• ◇

Discrete Structures Course Notes

Discrete Structures Course Notes

Mike Janssen
Dordt University

Website: https://prof.mkjanssen.org/ra/notes

Contents

1 Logic **1**

1.1 Introduction to Mathematical Statements 1
1.2 Logical Connectives and Equivalence 2
1.3 Quantifiers 7

2 Introduction to Graphs **10**

2.1 Toward a Definition of a Graph 10

3 Proof Techniques **12**

3.1 Direct Proof 12
3.2 Proof by Contradiction 12
3.3 Proof via Contrapositive 12
3.4 Induction 12

4 Set Theory **13**

4.1 The idea of a set 13
4.2 Operations with Sets 13
4.3 Families of Sets 13

5 Functions **14**

5.1 Introduction to Functions 14
5.2 Properties of Functions 14

6 Further Explorations in Graph Theory **15**

6.1 Trees and Cycles 15
6.2 Graph Isomorphisms 15
6.3 Graph Invariants 15

Appendices

A On Mathematical Writing **16**

Back Matter

Index **17**

Chapter 1

Logic

1.1 Introduction to Mathematical Statements

Guiding Questions.

In this section, we'll seek to answer the questions:

- What is a statement?
- What are some examples and non-examples of statements?
- What is an atomic statement?

In order to do mathematics, we need an idea of what sorts of questions we can explore. We will focus our energies on *statements*. This implicitly limits the domain of mathematics, but in doing so

Definition 1.1.1 A **statement** is a declarative sentence that is either true or false. ◊

Activity 1.1.2 Determine whether the following are statements. If so, explain why, and determine their truth value if you are able. If not, explain why not.

1. Quadratic equations have at most two real solutions.
2. There exists a function that is differentiable at 0 but not continuous at 0.
3. Winter is coming.
4. What time is it?
5. The sum of the first n positive integers is $\frac{n(n+1)}{2}$.
6. That is beautiful!
7. A hexagon has six sides, and two distinct lines in the plane are either parallel or meet in exactly one point.
8. The Green Bay Packers are the worst football team.
9. $3 + 4 = 7$.
10. This statement is false.

Activity 1.1.3 Give at least one example and one non-example of a statement.

Most of the statements we've explored so far might be considered **atomic** in the sense that they cannot be broken down into combinations of simpler statements. In Section 1.2, we will explore ways of building more complex statements out of simpler ones via logical connectives.

1.2 Logical Connectives and Equivalence

Guiding Questions.

In this section, we'll seek to answer the questions:

- How can we combine statements into more complex statements?
- What is a truth table?
- How can we determine if two statements are logically equivalent?

In this section, we will explore ways of combining statements to form new, more complex statements. A hallmark of our approach is that we will *define* the truth values of the statements completely formally (though generally the way you might expect). That is to say, the truth values of the combined statements will depend only on the truth values of the atomic statements of which they are composed. Deeper exploration will require a clear notion of logical equivalence, established via truth tables.

1.2.1 Logical Connectives

We begin with negation.

Definition 1.2.1 Suppose P is a statement. The **negation** of P, denoted $\neg P$ and read "not P", has the opposite truth value of P and is defined by the *truth table* found in Table 1.2.2.

Table 1.2.2 The negation of P, $\neg P$.

P	$\neg P$
T	F
F	T

◊

LaTeX Code 1.2.3 To typeset $\neg P$ in $\LaTeX$, use `\neg P` in math mode.

We observe that Table 1.2.2 is our first encounter with a *truth table*. A truth table lists all possible truth values for a (compound) statement given all possible combinations of truth values of its atomic statements. Since there are only two possible truth values for the atomic statement P, there are only two rows in Table 1.2.2.

Activity 1.2.4 Meaningfully negate the following propositions (without just saying "It is not the case that...").

1. e is a negative real number.
2. Iowa is the tenth largest state.
3. 17 is a prime number.

The next connective we introduce is the logical **and**, also known as **conjunction**. Then, we'll see the logical **inclusive or**, also known as **disjunction**.

Definition 1.2.5 The **conjunction** of P and Q, denoted $P \wedge Q$ and read "P and Q", is true when both P and Q are true, and false otherwise. See Table 1.2.6.

Table 1.2.6 The conjunction of P and Q.

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

◊

LaTeX Code 1.2.7 To typeset $P \wedge Q$ in LaTeX, use `P \land Q` in math mode.

Definition 1.2.8 The **disjunction** of P and Q, denoted $P \vee Q$ and read "P or Q", is true when P is true, Q is true, or both are true, and false otherwise. See Table 1.2.9.

Table 1.2.9 The disjunction of P and Q.

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

◊

LaTeX Code 1.2.10 To typeset $P \vee Q$ in LaTeX, use `P \lor Q` in math mode.

Problem 1.2.11 Determine the truth values of the following propositions.

1. Math 212 meets on Mondays and the capital of Iowa is Des Moines.
2. Math 212 meets on Mondays and the capital of Minnesota is Minneapolis.
3. Math 212 meets on Mondays or the capital of Minnesota is Minneapolis.
4. Math 212 meets on Mondays or the capital of Minnesota is St. Paul.

Observe that the "or" connective defined in Definition 1.2.8 is distinct from the so-called "exclusive or" that is often used in, e.g., computer science. The statement $P \vee Q$ is true so long as at least one of P, Q is true—or both!

There are two more connectives to consider. As a warmup, consider the following problem.

Problem 1.2.12 A mathematician[1] places a set of four cards on a table, each of which has a number on one side and a colored patch on the other side. She claims the following:

> If a card shows an even number on one face, then its opposite face is red.

The visible faces of the cards show 3, 8, red, and blue. Which card(s) must you turn over in order to test the truth of her claim? Carefully explain.

[1] This exploration due to Dr. L. Keough.

Definition 1.2.13 Let P and Q be statements. The **implication**, "P implies Q"[2] (or "if P, then Q") is denoted $P \Rightarrow Q$, and is false only when P is true but Q is false. See Table 1.2.14.

Table 1.2.14 The implication $P \Rightarrow Q$.

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

$\Diamond$

LaTeX Code 1.2.15 To typeset $P \Rightarrow Q$ in LaTeX, use `P \Rightarrow Q` in math mode.

Problem 1.2.16 Determine the truth values of the following statements. Identify which row of Table 1.2.14 you are in.

1. If x is a negative integer, then $-5 \cdot x$ is positive.
2. If we have Math 212 today, then it is Wednesday.
3. If $9 > 5$, then dogs do not have wings.
4. If $2 = 4$, then Calvin Coolidge is our greatest president.

The formalization of mathematical logic ramps up a bit when we consider conditional statements. It is important to remember that we define the truth value of the proposition $P \Rightarrow Q$ *formally* based on the structure of the conditional statement and the truth values of the constituents P and Q. That is to say that there need not be a causal relationship between P and Q!

The last two rows of Table 1.2.14 are also worth a moment of our time. They state that if the statement P[3] is false, then the *implication* $P \Rightarrow Q$ is *true*. Note that this is different than saying that Q[4] is true. When the implication $P \Rightarrow Q$ is true because P is false, we usually say that $P \Rightarrow Q$ is **vacuously true**.

Problem 1.2.17 Suppose Dr. Janssen promises that, if everyone get an A in the class, then he will bring Defender sandwiches (on pretzel buns, with everything, as God intended) to celebrate on the last day of class[5]. Unfortunately, a few students finish the course with an A–, so Dr. Janssen does not bring Defenders.

Decide the truth value of the implication

> If everyone gets an A in the class, Dr. Janssen will bring Defender sandwiches to celebrate.

Make sure you can clearly explain your thinking and be able to give reasons for why the truth value you chose makes sense.

Our final logical connective is the **biconditional** connective.

[2] We sometimes say that P is "sufficient for" Q.
[3] Often called the *antecedent*.
[4] Often called the *consequent*.
[5] This is purely hypothetical.

Definition 1.2.18 Let P and Q be statements. The biconditional statement joining P and Q, read "P if and only if Q"[6] and written $P \Leftrightarrow Q$, is true precisely when both P and Q are true, and false otherwise. ◊

LaTeX Code 1.2.19 To typeset $P \Leftrightarrow Q$ in LaTeX, use `P \Leftrightarrow Q` in math mode.

Problem 1.2.20 First, construct a truth table for $P \Leftrightarrow Q$ based on Definition 1.2.18. Then, determine the truth values of the following statements. Identify which row of your truth table you are in.

1. x is a negative integer if and only if $-5 \cdot x$ is positive.
2. We have Math 212 today if and only if today is Wednesday.
3. $9 > 5$ if and only if dogs do not have wings.
4. $2 = 4$ if and only if $\sin(\pi) = 1$.

1.2.2 Logical Equivalence

A fundamental skill in analyzing and proving mathematical statements is the ability to carefully describe their logical structure and, when appropriate, convert the statement into a **logically equivalent** statement. Ideally, the new equivalent form of the statement will have a structure more amenable to a particular type of proof or solution.

Definition 1.2.21 Two statements P and Q are said to be **logically equivalent** if they have the same truth table. In this case, we write $P \equiv Q$. ◊

Problem 1.2.22 Double Negation. Use a truth table to verify that $P \equiv \neg(\neg P)$.

Problem 1.2.23 Use a truth table to determine whether the following statements are logically equivalent.

- $P \Leftrightarrow Q$
- $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$

Given an implication $P \Rightarrow Q$, one may construct some related statements.

Definition 1.2.24 Let P and Q be statements.

- The **converse of the statement** $P \Rightarrow Q$ is the statement $Q \Rightarrow P$.
- The **contrapositive of the statement** $P \Rightarrow Q$ is the statement $(\neg Q) \Rightarrow (\neg P)$.
- Thhe **inverse of the statement** $P \Rightarrow Q$ is the statement $(\neg P) \Rightarrow (\neg Q)$.

◊

Activity 1.2.25 Choose at least two conditional statements we've mentioned in class and state the coverse, contrapositive, and inverse of each. Based on the statements you wrote, make a conjecture[7] about how the truth values of these four implications (the original conditional statement, together with its converse, contrapositive, and inverse) seem to relate.

Problem 1.2.26 Use truth tables to identify any logical equivalences between $P \Rightarrow Q$, $Q \rightarrow P$, $(\neg Q) \Rightarrow (\neg P)$, and $(\neg P) \Rightarrow (\neg Q)$.

[6] We sometimes say that P is "necessary and sufficient" for Q.
[7] A conjecture is a precise educated "guess" about what is true in general.

Problem 1.2.27 Show that $P \to Q \equiv (\neg P) \vee Q$.

The following pairs of logical equivalences, named for British mathematician Augustus De Morgan[8], allow us to convert conjunctions to disjunctions, and vice versa.

Problem 1.2.28 De Morgan's Laws. Show that the following pairs of statements are logically equivalent.

- $\neg(P \wedge Q)$ and $(\neg P) \vee (\neg Q)$
- $\neg(P \vee Q)$ and $(\neg P) \wedge (\neg Q)$

Problem 1.2.29 Distributive Laws. Let P, Q, R be statements. Use truth tables to verify the following equivalences, but be careful—how many rows should your truth table contain?

- $P \wedge (Q \vee R) \equiv P \vee (Q \wedge R)$
- $P \vee (Q \wedge R) \equiv P \wedge (Q \vee R)$

We will often interested in proving conditional statements in which our antecdents or consequents have multiple cases. The following equivalences will come in handy.

Problem 1.2.30 Use truth tables to verify the following.

- $P \Rightarrow (Q \wedge R) \equiv (P \wedge \neg Q) \Rightarrow R$
- $(P \vee Q) \Rightarrow R \equiv (P \Rightarrow R) \wedge (Q \Rightarrow R)$

To conclude this section, we mention two types of special statements: tautologies and contradictions.

Definition 1.2.31 A statement is called a **tautology** if it is always true. A statement is called a **contradiction** if it is never true. ◊

`https://imgs.xkcd.com/comics/honor_societies.png`

Figure 1.2.32 From XKCD 703: Honor Societies[9]

[8] `https://en.wikipedia.org/wiki/Augustus_De_Morgan`
[9] `https://xkcd.com/703/`

Activity 1.2.33 Let P and Q be statements. Determine, with proof, whether the following are tautologies, contradictions, or neither.

1. $P \wedge (\neg P)$

2. $P \vee (\neg P)$

3. $P \Rightarrow (P \Rightarrow (P \Rightarrow (P \Rightarrow Q)))$

4. $P \Rightarrow (\neg P \Rightarrow (P \Rightarrow (\neg P \Rightarrow Q)))$

1.3 Quantifiers

Guiding Questions.

In this section, we'll seek to answer the questions:

- What is an open statement?

- What are quantifiers?

- How can we negate quantified statements?

Many reasonable mathematical sentences do not meet our definition of a statement. For instance, $x + 5 = 7$ is a declarative sentence, but its truth value depends on x; when $x = 2$, the statement is true, and when $x = 57$ it is not. We say that such a sentence is **open**.

Definition 1.3.1 An **open sentence** (or **predicate**) is a sentence $P(x_1, x_2, \ldots, x_n)$ depending on variables $x_1, x_2, \ldots, x_n$ with the property that the sentence becomes a statement when values are assigned to the variables, or a **domain** is specified for each variable. ♢

That is to say, we can turn an open sentence into a statement in one of two ways. First, we could assign *specific* values to the variables. However, our typical approach will be to *quantify* the variables by assigning them to a particular domain, such as the real numbers or positive integers, and making one of two assertions: that the quantified statement is true *for every* object in the domain, or *for some* object in the domain.

Definition 1.3.2 The **existential quantifier**, denoted with the symbol $\exists$, is the phrase "there exists" (or equivalent). The **universal quantifier**, denoted with the symbol $\forall$, is the phrase "for every" (or equivalent). ♢

LaTeX Code 1.3.3 To typeset the existential quantifier, $\exists$, use the command `\exists` in math mode. To typeset the universal quantifier, $\forall$, use the command `\forall` in math mode.

When writing a careful mathematical statement, it is often helpful to use quantifiers and mathematical symbols to clearly and concisely communicate an idea, such as

$$(\forall x \in \mathbb{R})(x^2 \geqslant 0),$$

which states that for every real number x, the square of x is nonnegative.

It can also be tempting to use the quantifier notation as an abbreviation when working through the scratchwork necessary to solve a problem. As a rule, however, one should *not* use quantifier notation in a final write-up for a problem *unless the problem is about quantifiers.*

Finding the right balance between mathematical symbols and ordinary writing takes time and practice. For more suggetions, see Appendix A.

Activity 1.3.4 Translate each quantified statement below to English. As best you can tell, is the statement true or false? Why?

1. $(\exists a \in \mathbb{Z})(2 \cdot a = 2)$ (note that the symbols "$a \in \mathbb{Z}$" means that a is in the set $\mathbb{Z}$ of integers, i.e., positive and negative whole numbers, and 0)
2. $(\forall a \in \mathbb{Z})(2 \cdot a = 2)$
3. $(\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z})(x + y = 0)$
4. $(\exists y \in \mathbb{Z}, \forall x \in \mathbb{Z})(x + y = 0)$

LaTeX Code 1.3.5 To typeset $\mathbb{Z}$ or $\mathbb{R}$ in *LaTeX*, load the `amsmath` package using `\usepackage{amsmath}` in the preamble, and then type `\mathbb{Z}` or `\mathbb{R}`.

Activity 1.3.6 Translate the following from English to use mathematical notation in the style of Activity 1.3.4. Choose variables as appropriate.

1. The square of every nonzero integer is greater than or equal to 1.
2. There exists a real number whose square is 2.
3. The cube of every positive integer m is greater than its square.
4. There is an integer x for which there is an integer y such that $x \cdot y = -1$.
5. Every real number is positive, negative, or zero.

We will often find ourselves needing to negate quantified statements. This should be done with care, especially as the statements get more complicated.

Problem 1.3.7

1. Consider the statement "Every integer is positive." One way to negate this statement is to say "It is not the case that every integer is positive," but this style of negation is somewhat clumsy. Find a better way to negate the statement.
2. What does your answer to the previous question suggest about how to negate a universally quantified statement? Test your hypothesis on another univerally quantified statement of your own choosing.
3. Find an elegant negation of the statement "There is an integer x satisfying $2x = 1$" that uses a univeral quantifier.
4. Can the negation of any existentially quantified statement be written in terms of a universal quantifier?

Let's put our discoveries from Problem 1.3.7 to the test.

Activity 1.3.8 Negate the following statements. As best you can tell, is the statement true or false? Why?

1. $(\exists a \in \mathbb{Z})(2 \cdot a = 2)$
2. The square of every nonzero integer is greater than or equal to 1.
3. $(\forall a \in \mathbb{Z})(2 \cdot a = 2)$
4. The cube of every positive integer m is greater than its square.
5. $(\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z})(x + y = 0)$

6. There exists a real number whose square is 2.
7. There is an integer x for which there is an integer y such that $x \cdot y = -1$.
8. $(\exists y \in \mathbb{Z}, \forall x \in \mathbb{Z})(x + y = 0)$
9. Every real number is positive, negative, or zero.

Chapter 2

Introduction to Graphs

2.1 Toward a Definition of a Graph

Guiding Questions.

In this section, we'll seek to answer the questions:

- How is mathematical knowledge developed?
- What is a graph?
- How can we represent graphs?

Influenced by the ancient mathematician Euclid of Alexandria, mathematical knowledge has traditionally been developed via a careful litany of *definition*, *theorem*, and *proof*. The word **definition** means more or less what you expect: it is a precise description of the meaning of a term. The mathematical version places an emphasis on clarity and concision. This does not always make the definition of a mathematical term easy to parse, but *should* make it clear to *apply*.

A **theorem** is a true mathematical statement. If you read enough mathematics, you will also see words like *proposition*, *lemma*, and *corollary*. These are all true mathematical statements as well, but have slightly different connotations.

- A proposition is a true mathematical statement which is not perceived to be important enough to be called a theorem.
- A lemma is a true mathematical statement which may be somewhat technical to write, but is used to prove a theorem or proposition. One often has the experience of proving a theorem and realizing that a portion of the proof is substantive enough that it can be pulled out and written as the proof of a lemma, to which you can then refer to complete the proof of the theorem.
- A corollary is a true mathematical statement that follows immediately from the proof of a theorem[1].

How is the truth of a mathematical statement ascertained? Via *proof*. One of the goals of this course is develop proficiency writing proofs, and we will explore techniques for doing so in more detail in Chapter 3.

[1] For instance, suppose you've proved that all rectangles in the plane contain only right angles. A corollary to this theorem is that all squares in the plane contain only right angles.

For now, we will focus on the first part of the mathematical litany: definitions. Theorems (and their accompanying proofs) must be about *something*, so in this section, we introduce one of the basic ideas of the course: the **graph**.

Problem 2.1.1 Below there are several examples of objects called *graphs*. Develop a clear, concise definition of the word **graph** and explain why each of the objects below meets your definition. For the sake of consistency, we'll call the circles **vertices** (or **nodes**), and the lines connecting them **edges**.

Figure 2.1.2 A graph.

Figure 2.1.3 A second graph.

Figure 2.1.4 A third graph.

Figure 2.1.5 A fourth graph.

Figure 2.1.6 A fifth graph.

Figure 2.1.7 A sixth graph (no, this is not a mistake).

Problem 2.1.8 Below there are several examples of objects that are *not* graphs. If necessary, modify your definition from Problem 2.1.1, and explain why the objects below do not meet your (possibly new) definition—but make sure the objects in Problem 2.1.1 still do!

Figure 2.1.9 Not a graph.

Figure 2.1.10 Not a graph.

Figure 2.1.11 (Still) Not a graph.

Definition 2.1.12 A **graph** is .... ◊

Activity 2.1.13 Given the definition of a graph that we've come up with, exhibit two additional examples of a graph, as well as two non-examples.

Chapter 3

Proof Techniques

3.1 Direct Proof

Guiding Questions.

In this section, we'll seek to answer the questions:

- What is a proof?
- What is a direct proof?
- What are some ideas for getting started/unstuck when stuck on a proof?

3.2 Proof by Contradiction

3.3 Proof via Contrapositive

3.4 Induction

Chapter 4

Set Theory

4.1 The idea of a set

4.2 Operations with Sets

4.3 Families of Sets

Chapter 5

Functions

5.1 Introduction to Functions

5.2 Properties of Functions

Chapter 6

Further Explorations in Graph Theory

6.1 Trees and Cycles

6.2 Graph Isomorphisms

6.3 Graph Invariants

Appendix A

On Mathematical Writing

Index

$\LaTeX$ symbols$\Leftrightarrow$, 5
$\LaTeX$ symbols$\mathbb{R}$, 8
$\LaTeX$ symbols$\Rightarrow$, 4
$\LaTeX$ symbols$\mathbb{Z}$, 8
$\LaTeX$ symbols$\exists$, 7
$\LaTeX$ symbols$\forall$, 7
$\LaTeX$ symbols$\wedge$, 3
$\LaTeX$ symbols$\vee$, 3
$\LaTeX$ symbols$\neg$, 2

and, 3

biconditional, 5

conditional statements, 4, 5
conjunction, 3
contradiction, 6
contrapositive, 5
converse, 5

disjunction, 3

graph, 11

if and only if, 5
if-then, 4

implication, 4
implicationcontrapositive of, 5
implicationconverse of, 5
implicationinverse of, 5
inverse, 5

logical connectivesand, 3
logical connectivesbiconditional, 5
logical connectivesimplies, 4
logical connectivesnegation, 2
logical connectivesor, 3
logical equivalence, 5

negation, 2

open sentence, 7
or, 3

quantifiersexistential quantifier, $\exists$, 7
quantifiersuniversal quantifier, $\forall$, 7

statement, 1

tautology, 6