Activity 3.5.3. A television camera is positioned 4000 feet from the base of a rocket launching pad. The angle of elevation of the camera has to change at the correct rate in order to keep the rocket in sight. In addition, the auto-focus of the camera has to take into account the increasing distance between the camera and the rocket. We assume that the rocket rises vertically. (A similar problem is discussed and pictured dynamically at http://gvsu.edu/s/9t. Exploring the applet at the link will be helpful to you in answering the questions that follow.)

a. Draw a figure that summarizes the given situation. What parts of the picture are changing? What parts are constant? Introduce appropriate variables to represent the quantities that are changing.

b. Find an equation that relates the camera's angle of elevation to the height of the rocket, and then find an equation that relates the instantaneous rate of change of the camera's elevation angle to the instantaneous rate of change of the rocket's height (where all rates of change are with respect to time).

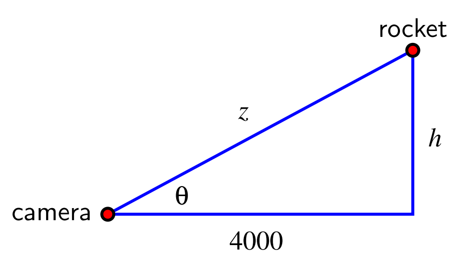
c. Find an equation that relates the distance from the camera to the rocket to the rocket's height, as well as an equation that relates the instantaneous rate of change of distance from the camera to the rocket to the instantaneous rate of change of the rocket's height (where all rates of change are with respect to time).

d. Suppose that the rocket's speed is 600 ft/sec at the instant it has risen 3000 feet. How fast is the distance from the television camera to the rocket changing at that moment? If the camera is following the rocket, how fast is the camera's angle of elevation changing at that same moment?

e. If from an elevation of 3000 feet onward the rocket continues to rise at 600 feet/sec, will the rate of change of distance with respect to time be greater when the elevation is 4000 feet than it was at 3000 feet, or less? Why?

Solution.

a. Letting θ be the camera's elevation angle, h the rocket's height, and z the distance from the camera to the rocket, we have the following situation at a given point in time:



b. To relate θ and h, observe that the tangent function is a good choice, since $\tan(\theta) = \frac{h}{4000}$, so that

$$h = 4000\tan(\theta).$$

Differentiating implicitly with respect to t, we find that

$$\frac{dh}{dt} = 4000\sec^2(\theta)\frac{d\theta}{dt}.$$

c. To relate z and h, the Pythagorean Theorem is natural to consider. By this famous result,

$$h^2 + 4000^2 = z^2.$$

Differentiating both sides implicitly with respect to t, it follows

$$2h\frac{dh}{dt} = 2z\frac{dz}{dt},$$

and thus $h\frac{dh}{dt} = z\frac{dz}{dt}$.

d. Using the given fact that the rocket's speed is 600 ft/sec at the instant it has risen 3000 feet, we know that $\frac{dh}{dt}\big|_{h=3000} = 600$. Note further in the triangle that when $h = 3000$, it follows $z = 5000$, since the base leg of the triangle is constant at 4000, by using the Pythagorean Theorem. Substituting this information from the instant $h = 3000$ into the equation that relates the rates of change of z and h found in (c), we find that

$$2 \cdot 3000 \cdot 600 = 2 \cdot 5000 \cdot \left.\frac{dz}{dt}\right|_{h=3000}.$$

Solving for $\frac{dz}{dt}\big|_{h=3000}$ we have

$$\left.\frac{dz}{dt}\right|_{h=3000} = \frac{1800}{5} = 360 \text{ feet/sec.}$$

To answer the question about how fast the camera angle is changing, we use the same information but now in the equation we found in (b) that relates the rates of change of θ and h:

$$\frac{dh}{dt} = 4000 \sec^2(\theta)\frac{d\theta}{dt}.$$

Observe that when $h = 3000$, in the 3000-4000-5000 right triangle, it follows that $\cos(\theta) = \frac{4}{5}$, so $\sec(\theta) = \frac{5}{4}$. Thus, using the instantaneous information,

$$600 = 4000 \cdot \frac{25}{16} \left.\frac{d\theta}{dt}\right|_{h=3000},$$

and thus

$$\left.\frac{d\theta}{dt}\right|_{h=3000} = \frac{6 \cdot 16}{40 \cdot 25} = \frac{12}{125},$$

which is measured in radians per second.

e. Recalling that $2h\frac{dh}{dt} = 2z\frac{dz}{dt}$, it follows that

$$\frac{dz}{dt} = \frac{h}{z}\frac{dh}{dt}.$$

Using this equation when $h = 3000$ and $\frac{dh}{dt} = 600$ led us to conclude that $\frac{dz}{dt}\big|_{h=3000} = \frac{3}{5} \cdot 600 = 360$ feet/sec. If we instead use $h = 4000$, it follows that $z = 4000\sqrt{2}$, so that

$$\left.\frac{dz}{dt}\right|_{h=4000} = \frac{4}{4\sqrt{2}} \cdot 600 \approx 424.26 \text{ feet/sec.}$$

Indeed, $\frac{dz}{dt}$ is an increasing function of h, provided that $\frac{dh}{dt}$ is constant, because we can write $h^2 + 4000^2 = z^2$, so $z = \sqrt{h^2 + 4000^2}$, making

$$\frac{dz}{dt} = \frac{h}{\sqrt{h^2 + 4000^2}}\frac{dh}{dt} = \frac{h}{\sqrt{h^2 + 4000^2}}600.$$

It is straightforward to verify that $\frac{h}{\sqrt{h^2+4000^2}}$ is an increasing function of h.