The first exam (on Friday, March 5) will cover everything we have discussed so far this semester. That means mostly counting (chapter 1), but don't forget about statements, sets, and functions (chapter 0).

- ☐ Statements:
  - ◯ Atomic and Molecular statements
  - ◯ Logical connectives and quantifiers
  - ◯ Implications: converse, contrapositive, and negation.
- ☐ Set Theory:
  - ◯ How to read and make sense of set theory notation.
  - ◯ Unions, intersections and compliments.
  - ◯ Elements vs. Subsets ($\in$ vs. $\subseteq$).
  - ◯ The power set.
  - ◯ Venn diagrams.
- ☐ Functions:
  - ◯ Injections, surjections and bijections
  - ◯ Interpret a real world situation as a function
  - ◯ Use a bijection to show two sets have the same size
- ☐ Additive and Multiplicative Principles
- ☐ Principle of Inclusion/Exclusion
- ☐ Binomial coefficients: $\binom{n}{k}$
- ☐ Combinations and Permutations
- ☐ Stars and Bars
- ☐ Things to count:
  - ◯ Bit strings
  - ◯ Subsets
  - ◯ Lattice Paths
  - ◯ Stacks vs Handfuls of chips
  - ◯ Ways to distribute items
  - ◯ Functions
- ☐ Combinatorial Proofs

The quizzes, in-class activities, and homework should give you a good idea of the types of questions to expect.

Additionally, the questions below would all make fine exam questions. Most of these are taken from the exercises in the textbook, where you will also find a few more good problems to try (also look at the chapter summary exercises).[1]

Sample Questions

1. Consider the statement, "For all sets A and B, if $|A| = |B|$ then $|A \cup B| = 2|A|$."
   (a) Write the converse and the contrapositive of the statement.
   (b) Is the original statement true? Explain.
   (c) Is the converse of the original statement true? Explain.
   (d) Consider another statement: "For $A = \{1, 2\}$ and $B = \{1\}$, if $|A| = |B|$ then $|A \cup B| = 2|A|$." Explain why this statement is true, but that does not tell us anything about the truth of the original statement or its converse.

2. Let A and B be sets with $|A| = 9$ and $|B| = 16$, and $|A \cup B| = 25$. Find $A \cap B$. Explain how you know your answer is correct.

3. If X is a finite set, and $f : X \to Y$ is a injective function, must it also be surjective?

4. If X is a finite set, and $f : X \to Y$ is both injective and surjective, what can you say about Y?

5. Give a counting question where the answer is $8 \cdot 3 \cdot 3 \cdot 5$. Give another question where the answer is $8 + 3 + 3 + 5$.

6. Suppose you own 7 bow ties and 2 fezzes. Let A be the *set* of all outfits you can make (using 1 bow tie and 1 fez).
   (a) Write down the set A by listing all its elements (you might want to use $b_1, b_2, \ldots$ for bow ties and f_1, f_2 for fezzes).
   (b) Find two disjoint sets B and C such that $A = B \cup C$. Explain how this illustrates the additive principle.
   (c) Find two sets D and E such that $A = D \times E$ (when written correctly). Explain how this illustrates the multiplicative principle.

7. Consider five digit numbers $\alpha = a_1a_2a_3a_4a_5$, with each digit from the set $\{1, 2, 3, 4\}$.
   (a) How many such numbers are there?
   (b) How many such numbers are there for which the *sum* of the digits is even?
   (c) How many such numbers contain more even digits than odd digits?

8. For how many $n \in \{1, 2, \ldots, 500\}$ is n a multiple of one or more of 5, 6, or 7? Hint: to find the number of n that are a multiple of, say 35, you can take 500, divide by 35, and round down.

[1]Disclaimer: Question on the actual exam may be easier or harder than those given her. There might be types of questions on this study guide not covered on the exam and questions on the exam not covered in this study guide. Questions on the exam might be asked in a different way than here. If solving a question lasts longer than four hours, contact your professor immediately.

9. Recall, by 8-bit strings, we mean strings of binary digits, of length 8.
   (a) How many 8-bit strings are there total?
   (b) How many 8-bit strings have weight 5?
   (c) How many subsets of the set $\{a, b, c, d, e, f, g, h\}$ contain exactly 5 elements?
   (d) Explain why your answers to parts (b) and (c) are the same. Why are these questions equivalent?

10. How many 8-letter words contain exactly 5 vowels (a,e,i,o,u)? What if repeated letters were not allowed?

11. For each of the following, find the number of shortest lattice paths from $(0, 0)$ to $(8, 8)$ which:
    (a) pass through the point $(2, 3)$.
    (b) avoid (do not pass through) the point $(7, 5)$.
    (c) either pass through $(2, 3)$ or $(5, 7)$ (or both).

12. You live in Grid-Town on the corner of 2nd and 3rd, and work in a building on the corner of 10th and 13th. How many routes are there which take you from home to work and then back home, but by a different route?

13. Suppose an exam has 10 questions on it, and you must answer 6 of them. In how many different ways could you complete the exam? There are actually two reasonable answers to this question. Give both of them and explain what the difference is and how they are related. Your explanation should include a justification for why the larger answer is larger and by how much.

14. Your favorite BBQ restaurant offers a pick-3 menu, in which you can choose 3 menu items from a larger list. You have narrowed you choices down to 4 that sound good. How many ways can you select 3 of these 4?
    (a) If you assume order matters, how many ways can you make your selection? Write down the set of all of these.
    (b) If you assume order doesn't matter, how many ways can you make your selection? Again, write down the set of all of these.
    (c) Show how the two sets of outcomes you gave in the parts above are related to each other. Use this to explain what we mean when we say "order matters" in counting problems.

15. How many 10-bit strings start with 111 or end with 101 or both?

16. How many 10-bit strings of weight 6 start with 111 or end with 101 or both?

17. How many lattice paths traveling only up and right, start at the origin and end on the line $x + y = 5$? Answer this question in two ways. What pattern in Pascal's triangle is this an example of?

18. Give a combinatorial proof for the identity

$$\binom{n}{k}\binom{n-k}{j} = \binom{n}{j}\binom{n-j}{k}.$$

19. Suppose you have 20 one-dollar bills to give out as prizes to your top 5 discrete math students. How many ways can you do this if:
    (a) each of the 5 students gets at least 1 dollar?
    (b) some students might get nothing?
    (c) each student gets at least 1 dollar but no more than 7 dollars?

20. How many functions $f : \{1, 2, 3, 4, 5\} \to \{a, b, c, d, e\}$ are there for which
    (a) $f(1) = a$ or $f(2) = b$ (or both)?
    (b) $f(1) \neq a$ or $f(2) \neq b$ (or both)?
    (c) $f(1) \neq a$ *and* $f(2) \neq b$, and are also injective?
    (d) Are injective and have at least one of $f(1) = a$, $f(2) = b$, or $f(3) = c$.

21. Consider functions $f : \{1, 2, 3, 4, 5\} \to \{0, 1, 2, \ldots, 9\}$.
    (a) How many of these functions are strictly increasing? Explain. (A function is strictly increasing provided if $a < b$, then $f(a) < f(b)$.)
    (b) How many of the functions are non-decreasing? Explain. (A function is non-decreasing provided if $a < b$, then $f(a) \leq f(b)$.)

22. How many permutations of $\{1, 2, 3, 4, 5\}$ leave exactly 1 element fixed?

23. How many functions map $\{1, 2, 3, 4, 5, 6\}$ *onto* $\{a, b, c, d\}$ (i.e., how many *surjections* are there)?

24. After a late night of math studying, you and your friends decide to go to your favorite tax-free fast food Mexican restaurant, *Burrito Chime*. You decide to order off of the dollar menu, which has 7 items. Your group has \$16 to spend (and will spend all of it).
    (a) How many different orders are possible? Explain. (The *order* in which the order is placed does not matter – just which and how many of each item that is ordered.)
    (b) How many different orders are possible if you want to get at least one of each item? Explain.
    (c) How many different orders are possible if you don't get more than 4 of any one item? Explain. Hint: get rid of the bad orders using PIE.
    (d) When you get back to your apartment, you give 3 items to your roommate (in a single bag). How many different collections of items could he receive provided he does not get more than one of any item? Explain.

25. While enjoying your "food," a commercial for *Burrito Chime* comes on TV advertising a special "Buy 5 for \$4" deal. The ad claims that this means that for \$4, you can choose from 2520 different meals. One of your friends says that this is too small (that it should be 16807), while another friend says the true number should be only 21. Who is right?