Proof

First note that

$$1 = \int f(y|\theta) dy.$$

If $\mathbf{Y}$ takes values in $\mathbb{Z}$ or $\mathbb{N}_0$, replace the integral with a sum. Differentiating both sides with respect to $\boldsymbol{\theta}$, we obtain

$$0 = \frac{d}{d\theta} \cdot 1 = \frac{d}{d\theta} \int f(y|\theta) dy = \int \frac{df(y|\theta)}{d\theta} dy.$$

Differentiating once again yields

$$\int \frac{d^2 f(y|\theta)}{d\theta^2} dy = 0.$$

Now apply these results to the distribution $f(y, \theta)$ in the exponential family:

$$f(y|\theta) = \exp[a(y)b(\theta) + c(\theta) + d(y)]$$

Then

$$\frac{df(y|\theta)}{d\theta} = [a(y)b'(\theta) + c'(\theta)]f(y|\theta)$$

which implies

$$0 = \int [a(y)b'(\theta) + c'(\theta)]f(y|\theta) dy = b'(\theta)\mathbb{E}[a(\mathrm{Y})] + c'(\theta),$$

and hence

$$\mathbb{E}[a(\mathrm{Y})] = -\frac{c'(\theta)}{b'(\theta)} \tag{1.5.5}$$