1.6 The second derivative

Activity 1.6.2. The position of a car driving along a straight road at time t in minutes is given by the function $y = s(t)$ that is pictured in Figure 1.6.11. The car's position function has units measured in thousands of feet. Remember that you worked with this function and sketched graphs of $y = v(t) = s'(t)$ and $y = v'(t)$ in Preview Activity 1.6.1.

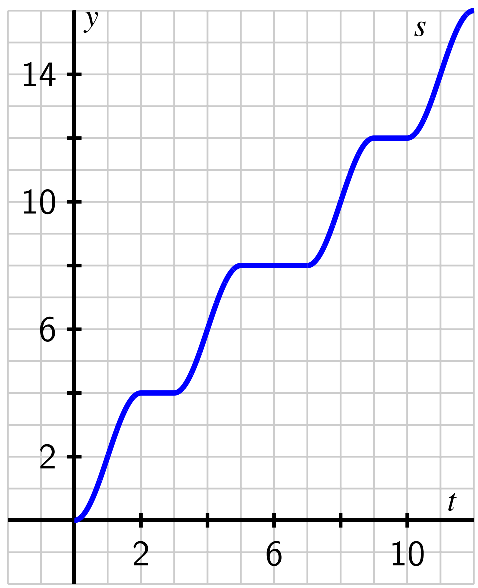

Figure 1.6.11: The graph of $y = s(t)$, the position of the car (measured in thousands of feet from its starting location) at time t in minutes.

a. On what intervals is the position function $y = s(t)$ increasing? decreasing? Why?

b. On which intervals is the velocity function $y = v(t) = s'(t)$ increasing? decreasing? neither? Why?

c. *Acceleration* is defined to be the instantaneous rate of change of velocity, as the acceleration of an object measures the rate at which the velocity of the object is changing. Say that the car's acceleration function is named $a(t)$. How is $a(t)$ computed from $v(t)$? How is $a(t)$ computed from $s(t)$? Explain.

d. What can you say about s'' whenever s' is increasing? Why?

e. Using only the words *increasing*, *decreasing*, *constant*, *concave up*, *concave down*, and *linear*, complete the following sentences. For the position function s with velocity v and acceleration a,

- on an interval where v is positive, s is ________.
- on an interval where v is negative, s is ________.
- on an interval where v is zero, s is ________.
- on an interval where a is positive, v is ________.
- on an interval where a is negative, v is ________.
- on an interval where a is zero, v is ________.
- on an interval where a is positive, s is ________.
- on an interval where a is negative, s is ________.

- on an interval where a is zero, s is _________.

Solution.

a. The position function $y = s(t)$ increasing on the intervals $0 < t < 2$, $3 < t < 5$, $7 < t < 9$, and $10 < t < 12$, because at every point in such intervals, $s'(t)$ is positive. For the provided function, $s(t)$ is never decreasing because its derivative is never negative.

b. The velocity function $y = v(t)$ appears to be increasing on the intervals $0 < t < 1$, $3 < t < 4$, $7 < t < 8$, and $10 < t < 11$ because the curve $y = s(t)$ is concave up which corresponds to an increasing first derivative $y = s'(t)$. Similarly, $y = v(t)$ appears to be decreasing on the intervals $1 < t < 2$, $4 < t < 5$, $8 < t < 9$, and $11 < t < 12$ because the curve $y = s(t)$ is concave down which corresponds to a decreasing first derivative $y = s'(t)$. On the intervals $2 < t < 3$, $5 < t < 7$, and $9 < t < 10$, the curve $y = s(t)$ is constant, and thus linear, so neither concave up nor concave down.

c. Since $a(t)$ is the instantaneous rate of change of $v(t)$, $a(t) = v'(t)$. And because $v(t) = s'(t)$, it follows that $a(t) = v'(t) = [s'(t)]' = s''(t)$, so acceleration is the second derivative of position.

d. Because $s''(t)$ is the first derivative of $s'(t)$, when $s'(t)$ is increasing, $s''(t)$ must be positive.

e. For the position function $s(t)$ with velocity $v(t)$ and acceleration $a(t)$,

   - on an interval where $v(t)$ is positive, $s(t)$ is *increasing*.
   - on an interval where $v(t)$ is negative, $s(t)$ is *decreasing*.
   - on an interval where $v(t)$ is zero, $s(t)$ is *constant*.
   - on an interval where $a(t)$ is positive, $v(t)$ is *increasing*.
   - on an interval where $a(t)$ is negative, $v(t)$ is *decreasing*.
   - on an interval where $a(t)$ is zero, $v(t)$ is *constant*.
   - on an interval where $a(t)$ is positive, $s(t)$ is *concave up*.
   - on an interval where $a(t)$ is negative, $s(t)$ is *concave down*.
   - on an interval where $a(t)$ is zero, $s(t)$ is *linear*.