binomial dist

$$f_Y(y) = \binom{n}{y} p^y (1-p)^{n-y} \qquad y = 0, 1, \dots, n$$

$$= \exp\left(\log \binom{n}{y} p^y (1-p)^{n-y} \right)$$

$$= \exp\left(\log \binom{n}{y} + y \log p + (n-y) \log(1-p) \right)$$

$$= \exp\left[\log \binom{n}{y} + y \underbrace{\left[\log p - \log(1-p) \right]}_{\log \frac{p}{1-p}} + n \log(1-p) \right]$$

$$= \exp\left[\underbrace{\log \binom{n}{y}}_{d(y)} + \underbrace{y}_{a(y) = y} \underbrace{\log \frac{p}{1-p}}_{b(p)} + \underbrace{n \log(1-p)}_{c(p)} \right]$$

canonical form

$b(p)$: natural parameter

normal dist

$$f_Y(y) = \underbrace{\frac{1}{\sqrt{2\pi\sigma^2}}}_{\exp\left(-\frac{1}{2}\log 2\pi\sigma^2\right)} e^{\frac{-1}{2\sigma^2}(y-\mu)^2}$$

$y \in \mathbb{R}$
$\mu \in \mathbb{R}$
$\sigma^2 \in \mathbb{R}^+$

$$= \exp\left[\frac{-1}{2}\left[\underbrace{\left(\frac{y-\mu}{\sigma}\right)^2}_{\frac{y^2-2\mu y+\mu^2}{\sigma^2}} + \log 2\pi\sigma^2\right]\right]$$

$E(a(Y))$

y: continuous $\quad \int_y f_Y(y;\theta)\,dy = 1 \quad (*)$

y: discrete $\quad \sum_y f_Y(y;\theta) = 1$

$(*) \quad \frac{d}{d\theta} \int_y f_Y(y;\theta)\,dy = \frac{d}{d\theta} 1 = 0$

change the order of $\frac{d}{d\theta}$ and $\int$ $\quad \int_y \frac{d}{d\theta} f_Y(y;\theta)\,dy = 0 \quad (**)$

$$\int_y \frac{d^2}{d\theta^2} f_Y(y;\theta)\,dy = 0$$

$$f_Y(y;\theta) = \exp(a(y)b(\theta) + c(\theta) + d(y))$$

$$\frac{d}{d\theta} f_Y(y,\theta) = (a(y)b'(\theta) + c'(\theta))\underbrace{\exp(\cdot\ \cdot\ \cdot\ \cdot)}_{f_Y(y;\theta)}$$

$(**) \quad 0 = \int (a(y)b'(\theta) + c'(\theta))\, f_Y(y;\theta)\,dy$

$= b'(\theta) E(a(Y)) + c'(\theta)$

$\Rightarrow E(a(Y)) = -\frac{c'(\theta)}{b'(\theta)}$

$E(g(X)) = \int g(x) f_X(x)\,dx$

$$U(Y) = a(Y)\, b'(\theta) + c'(\theta)$$

$$E(U(Y)) = \underbrace{E(a(Y))}_{-\frac{c'(\theta)}{b'(\theta)}}\, b'(\theta) + c'(\theta)$$

$$= 0$$

$$\mathrm{Var}(U(Y)) = \mathrm{Var}(a(Y)\, b'(\theta) + c'(\theta))$$

$$= \mathrm{Var}(a(Y)\, b'(\theta))$$

$$= (b'(\theta))^2\, \mathrm{Var}(a(Y))$$

$$= (b'(\theta))^2 \times \frac{b''(\theta)c'(\theta) - c''(\theta)b'(\theta)}{(b'(\theta))^3}$$

$$= \frac{b''(\theta)c'(\theta) - c''(\theta)b'(\theta)}{b'(\theta)}$$

Score function in binomial distribution $B(n, p)$

log-likelihood function

$$l(p) = y \log p + (n-y)\log(1-p) + \log\binom{n}{y}$$

$$U = \frac{\partial l(p)}{\partial p} = \frac{y}{p} - \frac{n-y}{1-p} = \frac{y-np}{p(1-p)}$$

we know $E(Y) = np$, $Var(Y) = np(1-p)$

$$E(U) = \frac{1}{p(1-p)}\left(E(y) - np\right) = 0 \checkmark$$

$$I = Var(U) = \frac{Var(Y)}{[p(1-p)]^2} = \frac{np(1-p)}{p^2(1-p)^2} = \frac{n}{p(1-p)}$$

$$\Longrightarrow \frac{U}{\sqrt{I}} = \frac{y-np}{\sqrt{np(1-p)}} \,\dot\sim\, N(0, 1)$$

$$\begin{cases} H_0 : M_0 \text{ fits the data well} \\ H_1 : M_1 \text{ " " "} \end{cases}$$

$$\Delta D = D_0 - D_1 \begin{cases} \text{large} \longrightarrow M_1 \\ \text{small} \longrightarrow M_0 \end{cases}$$