Skill Mastery Quizzes - Question Bank

Communicating in Mathematics (MTH 210-02)
Fall 2017

Skills

Logic

L1 Identify the hypothesis and conclusion of a conditional statement, determine its truth value, and apply it.

L1-1 Consider the following conditional statement:

If n is a prime number then n^2 has three positive factors.

Identify the hypothesis and conclusion.

The hypothesis is "n is a prime number" and the conclusion is "n^2 has three positive factors". Notice we leave off the if and the then when writing the hypothesis and conclusion.

Assume the above conditional statement is true. Assuming *only* the conditional statement and that 6 is not prime what can you conclude (if anything)? Explain your answer.

We are given that the conditional statement is true and that 6 is not prime. The fact that 6 is not prime tells us that our hypothesis is false. Given a false hypothesis, we know nothing about the conclusion. So we can not conclude anything. We can also look at this from a truth table perspective. Consider the truth table for the conditional statement $P \to Q$:

P	Q	$P \to Q$
T	T	T
T	F	F
F	T	T
F	F	T

In the third and fourth rows the hypothesis of the conditional statement is false (that is, n is not a prime number), and the conditional statement is true (that is, if n is a prime number then n^2 has 3 positive factors). Since Q (that is, n^2 has 3 positive factors), is true in the third row and false in the fourth row we can't conclude anything about the truth value of Q.

L1-2 Consider the following (true) conditional statement:

If the function f is continuous at a, then $\lim_{x \to a} f(x)$ exists.

Identify the hypothesis and conclusion of this conditional statement.

The hypothesis is "the function f is continuous at a" and the conclusion is "$\lim_{x \to a} f(x)$ exists".

Assume the above conditional statement is true. Assuming *only* the conditional statement and that a function f is not continuous at 7, what can you conclude (if anything)? Explain your answer.

Given that the function f is not continuous at 7 we know the hypothesis of the conditional statement is false. Thus we cannot conclude anything, since the statement makes no promises about what happens if a function is not continuous at a given a. See Quiz 1 solutions for an explanation with a truth table.

L1-2 Consider the following conditional statement:

If $p \neq 2$ and p is an even number, then p is not prime.

Identify the hypothesis and conclusion of this conditional statement.

Assume the above conditional statement is true. Assuming *only* the conditional statement and that 7 is not equal to 2 and is not even what can you conclude (if anything)? Explain your answer.

L1-3 If $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function at a real number a then f is a continuous function at a. State the hypothesis and conclusion of this conditional statement. Suppose you know a function $g : \mathbb{R} \to \mathbb{R}$ is continuous at 5. What can you conclude (if anything)? Explain your answer.

L1-token Consider the following conditional statement:

If p is a prime number then $p = 2$ or p is an odd number.

Identify the hypothesis and conclusion. Assume the above conditional statement is true. Assuming *only* the conditional statement and that 7 is odd what can you conclude (if anything)? Explain your answer.

L2 State precisely the definition of an even and odd integer and outline the proof of a statement using these terms.

L2-1 State the definition of even integer precisely. An integer n is even provided that...

there exists an integer q such that $n = 2q$.

Then outline a proof that if x is even and y is odd then $x+y$ is odd. (Make sure to include key details - like what things are integers.)

Suppose x is even and y is odd. Then there exist integers a, b such that $x = 2a$ and $y = 2b+1$. Then $x+y = (2a)+(2b+1)$ by substitution. By the distributive property $x+y = 2(a+b)+1$. Note that $a+b$ is an integer by closure of the integers under addition. Let $q = a+b$. Then $x+y = 2q+1$ for the integer q and so $x+y$ is an odd integer.

L2-2 State the definition of odd integer precisely. Then outline a proof that if $x, y \in \mathbb{Z}$ are odd integers then $x+y$ is even.

L2-3 State the definition of even integer precisely. Then outline a proof that if $x, y \in \mathbb{Z}$ are even integers then xy is even.

L2-token State the definition of odd integer precisely. Then outline a proof that if $x, y \in \mathbb{Z}$ are odd integers then xy is odd.

L3 Construct truth tables for statements that use the logical operators and, or, not, and implies.

1. Construct a truth table for $(\neg P \vee Q) \to R$.
2. Construct a truth table for $P \implies (Q \wedge R)$
3. Construct a truth table for $P \implies (Q \vee R)$.

L4 Write sets using set builder notation and interpret sets written in this notation.

L4-1 Write the set $\left\{\sqrt{2}, \left(\sqrt{2}\right)^3, \left(\sqrt{2}\right)^5, \dots\right\}$ in set builder notation.

L4-2 Describe what the set $\{x \in \mathbb{R} \mid 3 \leq x \leq 5\}$ is in words. Then write what the set is in roster notation or explain why you can not.

L4-3 Write the set $\{\dots, -5, -3, 1, 1, 3, 5, \dots\}$ using set builder notation.

L5 Negate a statement with and, or, not, implies, exists, and/or for all.

L5-1 Write a useful negation of the following statement:

$$\text{There exists } x \in \mathbb{Z} \text{ such that if } y \in \mathbb{Z} \text{ then } \frac{y}{x} \in \mathbb{Z}.$$

Useful negations don't start with "It is not true that..." and avoid the word not.

L5-2 Write a useful negation for the following statement:

$$\text{For every positive real number } \epsilon \text{ there exists a natural number } n \text{ with } \frac{1}{n} < \epsilon.$$

Useful negations don't start with "It is not true that..." and avoid the word not.

L5-3 For all integers n and m, if nm is even then n is even or m is even.

L5-token? Write a useful negation of the following statement:

$$\text{For all } m \in \mathbb{Z} \text{ there exists } n \in \mathbb{Z} \text{ such that } m > n.$$

Useful negations don't start with "It is not true that..." and avoid the word not.

Proofs

P1 Given a theorem, correctly state what will be assumed in a direct proof, proof by contradiction, and proof by contrapositive.

P1-1 Consider the following statement:

Every even integer greater than 2 can be expressed as the sum of two (not necessarily distinct) prime numbers.

State what you would assume in a direct proof. State what you would assume in a proof by contradiction.

P1-2 Consider the following statement:

For all natural numbers p and q, if p and q are twin primes other than 3 and 5 , then $pq+1$ is a perfect square and 36 divides $pq+1$.

State what you would assume in a direct proof. State what you would assume in a proof by contradiction.

P1-3 Consider the following statement

Let a and b be integers with $a \neq 0$. If a does not divide b then the equation $ax^3 + bx + (b+a) = 0$ does not have a solution that is a natural number.

State what you would assume in a direct proof. State what you would assume in a proof by contradiction.

P2 Identify situations in which it is appropriate to use induction and state the procedure for proving a statement by induction.

P2-1 For which of the following situations is it more appropriate to use induction. Explain.

(a) For all $a \in \mathbb{Z}$ the equation $ax^3 + ax + a = 0$ does not have a solution that is a natural number.

(b) Let a and b be integers and $n \in \mathbb{N}$. For all $m \in \mathbb{N}$ if $a \cong b \pmod{n}$ then $a^m \cong b^m \pmod{n}$.

For the statement you chose, state what your steps would be in a proof by induction.

P2-2 For which of the following situations is it more appropriate to use induction. Explain.

(a) For all integers a and b, $(a+b)^2 \equiv (a^2 + b^2) \pmod{2}$

(b) For each natural number n, 3 divides $4^n - 1$.

For the statement you chose, state what your steps would be in a proof by induction.

P2-3 For which of the following situations is it more appropriate to use induction. Explain.

(a) For all $a \in \mathbb{Z}$ the equation $ax^3 + ax + a = 0$ does not have a solution that is a natural number.

(b) For all $n \in \mathbb{N}$, $1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}$.

For the statement you chose, state what your steps would be in a proof by induction.

P3 Clearly and correctly disprove a statement using a counterexample.

P3-1 The following statement is incorrect:

$$\text{If } n \text{ is an integer then } n^2 \equiv 1 \pmod{3}.$$

Show the statement is false using a counterexample. You should clearly explain why the counterexample you found shows the statement is false.

P3-2 The following statement is incorrect:

The set of natural numbers is closed under subtraction.

Show the statement is false using a counterexample. You should clearly explain why the counterexample you found shows the statement is false.

P3-3 The following statement is incorrect:

$$\text{For each integer } n,\ (n^2 + 1) \text{ is a prime number.}$$

Show the statement is false using a counterexample. You should clearly explain why the counterexample you found shows the statement is false.

P4 Evaluate if a given proof is valid and adheres to our writing guidelines.

P4-1 Consider the following proposition and proof. Is the proof correct? If not, explain why not. If so, does the proof meet our writing guidelines?

Proposition. *If m is an odd integer then $m+6$ is an odd integer.*

Proof. For $m+6$ to be an odd integer there must exist an integer n such that

$$m + 6 = 2n + 1.$$

By subtracting 6 from both sides of this equation we obtain

$$\begin{aligned} m &= 2n - 6 + 1 \\ &= 2(n-3) + 1. \end{aligned}$$

By the closure properties of integers, $n-3$ is an integer, and hence, the last equation implies that m is an odd integer. This proves that if m is an odd integer then $m+6$ is an odd integer. □

P4-2 Considering the following proposition and proof. Is the proof correct? If not, explain why not. If so, does the proof meet our writing guidelines?

Theorem. *For each integer n, $3 \mid n^2+2$.*

Proof. We will consider two cases, $n \equiv 1 \pmod 3$ and $n \equiv 2 \pmod 3$. When $n \equiv 1 \pmod 3$, there exists an integer k such that $3k = n-1$. Then

$$n^2 + 2 = (3k+1)^2 + 2 = 9k^2 + 6k + 1 + 2 = 3(3k^2 + 2k + 1).$$

Since integers are closed under addition and multiplication $3k^2 + 2k + 1$ is an integer. Thus $3 \mid n^2 + 2$ in this case.
When $n \equiv 2 \pmod 3$ there exists an integer k such that $3k = n - 2$. Then

$$n^2 + 2 = (3k+2)^2 + 2 = 9k^2 + 12k + 4 + 2 = 3(3k^2 + 4k + 2).$$

Since integers are closed under addition and multiplication, $3k^2 + 4k + 2$ is an integer. Thus $3 \mid n^2 + 2$ in this case as well.
Since we've proven that $3 \mid n^2 + 2$ in all possible cases we have completed the proof. □

P4-3 Considering the following proposition and proof. Is the proof correct? If not, explain why not. If so, does the proof meet our writing guidelines?

Theorem. *For all integers m and n, if mn is an even integer, then m is even or n is even.*

Proof. For either m or n to be even there exists an integer k such that $m = 2k$ or $n = 2k$. So if we multiply m and n the product will contain a factor of 2 and, hence, mn will be even. □

Sets, Functions, and Equivalence Relations

S1 Use the symbols $\in, \notin, =, \neq, \subseteq, \nsubseteq, \subset, \not\subset$ correctly.

S1-1 Let $A = \{1, \{2\}, \{3, 4\}, 5\}$. Fill in a correct symbol for each of the following:

- $\{1\}$___A
- $\{2\}$___A
- $\{1, 2, 3, 4, 5\}$___A

S1-2 Let $A = \{1, 2, 4\}$ and $B = \{1, 2, 3, 5\}$. Fill in a correct symbol for each of the following:

- A___B
- $\emptyset$___A
- $\{4, 2, 1\}$___B

S1-3 Let $A = \{0, 1, 2, 3, \{4\}\}$. Fill in a correct symbol (from $\in$, $\subset$, $\subseteq$, $=$, $\neq$) for each of the following.

- $\{4\}$___A
- $\{2\}$___A
- $\{1, 2, 3\}$___A

S2 Given two sets and a universal set identify the union, intersection, complement, and set difference and find the power set of a given set.

S2-1 Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ be the universal set. Let $A = \{3, 4, 5, 6, 7\}$ and $B = \{1, 5, 7, 9\}$.

(a) Find $A \cap B$

(b) Find $A \cup B$

(c) Find A^C

(d) Find $A \setminus B$ (or $A - B$).

S2-2 Let $U = \mathbb{Z}$. Let $A = \{x \in \mathbb{Z} : x \geq 7\}$ and $B = \{x \in \mathbb{Z} : xisodd\}$. (Roster method is okay for your answers.)

(a) Find $A \cap B$

(b) Find $A \cup B$

(c) Find A^C

(d) Find $A \setminus B$ (or $A - B$).

S2-3 Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ be the universal set. Let $A = \{2, 4, 6, 8, 10\}$ and $B = \{1, 3\}$.

(a) Find $A \cap B$

(b) Find A^C

(c) Find $A - B$

(d) Find $A \cup B$.

S3 Correctly use function terminology such as domain, codomain, range, dependent variable, independent variable, image, and preimage.

S3-1 Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2 - 2x$.

- State the domain, codomain, and range of f. (Clearly state which one is which.)
- Find the image(s) of 3 under f.
- Find the preimage(s) of 0.

S3-2 Let $R^* = \{x \in \mathbb{R} : x \geq 0\}$. Let $s : \mathbb{R}^* \to \mathbb{R}^*$ be defined by $f(x) = x^2$.

- State the domain, codomain of f. (Clearly state which one is which.)
- Find the image(s) of 3 under f.
- Find the preimage(s) of 4.

S3-3 Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(m) = 3 - m$.

- State the domain, codomain, and range of f. (Clearly state which one is which.)
- Find the image(s) of 3 under f.
- Find the preimage(s) of 0.

S4 State the definition of injection, surjection, and bijection.

S4-1 Let A and B be sets. Carefully complete the definitions of the following terms:

- A function $f : A \to B$ is injective provided that...
- A function $f : A \to B$ is surjective provided that...
- A function $f : A \to B$ is bijective provided that...

S5 Prove or disprove that a given relation is reflexive, symmetric, and/or transitive.

1. For all $a, b \in \mathbb{Z}$ say $a \sim b$ if and only if $a \mid b$. Is $\sim$ an equivalence relation? Explain.
2. For all $a, b \in \mathbb{Z}$ say $a \sim b$ if and only if $a \leq b$. Is $\sim$ an equivalence relation? Explain.
3. For all $a, b \in \mathbb{R}$ say $a \sim b$ if and only if $|a - b| < 5$. Is $\sim$ an equivalence relation? Explain.

S6 State the definition of "a divides b" and "a is congruent to b modulo n" and correctly apply these definitions in examples.

1. Carefully state the definition of $a \mid b$ (for nonzero a) and $a \equiv b \pmod{n}$ (for nonzero n). Then give an example of integers a and b such that $a \cong b \pmod{15}$ and $b < 0$.
2. Carefully state the definition of $a \mid b$ (for nonzero a) and $a \equiv b \pmod{n}$ (for nonzero n). Then give an example of integers a and b such that $a \nmid b$.
3. Carefully state the definition of $a \mid b$ (for nonzero a) and $a \equiv b \pmod{n}$ (for nonzero n). Then give an example of integers a and b such that $a \not\equiv b \pmod{3}$.