# Proof

First note that the covariance matrix of the estimated residuals is

$$\begin{aligned}\mathbb{V}\mathrm{ar}(\mathbf{r}) &= \mathbb{V}\mathrm{ar}[(\mathbb{I}-\mathbf{H})\mathbf{y}] \\ &= \mathbb{V}\mathrm{ar}[(\mathbb{I}-\mathbf{H})(\mathbf{X}\boldsymbol{\beta}+\varepsilon)] \\ &= \mathbb{V}\mathrm{ar}[(\mathbb{I}-\mathbf{H})\varepsilon] \\ &= (\mathbb{I}-\mathbf{H})\mathbb{V}\mathrm{ar}[\varepsilon](\mathbb{I}-\mathbf{H})^\top \\ &= \sigma^2(\mathbb{I}-\mathbf{H})(\mathbb{I}-\mathbf{H})^\top \\ &= \sigma^2(\mathbb{I}-\mathbf{H}) \\ &= \sigma^2[\mathbb{I}-\mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top]\end{aligned}$$

(Note that the covariance matrix is singular.) Since $\mathbb{C}\boldsymbol{ov}[\boldsymbol{r}]$ is not diagonal, the residuals are not independent; since $\mathbb{V}\mathbf{ar}[r_i]=\boldsymbol{\sigma}^2(1-h_{ii})$, their variance is not constant.

If the model assumptions hold, $\mathbf{y}$ is a Gaussian random vector, and then (2.2.8) shows that $\boldsymbol{r}$ is a Gaussian random vector. Its mean is

$$\mathbb{E}[\mathbf{r}]=(\mathbb{I}-\mathbf{H})\mathbb{E}[\mathbf{y}]=(\mathbb{I}-\mathbf{H})\mathbf{X}\boldsymbol{\beta}=\mathbf{0}$$

and its covariance matrix is

$$\begin{aligned}\mathbb{C}ov[\mathbf{r}] &= \mathbb{C}ov[(\mathbb{I}-\mathbf{H})\mathbf{y}]=(\mathbb{I}-\mathbf{H})\mathbb{C}ov[\mathbf{y}](\mathbb{I}-\mathbf{H})^\top=(\mathbb{I}-\mathbf{H})(\sigma^2\mathbb{I})(\mathbb{I}-\mathbf{H}) \\ &= \sigma^2(\mathbb{I}-\mathbf{H})\end{aligned}$$

since $\mathbf{H}=\mathbf{H}$ (a linear projection) and $\mathbb{I}-\mathbf{H}$ is idempotent:

$$(\mathbb{I}-\mathbf{H})(\mathbb{I}-\mathbf{H})=\mathbb{I}(\mathbb{I}-\mathbf{H})-\mathbf{H}(\mathbb{I}-\mathbf{H})=\mathbb{I}-\mathbf{H}-\mathbf{H}+\mathbf{H}\mathbf{H}=\mathbb{I}-\mathbf{H}.$$