Parametrically Driven Nonlinear Schrödinger

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Abstract

We attempt to analyse a chain of torsionally coupled pendula driven, parametrically, at the frequency close to double the natural frequency of the pendula. The aim was to find the asymptotic expansion and then simulate the problem in MATLAB and compare the two, but the analysis proved too tough to complete in the time frame and so no results are presented.

1 Introduction

A chain of torsionally coupled pendula driven, parametrically, at the frequency close to double the natural frequency of the pendula is represented by the equation

$$\begin{aligned} \ddot{\theta}_n + 2\lambda\dot{\theta}_n - k(\theta_{n+1} - 2\theta_n + \theta_{n-1}) + \omega^2(t)\sin\theta_n = 0 \\ \omega^2(t) = 1 + 2\epsilon^2\cos[2(1+\epsilon^2\beta)t] \end{aligned} \tag{1.1}$$

Using the method of multiple scales, we can find an equation for the amplitudes of small amplitude, long wavelength oscillations of this chain. This equation can be analysed using an asymptotic expansion, as well as a numerical scheme in order to compare the two techniques.

2 Analysis

We start with equation (1.1):

$$\begin{aligned} \ddot{\theta}_n + 2\lambda\dot{\theta}_n - k(\theta_{n+1} - 2\theta_n + \theta_{n-1}) + \omega^2(t)\sin\theta_n = 0 \\ \omega^2(t) = 1 + 2\epsilon^2\cos[2(1+\epsilon^2\beta)t] \end{aligned}$$

In order to analyse this, we need it in the form of a continuous PDE in x and t. We can start by expanding $\theta n \pm 1$ in a Taylor series about θ_n, where the spacing between θ_n's is h:

$$\begin{aligned} \theta_{n+1} &= \theta_n + h\theta_n' + \frac{h^2}{2!}\theta_n'' + \dots \\ \theta_{n-1} &= \theta_n - h\theta_n' + \frac{h^2}{2!}\theta_n'' - \dots \\ \implies \theta_{n+1} + \theta_{n-1} &= 2\theta_n + h^2\theta_n'' + \mathcal{O}(h^4) \end{aligned}$$

So ignoring higher order terms, our equation becomes

$$\ddot{\theta}(x,t) + 2\lambda\dot{\theta}(x,t) - kh^2\theta''(x,t) + (1 + 2\epsilon^2\cos[2(1+\epsilon^2\beta)t])\sin(\theta(x,t)) = 0 \tag{2.1}$$

Let us simplify some things. We define $\tau = (1+\epsilon^2\beta)t$, $\kappa^2 = kh^2$, and assuming λ is of the order ϵ^2 we say $\lambda = \epsilon^2\gamma$. Putting these into equation (2.1) and using the chain rule to get the PDE in terms of x and the new and improved τ we see

$$\begin{aligned} &\ddot{\theta}(x,\tau) + 2\epsilon^2\gamma\dot{\theta}(x,\tau) - \kappa^2\theta''(x,\tau) + (1 + 2\epsilon^2\cos[2\tau])\sin(\theta(x,\tau)) = 0 \\ \implies &\left(\frac{d\tau}{dt}\right)^2 \frac{\partial^2}{\partial\tau^2}\theta + 2\epsilon^2\gamma\left(\frac{d\tau}{dt}\right)\frac{\partial}{\partial\tau}\theta - \kappa^2\theta'' + (1 + 2\epsilon^2\cos[2\tau])\sin\theta = 0 \\ \implies &\left(1+\epsilon^2\beta\right)^2\frac{\partial^2}{\partial\tau^2}\theta + 2\epsilon^2\gamma\left(1+\epsilon^2\beta\right)\frac{\partial}{\partial\tau}\theta - \kappa^2\theta'' + (1 + 2\epsilon^2\cos[2\tau])\sin\theta = 0 \end{aligned}$$

We will use the method of multiple scales to tackle this beast, which requires us to expand θ in terms of some small quantity, which we will choose to be ϵ. Since we are examining small amplitudes, we assume θ is small and thus

$$\theta = \epsilon\theta_1 + \epsilon^2\theta_3 + \epsilon^5\theta_5 + \dots \tag{2.2}$$

We then expand in both the time and space scales, where τ is the time and x is the space:

$$T_n = \epsilon^n\tau,\, D_n = \frac{\partial}{\partial T_n}$$

$$\implies \frac{\partial}{\partial\tau} = D_0 + \epsilon D_1 + \epsilon^2 D_2 + \dots,\ \frac{\partial^2}{\partial\tau^2} = D_0^2 + 2\epsilon D_0 D_1 + \epsilon^2(D_1^2 + 2D_0D_2) + \dots$$

$$X_n = \epsilon^n x,\, P_n = \frac{\partial}{\partial X_n}$$

$$\implies \frac{\partial}{\partial x} = P_0 + \epsilon P_1 + \epsilon^2 P_2 + \dots,\ \frac{\partial^2}{\partial x^2} = P_0^2 + 2\epsilon P_0 P_1 + \epsilon^2(P_1^2 + 2P_0P_2) + \dots$$

We can also expand the $\sin\theta$ in its Taylor series:

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \dots = \epsilon\theta_1 + \epsilon^3\theta_3 - \frac{\epsilon^3\theta_1^3}{3!} + \mathcal{O}(\epsilon^5)$$

All of this leads us to the PDE looking like

$$\begin{aligned}(1 + 2\epsilon^2\beta + \epsilon^4\beta^2)(D_0^2 + 2\epsilon D_0D_1 + \epsilon^2(D_1^2 + 2D_0D_2) + \dots)(\epsilon\theta_1 + \epsilon^2\theta_3 + \dots)\\ +2\gamma\epsilon^2(1 + \epsilon^2\beta)(D_0 + \epsilon D_1 + \epsilon^2 D_2 + \dots)(\epsilon\theta_1 + \epsilon^2\theta_3 + \dots)\\ -\kappa^2(P_0^2 + 2\epsilon P_0P_1 + \epsilon^2(P_1^2 + 2P_0P_2) + \dots)(\epsilon\theta_1 + \epsilon^2\theta_3 + \dots)\\ +(1 + 2\epsilon^2\cos(2\tau))(\epsilon\theta_1 + \epsilon^3\theta_3 - \frac{\epsilon^3\theta_1^3}{3!} + \mathcal{O}(\epsilon^5)) = 0\end{aligned} \tag{2.3}$$

The method of multiple scales requires we set the coefficients of like powers of ϵ to 0. There are no coefficients with ϵ^0, so we start with ϵ^1:

$$D_0^2\theta_1 - \kappa^2 P_0^2\theta_1 + \theta_1 = 0$$

Since we are looking for stationary waves, or at least slow waves, θ_1 does not depend on X_0, so $P_0\theta_1 = 0$. Then

$$D_0^2\theta_1 + \theta_1 = 0$$

which is just a linear oscillator, with the simple solution

$$\theta_1 = Ae^{iT_0} + A^*e^{-iT_0}$$

where A is some complex function of $T_1, T_2, T_3, \dots$ and A^* is its complex conjugate.

Now that we have a solution for one of the θ terms, we can use it going forward. We examine ϵ^2:

$$2D_0D_1\theta_1 - 2\kappa^2 P_0P_1\theta_1 = 0 \implies 2D_1(Ae^{iT_0} + A^*e^{-iT_0}) \qquad = 2\kappa^2 P_0P_1$$

In order to avoid resonance, we need to kill the secular terms, that is set $D_1A = 0$, and thus $D_1A^* = 0$ as a consequence. This means that $P_1\theta_1 = 0$.

The last term we need to look at are the ϵ^3 terms:

$$\begin{aligned}D_0^2\theta_3 + 2\beta D_0^2\theta_1 + (D_1^2 + 2D_0D_2)\theta_1 + 2\gamma D_0\theta_1\\ -\kappa^2[P_0^2\theta_3 + (P_1^2 + 2P_0P_2)\theta_1] - \frac{\theta_1^3}{3!} + 2\cos(2\tau)\theta_1 = 0\end{aligned}$$

Noting that θ_n does not depend on X_m where $n \neq m$, $P_0^3\theta_3$ and $2P_0P_2\theta_1$ go to zero, and inserting our solution for θ_1 we find

$$\begin{aligned} D_0^2\theta_3 + 2\beta(-Ae^{iT_0} - A^*e^{-iT_0}) + D_1^2(Ae^{iT_0} + A^*e^{-iT_0}) + 2D_2(iAe^{iT_0} - iA^*e^{-iT_0}) \\ +2\gamma(iAe^{iT_0} - iA^*e^{-iT_0}) - \kappa^2 P_1^2(Ae^{iT_0} + A^*e^{-iT_0}) \\ -\frac{(Ae^{iT_0} + A^*e^{-iT_0})^3}{3!} + (e^{2iT_0} + e^{-2iT_0})(Ae^{iT_0} + A^*e^{-iT_0}) = 0 \end{aligned}$$

We can deal with this $\frac{(Ae^{iT_0}+A^*e^{-iT_0})^3}{3!}$:

$$\frac{(Ae^{iT_0} + A^*e^{-iT_0})^3}{3!} = \frac{A^3e^{3iT_0} + A^{*3}e^{-3iT_0} + 3|A|^2(Ae^{iT_0} + A^*e^{-iT_0})}{3!}$$

and insert it back in, but to save space we will now just kill those terms which lead to resonance, leading to

$$2iD_2A + 2i\gamma A - \kappa^2 P_1^2 A - \frac{1}{2}A|A|^2 + A^* - 2\beta A = 0 \tag{2.4}$$

The A^* term arises from the expansion of $\cos(2\tau)$ combining with θ_1. The A^* and nonlinearity in equation (2.4) is a problem when it comes to finding the dispersion relation. To remedy this we drop the nonlinearity and let $A = u + iv$, so $A^* = u - iv$:

$$\begin{aligned} 2iD_2(u+iv) + 2i\gamma(u+iv) - \kappa^2P_1^2(u+iv) + (u-iv) - 2\beta(u+iv) = 0 \\ \implies 2iD_2u - 2D_2v + 2i\gamma u - 2\gamma v - \kappa^2P_1^2u - i\kappa^2P_1^2v + u - iv - 2\beta u - 2i\beta v = 0 \end{aligned}$$

We can split this into a real and imaginary part, both of which must equal zero:

$$\begin{aligned} -2D_2v - 2\gamma v - \kappa^2P_1^2u + u - 2\beta u = 0 \\ 2D_2u + 2\gamma u - \kappa^2P_1^2v - v - 2\beta v = 0 \end{aligned}$$

These are two coupled linear Schrödinger equations, so we need to find a way to untangle them from each other. Unfortunately, I couldn't manage to do this. After searching online and asking Charlotte, nothing worked and so I have nothing to analyse computationally or with the asymptotic expansion. As far as I'm aware, Tristan and Yola are also as stuck as I am. I'm confident that without this extremely tough analysis to do, I would've done well with the computational side of things, but that was not to be.