# Week 1- Slide 10

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$Y = f(\underset{\sim}{x}) + \underset{\text{error}}{\underset{\sim}{\varepsilon}}$

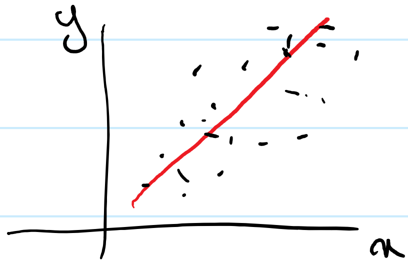
$E(\varepsilon) = 0$

$\underset{\sim}{x}$ and $\varepsilon$ are independent

$Y = f(X) \longrightarrow$ deterministic relationship

$Y = \beta_0 + \beta_1 x + \varepsilon$

$f(x) = \beta_0 + \beta_1 x$




$\hat{f}(x) = \hat{\beta}_0 + \hat{\beta}_1 x$

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Slide 10

$$E(Y - \hat{Y})^2 = E\left(f(\underset{\sim}{x}) + \varepsilon - \hat{f}(\underset{\sim}{x})\right)^2$$

$$= E\left(f(\underset{\sim}{x}) - \hat{f}(\underset{\sim}{x}) + \varepsilon\right)^2$$

$$= E\left\{\left(f(\underset{\sim}{x}) - \hat{f}(\underset{\sim}{x})\right)^2 + \varepsilon^2 + 2\left(f(x) - \hat{f}(\underset{\sim}{x})\right)\varepsilon\right\}$$

$$= E\left(f(\underset{\sim}{x}) - \hat{f}(\underset{\sim}{x})\right)^2 + E(\varepsilon^2) + 2\,\underbrace{E\left(f(x) - \hat{f}(x)\right)E(\varepsilon)}_{\text{indep of } \varepsilon \text{ and } x}$$

# Week 1- Slide 10 Cont.

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we know that $E(\varepsilon)=0$

$$E(Y-\hat{Y})^2 = E(f(x)-\hat{f}(x))^2 + E(\varepsilon^2)$$

$$Var(\varepsilon) = E(\varepsilon^2) - \underbrace{(E(\varepsilon))^2}_{0}$$

$$= \underbrace{(f(x)-\hat{f}(x))^2}_{\text{reducable error}} + \underbrace{Var(\varepsilon)}_{\text{irreducable error}}$$

# MLE and MLE of Poisson

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## MLE

$Y_1 \dots Y_n$ , $Y_i$ follows $f_i(y_i; \underset{\sim}{\theta})$ $\hookrightarrow \in \mathbb{R}^p$

joint density : $f(\underset{\sim}{y}; \underset{\sim}{\theta})$

we want to estimate $\underset{\sim}{\theta}$ :

likelihood function $L(\theta; y)$

maximize $L(\theta; y)$
$\theta \in \Theta \hookrightarrow$ parameter space

if $Y_i$'s are indep $f(\underset{\sim}{y}; \underset{\sim}{\theta}) = \prod_{i=1}^{n} f_i(y_i; \underset{\sim}{\theta})$

instead of working with $L(\underset{\sim}{\theta}; \underset{\sim}{y})$, try to maximize $\ln(L(\underset{\sim}{\theta}; \underset{\sim}{y}))$

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example Poisson dist (slide 15) $P(\theta)$

$$f_Y(y; \theta) = \frac{e^{-\theta} \theta^{y}}{y!} \qquad Y_1, \dots, Y_n \overset{iid}{\sim} P(\theta)$$

$$f(\underset{\sim}{y}; \theta) = \prod_{i=1}^{n} \frac{e^{-\theta} \theta^{y_i}}{y_i!}$$

# MLE of Poisson Cont.

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$$L(\theta) = \prod_{i=1}^{n} \frac{e^{-\theta}\theta^{y_i}}{y_i!} = \frac{e^{-n\theta}\,\theta^{\sum_{i=1}^{n} y_i}}{\prod y_i!}$$

$y_i \geq 0$ Since Poisson variable Counts the number of success in a given period of time or place

$$\ell(\theta) = \ln(L(\theta)) = -n\theta + \left(\sum_{i=1}^{n} y_i\right)\ln\theta - \ln \prod y_i!$$

$$\frac{\partial \ell(\theta)}{\partial\theta} = -n + \frac{\sum y_i}{\theta} = 0 \implies \hat{\theta} = \frac{\sum y_i}{n} = \bar{y}$$

$\downarrow$ MLE of $\theta$

$$\frac{\partial^2 \ell(\theta)}{\partial\theta^2} = -\frac{\sum y_i}{\theta^2} \leq 0$$

$\sum y_i \geq 0$, $\theta^2 \geq 0$

# Sufficient Statistics

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$\varepsilon_i \overset{iid}{\sim} N(0, \sigma^2)$

$Y_i = f(X_i) + \varepsilon_i \Rightarrow Y_i \sim N(\underbrace{f(x_i)}_{\beta_0 + \beta_1 x_i}, \sigma^2)$

sufficiency : $X_1 \dots X_n \overset{iid}{\sim} f(x_i, \theta)$, the statistic $T(X_1 \dots X_n)$ is sufficient for $\theta$ if it contains all the information that can be extracted from $X_1 \dots X_n$

e.g $X_i \sim$ Binomial$(p)$ MLE of $p$ $\hat{p} = \frac{\sum X_i}{n}$