

1.5 Exponential family of distributions

Exponential family of distributions



Definition:

Let Y be a random variable with values in \mathbb{R} (or \mathbb{N}_0 or \mathbb{Z}), and suppose its probability distribution $f(y; \theta)$ depends on a single parameter θ . Then the distribution belongs to the *exponential family* if it admits the form

$$f(y; \theta) = s(y)t(\theta)e^{a(y)b(\theta)}$$

for some functions a , b , s , and t .

Another way of writing this is

$$f(y; \theta) = \exp[a(y)b(\theta) + c(\theta) + d(y)] \quad (1.5.1)$$

where $s(y) = e^{d(y)}$ and $t(\theta) = e^{c(\theta)}$.

- If $a(y) = y$, the distribution of Y is called **canonical**
- If the distribution is in its canonical form, $b(\theta)$ is called the **natural parameter** of the distribution.
- If there are other parameters in addition to θ , they are usually called **nuisance parameters**, and they can be functions of a , b , c and d .

Notice that many well-known distributions belong to the exponential family.

Binomial distribution

Suppose $Y \sim \text{Bin}(n, p)$. The probability of getting exactly y successes in n trials is given by the **probability mass function**:

$$f(y|p) = \binom{n}{y} p^y (1-p)^{n-y},$$

for $y \in \{0, 1, \dots, n\}$.

Assume that p is the parameter of interest which indicates the probability of success in a single experiment, and that we know n .

Think: Does the Binomial belong to the Exponential family of distributions?

The probability function can be rewritten as

$$f(y|p) = \exp \left[y \log p - y \log(1-p) + n \log(1-p) + \log \binom{n}{y} \right]$$

this is of the form in (1.5.1) with

$$a(y) = y, \quad b(p) = \log \frac{p}{1-p}, \quad c(p) = n \log(1-p), \quad d(y) = \log \binom{n}{y}. \quad (1.5.2)$$

The binomial distribution is used as a model for observations of a process with binary outcomes:

1. The number of candidates who pass a test - the possible outcomes for each candidate being to pass or to fail;
2. The number of patients with some disease who are alive at a specified time since diagnosis - the possible outcomes being survival or death.

Practice: Draw the probability mass function of a Binomial random variable with $n = 40$ and $p = 0.5$ in R.

Practice using R

Draw the probability mass function of a Binomial random variable with $n = 40$ and $p = 0.5$ in R.

Normal distribution

A random variable Y with values in \mathbb{R} is **normal**, i.e.

$Y \sim N(\mu, \sigma^2)$, if it has **probability density function**

$$f(y|\mu) = \frac{1}{\sqrt{(2\pi\sigma^2)}} \exp\left(-\frac{1}{2\sigma^2}(y - \mu)^2\right).$$

Think: Does the Normal belong to the Exponential family of distributions?

The pdf can be rewritten in exponential family form:

$$f(y|\mu) = \exp\left[-\frac{y^2}{2\sigma^2} + \frac{y\mu}{\sigma^2} - \frac{\mu^2}{2\sigma^2} - \frac{1}{2}\log(2\pi\sigma^2)\right]$$

where

$$a(y) = y, \quad b(\mu) = \mu/\sigma^2, \quad c(\mu) = -\frac{\mu^2}{2\sigma^2} - \frac{1}{2}\log(2\pi\sigma^2), \quad d(y) = -\frac{y^2}{2\sigma^2} \quad (1.5.3)$$

(alternatively, the term $-\frac{1}{2}\log(2\pi\sigma^2)$ can be included in $d(y)$).

The normal distribution is used to model *continuous data with symmetric distribution*. The normal distribution is common in applications, since:

1. Many phenomena are well described by the normal distribution, ex. height or blood pressure of people;
2. Even if data are not normal, the average or total of a random sample of values will be approximately normally distributed (Central Limit Theorem);
3. Statistical theory is developed in a large extent for the Normal distribution.

Practice: Draw the probability density function of a Normal random variable with $\mu = 4$ and $\sigma = 1.5$ in R.

Practice using R

Draw the probability density function of a Normal random variable with $\mu = 4$ and $\sigma = 1.5$ in R.

Poisson distribution

A random variable Y with positive values is **Poisson** with mean $\lambda > 0$, i.e. $Y \sim \text{Pois}(\lambda)$, if it has **probability density function**,

$$f(y|\lambda) = \frac{\lambda^y}{e^\lambda y!}.$$

Think: Does the Normal belong to the Exponential family of distributions?

The pdf can be rewritten in exponential family form:

$$f(y|\lambda) = \exp(y \log \lambda - \lambda - \log y!)$$

where

$$a(y) = y, \quad b(\lambda) = \log \lambda, \quad c(\lambda) = -\lambda, \quad d(y) = -\log y!. \quad (1.5.4)$$

The Poisson distribution provides a suitable model for **count data** and expresses *the probability of a given number of events occurring in a fixed interval of time and/or space* if these events occur with a known average rate and independently of the time since the last event.

1. The number of medical conditions reported by a person;
2. The number of tropical cyclones during a season;
3. The number of spelling mistakes on the page of a newspaper;
4. The number of faulty components in a computer or in a batch of manufactured items.

Practice: Draw the probability density function of a Poisson random variable with $\lambda = 1$ in R.

Practice using R

Draw the probability density function of a Poisson random variable with $\lambda = 1$ in R.

Properties of distributions in the exponential family

The *expected value* and *variance* of $a(\mathbf{Y})$ are given by

$$\mathbb{E}[a(\mathbf{Y})] = -\frac{c'(\theta)}{b'(\theta)} \quad (1.5.5)$$

and

$$\mathbb{V}ar[a(\mathbf{Y})] = \frac{b''(\theta)c'(\theta) - c''(\theta)b'(\theta)}{b'(\theta)^3} \quad (1.5.6)$$

You may want to see how these results were derived.

Score and information

We now formally introduce the statistics **score** and **information**.

The **log-likelihood function** for the exponential family is

$$\ell(\theta; y) = a(y)b(\theta) + c(\theta) + d(y).$$

The **score statistic** is defined as

$$U(\theta; y) = \frac{d\ell(\theta; y)}{d\theta} = a(y)b'(\theta) + c'(\theta).$$

It depends on y and may hence be interpreted as a random variable:

$$U := U(\theta; Y) = a(Y)b'(\theta) + c'(\theta).$$

The *expected value of the score statistic* U is

$$\mathbb{E}(U) = b'(\theta)\mathbb{E}[a(Y)] + c'(\theta) = 0 \quad (1.5.7)$$

because, if we use (1.5.5),

$$\mathbb{E}(U) = b'(\theta) \left[-\frac{c'(\theta)}{b'(\theta)} \right] + c'(\theta) = 0$$

The *variance of the score statistic* U is called the **information**, denoted by \mathcal{I} :

$$\mathcal{I} = \text{Var}(U) = b'(\theta)^2 \text{Var}[a(Y)] = \frac{b''(\theta)c'(\theta)}{b'(\theta)} - c''(\theta). \quad (1.5.8)$$

Another property of the score function U is

$$\mathbb{E}(U^2) = \text{Var}(U) = -\mathbb{E} \left(\frac{dU}{d\theta} \right).$$

The first equality follows from the general result $\text{Var}(X) = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2$, valid for any r.v., and $\mathbb{E}(U) = 0$.

To see the second equality, note that

$$\begin{aligned}\mathbb{E}\left(\frac{dU}{d\theta}\right) &= \mathbb{E}(a(Y)b''(\theta) + c''(\theta)) = b''(\theta)\mathbb{E}[a(Y)] + c''(\theta) \\ &= b''(\theta)\left[-\frac{c'(\theta)}{b'(\theta)}\right] + c''(\theta) = -\mathbb{V}\text{ar}(\mathbf{U}) = -\mathcal{I}\end{aligned}$$

Pressure example

Let's consider back the example with the times to failure of Kevlar epoxy strand pressure.

The **Weibull distribution** belongs to the exponential family since its pdf

$$f(y; \lambda, \theta) = \frac{\lambda y^{\lambda-1}}{\theta^\lambda} \exp \left[- \left(\frac{y}{\theta} \right)^\lambda \right]$$

can be rewritten as

$$f(y; \theta) = \exp \left[\log \lambda + (\lambda - 1) \log y - \lambda \log \theta - \left(\frac{y}{\theta} \right)^\lambda \right]$$

with

$$a(y) = y^\lambda, \quad b(\theta) = -\theta^{-\lambda}, \quad c(\theta) = \log \lambda - \lambda \log \theta, \quad d(y) = (\lambda - 1) \log y$$

where λ is a nuisance parameter.

Check your understanding

This is a non-assessed self-practice. Work through each of the questions and press submit to be able to see the solution.

Question 1 *Submitted Mar 8th 2023 at 5:41:28 pm*

[See slide: [Exponential family of distributions](#)]

Use **R** to complete the following tasks:

1. Draw the probability mass function of a Binomial random variable with $n = 40$ and $p = 0.5$.
2. Draw the probability density function of a Normal random variable with $\mu = 4$ and $\sigma = 1.5$.
3. Draw the probability density function of a Poisson random variable with $\lambda = 1$.

```
# Probability density function of a Normal random variable with mu=4 and sigma=1.5
```

```
xseq <- seq(from=-10, to=10, length=100)
plot(xseq, dnorm(xseq, mean=4, sd=1.5), type="l", ylab=expression(f(x)))
```

```
# Probability density function of a Poisson random variable with mu=4 and sigma=1.5
```

```
xseq <- 0:10
plot(xseq, dpois(xseq, lambda=1), ylab=expression(f(x)), pch=16)
```

```
# Probability mass function of a Binomial random variable with n=40 and p=0.5
```

```
xseq <- 0:40
plot(xseq, dbinom(xseq, size=40, prob=0.5), ylab=expression(f(x)), pch=16)
```

Question 2 *Submitted Mar 8th 2023 at 5:58:07 pm*

[Dobson and Barnett (2018, Exercise 3.2)]

If the random variable Y has the Gamma distribution with scale parameter β , which is the parameter of interest, and a known shape parameter α , then its probability density function is:

$$f(y; \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-y\beta}.$$

Show that this distribution belongs to the exponential family and find the natural parameter. Find also $\mathbb{E}(Y)$ and $\mathbb{V}\text{ar}(Y)$.

