Activity 2.7.3. Consider the curve defined by the equation $y(y^2-1)(y-2) = x(x-1)(x-2)$, whose graph is pictured in Figure 2.7.6. Through implicit differentiation, it can be shown that

$$\frac{dy}{dx} = \frac{(x-1)(x-2) + x(x-2) + x(x-1)}{(y^2-1)(y-2) + 2y^2(y-2) + y(y^2-1)}.$$

Use this fact to answer each of the following questions.

a. Determine all points (x, y) at which the tangent line to the curve is horizontal. (Use technology appropriately to find the needed zeros of the relevant polynomial function.)

b. Determine all points (x, y) at which the tangent line is vertical. (Use technology appropriately to find the needed zeros of the relevant polynomial function.)

c. Find the equation of the tangent line to the curve at one of the points where $x = 1$.

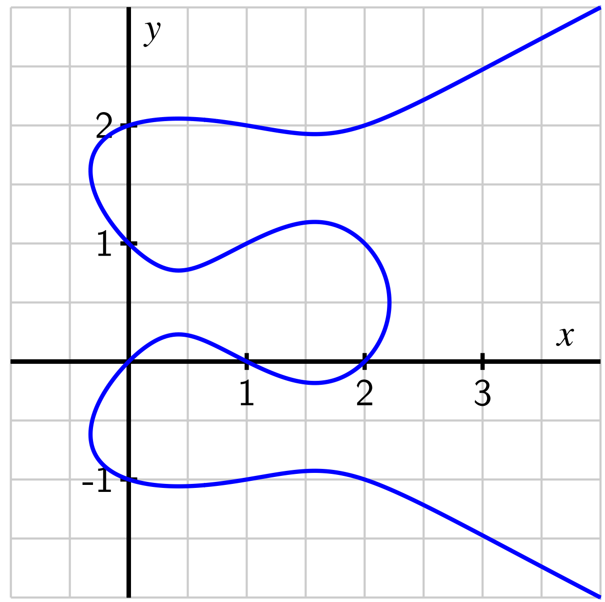

Figure 2.7.6: $y(y^2-1)(y-2) = x(x-1)(x-2)$.

Solution.

a. To find where the tangent line to the curve is horizontal, we set $\frac{dy}{dx} = 0$, which requires that the numerator be zero, or in other words that

$$(x-1)(x-2) + x(x-2) + x(x-1) = 0.$$

Expanding and combining like terms, we find that $3x^2 - 6x + 2 = 0$, which occurs where $x = \frac{3\pm\sqrt{3}}{3} \approx 0.42265, 1.57735$. From the graph in Figure 2.7.6, we observe that at each such x-value, there are several corresponding y-values for which the tangent line will be horizontal. For instance, when $x = 0.42265$, then y must satisfy the equation

$$y(y^2-1)(y-2) = 0.42265(0.42265-1)(0.42265-2) = 0.384900.$$

Because this is a quartic equation (degree 4) equation in y, we use computational technology to help us find the solutions. Doing so, we find four approximate values for y, $y \approx -1.05782, 0.229478, 0.770522, 2.05782$, and thus our estimates for four points at which the tangent line is horizontal are

$$(0.42265, -1.05782); (0.42265, 0.229478); (0.42265, 0.770522); (0.42265, 2.05782).$$

Similar work can be done to find the four points at which the tangent line is horizontal when $x \approx 1.57735$.

b. The tangent line to the curve is vertical wherever $\frac{dy}{dx}$ is undefined, which occurs precisely where

$$(y^2-1)(y-2) + 2y^2(y-2) + y(y^2-1) = 0.$$

Expanding and combining like terms, we see that we need to solve the cubic equation $4y^3 - 6y^2 - 2y + 2 = 0$; using a computer algebra system, we find that this occurs when $y = \frac{1}{2}, \frac{1 \pm \sqrt{5}}{2}$. It now remains to find the x-coordinate that corresponds to each such y-value. For instance, when $y = \frac{1}{2}$, x must satisfy

$$\frac{1}{2}(\frac{1}{4} - 1)(\frac{1}{2} - 2) = x(x - 1)(x - 2),$$

or in other words, $x^3 - 3x^2 + 2x = \frac{9}{16}$. Here, too, we use technology to determine that there is only one such x, and $x \approx 2.21028$. Similar work can be done to find the x-values that correspond to $y = \frac{1 \pm \sqrt{5}}{2}$.

c. There are four points on the curve where $x = 1$, which correspond to the y-values that satisfy $y(y^2-1)(y-2) = 0$: $(1,0)$, $(1,1)$, $(1,-1)$, $(1,2)$. We choose the point $(1,1)$ and evaluate $\frac{dy}{dx}$ at this point. Doing so,

$$\left.\frac{dy}{dx}\right|_{(1,1)} = \frac{(1-1)(1-2) + 1(1-2) + 1(1-1)}{(1^2-1)(1-2) + 2 \cdot 1^2(1-2) + 1(1^2-1)} = \frac{-1}{-2} = \frac{1}{2}.$$

Thus, the equation of the tangent line to the curve at $(1,1)$ is $y - 1 = \frac{1}{2}(x - 1)$.

Activity 2.7.4. For each of the following curves, use implicit differentiation to find dy/dx and determine the equation of the tangent line at the given point.

a. $x^3 - y^3 = 6xy$, $(-3, 3)$

b. $\sin(y) + y = x^3 + x$, $(0, 0)$

c. $3xe^{-xy} = y^2$, $(0.619061, 1)$

Solution.

a. Differentiating with respect to x,

$$\frac{d}{dx}[x^3 - y^3] = \frac{d}{dx}[6xy],$$

so that by the chain and product rules we have

$$3x^2 - 3y^2\frac{dy}{dx} = 6x\frac{dy}{dx} + 6y.$$

Rearranging to get all terms with $\frac{dy}{dx}$ on the same side, it follows that

$$-3y^2\frac{dy}{dx} - 6x\frac{dy}{dx} = 6y - 3x^2,$$

and thus

$$\frac{dy}{dx}(-3y^2 - 6x) = 6y - 3x^2.$$

Finally, we have established that

$$\frac{dy}{dx} = \frac{6y - 3x^2}{-3y^2 - 6x},$$

so evaluating at $(-3, 3)$, we have $\frac{dy}{dx}\Big|_{(-3,3)} = \frac{6(3)-3(-3)^2}{-3(3)^2-6(-3)} = 1$. Thus, the tangent line has equation $y - 3 = 1(x + 3)$.

b. After differentiating with respect to x, we have

$$\cos(y)\frac{dy}{dx} + \frac{dy}{dx} = 3x^2 + 1.$$

Taking the usual steps to solve for $\frac{dy}{dx}$, we find that

$$\frac{dy}{dx} = \frac{3x^2 + 1}{\cos(y) + 1}.$$

Evaluating the slope of the tangent line at $(0, 0)$, we have $\frac{dy}{dx}\Big|_{(0,0)} = \frac{1}{2}$, and thus the tangent line at $(0, 0)$ has equation $y = \frac{1}{2}x$.

c. When we differentiate both sides with respect to x,

$$\frac{d}{dx}[3xe^{-xy}] = \frac{d}{dx}[y^2],$$

we first observe that the product rule is needed on the left and the chain rule on the right. Applying those rules, we have

$$3x\frac{d}{dx}[e^{-xy}] + 3e^{-xy} = 2y\frac{dy}{dx}.$$

Next, we apply the chain rule to differentiate e^{-xy}, which yields

$$3xe^{-xy}\frac{d}{dx}[-xy] + 3e^{-xy} = 2y\frac{dy}{dx}.$$

Finally, to complete the process of differentiation, we use the product rule and get

$$3xe^{-xy}(-x\frac{dy}{dx} - y) + 3e^{-xy} = 2y\frac{dy}{dx}.$$

To solve for $\frac{dy}{dx}$, we first expand to have

$$3x^2e^{-xy}\frac{dy}{dx} - 3xye^{-xy} + 3e^{-xy} = 2y\frac{dy}{dx},$$

and then the usual algebraic work may be done to deduce that

$$\frac{dy}{dx} = \frac{3xye^{-xy} - 3e^{-xy}}{3x^2e^{-xy} - 2y}.$$

Evaluating at the point $(0.571433, 1)$, it follows that the slope of the tangent line is

$$\left.\frac{dy}{dx}\right|_{(0.571433,1)} = \frac{3 \cdot 0.571433e^{-0.571433} - 3e^{-0.571433}}{3(0.571433)^2e^{-0.571433} - 2} \approx 0.501835.$$

Thus, the tangent line is given by $y - 1 = 0.501835(x - 0.571433)$.