Activity 2.4.4. Answer each of the following questions. Where a derivative is requested, be sure to label the derivative function with its name using proper notation.

a. Let $f(x) = 5\sec(x) - 2\csc(x)$. Find the slope of the tangent line to f at the point where $x = \frac{\pi}{3}$.

b. Let $p(z) = z^2 \sec(z) - z\cot(z)$. Find the instantaneous rate of change of p at the point where $z = \frac{\pi}{4}$.

c. Let $h(t) = \dfrac{\tan(t)}{t^2+1} - 2e^t \cos(t)$. Find $h'(t)$.

d. Let $g(r) = \dfrac{r\sec(r)}{5^r}$. Find $g'(r)$.

e. When a mass hangs from a spring and is set in motion, the object's position oscillates in a way that the size of the oscillations decrease. This is usually called a *damped oscillation*. Suppose that for a particular object, its displacement from equilibrium (where the object sits at rest) is modeled by the function

$$s(t) = \frac{15\sin(t)}{e^t}.$$

Assume that s is measured in inches and t in seconds. Sketch a graph of this function for $t \geq 0$ to see how it represents the situation described. Then compute ds/dt, state the units on this function, and explain what it tells you about the object's motion. Finally, compute and interpret $s'(2)$.

Solution.

a. Using the sum and constant multiple rules along with the formulas for the derivatives of $\sec(x)$ and $\csc(x)$, we find that

$$f'(x) = 5\sec(x)\tan(x) + 2\csc(x)\cot(x).$$

Therefore, the slope of the tangent line to f at the point where $x = \frac{\pi}{3}$ is given by

$$\begin{aligned} m = f'(\frac{\pi}{3}) &= 5\sec(\frac{\pi}{3})\tan(\frac{\pi}{3}) + 2\csc(\frac{\pi}{3})\cot(\frac{\pi}{3}) \\ &= 5\cdot 2\cdot\sqrt{3} + 2\cdot\frac{2}{\sqrt{3}}\cdot\frac{1}{\sqrt{3}} = 10\sqrt{3} + \frac{4}{3}. \end{aligned}$$

b. By the sum rule and two applications of the product rule, we have

$$\begin{aligned} p'(z) &= \frac{d}{dz}[z^2\sec(z)] - \frac{d}{dz}[z\cot(z)] \\ &= [z^2\sec(z)\tan(z) + \sec(z)\cdot 2z] - [z(-\csc^2(z)) + \cot(z)\cdot 1] \\ &= z^2\sec(z)\tan(z) + 2z\sec(z) + z\csc^2(z) - \cot(z). \end{aligned}$$

Thus, the instantaneous rate of change of p at the point where $z = \frac{\pi}{4}$ is

$$\begin{aligned} p'(\frac{\pi}{4}) &= (\frac{\pi}{4})^2\sec(\frac{\pi}{4})\tan(\frac{\pi}{4}) + 2\frac{\pi}{4}\sec(\frac{\pi}{4}) + \frac{\pi}{4}\csc^2(\frac{\pi}{4}) - \cot(\frac{\pi}{4}) \\ &= \frac{\pi^2}{16}\sqrt{2} + 2\frac{\pi}{4}\sqrt{2} + \frac{\pi}{4}2 - 1 \\ &= \frac{\pi^2}{16}\sqrt{2} + \frac{\sqrt{2}\pi}{2} + \frac{\pi}{2} - 1 \end{aligned}$$

c. Using the sum and constant multiple rules, followed by the quotient rule on the first term and the product rule on the second, we find that

$$\begin{aligned} h'(t) &= \frac{d}{dt}\left[\frac{\tan(t)}{t^2+1}\right] - 2\frac{d}{dt}\left[e^t\cos(t)\right] \\ &= \frac{(t^2+1)\sec^2(t) - \tan(t)(2t)}{(t^2+1)^2} - 2(e^t(-\sin(t)) + \cos(t)e^t) \end{aligned}$$

$$= \frac{(t^2+1)\sec^2(t) - 2t\tan(t)}{(t^2+1)^2} + 2e^t\sin(t) - 2e^t\cos(t)$$

d. Note that g is fundamentally a quotient, so we need to use the quotient rule. But the numerator of g is a product, so the product rule will be required to compute the derivative of the top function. Executing the quotient rule and proceeding, we find that

$$\begin{aligned} g'(r) &= \frac{5^r \frac{d}{dr}[r\sec(r)] - r\sec(r)\cdot 5^r\ln(5)}{(5^r)^2} \\ &= \frac{5^r[r\sec(r)\tan(r) + \sec(r)\cdot 1] - r\sec(r)\cdot 5^r\ln(5)}{(5^r)^2} \\ &= \frac{r\sec(r)\tan(r) + \sec(r) - r5^r\sec(r)}{5^r} \end{aligned}$$

e. By the quotient rule,

$$\frac{ds}{dt} = \frac{e^t\cdot 15\cos(t) - 15\sin(t)\cdot e^t}{(e^t)^2} = \frac{15\cos(t) - 15\sin(t)}{e^t}.$$

The function $\frac{ds}{dt} = s'(t)$ measures the instantaneous vertical velocity of the mass that is attached to the spring. In particular, $s'(2) = \frac{15\cos(2)-15\sin(2)}{e^2} \approx -2.69$ inches per second, which tells us at the instant $t = 2$, the mass is moving downward at an instantaneous rate of 2.69 inches per second.