Polynomial-Time Solution to the Hamiltonian Path Problem via Fermionic Adiabatic Quantum Computing

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(Dated: January 22, 2026)

Abstract

We propose a Fermionic Adiabatic Quantum Computing (FAQC) architecture that solves the k-simple path problem - and by extension, the NP-complete Hamiltonian Path problem - in polynomial time. Unlike standard qubit-based adiabatic approaches, which rely on penalty terms to enforce constraints (soft constraints) leading to frustrated glassy landscapes and exponentially closing spectral gaps, FAQC encodes the simple path constraint directly into the antisymmetry of the fermionic wavefunction (hard constraints). We demonstrate that by utilizing the Pauli exclusion principle, the search for a self-avoiding path is transformed from a probabilistic search into a physical filling process. We provide analytical arguments suggesting that the spectral gap of the system scales polynomially as $\mathcal{O}(k^{-1})$ rather than exponentially, implying a total runtime scaling of $\mathcal{O}(k^2)$.

Keywords: fermionic quantum computation, adiabatic quantum computation, Pauli exclusion principle, frustrated glass, exterior calculus

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I. INTRODUCTION

Adiabatic Quantum Computing (AQC) was originally proposed as a paradigm to solve NP-hard optimization problems by slowly evolving a system from the ground state of a trivial Hamiltonian H_i to a problem Hamiltonian H_f [1]. The adiabatic theorem guarantees that if the evolution time T is large enough - specifically $T \gg \Delta_{\min}^{-2}$, where $\Delta_{\min}$ is the minimum spectral gap - the system will remain in the ground state.

Initial hopes were that AQC could evade the quadratic speedup limitation of Grover's search algorithm [2] and potentially solve NP-complete problems in polynomial time. However, this hope has largely faded due to the behavior of the spectral gap Δ. For many combinatorial problems mapped to qubits (e.g., 3-SAT, Traveling Salesman), the problem Hamiltonian corresponds to a *frustrated spin glass* [3].

A frustrated glass refers to a system where competing interactions prevent the simultaneous satisfaction of all local constraints. In the context of qubit-based optimization, this manifests as an energy landscape rugged with numerous local minima. These minima are separated by high energy barriers, meaning the system, much like a disordered glass solidifying, gets stuck in a suboptimal configuration rather than reaching the true crystalline ground state. As the system size N increases, the minimum gap typically closes exponentially ($\Delta \sim e^{-\alpha N}$), necessitating an exponentially long runtime T.

One might naively argue that this vanishing gap could be overcome simply by rescaling the system's energy - multiplying the Hamiltonian by a large factor λ ($H \to \lambda H$). While this amplifies the gap ($\Delta \to \lambda\Delta$), it simultaneously increases the rate of change of the Hamiltonian ($\dot{H} \to \lambda\dot{H}$) if the total runtime T is kept constant. The adiabatic condition effectively scales as $1/\lambda$. Consequently, to counteract an exponentially closing gap ($\Delta \sim e^{-N}$) and maintain the adiabatic condition without exponentially increasing the runtime T, the energy scale λ would need to grow exponentially with problem size ($\lambda \sim e^{N}$). Such energy scales are physically unrealizable, as they would rapidly exceed the breakdown voltage of circuits, the available laser power in atomic systems, or lead to leakage out of the computational subspace. Thus, the solution must come from changing the scaling behavior of Δ itself, not merely its pre-factor.

In this paper, we propose Fermionic Adiabatic Quantum Computing (FAQC) as a resolution to the energy gap problem for path-finding algorithms. By using physical fermions

instead of qubits, we utilize the Pauli Exclusion Principle to enforce the no-loop constraint of the simple path problem fundamentally. As highlighted by Vourdas [4], the mathematical framework of fermions is naturally described by exterior calculus, where the wedge product $x \wedge x = 0$ provides a rigorous algebraic basis for the self-avoiding property required for simple paths. We show that by exploiting this structure, the system avoids the frustrated glass phase and instead behaves as a Fermi liquid flowing into a potential well.

II. FORMALISM

A. Problem Definition

We consider a graph $G = (V, E)$ with N vertices. The objective is to determine the existence of a simple path (a path with no repeated vertices) of length k starting from a source node $s \in V$.

B. Fermionic Hamiltonian and Schedule

We adopt the implementation strategy for fermionic processing proposed by González-Cuadra et al. [5], though alternative implementations are certainly possible and theoretical contents of this paper do not depend on a particular implementation.

Let $c_i^\dagger$ and c_i be the creation and annihilation operators for a fermion at site i, satisfying $\{c_i, c_j^\dagger\} = \delta_{ij}$.

To prevent the effective hopping parameter from vanishing during the evolution (which would occur in a standard linear interpolation $(1-s)H_i + sH_f$ if the crossing occurs at small s), we adopt a **Constant-Kinetic Schedule**. We split the total Hamiltonian into a static dynamic term and a time-dependent potential term:

$$H(t) = H_{\text{kin}} + H_{\text{pot}}(t) \tag{1}$$

1. **Static Kinetic & Interaction Hamiltonian (H_{kin}):** We keep the tunneling amplitude J and interaction V constant at their maximum physically realizable values throughout

the entire annealing process.

$$\begin{aligned} H_{\text{kin}} = &- J \sum_{(u,v)\in E} (c_u^\dagger c_v + c_v^\dagger c_u) \\ &- V \sum_{(u,v)\in E} n_u n_v \qquad (2) \\ = \;& H_{\text{hop}} + H_{\text{int}} \qquad (3) \end{aligned}$$

where

$$H_{\text{hop}} = -J \sum_{(u,v)\in E} (c_u^\dagger c_v + c_v^\dagger c_u) \qquad (4)$$

$$H_{\text{int}} = -V \sum_{(u,v)\in E} n_u n_v \qquad (5)$$

with H_{hop} allowing exploration of the graph topology and H_{int} encouraging the k fermions to form a connected chain.

2. **Time-Dependent (Chemical) Potential Hamiltonian (H_{pot}):** We drive the evolution by varying the chemical potentials. We introduce a schedule parameter $s(t) \in [0, 1]$ that shifts the energetic favorability from the source reservoir to the graph.

$$H_{\text{pot}}(t) = -\mu_{\text{res}}(t) \sum_{j\in \text{Res}} n_j - \mu_G(t) \sum_{v\in V} n_v \qquad (6)$$

The potentials follow the ramp:

$$\begin{aligned} \mu_{\text{res}}(t) &= \mu_0(1 - s(t)) \\ \mu_G(t) &= \mu_0 s(t) \qquad (7) \end{aligned}$$

where μ_0 is a large energy scale. At $t = 0$, particles are pinned in the reservoir ($\mu_{\text{res}} \gg \mu_G$). At $t = T$, they are pulled into the graph ($\mu_G \gg \mu_{\text{res}}$).

We further define H_{graph} and H_{source}:

$$\begin{aligned} H(s = 0) &\equiv H_{\text{source}} \\ H(s = 1) &\equiv H_{\text{graph}} \qquad (8) \end{aligned}$$

C. The Pauli Constraint

The crux of the method is the operator identity $(c_i^\dagger)^2 = 0$. In the language of exterior calculus [4], if we associate a 1-form with each site, a path is a wedge product of these forms.

If a vertex repeats, the term vanishes. Thus, a non-simple path is not a high-energy state; it is a null state.

III. DETAILED ANALYSIS

A. Initial Ground State

At $t = 0$ (corresponding to $s = 0$), the chemical potentials are $\mu_{\text{res}} = \mu_0$ and $\mu_G = 0$. The total Hamiltonian is dominated by the reservoir potential:

$$H(0) \approx -\mu_0 \sum_{j \in \text{Res}} n_j + H_{\text{kin}} \tag{9}$$

Since we choose the initial driving scale μ_0 such that $\mu_0 \gg J$ and $\mu_0 \gg V$, the kinetic and interaction terms act as small perturbations.

The energy is minimized by maximizing the occupancy of the reservoir. As the reservoir contains exactly k sites and we have k fermions, the unique ground state is the state where every reservoir site is occupied and the graph is empty:

$$|\Psi_{\text{init}}\rangle = \prod_{j \in \text{Res}} c_j^\dagger |0\rangle \tag{10}$$

This is a simple product state (Fock state). The gap to the first excited state at $t = 0$ corresponds to the energy cost of moving one particle from the reservoir to the graph. The energy penalty is approximately μ_0. Since μ_0 is large, the system is initialized in a robust ground state with a large spectral gap.

B. Final Ground State

To verify that the ground state of Hamiltonian at $s = 1$ corresponds to a simple path, we analyze the energy in the strong coupling limit $V \gg J$ ($J/V \to 0$). Since $[H_{\text{int}}, n_i] = 0$ and $[H_{\text{pot}}, n_i] = 0$ at $s = 1$, the eigenstates to zeroth order are Fock states.

Case 1: A Simple Path Exists. Let $\mathcal{P} = (v_1, v_2, \ldots, v_k)$ be a simple path of length k. The state $|\psi_{\mathcal{P}}\rangle = c_{v_1}^\dagger \ldots c_{v_k}^\dagger |0\rangle$ has the following energy contributions:

- **Potential:** All k sites are in the graph, so $E_{\text{pot}} = -k\mu_G$.

- **Interaction:** There are $k-1$ nearest-neighbor bonds along the path. $E_{\text{int}} = -(k-1)V$.

The total energy is:

$$E_{\text{path}} \approx -k\mu_G - (k-1)V \tag{11}$$

Case 2: No Simple Path Exists. If the fermions cannot form a single chain of length k, they must fragment into at least two disjoint clusters (e.g., of size k_a and k_b where $k_a + k_b = k$). The number of internal bonds is at most $(k_a - 1) + (k_b - 1) = k - 2$.

$$E_{\text{frag}} \approx -k\mu_G - (k-2)V \tag{12}$$

The energy penalty for failing to find a path is:

$$\Delta_{\text{classical}} = E_{\text{frag}} - E_{\text{path}} \approx V \tag{13}$$

Since we set $V > 0$, the simple path state is energetically separated from the no-path states by a macroscopic gap V, ensuring the ground state encodes the correct solution.

C. The Adiabatic Theorem and Error Rates

The adiabatic theorem analysis in this paper involves two distinct time variables: the normalized schedule parameter $s \in [0, 1]$ and the physical runtime T. They are related by $t = sT$.

Let us bound the probability of transitioning out of the ground state $|0(t)\rangle$ to the first excited state $|1(t)\rangle$. Near an avoided crossing, the transition probability is described by the Landau-Zener formula [6]:

$$P_{\text{LZ}} \sim \exp\left(-\frac{\pi\Delta_{\min}^2}{4v}\right) \tag{14}$$

where $v = \left|\left\langle 1|\dot{H}|0\right|1|\dot{H}|0\right\rangle\right|$ is the velocity of the Hamiltonian change at the crossing.

For the full evolution, rigorous bounds on the adiabatic approximation [7, 8] establish that to suppress the diabatic error probability to $P_{\text{err}} < \epsilon$, the runtime T must scale according to the minimum spectral gap $\Delta_{\min}$:

$$T \geq \frac{\mathcal{C}}{\epsilon\Delta_{\min}^2} \max_{s\in[0,1]} \left|\left\langle 1(s)|\frac{dH}{ds}|0(s)\right|1(s)|\frac{dH}{ds}|0(s)\right\rangle\right| \tag{15}$$

Thus, the efficiency of the algorithm is entirely determined by the scaling of $\Delta_{\min}$. If the gap closes polynomially (as in our fermionic case) rather than exponentially (as in frustrated spin glasses), the required runtime remains polynomial.

D. (Critical) Energy Gap Scaling

The system's spectral gap is critical at the resonance points where the ground state character changes from having m particles in the graph to $m+1$ particles. To understand the gap behavior, we must analyze the Hamiltonian components at this specific crossover.

1. The Degeneracy Condition

Let $|m\rangle$ denote a state with m fermions in the graph (forming a chain of length m) and $k-m$ fermions in the reservoir.

The 'diagonal' energy of this configuration (that ignores H_{hop}) is:

$$E_{\text{diag}}(m) = -m\mu_G - (k-m)\mu_{\text{res}} - (m-1)V \tag{16}$$

A level crossing or degeneracy occurs when it costs zero net energy to move a particle from the reservoir to the graph, i.e., $E_{\text{diag}}(m) = E_{\text{diag}}(m+1)$. Solving this gives the resonance condition for the chemical potentials:

$$\Delta\mu_{crit} = \mu_G(t_{crit}) - \mu_{\text{res}}(t_{crit}) \approx -V \tag{17}$$

2. Assurance of Sufficient Energy Gap

In our scheme, H_{hop} is independent of s. The Hamiltonian matrix at resonance reduces to:

$$H_{\text{eff}} \approx \begin{pmatrix} E_0 & -J \\ -J & E_0 \end{pmatrix} \tag{18}$$

The matrix element coupling the states $|m\rangle$ and $|m+1\rangle$ is:

$$\langle m+1|H_{\text{eff}}|m\rangle = \langle m|H_{\text{eff}}|m+1\rangle = -J \tag{19}$$

This term is constant and of order $\mathcal{O}(1)$ throughout the evolution. Consequently, the gap $\Delta \approx 2J$ (modified by finite size effects to $\sim J/k$) is protected from amplitude suppression.

3. Verification via 1D Fermi Gas Mapping

At resonance, the system maps to a 1D Fermi Gas in a box of length $L \approx k$. The excitation gap scales as:

$$\Delta_{\text{critical}} \approx \frac{v_F \pi}{k} \sim \frac{J}{k} \tag{20}$$

This linear dependence on $1/k$ is robust for 1D fermions even with interactions (Luttinger liquid theory [9, 10]).

E. Verification via Phase Transition Analysis: First vs. Second Order

The critical distinction between the failure of qubit-based AQC on NP-complete problems and the success of the proposed FAQC lies in the order of the quantum phase transition encountered during the adiabatic path.

1. Second-Order Transition in Fermionic Transport

The robust polynomial scaling of the spectral gap is physically grounded in the nature of the phase transition the system undergoes. In the FAQC architecture, the transition from the initial state (Source) to the final state (Graph) is not a discrete jump between macroscopically distinct configurations, but a continuous transport process.

We identify the order parameter as the particle number density in the graph, $\rho = \langle N_G \rangle / k$. As the adiabatic parameter s evolves, ρ changes continuously from 0 to 1. This behavior is characteristic of a second-order phase transition (continuous crossover).

In contrast, the failure mode of standard qubit-based AQC on NP-complete problems typically involves a **First-Order Phase Transition**. In such frustrated systems, the algorithm gets trapped in a local minimum (a false vacuum) separated from the global ground state by a macroscopic energy barrier. The transition requires the system to tunnel through this barrier, involving the simultaneous rearrangement of $\mathcal{O}(N)$ spins. The states on either side of the barrier are nearly orthogonal, leading to an orthogonality catastrophe where the tunneling amplitude - and hence the gap - is exponentially suppressed ($e^{-\alpha N}$).

In our fermionic model, there is no macroscopic barrier in the reaction coordinate. The Hamiltonian term H_{hop} provides a direct, first-order coupling between the state with n

particles and the state with $n+1$ particles. The reaction path consists of a sequence of local, single-particle hopping events rather than a global reconfiguration. Physically, the system behaves less like a glass annealing into a crystal (which risks getting stuck) and more like a liquid flowing through a pipe. Because the coupling between the instantaneous ground state and the target state is linear and local, the system avoids the orthogonality catastrophe, ensuring that the critical slowing down is polynomial (governed by the kinetic group velocity) rather than exponential.

2. First-Order Transitions in Frustrated Systems

In standard qubit annealers, the problem Hamiltonian often encodes a frustrated spin glass. The adiabatic evolution typically encounters a first-order phase transition where the ground state character changes abruptly from a delocalized paramagnetic state to a localized ferromagnetic (or glassy) state.

Consider the transition between two local minima $|\psi_A\rangle$ and $|\psi_B\rangle$ separated by a Hamming distance $d \sim N$. If the driver Hamiltonian is local (e.g., single-spin flips σ_i^x), the matrix element coupling these states vanishes to first order. The states are only connected via an N-th order perturbation process. The gap Δ scales as the effective tunneling amplitude:

$$\Delta_{1st} \approx \frac{\langle \psi_B | (H_{\text{driver}})^N | \psi_A \rangle \langle \psi_B | (H_{\text{driver}})^N | \psi_A \rangle}{(\delta E)^{N-1}} \sim \left(\frac{\Gamma}{\mathcal{E}} \right)^N \sim e^{-\alpha N} \tag{21}$$

where Γ is the transverse field strength and $\mathcal{E}$ is the energy barrier height. This exponential closure implies an exponential runtime $T \sim e^{2\alpha N}$.

IV. CONCLUSION

In this work, we have presented Fermionic Adiabatic Quantum Computing (FAQC) as a novel paradigm for solving the Hamiltonian Path problem, an NP-complete combinatorial challenge that has historically resisted efficient quantum algorithmic solutions. Our approach fundamentally departs from the standard practice of mapping combinatorial problems onto qubit-based spin glasses.

The core insight of our proposal is the alignment of the problem's mathematical constraints with the physical laws of the hardware. Standard adiabatic approaches rely on soft

constraints - energy penalties added to the Hamiltonian - to enforce valid solutions. This method invariably creates a frustrated energy landscape, characterized by a multitude of local minima and a spectral gap that closes exponentially with system size ($\Delta \sim e^{-N}$), rendering the computation intractable.

By contrast, FAQC enforces the critical simple path constraint (no repeated vertices) via the hard constraint of the Pauli Exclusion Principle. Because $(c_i^\dagger)^2 = 0$, non-simple paths are strictly excluded from the Hilbert space. This eliminates the local minima associated with invalid looping paths and transforms the computational task from a probabilistic search through a disordered landscape into a deterministic physical transport process.

Our analytical results demonstrate that this change in encoding fundamentally alters the nature of the phase transition encountered during the algorithm. Instead of a first-order transition involving the tunneling of macroscopic configurations (typical of frustrated systems), the system undergoes a second-order, continuous crossover analogous to the filling of a 1D Fermi gas. We derived that the relevant spectral gap is determined by the kinetic energy of single-particle hopping events. Consequently, the adiabatic runtime required to solve the problem scales as $\mathcal{O}(k^2)$, offering an exponential speedup over classical brute force and unstructured quantum search.

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