1.7 Deviance

Deviance

One way of assessing the adequacy of a model is to compare it with a more general model with the maximum number of parameters that can be estimated. This is called the **saturated** model.

If there are N observations $Y_i, i = 1, \ldots, N$, then a saturated model can be specified with N parameters. Also called the **maximal** or **full model**. (In general, however, the maximum number m of parameters that can be estimated can be smaller than N, e.g., if observations are repeated.)

We write $\boldsymbol{\theta}_{\mathrm{max}}$ *for the parameter vector of the saturated model* and $\hat{\boldsymbol{\theta}}_{\mathrm{max}}$ for the maximum likelihood estimator of $\boldsymbol{\theta}_{\mathrm{max}}$.

The likelihood for the saturated model $L(\hat{\boldsymbol{\theta}}_{\mathrm{max}}; \mathbf{y})$ will be *larger* than any other likelihood function for these observations, with the same assumed distribution and link function, because it provides the most complete description of the data.

Let $L(\hat{\boldsymbol{\theta}}; \mathbf{y})$ denote the maximum value of the likelihood function for the model of interest. Then the **likelihood ratio**

$$\lambda = \frac{L(\hat{\boldsymbol{\theta}}_{max}; \mathbf{y})}{L(\hat{\boldsymbol{\theta}}; \mathbf{y})}$$

is a way of assessing the goodness of fit for the model. In practice

$$\log \lambda = \ell(\hat{\boldsymbol{\theta}}_{max}; \mathbf{y}) - \ell(\hat{\boldsymbol{\theta}}; \mathbf{y})$$

is used. *Large values of* $\log \lambda$ *suggest that the model of interest is a poor description of the data relative to the saturated model.*

The **Deviance** or **log likelihood ratio statistic** is defined as

$$D = 2[\ell(\hat{\boldsymbol{\theta}}_{max}; \mathbf{y}) - \ell(\hat{\boldsymbol{\theta}}; \mathbf{y})].$$

We know that

$$\ell(\boldsymbol{\theta}; \mathbf{y}) - \ell(\hat{\boldsymbol{\theta}}; \mathbf{y}) = -\frac{1}{2}(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}})^{\top} \mathcal{I}(\hat{\boldsymbol{\theta}})(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}})$$

or, equivalently,

$$2[\ell(\hat{\boldsymbol{\theta}}; \mathbf{y}) - \ell(\boldsymbol{\theta}; \mathbf{y})] = (\boldsymbol{\theta} - \hat{\boldsymbol{\theta}})^{\top}\mathcal{I}(\hat{\boldsymbol{\theta}})(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}})$$

and consequently the asymptotic distribution of this statistic is

$$2[\ell(\hat{\boldsymbol{\theta}}; \mathbf{y}) - \ell(\boldsymbol{\theta}; \mathbf{y})] \stackrel{\cdot}{\sim} \chi_p^2$$

We now have:

$$\begin{aligned}
D &= 2[\ell(\hat{\boldsymbol{\theta}}_{max}; \mathbf{y}) - \ell(\hat{\boldsymbol{\theta}}; \mathbf{y})] \\
&= 2[\ell(\hat{\boldsymbol{\theta}}_{max}; \mathbf{y}) \pm \ell(\boldsymbol{\theta}_{max}; \mathbf{y}) \pm \ell(\boldsymbol{\theta}; \mathbf{y}) - \ell(\hat{\boldsymbol{\theta}}; \mathbf{y})] \\
&= 2[\ell(\hat{\boldsymbol{\theta}}_{max}; \mathbf{y}) - \ell(\boldsymbol{\theta}_{max}; \mathbf{y})] - 2[\ell(\hat{\boldsymbol{\theta}}; \mathbf{y}) - \ell(\boldsymbol{\theta}; \mathbf{y})] \\
&\quad + 2[\ell(\boldsymbol{\theta}_{max}; \mathbf{y}) - \ell(\boldsymbol{\theta}; \mathbf{y})]
\end{aligned}$$

- The **first term** $2[\ell(\hat{\boldsymbol{\theta}}_{max}; \mathbf{y}) - \ell(\boldsymbol{\theta}_{max}; \mathbf{y})]$ has distribution χ_m^2, where m is the number of parameters in the saturated model
- The **second term** $2[\ell(\hat{\boldsymbol{\theta}}; \mathbf{y}) - \ell(\boldsymbol{\theta}; \mathbf{y})]$ has distribution χ_p^2, where p is the number of parameters in the model of interest
- The **third term** $2[\ell(\boldsymbol{\theta}_{max}; \mathbf{y}) - \ell(\boldsymbol{\theta}; \mathbf{y})]$ is a positive constant, which is zero if the model of interest has a fit which is as good as the saturated model; it can be considered a (usually negligible) non-centrality parameter

Therefore, the sampling distribution of the deviance is

$$D \stackrel{\cdot}{\sim} \chi_{m-p}^2(\nu) \tag{1.7.1}$$

where $\nu = 2[\ell(\boldsymbol{\theta}_{\max}; \mathbf{y}) - \ell(\boldsymbol{\theta}; \mathbf{y})]$ is a non-centrality parameter.

Remarks:

- The distribution is exact if the response variable is normally distributed
- For some other distributions, D can be calculated and used directly as a goodness of fit statistic

Example: Binomial distribution

If the response variables $Y_1, \ldots, Y_N$ are independent and $Y_i \sim \mathrm{Bin}(n_i, p_i)$, then the log-likelihood is

$$\ell(\mathbf{p}; \mathbf{y}) = \sum_{i=1}^{N} \left[Y_i \log p_i - Y_i \log(1 - p_i) + n_i \log(1 - p_i) + \log \binom{n_i}{Y_i} \right]$$

For a **saturated** model, the p_i's are all different.

The MLE are $\hat{p}_i = \frac{Y_i}{n_i}$ and

$$\ell(\hat{\mathbf{p}}_{max}; \mathbf{y}) = \sum_{i=1}^{N} \left[Y_i \log \left(\frac{Y_i}{n_i} \right) - Y_i \log \left(\frac{n_i - Y_i}{n_i} \right) + n_i \log \left(\frac{n_i - Y_i}{n_i} \right) + \log \binom{n_i}{Y_i} \right]$$

For any other model, the dimension of the parameter is $p < N$; let's call $\hat{p}_i^*$ *the MLE for a non-saturated model* and $\hat{Y}_i = n_i \hat{p}_i^*$ *the fitted values*; then

$$\ell(\hat{\mathbf{p}}^*; \mathbf{y}) = \sum \left[Y_i \log \left(\frac{\hat{Y}_i}{n_i} \right) - Y_i \log \left(\frac{n_i - \hat{Y}_i}{n_i} \right) + n_i \log \left(\frac{n_i - \hat{Y}_i}{n_i} \right) + \log \binom{n_i}{Y_i} \right]$$

And the **deviance** is

$$D = 2 \sum_{i=1}^{N} \left[Y_i \log \left(\frac{Y_i}{\hat{Y}_i} \right) + (n_i - Y_i) \log \left(\frac{n_i - Y_i}{n_i - \hat{Y}_i} \right) \right].$$

Nested model

We say that *model M_0 is nested in model M_1 if M_0 results as a special case of M_1.*

For instance, if we partition $\boldsymbol{\theta}$ as

$$\boldsymbol{\theta}^\top = \left(\boldsymbol{\theta}^{(1)^\top}, \boldsymbol{\theta}^{(2)^\top}\right)$$

where $\boldsymbol{\theta}$ has length p and $\boldsymbol{\theta}^{(1)}$ has length q,

Then model M_1 could assume unrestricted $\boldsymbol{\theta}$, whereas M_0 restricts, e.g, $\boldsymbol{\theta}^{(2)} = \mathbf{0}$.

The **scaled deviance** can be used for model comparison.

For two nested linear models, the difference ΔD between the two deviance statistics generally follows a χ^2 distribution.
The degrees of freedom equal the difference in the dimensions of the two models, that is:

$$\Delta D = D_0 - D_1 = 2[\ell(\hat{\boldsymbol{\theta}}_{\max}; \mathbf{y}) - \ell(\hat{\boldsymbol{\theta}}_0; \mathbf{y})] - 2[\ell(\hat{\boldsymbol{\theta}}_{\max}; \mathbf{y}) - \ell(\hat{\boldsymbol{\theta}}_1; \mathbf{y})] = 2[\ell(\hat{\boldsymbol{\theta}}_1; \mathbf{y}) - \ell(\hat{\boldsymbol{\theta}}_0; \mathbf{y})]$$

Then $D_0 \sim \chi^2(N - q)$ and $D_1 \sim \chi^2(N - p)$, then

$$\Delta D \sim \chi^2(p - q)$$

when N is large.

If the values of ΔD is in the critical region, then we would reject H_0 in favour of H_1 on the grounds that model M_1 provides a significantly better description of the data.

This is a non-assessed self-practice. Attempt the question below and press submit to be able to see the solution.

Question *Submitted Jan 22nd 2024 at 9:04:41 am*

[Dobson and Barnett (2018, Exercise 5.1)]

Consider the single response variable Y with $Y \sim \text{Bin}(n, \pi)$.

1. Find the Wald statistic $(\hat{\pi} - \pi)^\top \mathcal{I}(\hat{\pi} - \pi)$, where $\hat{\pi}$ is the maximum likelihood estimator of π and $\mathcal{I}$ the information.
2. Verify that the Wald statistic is the same as the score statistics $U^\top \mathcal{I}^{-1} U$.
3. Find the deviance $2 \left(\ell(\hat{\pi}; y) - \ell(\pi; y) \right)$. Note that this is an adaptation of the deviance for the case where there is only one predictor and therefore no saturated/non-saturated models.
4. For large sample. both the Wald/score statistic and the deviance approximately have the χ_1^2 distribution. For $n = 10$ and $y = 3$, use both statistics to assess the adequacy of the models:
   - $\pi = 0.1$;
   - $\pi = 0.3$;
   - $\pi = 0.5$.

   Do the two statistics lead to the same conclusions?

1. The log-likelihood is given by

$$\ell(\pi; y) = y \log(\pi) + (n - y) \log(1 - \pi) + \log \binom{n}{y},$$

and consequently the score is

$$U = \frac{y}{\pi} - \frac{n - y}{1 - \pi} = \frac{y - n\pi}{\pi(1 - \pi)},$$

Since $\mathbb{E}(Y) = n\pi$ and $\mathrm{Var}(Y) = n\pi(1 - \pi)$, we have $\mathbb{E}(U) = 0$ and

$$\mathcal{I} = \mathrm{Var}(U) = \frac{\mathrm{Var}(Y)}{\pi^2(1 - \pi)^2} = \frac{n}{\pi(1 - \pi)}.$$

As a consequence

$$(\hat{\pi} - \pi)^{\top} \mathcal{I}(\hat{\pi} - \pi) = (\hat{\pi} - \pi)^2 \frac{n}{\pi(1 - \pi)}$$

$$= \left(\frac{y}{n} - \pi\right)^2 \frac{n}{\pi(1 - \pi)}$$

$$= \frac{(y - n\pi)^2}{n\pi(1 - \pi)}.$$

2. Using the above information we have

$$U^{\top} \mathcal{I}^{-1} U = \left(\frac{y - n\pi}{\pi(1 - \pi)}\right)^2 \left(\frac{n}{\pi(1 - \pi)}\right)^{-1}$$

$$= \frac{(y - n\pi)^2}{n\pi(1 - \pi)}.$$

3. The deviance is given by

$$2 \left(\ell(\hat{\pi}; y) - \ell(\pi; y)\right) = 2 \left(y \log\left(\frac{\hat{\pi}}{\pi}\right) + (n - y) \log\left(\frac{1 - \hat{\pi}}{1 - \pi}\right)\right).$$

4. Assume that the size of the test is $\alpha = 0.05$, the $(1 - \alpha)\%$ quantile of the χ_1^2 is $\chi_{1,0.95}^2 = 3.84$.

```
wald <- function(y,n,p){
  (y-n*p)^2 / (n*p*(1-p))
}
```

```
deviance <- function(y,n,p){
  2 * (y * log(y/(n*p)) + (n-y)* log((1-y/n)/(1-p)) )
}

ps <- c(0.1,0.3,0.5)
alpha <- 0.05
crit <- qchisq(1-alpha, df=1)

for(i in 1:length(ps)){

  w <- wald(y=3, n=10, p=ps[i])
  d <- deviance(y=3, n=10, p=ps[i])

  if(w < crit){res.w <- "accept"}else{res.w <- "reject"}
  if(d < crit){res.d <- "accept"}else{res.d <- "reject"}
  cat("For p=", ps[i],":\n" )
  cat("we ", res.w, "H0 since Wald =", w, "and the critical value is ",
      crit, "\n" )
  cat("we ", res.d, "H0 since Deviance =", d, "and the critical value is ",
      crit, "\n\n" )

}

# For p= 0.1 :
# we   reject H0 since Wald = 4.444444 and the critical value is  3.841459
# we   accept H0 since Deviance = 3.073272 and the critical value is  3.841459

# For p= 0.3 :
# we   accept H0 since Wald = 0 and the critical value is  3.841459
# we   accept H0 since Deviance = 0 and the critical value is  3.841459

# For p= 0.5 :
# we   accept H0 since Wald = 1.6 and the critical value is  3.841459
# we   accept H0 since Deviance = 1.645658 and the critical value is  3.841459
```