4. We begin by considering the Schrödinger equation for the harmonic oscillator:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}+\frac{m\omega^2x^2}{2}\psi=E\psi \tag{1}$$

Firstly, to make some things easier, we will consider the values of $\hbar$, ω, and m to be 1. Now in order to approximate the second derivative in the equation above, we will use a finite difference scheme, namely a second order, centred difference. To do this, we first need to create a grid of x points, which we will consider on the interval $[-5,5]$ with 1000 grid points x_i. We will use the notation $\psi(x_i)=\psi_i$. Now we can discuss the approximation of the derivative:

$$\frac{d^2\psi}{dx^2}\approx\frac{\psi_{i-1}-2\psi_i+\psi i+1}{\Delta x^2} \tag{2}$$

where Δx is 10/1000, the distance between grid points.

Now we can rewrite our Schrödinger equation in its discretised form:

$$-\frac{1}{2}\frac{\psi_{i-1}-2\psi_i+\psi i+1}{\Delta x^2}+\frac{x^2}{2}\psi_i=E\psi_i \tag{3}$$

We can simply inspect the equation to find the matrix that can act on ψ to give us $E\psi$, finding it to be

$$H=\begin{pmatrix} \frac{1}{\Delta x^2}+\frac{x^2}{2} & -\frac{1}{2\Delta x^2} & 0 & \cdots & 0 \\ -\frac{1}{2\Delta x^2} & \frac{1}{\Delta x^2}+\frac{x^2}{2} & -\frac{1}{2\Delta x^2} & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & -\frac{1}{2\Delta x^2} & \frac{1}{\Delta x^2}+\frac{x^2}{2} & -\frac{1}{2\Delta x^2} \\ 0 & \cdots & 0 & -\frac{1}{2\Delta x^2} & \frac{1}{\Delta x^2}+\frac{x^2}{2} \end{pmatrix} \tag{4}$$

We can use any method to solve this system for its eigenvalues and eigenvectors, and we chose to use the `Eigensystem` method from Mathematica. This method spits out all N eigenvalues and eigenvectors, for N being the size of the matrix, in our case 1000, so we simply chose the last 4, which were

$$E = 0.499997 \quad 1.49998 \quad 2.49996 \quad 3.49992$$

This is fairly close to the expected $0.5, 1.5, 2.5, 3.5$, in units of $\hbar\omega$ of course, and can be improved by increasing the number of grid points. The corresponding eigenvectors were