Extracting energy from black holes (theoretically)

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Abstract

In this paper

1 Introduction

The underlying phenomenon that this paper investigates, that of black holes losing energy, is a phenomenon that at first glance doesn't seem to make much sense. We are told that black holes are the great attractor, letting nothing escape their grasp, not even light. Yet it turns out that the mathematics that led us to the concept of black holes, Einstein's general theory of relativity, also leads us towards the idea that, under the right circumstances, black holes can actually lose their energy.

In order to study this phenomenon, first we must build up the framework of general relativity that describes the specific circumstances where this effect can be seen.

2 Building the coordinate system

Since we are dealing with a system that is spherically symmetric, it's convenient for us to use a coordinate system that simplifies things. The simplest choice is spherical polar coordinates, where the transformation from cartesian coordinates is described by

$$x = r\sin\theta\cos\phi \qquad y = r\sin\theta\sin\phi \qquad z = r\cos\theta \tag{2.1}$$

$$r = \sqrt{x^2+y^2+z^2} \qquad \theta = \arctan\left(\frac{\sqrt{x^2+y^2}}{z}\right) \qquad \phi = \arctan(y/x) \tag{2.2}$$

We can then find the transformation matrix $\Lambda^{\bar{\alpha}}_{\beta}$ such that $dx^{\bar{\alpha}} = \Lambda^{\bar{\alpha}}_{\beta} dx^{\beta}$. We note that for neighbouring points we have

$$\begin{aligned}
dr &= \frac{\partial r}{\partial x}dx + \frac{\partial r}{\partial y}dy + \frac{\partial r}{\partial z}dz \\
d\theta &= \frac{\partial \theta}{\partial x}dx + \frac{\partial \theta}{\partial y}dy + \frac{\partial \theta}{\partial z}dz \\
d\phi &= \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz
\end{aligned}$$

The calculation of the derivatives takes a bit of work, but it isn't too hard to see that it simplifies to

$$\begin{aligned}
dr &= \sin\theta\cos\phi dx + \sin\theta\sin\phi dy + \cos\theta dz \\
d\theta &= \frac{\cos\theta\cos\varphi}{r}dx + \frac{\cos\theta\sin\varphi}{r}dy - \frac{\sin\theta}{r}dz \\
d\phi &= -\frac{\sin\varphi}{r\sin\theta}dx + \frac{\cos\varphi}{r\sin\theta}dy + 0dz
\end{aligned}$$

This gives us our components of the $\Lambda^{\bar{\alpha}}_{\beta}$ matrix:

$$\Lambda^{\bar{\alpha}}_{\beta} = \begin{pmatrix} \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ \frac{\cos\theta\cos\varphi}{r} & \frac{\cos\theta\sin\varphi}{r} & -\frac{\sin\theta}{r} \\ -\frac{\sin\varphi}{r\sin\theta} & \frac{\cos\varphi}{r\sin\theta} & 0 \end{pmatrix} \tag{2.3}$$

We can also define the one-form for an arbitrary scalar field φ:

$$\tilde{\mathbf{d}}\varphi \to \left(\frac{\partial\varphi}{\partial r}, \frac{\partial\varphi}{\partial\theta}, \frac{\partial\varphi}{\partial\phi}\right) \tag{2.4}$$

where we have

$$\begin{aligned}
\frac{\partial\varphi}{\partial r} &= \frac{\partial\varphi}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial\varphi}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial\varphi}{\partial z}\frac{\partial z}{\partial r} \\
&= \sin\theta\cos\phi\frac{\partial\varphi}{\partial x} + \sin\theta\sin\phi\frac{\partial\varphi}{\partial y} + \cos\theta\frac{\partial\varphi}{\partial z} \\
\frac{\partial\varphi}{\partial\theta} &= \frac{\partial\varphi}{\partial x}\frac{\partial x}{\partial\theta} + \frac{\partial\varphi}{\partial y}\frac{\partial y}{\partial\theta} + \frac{\partial\varphi}{\partial z}\frac{\partial z}{\partial\theta} \\
&= r\cos\theta\cos\phi\frac{\partial\varphi}{\partial x} + r\cos\theta\sin\phi\frac{\partial\varphi}{\partial y} - r\sin\theta\frac{\partial\varphi}{\partial z} \\
\frac{\partial\varphi}{\partial\phi} &= \frac{\partial\varphi}{\partial x}\frac{\partial x}{\partial\phi} + \frac{\partial\varphi}{\partial y}\frac{\partial y}{\partial\phi} + \frac{\partial\varphi}{\partial z}\frac{\partial z}{\partial\phi} \\
&= -r\sin\theta\sin\phi\frac{\partial\varphi}{\partial x} + r\sin\theta\cos\phi\frac{\partial\varphi}{\partial y} + 0\partial\varphi\partial z
\end{aligned}$$

The one-form transforms like

$$\frac{\partial\varphi}{\partial x^{\bar\beta}} = \Lambda^{\alpha}_{\bar\beta}\frac{\partial\varphi}{\partial x^{\alpha}}$$

so we can construct $\Lambda^{\alpha}_{\bar\beta}$:

$$\Lambda^{\alpha}_{\bar\beta} = \begin{pmatrix} \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\theta & -r\sin\theta & 0 \end{pmatrix} \tag{2.5}$$

We expect these two matrices to be related in the form $\Lambda^{\bar\alpha}_{\beta}\Lambda^{\alpha}_{\bar\beta} = \mathbb{1}$, the identity matrix [1]. We can now also find the metric tensor for this coordinate system:

$$g_{\alpha\beta} = \Lambda^{\gamma}_{\bar\alpha}\Lambda^{\delta}_{\bar\beta}g_{\gamma\delta} \tag{2.6}$$

In this case, $g_{\gamma\delta}$ is the metric for flat space. Note that we are not yet considering a time coordinate but at this point, with no mass in the system, it is identical to the time coordinate in Minkowski spacetime. We find our metric to be

$$g_{\bar\alpha\bar\beta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix} \tag{2.7}$$

Since it is diagonal we can simply find the inverse metric by taking the inverse of each component:

$$g^{\bar\alpha\bar\beta} = (g_{\bar\alpha\bar\beta})^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & \frac{1}{r^2\sin^2\theta} \end{pmatrix} \tag{2.8}$$

3 The wave equation

The mechanism by which energy will be extracted from this black hole involves the passing of a scalar wave through a part of space near the black hole. The details of this will be fleshed out later but for now let us investigate the wave equation in these coordinates. In a general reference frame and coordinate system, the wave equation is given by

$$g^{\alpha\beta}\nabla_\alpha\nabla_\beta\varphi = g^{\alpha\beta}\nabla_\alpha\frac{\partial\varphi}{\partial x^{\beta}} = 0 \tag{3.1}$$

where we are considering covariant derivatives. φ is scalar field, so its first derivative acts like a normal derivative. For convenience, we will define this first derivative as $\frac{\partial \varphi}{\partial x^\beta} = A_\beta$. The second derivative, however, needs to be dealt with properly, using the definition of the covariant derivative, from [1]:

$$V_{\alpha;\beta} = V_{\alpha,\beta} + V_\mu \Gamma^\alpha_{\mu\beta} \tag{3.2}$$

Our wave equation becomes

$$g^{\alpha\beta}\left(A_{\beta,\alpha} + A_\mu \Gamma^\alpha_{\mu\beta}\right) = 0$$

and the problem has been reduced to simply finding the Christoffel symbols $\Gamma^\alpha_{\mu\beta}$. We will show the calculation of one of these for completeness, but the actual calculation was done with symbolic computing, using the `GRQUICK` module for Mathematica.

The Christoffel symbols are given by

$$\Gamma^\alpha_{\mu\beta} = \frac{1}{2} g^{\alpha\delta}[g_{\mu\delta,\beta} + g_{\beta\delta,\mu} - g_{\mu\beta,\delta}] \tag{3.3}$$

So, choosing a combination of the indices that we know gives a non-zero result (thanks to Mathematica), we find

$$\begin{aligned}\Gamma^r_{\phi\phi} &= \frac{1}{2} g^{r\delta}[g_{\phi\delta,\phi} + g_{\phi\delta,\phi} - g_{\phi\phi,\delta}] \\ &= \frac{1}{2} g^{rr}[g_{\phi r,\phi} + g_{\phi r,\phi} - g_{\phi\phi,r}]\end{aligned}$$

since the only non-zero component of the inverse metric, in the r row, is the rr component, which is 1. We also notice that in the ϕ row of the metric, the only non-zero component is the $\phi\phi$ component, meaning the first two terms in the square bracket are zero.

$$\begin{aligned}\therefore \Gamma^r_{\phi\phi} &= \frac{1}{2}[0 + 0 - \frac{\partial}{\partial r}(r^2 \sin^2\theta)] \\ &= \frac{-1}{2} 2r\sin^2\theta \\ &= -r\sin^2\theta\end{aligned}$$

The other non-zero components were

$$\Gamma^\phi_{\theta\phi} = \cot\theta, \quad \Gamma^\phi_{r\phi} = \frac{1}{r}, \quad \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta, \quad \Gamma^\theta_{r\theta} = \frac{1}{r}, \quad \Gamma^r_{\theta\theta} = -r$$

Putting all these together, recalling our definition of A_β, and introducing the time component of the metric, which is simply -1, the wave equation becomes

$$\begin{aligned}0 &= g^{tt}\left(\frac{\partial^2\varphi}{\partial t^2} + \frac{\partial\varphi}{\partial x^\mu}\Gamma^t_{\mu t}\right) + g^{rr}\left(\frac{\partial^2\varphi}{\partial r^2} + \frac{\partial\varphi}{\partial x^\mu}\Gamma^r_{\mu r}\right) + g^{\theta\theta}\left(\frac{\partial^2\varphi}{\partial \theta^2} + \frac{\partial\varphi}{\partial x^\mu}\Gamma^\theta_{\mu\theta}\right) + g^{\phi\phi}\left(\frac{\partial^2\varphi}{\partial \phi^2} + \frac{\partial\varphi}{\partial x^\mu}\Gamma^\phi_{\mu\phi}\right) \\ &= -(\varphi_{tt} + 0) + (\varphi_{rr} + 0) + \frac{1}{r^2}\left(\varphi_{\theta\theta} + \varphi_r\frac{1}{r}\right) + \frac{1}{r^2\sin^2\theta}\left(\varphi_{\phi\phi} + \varphi_r\frac{1}{r} + \varphi_\theta\cot\theta\right)\end{aligned}$$

Note that the metric and its inverse are diagonal, so we only get components from them when the index is the same. Finally we have

$$-\varphi_{tt} + \varphi_{rr} + \frac{1}{r^2}\left(\varphi_{\theta\theta} + \varphi_r\frac{1}{r}\right) + \frac{1}{r^2\sin^2\theta}\left(\varphi_{\phi\phi} + \varphi_r\frac{1}{r} + \varphi_\theta\cot\theta\right) \tag{3.4}$$

4 Simplifying the wave equation

Equation (3.4) would work for our purposes, but there are a few things that we can do to simplify this calculation that actually remove the need to compute the Christoffel symbols entirely. We begin by defining the determinant of the metric as $det||g_{\alpha\beta}|| = g$.

Now, for any diagonalisable matrix A, [2] tells us that

$$\det\left(e^{A}\right) = e^{\operatorname{tr}(A)} \tag{4.1}$$

So, letting $B = \ln(A)$, we see

$$\begin{aligned}
\det(B) &= e^{\operatorname{tr}(\ln(B))} \\
\implies \ln(\det(B)) &= \operatorname{tr}(\ln(B)) \\
\implies \frac{\partial}{\partial x^{\alpha}} \ln(\det(B)) &= \frac{\partial}{\partial x^{\alpha}} \operatorname{tr}(\ln(B)) \\
&= \operatorname{tr}\left(\frac{\partial}{\partial x^{\alpha}} \ln(B)\right) \\
&= \operatorname{tr}\left(B^{-1} \frac{partial}{\partial x^{\alpha}} B\right)
\end{aligned}$$

Then using the fact that $\operatorname{tr}(A \cdot B) = A_{\mu\nu} B_{\mu\nu}$ and $(B_{\mu\nu})^{-1} = B^{\mu\nu}$ we have, substituting the metric in for B, our first identity:

$$\frac{\partial}{\partial x^{\alpha}} \ln(g) = g^{\mu\nu} g_{\mu\nu,\alpha} \tag{4.2}$$

Next up, we want to find a simplification of the Christoffel symbols. Recalling the definition of the Christoffel symbols and noting that in the wave equation we always get Christoffel symbols of the form $\Gamma^{\alpha}_{\alpha\beta}$, we can write

$$\Gamma^{\alpha}_{\alpha\beta} = \frac{1}{2} g^{\alpha\delta} [g_{\alpha\delta,\beta} + g_{\beta\delta,\alpha} - g_{\alpha\beta,\delta}]$$

Now notice that, if our metric is symmetric, all the off-diagonal terms will cancel with each other while the only contributions will come from the diagonal terms. In our case that means we only have non-zero terms where $\alpha = \delta$, so the second two terms end up cancelling, leaving

$$\Gamma^{\alpha}_{\alpha\beta} = \frac{1}{2} g^{\alpha\delta} g_{\alpha\delta,\beta}$$

Using equation (4.2) we can see that

$$\begin{aligned}
\frac{1}{2} g^{\alpha\delta} g_{\alpha\delta,\beta} &= \frac{1}{2} \frac{\partial}{\partial x^{\alpha}} \ln(g) \\
&= \frac{\partial}{\partial x^{\alpha}} \frac{1}{2} \ln(g) \\
&= \frac{\partial}{\partial x^{\alpha}} \ln\left(g^{1/2}\right)
\end{aligned}$$

Combining this we have a second identity:

$$\Gamma^{\alpha}_{\alpha\beta} = \frac{\partial}{\partial x^{\alpha}} \ln\left(\sqrt{|g|}\right) \tag{4.3}$$

Lastly, bringing it all back to the wave equation, we can write the wave equation as

$$g^{\alpha\beta} \nabla_{\alpha} \nabla_{\beta} \varphi = \nabla_{\alpha} \nabla^{\alpha} \varphi = \nabla_{\alpha} A^{\alpha} = 0 \tag{4.4}$$

where, similar to before, we have defined $A^\alpha = \nabla^\alpha \varphi$. The covariant derivative of a vector is given by equation (3.2) and switching that β to an α we have the wave equation as given in equation (4.4):

$$\begin{aligned}
\nabla_\alpha A^\alpha &= A^\alpha_{,\alpha} + A^\mu \Gamma^\alpha_{\mu\alpha} \\
&= A^\alpha_{,\alpha} + A^\mu \frac{\partial}{\partial x^\alpha} \ln\Big(\sqrt{|g|}\Big) \\
&= A^\alpha_{,\alpha} + \frac{A^\mu}{\sqrt{|g|}} \frac{\partial}{\partial x^\alpha} \sqrt{|g|} \\
&= \frac{1}{\sqrt{|g|}} (\sqrt{|g|} A^\alpha)_{,\alpha}
\end{aligned}$$

In the last line we have renamed the μ to α as it is a dummy index and then used product rule in reverse. Substituting the scalar field φ back in, we have a new form of the wave equation:

$$\frac{1}{\sqrt{|g|}} (\sqrt{|g|} \nabla^\alpha \varphi)_{,\alpha} = \frac{1}{\sqrt{|g|}} (\sqrt{|g|} g^{\alpha\beta} \nabla_\beta \varphi)_{,\alpha} = 0 \tag{4.5}$$

This does not require the calculation of endless Christoffel symbols, merely the determinant of the metric, which is fairly easily done. Note that this equation (4.5) was derived using only the fact that the metric is diagonalisable and symmetric, which is a fairly common set of conditions for a variety of metrics.

5 The Kerr-Newmann metric

The class of black hole that we will be studying is a rotating, uncharged black hole. This is described by the Kerr-Newmann metric [WOULD LIKE A REFERENCE FOR THIS]. We will also be using so-called geometrised units, where $G = c = 1$, in order to simplify our calculations. The Kerr-Newmann metric is defined as follows

$$g_{\alpha\beta} = \begin{pmatrix} (-1 + \frac{2mr}{\Sigma}) & 0 & 0 & \frac{-2amr\sin^2\theta}{\Sigma} \\ 0 & \frac{\Sigma}{\Delta} & 0 & 0 \\ 0 & 0 & \Sigma & 0 \\ \frac{-2amr\sin^2\theta}{\Sigma} & 0 & 0 & \left(r^2 + a^2 + \frac{2mra^2\sin^2\theta}{\Sigma}\right)\sin^2\theta \end{pmatrix} \tag{5.1}$$

where we have defined $\Sigma = r^2 + a^2\cos^2\theta$, $\Delta = r^2 + a^2 - 2mr$, m is the mass of the black hole, and a is its angular momentum.

Before we get into anything to do with wave equations, let us calculate some things that we will need. First off, the inverse metric is fairly simple to find. For the rr and $\theta\theta$ components we can simply invert them as we did with the spherical polar coordinates metric. This comes about simply from linear algebra as those terms have no off-diagonal elements that interfere. The other components are a different story, but they are also fairly easy to deal with as we can represent them as a 2×2 matrix, for which finding the inverse is simply given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

where we are considering the matrix

$$\tilde{g}_{\alpha\beta} = \begin{pmatrix} (-1 + \frac{2mr}{\Sigma}) & \frac{-2amr\sin^2\theta}{\Sigma} \\ \frac{-2amr\sin^2\theta}{\Sigma} & \left(r^2 + a^2 + \frac{2mra^2\sin^2\theta}{\Sigma}\right)\sin^2\theta \end{pmatrix}$$

This has determinant $-\Delta \sin^2\theta$, so the inverse is given by

$$\begin{aligned}
\tilde{g}^{\alpha\beta} &= \frac{1}{-\Delta \sin^2\theta}\begin{pmatrix} \left(r^2+a^2+\frac{2mra^2\sin^2\theta}{\Sigma}\right)\sin^2\theta & \frac{2amr\sin^2\theta}{\Sigma} \\ \frac{2amr\sin^2\theta}{\Sigma} & (-1+\frac{2mr}{\Sigma}) \end{pmatrix} \\
&= \begin{pmatrix} -\frac{1}{\Delta}\left(r^2+a^2+\frac{2mra^2\sin^2\theta}{\Sigma}\right) & \frac{-2mra}{\Sigma\Delta} \\ \frac{-2mra}{\Sigma\Delta} & \frac{\Delta - a^2\sin^2\theta}{\Sigma\Delta\sin^2\theta} \end{pmatrix}
\end{aligned}$$

We can now construct the inverse of the Kerr-Newman metric by returning the rr and $\theta\theta$ components to the matrix above after inverting them:

$$g^{\alpha\beta} = \begin{pmatrix} -\frac{1}{\Delta}\left(r^2+a^2+\frac{2mra^2\sin^2\theta}{\Sigma}\right) & 0 & 0 & \frac{-2mra}{\Sigma\Delta} \\ 0 & \frac{\Delta}{\Sigma} & 0 & 0 \\ 0 & 0 & \frac{1}{\Sigma} & 0 \\ \frac{-2mra}{\Sigma\Delta} & 0 & 0 & \frac{\Delta - a^2\sin^2\theta}{\Sigma\Delta\sin^2\theta} \end{pmatrix} \tag{5.2}$$

We can check that this is in fact the inverse by multiplying equation (5.1) by equation (5.2) and after a fair amount of algebra, which we will not get into here, it does return the identity matrix.

We will also need the determinant of the Kerr-Newman metric. This calculation is, as we have seen a few times, fairly straightforward but tedious. For completeness we will show some of the calculation here.

$$\begin{aligned}
g = det(g_{\alpha\beta}) =& (-1+\frac{2mr}{\Sigma})\left(\Sigma\left(\frac{\Sigma}{\Delta}\sin^2\theta\left(r^2+a^2+\frac{2mra^2\sin^2\theta}{\Sigma}\right)-0\right)\right) \\
&-0+0-\left(\frac{-2amr\sin^2\theta}{\Sigma}\left(0-\frac{\Sigma}{\Delta}\left(0-\Sigma\frac{-2amr\sin^2\theta}{\Sigma}\right)\right)\right) \\
=& \frac{-\Sigma^2\sin^2\theta}{\Delta}\left[\left(a^2+r^2+\frac{2amr\sin^2\theta}{\Sigma}\right)+\left(\frac{2mra^2}{\Sigma}+\frac{2mr^3}{\Sigma}\right)\right] \\
=& \frac{-\Sigma^2\sin^2\theta}{\Delta}\left[a^2+r^2-2mr\frac{r^2+a^2\cos^2\theta}{\Sigma}\right] \\
=& -\Sigma^2\sin^2\theta
\end{aligned}$$

6 The wave equation for the Kerr-Newmann metric

We finally have the tools we need to tackle the wave equation. Recall the form of the wave equation in equation (4.5), we can expand the derivative by product rule to write

$$\begin{aligned}
0 &= \frac{1}{\sqrt{|g|}}(\sqrt{|g|}g^{\alpha\beta}\nabla_\beta\varphi)_{,\alpha} \\
&= \frac{1}{\sqrt{|g|}}\sqrt{|g|}g^{\alpha\beta}\varphi_{,\beta\alpha}+\frac{1}{\sqrt{|g|}}\sqrt{|g|}g^{\alpha\beta}_{,\alpha}\varphi_{,\beta}+\frac{1}{\sqrt{|g|}}\frac{-1}{2\sqrt{|g|}}g_{,\alpha}g^{\alpha\beta}\varphi_{,\beta} \\
&= g^{\alpha\beta}\varphi_{,\beta\alpha}+g^{\alpha\beta}_{,\alpha}\varphi_{,\beta}-\frac{1}{2|g|}g_{,\alpha}g^{\alpha\beta}\varphi_{,\beta}
\end{aligned}$$

With this, we can formulate the wave equation for the Kerr-Newmann system. We begin by noting some useful things: The components of the inverse metric depend only on r and θ, so any derivatives of the inverse metric with respect to t or ϕ vanish. For this reason, the second term in the equation above is non-zero only for $\alpha = \beta = (r, \theta)$, so there are only derivatives of φ with respect to r and θ in that term. The same is true for the determinant of the metric, resulting in the gradient of the determinant only having r and θ terms. This results in a similar simplification of the third term. Let us expand the wave equation considering these simplifications:

$$\begin{aligned}
0 = &\left[g^{tt}\frac{\partial^2\varphi}{\partial t^2} + g^{rr}\frac{\partial^2\varphi}{\partial r^2} + g^{\theta\theta}\frac{\partial^2\varphi}{\partial\theta^2} + g^{\phi\phi}\frac{\partial^2\varphi}{\partial\phi^2} - g^{\phi t}\frac{\partial^2\varphi}{\partial t\partial\phi} - g^{t\phi}\frac{\partial^2\varphi}{\partial\phi\partial t}\right] \\
&+ \left[\frac{\partial g^{rr}}{\partial r}\frac{\partial\varphi}{\partial r} + \frac{\partial g^{\theta\theta}}{\partial\theta}\frac{\partial\varphi}{\partial\theta}\right] - \frac{1}{2|g|}\left[g_{,r}g^{rr}\frac{\partial\varphi}{\partial r} + g_{,\theta}g^{\theta\theta}\frac{\partial\varphi}{\partial\theta}\right]
\end{aligned}$$

We can now calculate some of the terms we are missing:

$$\begin{aligned}
\frac{\partial g^{rr}}{\partial r} &= \frac{\partial}{\partial r}\frac{\Delta}{\Sigma} \\
&= \frac{(2r-2m)\Sigma - \Delta(2r)}{\Sigma^2} \\
\frac{\partial g^{\theta\theta}}{\partial\theta} &= \frac{\partial}{\partial\theta}\frac{1}{\Sigma} \\
&= \frac{-2a^2\cos\theta\sin\theta}{\Sigma^2} \\
g_{,r} &= \frac{\partial}{\partial r}(-\Sigma^2\sin^2\theta) \\
&= -4r\Sigma\sin^2\theta \\
g_{,\theta} &= \frac{\partial}{\partial\theta}(-\Sigma^2\sin^2\theta) \\
&= 4\Sigma\sin^3\theta a^2\cos\theta - \Sigma^2 2\sin\theta\cos\theta
\end{aligned}$$

And finally substitute everything into the wave equation:

$$\begin{aligned}
0 = &\left[-\frac{1}{\Delta}\left(r^2 + a^2 + \frac{2mra^2\sin^2\theta}{\Sigma}\right)\frac{\partial^2\varphi}{\partial t^2} + \frac{\Delta}{\Sigma}\frac{\partial^2\varphi}{\partial r^2} + \frac{1}{\Sigma}\frac{\partial^2\varphi}{\partial\theta^2} + \frac{\Delta - a^2\sin^2\theta}{\Sigma\Delta\sin^2\theta}\frac{\partial^2\varphi}{\partial\phi^2} - \frac{4mra}{\Sigma\Delta}\frac{\partial^2\varphi}{\partial t\partial\phi}\right] \\
&+ \left[\frac{(2r-2m)\Sigma - \Delta(2r)}{\Sigma^2}\frac{\partial\varphi}{\partial r} + \frac{-2a^2\cos\theta\sin\theta}{\Sigma^2}\frac{\partial\varphi}{\partial\theta}\right] \\
&- \frac{1}{2\Sigma^2\sin^2\theta}\left[(-4r\Sigma\sin^2\theta)\frac{\Delta}{\Sigma}\frac{\partial\varphi}{\partial r} + (4\Sigma\sin^3\theta a^2\cos\theta - \Sigma^2 2\sin\theta\cos\theta)\frac{1}{\Sigma}\frac{\partial\varphi}{\partial\theta}\right]
\end{aligned}$$

We can multiply this whole equation by Σ and then find that the second and third term cancel most of their components. We find

$$\begin{aligned}
0 = &\left(-\frac{(r^2+a^2)^2}{\Delta} + a^2\sin^2\theta\right)\frac{\partial^2\varphi}{\partial t^2} + \frac{\partial}{\partial r}\left(\Delta\frac{\partial\varphi}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\varphi}{\partial\theta}\right) \\
&+ \left(\frac{1}{\sin^2\theta} - \frac{a^2}{\Delta}\right)\frac{\partial^2\varphi}{\partial\phi^2} - \frac{4mra}{\Delta}\frac{\partial^2\varphi}{\partial t\partial\phi}
\end{aligned} \tag{6.1}$$

The most important things to note when getting to this result are noticing the product rule terms for the derivatives with respect to r and to θ, as well as manipulating the term multiplying the time derivative.

7 Solving the wave equation

Now that we have the form of the wave equation in spacetime containing our rotating, uncharged black hole, we can begin to investigate the properties of waves in this spacetime. In order to do this, we will use separation of variables. We assume the scalar field can be represented as a product of components which each independently depend on a single coordinate:

$$\varphi(t, r, \theta, \phi) = e^{-i\omega t} e^{im\phi} R(r) S(\theta) \tag{7.1}$$

We have assumed the form of the t- and ϕ-dependent parts as they are of less interest to us than the r- or θ-dependent parts. Substituting this ansatz into equation (6.1), we can begin to simplify things:

$$\begin{aligned} 0 = &\left(-\frac{(r^2+a^2)^2}{\Delta} + a^2 \sin^2\theta\right)(-\omega^2)(e^{-i\omega t} e^{im\phi} R(r) S(\theta)) + \frac{\partial}{\partial r}\left(\Delta \frac{\partial R(r)}{\partial r}\right)(e^{-i\omega t} e^{im\phi} S(\theta)) \\ &+ \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial S(\theta)}{\partial\theta}\right)(e^{-i\omega t} e^{im\phi} R(r)) + \left(\frac{1}{\sin^2\theta} - \frac{a^2}{\Delta}\right)(-m^2)(e^{-i\omega t} e^{im\phi} R(r) S(\theta)) \\ &- \frac{4mra}{\Delta}(m\omega)(e^{-i\omega t} e^{im\phi} R(r) S(\theta)) \end{aligned}$$

Assuming $R(r)$ and $S(\theta)$ are non-zero as that would be the trivial solution, we can divide through by $e^{-i\omega t} e^{im\phi} R(r) S(\theta)$ and then separating all terms with r dependence from terms with θ dependence:

$$\frac{1}{R}\frac{\partial}{\partial r}\left(\Delta \frac{\partial R}{\partial r}\right) + \frac{\omega^2(r^2+a^2)^2}{\Delta} + \frac{m^2 a^2}{\Delta} - \frac{4m^2 ra\omega}{\Delta} = \omega^2 a^2 \sin^2\theta - \frac{1}{S\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial S(\theta)}{\partial\theta}\right) + \frac{m^2}{\sin^2\theta}$$

The radial part of this equation is of most interest to us, so it is where we will focus our efforts. Without loss of generality, we can equate it to some constant, which we will call $A \in \mathbb{R}$.

$$\frac{1}{R}\frac{\partial}{\partial r}\left(\Delta \frac{\partial R}{\partial r}\right) + \frac{\omega^2(r^2+a^2)^2 + m^2 a^2 - 4m^2 ra\omega}{\Delta} = A \tag{7.2}$$

To simplify further, we will introduce a change of variable, called the tortoise coordinate r^* such that

$$\frac{dr^*}{dr} = \frac{r^2+a^2}{\Delta} \tag{7.3}$$

Equation (7.2) becomes

$$\begin{aligned} 0 &= \frac{\partial}{\partial r}\left(\Delta \frac{\partial R}{\partial r}\right) + \left[\frac{\omega^2(r^2+a^2)^2 + m^2 a^2 - 4m^2 ra\omega}{\Delta} - A\right] R \\ &= \frac{\partial r^*}{\partial r}\frac{\partial}{\partial r^*}\left(\Delta \frac{\partial R}{\partial r^*}\frac{\partial r^*}{\partial r}\right) + \left[\frac{\omega^2(r^2+a^2)^2 + m^2 a^2 - 4m^2 ra\omega}{\Delta} - A\right] R \\ &= \frac{r^2+a^2}{\Delta}\frac{\partial}{\partial r^*}\left(\Delta \frac{r^2+a^2}{\Delta}\frac{\partial R}{\partial r^*}\right) + \left[\frac{\omega^2(r^2+a^2)^2 + m^2 a^2 - 4m^2 ra\omega}{\Delta} - A\right] R \\ &= \frac{(r^2+a^2)^2}{\Delta}\frac{\partial^2 R}{\partial r^{*2}} + 2r\frac{\partial R}{\partial r^*} + \left[\frac{\omega^2(r^2+a^2)^2 + m^2 a^2 - 4m^2 ra\omega}{\Delta} - A\right] R \end{aligned}$$

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