

Grammars

Rule in form $\alpha \rightarrow \beta$, α, β are some words

Example:

$$\left[\begin{array}{l} S \rightarrow aSb \\ aSb \rightarrow aaSbb \\ S \rightarrow \varepsilon \end{array} \right.$$

$$\Sigma = \{a, b\}$$

$$N = \{S\}$$

$$aabb \in L(G)$$

1. Σ alphabet of Terminal symbols

2. N alphabet of Nonterminal symbols

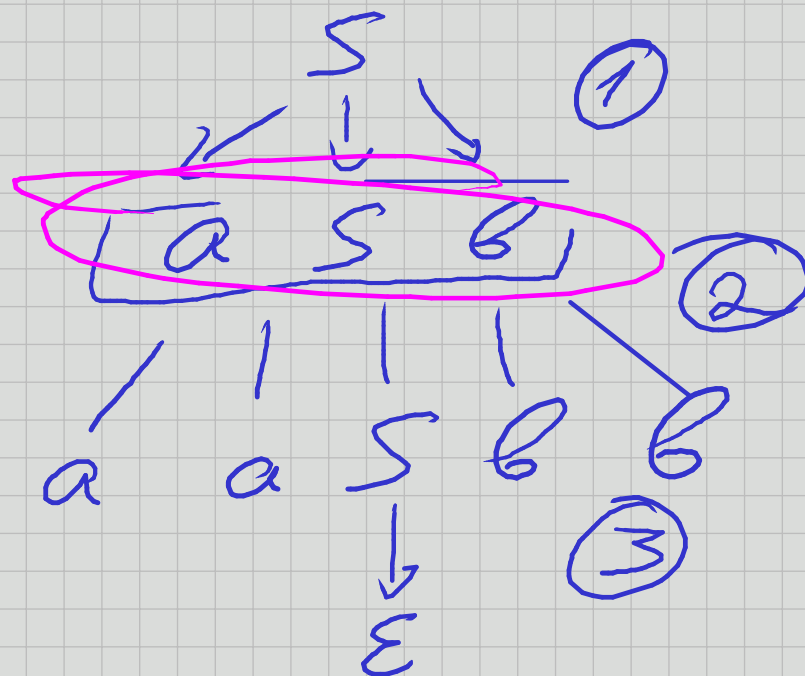
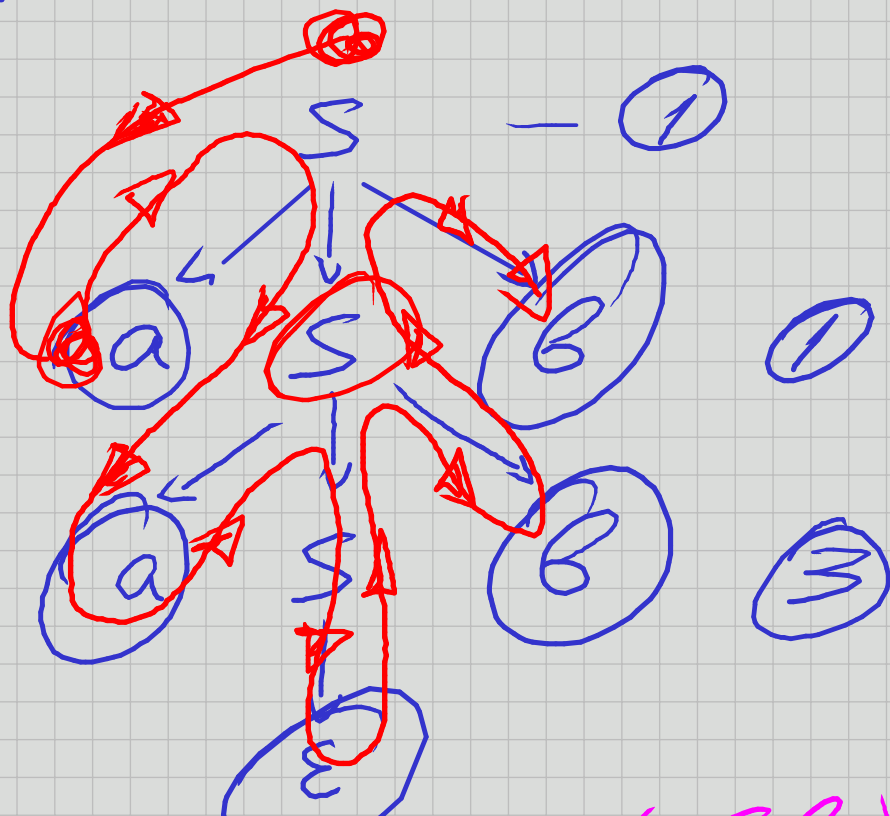
3. $S \in N$ axiom

4. $P \subseteq \{(\alpha, \beta) : \alpha, \beta \in (N \cup \Sigma)^*\}$ rules

Дерево вывода для слова из грамматики Derivation (?) tree

- ① $S \rightarrow aSb$
- ② $aSb \rightarrow aaSbb$
- ③ $S \rightarrow \varepsilon$

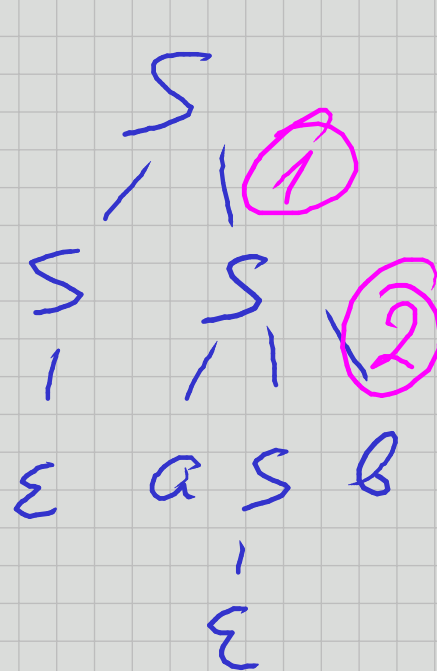
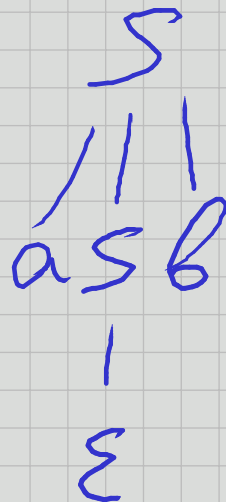
$w = aabbb$



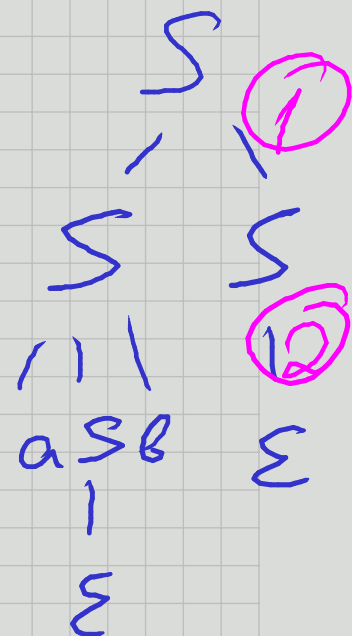
$S \rightarrow aSb \quad a(aSb)b \quad aa\varepsilon bb \quad aabbb$

- ① $S \rightarrow SS$
- ② $S \rightarrow aSb$
- ③ $S \rightarrow \varepsilon$

$w = ab$



$\varepsilon a \varepsilon b$ $a \varepsilon b \varepsilon$



Type 1 Grammars

$$S \rightarrow \Sigma$$

1. $(S \rightarrow e) \Rightarrow S$ shouldn't be on the right side of rules

2. for other \rightarrow rules

$$|B| \geq |A|$$

$$\alpha \rightarrow \beta$$

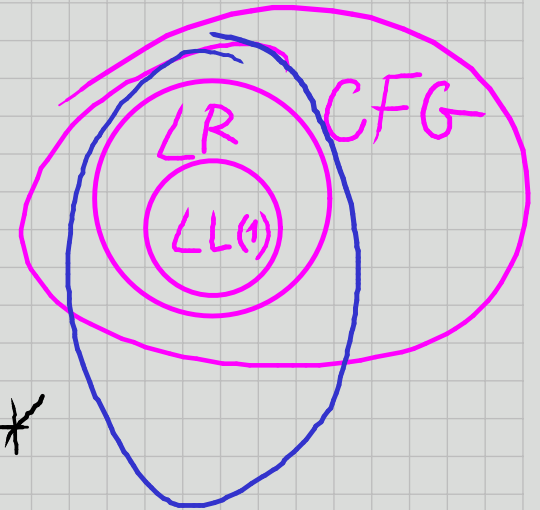
Context dependant Languages

Type 2 Grammars

$$A \rightarrow \beta, A \in N, \beta \in (N \cup \Sigma)^*$$

Context free languages

Fortran, Lisp
C, Java



Type 3 Grammars

All rules :

$$A \rightarrow wB \quad \text{or} \quad A \rightarrow w$$

$$A, B \in N$$

$$w \in \Sigma^*$$

$$\begin{array}{l} S \rightarrow Sa \mid Sb \\ S \rightarrow Ta \\ T \rightarrow b \end{array} \quad \text{left linear}$$

Right-linear Grammars

$$L(G_1) = L(G_2)$$

Regular Languages

$$S \rightarrow aT$$

$$T \rightarrow Sb$$

$$S \rightarrow \epsilon$$

$$S \rightarrow aSb$$

$$\begin{aligned} S &\Rightarrow aT \Rightarrow aSb \Rightarrow aaTb \Rightarrow \\ &\Rightarrow aaSbb \Rightarrow aaabbb \end{aligned}$$

$$S^* \Rightarrow aSb \Rightarrow aaSbb$$

$$\varepsilon \in L(G)$$

$$\varepsilon, ab, aabb$$

$$\underline{S} \Rightarrow \varepsilon$$

$$aaSbb \Rightarrow aa(aT)bb \Rightarrow aaa(\overset{\nearrow \varepsilon}{S}b)bb \Rightarrow \underline{aaaa}bb$$

$$aaaaabb \in L(G) = \{a^n b^n\} \in REG?$$

example:

Grammar for language of correct scope structures

$$S \rightarrow SS \mid (S) \mid \varepsilon$$

$$(()) \quad ((() ()) ())$$

$$S \rightarrow aSb$$

$$\left[\begin{array}{l} S \rightarrow (S) \\ S \rightarrow [S] \\ S \rightarrow \{S\} \end{array} \right.$$

$$L(G) = \{ w \mid S \xRightarrow{*} w \}$$

equivalent
models
of
computation

Type 0: рекурсивно перечислимые
recursively enumerable languages

Turing Machine accepts words from this language

слова принимаются, но не распознаются

Type 1: Context dependant

Turing Machine accepts words from this language
but we are not allowed to use more memory cells
that was used to write down input word

Type 2: Context Free (CF)
NFA + Stack = PDA

Type 3: Regular
NFA

Однозначные и неоднозначные грамматики

unambiguous and ambiguous grammars

deliberately ambiguous grammars

Grammar is ambiguous if there exists a word (in the language of grammar) that has more than one derivation path.

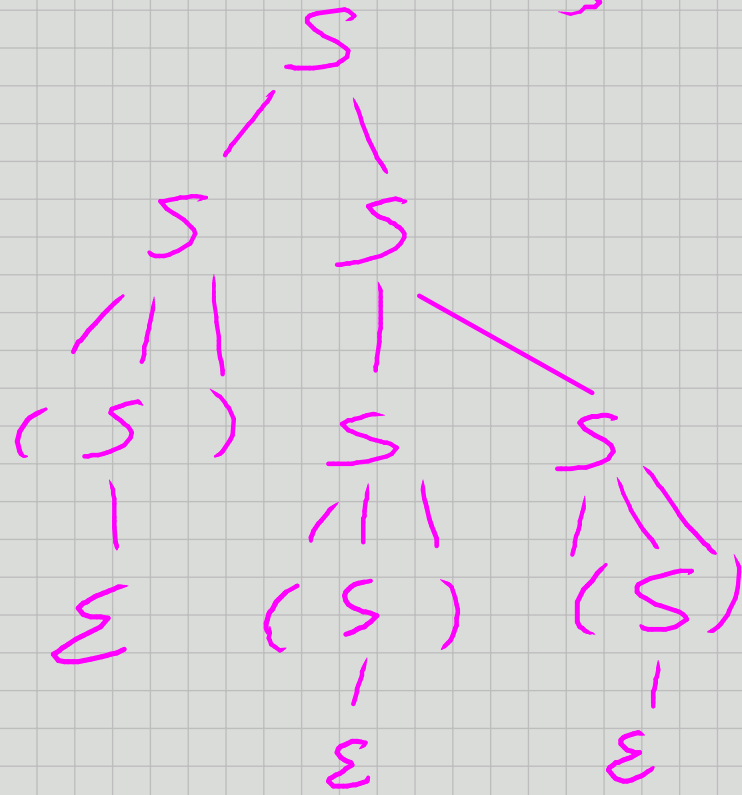
$$S \rightarrow (S)S \mid \varepsilon$$

example $S \rightarrow SS \mid (S) \mid \varepsilon$

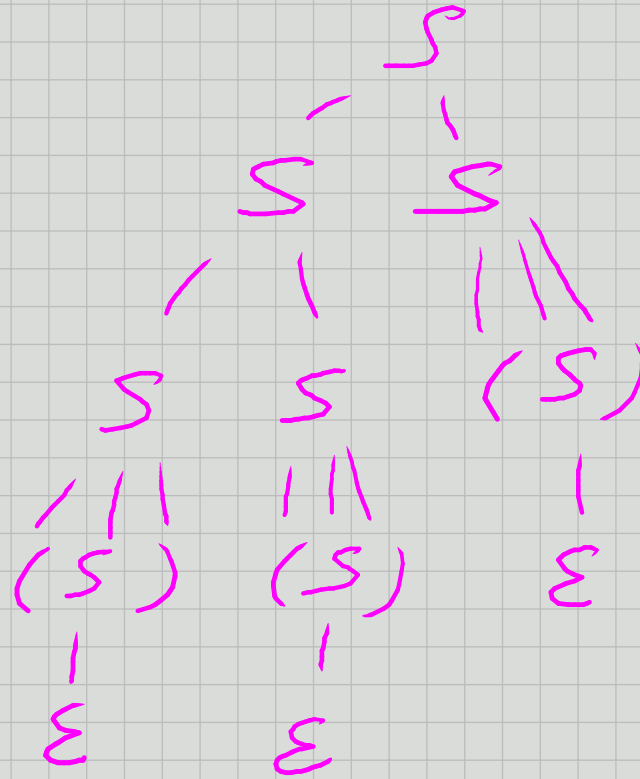
$$\begin{aligned} 1) S &\Rightarrow \underline{SS} \Rightarrow (S)\underline{S} \Rightarrow (S)SS \Rightarrow \dots \Rightarrow \\ &\Rightarrow ()()() \end{aligned}$$

$$\begin{aligned} 2) S &\Rightarrow S\underline{S} \Rightarrow S(S) \Rightarrow SS(S) \Rightarrow \dots \Rightarrow \\ &\Rightarrow ()()() \end{aligned}$$

2



?



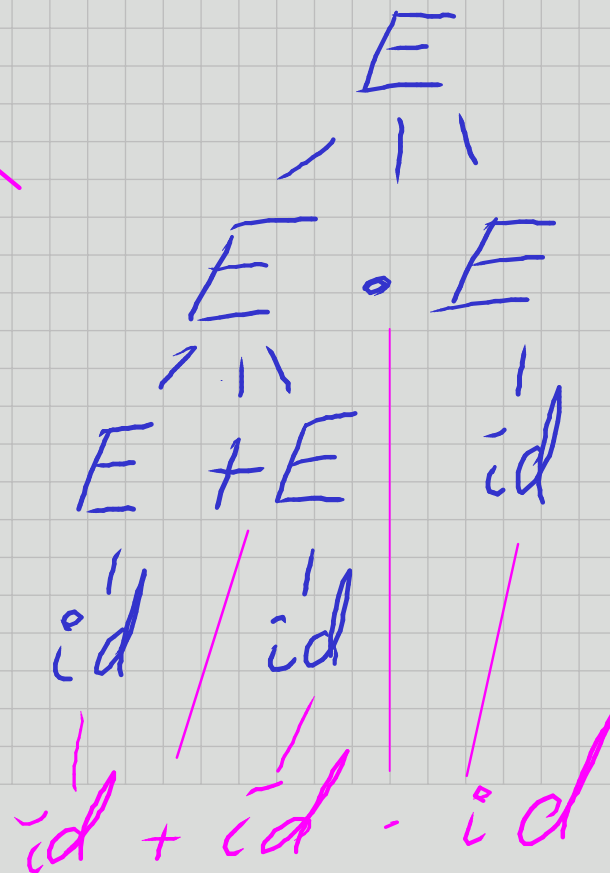
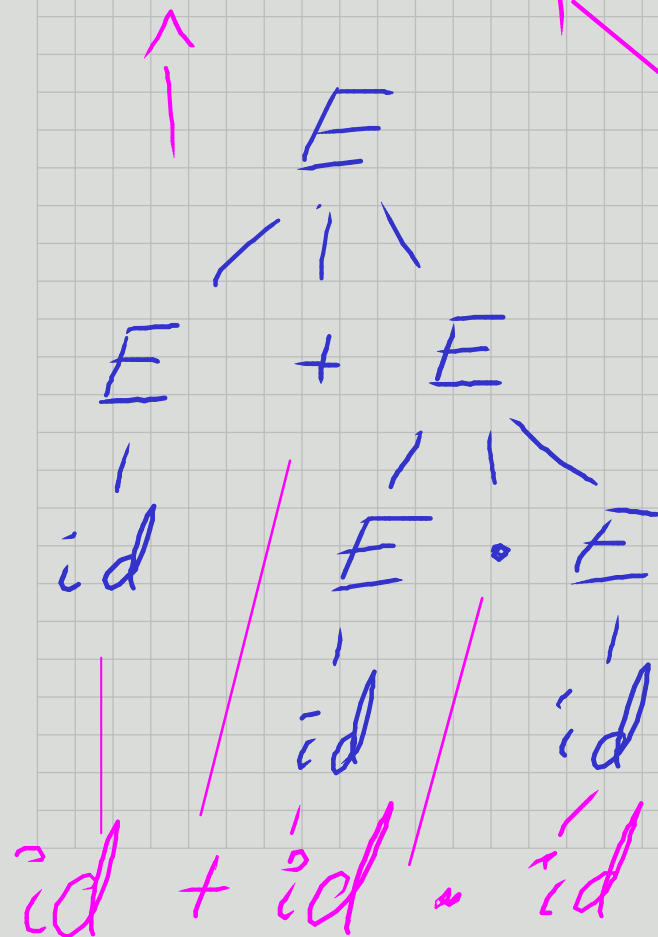
example

$E \rightarrow E + E \mid E \cdot E \mid id$

$id + id \cdot id$

$x + y \cdot z$
 $0 \times 10 \leq 10$

~~2×2~~
 ~~2×2~~
 ~~2×2~~



$2 + 2 \cdot 2$

$2 \cdot 2 = 4$

$4 + 2 = 6$

$2 + 2 = 4$

$4 \cdot 2 = 8$

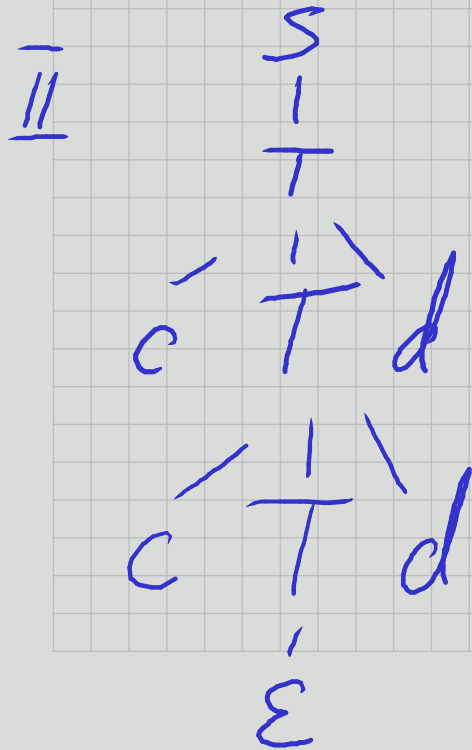
Übungsblatt 5-3

Gegeben sei die KFG $G = (N, \Sigma, P, S)$ mit $N = \{S, T\}$, $\Sigma = \{a, b, c, d\}$ und $P = \{S \rightarrow aSb | T, T \rightarrow cTd | \varepsilon\}$.

- Geben Sie die Ableitung für $aacdbb$ an.
- Geben Sie den Ableitungsbaum für $ccdd$ an.
- Beschreiben Sie die von G erzeugte Sprache formal.

1 $S \rightarrow aSb$
 2 $S \rightarrow T$
 3 $T \rightarrow cTd$
 4 $T \rightarrow \varepsilon$

$I \quad S \xrightarrow{1} aSb \xrightarrow{2} aaSbb \xrightarrow{3} aacTdbb \xrightarrow{4} aacdbb$



III $S \rightarrow aSb$
 $S \xrightarrow{*} a^n S b^n$
 $T \xrightarrow{*} c^m T d^m$

$S \xrightarrow{*} (a^n c^m T d^m b^n)$

$L = \{a^n c^m d^m b^n \mid n, m \geq 0\}$

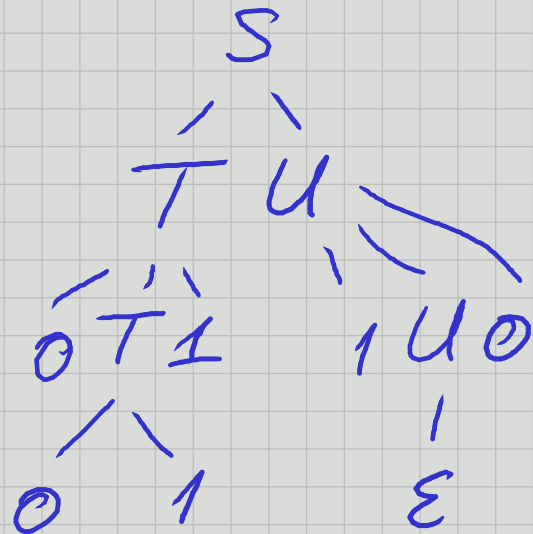
Übungsblatt 5-4

Aufgabe 4:

Gegeben sei die KFG $G = (N, \Sigma, P, S)$ mit $N = \{S, T, U\}$, $\Sigma = \{0, 1\}$ und $P = \{S \rightarrow TU, T \rightarrow 0T1|01, U \rightarrow 1U0|\varepsilon\}$.

- Geben Sie die Ableitung und den Ableitungsbaum für 001110 an.
- Beschreiben Sie die von G erzeugte Sprache formal.

$$\begin{array}{l} 1 \ S \rightarrow TU \\ 2 \ T \rightarrow 0T1 \\ 3 \ T \rightarrow 01 \\ 4 \ U \rightarrow 1U0 \\ 5 \ U \rightarrow \varepsilon \end{array} \quad \text{I} \quad S \xrightarrow{1} TU \xrightarrow{2} 0T1U \xrightarrow{3} 0011U \xrightarrow{4} 00111U0 \xrightarrow{5} 001110$$



$$L = \{0^m 1^m 1^k 0^k \mid m \geq 1, k \geq 0\}$$

$$\text{II} \quad T \xrightarrow{*} 0T1 \xrightarrow{*} 0^n T 1^n \xrightarrow{*} 0^m 1^m$$

$$U \xrightarrow{*} 1U0 \xrightarrow{*} 1^n U 0^n \xrightarrow{*} 1^k 0^k$$

$$S \xrightarrow{*} TU \xrightarrow{*} 0^m 1^m 1^k 0^k$$

Aufgabe 5:

Gegeben sei die Sprache $L = \{a^x b^y \mid x, y \geq 0\}$ über dem Alphabet $\Sigma = \{a, b\}$.

- (i) Geben Sie eine KFG G an, sodass $L = L(G)$.
- (ii) Geben Sie den Ableitungsbaum für $abbb$ an.
- (iii) Falls ε Teil der Sprache ist, geben Sie Ableitung und Ableitungsbaum an.

$$T \xRightarrow{*} w$$

$$U \xRightarrow{*} v$$

$$S \Rightarrow T U \xRightarrow{*} w v$$

$$L = a^n b^m$$

$$a^n : S \rightarrow a S \mid \varepsilon$$

$$b^m : S \rightarrow b S \mid \varepsilon$$

$$S \rightarrow a S \mid T \mid \varepsilon$$

$$T \rightarrow b T \mid \varepsilon$$

$$S \rightarrow S_1 S_2$$

$$S_1 \rightarrow a S_1 \mid \varepsilon$$

$$S_2 \rightarrow b S_2 \mid \varepsilon$$

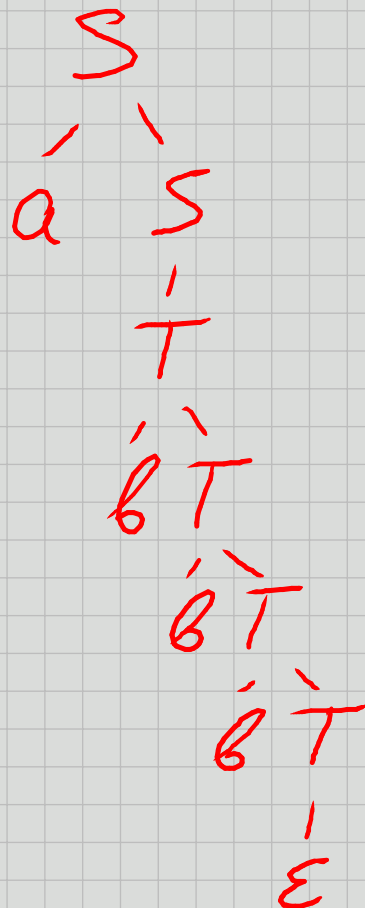
$$S \rightarrow a S \mid S b \mid \varepsilon$$

$$S \rightarrow aS | T | \epsilon$$

$$T \rightarrow bT | \epsilon$$

II $abbb$

- 1 $S \rightarrow aS$
- 2 $S \rightarrow T$
- 3 $S \rightarrow \epsilon$
- 4 $T \rightarrow bT$
- 5 $T \rightarrow \epsilon$



III ϵ

$$S \xRightarrow{3} \epsilon$$

S
|
 ϵ

Aufgabe 6:

Gegeben sei die kontextfreie Sprache $L = \{a^{2n}b^{3n} \mid n \geq 1\}$.

- (a) Geben Sie eine KFG G an, sodass $L = L(G)$.
- (b) Geben Sie Ableitung und Ableitungsbaum für $aaaabbbbbb$ an.

$aa\ bbb$

$a^n b^n \quad S \rightarrow aSb \mid \varepsilon$

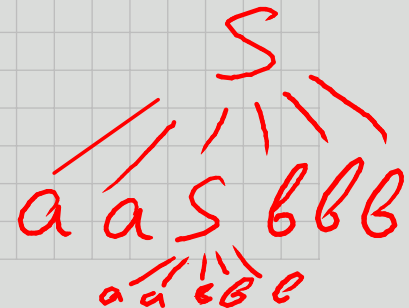
$S \rightarrow aaS \mid T$
 $T \rightarrow aaTb$

$T \xRightarrow{*} a^{2n} T b^n$

$S \rightarrow aaSbbb \mid aaabbbb$

$S \xRightarrow{*} a^{2n} S b^{3n}$

$S \xRightarrow{1} aaSbbb \xRightarrow{2} aaaabbbbbb$



$$L = \{ a^{2n+5} b^{3n} \} \quad n \geq 0$$

$$L = \underbrace{a^5}_{L_1} \circ \underbrace{a^{2n} b^{3n}}_{L_2}$$

$$L_2: S_2 \rightarrow aaS_2bbb / \epsilon$$

$$L_1: S_1 \rightarrow aaaaaa$$

$$S \rightarrow S_1 S_2$$

$$S \rightarrow aaSbbb / aaaaaa$$

REG \xleftrightarrow{DFA} NFA

Pumping Lemma

$$xyz \in L \wedge |xyz| > p \implies xy^iz \in L$$

Классы эквивалентности Майхилла-Нероуда

$a^n b^n$

ww

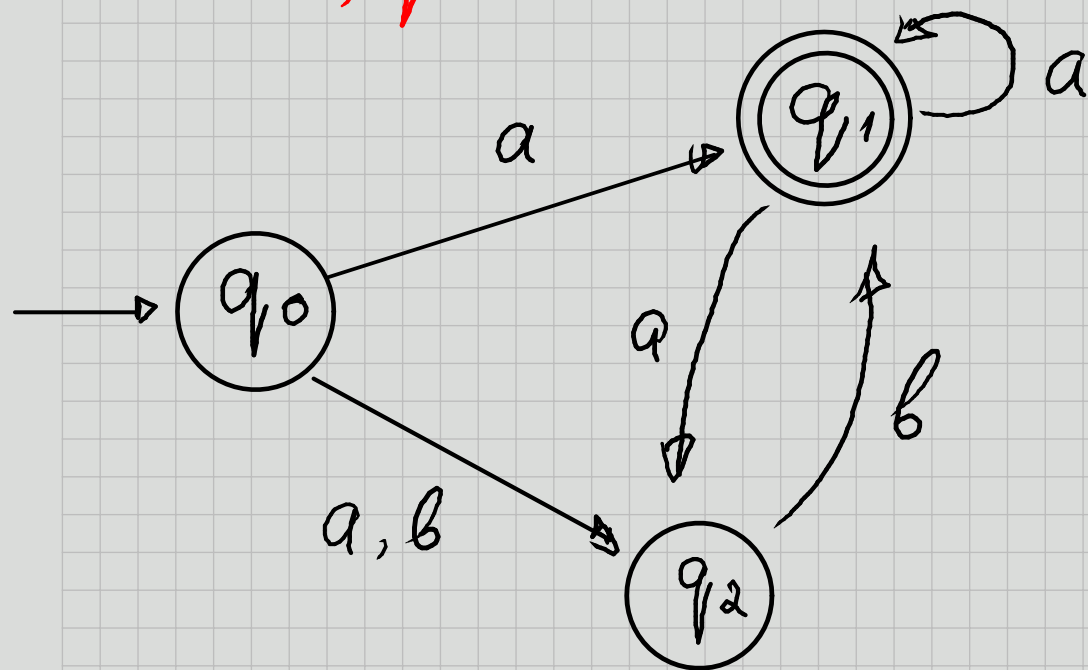
ww^R

$$|w|_a = |w|_b$$

$\Sigma^* / a^n b^n$

Σ^* / ww^R

NFA $\rightarrow G(N, \Sigma, S, P)$
 $(Q, \Sigma, \delta, q_0, F)$



1. $N = Q$

2. $\Sigma = \Sigma$

3. $S = q_0$

4. $q_i \xrightarrow{\alpha} q_j \quad \alpha \in \Sigma \cup \{\epsilon\}$
 $Q_i \rightarrow \alpha Q_j$

$N = \{Q_0, Q_1, Q_2\}, S = Q_0$

$Q_0 \rightarrow aQ_1 \mid aQ_2 \mid bQ_2$

$Q_1 \rightarrow aQ_1 \mid aQ_2 \mid \epsilon$

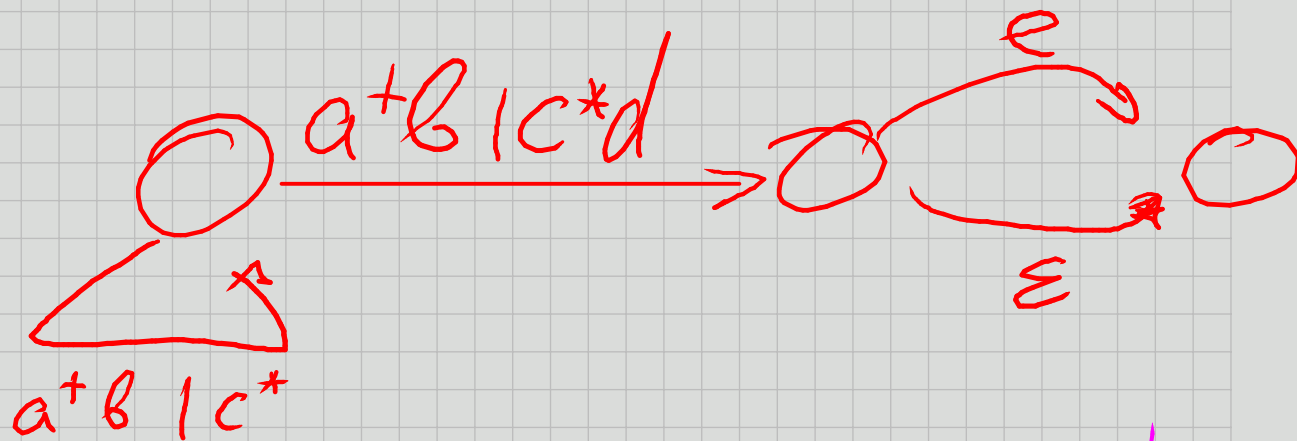
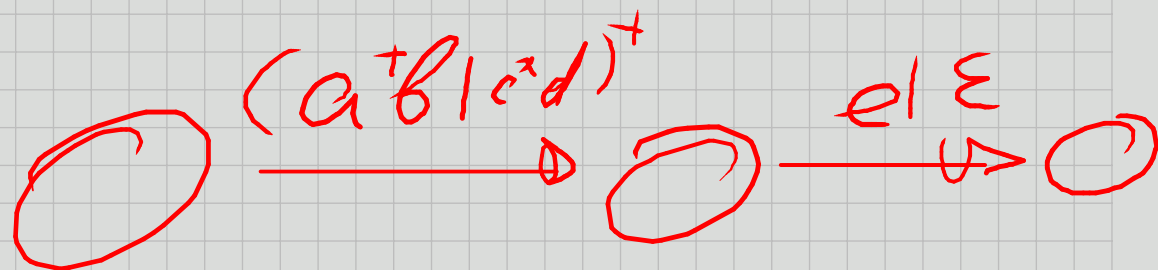
$Q_2 \rightarrow bQ_1$

$q_f \in F \Rightarrow Q_f \rightarrow \epsilon$

$RE \rightarrow G$

\downarrow NFA \nearrow

$(a^+b|c^*d)^+(e|\epsilon)$



$S \rightarrow a^+b A_1$

$S \rightarrow a S | a b A_1$

$S \rightarrow (a^+b|c^*d) S | \epsilon$

$S \rightarrow (a^+b|c^*d)^+ A_1$

$A_1 \rightarrow e | \epsilon$

$S \rightarrow a^+b A_1 | c^*d A_1$

$G \rightarrow NFA$

$A \rightarrow \alpha B \mid \epsilon$
 $\alpha \in \Sigma$

- 1) $A \rightarrow w$, $w \in \Sigma^*$
- 2) $A \rightarrow x B$, $x \in \Sigma^*$

If our rules are complex* then we rewrite our Grammar and simplify it.

① $A \rightarrow w \iff \begin{matrix} R \rightarrow \epsilon \\ A \rightarrow wR \end{matrix}$

② $A \rightarrow x_1 x_2 x_3 \dots x_n B$
 $A \rightarrow x_1 [x_2 x_3 \dots x_n B]$ "C"
 $\iff \begin{matrix} A \rightarrow x_1 C \\ C \rightarrow x_2 [x_3 \dots x_n B] \end{matrix}$ "D"

example $G \rightarrow$ NFA

$S \rightarrow abbS \mid T$

$T \rightarrow bba \mid T$

$S \rightarrow abbS$

$S \rightarrow T$

$T \rightarrow bba$

$T \rightarrow T$

\rightarrow $\left[\begin{array}{l} R \rightarrow \epsilon \\ T \rightarrow bbaR \\ S \rightarrow abbS \\ S \rightarrow \epsilon T \\ T \rightarrow \epsilon T \end{array} \right. \rightarrow$

$\begin{array}{l} R \rightarrow \epsilon \\ S \rightarrow \epsilon T \\ T \rightarrow \epsilon T \end{array}$

$T \rightarrow bbaR$
 $\underbrace{\quad}_{A}$
 $\underbrace{\quad}_{B}$

$S \rightarrow a b b S$
 $\underbrace{\quad}_{C}$
 $\underbrace{\quad}_{D}$

$T \rightarrow bB$

$B \rightarrow bA$

$A \rightarrow aR$

$S \rightarrow aD$

$D \rightarrow bC \quad C \rightarrow bS$

$R \rightarrow \epsilon \checkmark$
 $S \rightarrow \epsilon T \checkmark$
 $T \rightarrow \epsilon T \checkmark$

$T \rightarrow bB \checkmark$

$B \rightarrow bA \checkmark$

$A \rightarrow aR \checkmark$

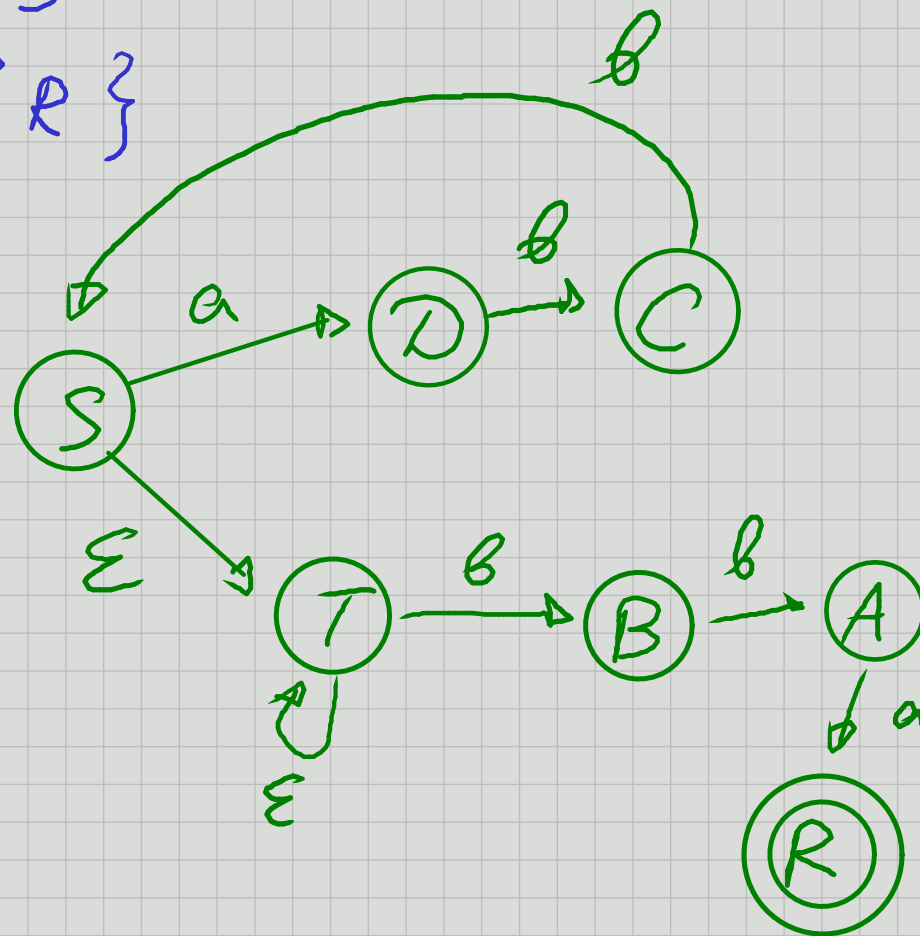
$S \rightarrow aD \checkmark$

$D \rightarrow bC \checkmark$ $C \rightarrow bS \checkmark$

$Q = \{S, R, T, A, B, C, D\}$

$q_0 = S$

$F = \{R\}$



Übungsblatt - 7

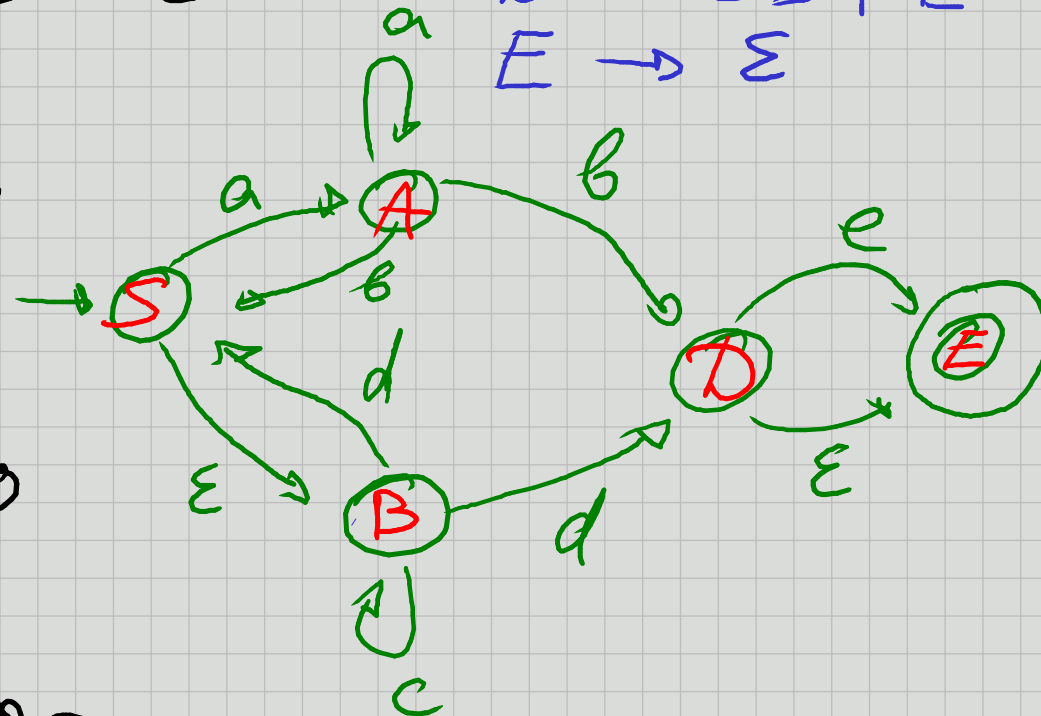
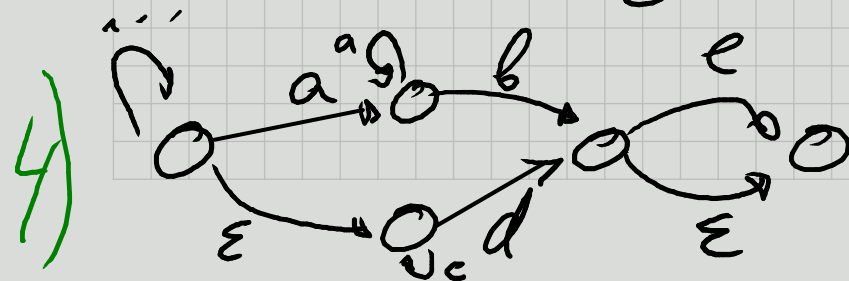
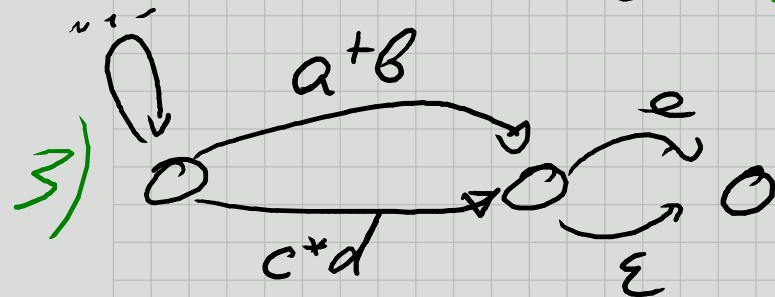
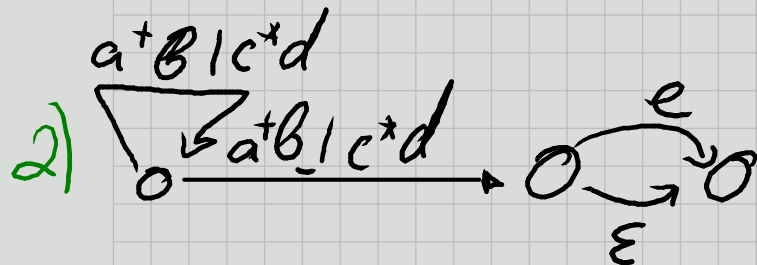
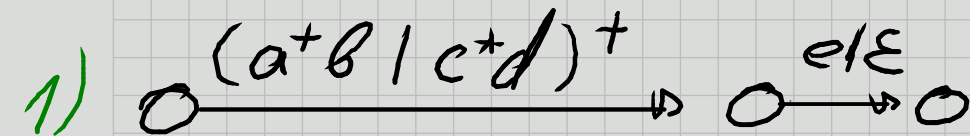
Aufgabe 3:

Sei r ein regulärer Ausdruck der Form $(a^+b|c^*d)^+(e|\varepsilon)$.

- Konstruieren Sie einen Automaten A , sodass $L(A) = L(r)$.
Ist der Automat deterministisch?
- Spezifizieren Sie eine reguläre Grammatik G , sodass $L(G) = L(r)$.

$RE \rightarrow NFA \rightarrow G$

$S \rightarrow aA | B$
 $A \rightarrow aA | bD | bS$
 $B \rightarrow cB | dD | dS$
 $D \rightarrow eE | E$
 $E \rightarrow \varepsilon$



Data Structures - as a way to organize data/memory

1) Array / Vector

linear memory layout

2) Set / Map

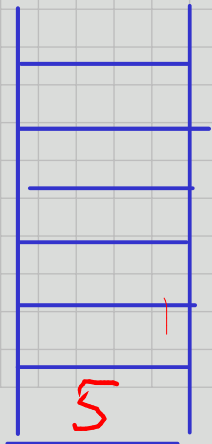
Hashing

3) Stack

LIFO - Last In First Out

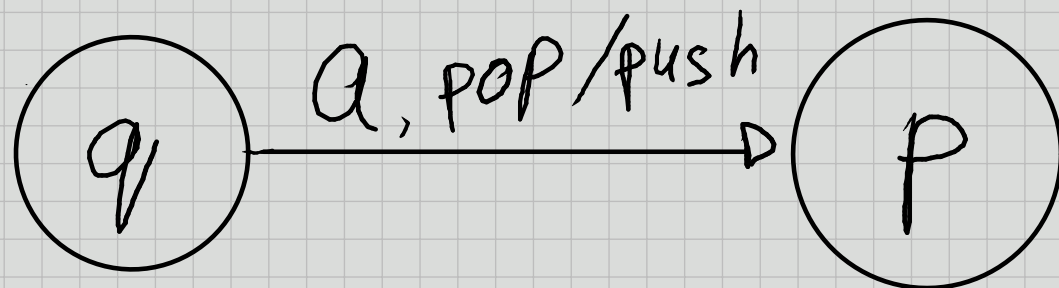
4) Queue

FIFO - First In First Out



push -
pop -

Push Down Automaton = Finite Automaton + Stack



$$M = (Q, q_0, \Sigma, \Gamma, Z_0, \delta, F)$$

stack alphabet

null protector
marker q_{na}

sometimes
we don't
have it

Configuration: $C \in Q \times \Sigma^* \times \Gamma^*$

(q_0, w, Z_0)

part of input

$x_1 x_2 \dots x_n Z_0 \in \Gamma^*$

$$\delta: Q \times (\Sigma \cup \{\epsilon\}) \times \Gamma \rightarrow 2^{Q \times \Gamma^*}$$

we start in some state

we read next symbol of the word (or we don't)

we move to a new state

we can put something on the stack (zero or several symbols)

we look at the top of the stack

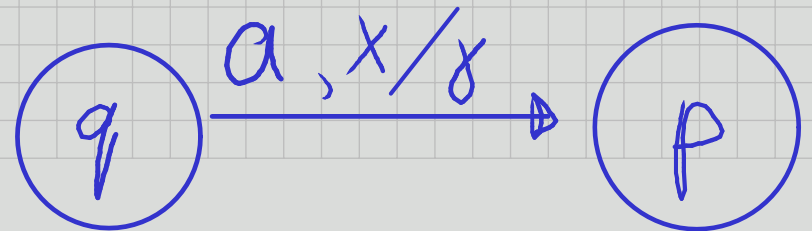
2^M

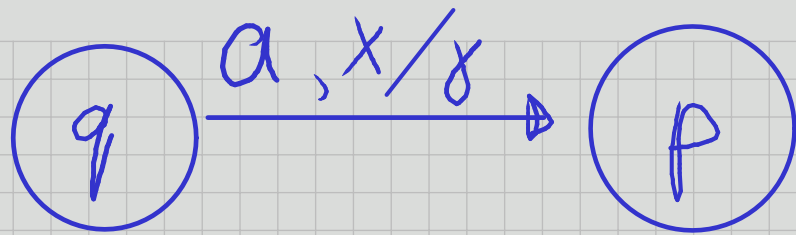
set of all possible subsets

basically implies non-determinism for transitions

$$\delta(q, a, x) \vdash (p, \gamma)$$

$$(p, \gamma) \in \delta(q, a, x)$$





ϵ / δ
 x / ϵ

x
x_n
\vdots
x_1
z_0

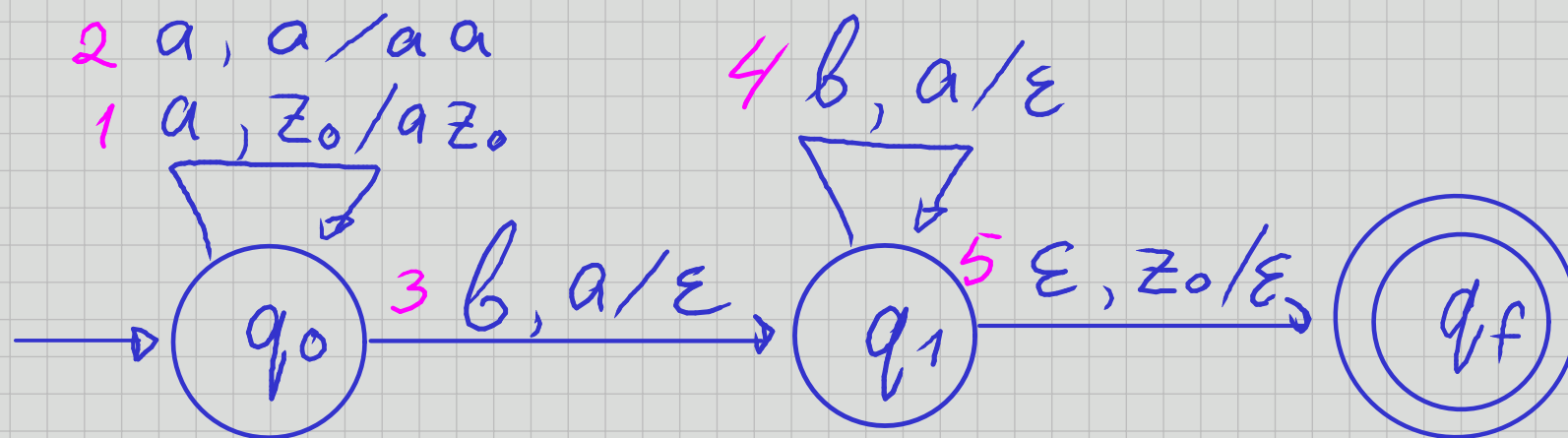
δ_2
δ_1
x_n
\vdots
x_1
z_0

$$\delta = \delta_2 \delta_1$$

configurations

$$(\underline{q}, \underline{a}u, \underline{x}x_n \dots x_1 z_0) \vdash (\underline{p}, \underline{u}, \underline{\delta}x_n \dots x_1 z_0)$$

$$L = \{a^n b^n\}$$



configuration

aa bb

↓

1

$(q_0, aabb, z_0) \vdash (q_0, abb, az_0) \vdash$

2

$\vdash (q_0, bb, aa z_0) \vdash$

3

$(q_1, b, a z_0) \vdash$

4

$\vdash (q_1, \epsilon, z_0) \vdash$

5

$(q_f, \epsilon, \epsilon)$

Automaton accepts word by accepting state

$$w \in L(M) \iff (q_0, w, z_0) \xrightarrow{*} (q_f, \varepsilon, \alpha)$$

$$q_f \in F$$

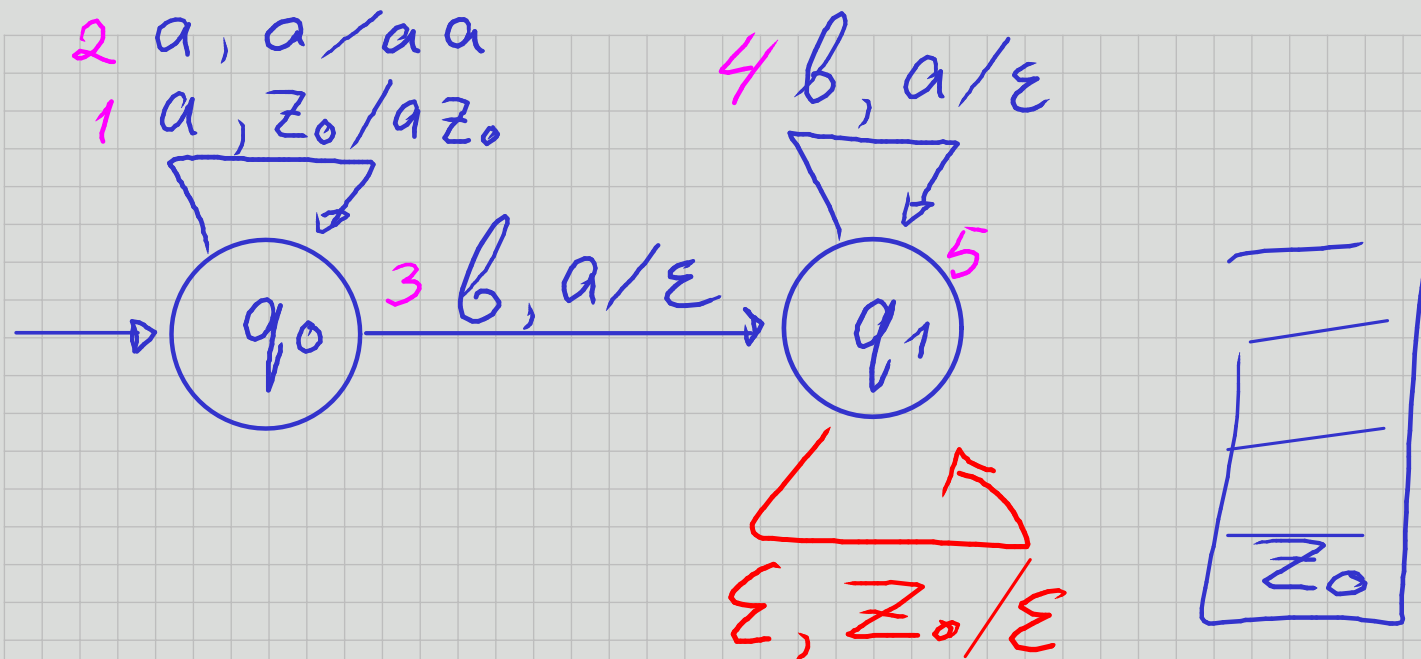
$$\alpha \in \Gamma^*$$

Word belongs to Language (of the automaton) if and only if there exists a computation path from initial configuration to accepting configuration

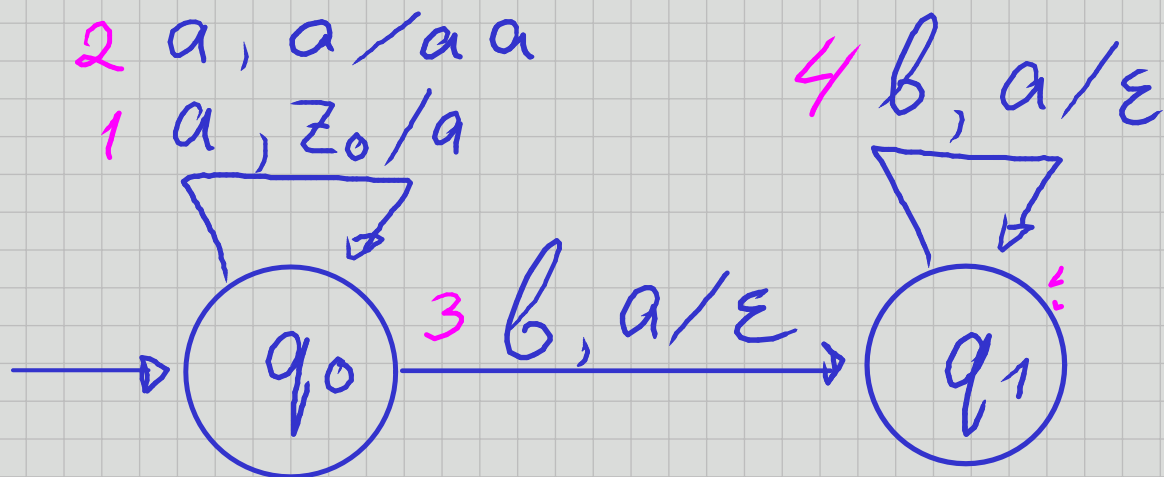
Automaton accepts by empty stack

$$w \in L(M) \iff (q_0, w, z_0) \xrightarrow{*} (p, \varepsilon, \varepsilon)$$

$$p \in Q$$

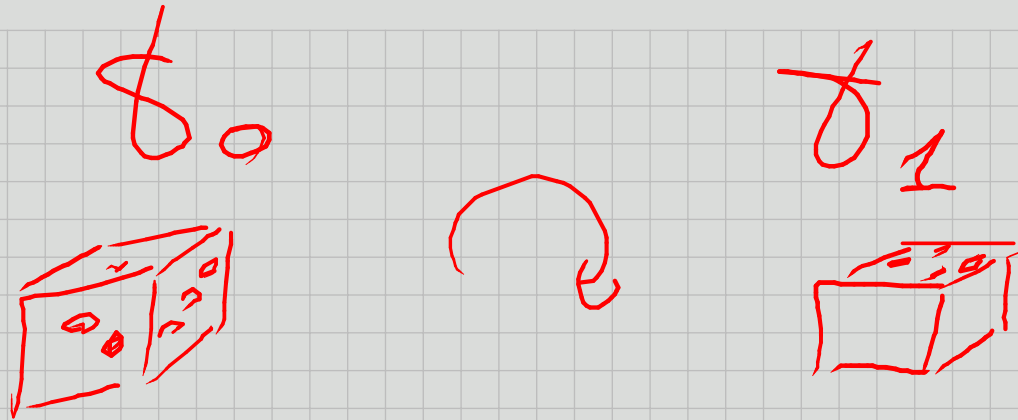


$(q_0, aabbb, z_0) \vdash (q_0, abb, az_0) \vdash (q_0, bbb, aaz_0) \vdash$
 $\vdash (q_1, b, az_0) \vdash (q_1, \epsilon, z_0) \vdash (q_1, \epsilon, \epsilon)$



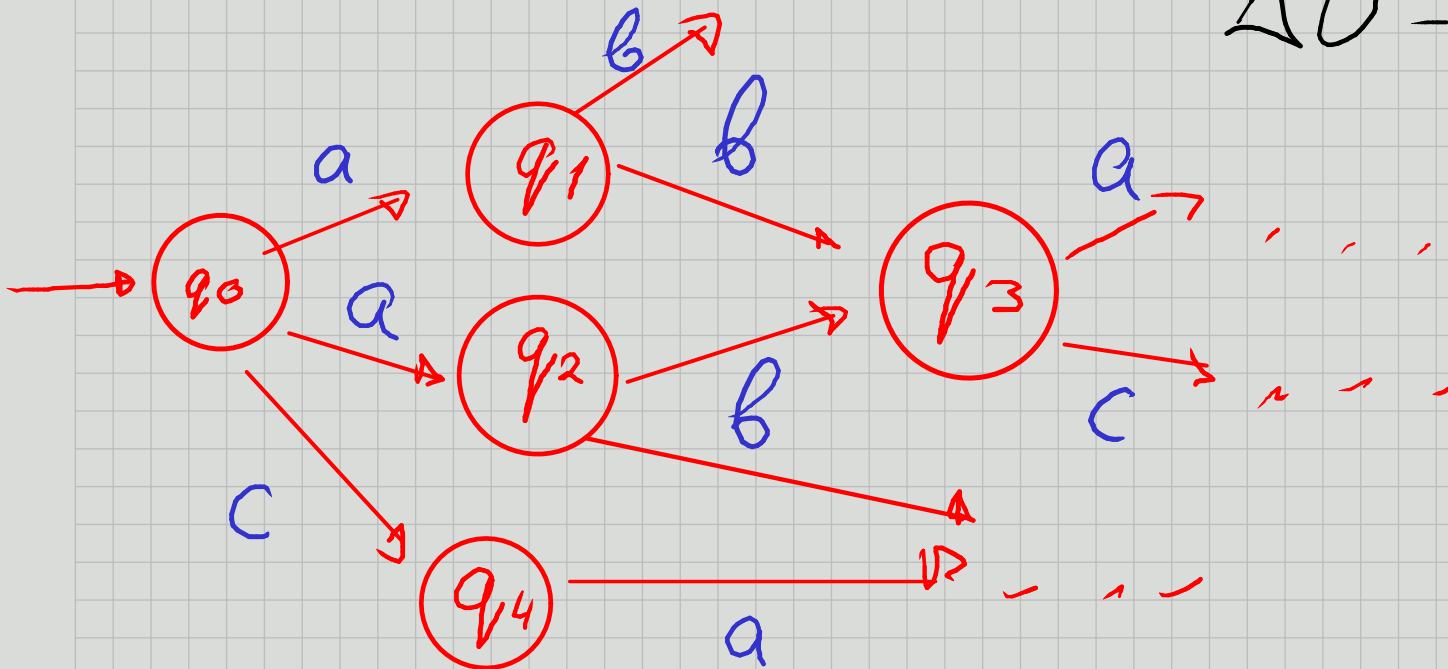
$(q_0, aabbb, z_0) \vdash (q_0, abb, a) \vdash (q_0, bbb, aa) \vdash$
 $\vdash (q_1, b, a) \vdash (q_1, \varepsilon, \varepsilon)$

Determinism

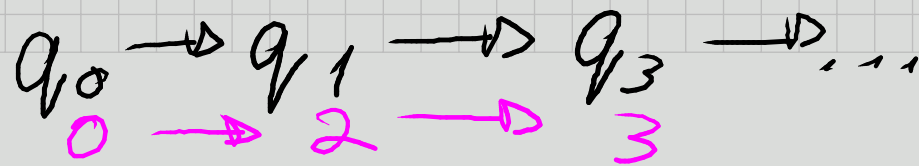


Finite automaton

$$\Delta t = 1s$$



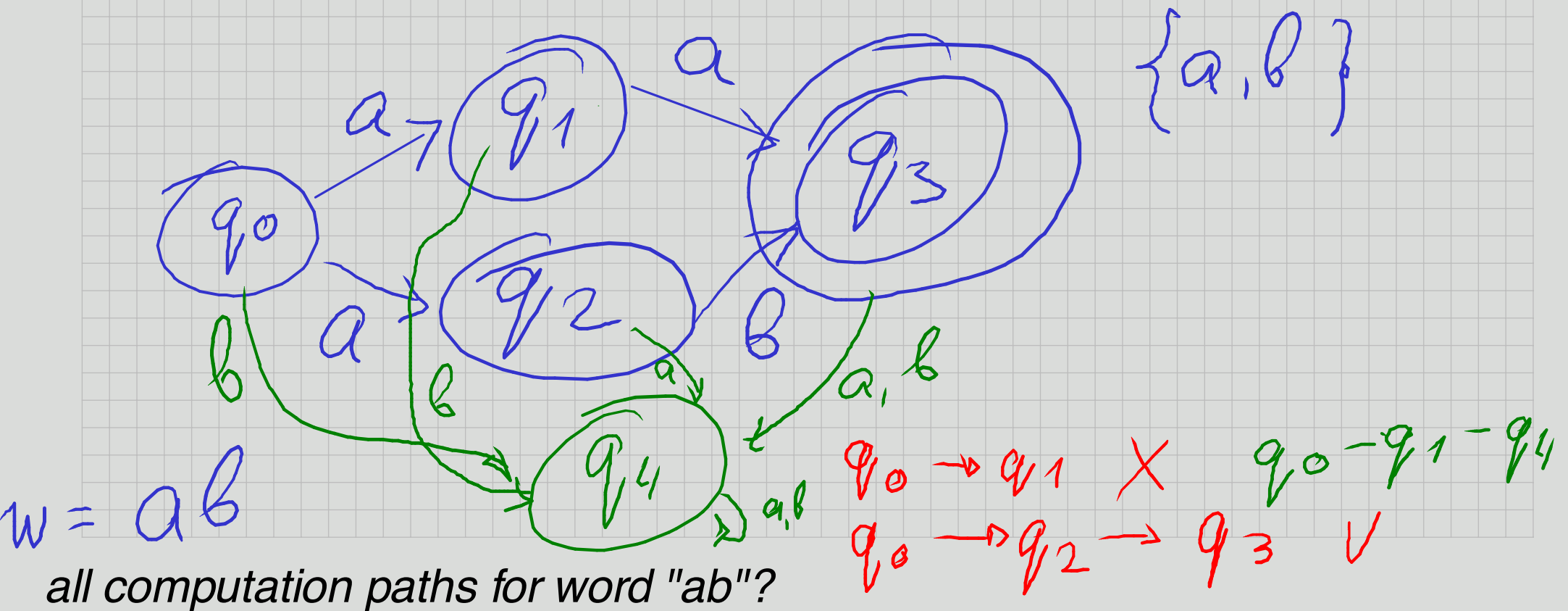
"a b c ..."



Недетерминизм: это когда существуют слова у которых есть альтернативные пути вычисления

Детерминизм: это когда для любого слова существует один единственный путь вычисления

For each (automaton state, alphabet symbol): are transitions unique?



Determinism in PDA ?

1. $\forall a \in \Sigma, \forall x \in \Gamma, \forall q \in Q:$

$$|\delta(q, a, x)| \leq 1$$

2. $\delta(q, \varepsilon, x) \neq \emptyset \Rightarrow$

$$|\delta(q, \varepsilon, x)| = 1$$

$$\left\{ \begin{array}{l} \forall a \in \Sigma \quad \delta(q, a, x) = \emptyset \end{array} \right.$$

NFA \rightarrow DFA

$$L(\text{NFA}) = L(\text{DFA})$$

PDA:

$$L(\text{n-PDA}) \neq L(\text{d-PDA})$$

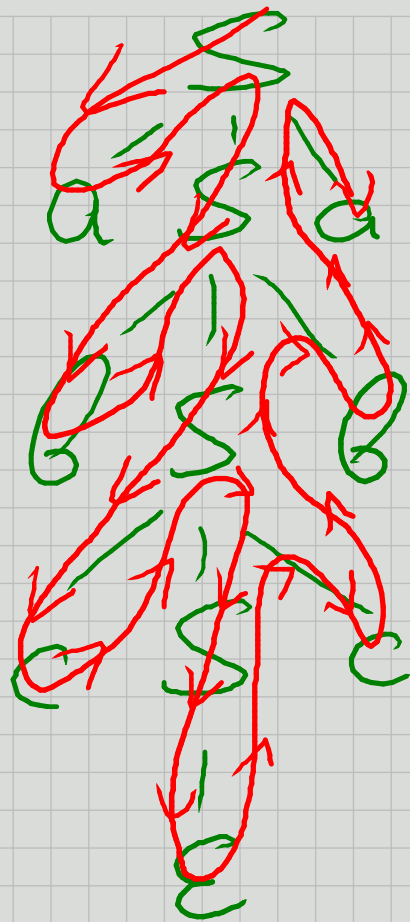
$$L = \{ ww^R \mid w \in \Sigma^* \}$$

$\underbrace{abc}_w \underbrace{cba}_{w^R}$

$$S \rightarrow aSa \mid bSb \mid cSc \mid \varepsilon$$

$$S \rightarrow \alpha S \alpha \mid \varepsilon, \alpha \in \Sigma$$

$$S \Rightarrow a[S]a \Rightarrow a[bSb]a \Rightarrow abcScba \Rightarrow abcba$$



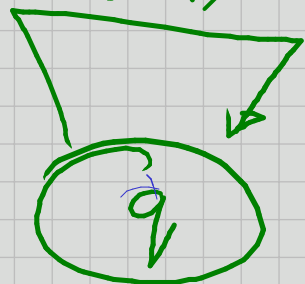
$S \rightarrow aSa$

$a, a/\epsilon$ —

when we read
the last "a"

$a, x/ax$ —

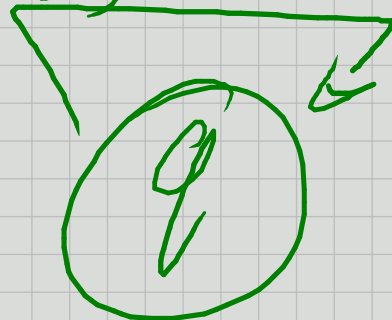
when we read
the first "a"



$S \rightarrow bSb$

$b, b/\epsilon$

$b, x/bx$



$S \rightarrow cSc$

$c, c/\epsilon$

$c, x/cx$

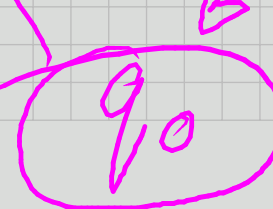
$aaabcc$
 $abccba$

~~$c, c/\epsilon$
 $c, x/cx$~~

~~$b, b/\epsilon$~~

~~$b, x/bx$~~

~~$a, a/\epsilon$
 $a, x/ax$~~



abba cc

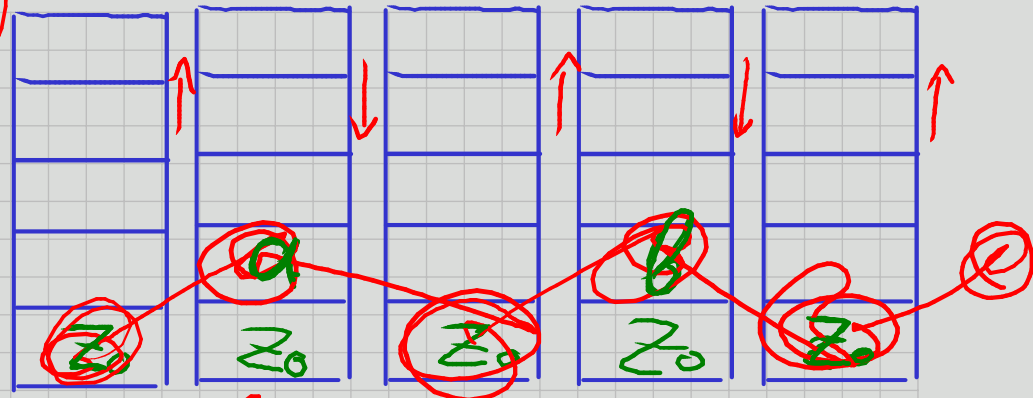
pop c, c/ε
 push c, x/cx
 pop b, b/ε
 push b, x/bx
 pop a, a/ε
 push a, x/ax

pop
 push

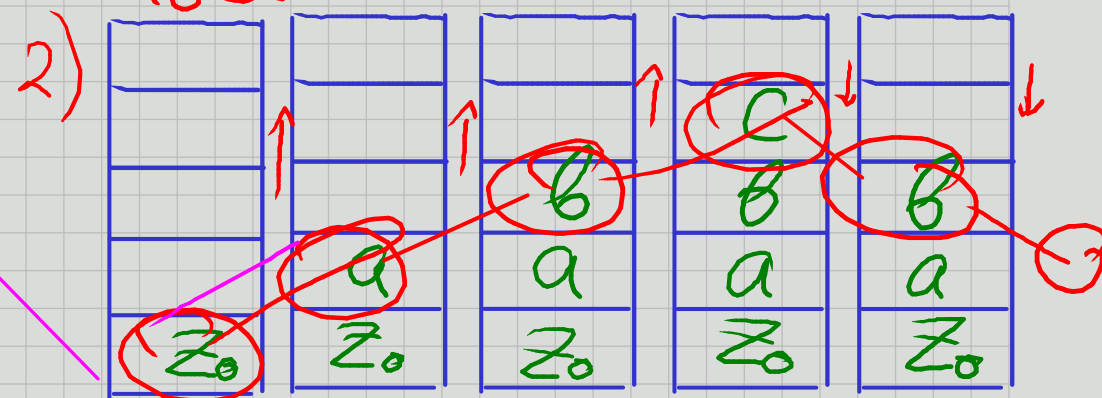
1. aabbcc
2. abccba

q₀

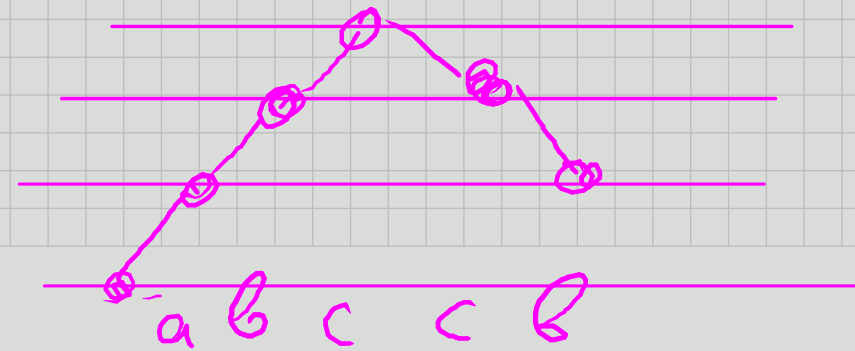
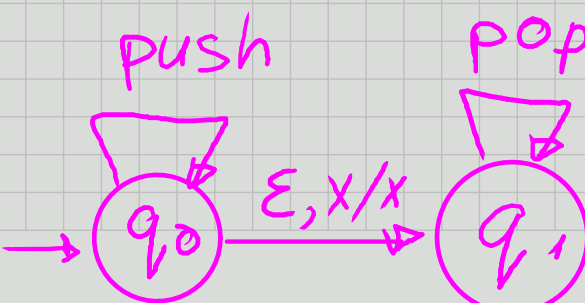
1) aabbcc



abccba



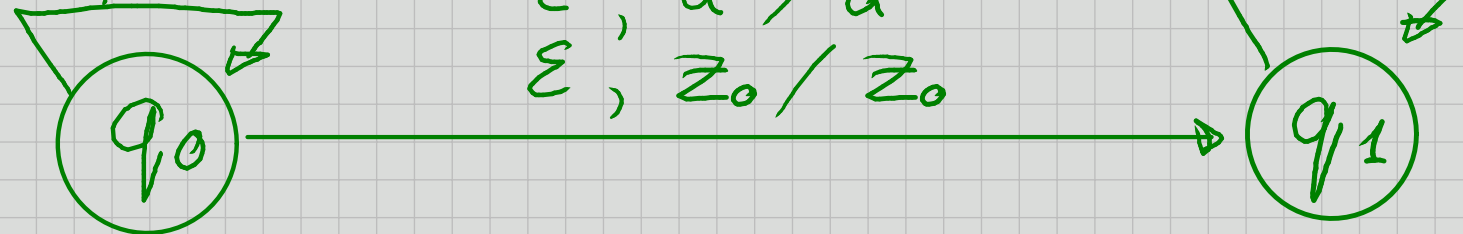
Stack level: does it change monotonically?



$c, c/cc$
 $c, b/cb$
 $c, a/ca$
 $c, z_0/cz_0$
 $b, c/bc$
 $b, b/bb$
 $b, a/ba$
 $b, z_0/bz_0$
 $a, c/ac$
 $a, b/ab$
 $a, a/aa$
 $a, z_0/az_0$

$\epsilon, c/c$
 $\epsilon, b/b$
 $\epsilon, a/a$
 $\epsilon, z_0/z_0$

$\epsilon, z_0/\epsilon$
 $c, c/\epsilon$
 $b, b/\epsilon$
 $a, a/\epsilon$



$$\Sigma = \{a, b, c\}$$

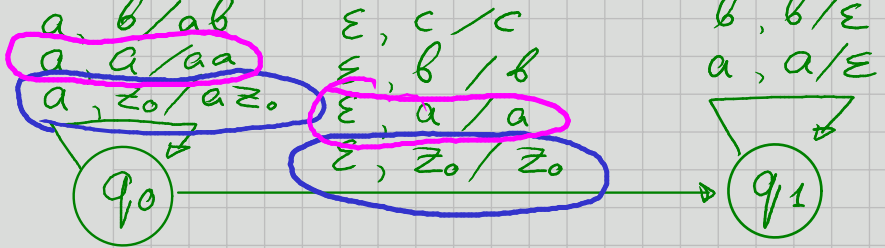
$$\sigma, x / \sigma x \quad \sigma \in \Sigma \quad x \in \Gamma$$

$c, c/cc$
 $c, b/cb$
 $c, a/ca$
 $c, z_0/cz_0$
 $b, c/bc$
 $b, b/bb$
 $b, a/ba$
 $b, z_0/bz_0$
 $a, c/ac$
 $a, b/ab$
 $a, a/aa$
 $a, z_0/az_0$

$$\Sigma = \{a, b, c\}$$

$\epsilon, c/c$
 $\epsilon, b/b$
 $\epsilon, a/a$
 $\epsilon, z_0/z_0$

$\epsilon, z_0/\epsilon$
 $c, c/\epsilon$
 $b, b/\epsilon$
 $a, a/\epsilon$



$$\vdash (q_1, bbbaa, z_0)$$

$$\vdash (q_1, abbaa, az_0)$$

$$(q_0, aabbaa, z_0) \vdash (q_0, abbaa, az_0) \vdash (q_0, bbbaa, aaz_0)$$

$$\quad \quad \quad \nwarrow \quad \quad \quad \swarrow$$

$$\quad \quad \quad (q_1, aabbaa, z_0)$$

Determinism: let's take a look at following rules:

$$(q_0, a, z_0) \vdash (q_0, az_0) \text{ and } (q_0, \epsilon, z_0) \vdash (q_1, z_0)$$

Since we have epsilon-rule, determinism would require that "z0" doesn't appear in the configurations corresponding to state q_0
 But we have a rule $(q_0, a, z_0) \vdash (q_0, az_0)$
 which is a contradiction.

Theorem:

$$G \in \text{Type 2} \Rightarrow \exists M \text{ PDA} \text{ s.t. } L(M) = L(G)$$

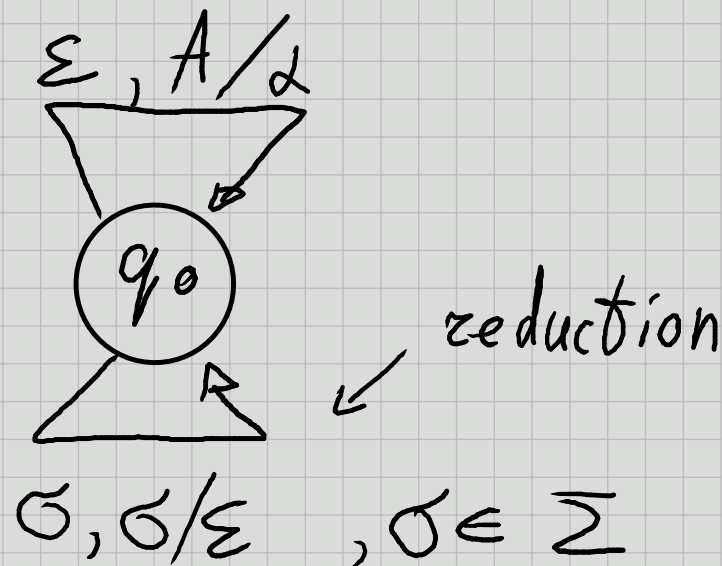
CF

- M accepts by empty stack

- $Z_0 = S$

- $Q = \{q_0\}$

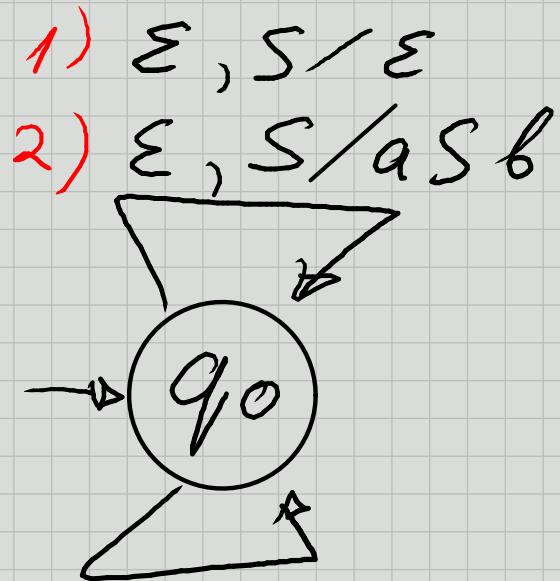
- $A \rightarrow \alpha \in P$



example: $L = \{a^n b^n\}$

$G: S \rightarrow aSb$
 $S \rightarrow \varepsilon$

$z_0 = S$



$w = aaaa bbbb$

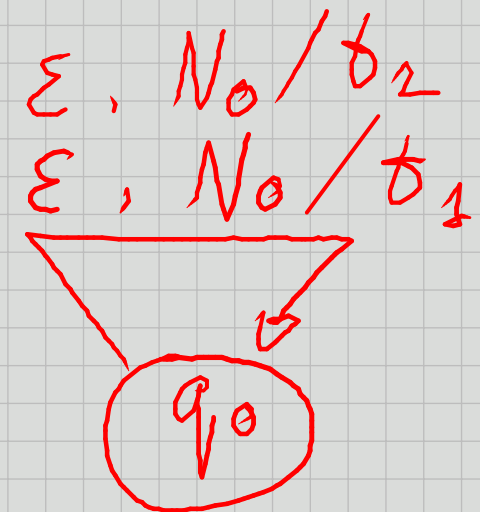
3) $a, a/\varepsilon$
 4) $b, b/\varepsilon$

$(q_0, aaaa bbbb, S) \xrightarrow{2} (q_0, aaaa bbbb, aSb) \xrightarrow{3} (q_0, aabb bbb, Sb) \xrightarrow{2}$
 $\xrightarrow{2} (q_0, aabb bbb, aSbb) \xrightarrow{3} (q_0, abbb, Sb b) \xrightarrow{2}$
 $\xrightarrow{2} (q_0, abbb, aSbbb) \xrightarrow{2} (q_0, bbb, Sb b b) \xrightarrow{2} (q_0, bbb, bbb)$
 $\xrightarrow{2} (q_0, bb, bb) \xrightarrow{2} (q_0, b, b) \xrightarrow{2} (q_0, \varepsilon, \varepsilon)$

$$\exists N_0 \in N$$

$$N_0 \rightarrow t_1 \mid t_2$$

$$N_0 \rightarrow t_1$$



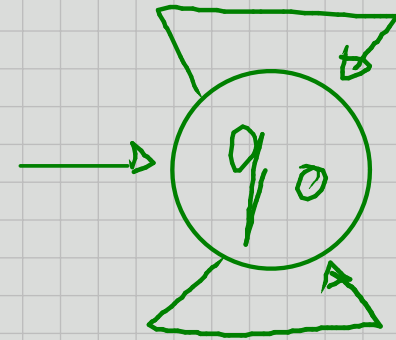
If our Language consists of more than one word, there should be a "fork" in Grammar's rules.

$$L = \{ w w^R \mid w \in \Sigma^* \}$$

G: $S \rightarrow a S a$
 $S \rightarrow b S b$
 $S \rightarrow c S c$
 $S \rightarrow \epsilon$

M:

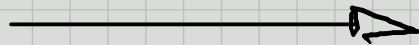
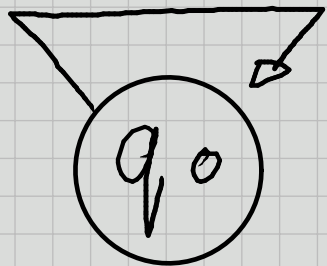
$\epsilon, S / \epsilon$
 $\epsilon, S / c S c$
 $\epsilon, S / b S a$
 $\epsilon, S / a S a$



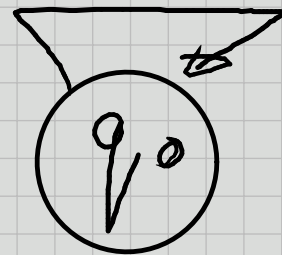
$a, a / \epsilon$
 $b, b / \epsilon$
 $c, c / \epsilon$

Hint: $M(ww^R) \sim M(a^n b^n)$

$b, b / \varepsilon$ pop
 $b, x / bx$ push
 $a, a / \varepsilon$ pop
 $a, x / ax$ push

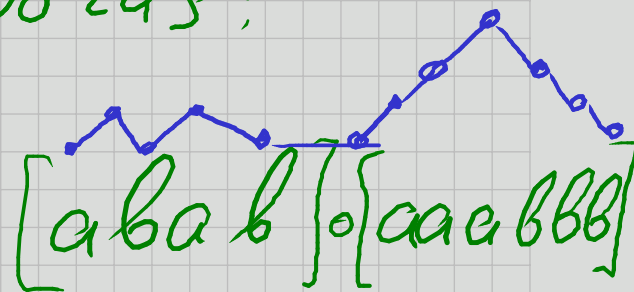


$\varepsilon, z_0 / \varepsilon$
 $b, a / \varepsilon$ pop
 $a, x / ax$ push



examples of words:

$aaabbb$
 $abab$



$a^n b^n \mid n=1$



$(q_0, [abab], (z_0)) \vdash (q_0, bab, az_0) \vdash (q_0, \downarrow ab, (z_0)) \vdash$
 $\vdash (q_0, b, az_0) \vdash (q_0, \varepsilon, z_0)$

$$1) \begin{matrix} u \in L(M) \\ v \in L(M) \end{matrix} \Rightarrow uv, vu \in L(M)$$

$$2) u, v \sim a^n b^n$$

Grammar?

$$2) u, v \sim a^n b^n$$

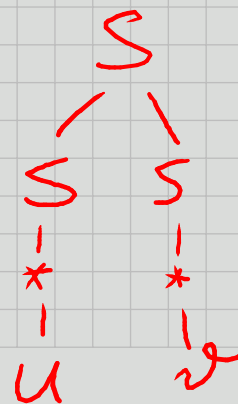
$$S \rightarrow aSb \mid \varepsilon$$

$$1) u, v \in L \Rightarrow \begin{matrix} uv \\ vu \end{matrix} \in L$$

$$S \rightarrow SS$$

$$S \rightarrow SS \mid aSb \mid \varepsilon$$

Dyck
Language

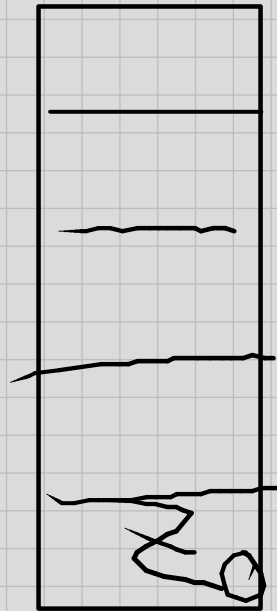
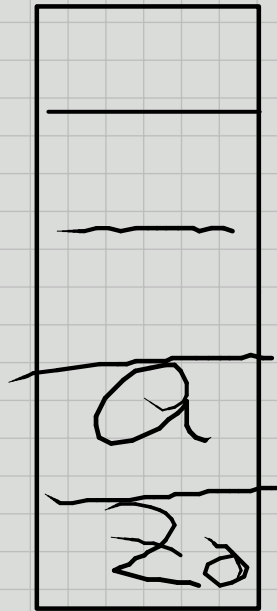
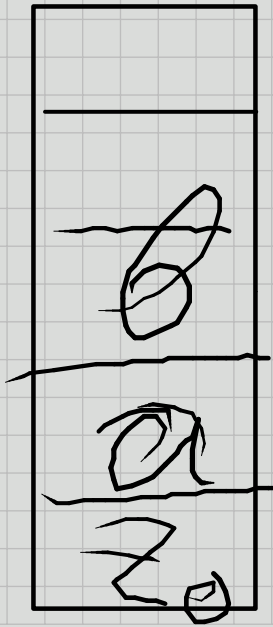
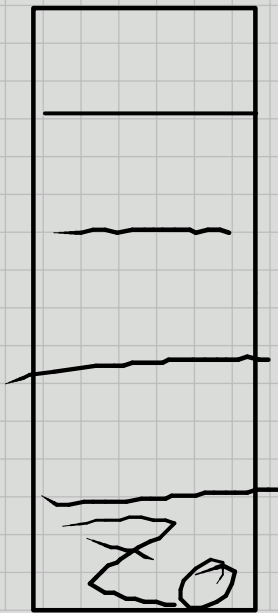


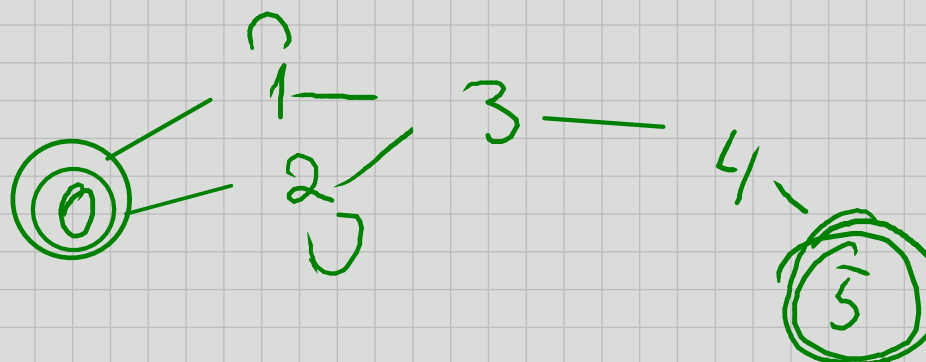
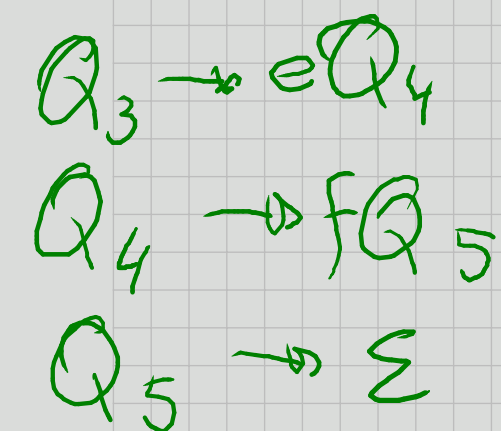
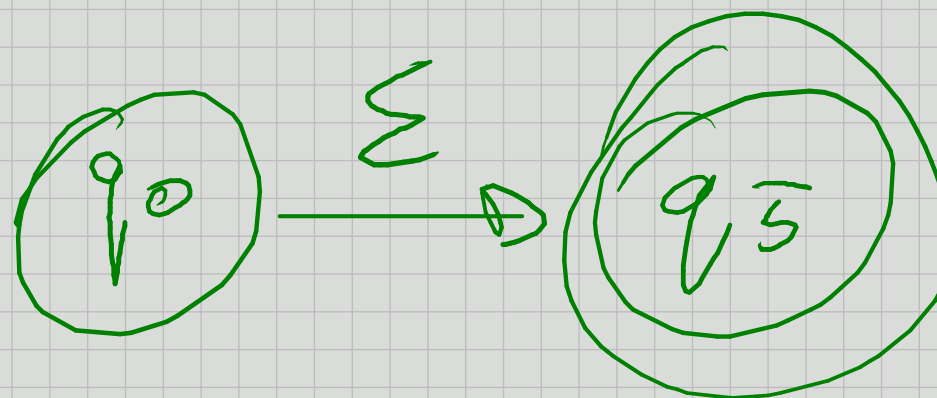
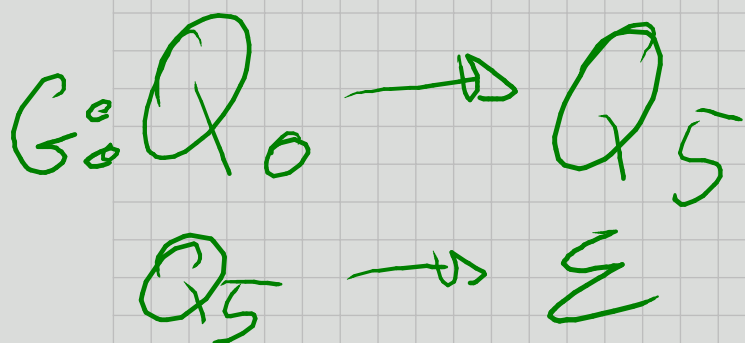
- 1) $u, v \in L \Rightarrow uv \in L$
- 2) $ww^R \in L$

ww^R

$abba$

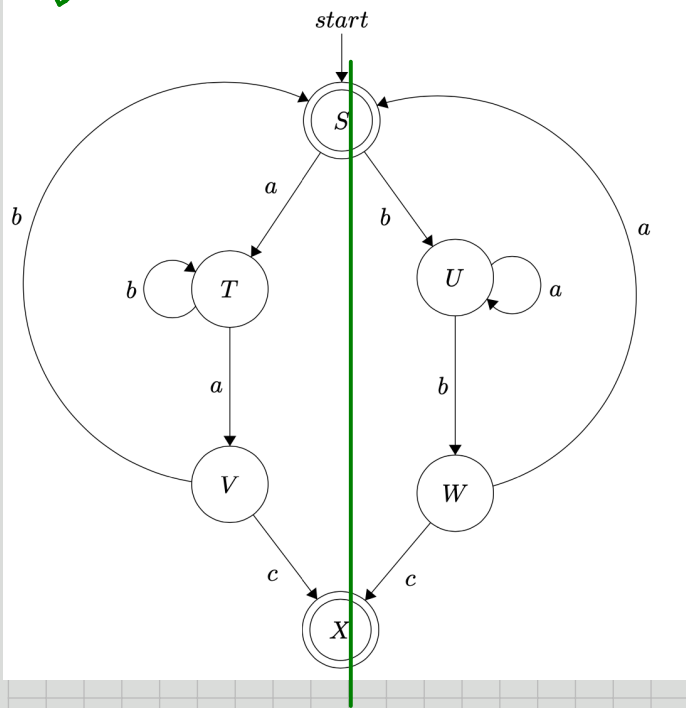
$[aa\ bb]$



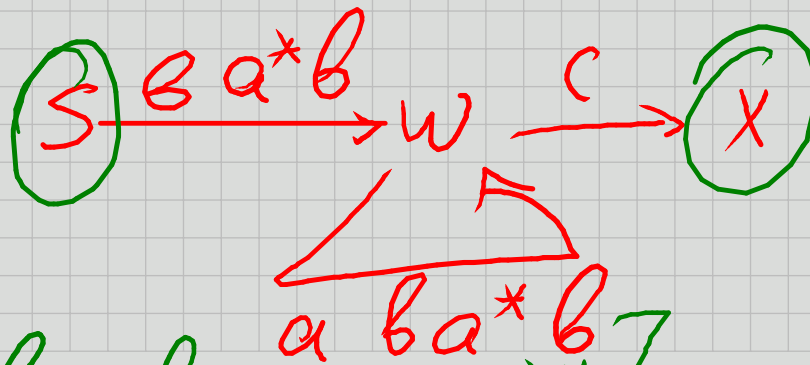
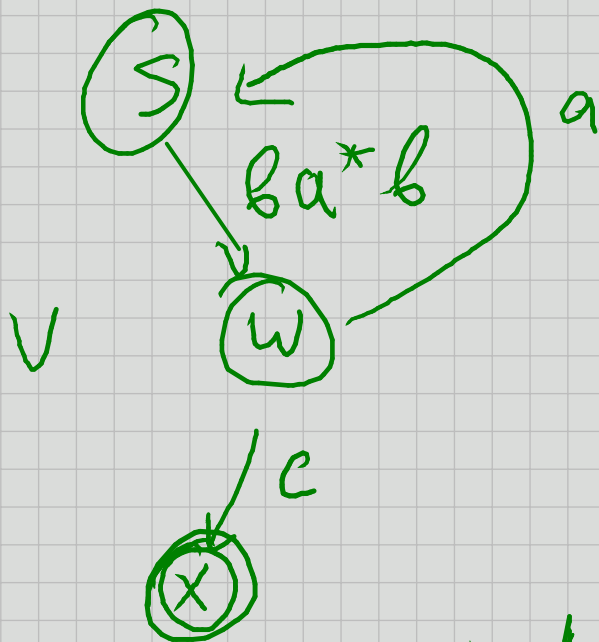


Lets rewrite our grammar in more convinient (equivalent) form

7-2



$S \rightarrow bU \mid aT \mid \epsilon$
 $U \rightarrow aU \mid bW$
 $T \rightarrow bT \mid aV$
 $W \rightarrow aS \mid cX$
 $V \rightarrow bS \mid cX$
 $X \rightarrow \epsilon$



$[ab^*a(bab^*a)^* \mid ba^*b(aba^*b)^*]c$

$\{a, b, c\}$ $aa[baa[baaaCaaaaaa]$
 $\{a, \cdot, =\}$

$$a^n = a^n$$

$$a^n b^n$$

$$G: S \rightarrow aSb \mid \varepsilon$$

$$a^n = b^n \quad G: S \rightarrow aSb \mid =$$

$$S \rightarrow aSb \rightarrow aaSbb \rightarrow aaaSbbb \rightarrow aaaa = bbbb$$

$$a^n = a^n$$

$$G: S \rightarrow aSa \mid =$$

$$\underbrace{ba^n}_{L_1} = \underbrace{a^n}_{L_2}$$

$$L_1 \quad L_2$$

$$L = L_1 L_2 \rightarrow G = S \rightarrow S_1 S_2$$

$$G: S \rightarrow S_1 S_2$$

$$S_1 \rightarrow b$$

$$S_2 \rightarrow aS_2 a \mid =$$

$$S \rightarrow bT$$

$$T \rightarrow aTa \mid =$$

$$a^n \sim a^n$$

$$\bullet a^n = a^n$$

$$a^n \cdot a^k = a^k a^n$$

$$G: S \rightarrow aSa \mid \varepsilon$$

$$S \rightarrow \cdot T$$

$$T \rightarrow aTa \mid =$$

$$\left[\begin{array}{l} S \rightarrow aSa \mid w \\ w \rightarrow \cdot T \\ T \rightarrow aTa \mid = \end{array} \right.$$

$$\begin{array}{l} S \rightarrow aSa \mid \cdot T \\ T \rightarrow aTa \mid = \end{array}$$

$$a^n \sim a^n$$

$$\bullet a^m \cdot a^k = a^k a^m$$

$$a^n \cdot a^m \cdot a^k = a^k a^m a^n$$

$$\left[\begin{array}{l} S \rightarrow aSa \mid \cdot T \\ T \rightarrow aTa \mid \cdot U \\ U \rightarrow aUa \mid = \end{array} \right.$$

$$\begin{aligned}
 S &\Rightarrow aSa \Rightarrow a.Ta \Rightarrow a.aTa \Rightarrow \\
 &\Rightarrow a.a.Ua \Rightarrow a.a.alLa \Rightarrow \\
 &\Rightarrow a.a.aalLa \Rightarrow \\
 &\Rightarrow a.a.a = aaaa
 \end{aligned}$$

8-5

$$L = \{a^m b^n c^n a^{m+n}\} = \{a^m \underbrace{b^n c^n a^n} a^m\}$$

$$\Sigma^+ a^n b^n$$

$$b \Sigma^+ \Sigma^+ a$$

$$a^+ a^n b^n$$

$$a^n b^n b^+$$

$$\Sigma^+ b a \Sigma^+$$

REG? %



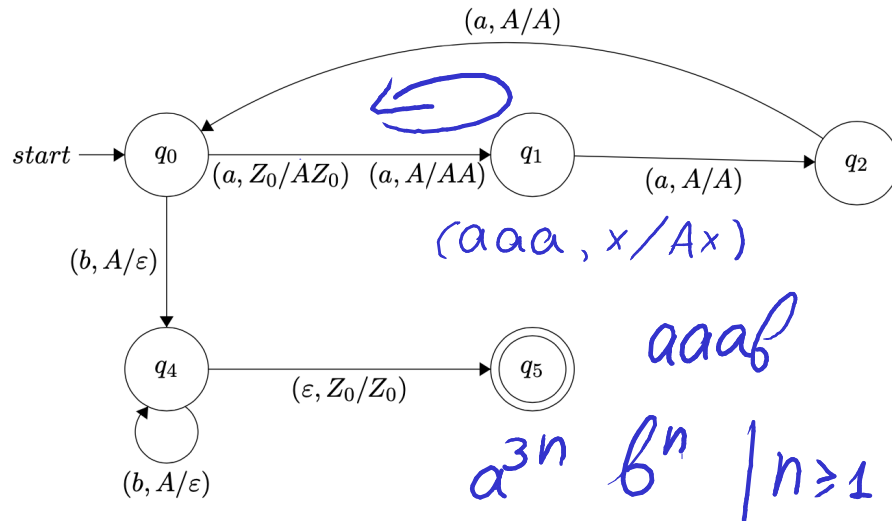
Übungsblatt 9

Aufgabe 1:

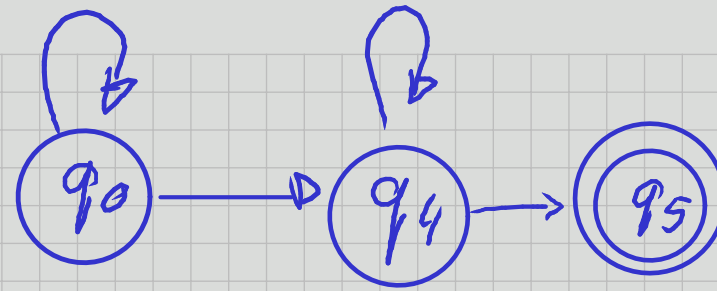
Gegeben sei ein deterministischer Kellerautomat

$$KA = (\{q_0, q_1, q_2, q_3, q_4, q_5\}, \{a, b\}, \{A, Z_0\}, \delta, q_0, Z_0, \{q_5\})$$

mit Übergangsfunktion δ wie folgt:



Welche Sprache akzeptiert KA durch Endzustand? Erklären sie die Arbeitsweise des Kellerautomaten durch Analyse eines Wortes (z.B. Wort $aaab$: $(q_0, aaab, Z_0) \vdash \dots$).



a^{3n}

b^{3n}

$(q_0, aaab, z_0) \vdash (q_1, aab, AZ_0) \vdash (q_2, ab, AZ_0) \vdash$
 $\vdash (q_0, b, AZ_0) \vdash (q_4, \varepsilon, Z_0) \vdash (q_5, \varepsilon, \varepsilon)$

Übungsblatt 9

Aufgabe 2:

Konstruieren Sie einen Kellerautomaten, der die Sprache

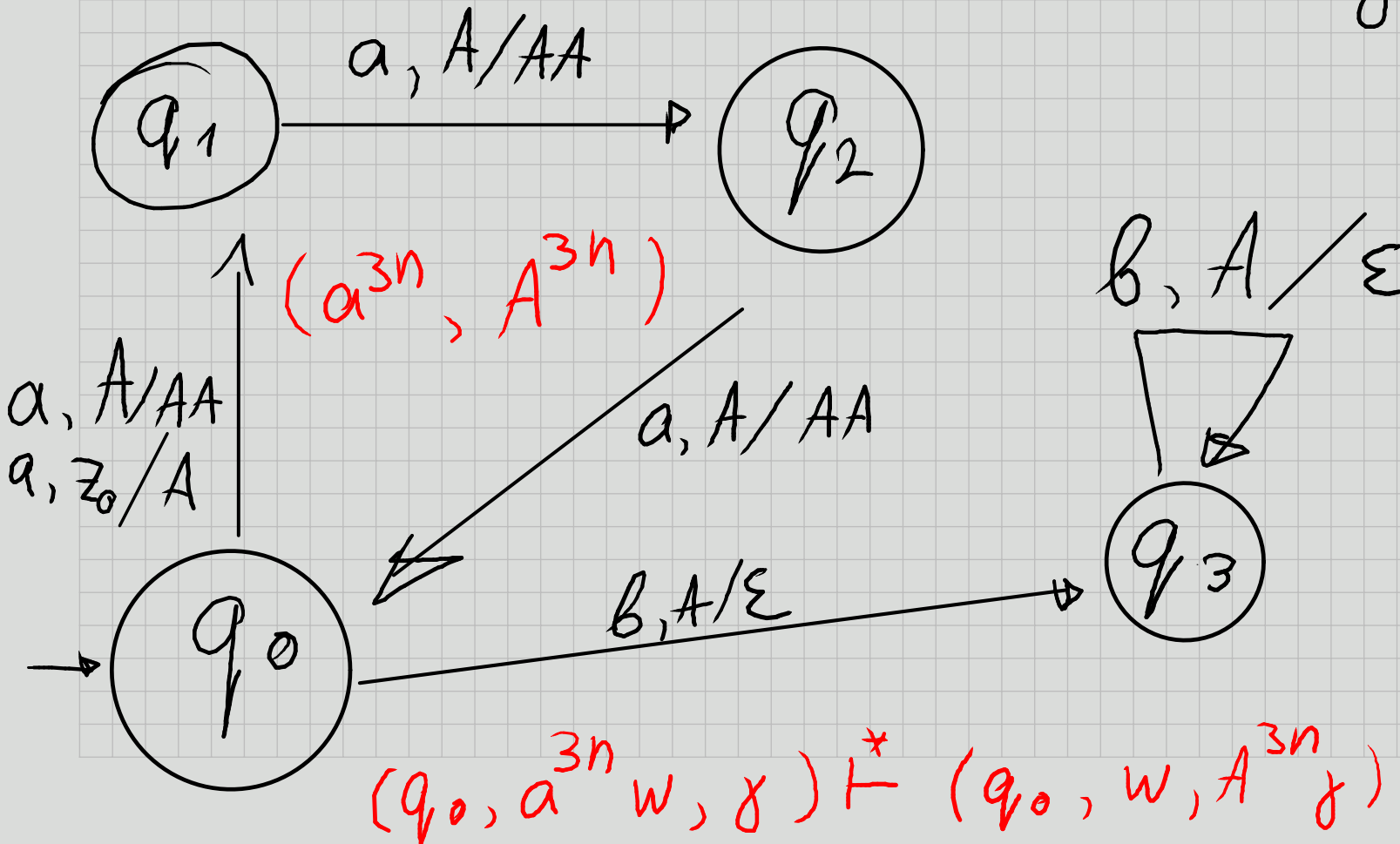
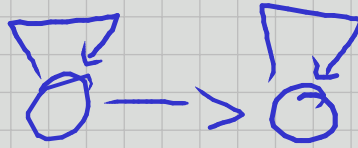
$$L = \{a^n b^n \mid n \bmod 3 = 0 \wedge n > 0\}$$

durch Endzustand akzeptiert. Ist Ihr Kellerautomat deterministisch?

$$n = 3, 6, 9, 12, \dots$$

PDA accepts
by empty stack

$$a^n b^n$$



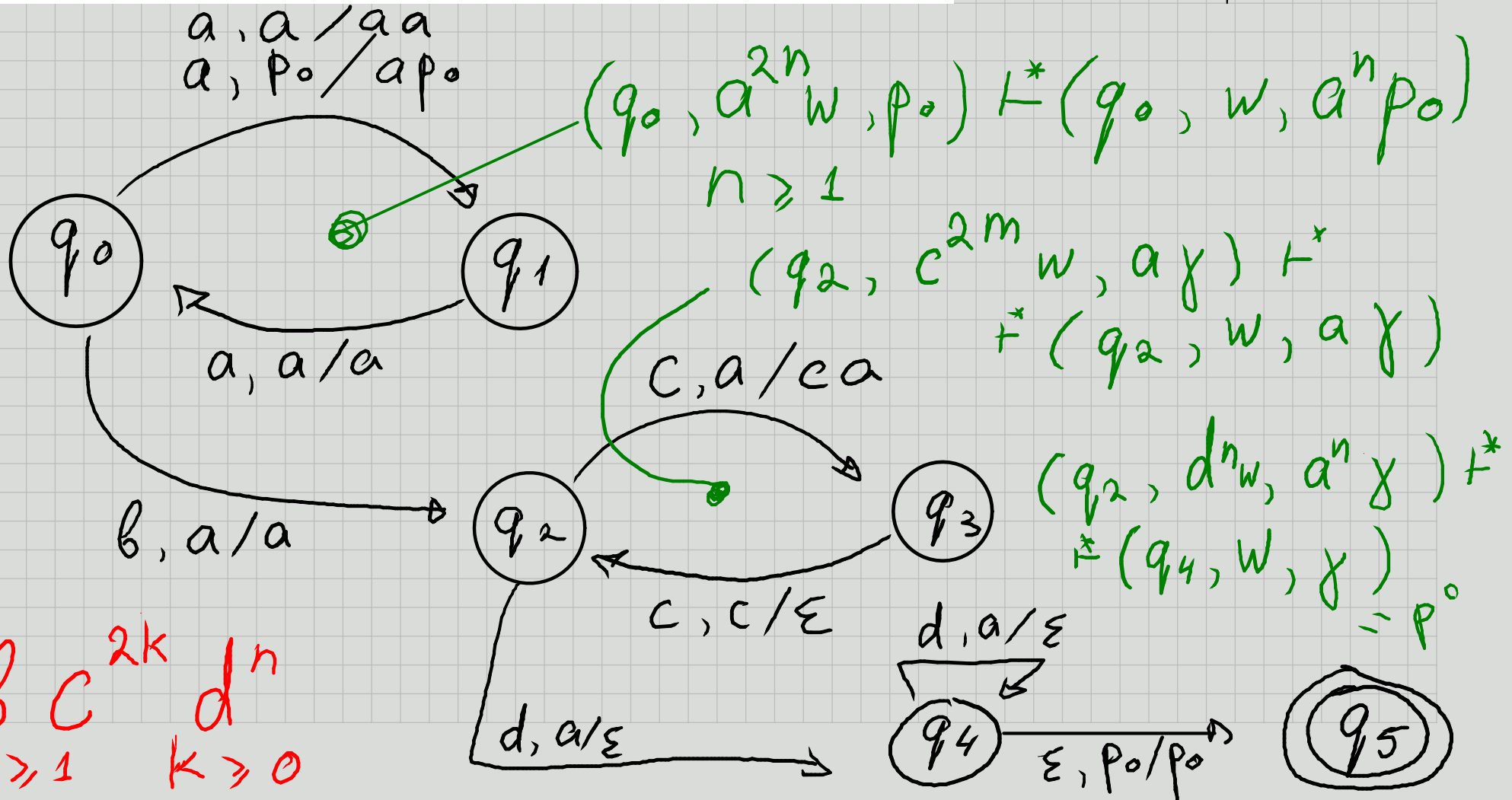
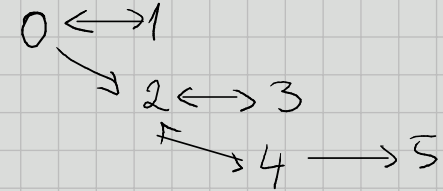
Übungsblatt 9

Aufgabe 3:

Gegeben sei ein Kellerautomat $KA = (\{q_0, q_1, q_2, q_3, q_4, q_5\}, \{a, b, c, d\}, \{p_0, a, c\}, \delta, q_0, p_0, \{q_5\})$ mit Übergangsfunktion δ wie folgt:

- | | |
|---|--|
| (1) $\delta(q_0, a, p_0) = \{(q_1, ap_0)\}$ | (6) $\delta(q_3, c, c) = \{(q_2, \varepsilon)\}$ |
| (2) $\delta(q_1, a, a) = \{(q_0, a)\}$ | (7) $\delta(q_2, d, a) = \{(q_4, \varepsilon)\}$ |
| (3) $\delta(q_0, a, a) = \{(q_1, aa)\}$ | (8) $\delta(q_4, d, a) = \{(q_4, \varepsilon)\}$ |
| (4) $\delta(q_0, b, a) = \{(q_2, a)\}$ | (9) $\delta(q_4, \varepsilon, p_0) = \{(q_5, p_0)\}$ |
| (5) $\delta(q_2, c, a) = \{(q_3, ca)\}$ | |

Welche Sprache akzeptiert KA durch Endzustand?



Übungsblatt 9

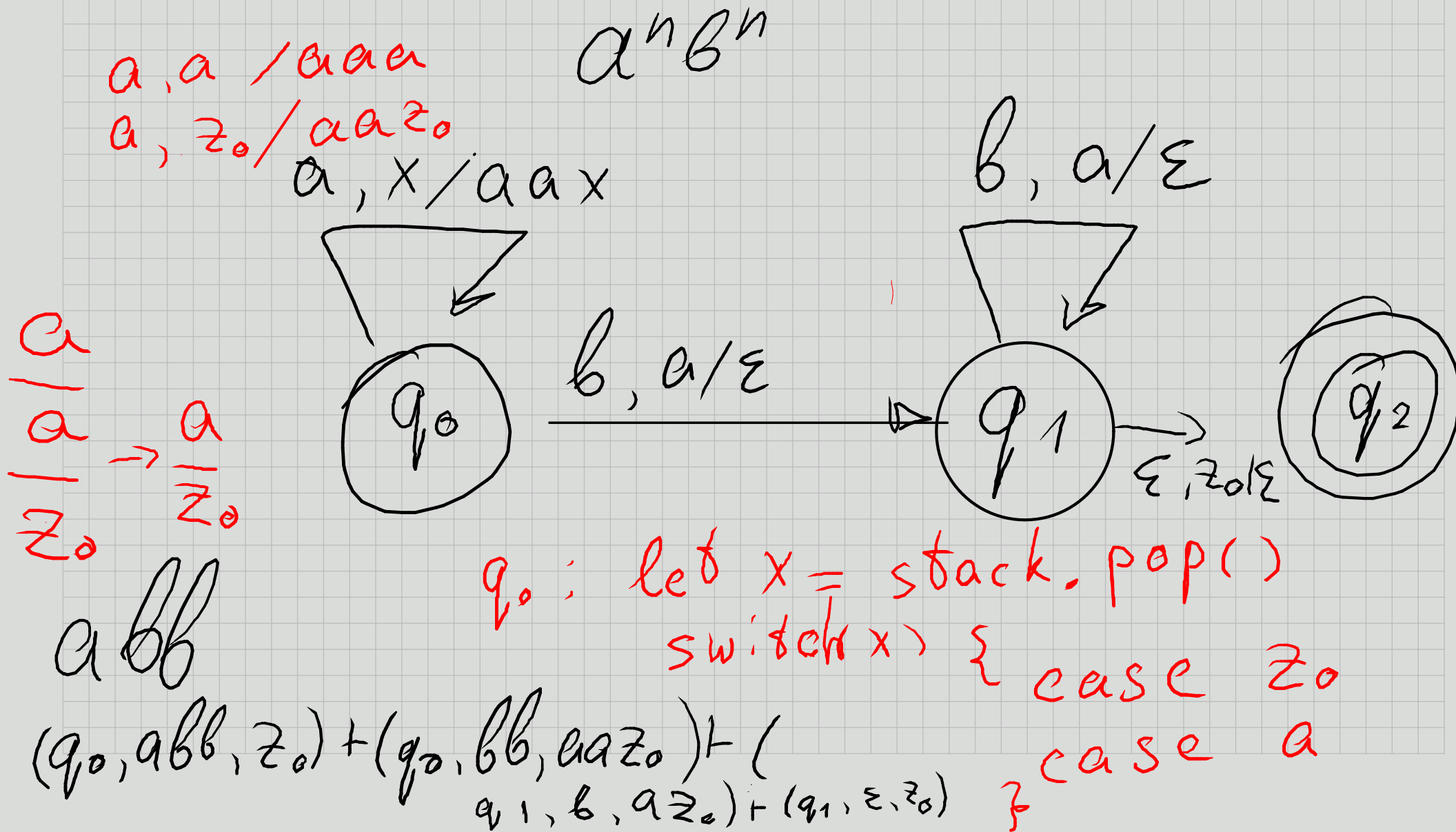
Aufgabe 4:

Konstruieren Sie einen deterministischen Kellerautomaten, der die Sprache

$$L = \{a^n b^{2n} \mid n \geq 1\}$$

durch Endzustand akzeptiert. Erklären sie die Arbeitsweise des Kellerautomaten durch Analyse eines Wortes (z.B. Wort $aabbbb$: $(q_0, aabbbb, Z_0) \vdash \dots$).

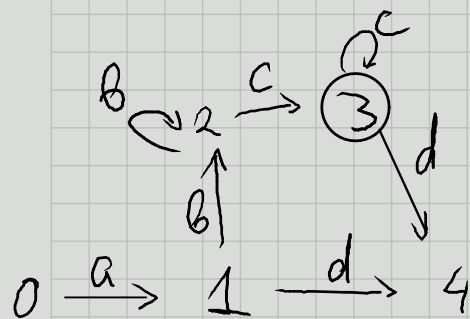
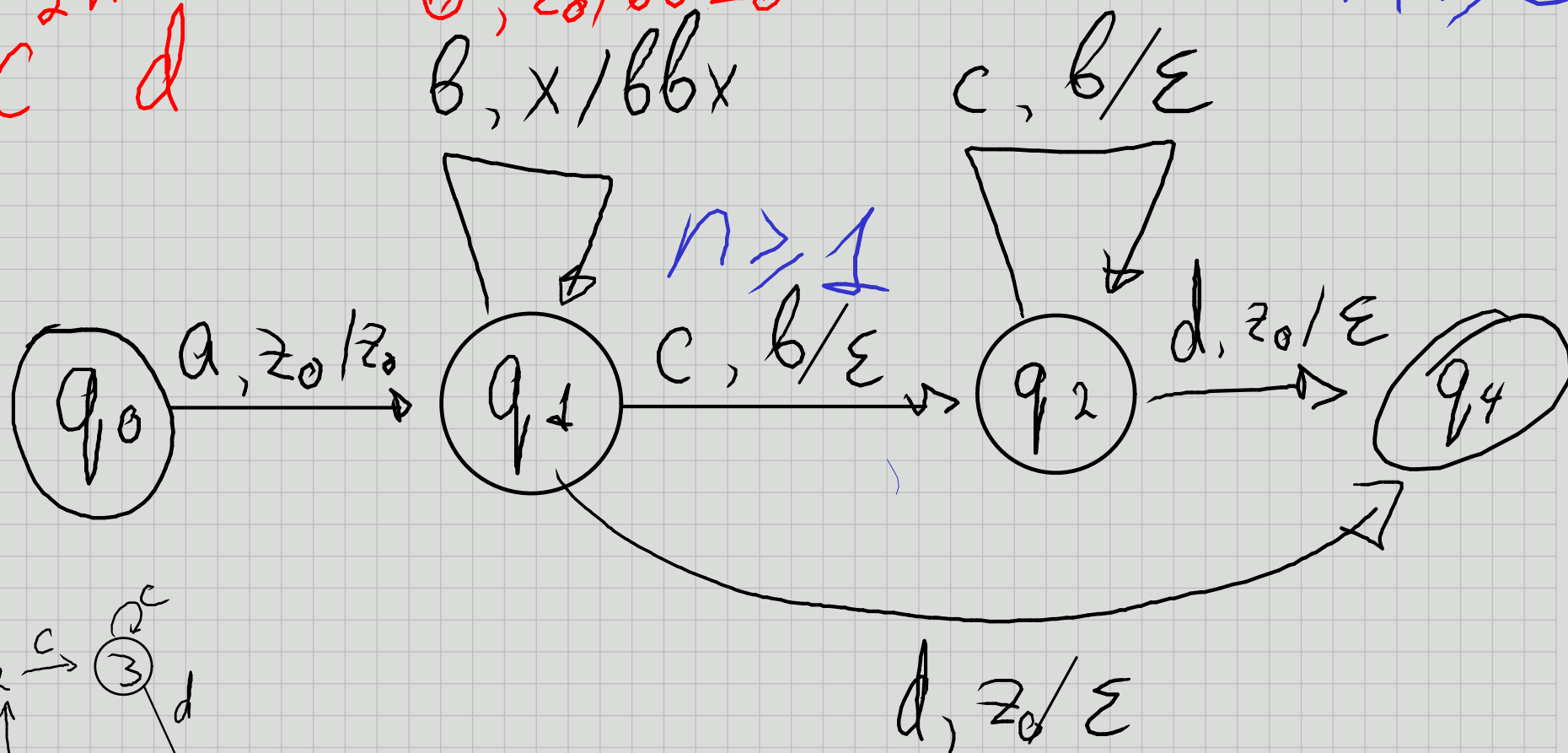
$$n \geq 1$$



$a b^n c^{2n} d$

$b, b / bbb$
 $b, z_0 / bbb z_0$
 $b, x / bbb x$

$n \geq 0$



$(q_0, abcccd, z_0) \vdash (q_1, bcccd, z_0) \vdash (q_1, ccd, bbb z_0) \vdash$
 $\vdash (q_2, cd, b z_0) \vdash (q_2, d, z_0) \vdash (q_4, \epsilon, \epsilon)$

$$L = \{a^m b^n \mid m > n, n \geq 0\} \cup \{a^m c^n d^n a^m \mid m \geq 1, n \geq 0\}$$

$$a^m b^n$$

~~REG~~

S_1

$$a^m c^n d^n a^m$$

CF

S_2

$$S \rightarrow S_1 \mid S_2$$

~~$S_1 \rightarrow a S_1 \mid T_1$
 $T_1 \rightarrow b T_1 \mid \epsilon$~~

$$S_2 \rightarrow a S_2 a \mid a T_2 a$$

$$T_2 \rightarrow c T_2 d \mid \epsilon$$

$$a^m b^n \equiv a^+ (a^n b^n)$$

$$S \rightarrow a S \mid a T$$

$$T \rightarrow a T b \mid \epsilon$$

$$S \rightarrow a U a$$

$$U \rightarrow a U a \mid T$$

$$T \rightarrow c T d \mid \epsilon$$

Übungsblatt 9

Aufgabe 5:

Konstruieren Sie einen Kellerautomaten über $\Sigma = \{a, b\}$, der die Sprache

$$L = \{w \in \{a, b\}^* : \text{Anzahl der } a \text{ und } b \text{ sind gleich}\}$$

durch Endzustand akzeptiert. Ist Ihr Kellerautomat deterministisch?

intuition:

$$a^n b^n$$

$((()))$ - Dyck Language

$a a b a b a b b$

$$S \rightarrow SS \mid \underbrace{a S b}_{\Downarrow^* a^n b^n} \mid \epsilon$$

Hint: $w = b^n a^n$

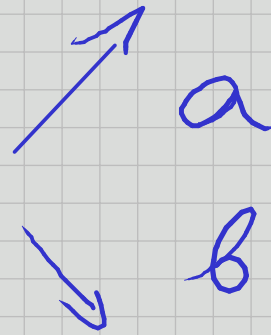
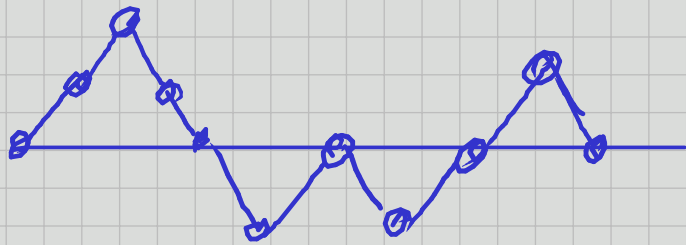
$w = b a b a b a$

$$S \rightarrow b S a \mid \epsilon \mid SS$$

What if we unite these?

$$S \rightarrow SS \mid a S b \mid b S a \mid \epsilon$$

Since all rules that generate terminal symbols generate them in balanced way, it is easy to see that our new grammar only generates "good" words. The only question is: does the grammar generate ALL words from the Lang?

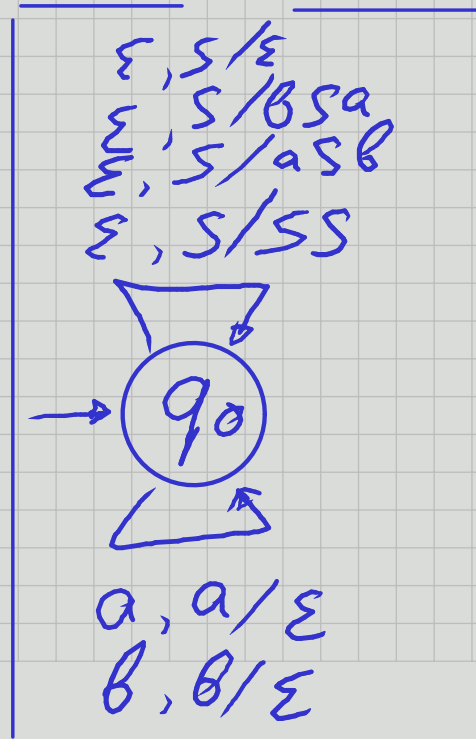


aabbabaab

lets apply algorithm

$S \rightarrow SS \mid aSb \mid bSa \mid \epsilon$

$G \rightarrow n\text{-PDA}$

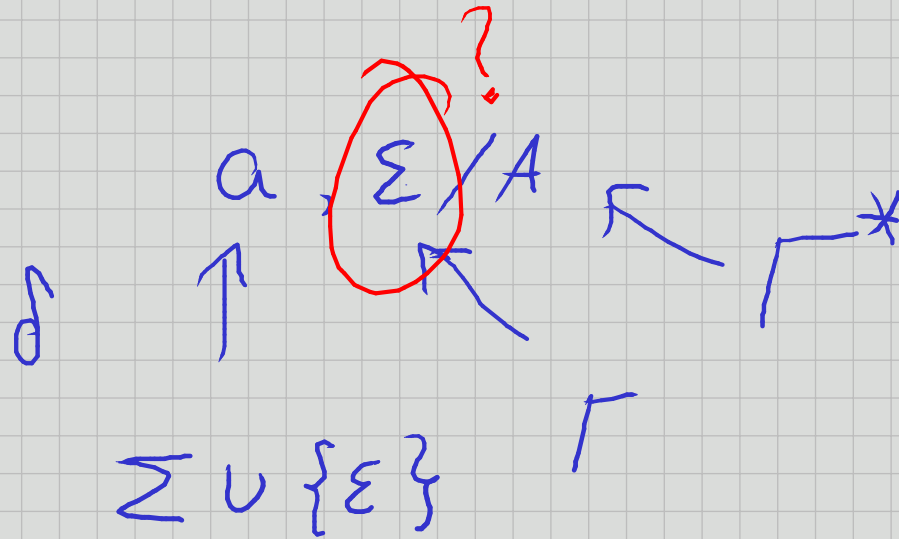
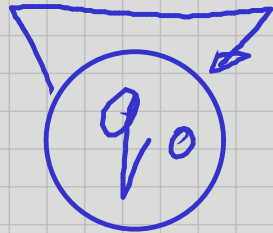


$z_0 = S$

$\Sigma = \{a, b\}$

Alternative solution

$a, B/\varepsilon$
 $b, \varepsilon/B$
 $b, A/\varepsilon$
 $a, \varepsilon/A$



$\Rightarrow \left\{ \begin{array}{l} a, x/Ax \\ a, y/Ay \\ a, z/Az \\ \dots \end{array} \right.$

$\Gamma = \{z_0, A, B\}$



$a, \varepsilon/A \Rightarrow$

$a, z_0/Az_0$
 $a, A/AA$
 $a, B/AB$

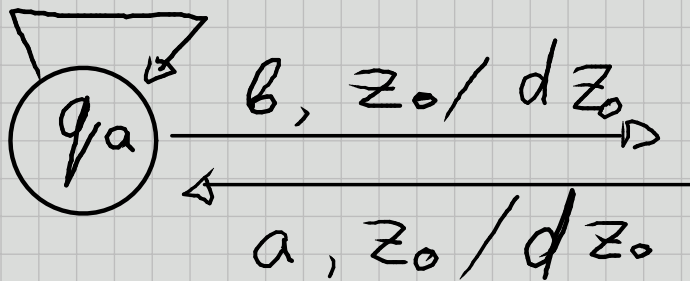
$b, \varepsilon/B \Rightarrow$

$b, z_0/Bz_0$
 $b, A/BA$
 $b, B/BB$

$a, B/\varepsilon$
 $(q_0, aw, B\gamma) \vdash (q_0, w, \gamma)$
 $(q_0, w, B\gamma) \vdash (q_0, w, AB\gamma)$

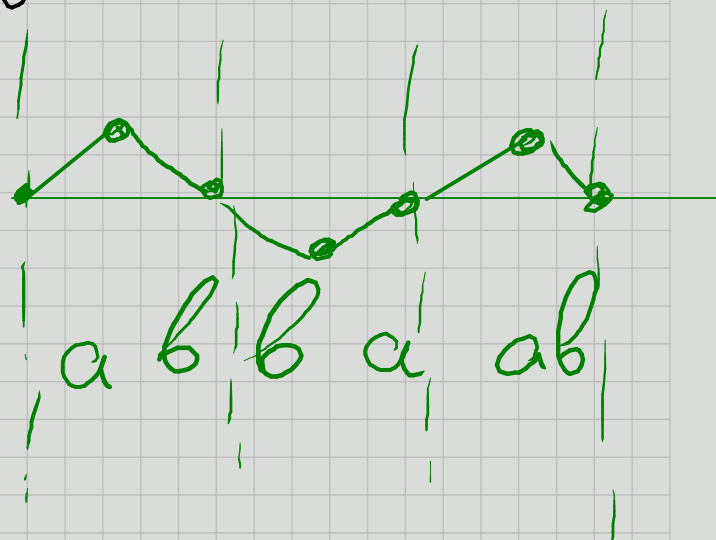
$b, A/\varepsilon$

$b, d / \varepsilon$
 $a, d / dd$
 $a, z_0 / dz_0$



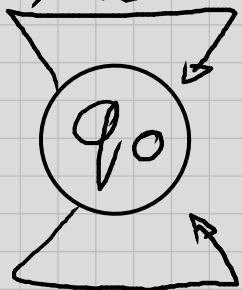
$a, d / \varepsilon$
 $b, d / dd$
 $b, z_0 / dz_0$

$\Gamma = \{z_0, d\}$



q_a

$b, a / \varepsilon$
 $a, a / aa$
 $a, z_0 / az_0$



(q_0, aw, z_0)

(q_0, bw, z_0)

$\Gamma = \{z_0, a, b\}$

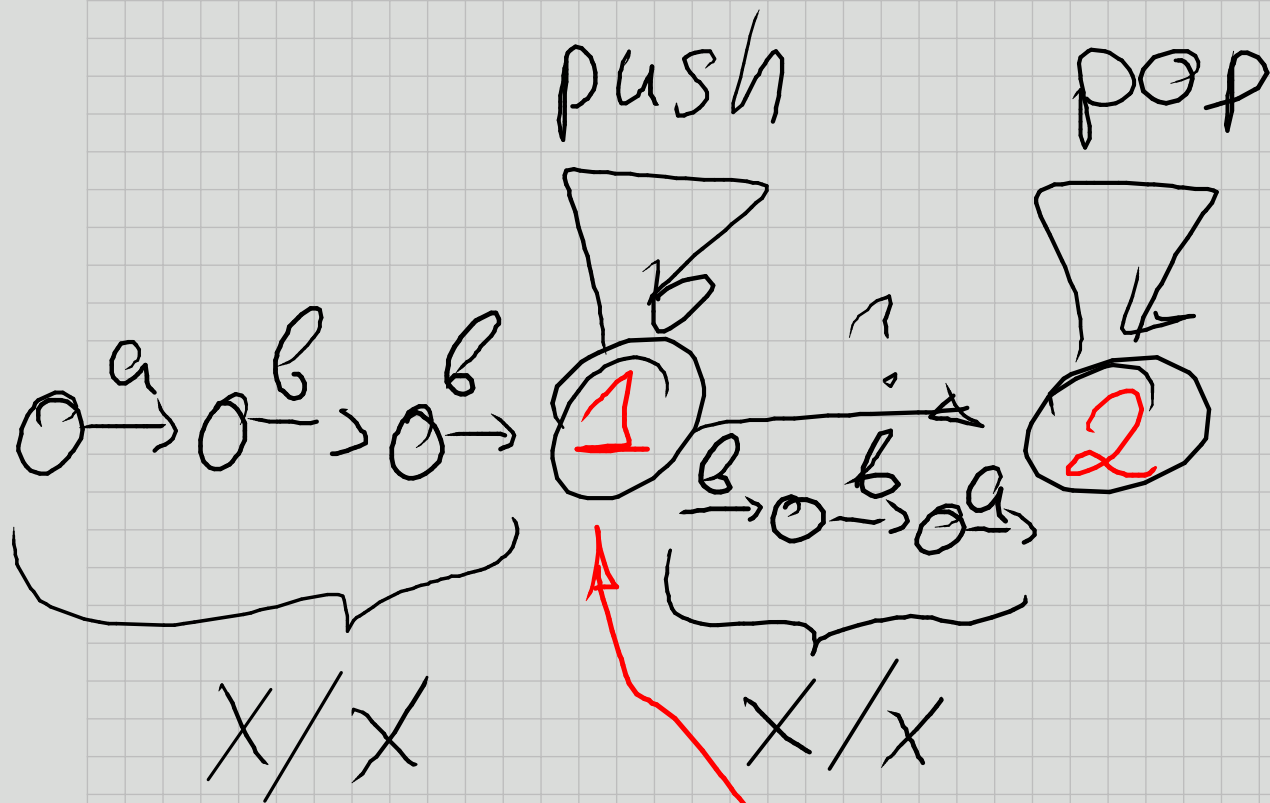
ab

q_b

$b, z_0 / bz_0$
 $b, b / bb$
 $a, b / \varepsilon$

ba

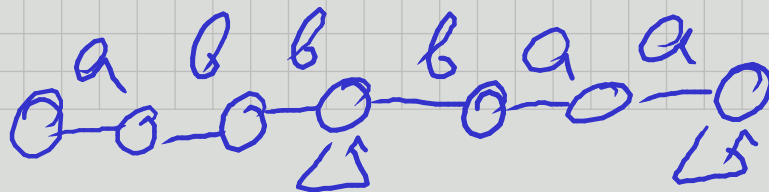
$abb(a^n)ba(a^n)$



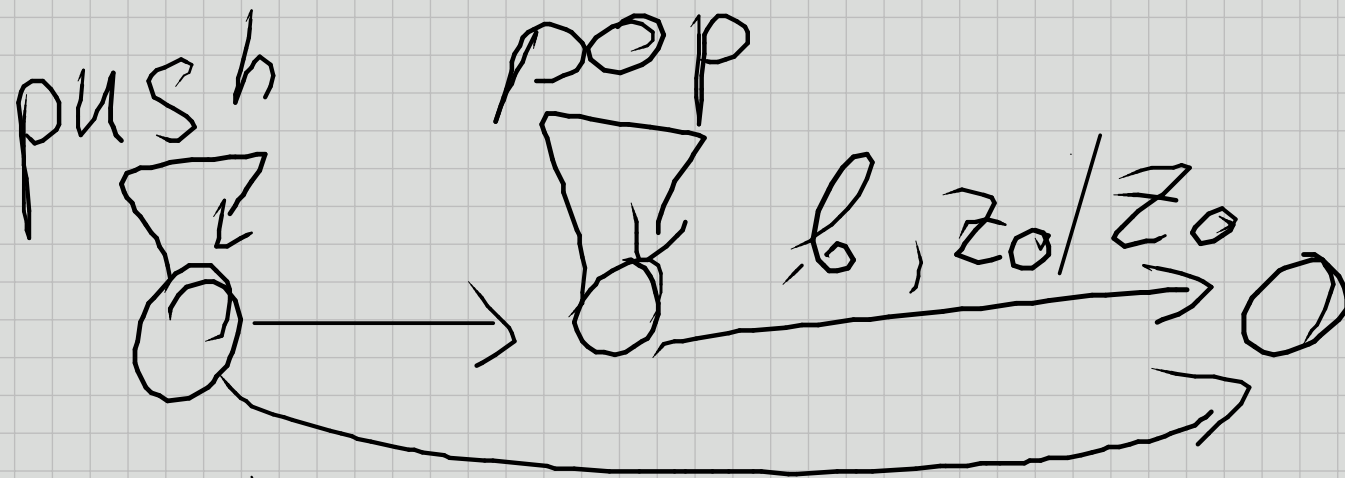
$X = ?$

$(q_0, w, z_0) \vdash^* (q_1, w', a^n z_0) \vdash^* (q_2, w'', \underline{a^n z_0})$

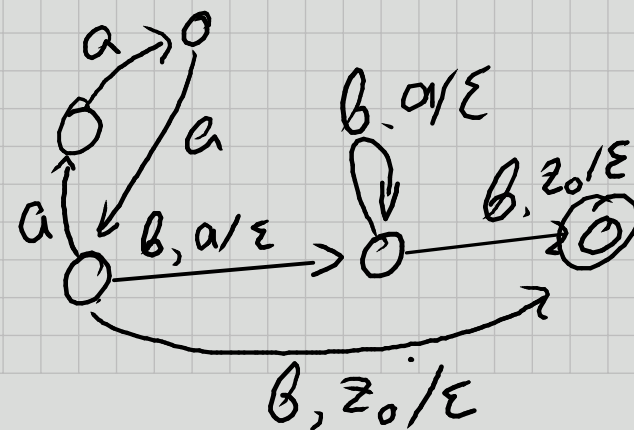
z_0/z_0



$$a^{3n} b^{n+1} \equiv (a^{3n} b^n) b$$



$\Sigma, z_0/z_0$
 $\Sigma, a/a$ $n \geq 0$
 det



$$S \rightarrow aaSbb \mid T$$

$$T \rightarrow cT \mid cU$$

$$U \rightarrow dU \mid ab$$

$$1) cdd$$

$$2) cdddd$$

$$aaabbbbbb$$

$$3) cddd$$

$$S \Rightarrow \dots$$

$$G : \Rightarrow$$

$$M : \vdash$$

$$S \rightarrow S_1 \mid S_2$$

$$S_1 \rightarrow aS_1bb \mid \emptyset$$

$$S_2 \rightarrow cS_2 \mid u$$