The score function

To obtain the maximum likelihood estimator for the parameter β_j we derive the score function using the chain rule:

$$\frac{d\ell}{d\beta_j} = \mathrm{U}_j = \sum_{i=1}^{N} \left[\frac{d\ell_i}{d\beta_j} \right] = \sum_{i=1}^{N} \left[\frac{d\ell_i}{d\theta_i} \cdot \frac{d\theta_i}{d\mu_i} \cdot \frac{d\mu_i}{d\beta_j} \right]. \tag{3.1.3}$$

Now, let's consider each term separately

- $\frac{d\ell_i}{d\theta_i} = \mathrm{Y}_i b'(\theta_i) + c'(\theta_i) = b'(\theta_i)(\mathrm{Y}_i - \mu_i)$
- $\frac{d\theta_i}{d\mu_i} = \left(\frac{d\mu_i}{d\theta_i} \right)^{-1} = \left(\frac{-c''(\theta_i)}{b'(\theta_i)} + \frac{c'(\theta_i) b''(\theta_i)}{[b'(\theta_i)]^2} \right)^{-1} = (b'(\theta_i) \mathbb{V}\mathrm{ar}(\mathrm{Y}_i))^{-1}$
- $\frac{d\mu_i}{d\beta_j} = \frac{d\mu_i}{d\eta_i} \cdot \frac{d\eta_i}{d\beta_j} = \frac{d\mu_i}{d\eta_i} \mathrm{X}_{ij}$

Hence, the score function is

$$U_j = \sum_{i=1}^{N} \left[\frac{(y_i - \mu_i)}{\mathbb{V}ar(\mathrm{Y}_i)} \mathrm{X}_{ij} \left(\frac{d\mu_i}{d\eta_i} \right) \right] \tag{3.1.4}$$