3.2 Using derivatives to describe families of functions

Activity 3.2.2. Consider the family of functions defined by $p(x) = x^3 - ax$, where $a \neq 0$ is an arbitrary constant.

a. Find $p'(x)$ and determine the critical numbers of p. How many critical numbers does p have?

b. Construct a first derivative sign chart for p. What can you say about the overall behavior of p if the constant a is positive? Why? What if the constant a is negative? In each case, describe the relative extremes of p.

c. Find $p''(x)$ and construct a second derivative sign chart for p. What does this tell you about the concavity of p? What role does a play in determining the concavity of p?

d. Without using a graphing utility, sketch and label typical graphs of $p(x)$ for the cases where $a > 0$ and $a < 0$. Label all inflection points and local extrema.

e. Finally, use a graphing utility to test your observations above by entering and plotting the function $p(x) = x^3 - ax$ for at least four different values of a. Write several sentences to describe your overall conclusions about how the behavior of p depends on a.

Solution.

a. We first note that $p'(x) = 3x^2 - a$, so to find critical numbers we set $p'(x) = 0$ and solve for x. This leads to the equation $3x^2 - a = 0$, which implies

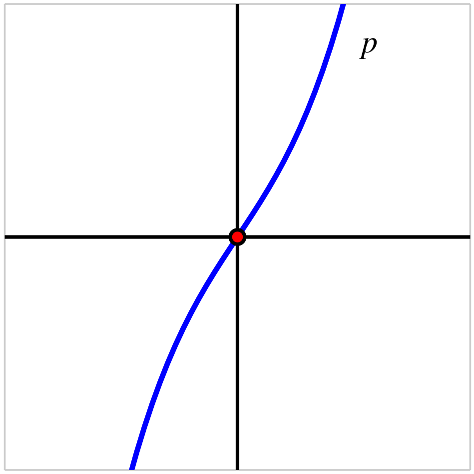
$$x^2 = \frac{a}{3}.$$

If $a > 0$, then the solutions to this equation are $x = \pm\sqrt{\frac{a}{3}}$; if $a < 0$, then the equation has no solution. Hence, p has two critical numbers ($x = \pm\sqrt{\frac{a}{3}}$) whenever $a > 0$ and no critical numbers when $a < 0$.

b. For the case when $a < 0$, we observe that $p'(x) = 3x^2 - a$ is positive for every value of x, and thus p is always increasing and has no relative extreme values. (There are no critical numbers to place on the first derivative sign chart, and p' is always positive.) For the case when $a > 0$, we observe that $p'(x) = 3x^2 - a$ is a concave up parabola with zeros at $x = -\sqrt{\frac{a}{3}}$ and $x = +\sqrt{\frac{a}{3}}$. It follows that for $x < -\sqrt{\frac{a}{3}}$, $p'(x) > 0$ (so p is increasing); for $-\sqrt{\frac{a}{3}} < x < \sqrt{\frac{a}{3}}$, $p'(x) > 0$ (so p is decreasing); and for $x > \sqrt{\frac{a}{3}}$, $p'(x) > 0$ (so p is again increasing). In this situation, we see that p has a relative maximum at $x = -\sqrt{\frac{a}{3}}$ and a relative minimum at $x = +\sqrt{\frac{a}{3}}$.

c. Since $p'(x) = 3x^2 - a$ and a is constant, it follows that $p''(x) = 6x$. Note that $p''(x) = 0$ when $x = 0$ and that $p''(x) < 0$ for $x < 0$ and $p''(x) > 0$ for $x > 0$. Hence p is CCD for $x < 0$ and p is CCU for $x > 0$, making $x = 0$ an inflection point.

d. Below, we show the two possible situations. At left, for the case when $a < 0$ and p is always increasing with an inflection point at $x = 0$, and at right for when $a > 0$ and p has a relative maximum at $x = -\sqrt{\frac{a}{3}}$ and a relative minimum at $x = +\sqrt{\frac{a}{3}}$, again with an inflection point at $x = 0$. Note, too, that p has its x-intercepts at $x = \pm\sqrt{a}$.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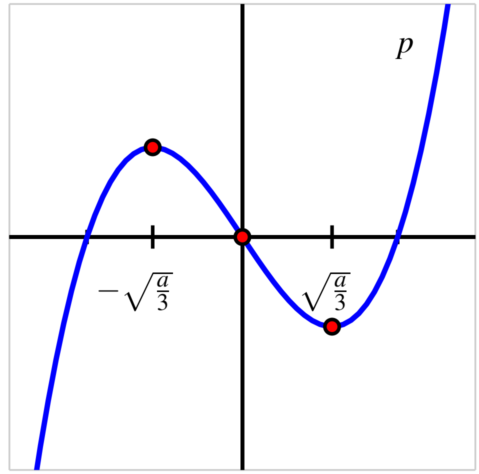
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Activity 3.2.3. Consider the two-parameter family of functions of the form $h(x) = a(1 - e^{-bx})$, where a and b are positive real numbers.

a. Find the first derivative and the critical numbers of h. Use these to construct a first derivative sign chart and determine for which values of x the function h is increasing and decreasing.

b. Find the second derivative and build a second derivative sign chart. For which values of x is a function in this family concave up? concave down?

c. What is the value of $\lim_{x\to\infty} a(1 - e^{-bx})$? $\lim_{x\to-\infty} a(1 - e^{-bx})$?

d. How does changing the value of b affect the shape of the curve?

e. Without using a graphing utility, sketch the graph of a typical member of this family. Write several sentences to describe the overall behavior of a typical function h and how this behavior depends on a and b.

Solution.

a. Since $h(x) = a - ae^{-bx}$, we have by the constant multiple and chain rules that

$$h'(x) = -ae^{-bx}(-b) = abe^{-bx}.$$

Since a and b are positive constants and $e^{-bx} > 0$ for all x, we see that $h'(x)$ is never zero (nor undefined), and indeed $h'(x) > 0$ for all x. Hence h is an always increasing function.

b. Because $h'(x) = abe^{-bx}$, we have that $h''(x) = abe^{-bx}(-b) = -ab^2e^{-bx}$. As with h', we recognize that a, b^2, and e^{-bx} are always positive, and thus $h''(x) = -ab^2e^{-bx} < 0$ for all values of x, making h always concave down.

c. As $x \to \infty$, $e^{-bx} \to 0$. Thus,

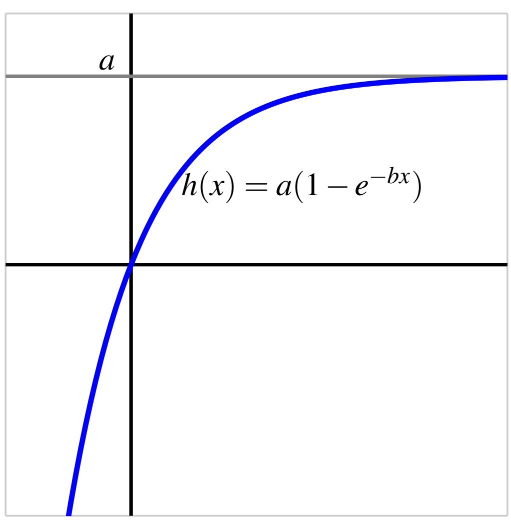
$$\lim_{x\to\infty} a(1 - e^{-bx}) = \lim_{x\to\infty} a - ae^{-bx} = a - 0 = a.$$

This shows that h has a horizontal asymptote at $y = a$ as we move rightward on its graph. As $x \to -\infty$, $e^{-bx} \to \infty$. Thus,

$$\lim_{x\to\infty} a(1 - e^{-bx}) = \lim_{x\to\infty} a - ae^{-bx} = -\infty.$$

d. Noting that $h'(x) = abe^{-bx}$, we see that if we consider different values of b, the slope of the graph changes. If b is large and x is close to zero, $h'(x) \approx ab$ (since $e^0 = 1$), so $h'(x)$ is relatively large near $x = 0$. At the same time, for large b, e^{-bx} approaches zero quickly as x increases, so the curve's slope will quickly approach zero as x increases. If b is small, the graph is less steep near $x = 0$ and its slope goes to zero less quickly as x increases.

e. Observing that $h(0) = 0$ and $\lim_{x\to\infty} h(x) = a$, along with the facts that h is always increasing and always concave down, we see that a typical member of this family looks like the following graph.



Activity 3.2.4. Let $L(t) = \frac{A}{1+ce^{-kt}}$, where A, c, and k are all positive real numbers.

a. Observe that we can equivalently write $L(t) = A(1 + ce^{-kt})^{-1}$. Find $L'(t)$ and explain why L has no critical numbers. Is L always increasing or always decreasing? Why?

b. Given the fact that

$$L''(t) = Ack^2e^{-kt}\frac{ce^{-kt} - 1}{(1 + ce^{-kt})^3},$$

find all values of t such that $L''(t) = 0$ and hence construct a second derivative sign chart. For which values of t is a function in this family concave up? concave down?

c. What is the value of $\lim_{t\to\infty} \frac{A}{1+ce^{-kt}}$? $\lim_{t\to-\infty} \frac{A}{1+ce^{-kt}}$?

d. Find the value of $L(x)$ at the inflection point found in (b).

e. Without using a graphing utility, sketch the graph of a typical member of this family. Write several sentences to describe the overall behavior of a typical function L and how this behavior depends on A, c, and k number.

f. Explain why it is reasonable to think that the function $L(t)$ models the growth of a population over time in a setting where the largest possible population the surrounding environment can support is A.

Solution.

a. By the chain rule and treating A, c, and k as constants, we find that

$$L'(t) = A(-1)(1 + ce^{-kt})^{-2}ce^{-kt}(-k) = Acke^{-kt}(1 + ce^{-kt})^{-2}.$$

Since A, c, and k are all positive and $e^{-kt} > 0$ for all values of t, it is apparent that $L'(t)$ is never zero, and indeed is positive for every value of t. Thus, L is an always increasing function.

b. Given that

$$L''(t) = Ack^2e^{-kt}\frac{ce^{-kt} - 1}{(1 + ce^{-kt})^3},$$

the only way $L''(t) = 0$ is if $ce^{-kt} - 1 = 0$. Solving $ce^{-kt} - 1 = 0$ for t, we first write $e^{-kt} = \frac{1}{c}$. Taking the natural logarithm of both sides, $-kt = \ln(\frac{1}{c})$, so that

$$t = -\frac{1}{k}\ln\left(\frac{1}{c}\right)$$

is the only value of t for which $L''(t) = 0$. Now, observe that since $ce^{-kt} \to 0$ as $t \to \infty$, the quantity $ce^{-kt} - 1$ will be positive to the left of where it is zero and negative to the right of where it is zero. Since this is the only term in $L''(t)$ that can change sign, it follows that $L''(t) > 0$ for $t < -\frac{1}{k}\ln\left(\frac{1}{c}\right)$ and $L''(t) > 0$ for $t > -\frac{1}{k}\ln\left(\frac{1}{c}\right)$, making L concave up to the left of the noted inflection point and concave down thereafter.

c. Recalling that $e^{-kt} \to 0$ as $t \to \infty$, we observe that

$$\lim_{t\to\infty}\frac{A}{1 + ce^{-kt}} = \frac{A}{1 + 0} = A,$$

so L has a horizontal asymptote of $y = A$ as $t \to \infty$. On the other hand, since $e^{-kt} \to \infty$ as $t \to -\infty$, this causes the denominator of L to grow without bound (while the numerator remains constant), and therefore

$$\lim_{t\to\infty}\frac{A}{1 + ce^{-kt}} = 0,$$

which means L has a horizontal asymptote of $y = 0$ as $t \to -\infty$.

d. From (b), we know that $t = -\frac{1}{k}\ln\left(\frac{1}{c}\right)$ is the location of the inflection point of L. Evaluating the function at this point, we find that

$$L\left(-\frac{1}{k}\ln\left(\frac{1}{c}\right)\right) = \frac{A}{1 + ce^{-k[-\frac{1}{k}\ln\left(\frac{1}{c}\right)]}} = \frac{A}{1 + ce^{\ln\left(\frac{1}{c}\right)}} = \frac{A}{1 + c\cdot\frac{1}{c}} = \frac{A}{2}.$$

Thus, the inflection point on the graph of L is located at $(-\frac{1}{k}\ln\left(\frac{1}{c}\right), \frac{A}{2})$.

e. We have shown that L is an always increasing function that has horizontal asymptotes at $y = 0$ and $y = A$, as well as an inflection point at $(-\frac{1}{k}\ln\left(\frac{1}{c}\right), \frac{A}{2})$, which we note lies vertically halfway between the asymptotes. In addition, we see that $L(0) = \frac{A}{1+c}$. The combination of all of this information shows us that a typical graph in this family of functions is given by the following figure.