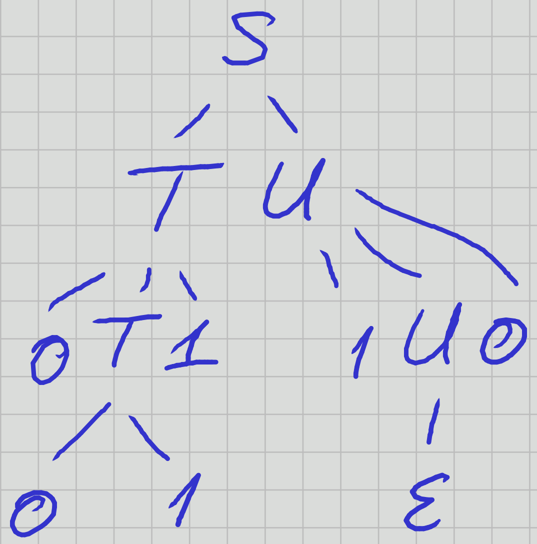
(i) Geben Sie die Ableitung und den Ableitungsbaum für 001110 an.

(ii) Beschreiben Sie die von G erzeugte Sprache formal.

1 $S \to TU$
2 $T \to 0T1$
3 $T \to 01$
4 $U \to 1U0$
5 $U \to \varepsilon$

I $S \overset{1}{\Rightarrow} TU \overset{2}{\Rightarrow} 0T1U \overset{3}{\Rightarrow} 0011U \overset{4}{\Rightarrow}$

$\overset{4}{\Rightarrow} 00111U0 \overset{5}{\Rightarrow} 001110$



$L = \{0^m 1^m 1^k 0^k \mid m \geq 1, k \geq 0\}$

II $T \Rightarrow 0T1 \overset{*}{\Rightarrow} 0^n T 1^n \overset{*}{\Rightarrow} 0^m 1^m$

$U \Rightarrow 1U0 \overset{*}{\Rightarrow} 1^n U 0^n \Rightarrow 1^k 0^k$

$S \Rightarrow TU \overset{*}{\Rightarrow}$

$\overset{*}{\Rightarrow} 0^m 1^m 1^k 0^k$

Aufgabe 5:

Gegeben sei die Sprache $L = \{a^x b^y \mid x, y \geq 0\}$ über dem Alpabet $\Sigma = \{a, b\}$.

(i) Geben Sie eine KFG G an, sodass $L = L(G)$.

(ii) Geben Sie den Ableitungsbaum für *abbb* an.

(iii) Falls ε Teil der Sprache ist, geben Sie Ableitung und Ableitungsbaum a

$T \overset{*}{\Longrightarrow} w$

$U \overset{*}{\Longrightarrow} V$

$S \Longrightarrow TU \overset{*}{\Longrightarrow} wV$

$L = a^n b^m$

$a^n : S \longrightarrow aS \mid \varepsilon$

$b^m : S \longrightarrow bS \mid \varepsilon$

$S \rightarrow aS \mid T / \varepsilon$

$T \rightarrow bT \mid \varepsilon$

$S \longrightarrow S_1 S_2$

$S_1 \longrightarrow aS_1 \mid \varepsilon$

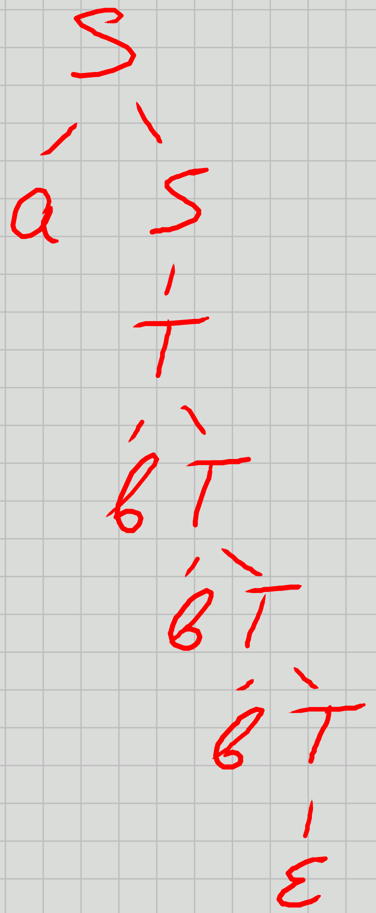
$S_2 \longrightarrow bS_2 \mid \varepsilon$

$S \rightarrow aS \mid Sb \mid \varepsilon$

$S \to aS \mid T \mid \varepsilon$
$T \to bT \mid \varepsilon$

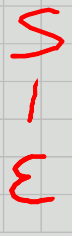
1 $S \to aS$
2 $S \to T$
3 $S \to \varepsilon$
4 $T \to bT$
5 $T \to \varepsilon$

II abbb



III ε

$S \overset{3}{\Rightarrow} \varepsilon$



Aufgabe 6:

Gegeben sei die kontextfreie Sprache $L = \{a^{2n}b^{3n} \mid n \geq 1\}$.

(a) Geben Sie eine KFG G an, sodass $L = L(G)$.

(b) Geben Sie Ableitung und Ableitungsbaum für *aaaabbbbbb* an.

aa bbb

$a^n b^n$ $S \to aSb \mid \varepsilon$

$S \to aaS \mid T$

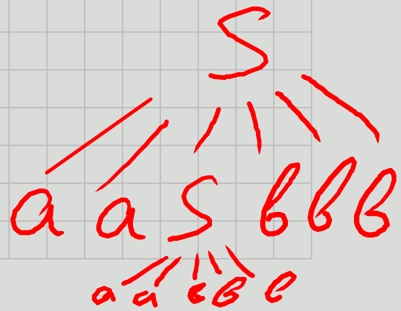
$\boxed{T \to aaTb}$

$T \overset{*}{\Rightarrow} a^{2n} T b^n$

$S \to aaSbbb \mid aabbb$

$S \overset{*}{\Rightarrow} a^{2n} S b^{3n}$

$S \overset{1}{\Rightarrow} aaSbbb \overset{2}{\Rightarrow} aaaabbbbbb$



$L = \{ a^{2n+5} b^{3n} \} \quad n \geq 0$

$$L = \underbrace{a^5}_{L_1} \circ \underbrace{a^{2n} b^{3n}}_{L_2}$$

$L_2 : \; S_2 \to aaS_2bbb \mid \varepsilon$

$L_1 : \; S_1 \to aaaaa$

$S \to S_1 S_2$

$S \to aaSbbb \mid aaaaa$

DFA

REG $\longleftrightarrow$ NFA

Pumping Lemma

$xyz \in L \Longleftrightarrow xy^iz \in L$

$|xyz| \geqslant p$

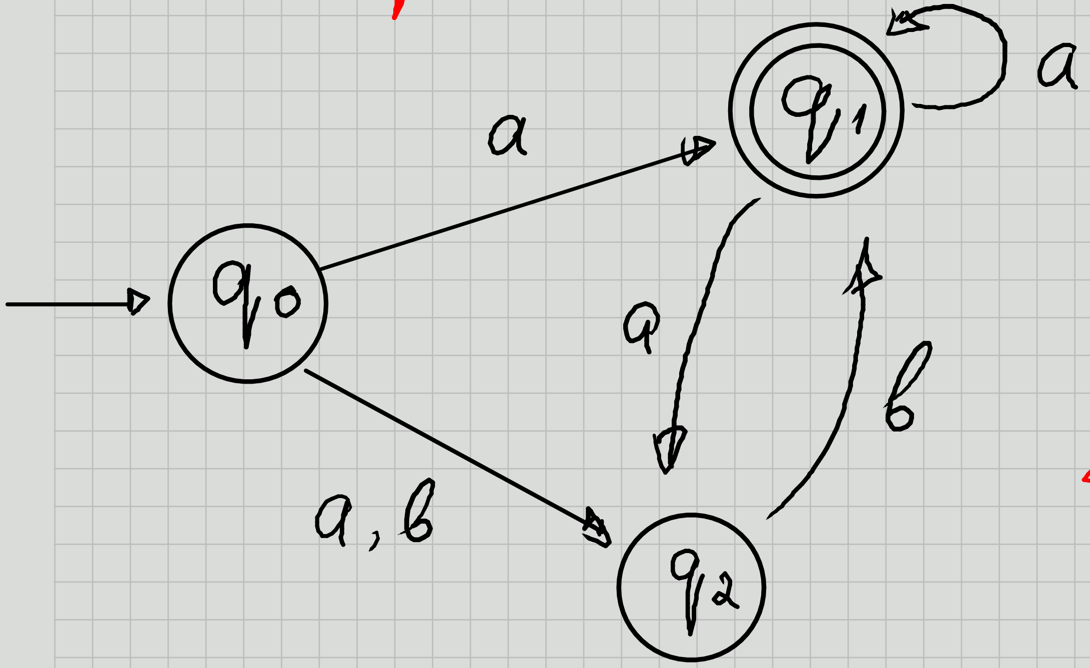
Классы эквивалентности Майхилла-Нероуда

a^nb^n ww ww^R $|w|_a = |w|_b$

Σ^*/a^nb^n Σ^*/ww^R

$NFA \longrightarrow G\,(N, \Sigma, S, P)$

$(Q, \Sigma, \delta, q_0, F)$



1. $N = Q$
2. $\Sigma = \Sigma$
3. $S = q_0$
4. $q_i \xrightarrow{\alpha} q_j \quad \alpha \in \Sigma \cup \{\varepsilon\}$

$Q_i \longrightarrow \alpha Q_j$

$q_f \in F \Rightarrow Q_f \rightarrow \varepsilon$

$N = \{Q_0, Q_1, Q_2\}, S = Q_0$

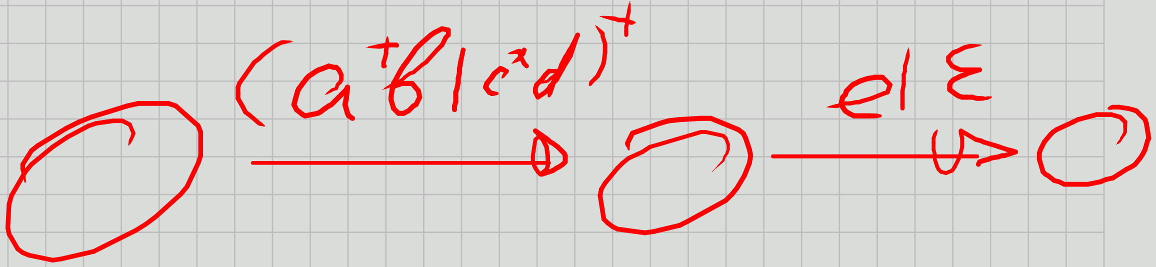
$Q_0 \rightarrow aQ_1 \mid aQ_2 \mid bQ_2$

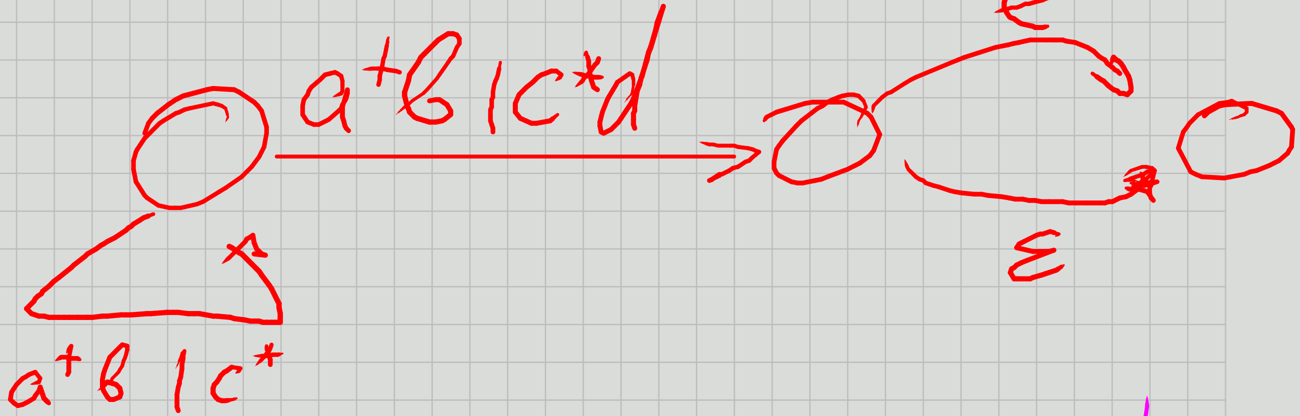
$Q_1 \rightarrow aQ_1 \mid aQ_2 \mid \varepsilon$

$Q_2 \rightarrow bQ_1$

$$\begin{tikzcd}
RE \arrow[rr] \arrow[dr] & & G \\
& NFA \arrow[ur] &
\end{tikzcd}$$

$$(a^+b|c^*d)^+(e|\varepsilon)$$





$S \rightarrow a^+b A_1$

$S \rightarrow aS \mid abA_1$

$S \rightarrow (a^+b \mid c^*d)S \mid \ldots$

$S \rightarrow (a^+b \mid c^*d)^+ A_1$

$A_1 \rightarrow e \mid \varepsilon$

$S \rightarrow a^+b A_1 \mid c^*d A_1$

$G \longrightarrow NFA$

$A \longrightarrow \alpha B \mid \varepsilon$

$\alpha \in \Sigma$

1) $A \longrightarrow w \quad , \quad w \in \Sigma^*$

2) $A \longrightarrow x B \quad , \quad x \in \Sigma^*$

If our rules are complex* then we rewrite our Grammar and simplify it.

① $A \longrightarrow w \quad \Longleftrightarrow \quad \begin{array}{l} R \longrightarrow \varepsilon \\ A \longrightarrow wR \end{array}$

② $A \longrightarrow x_1 x_2 x_3 \dots x_n B \quad \Longleftrightarrow \quad \begin{array}{l} A \longrightarrow x_1 C \\ C \longrightarrow x_2 \underbrace{[x_3 \dots x_n B]}_{D} \end{array}$

$A \longrightarrow x_1 \underbrace{[x_2 x_3 \dots x_n B]}_{C}$

example G → NFA

$S \to abbS \mid T$

$T \to bba \mid T$

$S \to abbS$

$S \to T$

$T \to bba$

$T \to T$

$\longrightarrow$

$R \to \varepsilon$

$T \to bbaR$

$S \to abbS$

$S \to \varepsilon T$

$T \to \varepsilon T$

$\longrightarrow$

$R \to \varepsilon$

$S \to \varepsilon T$

$T \to \varepsilon T$

$T \to b\,\underbrace{b\,\underbrace{aR}_{A}}_{B}$

$S \to a\,\underbrace{b\,\underbrace{bS}_{C}}_{D}$

$T \to bB$

$B \to bA$

$A \to aR$

$S \to aD$

$D \to bC \qquad C \to bS$

$R \to \varepsilon$ ✓

$S \to \varepsilon T$ ✓

$T \to \varepsilon T$ ✓

$T \to bB$ ✓

$B \to bA$ ✓

$A \to aR$ ✓

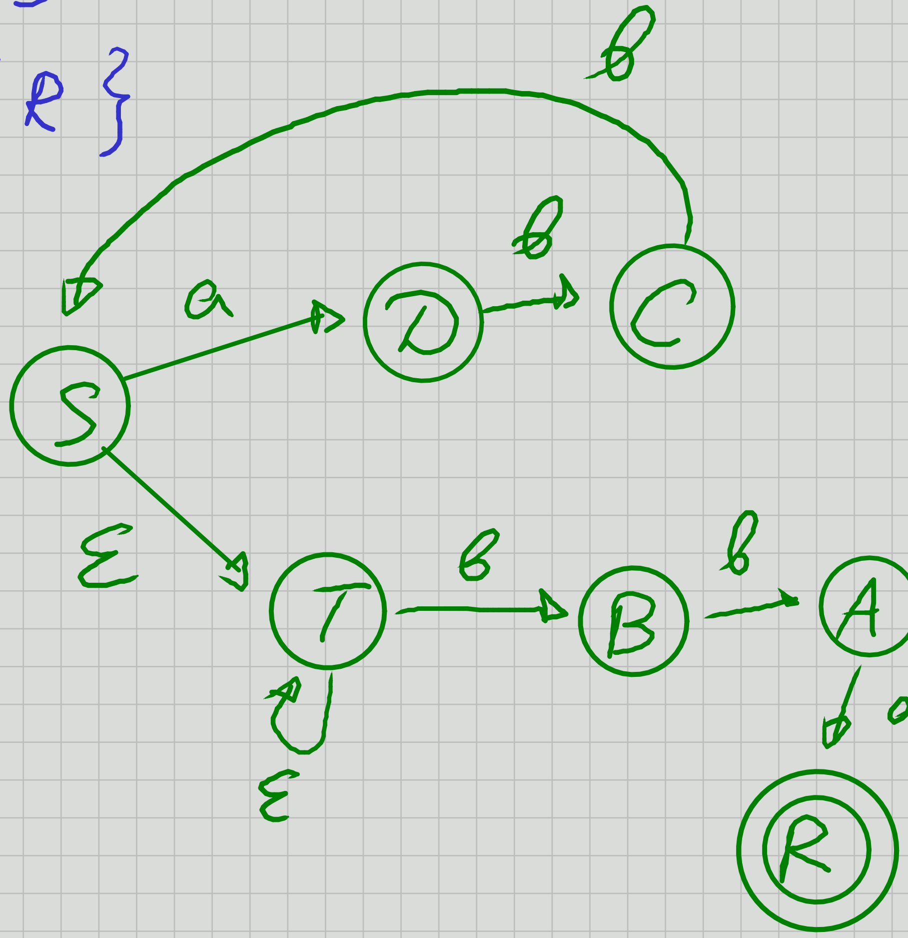
$S \to aD$ ✓

$D \to bC$ ✓ $C \to bS$ ✓

$Q = \{S, R, T, A, B, C, D\}$

$q_0 = S$

$F = \{R\}$



Uebungsblatt - 7

Aufgabe 3:
Sei r ein regulärer Ausdruck der Form $(a^+b|c^*d)^+(e|\varepsilon)$.

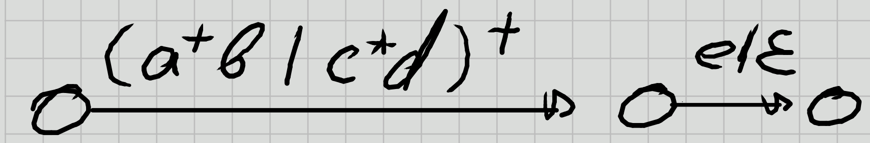
(i) Konstruieren Sie einen Automaten A, sodass $L(A) = L(r)$.
Ist der Automat deterministisch?

(ii) Spezifizieren Sie eine reguläre Grammatik G, sodass $L(G) = L(r)$.

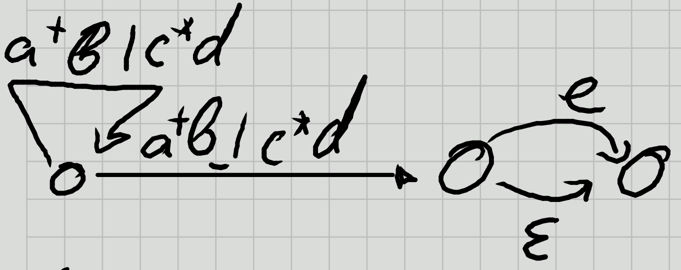
RE → NFA → G

$S \to aA \mid B$
$A \to aA \mid bD \mid bS$
$B \to cB \mid dD \mid dS$
$D \to eE \mid E$
$E \to \varepsilon$

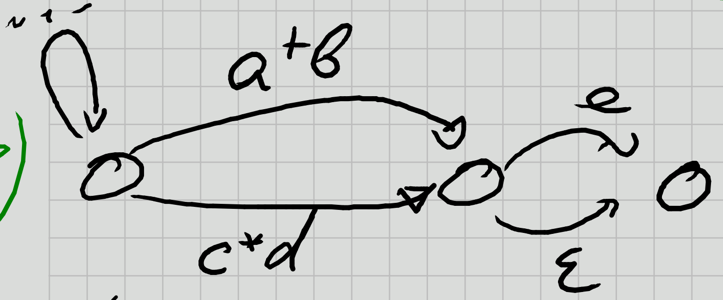
1)



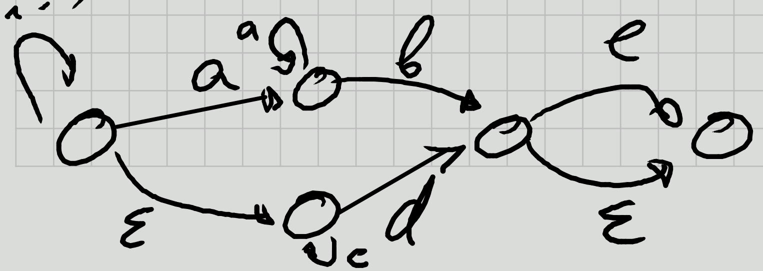
2)

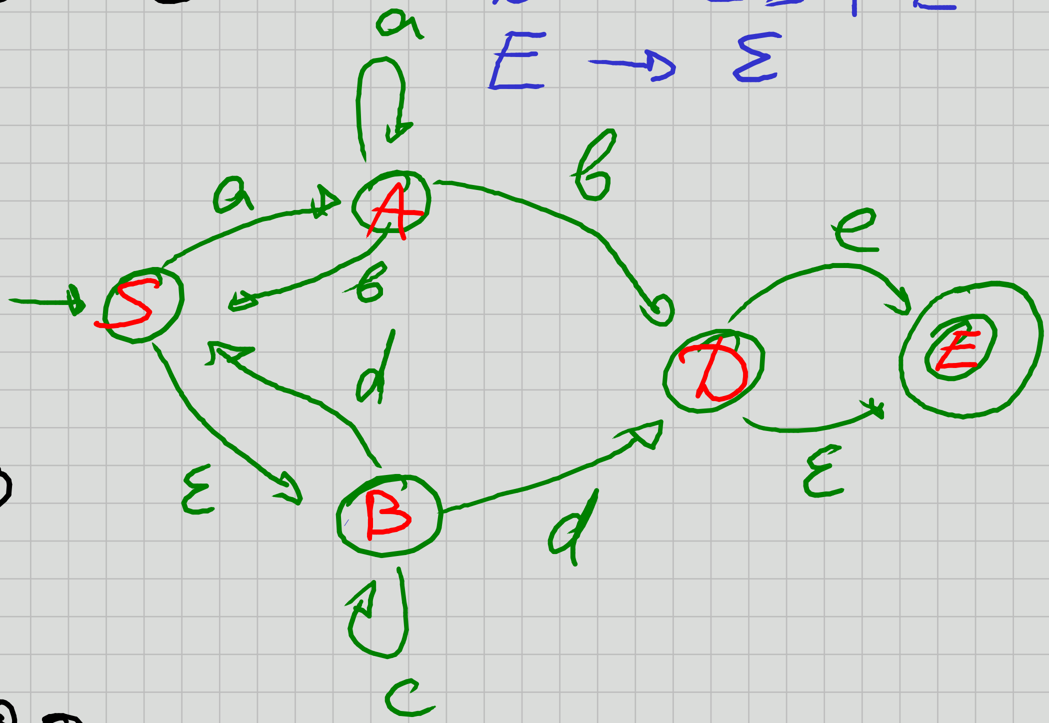


3)



4)





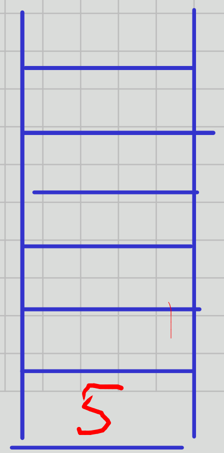
Data Structures - as a way to organize data/memory

1) Array / Vector — linear memory layout

2) Set / Map — Hashing

3) Stack — LIFO - Last In First Out

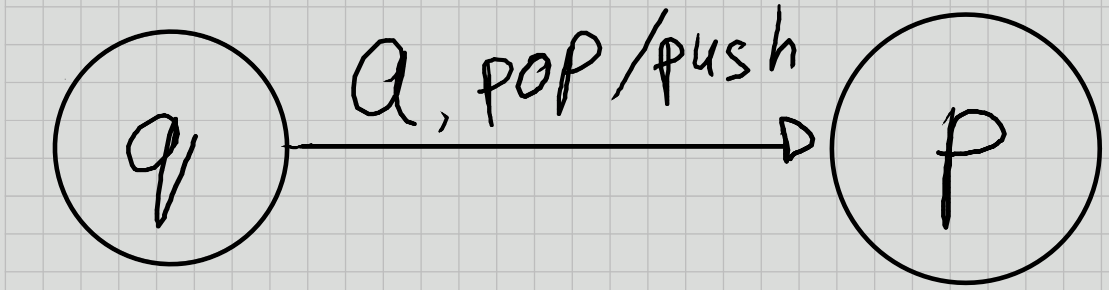
4) Queue — FIFO - First In First Out



push -

pop -

Push Down Automatons = Finite Automatons + Stack



$$M = (Q, q_0, \Sigma, \Gamma, Z_0, \delta, F)$$

stack alphabet

null protector
маркер дна

sometimes we don't have it

Configuration: $C \in Q \times \Sigma^* \times \Gamma^*$

part of input

$x_1 x_2 \ldots x_n Z_0 \in \Gamma^*$

(q_0, w, Z_0)

$$\delta : Q \times (\Sigma \cup \{\varepsilon\}) \times \Gamma \to 2^{Q \times \Gamma^*}$$

we start in some state

we read next symbol of the word (or we don't)

we look at the top of the stack

we move to a new state

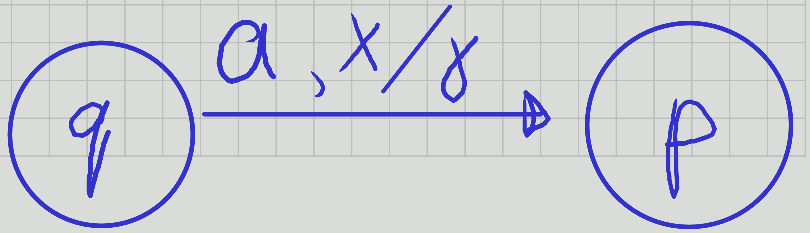
we can put someting on the stack (zero or several symbols)

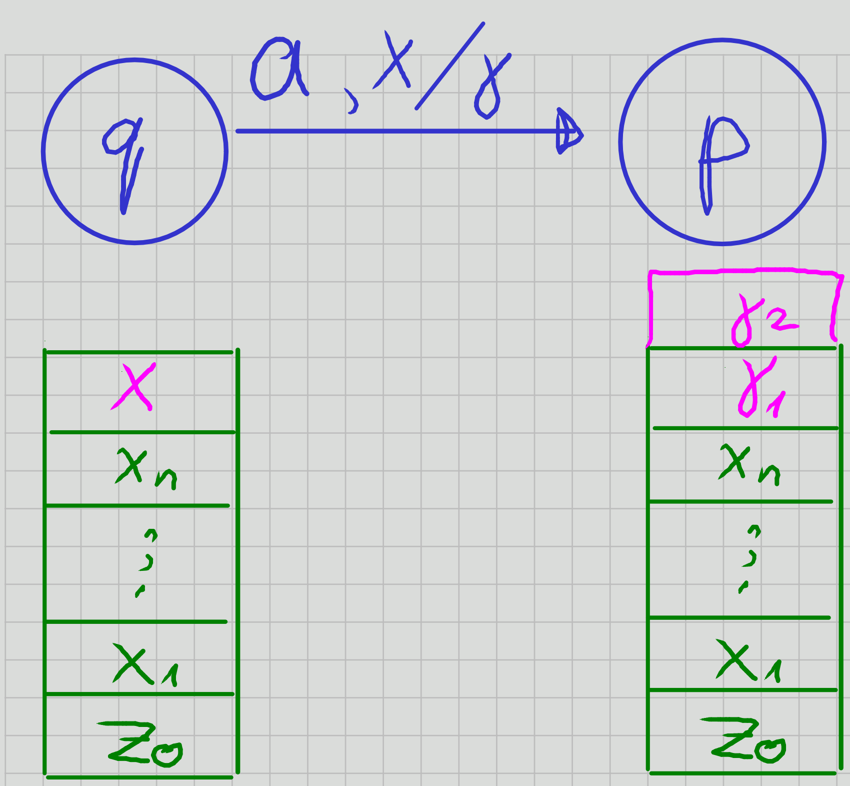
2^M set of all possible subsets

basically implies non-determinism for transitions

$$\delta(q, a, x) \vdash (p, \gamma)$$

$$(p, \gamma) \in \delta(q, a, x)$$





$γ = γ_2 γ_1$

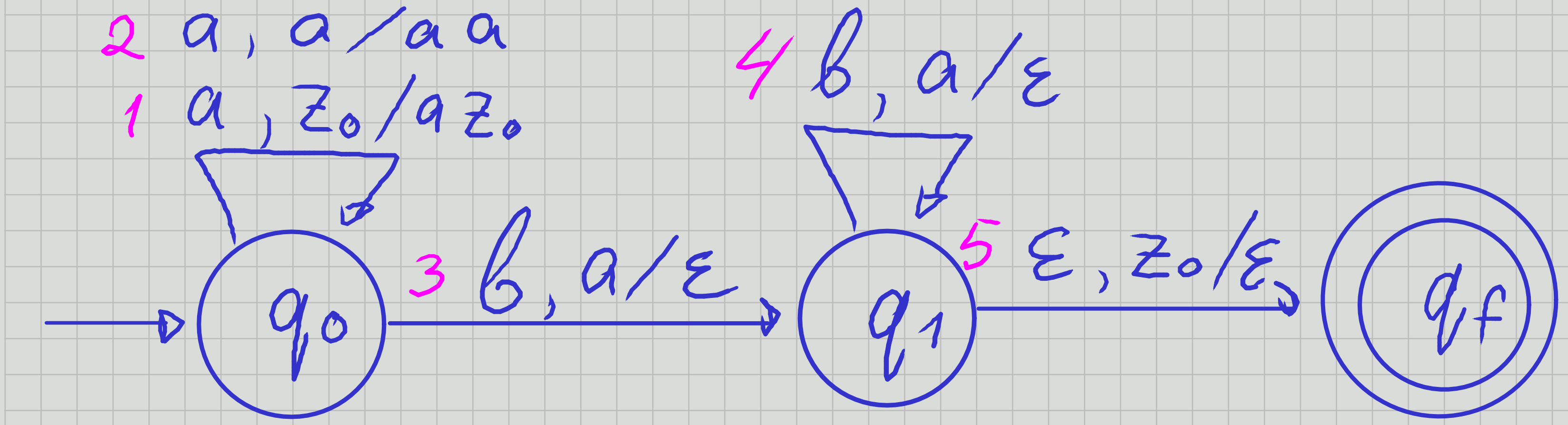
$ε / γ$

$X / ε$

configurations

$$(\underline{q}, \underline{a}u, \underline{X}X_n \dots X_1 Z_0) \vdash (\underline{p}, u, \underline{γ}X_n \dots X_1 Z_0)$$

$$L = \{ a^n b^n \}$$



aabb

configurations

$$(q_0, aabb, Z_0) \overset{1}{\vdash} (q_0, abb, aZ_0) \overset{2}{\vdash}$$

$$\overset{2}{\vdash} (q_0, bb, aaZ_0) \overset{3}{\vdash} (q_1, b, aZ_0) \overset{4}{\vdash}$$

$$\overset{4}{\vdash} (q_1, \varepsilon, Z_0) \overset{5}{\vdash} (q_f, \varepsilon, \varepsilon)$$

Automaton accepts word by accepting state

$$w \in L(M) \Longleftrightarrow (q_0, w, Z_0) \overset{*}{\vdash} (q_f, \varepsilon, \alpha)$$

$q_f \in F$

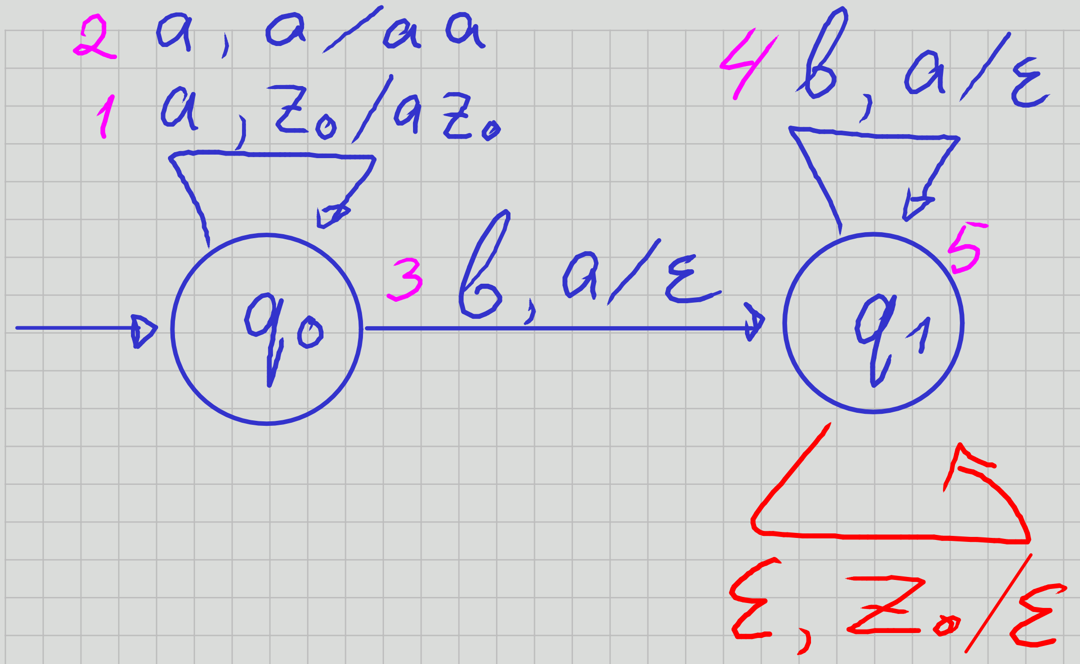
$\alpha \in \Gamma^*$

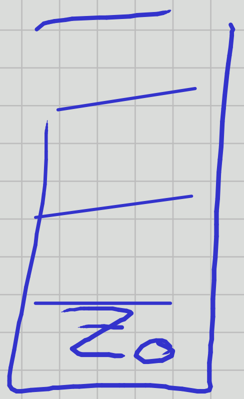
Word belongs to Language (of the automaton) if and only if there exists a computation path from initial configuration to accepting configuration

Automaton accepts by empty stack

$$w \in L(M) \Longleftrightarrow (q_0, w, Z_0) \overset{*}{\vdash} (p, \varepsilon, \varepsilon)$$

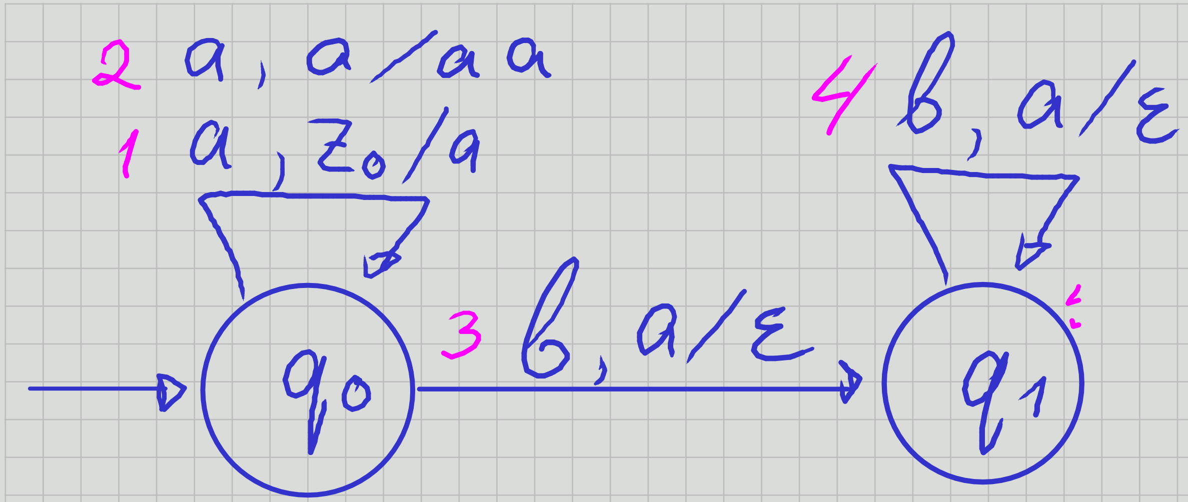
$p \in Q$





$(q_0, aabb, Z_0) \vdash (q_0, abb, aZ_0) \vdash (q_0, bb, aaZ_0) \vdash$

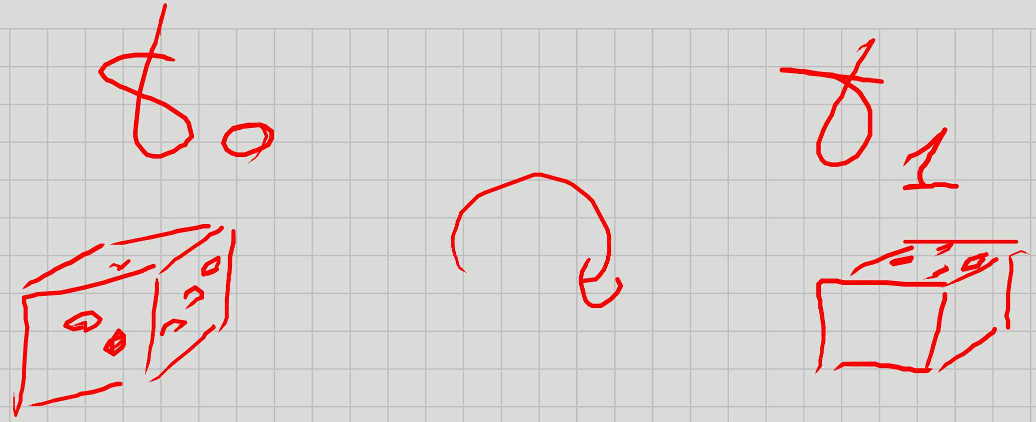
$\vdash (q_1, b, aZ_0) \vdash (q_1, \varepsilon, Z_0) \vdash (q_1, \varepsilon, \varepsilon)$



$$(q_0, aabb, Z_0) \vdash (q_0, abb, a) \vdash (q_0, bb, aa) \vdash$$

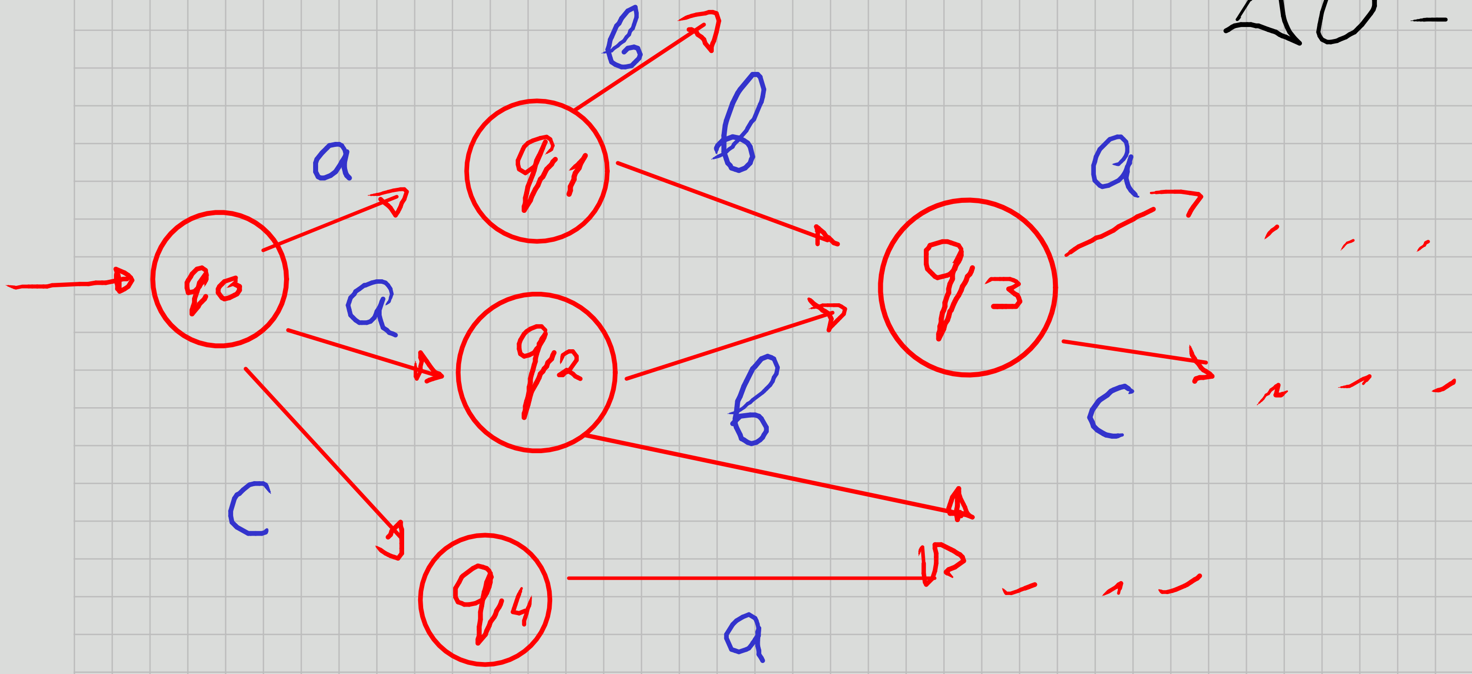
$$\vdash (q_1, b, a) \vdash (q_1, \varepsilon, \varepsilon)$$

Determinism



Finite automaton

$\Delta t = 1s$



"a b c ...."

$q_0 \longrightarrow q_1 \longrightarrow q_3 \longrightarrow \ldots$

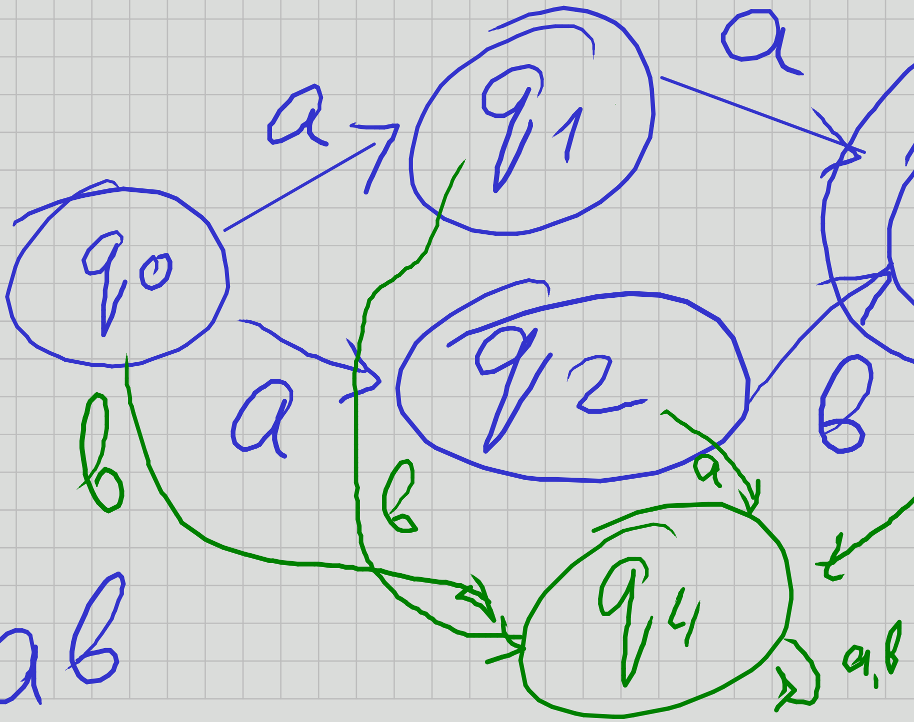
$0 \longrightarrow 2 \longrightarrow 3$

Недетерминизм: это когда существуют слова у которых есть альтернативные пути вычисления

Детерминизм: это когда для любого слова существует один единственный путь вычисления

For each (automaton state, alpahebet symbol): are transitions unique?

$\{a, b\}$



$w = ab$

$q_0 \to q_1$ X $\quad q_0 - q_1 - q_4$

$q_0 \to q_2 \to q_3$ V

all computation paths for word "ab"?

Determinism in PDA ?

1. $\forall a \in \Sigma, \forall x \in \Gamma, \forall q \in Q:$

$$|\delta(q, a, x)| \leq 1$$

2. $\delta(q, \varepsilon, x) = \emptyset \implies$

$$\begin{cases} |\delta(q, \varepsilon, x)| = 1 \\ \forall a \in \Sigma \quad \delta(q, a, x) = \emptyset \end{cases}$$

$NFA \longrightarrow DFA$

$L(NFA) = L(DFA)$

PDA:

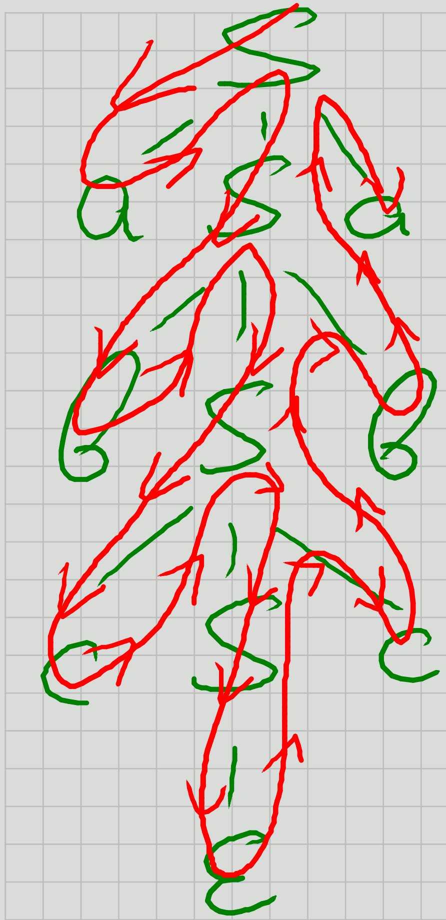
$L(n\text{-}PDA) \neq L(d\text{-}PDA)$

$L = \{ ww^R \mid w \in \Sigma^* \}$ $\qquad \underbrace{abc}_{w}\underbrace{cba}_{w^R}$

$S \rightarrow aSa \mid bSb \mid cSc \mid \varepsilon$

$S \rightarrow \alpha S \alpha \mid \varepsilon \; , \; \alpha \in \Sigma$

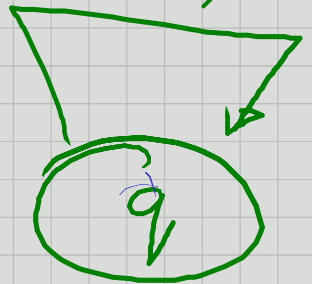
$S \Rightarrow a[S]a \Rightarrow a[bSb]a \Rightarrow abcScba \Rightarrow abccba$



$S \rightarrow aSa$

$a, a/\varepsilon$ — when we read the last "a"

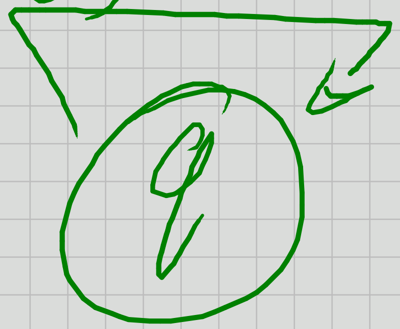
$a, X/aX$ — when we read the first "a"



$S \rightarrow bSb$

$b, b/\varepsilon$

$b, X/bX$



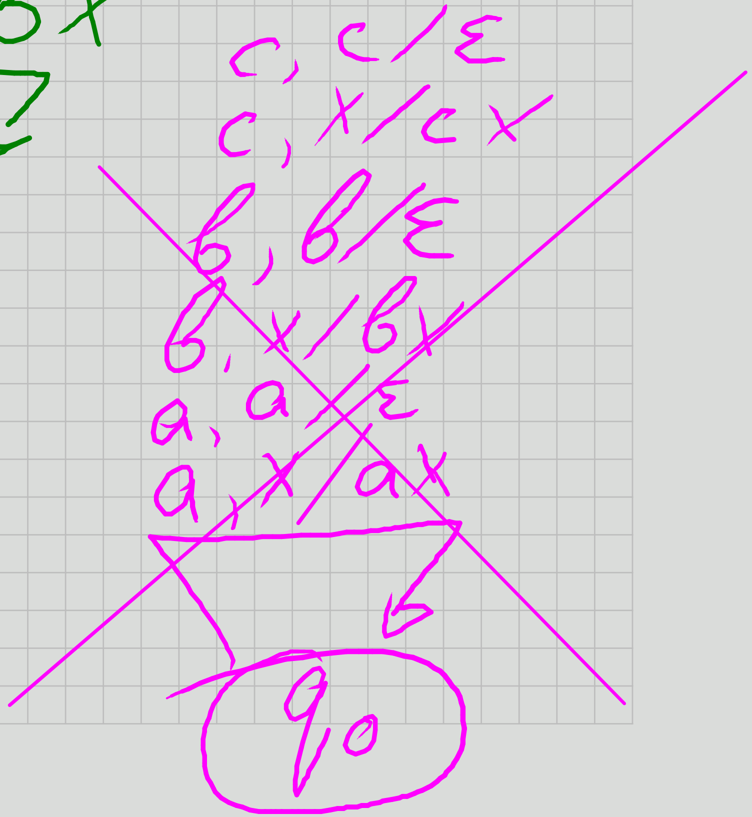
$S \rightarrow cSc$

$c, c/\varepsilon$

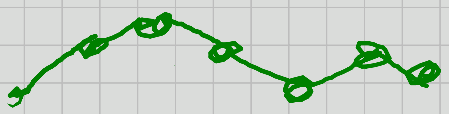
$c, X/cX$

aabbcc

abccba

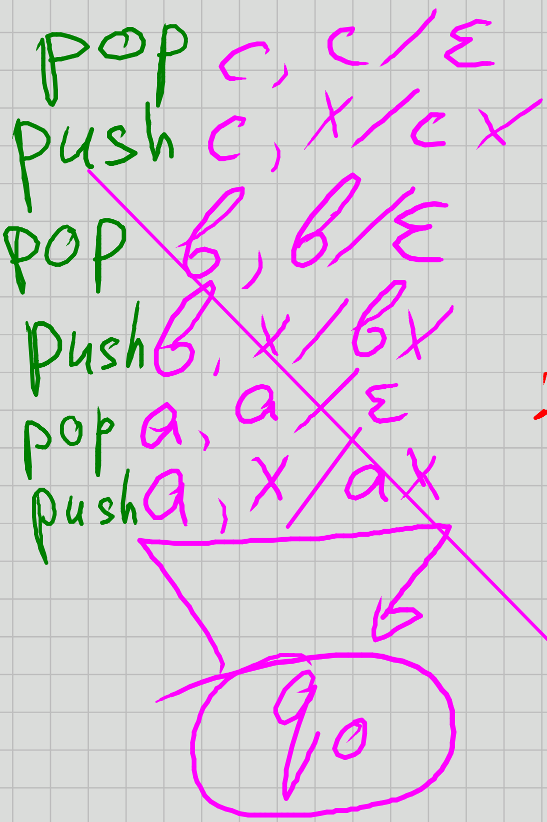


abbacc



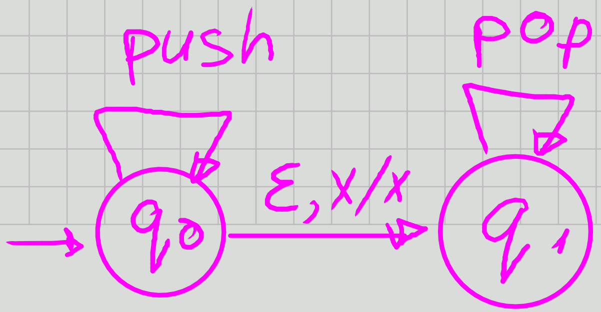
pop

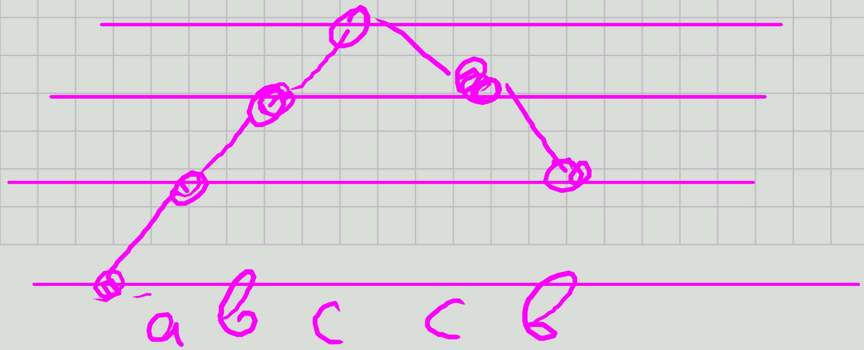
push



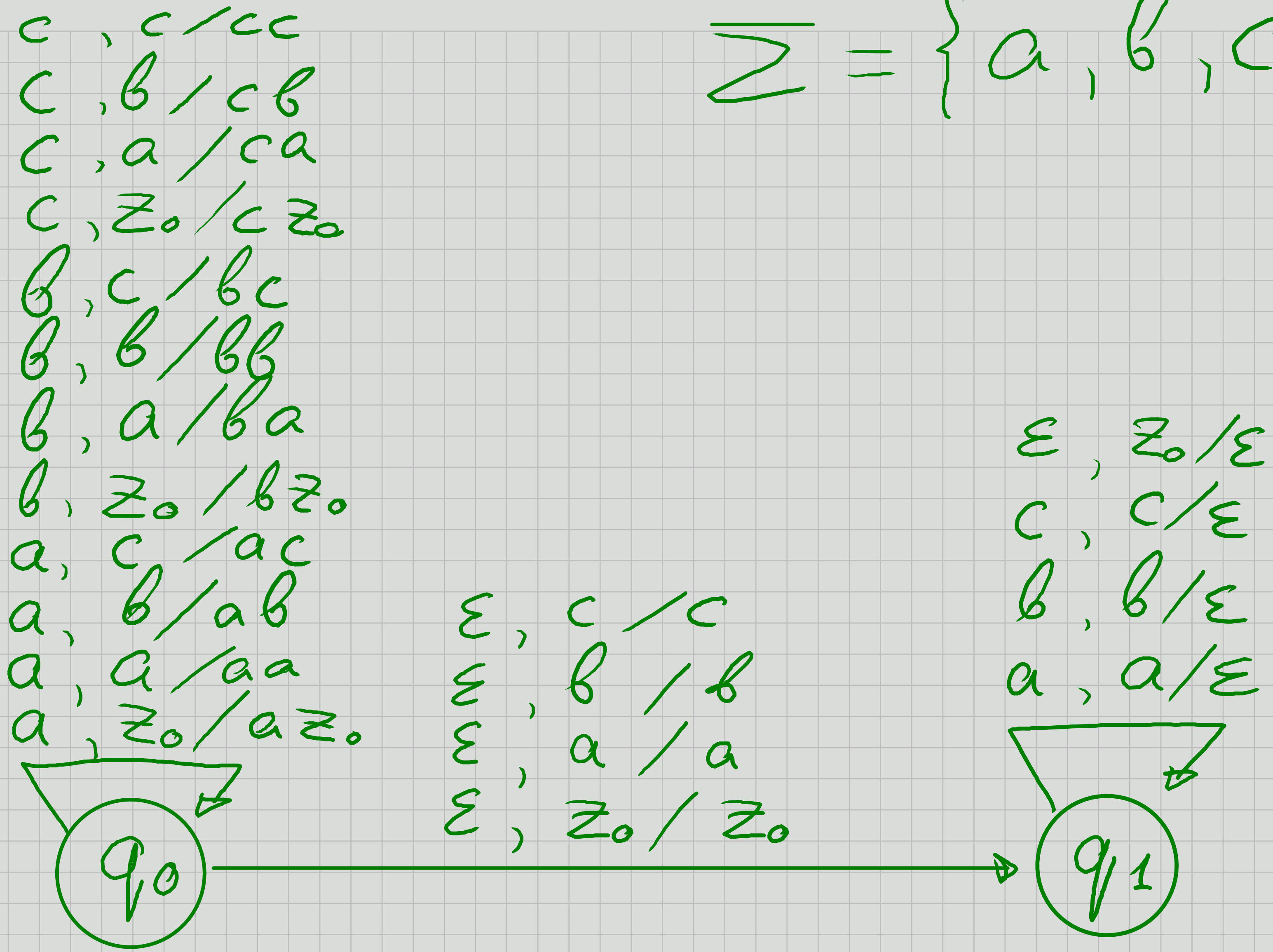
1. aabbcc
2. abccba

Stack level: does it change monotonically?



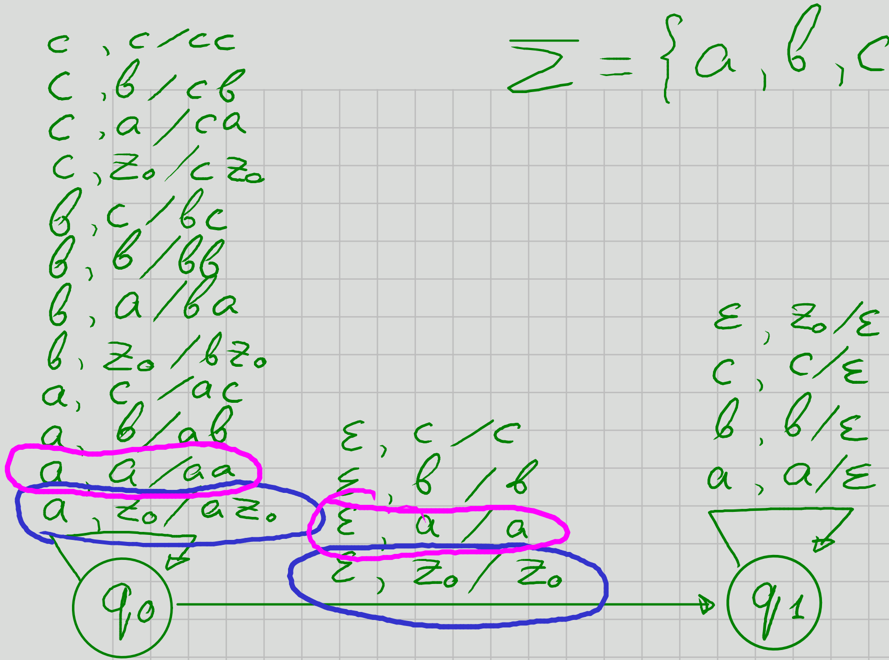


$\Sigma = \{a, b, c\}$



$\sigma, X / \sigma X \quad \sigma \in \Sigma \quad X \in \Gamma$

$\Sigma = \{a, b, c\}$



$(q_0, aabbaa, z_0) \vdash (q_0, abbaa, az_0) \vdash (q_0, bbaa, aaz_0)$

$\vdash (q_1, aabbaa, z_0)$

$\vdash (q_1, abbaa, az_0)$

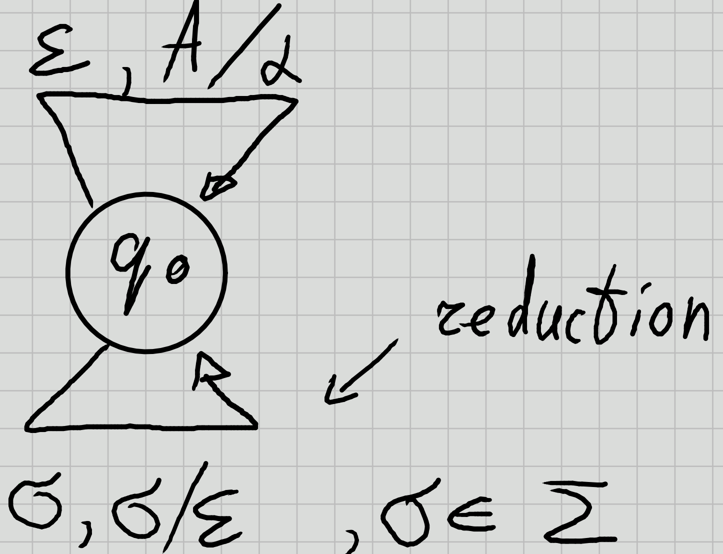
$\vdash (q_1, bbaa, z_0)$

Determinism: lets take a look at following rules: $(q_0, a, z_0) \vdash (q_0, az_0)$ and $(q_0, \varepsilon, z_0) \vdash (q_1, z_0)$

Since we have epsilon-rule, determinism would require that "z0" doesn't appear in the configurations corresponding to state q0
But we have a rule $(q_0, a, z_0) \vdash (q_0, az_0)$
which is a contradiction.

Theorem: $G \in \underset{\text{CF}}{\text{Type } 2} \Rightarrow \exists \underset{\text{PDA}}{M}$ ↳ $L(M)=L(G)$

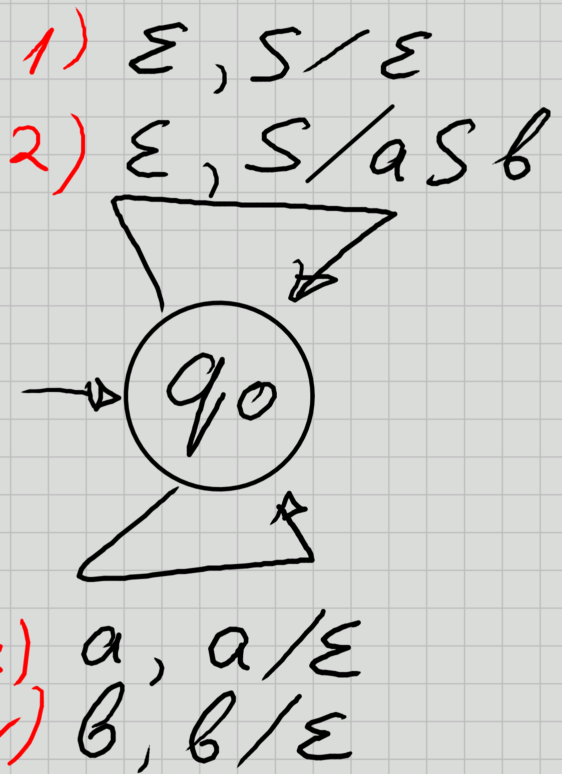
- M accepts by empty stack
- $Z_0 = S$
- $Q = \{q_0\}$
- $A \to \alpha \in P$



example: $L = \{a^n b^n\}$

$G: \quad S \rightarrow aSb$
$\quad\quad\; S \rightarrow \varepsilon$

$Z_0 = S$



w: aaabbb

$(q_0, aaabbb, S) \overset{2}{\vdash} (q_0, aaabbb, aSb) \overset{3}{\vdash} (q_0, aabbb, Sb) \overset{2}{\vdash}$

$\overset{2}{\vdash} (q_0, aabbb, aSbb) \overset{3}{\vdash} (q_0, abbb, Sbb) \vdash$

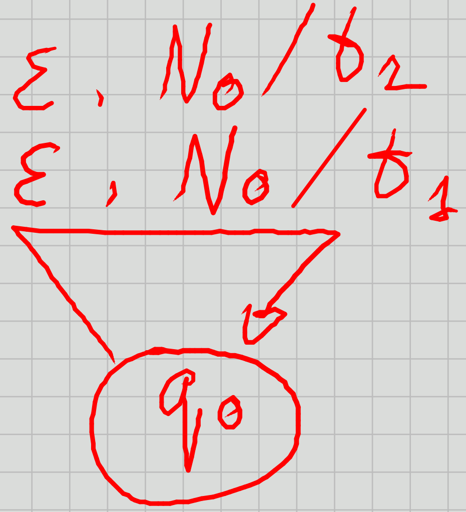
$\vdash (q_0, abbb, aSbbb) \vdash (q_0, bbb, Sbbb) \vdash (q_0, bbb, bbb)$

$\vdash (q, bb, bb) \vdash (q_0, b, b) \vdash (q_0, \varepsilon, \varepsilon)$

$$\exists\ N_0 \in N$$

$$N_0 \rightarrow t_1 \mid t_2$$

$$N_0 \rightarrow t_1$$



If our Language consists of more than one word, there should be a "fork" in Grammar's rules.

$L = \{ w w^R \mid w \in \Sigma^* \}$

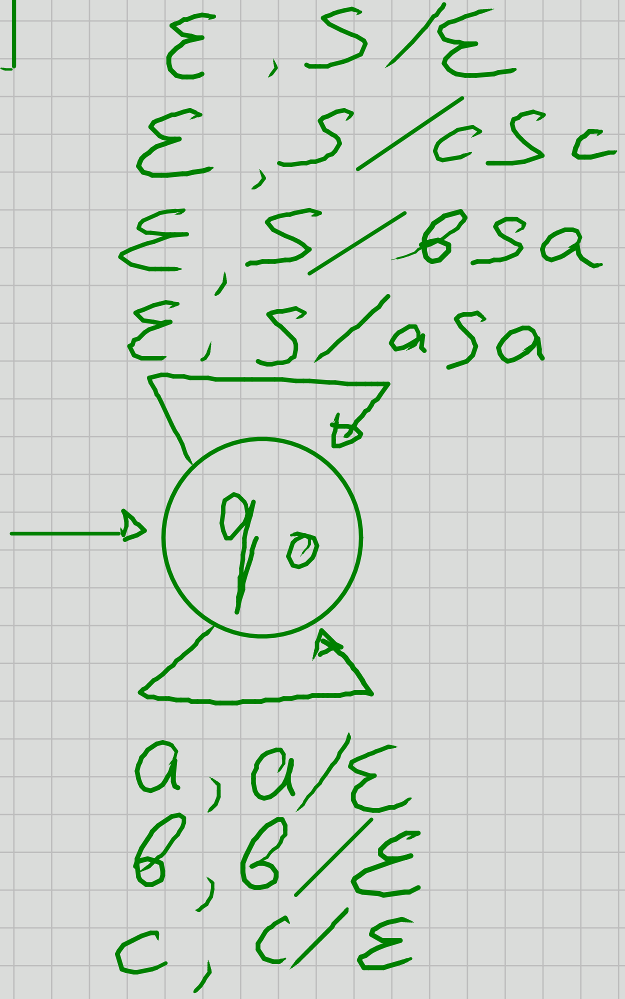
G: $S \rightarrow aSa$

$S \rightarrow bSb$

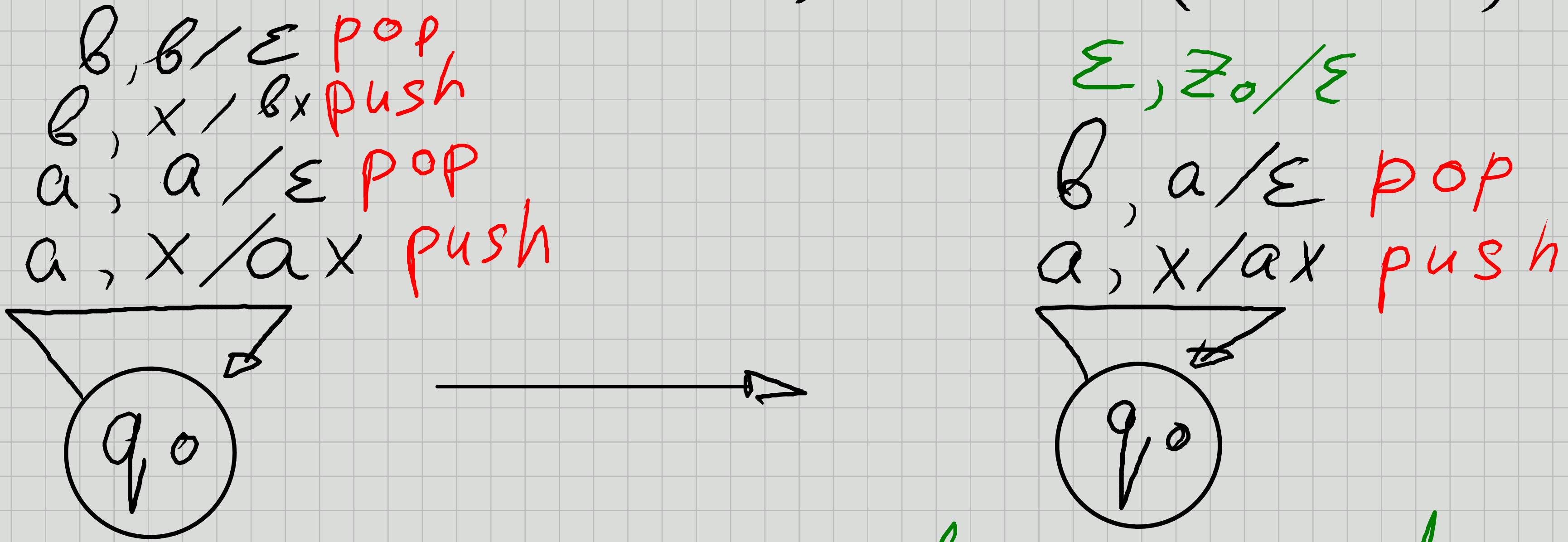
$S \rightarrow cSc$

$S \rightarrow \varepsilon$

M:



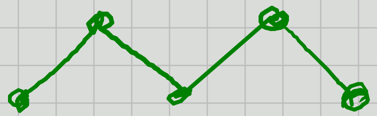
Hint: $M(ww^R) \sim M(a^n b^n)$



examples of words:

aaabbb

abab



$[abab] \circ [aaabbb]$

$a^n b^n \mid n = 1$

$(q_0, \boxed{ab}ab, \enclose{circle}{Z_0}) \vdash (q_0, bab, aZ_0) \vdash (q_0, \downarrow ab, \enclose{circle}{Z_0}) \vdash$
$\vdash (q_0, b, aZ_0) \vdash (q_0, \varepsilon, Z_0)$

1) $\left.\begin{array}{l} u \in L(M) \\ v \in L(M) \end{array}\right| \Longrightarrow uv,\ vu \in L(M)$

2) $u, v \sim a^n b^n$

Grammar ?

2) $u, v \sim a^n b^n$ $\boxed{S \to aSb \mid \varepsilon}$

1) $u, v \in L \Longrightarrow \begin{array}{l} uv \\ vu \end{array} \in L$ $\boxed{S \to SS}$

$\boxed{S \to SS \mid aSb \mid \varepsilon}$ Dyck languag

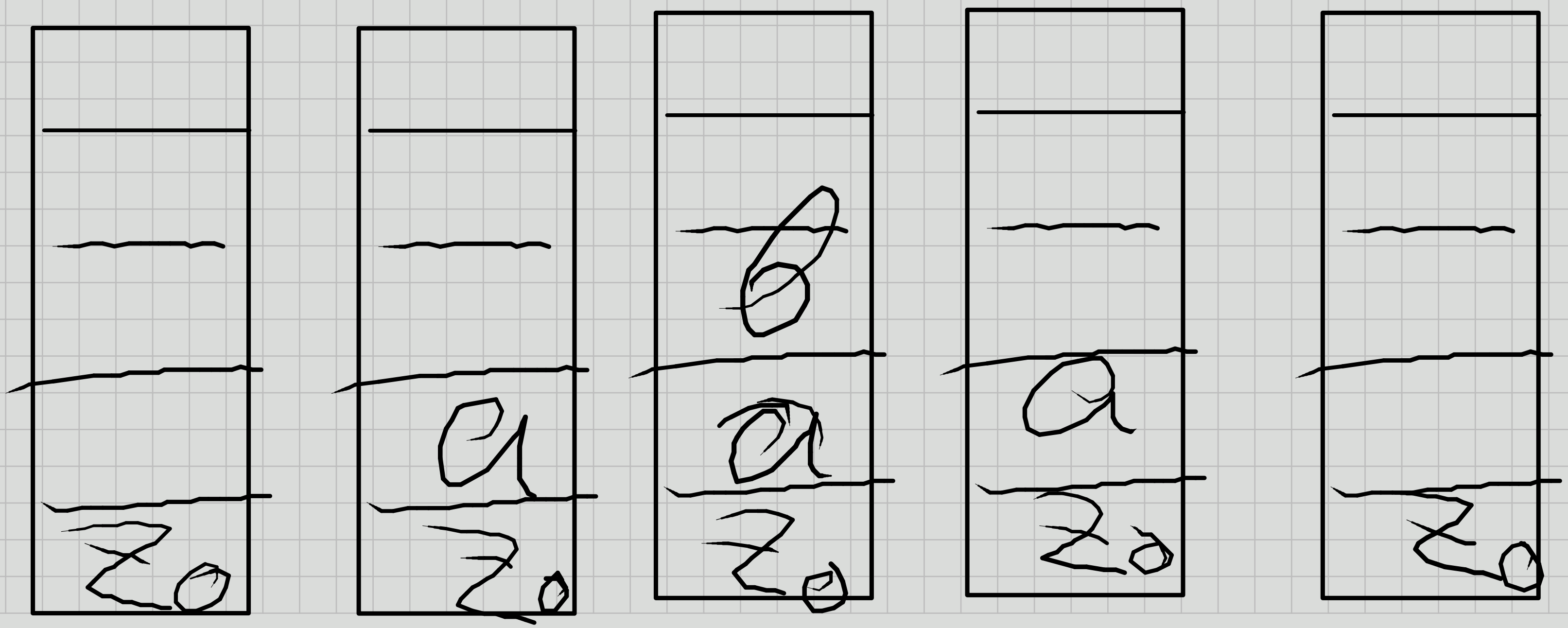
1) $u, v \in L \Longrightarrow u \circ v \in L$

2) $ww^R \in L$

ww^R

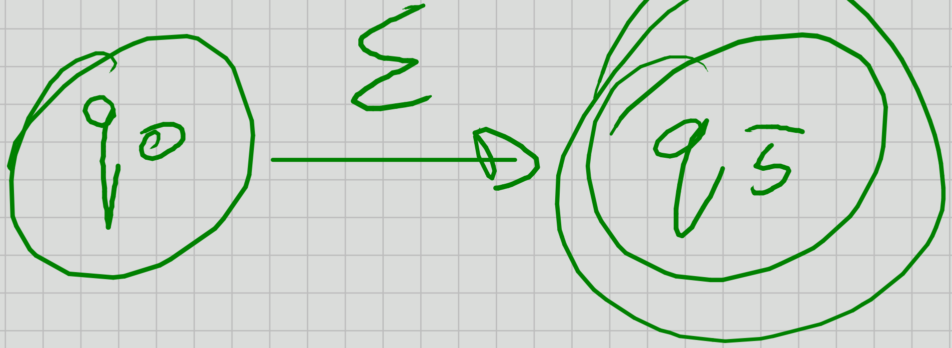
abba

[aabb]



$G: Q_0 \to Q_5$

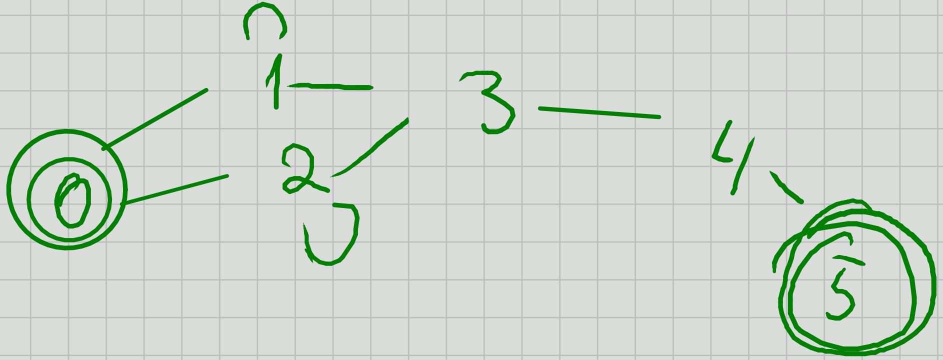
$Q_5 \to \varepsilon$



$Q_3 \to eQ_4$

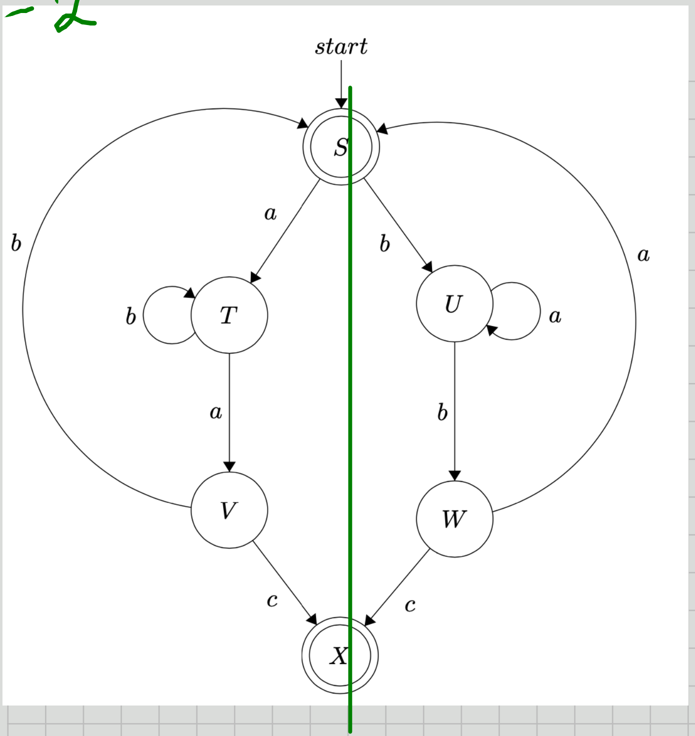
$Q_4 \to fQ_5$

$Q_5 \to \varepsilon$



Lets rewrite our grammar in more convinient (equivalent) form

7-2



$S \rightarrow bU \mid aT \mid \varepsilon$

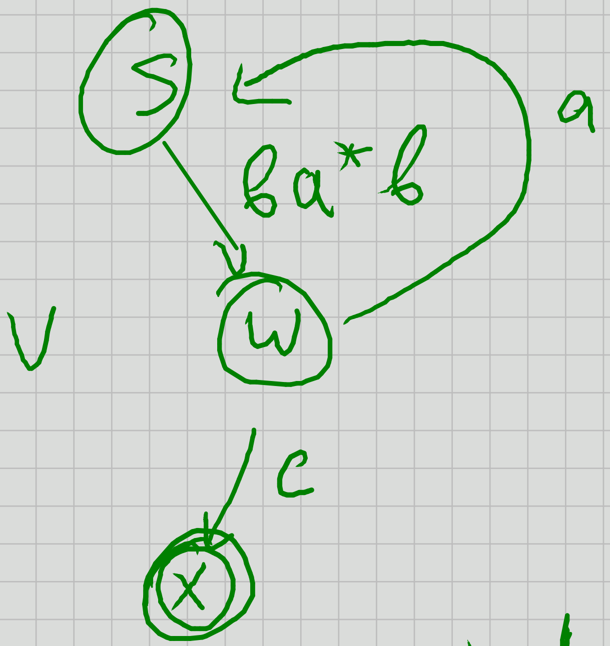
$U \rightarrow aU \mid bW$

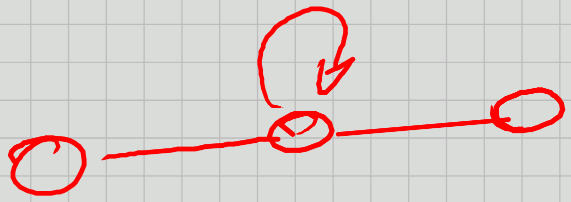
$T \rightarrow bT \mid aV$

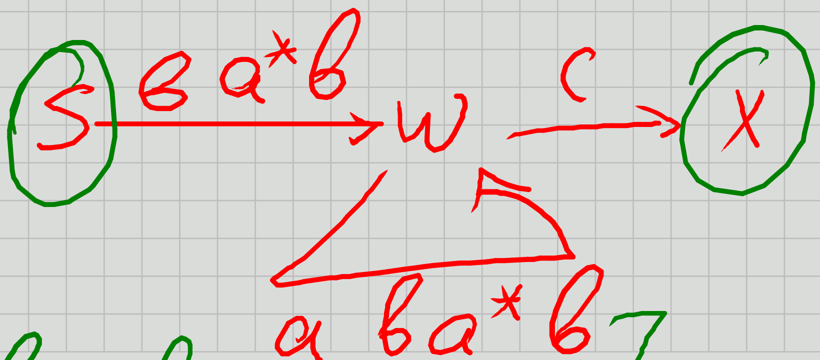
$W \rightarrow aS \mid cX$

$V \rightarrow bS \mid cX$

$X \rightarrow \varepsilon$







$[ab^*a(bab^*a)^* \mid ba^*b(aba^*b)^*]c$

$\{a, b, c\}$ $\quad$ $aa[baa[baaaCaaa]aa]aa$

$\{a, \bullet, =\}$ $\quad$ $\bullet \quad \bullet \quad =$

$a^n = a^n$

$a^n b^n \qquad G: S \to aSb \mid \varepsilon$

$a^n = b^n \qquad G: S \to aSb \mid =$

$S \to aSb \to aaSbb \to aaaSbbb \to aaa=bbb$

$a^n = a^n \qquad G: S \to aSa \mid =$

$\underbrace{b}_{L_1}\underbrace{a^n = a^n}_{L_2} \qquad G: S \to S_1 S_2$

$S_1 \to b$

$S_2 \to aS_2a \mid =$

$S \to bT$

$T \to aTa \mid =$

$L = L_1 \circ L_2 \hookrightarrow G = S \to S_1 S_2$

$a^n \frown a^n$ $\qquad$ $a^n . a^k = a^k a^n$

$. a^n = a^n$

$G: \quad S \rightarrow aSa \mid \varepsilon$

$S \rightarrow .T \qquad T \rightarrow aTa \mid =$

$$\left[\begin{array}{l} S \rightarrow aSa \mid W \\ W \rightarrow .T \\ T \rightarrow aTa \mid = \end{array}\right.$$

$$\begin{array}{l} S \rightarrow aSa \mid .T \\ T \rightarrow aTa \mid = \end{array}$$

$a^n \frown a^n$ $\qquad$ $a^n . a^m . a^k = a^k a^m a^n$

$. a^m . a^k = a^k a^m$

$$\left[\begin{array}{l} S \rightarrow aSa \mid .T \\ T \rightarrow aTa \mid .U \\ U \rightarrow aUa \mid = \end{array}\right.$$

$$S \Rightarrow aSa \Rightarrow a.Ta \Rightarrow a.aTaa \Rightarrow$$

$$\Rightarrow a.a.Uaa \Rightarrow a.a.aUa\ aa \Rightarrow$$

$$\Rightarrow a.a.aaUa\ aaa \Rightarrow$$

$$\Rightarrow a.a.aa = aaaa$$

---

8 - 5

$$L = \{a^m b^n c^n a^{m+n}\} = \{a^m \underbrace{b^n c^n a^n} a^m\}$$

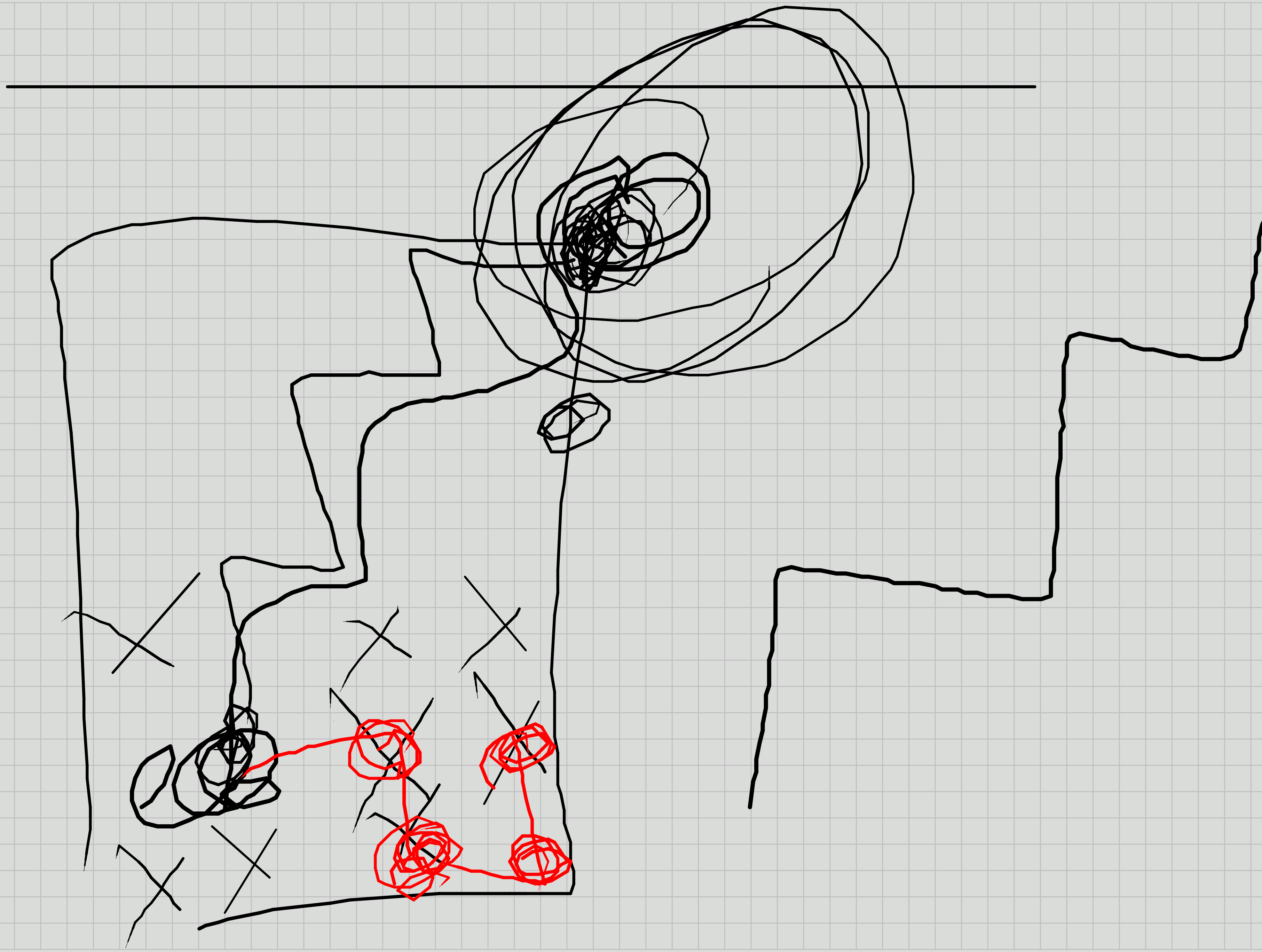
Σ^*/a^nb^n $b\Sigma^*$ Σ^*a

$a^+a^nb^n$

$a^nb^nb^+$

$\Sigma^*ba\Sigma^*$

REG ? 0/6



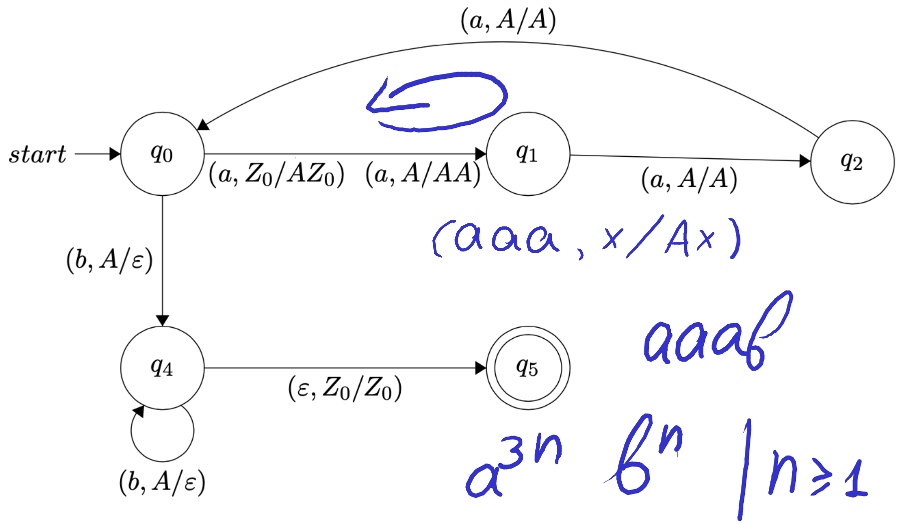
Uebungsblatt 9

Aufgabe 1:

Gegeben sei ein deterministischer Kellerautomat

$$KA = (\{q_0, q_1, q_2, q_3, q_4, q_5\}, \{a, b\}, \{A, Z_0\}, \delta, q_0, Z_0, \{q_5\})$$

mit Übergangsfunktion δ wie folgt:

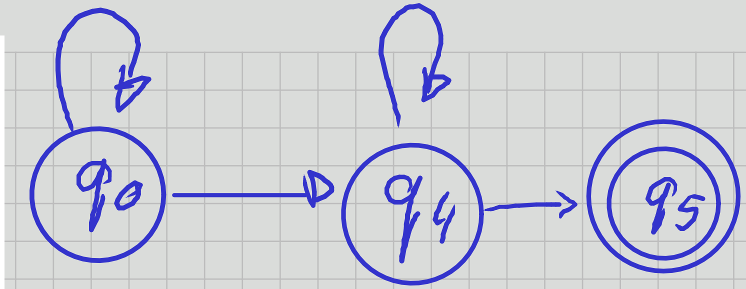


$(aaa, x/Ax)$

aaab

$a^{3n} b^n \mid n \geq 1$

Welche Sprache akzeptiert KA durch Endzustand? Erklären sie die Arbeitsweise des Kellerautmaten durch Analyse eines Wortes (z.B. Wort *aaab*: $(q_0, aaab, Z_0) \vdash$ ...).



a^{3n}

A^{3n}

$(q_0, aaab, Z_0) \vdash (q_1, aab, AZ_0) \vdash (q_2, ab, AZ_0) \vdash$

$\vdash (q_0, b, AZ_0) \vdash (q_4, \varepsilon, Z_0) \vdash (q_5, \varepsilon, \varepsilon)$

Uebungsblatt 9

Aufgabe 2:
Konstruieren Sie einen Kellerautomaten, der die Sprache

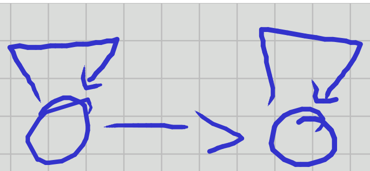
$$L = \{a^n b^n \mid n \; mod \; 3 = 0 \wedge n > 0\}$$

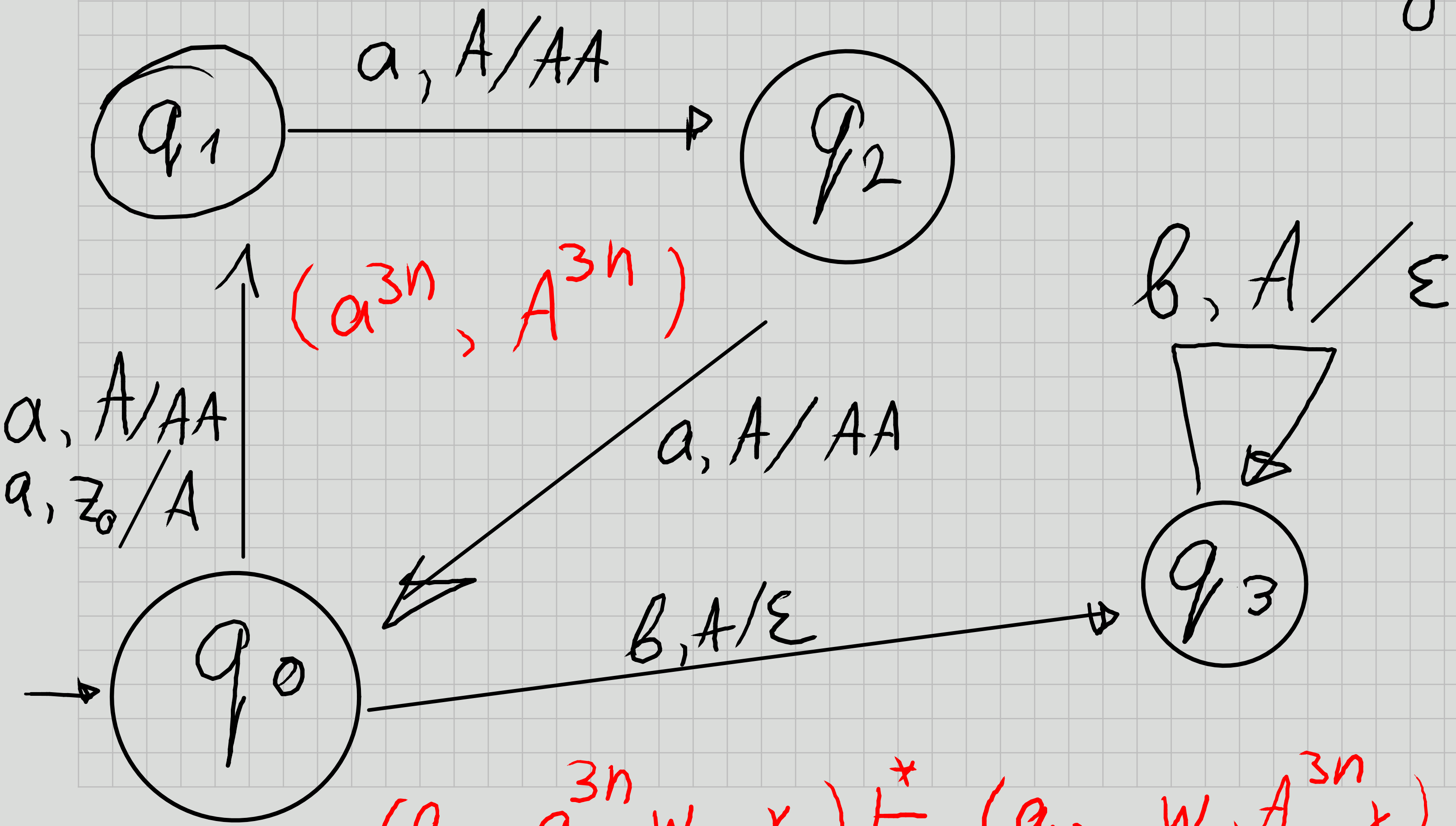
durch Endzustand akzeptiert. Ist Ihr Kellerautomat deterministisch?

$n = 3, 6, 9, 12, \ldots$

PDA accepts by empty stack

$a^n b^n$





$$(q_0, a^{3n} w, \gamma) \vdash^* (q_0, w, A^{3n} \gamma)$$

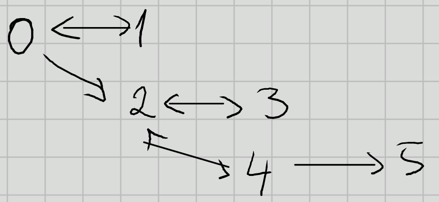
Uebungsblatt 9

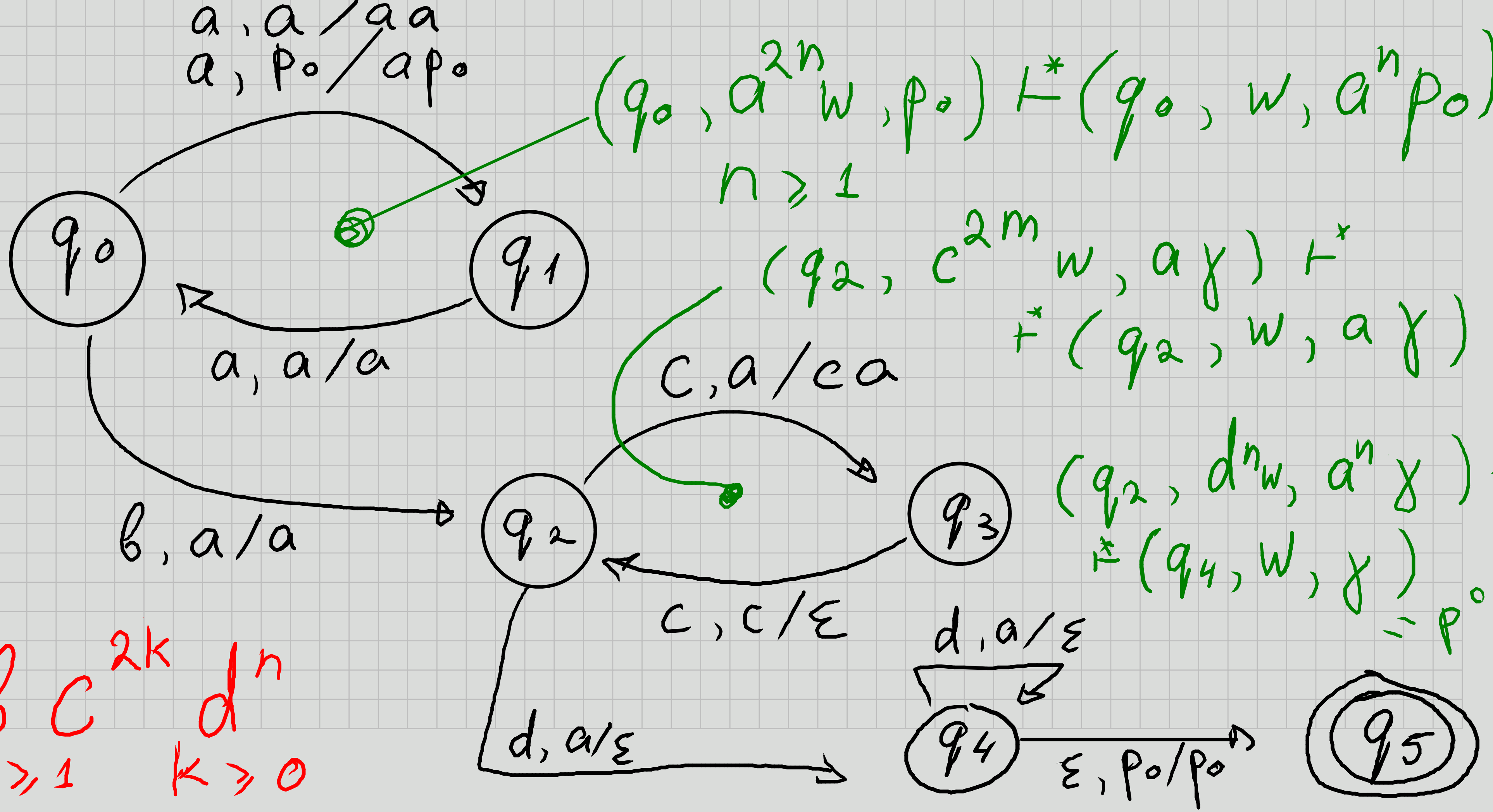
Aufgabe 3:

Gegeben sei ein Kellerautomat $KA = (\{q_0, q_1, q_2, q_3, q_4, q_5\}, \{a, b, c, d\}, \{p_0, a, c\}, \delta, q_0, p_0, \{q_5\})$ mit Übergangsfunktion δ wie folgt:

(1) $\delta(q_0, a, p_0) = \{(q_1, ap_0)\}$
(2) $\delta(q_1, a, a) = \{(q_0, a)\}$
(3) $\delta(q_0, a, a) = \{(q_1, aa)\}$
(4) $\delta(q_0, b, a) = \{(q_2, a)\}$
(5) $\delta(q_2, c, a) = \{(q_3, ca)\}$
(6) $\delta(q_3, c, c) = \{(q_2, \varepsilon)\}$
(7) $\delta(q_2, d, a) = \{(q_4, \varepsilon)\}$
(8) $\delta(q_4, d, a) = \{(q_4, \varepsilon)\}$
(9) $\delta(q_4, \varepsilon, p_0) = \{(q_5, p_0)\}$

Welche Sprache akzeptiert KA durch Endzustand?





$(q_0, a^{2n}w, p_0) \vdash^* (q_0, w, a^n p_0)$

$n \geq 1$

$(q_2, c^{2m}w, a\gamma) \vdash^*$
$\vdash^* (q_2, w, a\gamma)$

$(q_2, d^n w, a^n\gamma) \vdash^*$
$\vdash^* (q_4, w, \gamma)$

$= p^0$

$a^{2n} b c^{2k} d^n$

$n \geq 1 \quad k \geq 0$

Uebungsblatt 9

Aufgabe 4:
Konstruieren Sie einen deterministischen Kellerautomaten, der die Sprache

$$L = \{a^n b^{2n} \mid n \geq 1\}$$

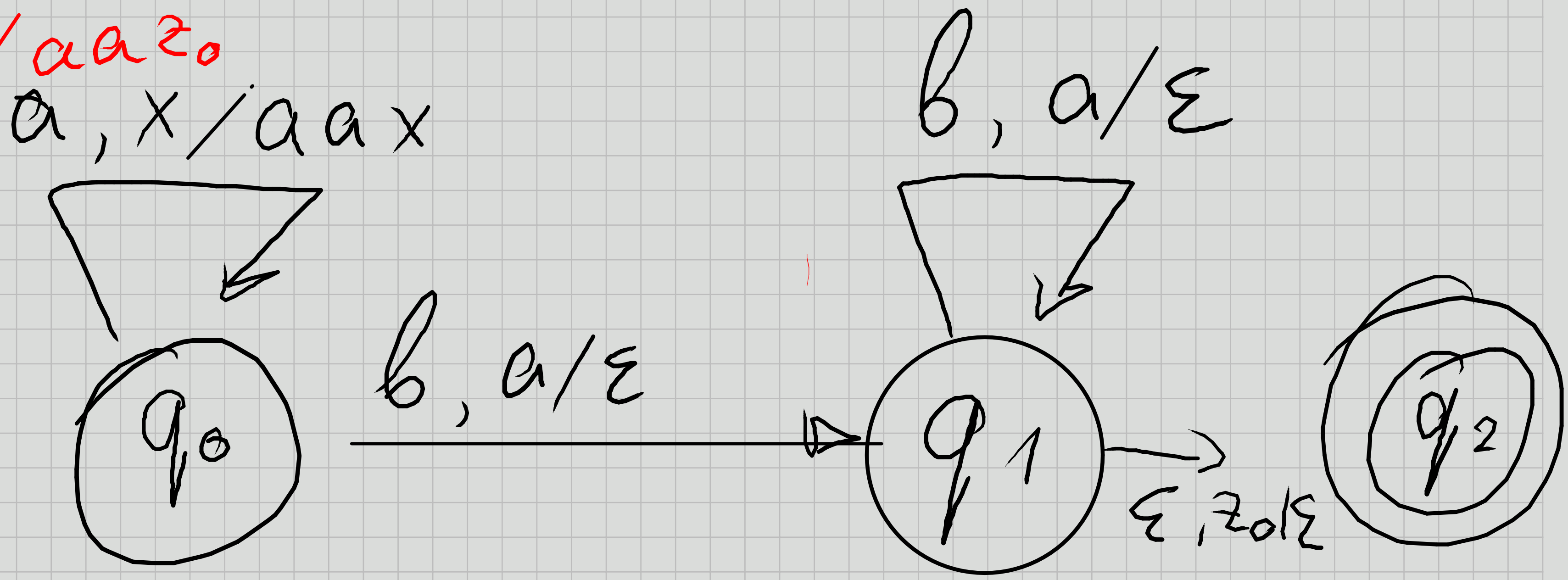
durch Endzustand akzeptiert. Erklären sie die Arbeitsweise des Kellerautmaten durch Analyse eines Wortes (z.B. Wort *aabbbb*: $(q_0, aabbbb, Z_0) \vdash$ ...).

$n \geq 1$

$a^n b^n$

$a, a / aaa$

$a, Z_0 / aaZ_0$



a
a
Z0
→
a
Z0

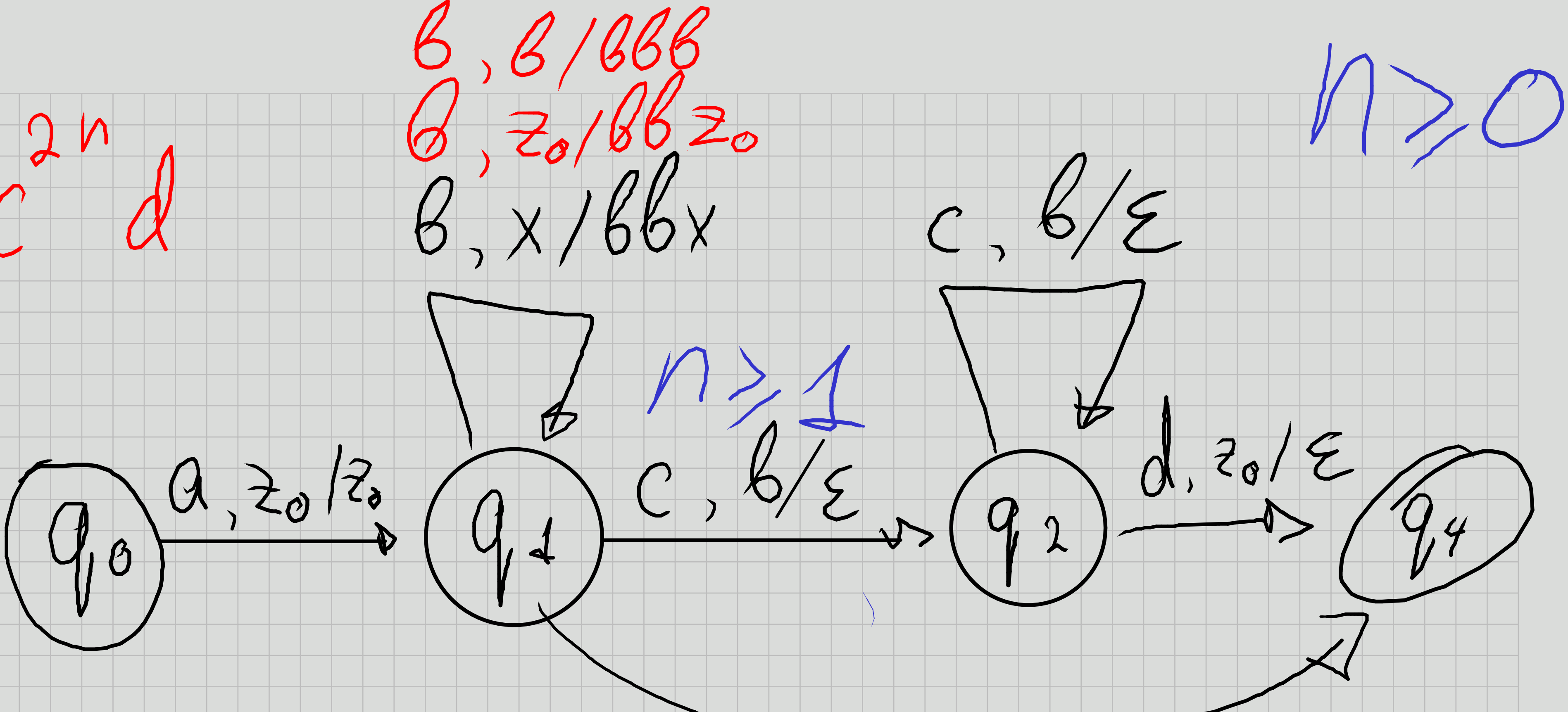
abb

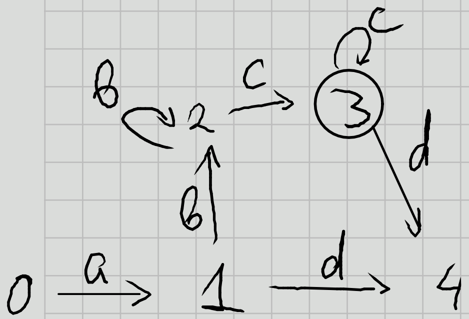
$(q_0, abb, Z_0) \vdash (q_0, bb, aaZ_0) \vdash (q_1, b, aZ_0) \vdash (q_1, \varepsilon, Z_0)$

q_0 : let x = stack.pop()
switch x) { case Z_0
case a
}

$ab^nc^{2n}d$

$n \geq 0$





$(q_0, abccd, z_0) \vdash (q_1, bccd, z_0) \vdash (q_1, ccd, bbz_0) \vdash$

$\vdash (q_2, cd, bz_0) \vdash (q_2, d, z_0) \vdash (q_4, \varepsilon, \varepsilon)$

$$L = \{a^m b^n \mid m > n, n \geq 0\} \cup \{a^m c^n d^n a^m \mid m \geq 1, n \geq 0\}$$

$a^m b^n$ $\qquad$ $a^m c^n d^n a^m$

~~REG~~ $\quad \cup \quad$ CF

S_1 $\qquad$ S_2

$S \to S_1 \mid S_2$

~~$S_1 \to aS_1 \mid T_1$~~

~~$T_1 \to bT_1 \mid \varepsilon$~~

$S_2 \to aS_2a \mid aT_2a$

$T_2 \to cT_2d \mid \varepsilon$

$a^m b^n \equiv a^+ (a^n b^n)$

$S \to aS \mid aT$

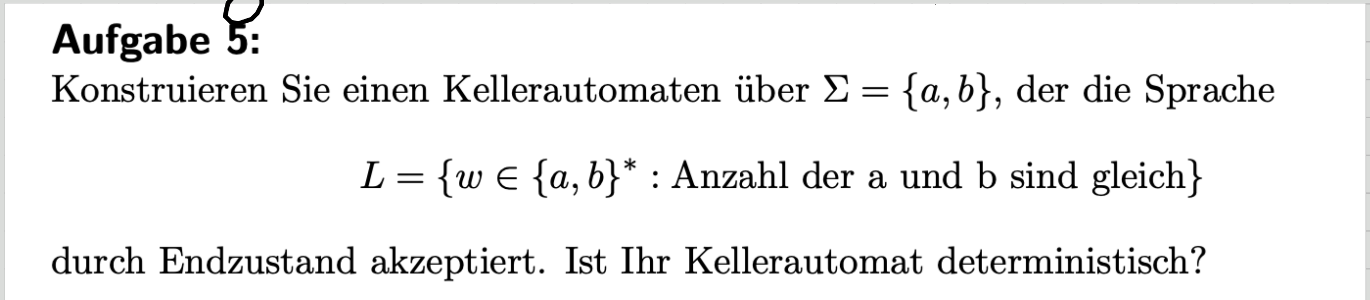
$T \to aTb \mid \varepsilon$

$S \to aUa$

$U \to aUa \mid T$

$T \to cTd \mid \varepsilon$

Uebungsblatt 9



Aufgabe 5:
Konstruieren Sie einen Kellerautomaten über $\Sigma = \{a, b\}$, der die Sprache

$$L = \{w \in \{a,b\}^* : \text{Anzahl der a und b sind gleich}\}$$

durch Endzustand akzeptiert. Ist Ihr Kellerautomat deterministisch?

intuition:

$a^n b^n$

(( )( )( )) - Dyck Language

a a b a b a b b

$$S \to SS \mid \underbrace{aSb}_{\Downarrow^* \, a^n b^n} \mid \varepsilon$$

Hint: $w = b^n a^n$

$w = bababa$ $\quad S \to bSa \mid \varepsilon \mid SS$

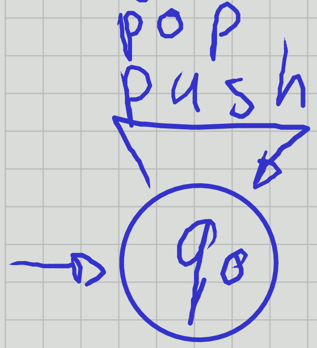
What if we unite these?

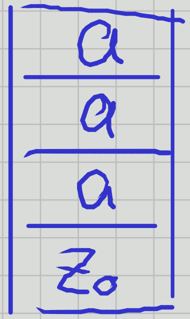
$$S \to SS \mid aSb \mid bSa \mid \varepsilon$$

Since all rules that generate terminal symbols generate them in balanced way, it is easy to see that our new grammar only generates "good" words. The only question is: does the grammar generate ALL words from the Lang?

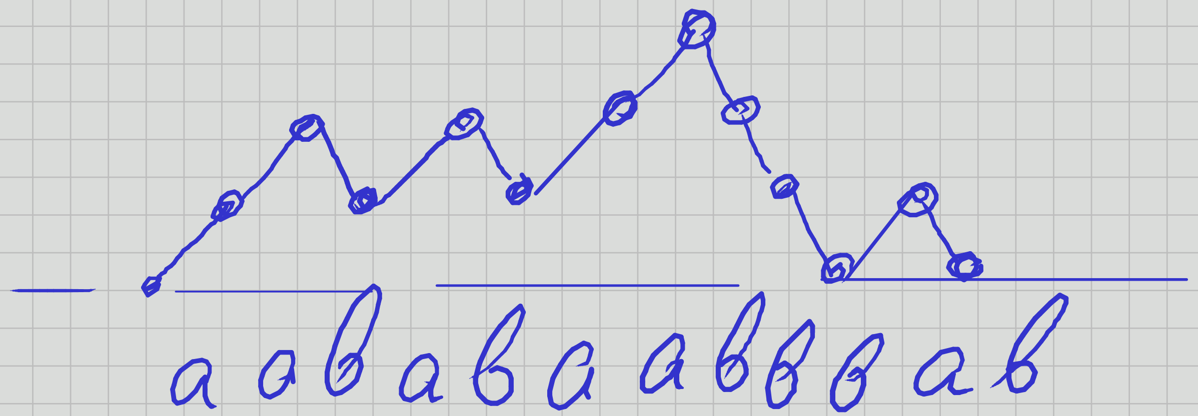
Dyck :

$\Gamma = \{a, Z_0\}$

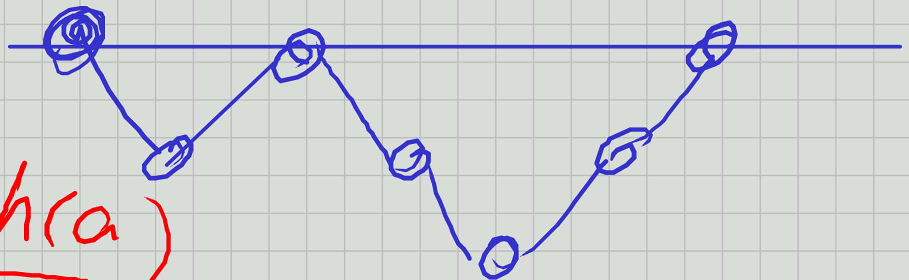




stack level

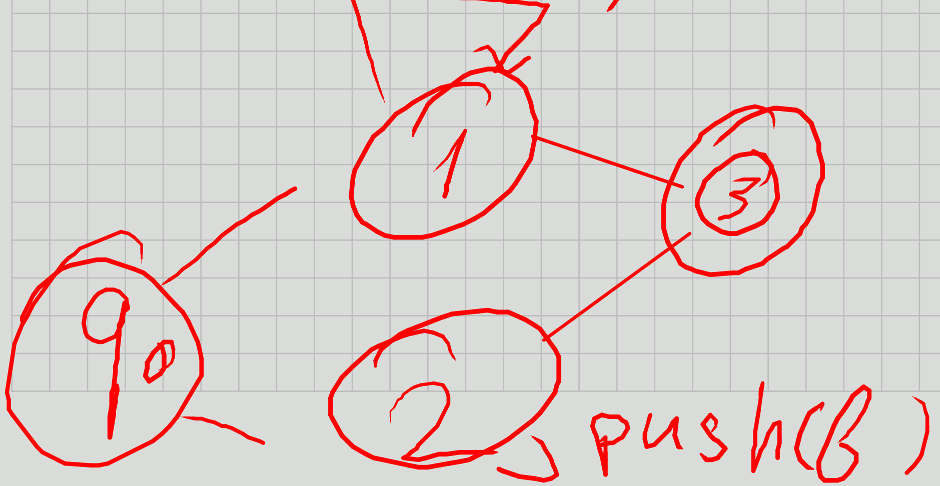


w = ba ?

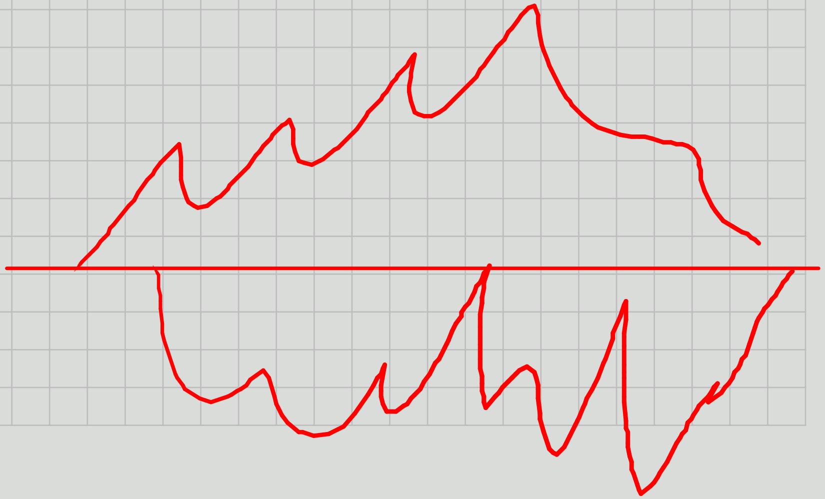


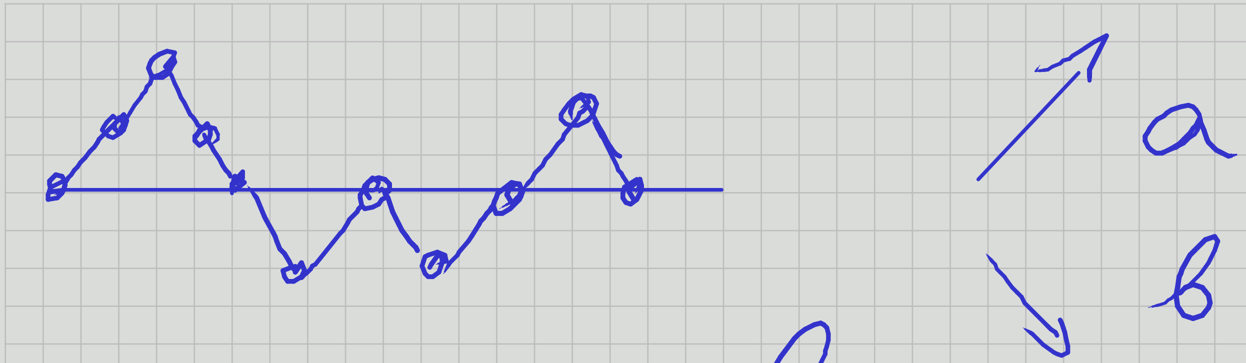
?

push(a)



?



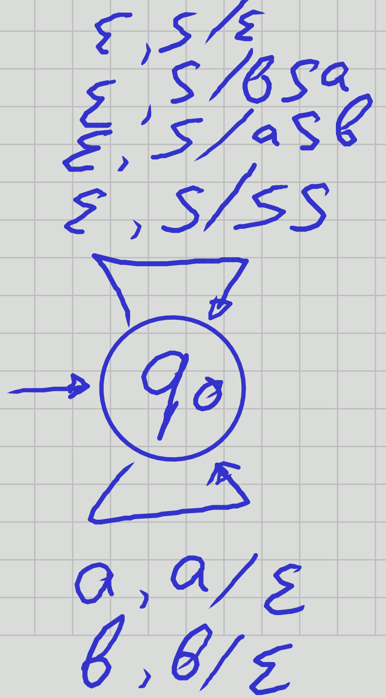


a abbbabaab

---

lets apply algorithm G → n-PDA

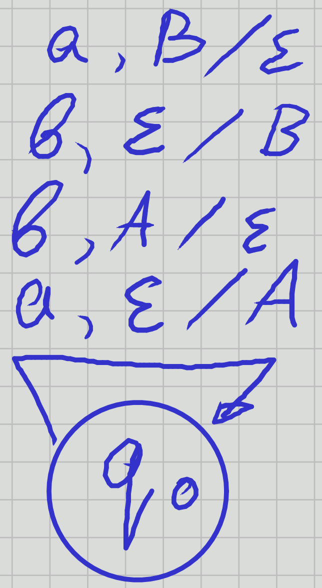
$S \to SS \mid aSb \mid bSa \mid \varepsilon$



$z_0 = S$

$\Sigma = \{a, b\}$

Alternative solution



δ $\quad$ a , ε ? / A

$\Sigma \cup \{\varepsilon\}$ $\quad$ Γ $\quad$ Γ^*

$\Longleftrightarrow \begin{cases} a, x/Ax \\ a, y/Ay \\ a, z/Az \\ \dots \end{cases}$

✓

$\Gamma = \{z_0, A, B\}$

$a, \varepsilon/A \Longrightarrow$

$a, z_0/Az_0$
$a, A/AA$
$a, B/AB$

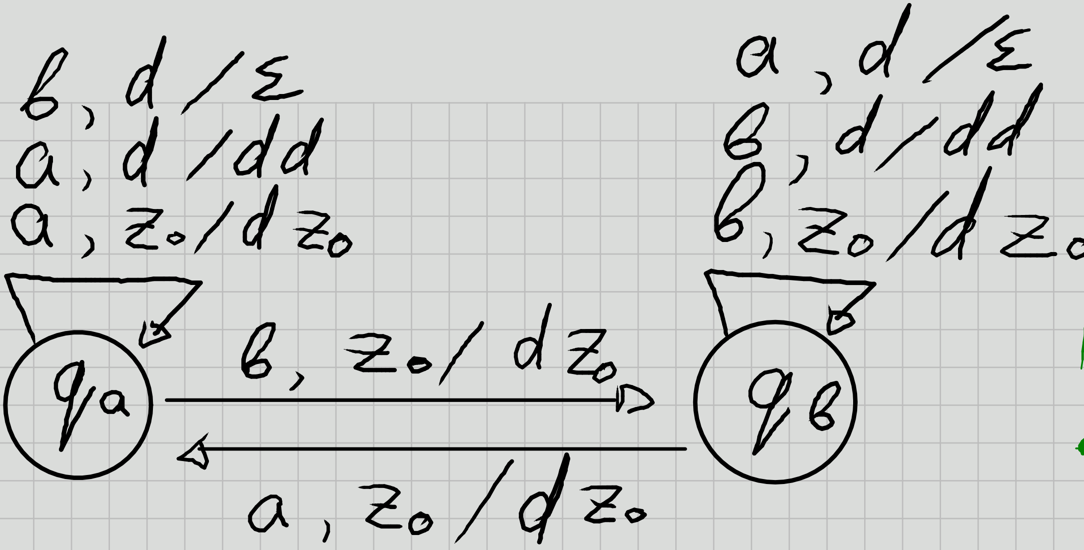
$a, B/\varepsilon$

$(q_0, aw, B\gamma) \vdash (q_0, w, \gamma)$
$\vdash (q_0, w, AB\gamma)$

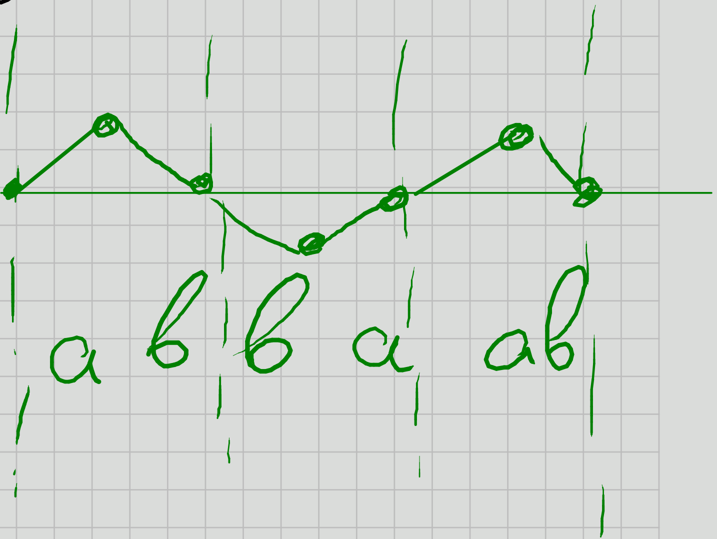
$b, \varepsilon/B \Longrightarrow$

$b, z_0/Bz_0$
$b, A/BA$
$b, B/BB$

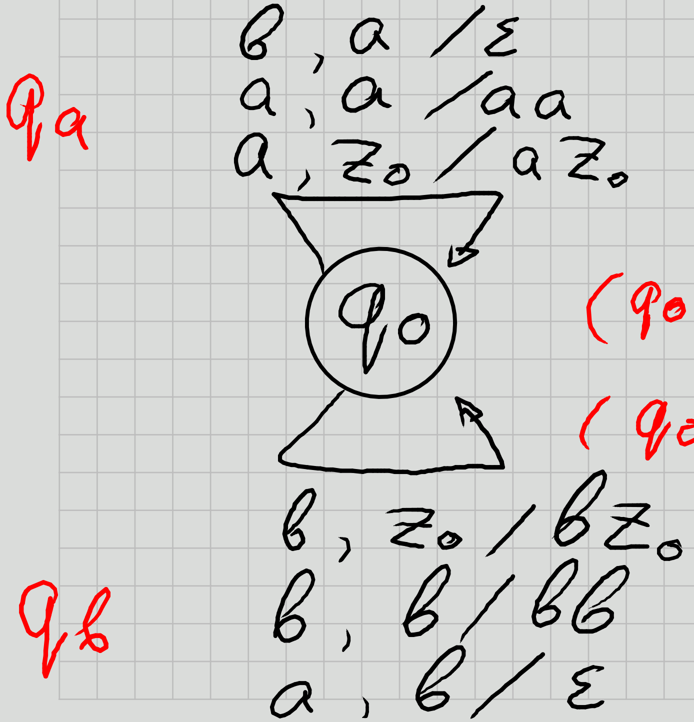
$b, A/\varepsilon$



$\Gamma = \{Z_0, d\}$



$\Gamma = \{Z_0, a, b\}$



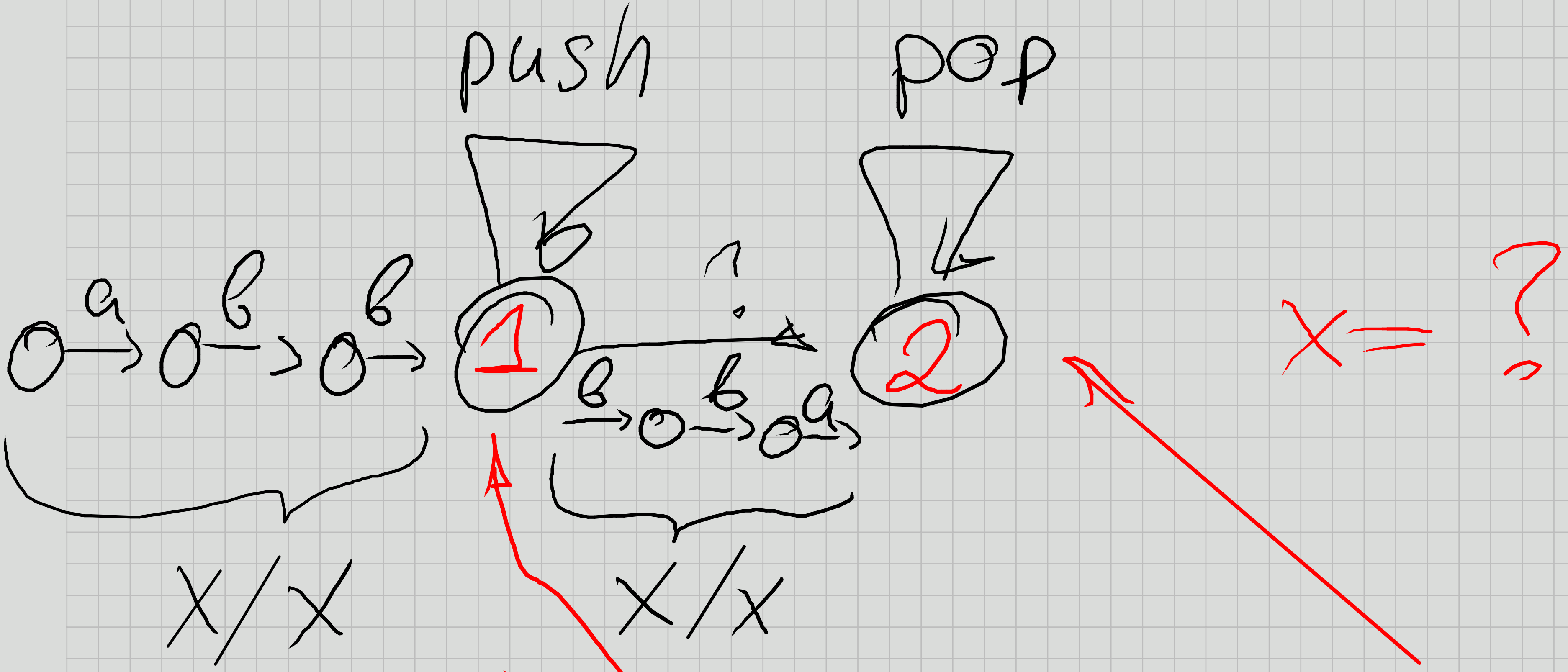
(q_0, aw, Z_0)

(q_0, bw, Z_0)

a b

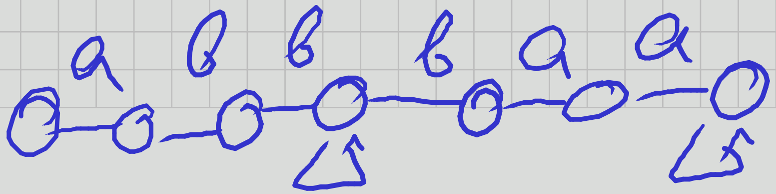
b a

$abb(a^n)\ bba\ (a^n)$

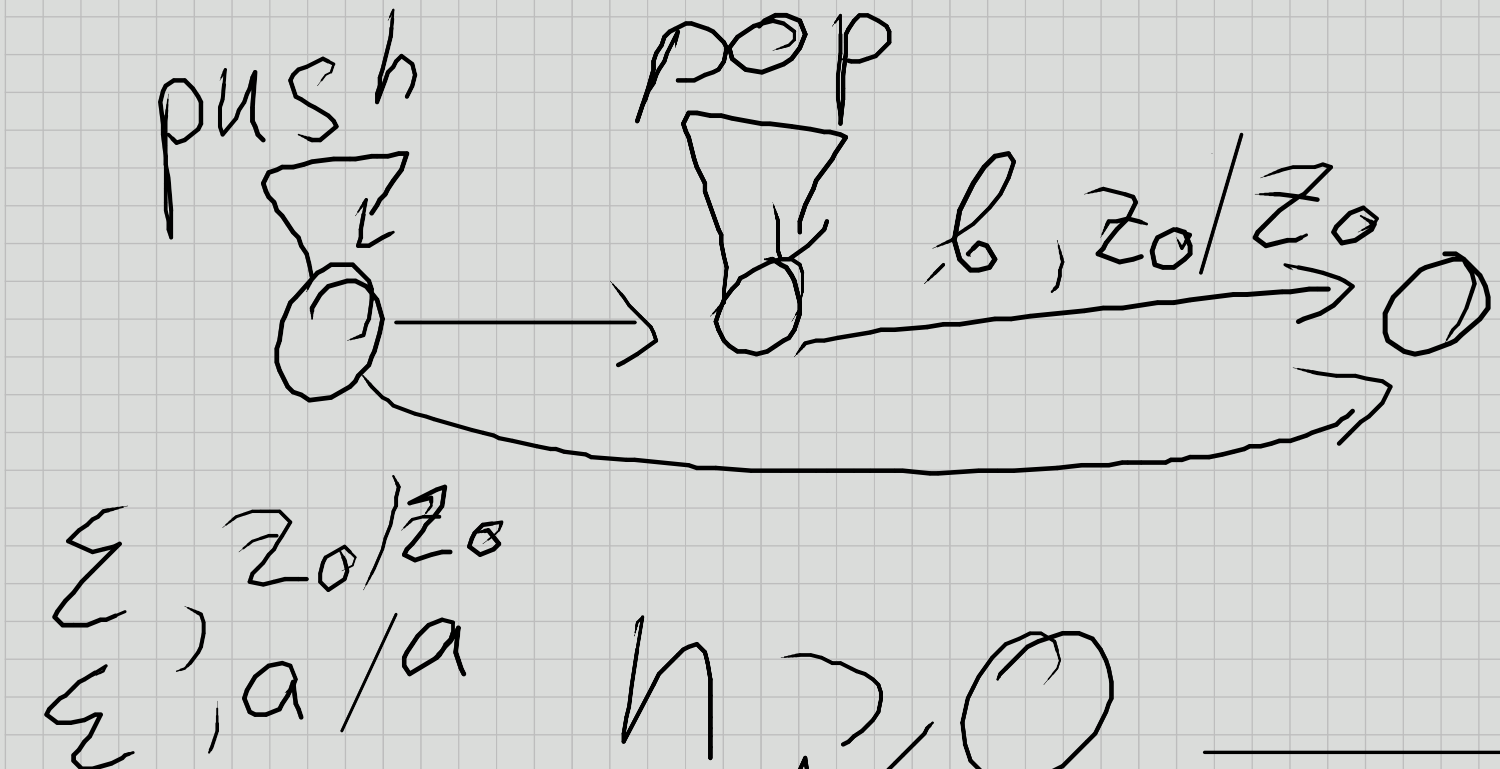


$(q_0, w, z_0) \vdash^* (q_1, w', \underline{a^n z_0}) \vdash^* (q_2, w'', \underline{a^n z_0})$

Z_0/Z_0

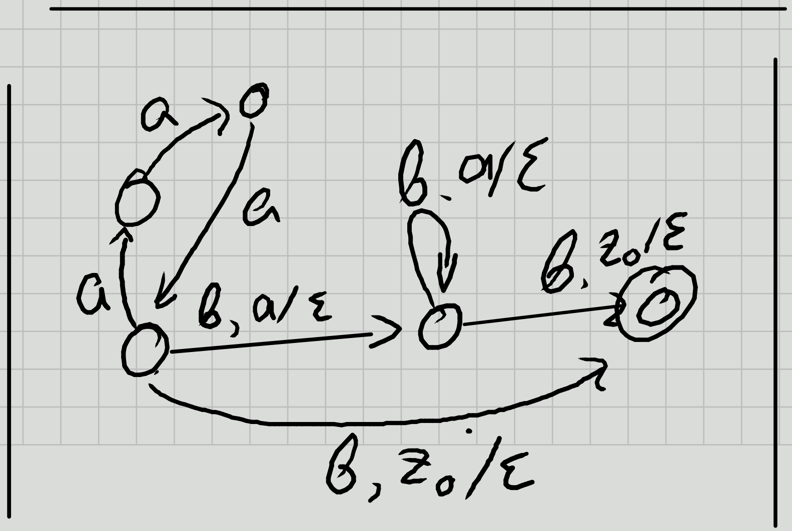


$$a^{3n} b^{n+1} \equiv (a^{3n} b^{n}) b$$



$n \geq 0$

det



$S \to aaSbb \mid T$

$T \to cT \mid cU$

$U \to dU \mid ab$

1) cdd

2) cdddd

3) cddd

---

aaabbbbbbb

$S \Rightarrow \ldots$

$G: \Rightarrow$

$M: \vdash$

$S \to S_1 \mid S_2$

$S_1 \to aS_1bb \mid \varepsilon$

$S_2 \to cS_2 \mid U$