Broadband Quantum Cooling for Hamiltonian Path Finding via Harmonic Reservoirs

Minseong Kim*

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Abstract

We propose a robust quantum algorithm for solving the directed Hamiltonian Path problem using a "Broadband Sequential Cooling" protocol. We define a problem Hamiltonian H_{path} with integer-quantized penalty coefficients, ensuring a discrete energy spectrum where all gaps are commensurate with a fundamental frequency ε. We couple this system to a harmonic reservoir with a frequency comb spectrum $\epsilon_q = q\varepsilon$. This architecture allows for simultaneous, multi-resonant inelastic scattering, where any relaxation event in the system is automatically resonant with a specific mode in the reservoir. This eliminates the need for sequential tuning of reservoir parameters and significantly accelerates the ground-state search compared to standard adiabatic or single-mode cooling techniques.

* mkimacad@gmail.com

I. INTRODUCTION

The Hamiltonian Path problem is a core NP-complete challenge. Traditional Quantum Adiabatic Computing (AQC) approaches struggle with the vanishing spectral gap and metastability in the optimization landscape. While single-mode cooling can target specific defects, it requires precise, iterative tuning of the reservoir frequency to cascade down the energy ladder.

In this work, we introduce **Harmonic Reservoir Cooling**, building upon and providing a specific (though modified) instance of the ideas in [1]. By quantizing the penalty structure of the Hamiltonian and coupling it to a multi-mode bath (a frequency comb), we engineer a system where *every* energetically favorable transition is resonant. This allows the system to "fall" through its configuration space via multiple decay channels simultaneously, rapidly reaching the ground state.

II. QUANTIZED SYSTEM HAMILTONIAN

We consider a directed graph $G = (V, E)$ and a temporal lattice of size $N \times N$, with bosonic operators $b^\dagger_{v,t}, b_{v,t}$. To ensure commensurability with the reservoir, we define the penalty coefficients A, B, C as integer multiples of a fundamental energy unit ε:

$$A = k_A\varepsilon, \quad B = k_B\varepsilon, \quad C = k_C\varepsilon \tag{1}$$

where $k_A, k_B, k_C \in \mathbb{N}^+$. Typically, we set $k_C = 1$ (base unit) and $k_A, k_B > k_C$ to prioritize satisfying hard constraints (valid permutation) over soft constraints (valid edges).

The Hamiltonian is:

$$H_{\text{path}} = k_A\varepsilon \sum_t \left(\sum_v \hat{n}_{v,t} - 1 \right)^2 \tag{2}$$

$$+ k_B\varepsilon \sum_v \left(\sum_t \hat{n}_{v,t} - 1 \right)^2 \tag{3}$$

$$+ k_C\varepsilon \sum_t \sum_{(u,w)\notin E} \hat{n}_{u,t}\hat{n}_{w,t+1}. \tag{4}$$

A. Discrete Energy Gaps

Because the Hamiltonian is a sum of number operators with integer coefficients, the eigenvalues are:

$$E(\{n\}) = (M_A k_A + M_B k_B + M_C k_C)\,\varepsilon \tag{5}$$

where $M_{A,B,C}$ are non-negative integers determined by the defect configuration. Crucially, the energy gap ΔE between *any* two states is strictly an integer multiple of ε:

$$\Delta E_{if} = E_i - E_f = m \cdot \varepsilon, \quad m \in \mathbb{Z}. \tag{6}$$

III. HARMONIC RESERVOIR PROTOCOL

Instead of a single-mode bath, we employ a multi-mode reservoir R with a harmonic spectrum.

A. Reservoir Spectrum

The reservoir consists of a set of independent bosonic modes (harmonic oscillators) indexed by $q \in \mathbb{N}^+$:

$$H_{\text{res}} = \sum_{q=1}^{Q_{\text{cutoff}}} (q\varepsilon)\, d_q^\dagger d_q. \tag{7}$$

Mode q has frequency exactly q times the fundamental ε. This spectrum is physically realizable in:

- **1D Cavities:** Longitudinal modes of a transmission line resonator ($f_n = n \cdot f_0$).
- **Frequency Combs:** Optical frequency combs generated by mode-locked lasers.

B. Broadband Interaction

The interaction couples the system's hopping to *all* reservoir modes:

$$H_{\text{int}} = \sum_{\alpha,\beta} \sum_{q} g_q \left(b_\alpha^\dagger b_\beta d_q^\dagger + b_\beta^\dagger b_\alpha d_q \right). \tag{8}$$

Here, $b_\alpha^\dagger b_\beta d_q^\dagger$ represents a process where the system rearranges (hops $\beta \to \alpha$), releasing energy ΔE, which creates a photon in reservoir mode q.

For analytical convenience without loss of generality, we may assume $g_q = g_{\max}$, a constant.

IV. TRANSITION PROBABILITY

We analyze the transition dynamics in the interaction picture in the first-order ($O(g)$) perturbation theory and then derive the conditions necessary for the first-order approximation of transition probability.

A. Interaction Picture Derivation

The total Hamiltonian is $H = H_0 + H_{\text{int}}$, where $H_0 = H_{\text{path}} + H_{\text{res}}$. We move to the interaction picture defined by the unitary $U_0(t) = e^{-iH_0 t}$. The state vector $|\Psi(t)\rangle_I$ evolves according to:

$$i\frac{d}{dt}|\Psi(t)\rangle_I = H^I_{\text{int}}(t)\,|\Psi(t)\rangle_I \tag{9}$$

where $H^I_{\text{int}}(t) = e^{iH_0 t} H_{\text{int}} e^{-iH_0 t}$.

Consider a transition from an initial state $|I\rangle = |i\rangle_S \otimes |0\rangle_R$ to a final state $|F\rangle = |f\rangle_S \otimes |1_q\rangle_R$. The corresponding energies are $E_I = E_i$ and $E_F = E_f + q\varepsilon$. The interaction term coupling these states is:

$$\langle F|H^I_{\text{int}}(t)|I\rangle = e^{i(E_F - E_I)t}\,\langle F|H_{\text{int}}|I\rangle\,. \tag{10}$$

Let the static matrix element be $\mathcal{M}^q_{fi} = \langle f, 1_q|H_{\text{int}}|i, 0\rangle$. Using first-order time-dependent perturbation theory, the amplitude $c_F(t)$ at time Δt is:

$$\begin{aligned} c_F(\Delta t) &= -i\int_0^{\Delta t} dt\,\mathcal{M}^q_{fi}\,e^{i(E_F - E_I)t} \\ &= -i\mathcal{M}^q_{fi}\left[\frac{e^{i(E_F - E_I)\Delta t} - 1}{i(E_F - E_I)}\right]. \end{aligned} \tag{11}$$

The transition probability $P^{(q)}_{i\to f} = |c_F(\Delta t)|^2$ is:

$$P^{(q)}_{i\to f}(\Delta t) = 4|\mathcal{M}^q_{fi}|^2 \frac{\sin^2\left(\frac{(E_f + q\varepsilon - E_i)\Delta t}{2}\right)}{(E_f + q\varepsilon - E_i)^2}. \tag{12}$$

B. Bound for Energy Gap ε

In the resonant case (conservation of total non-interaction energy, $E_f = E_i - q\varepsilon$), we have:

$$P_{i\to f}^{(q,dominant)}(\Delta t) = |\mathcal{M}_{fi}^{q}\Delta t|^2. \tag{13}$$

The goal is to ensure that the resonant term dominate over the non-resonant terms. The only degree of freedom, having fixed i and f, that non-resonant terms has is q, and $1 \leq q \leq Q_{\text{cutoff}}$. Additionally, in first-order, only one particle can be moved, which gives us N^3 possible Fock states for $|f\rangle$ that can be reached from the initial system Fock state $|i\rangle$.

Assuming $\Delta t \sim \mathcal{O}(1)$, this suggest the following condition up to the question of $\mathcal{M}_{fi}^{q}$:

$$\varepsilon \gg N^3 Q_{\text{cutoff}} \tag{14}$$

Due to the issue of $\mathcal{M}_{fi}^{q}$, we add the safety margin of N such that the condition becomes:

$$\varepsilon \gg N^4 Q_{\text{cutoff}} \tag{15}$$

Setting $Q_{\text{cutoff}} = \mathcal{O}(1)$:

$$\varepsilon \sim \mathcal{O}(N^5) \tag{16}$$

This ensures that the decay factor of at least $1/\varepsilon^2$ counteracts the number of possible configurations. When the resonant term dominates, $q > 0$ ensures that interaction mostly goes in the direction of cooling unless $|i\rangle$ is already the ground state.

1. *Alternative: Bound for Interaction Time Δt*

Instead of assuming $\Delta t \sim \mathcal{O}(1)$, we may instead assume $\varepsilon \sim \mathcal{O}(1)$ and bound Δt instead as follows:

$$\Delta t \sim \mathcal{O}(N^3) \tag{17}$$

This relies on the fact that $\sin kx/x^2 \leq 1$ whenever $x^2 \geq 1$ with squared interaction time scaling of dominant (resonant) transition probability.

C. Derivation of the Matrix Element

We explicitly calculate the matrix element $\mathcal{M}_{fi}^{q} = \langle F|H_{\text{int}}|I\rangle$ which governs the transition amplitude.

Substituting the definitions:

$$\begin{aligned}\mathcal{M}_{fi}^{q} &= \left(\langle f|_S \otimes \langle 1_q|_R\right) H_{\text{int}} \left(|i\rangle_S \otimes |0\rangle_R\right) \\ &= \sum_{\alpha,\beta,k} g_k \Big[\langle f|_S \, b_\alpha^\dagger b_\beta \, |i\rangle_S \, \langle 1_q|d_k^\dagger|0\rangle_R \\ &\quad + \langle f|_S \, b_\beta^\dagger b_\alpha \, |i\rangle_S \, \langle 1_q|d_k|0\rangle_R\Big] . \end{aligned} \tag{18}$$

We evaluate the reservoir terms first:

1. **Heating Term:** The term containing d_k acts on the reservoir vacuum state $|0\rangle_R$. Since $d_k \, |0\rangle_R = 0$, this term vanishes identically. This confirms that the system cannot absorb energy from an empty reservoir.

2. **Cooling Term:** The term containing $d_k^\dagger$ acts as $d_k^\dagger \, |0\rangle_R = |1_k\rangle_R$. The inner product becomes:

$$\langle 1_q|1_k\rangle_R = \delta_{qk}. \tag{19}$$

   This Kronecker delta collapses the sum over k, selecting only the mode $k = q$ corresponding to the final reservoir state.

The expression simplifies to:

$$\mathcal{M}_{fi}^{q} = g_q \sum_{\alpha,\beta} \langle f|_S \, b_\alpha^\dagger b_\beta \, |i\rangle_S . \tag{20}$$

Since we are in the first-order perturbation theory where only one particle can be moved, if i, f refer to Fock states, then:

$$\mathcal{M}_{fi}^{q} = \mathcal{M}_{\alpha\beta}^{q} = g_q \, \langle f|_S \, b_\alpha^\dagger b_\beta \, |i\rangle_S . \tag{21}$$

This justifies the additional N factor in Equation (15) such that as in Equation (16):

$$\varepsilon \sim \mathcal{O}(N^5)$$

with $\Delta t \sim \mathcal{O}(1)$ or as in Equation (17),

$$\Delta t \sim \mathcal{O}(N^3)$$

with $\varepsilon \sim \mathcal{O}(1)$.

D. Validity Regime (Weak Coupling Constant) of Perturbation Theory

Our derivation of the transition probability relies on first-order time-dependent perturbation theory. We now quantify the bounds on the coupling strength g_q imposed by this requirement. As before, we assume without loss of generality that $g_q = g_{\max}$ for all q.

This is easy to quantify. For each additional perturbation order, additional H_{int} moves an additional particle. This gives us the additional multiplicative factor of $N^4 Q_{\text{cutoff}} \sim \mathcal{O}(N^4)$ to compensate for, as with the bound for ε. The following bound is therefore obtained:

$$g \ll \frac{1}{N^4} \Rightarrow g \sim \mathcal{O}(N^{-5}) \tag{22}$$

E. Number of Reservoirs

With conditions in either Equation (16) or (17) and Equation (22), the cooling failure probability for an excited initial system state is:

$$P_{fail} = \mathcal{O}(1/N^{-1}) \tag{23}$$

To achieve almost perfect success for cooling by at least energy ε, this requires interacting sequentially with N reservoirs, each with interaction time of Δt.

The maximum energy of N particles in the system (with Hamiltonian H_{path}) that may needs to be cooled off is:

$$E_{\max}/\varepsilon = \mathcal{O}(N^7) \tag{24}$$

This implies that we would need total of $\mathcal{O}(N^7)$ reservoirs to cool off the system into the ground state to discover the Hamiltonian path.

F. Perfect Reservoir Problem

While we avoid exponential runtime in adiabatic quantum computation, we have assumed a perfect reservoir in its ground state. In practice, this is not easy to achieve. There are exponentially many states occupying $\mathcal{O}(N^7)$ excited states. This means that if a reservoir is in some finite temperature of $T \sim \mathcal{O}(1)$, then ground state cooling will not work, as thermal equilibration brings the system in equal temperature with the reservoir. We need extremely small temperature for reservoir for ground state cooling to work.

V. PHYSICAL IMPLEMENTATION

A. Superconducting Circuit Realization

We map the Hard-Core Bosons to Transmon qubits. The harmonic reservoir is realized naturally by a multi-mode superconducting transmission line resonator.

A coplanar waveguide resonator of length L supports modes at frequencies $f_n = n \cdot \frac{v}{2L}$. By designing the fundamental mode $f_1 = \varepsilon$, the higher harmonics provide the required spectrum $q\varepsilon$.

The interaction is mediated by a flux-tunable coupler (SQUID) modulated at a sideband frequency. To couple to all modes simultaneously, one can apply a broadband pulse or a "comb" drive that activates the parametric exchange constraints for all q.

B. Fermionic Implementation (Ultracold Atoms)

For a fermionic realization, we utilize ultracold neutral atoms (e.g., ^{6}Li or ^{40}K) trapped in a deep 2D optical lattice. This platform is naturally suited for simulating Fermi-Hubbard-like models.

1. Engineering the Integer Spectrum

To implement the integer-quantized Hamiltonian H_{path}, we require programmable long-range interactions. We employ **Rydberg Dressing**, where the ground state atoms are weakly coupled to a high-lying Rydberg state via a far-detuned laser. This induces a soft-core interaction potential $V(r)$.

- **Digitized Potential:** By spatially modulating the dressing laser intensity using a Digital Micromirror Device (DMD), we can engineer the interaction strength V_{ij} between any two sites to take discrete values.

- **Quantization:** We calibrate the laser intensity such that the interaction energy for a "broken edge" is $V_{\text{edge}} = \varepsilon$, and the penalties for multiple occupancy are set to $V_{\text{collision}} = k_A\varepsilon$.

2. Harmonic Reservoir via Optical Frequency Combs

To realize the "Broadband Reservoir" where decay channels exist at all integer harmonics $q\varepsilon$, we replace the standard Raman cooling beams with an Optical Frequency Comb.

An optical frequency comb consists of a series of phase-coherent laser lines spaced exactly by a repetition rate f_{rep}. We set the repetition rate to match our fundamental energy unit:

$$f_{\text{rep}} = \varepsilon/h. \tag{25}$$

The Cooling Mechanism: We drive stimulated Raman transitions using the comb. A fermion at site β can hop to site α (lowering the system energy by $\Delta E = m\varepsilon$) by scattering a photon from comb line n to comb line $n - m$.

- The energy difference between these two comb lines is exactly $h(n\cdot f_{\text{rep}} - (n-m)\cdot f_{\text{rep}}) = m \cdot hf_{\text{rep}} = m\varepsilon$.
- This process satisfies energy conservation for *any* integer decay m.
- The scattered photon eventually decays into the vacuum, rendering the process irreversible (cooling).

This setup effectively couples the fermions to a bath with a discrete spectral density at every harmonic of ε, perfectly matching the requirements of the algorithm without needing to sweep a single-frequency laser.

The fermionic approach greatly reduces the Hilbert space and increases efficiency, since only one fermion can occupy one site at maximum.

VI. CONCLUSION

We have presented a "Broadband Quantum Cooling" algorithm. By quantizing the Hamiltonian Path problem such that all solution defects correspond to integer energy quanta, and coupling the system to a harmonic frequency comb reservoir, we enable a massively parallel relaxation process. This removes the control overhead of sequential tuning and robustly

guides the system to the ground state via automatic resonance matching.

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[1] M. Kim, Quantum Information Processing **23** (2024), 10.1007/s11128-024-04413-x.