Activity 4.2.3. Suppose that an object moving along a straight line path has its velocity in feet per second at time t in seconds given by $v(t) = \frac{2}{9}(t-3)^2 + 2$.

a. Carefully sketch the region whose exact area will tell you the value of the distance the object traveled on the time interval $2 \le t \le 5$.

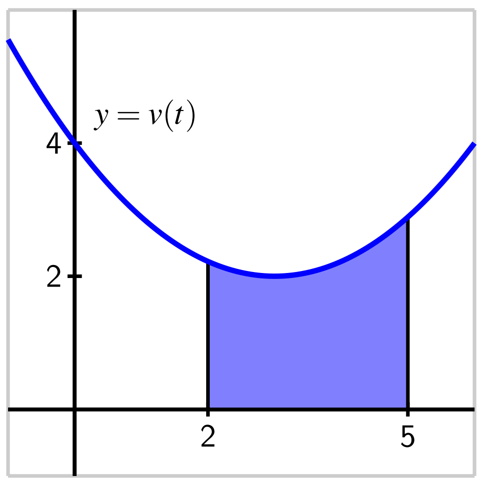
b. Estimate the distance traveled on $[2, 5]$ by computing L_4, R_4, and M_4.

c. Does averaging L_4 and R_4 result in the same value as M_4? If not, what do you think the average of L_4 and R_4 measures?

d. For this question, think about an arbitrary function f, rather than the particular function v given above. If f is positive and increasing on $[a, b]$, will L_n over-estimate or under-estimate the exact area under f on $[a, b]$? Will R_n over- or under-estimate the exact area under f on $[a, b]$? Explain.

Solution.

a. The region whose exact area tells us the value of the distance the object traveled on the time interval $2 \le t \le 5$ is shown below.



b. $L_4 = \frac{311}{48} \approx 6.47917$, $R_4 = \frac{335}{48} \approx 6.97917$, and $M_4 = \frac{637}{96} \approx 6.63542$.

c. The average of L_4 and R_4 is

$$\frac{L_4 + M_4}{2} = \frac{311 + 335}{96} = \frac{646}{96} \neq \frac{637}{96} = M_4.$$

This average actually measures what would result from using four trapezoids, rather than rectangles, to estimate the area on each subinterval. One reason this is so is because the area of a trapezoid is the average of the bases times the width, and the "bases" are given by the function values at the left and right endpoints.

d. If f is positive and increasing on $[a, b]$, L_n will under-estimate the exact area under f on $[a, b]$. Because f is increasing, its value at the left endpoint of any subinterval will be lower than every other function value in the interval, and thus the rectangle with that height lies exclusively below the curve. In a similar way, R_n over-estimates the exact area under f on $[a, b]$.

Activity 4.2.4. Suppose that an object moving along a straight line path has its velocity v (in feet per second) at time t (in seconds) given by

$$v(t) = \frac{1}{2}t^2 - 3t + \frac{7}{2}.$$

a. Compute M_5, the middle Riemann sum, for v on the time interval $[1, 5]$. Be sure to clearly identify the value of Δt as well as the locations of t_0, t_1, $\cdots$, t_5. In addition, provide a careful sketch of the function and the corresponding rectangles that are being used in the sum.

b. Building on your work in (a), estimate the total change in position of the object on the interval $[1, 5]$.

c. Building on your work in (a) and (b), estimate the total distance traveled by the object on $[1, 5]$.

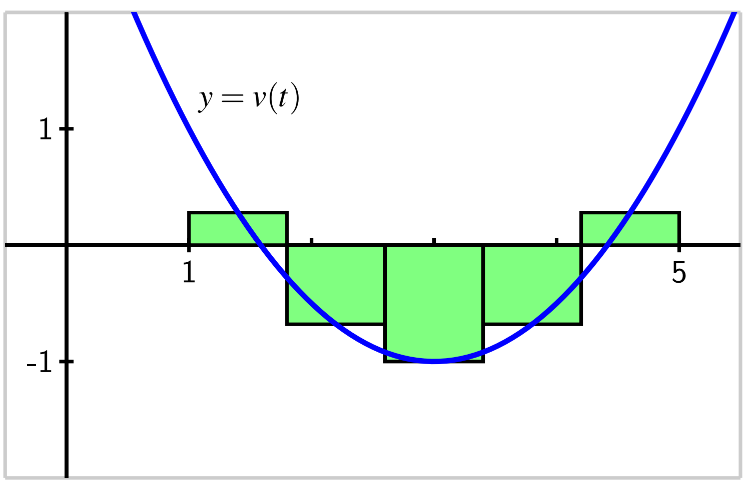
d. Use appropriate computing technology[1] to compute M_{10} and M_{20}. What exact value do you think the middle sum eventually approaches as n increases without bound? What does that number represent in the physical context of the overall problem?

Solution.

a. For this Riemann sum with five subintervals, $\Delta t = \frac{5-1}{5} = \frac{4}{5}$, so $t_0 = 1$, $t_1 = 1.8$, $t_2 = 2.6$, $t_3 = 3.4$, $t_4 = 4.2$ and $t_5 = 4$. It follows that

$$\begin{aligned} M_5 &= v(1.4)\Delta t + v(2.2)\Delta t + v(3.0)\Delta t + v(3.8)\Delta t + v(4.6) \\ &= \frac{7}{25}\cdot\frac{4}{5} - \frac{17}{25}\cdot\frac{4}{5} - 1\cdot\frac{4}{5} - \frac{17}{25}\cdot\frac{4}{5} + \frac{7}{25}\cdot\frac{4}{5} \\ &= -\frac{36}{25} = -1.44 \end{aligned}$$

A sketch is shown below.



b. Since the net signed area bounded by v on $[1, 5]$ represents the total change in position of the object on the interval $[1, 5]$, it follows that M_5 estimates the total change in position. Hence, the change in position is approximately -1.44 feet.

c. To estimate the total distance traveled by the object on $[1, 5]$, we have to calculate the total area between the curve and the t-axis. Thus,

$$D \approx \frac{7}{25}\cdot\frac{4}{5} + \frac{17}{25}\cdot\frac{4}{5} + 1\cdot\frac{4}{5} + \frac{17}{25}\cdot\frac{4}{5} + \frac{7}{25}\cdot\frac{4}{5} = \frac{292}{125} \approx 2.336.$$

[1]For instance, consider the applet at http://gvsu.edu/s/a9 and change the function and adjust the locations of the blue points that represent the interval endpoints a and b.

d. Using appropriate technology, $M_{10} = -1.36$ and $M_{20} = -1.34$. Further calculations suggest that $M_n \to -\frac{4}{3} = -1.\overline{33}$ as $n \to \infty$, and this number represents the object's total change in position on $[1, 5]$.