1. Using the Lagrange interpolation method with the Regula Falsi root finding method, we were able to find the root of the function $f(x) = e^x \ln x - x^2$.
   To do this, we simply generated data by evaluating $y_i = f(x_i)$ for $x_i = 1.0, 1.1, \ldots, 2.0$ and then, choosing our starting points as the leftmost and rightmost points of the interval, used linear interpolation to find the x-value for which the line connecting the function evaluation at those two points crosses the $x - axis$:

$$y(x) = y_1 + \frac{(y_2 - y_1)(x - x_1)}{(x_2 - x_1)} \stackrel{!}{=} 0$$
$$\implies x = x_1 - y_1 \frac{x_2 - x_1}{y_2 - y_1}$$

   This then replaced the point on the same side of the root as it and we found the function evaluation at that point by interpolation. We ran this algorithm until a tolerance was reached, in our case until the function value at the root was $|y_0| < 1 \times 10^{-5}$.

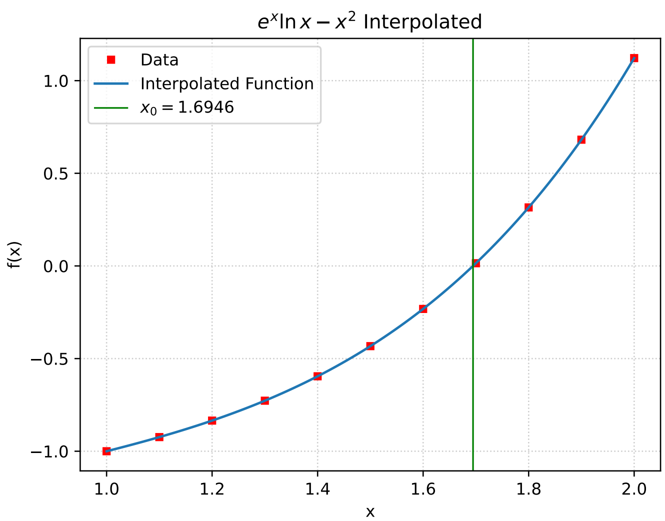

Figure 1: Data generated from $y_i = f(x_i) = e^{x_i} \ln(x_i) - x_i^2$ with $x_i = 1.0, 1.1, \ldots, 2.0$, interpolated using the Lagrange interpolation method on 100 points. The root of the function x_0 was found using Regula Falsi and Lagrange interpolation to a tolerance of 1×10^{-5}.

2. In our implementation of Smoothed Particle Interpolation, our two parameters were number of "particles" N and smoothing length h. We are able to vary N here, but in practice N is most likely going to be fixed as the "particles" we choose are most likely going to be the data that we have.
   We began by generating our data from the function $f(x) = 3x^4 - 3x^2$ using $N = 50$ evenly spaced data points on the interval $[-10, 10]$. Since SPI struggles at the boundaries of intervals, we chose to only interpolate over a smaller interval $[-5, 5]$. We used the Gaussian kernel

function and its derivatives

$$W(x,x';h) = \frac{1}{h\sqrt{\pi}} \exp\left(-\left(\frac{x'-x}{h}\right)^2\right) \tag{1}$$

$$W'(x,x';h) = \frac{-2(x'-x)}{h^3\sqrt{\pi}} \exp\left(-\left(\frac{x'-x}{h}\right)^2\right) \tag{2}$$

$$W''(x,x';h) = \frac{-2}{h^3\sqrt{\pi}} \exp\left(-\left(\frac{x'-x}{h}\right)^2\right) + \frac{4(x'-x)^2}{h^5\sqrt{\pi}} \exp\left(-\left(\frac{x'-x}{h}\right)^2\right) \tag{3}$$

where x is the point at which we evaluate the function, x' are the data points, which we sum over for each x, and h is the smoothing length, i.e. the width of the Gaussian. Using these, the interpolation for the function and its first 2 derivatives was

$$f(x) \approx \sum_i \Delta x_i f(x_i) W(x, x_i; h) \tag{4}$$

$$f'(x) \approx -\sum_i \Delta x_i f(x_i) W'(x, x_i; h) \tag{5}$$

$$f''(x) \approx \sum_i \Delta x_i f(x_i) W''(x, x_i; h) \tag{6}$$

where the x_i are our data points, or "particles".
With these approximations, using $N = 50$ and $h = 0.5$, we found were able to interpolate 1 000 points on the interval $[-5, 5]$ and get the plots in figure 2. These can be compared to the exact values of the function and its derivatives in figure 3.

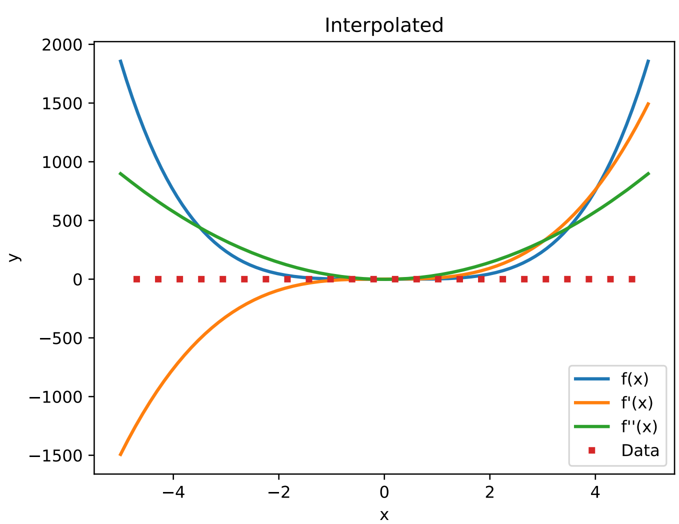

Figure 2: Data generated from $y_i = f(x_i) = 3x_i^4 - 3x_i^2$ with $x_i \in [-10, 10]$, interpolated using SPI on 1 000 points in the interval $[-5, 5]$ to find the function value as well as its first two derivatives.

What is of interest is the error between the interpolated functions and the exact value, so plotted

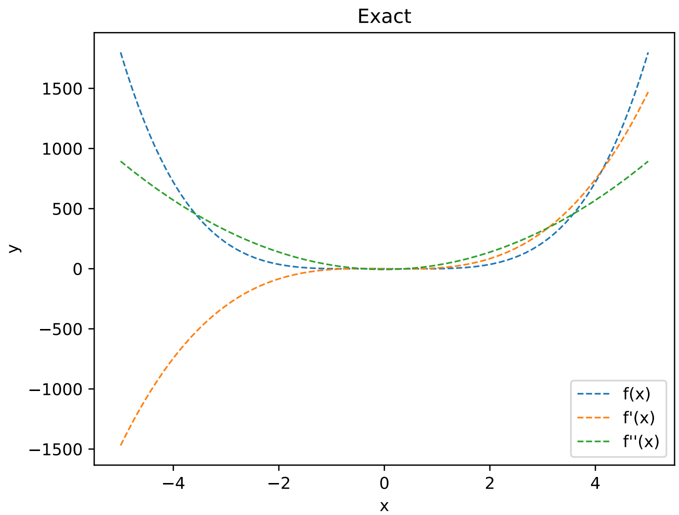

Figure 3: The exact form of the function $f(x) = 3x^4 - 3x^2$ and its first two derivatives, evaluated on 1 000 points in the interval $[-5, 5]$.

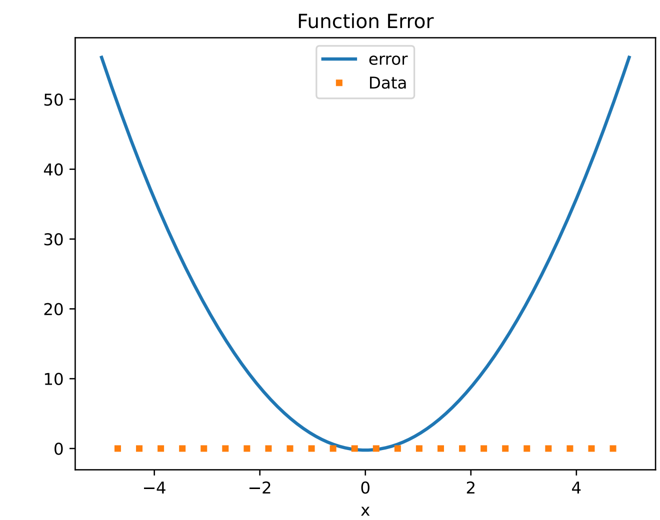

Figure 4: Error between the interpolated values and the exact values for $f(x) = 3x^4 - 3x^2$ on 1 000 points in the interval $[-5, 5]$.

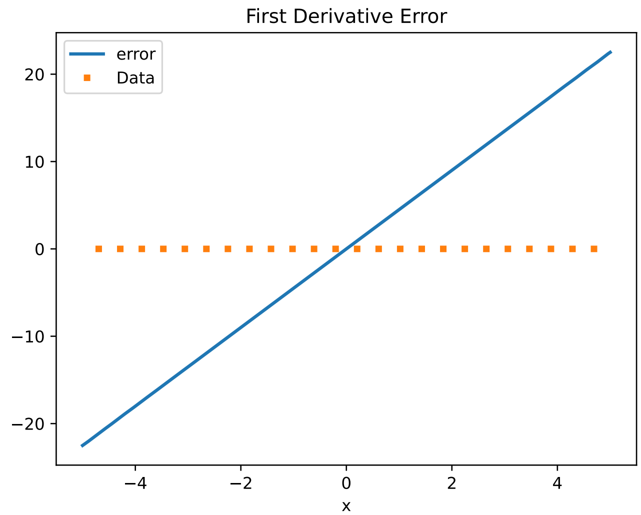

Figure 5: Error between the interpolated values and the exact values for $f'(x) = 12x^3 - 6x$ on 1 000 points in the interval $[-5, 5]$.

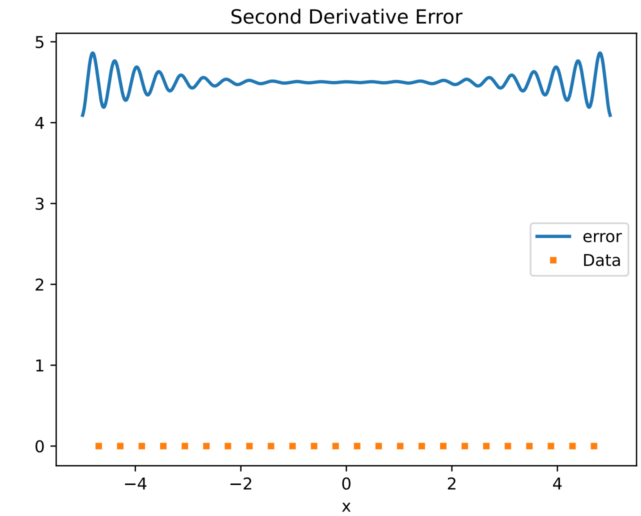

Figure 6: Error between the interpolated values and the exact values for $f''(x) = 36x^2 - 6$ on 1 000 points in the interval $[-5, 5]$.