Parametrically driven, damped nonlinear Schrödinger equation

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Abstract

1 Introduction

The equation of interest to us is the following:

$$i\psi_t + \psi_{xx} + 2|\psi|^2\psi - \psi = h\psi^* - i\gamma\psi = 0 \tag{1.1}$$

Here $\gamma > 0$ is the damping coefficient and $h > 0$ is the amplitude of the driving term.

2 Finding the dispersion relation

To begin, since ψ is a complex number, let us express it in terms of its components, letting

$$\psi = u + iv \quad \implies \quad \psi^* = u - iv \tag{2.1}$$

where u and v are both real numbers. Doing this, we are able to re-express equation (1.1) in this form, leading to

$$iu_t - v_t + u_{xx} + iv_{xx} + 2(u^2 + v^2)(u + iv) - u - iv = hu - hiv - i\gamma u + \gamma v = 0 \tag{2.2}$$

This has very neatly handled the problem of the complex conjugate, as otherwise that would've been a nasty guy to try and tackle. The other problem we have with this equation is the nonlinear part, stemming from the $2|\psi|^2\psi$ term in equation (1.1). As we are seeking a dispersion relation and nonlinearity tends to rain on the parade of dispersion relations, we will simply let this term wander off into the night, never to be seen again. This can be done if we are considering small values of u and v. (This translates into small amplitude waves when we make the substitution in equations (2.5) and (2.6), but let's not get ahead of ourselves.)

Since this equation (2.2) is equal to zero, both its real and imaginary parts must be equal to zero, handily leading us to two separate, but coupled, equations.

$$(\partial_{xx} - 1 - h)u + (-\partial_t - \gamma)v = 0 \tag{2.3}$$

$$(\partial_t + \gamma)u + (\partial_{xx} - 1 + h)v = 0 \tag{2.4}$$

These are a pair of coupled Schrödinger equations and we know what kind of solution Schrödinger equations have: travelling waves. Let us guess two waves for u and v and see what happens:

$$u = Ue^{i(\omega t - kx)} \tag{2.5}$$

$$v = Ve^{i(\omega t - kx)} \tag{2.6}$$

Substituting these back into our coupled equations, and noting that space derivatives will yield simply $-ik$ and time derivatives will yield $-\omega$, we find

$$(-k^2 - 1 - h)Ue^{i(\omega t - kx)} + (i\omega - \gamma)Ve^{i(\omega t - kx)} = 0$$

$$(i\omega + \gamma)Ue^{i(\omega t - kx)} + (-k^2 - 1 + h)Ve^{i(\omega t - kx)} = 0$$

Thankfully, we can divide through both equations by the exponential, leaving us with a simple algebraic system of equations with unknowns U and V

$$
\begin{aligned}
-(k^2+1+h)U-(i\omega+\gamma)V&=0\\
(i\omega+\gamma)U-(k^2+1-h)V&=0
\end{aligned}
$$

which can be re-expressed as a matrix equation:

$$
\begin{pmatrix} -(k^2+1+h) & -(i\omega+\gamma) \\ (i\omega+\gamma) & -(k^2+1-h) \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix} = 0 \tag{2.7}
$$

We know from linear algebra that this system of equations has a nontrivial solution only if the determinant of this matrix is zero. So we have

$$
\begin{aligned}
0 &= (k^2+1+h)(k^2+1-h)+(i\omega+\gamma)^2\\
&= k^4+k^2-k^2h+k^2+1-h+k^2h+h-h^2-\omega^2+2i\omega\gamma+\gamma^2\\
&= \omega^2-2i\gamma\omega-k^4-2k^2+h^2-\gamma^2-1
\end{aligned}
$$

We can now solve for ω with the quadratic formula:

$$
\begin{aligned}
\omega &= \frac{2i\gamma \pm \sqrt{(-2i\gamma)^2+4(k^4-2k^2+h^2-\gamma^2-1)}}{2}\\
&= i\gamma \pm \sqrt{k^4+2k^2-h^2+1}
\end{aligned}
$$

where we will label the two branches with $\omega_\pm$.

The first branch we will consider is ω_+, which has a positive real part if $h^2 < k^4+2k^2+1$ or no real part if $h^2 \geq k^4+2k^2+1$. This means ω_+ is unstable for $h^2 < k^4+2k^2+1$ and marginally stable for $h^2 \geq k^4+2k^2+1$. ω_- cannot have a positive real part as the square root term cannot be negative, so it is stable for $h^2 < k^4+2k^2+1$ and marginally stable for when there is no real part: $h^2 \geq k^4+2k^2+1$.

Note that for both branches, γ only affects the imaginary part of the dispersion relation and so has no influence on the stability of the system; we are free to choose whatever γ we like.

References

[1] M. Bondila, *Numerical study of the parametrically driven damped nonlinear Schrödinger equation*, University of Cape Town, (1995)