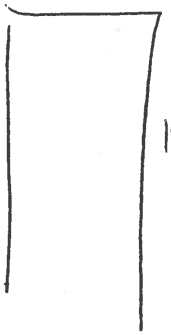
Proof Portfolio Problem # 1– Example–

–Dr. Keough –

Here's the problem: You work for a cell phone company which has just invented a new cell phone protector and wants to advertise that it can be dropped from the n^{th} floor without breaking. If you are given 2 phones and a 100 story building, how do you guarantee you know the highest floor it won't break with the smallest number of trial drops?

You might want to try this problem before you read the proof. On the next page I include my scratchwork.



Guess: Drop from the 50th floor

breaks → only 1 drop left → try from 1-49

doesn't break → try from 100th?

- breaks → try from ~~1-50~~ 51-~~99~~ 99
- doesn't break

at least 50 drops

Can do better? Drop from 10th floor

breaks → drop from 1-9 → ≤ 10 total

doesn't break → drop from 20

- breaks → drop from 11-19 → ≤ 11 total
- doesn't break → drop from 30
  - breaks → drop from 21-29 → 12 total
  - doesn't break → drop from 40

gang up every time

40th breaks 13 times

50th breaks 14

60th → 15

70th → 18

80th → 19

90th → 20

100th → 21

So can do in fewer than 21 times?

Seems like could do better w/o making it increase every time? factor in that you're "losing" a drop every time.

Start on 14

- breaks 14 drops (14 + 1-13)
- doesn't, go up 13 floors to 27
  - breaks → 14, 27, 15-26 → 14 drops again
  - doesn't break → go up 12 floors → 14, 27, 39, 28-38 (11) → 14 drops again

get to top in 14 drops?

14, 27, 39, 50, 60, 69, 77, 84, 90, 95, 99, 100 (+13, +12, +11, +10, +9, +8, +7, +6, +5, +4)

yes, this is in 12 drops, breaks at any stage then max of 14 drops.

can I do better? think no?

Theorem. *If there are 2 cell phones and a 100 story building available then one can test dropping cell phones from 14 stories and find the maximum number of stories the cell phone can be dropped from without breaking.*

Proof. We will describe an algorithm for needing only to drop from 14 stories. Our first step will be to drop the first phone from the 14^{th} floor. If the phone breaks from this floor, we will test floors 1 through 13, in order, until the cell phone breaks, giving us up to a total of 14 drops. If the phone does not break on a drop from the 14^{th} floor, we will drop the phone from the 27^{th} floor. If the phone breaks from the 27^{th} floor, we'll need to test floors 15 through 26. This is 12 more floors, in addition to the drop from the 14^{th} and the drop from the 27^{th} again giving us 14 drops.

We'll continue this process by dropping from floors $39, 50, 60, 69, 77, 84, 90, 95$, and finally 99. If we make it all the way to the 99^{th} floor then we will have done 11 drops. In any other case, we will do 14 total drops as seen by the following cases, which consider the first floor the phone breaks on from the list $39, 50, 60, 69, 77, 84, 90, 95$:

- The phone first breaks on the 39^{th} floor: In this case we test floors $14, 27, 39$ and floors $28 - 38$ giving 14 total drops.
- The phone first breaks on the 50^{th} floor: In this case we test floors $14, 27, 39, 50$ and floors $40 - 49$ giving 14 total drops.
- The phone first breaks on the 60^{th} floor: In this case we test floors $14, 27, 39, 50, 60$ and floors $51 - 59$ giving 14 total drops.
- The phone first breaks on the 69^{th} floor: In this case we test floors $14, 27, 39, 50, 60, 69$ and floors $60 - 68$ giving 14 total drops.
- The phone first breaks on the 77^{th} floor: In this case we test floors $14, 27, 39, 50, 60, 69, 77$ and floors $70 - 76$ giving 14 total drops.
- The phone first breaks on the 84^{th} floor: In this case we test floors $14, 27, 39, 50, 60, 69, 77, 84$ and floors $78 - 83$ giving 14 total drops.
- The phone first breaks on the 90^{th} floor: In this case we test floors $14, 27, 39, 50, 60, 69, 77, 84, 90$ and floors $85 - 89$ giving 14 total drops.
- The phone first breaks on the 95^{th} floor: In this case we test floors $14, 27, 39, 50, 60, 69, 77, 84, 90, 95$ and floors $91 - 94$ giving 14 total drops.

Thus we see that we can figure out the maximum floor where the phone breaks in 14 total drops. □

A Couple of Notes

1. Note the proof does not claim that there isn't a better solution. Don't feel like you need to do every piece of a problem. Figure out something cool, and try to prove it.

2. On that note it's best to check your conjectures with me. I can help you if they are too hard, too obvious, too false, and also brainstorm proof ideas.

3. Remember that a proof, at its heart, is just an explanation that other mathematicians believe. Don't feel like you need to adhere to strictly to a specific proof technique. Give a convincing explanation.

4. Pick a problem you like and start early so that this can be fun and not something you're worried about in the last week of the semester.