Proof

Now to find $\mathbb{V}\mathbf{ar}(a(\mathrm{Y}))$, we calculate

$$\frac{d^2 f(y|\theta)}{d\theta^2} = [a(y)b''(\theta) + c''(\theta)]f(y|\theta) + [a(y)b'(\theta) + c'(\theta)]^2 f(y|\theta).$$

Using $(1.5.5)$, the second term on the right hand side can be written as

$$b'(\theta)^2\{a(y) - \mathbb{E}[a(\mathrm{Y})]\}^2 f(y|\theta)$$

Consequently, we have

$$0 = \int \frac{d^2 f(y|\theta)}{d\theta^2}dy = b''(\theta)\mathbb{E}[a(\mathrm{Y})] + c''(\theta) + [b'(\theta)]^2\mathbb{V}ar(a(\mathrm{Y}))]$$

since $\int\{a(y) - \mathbb{E}[a(\mathrm{Y})]\}^2 f(y;\theta)dy = \mathbb{V}ar[a(\mathrm{Y})]$. Hence

$$\mathbb{V}ar[a(\mathrm{Y})] = \frac{b''(\theta)c'(\theta) - c''(\theta)b'(\theta)}{b'(\theta)^3} \tag{1.5.6}$$