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Explorations in Modern Mathematics

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Introduction

In January 2017, Dr. Francis E. Su, president of the Mathematical Association of America, gave his retiring presidential address, *Mathematics for Human Flourishing*[2][3]. In this address and the subsequent book [1], Su describes ways that the practice of mathematics can help human beings cultivate virtues which in turn enable us live the good life. He identifies several desires that encourage the cultivation of virtue, among them *exploration*, *play*, *truth*, *justice*, and *beauty*.

This book uses guided exploration to investigate mathematical topics that specifically meet those desires. We will explore the Rubik's cube and other puzzles and thus *play* with mathematics. We'll see how deductive reasoning can be used to argue for *truth*, while also exploring its limitations. *Justice* will be explored by application of mathematics to democracy and the networks used to exploit our fellow image-bearers. And we'll finish the semester with projects that explore *beauty* in mathematics. ReferencesReferencesg:references:idp105544866358288 F. E. Su, C. Jackson, *Mathematics for Human Flourishing.* Yale University Press, 2020.

[2] https://mathyawp.wordpress.com/2017/01/08/mathematics-for-human-flourishing/

[3] In 2018, he gave an expanded version of this talk as a First Mondays address at Dordt College.

A Note to Students

This work was compiled with some strong opinions.

The first is that, as with a sport, instrument, or nearly any other skill, you can't learn mathematics without doing mathematics. Thus, there are few worked-out examples in this book. We would rather spend a long time in productive struggle to understand an idea deeply than be spoon-fed solutions.

The second is that most people's ideas of what counts as a mathematical question are far too narrow. Mathematics is not merely geometry, algebra, and calculus (maybe with a dash of statistics for good measure). Mathematics is a way of thinking; it's about abstraction, deductive reasoning, and problem-solving. This way of thinking can be applied in surprising places, and can lead to delight and wonder at the world around us. We will explore some of these questions this semester.

A Note to Instructors

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Part I

Play

Chapter 1

The Cube

We begin our mathematical explorations in a place that may seem unusual. The Rubik's Cube, invented[1] in the 1970s by Hungarian architecture professor Erno Rubik, is the best-selling toy of all time. It is also a rich mathematical playscape, a tactile means of exploring and challenging our fundamental ideas around what counts as an arithmetic operation.

The Cube's colorful, playful nature also underscores our purpose in beginning with it. Specifically, exploring the Cube will help us develop some of the virtues Francis Su identifies in [1] under the desire of *play*: exploring the Cube will hopefuly pique your *curiosity* and build your *patience* and *perseverance*. Solving it will require you to *change perspectives* and build *confidence in struggle*. And, I hope, the satisfaction and joy you experience in finally solving it will engender an *openness of spirit* that we will carry into further explorations for the rest of the text.

As we begin, I also wish to acknowledge the efforts of the *Discovering the Art of Mathematics*[2] project, especially the Games and Puzzles[3] book. I learned the Rubik's Cube by reading and teaching their work, and it is impossible to overstate their influence on this chapter.

1.1 Introduction and Background

Motivating Questions.

In this section, we will explore the following questions.

1. What does it mean to play?
2. What are the components of the Cube?
3. What are some of the elementary mathematical properties of the Cube?

[1] Or, to use Rubik's word, *discovered*.
[2] `https://www.artofmathematics.org/`
[3] `https://www.artofmathematics.org/books/games-and-puzzles`

1.1.1 Thinking about Play

Exploration 1.1.1 First, introduce yourself to the members of your group. Then, consider the questions:

1. What are the essential qualities of play? That is, what makes one activity *play*, and another not?

2. What does it mean to be *playful* in your own major disciplines?

3. What does it mean to be an *explorer* in your major disciplines?

4. What are you excited about this semester? Your answer can be from one of your classes or outside of your classes.

Write your answers in your notes. We'll discuss your responses before working on Investigation 1.1.2.

The Dutch historian and cultural theorist Johan Huizinga identified four elements of play: it should be *voluntary*; it should be distinct from ordinary life, taking place within its circumscribed time and locality, since dubbed the "magic circle"; and it, like a game of chess or solve of a Rubik's cube, can be repeated.

It is the goal of this text to make mathematics playful, inasmuch as is possible. We recognize that you may be reading this for a class which was not voluntary, and you are constrained in your mathematical play by the requirements put upon you by your instructor. However, we encourage you to approach not only the solution of the Rubik's cube from a playful posture, but subsequent explorations as well.

1.1.2 Exploring the Cube

Investigation 1.1.2 In your groups, investigate your Cubes. What do you notice? What do you wonder? Make a list of as many observations and questions as you can, and write them in your notes.

Definition 1.1.3 The little cubes which make up the Cube are often called **cubies**. The cubies located at the corners are imaginatively called **corner cubies**, the cubies located at the centers **center cubies**, and the others **edge cubies**. ◊

Investigation 1.1.4 In this investigation we'll consider the corner cubies. Hold your Cube with the white center cubie facing up and the red center cubie facing you.

1. How many corners are there?

2. How many stickers does a corner cubie have?

3. Can you move a corner cubie to a position other than a corner? Explain your reasoning.

4. Identify all the positions on the Cube to which a corner cubie can be moved while keeping the white center cubie on top and red in front.

As always, make sure you can explain your answers.

Investigation 1.1.5 In this investigation we'll consider the edge cubies. Hold your Cube with the white center cubie facing up and the red center cubie facing you.

1. How many edge cubies are there?
2. How many stickers does an edge cubie have?
3. Can you move an edge cubie to a position other than an edge? Explain your reasoning.
4. Identify all the positions on the Cube to which an edge cubie can be moved while keeping the white center cubie on top and red in front.

As always, make sure you can explain your answers.

Investigation 1.1.6 In this investigation we'll consider the center cubies. Hold your Cube with the white center cubie facing up and the red center cubie facing you.

1. How many center cubies are there?
2. How many stickers does an center cubie have?
3. Can you move a center cubie to a position other than a center? Explain your reasoning.
4. Identify all the positions on the Cube to which a center cubie can be moved while keeping the white center cubie on top and red in front.

As always, make sure you can explain your answers.

Question.

How can you tell when a cubie is in the correct *location*? How can you tell when it is *oriented* correctly? Is there a difference?[1]

1.2 Exploring and Describing the Cube

Motivating Questions.

In this section, we will explore the following questions.

1. What are faces and layers, and what methods exist for solving them?
2. How can we describe a complex series of Cube moves?
3. What is the order of a Cube move?

In this section, we'll define a few terms and work on an initial exploration of the Cube's white layer. Then, we'll discuss the need for a precise method communicating about our Cubes, and introduce some standard notation.

1.2.1 Faces and Layers

We begin with a definition.

[1] Does it depend on the cubie?

Definition 1.2.1 A **face** of the Cube refers to one of its sides. We say a face is **solved** if all the stickers on that side are the same color. ◊

There are many ways to solve the Cube, and we'll explore one approach later in this chapter. A crucial step in learning to solve the Cube is learning to solve one face.

Challenge.

Scramble your Cube. Then solve the white face. If you have not done this before (or even if you have), this may take hours or days, not minutes! Avoid relying on any outside resources (including websites, friends, etc.). When you solve it, congratulations! Then scramble the face and solve it again. Repeat this process until you can reliably do it in just a few minutes. Once you can reliably solve the white face, describe, in writing, your methods as clearly and precisely as you can. What challenges did you have to overcome? How did you overcome them?

Congratulations on solving a face of your Cube! But a question comes to mind. When your white face is in its solved state, are all the cubies on the white face in the correct location?

Definition 1.2.2 A **layer** of the Cube consists of a face and all of the stickers on Cubies which compose the face. A layer is **solved** if all of the Cubies in the layer are in the correct location with the correct orientation. ◊

Challenge.

Scramble your Cube and then solve the white layer. As before, this may take hours or days. When you solve it, congratulations! Scramble it and do it again. Once you can reliably solve the white layer, describe, in writing, your methods as clearly and precisely as you can.

1.2.2 The Need for Notation

Discussion 1.2.3 *Find a partner. Take turns describing verbally and without pointing to your Cube your method for moving a corner cubie to its proper position. Similarly, verbally describe your method for moving an edge cubie to its proper position.*

Discussion 1.2.4 *As you probably noticed, it is challenging to orally describe complex/technical ideas in much detail without becoming lost. Is it easier to describe your cubie movements in writing? Why or why not?*

Every area of inquiry has an associated collection of terminology and shorthand notation. At its best, this terminology enables efficient and clear communication. However, terminology can often be a barrier. Thus, we want to avoid introducing unnecessary or confusing jargon whenever possible. However, as I hope Discussion 1.2.3 and Discussion 1.2.4 underscore, a lack of terminology or shorthand notation severely impairs our ability to communicate deep technical ideas. It is with this background that we introduce the following notation, which has become standard in the Cubing community.

Definition 1.2.5 Hold the Cube in such a way that you are looking at one of the faces.

- The face you are looking at is referred to as the **front** (F) face.

- The face on the side opposite the front is referred to as the **back** (B) face.
- The face on the right side is referred to as the **right** (R) face.
- The face opposite the right is the **left** (L) face.
- The face on the top of the Cube is the **up** (U) face.
- The face on the bottom of the Cube is the **down** (D) face.

A graphical version appears in Figure 1.2.6.

Figure 1.2.6

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Activity 1.2.7 List the colors of each of the six faces in Figure 1.2.6.

As we we will see momentarily, the names described in Definition 1.2.5 not only help us refer to faces, but also moves of the Cube which help us solve it. In order to understand what a given move does to the Cube, we will need to refer to certain cubies by location. The following definition enables this.

Definition 1.2.8 A cubie is named by the face(s) on which it sits, using lowercase letters. ◊

Activity 1.2.9 Consider the scrambled Cube in Figure 1.2.10.

1. What colors are the $\ell f u$ cubie?
2. Give the colors of the br cubie.
3. How can you tell whether the cubie in the named location is a corner or edge cubie?
4. How many letters are required to name a center cubie?
5. Where is the edge cubie that is blue and orange?

Figure 1.2.10 A scrambled cube.

We are most interested in using our new notation to describe *moves* of the various faces. We thus make the following definition.

Definition 1.2.11 A given face name, e.g., R, describes a 90° clockwise turn of that face, *as you look at the face.* A given face name with prime symbol, e.g., R', denotes a 90° counterclockwise turn of that face, *as you look at the face.* A sequence of face moves is written multiplicatively, left to right. We may use parentheses and exponents to write our moves more compactly. ◊

Exploration 1.2.12 Consider the following questions in the context of a completely solved cube as pictured in Figure 1.2.13.

1. After performing the move R, what color(s) will be on the Up face?
2. Again starting from a solved cube, perform the move L. What color(s) are on the Up face?
3. Explain the difference in your answers to the first two questions.
4. How are F^3 and F' related?
5. Consider the Cube move $RF^2U(LDR)^3$. Which face will you turn first: the left face, or the front face?

6. Again consider the Cube move $RF^2U(LDR)^3$. How many times in this move will we turn the right face?

7. What would it mean for two cube moves to be **equal**? Test your definition by determining whether $LD = DL$.

8. Suppose X and Y are two cube moves. When, if ever, is $XY = YX$?

Figure 1.2.13 A solved cube.

1.2.3 Order

A useful algebraic concept for describing Cube moves is that of *order*.

Definition 1.2.14 The **order** of a cube move M is the least number of times $n > 1$ that M can be repeated before a solved Cube is solved again. We write $|M| = n$. ◊

Example 1.2.15 The order of R is 4, since 4 clockwise quarter-turns of the right face of a solved Cube results in a solved Cube, and no fewer number of turns results in a solved Cube. □

Exploration 1.2.16

1. What is $|U^2|$?

2. Calculate $|R^3|$. Why does this make sense?

3. Calculate $|RUR'U'|$.

Conclusion. In this section, we began a systematic exploration of the properties of the Cube. First, we described the need for a notational shorthand in solving the white face and layer.

The standard notation refers to each face *as you look at it*, ignoring color, and describes a 90° clockwise turn of the face. So, FR means to first turn the front face 90° clockwise, then the right face 90° clockwise. We then noted that we can refer to a cubie by describing the face(s) on which it sits (using lowercase letters to distinguish cubies from faces): three letters refer to a corner, as it sits at the intersection of three faces; two letters refer to an edge, and one to a center.

We concluded by introducing the idea of order, which will help us analyze and understand more complex sequences of Cube moves in the next section.

1.3 Moving Cubies

Motivating Questions.

In this section, we will explore the following questions.

1. How can we move the corner cubies so they are in the correct location?

2. How can we describe the movement of the middle "slice" of our Cube?

3. How can we move the edge cubies so they are in the correct location?

Our goal in this chapter is to explore, and eventually solve, the Cube. To do this, every cubie must be in the correct *location* with the correct *orientation*. In this section, we focus on two moves which will allow us to put the cubies in the correct location. Once they are in the correct location, we'll see moves in the next section which will help us orient them correctly, enabling us to (finally) solve the Cube.

We begin by considering Cube Move 1.3.1.

Cube Move 1.3.1 Moving Corners. *Consider the following sequence of Cube moves.*

$$U'R'D'RUR'DR$$

Exploration 1.3.2 The Cube move described in Cube Move 1.3.1 moves some of the corners on the front face of the Cube.

1. By performing this move several times, identify on the blank Cube below what is happening to the front face of the Cube.
2. Using the cubie notation described in Definition 1.2.8, describe what happens to the front face of the Cube after performing this move.
3. Given what happens to the front face, exercise your human creativity and suggest a short name/abbreviation for this move.
4. What is the order of the move?
5. Practice the move until you can reliably execute it.

Figure 1.3.3 A blank cube.

Activity 1.3.4 Now that you have identified exactly what Cube Move 1.3.1 does, use it to solve as much of your Cube it enables.

In order to move the edges, it will be helpful to add one more type of fundamental Cube move to our repertoire.

Definition 1.3.5 By S_R we mean a clockwise rotation of the (vertical) middle slice *as we look at the right face.*

$$\xrightarrow{S_R}$$

$\Diamond$

Cube Move 1.3.6 Moving Edges. *Consider the following sequence of Cube moves.*

$$S_R^2U'S_R'U^2S_RU'S_R^2$$

Exploration 1.3.7 The Cube move described in Cube Move 1.3.6 moves some of the edges on the up face of the Cube.

1. By performing this move several times, identify on the blank Cube below what is happening to the up face of the Cube.
2. Using the cubie notation described in Definition 1.2.8, describe what is happens to the up face of the Cube after performing this move.
3. Given what happens to the up face, exercise your human creativity and suggest a short name/abbreviation for this move.
4. What is the order of the move?
5. Practice the move until you can reliably execute it.

Figure 1.3.8 A blank cube.

Activity 1.3.9 Now that you have identified exactly what Cube Move 1.3.1 and Cube Move 1.3.6 do, use them to put every cubie on your Cube in its correct location.

Conclusion. In this section, we focused on two moves that enable us to put our cubies in the correct location. We first found Cube Move 1.3.1, which affects corners on the front face, and no other cubies. Similarly, Cube Move 1.3.6 affects edges on the up face, and no other cubies. We were then able to put every cubie on our Cube in the correct location. Hurray!

Exercises

1. Consider the state of Lila's Cube in Figure 1.3.10. She is so close to solving it! Can you help her finish?

Figure 1.3.10

2. Sam is nearly done with his Cube; it's pictured in Figure 1.3.11. Can you help him finish?

Figure 1.3.11

3. Thanks to your help, Lila solved her Cube in Exercise 1.3.1, but now she needs more help! Can you help her finish the Cube pictured in Figure 1.3.12?

Figure 1.3.12

4. Consider the situation in Figure 1.3.13. Why will Cube Move 1.3.1 alone be insufficient for putting the corners in their correct locations? Devise a strategy for putting the corners in the correct locations.

Figure 1.3.13

1.4 Reorienting Cubies

Motivating Questions.

In this section, we will explore the following questions.

1. How can we change the orientation of the corner cubies?
2. How can we change the orientation of the edge cubies?
3. How can we apply our four Cube moves to solve the Cube?

In Section 1.3, we considered ways of moving certain edge and corner cubies. The key that allows us to use Cube Move 1.3.1 and Cube Move 1.3.6 to put your cubies in the correct location is that each move affects precisely three cubies. All others are left unmoved. By using just Cube Move 1.3.1 and Cube Move 1.3.6, we can thus put each cubie in the correct location. However, even if a cubie is in the correct location, it may be that its stickers are on the wrong faces—in this case, it needs to be *reoriented*.

In this section, we will see how to reorient cubies once they are in the correct location. To do so, we'll learn two moves: one that reorients corners, and one that reorients edges. Once we are able to put each cubie in the correct location, and then orient it correctly, each scrambled Cube becomes a puzzle, solvable by clever application of these moves.

We begin by exploring a move that reorients two corners.

Cube Move 1.4.1 *Consider the following sequence of Cube moves.*

$$(R'D^2RB'U^2B)^2$$

Exploration 1.4.2 Reorienting Corners. The Cube move described in Cube Move 1.4.1 reorients two corners: one on the Up face, and one on the Down face.

1. By performing this move several times, identify on the blank Cube below what is happening to the Up face of the Cube. Describe what is happening to the Down face as well.
2. Using the cubie notation described in Definition 1.2.8, describe what is happens to the Up and Down faces of the Cube after performing this move.
3. Given what this move does, exercise your human creativity and suggest a short name/abbreviation for this move.
4. What is the order of the move?
5. Practice the move until you can reliably execute it.

Figure 1.4.3 A blank cube.

Our last Cube move will reorient two edge cubies. Recall Definition 1.3.5.

Cube Move 1.4.4 *Consider the following sequence of Cube moves.*

$$(S_RU)^3U(S_R'U)^3U$$

Exploration 1.4.5 Reorienting Edges. The Cube move described in Cube Move 1.4.1 reorients two edges on the Up face.

1. By performing this move several times, identify on the blank Cube below what is happening to the Up face of the Cube.
2. Using the cubie notation described in Definition 1.2.8, describe what is happens to the Up face of the Cube after performing this move.
3. Given what this move does, exercise your human creativity and suggest a short name/abbreviation for this move.
4. What is the order of the move?
5. Practice the move until you can reliably execute it.

Figure 1.4.6 A blank cube.

Solving the Cube.

When you can consistently perform Cube Move 1.3.1, Cube Move 1.4.1, Cube Move 1.3.6, and Cube Move 1.4.4, you can use them to solve the Cube. You can use any strategy you want, but here is one to consider.

1. Put the corners in the correct *location*.

2. Put the edges in the correct *location*.
3. *Reorient* any corners that need reorienting.
4. *Reorient* any edges that need reorienting.
5. *Celebrate!*
6. Scramble the Cube and do it again.

Conclusion. In this section, we explored the last moves we need to solve the Cube. We also described a strategy for solving the Cube. Our method of solving the Cube is based on the *corners-first* (CF) method, which is distinct from the layer-by-layer (LL) method which is often the first solving method people learn. The first solution to the Cube, by Erno Rubik himself, was corners-first, and the first world speed-cubing record (22.95 seconds) was done corners-first.

There are advantages and disadvantages to all solution methods. Advantages to corners-first are:

- The move sequences are generally shorter in CF
- There are just a few important move sequences to memorize
- You generally don't "break" your existing work and so can recover from mistakes more easily
- The solution can scale from a leisurely solve of a few minutes or more to quite fast solves

Exercises

1. Sam is nearly done with his Cube! Which of the Cube moves from this section does he need? How many times will he have to perform it?

Figure 1.4.7

2. Lila is nearly done with her Cube! Which move from this section will be helpful? How should she use it?

Figure 1.4.8

3. Sam has gotten himself into a bit of a pickle. How can he apply our moves to solve his Cube?

Figure 1.4.9

4. This configuration is known as the **superflip**. Describe what you see. Can you achieve this on your Cube?

Figure 1.4.10

Part II

Truth

As Su identifies in [1], humans have an innate desire for truth. Defining just what we mean by **truth** is tricky, so we'll follow Su's lead and just say that "true statements are ones that align with reality". In an age of rampant disinformation, we seek to know what is actually true.

Mathematics is often thought of as one of the last bastions of objective truth. One widely accepted cultural norm is that if a statement is quantitative, or can be arrived at via a sequence of logical deductions, it is often considered "true", whereas statements that cannot be presented in such a way live in the realm of opinion. Yet, mathematics is a human endeavor; and what do mathematicians mean when they talk of a statement being "true", anyway? The answer might surprise you.

In this unit, we'll explore inductive and deductive reasoning, as well as formal logic. We'll explore the shortcomings of mathematical thinking in addition to its strengths.

This chapter is heavily influenced by *Discovering the Art of Mathematics: Truth, Reasoning, Certainty, and Proof*[1] by Fleron, Hotchkiss, Ecke, and von Renesse.

[1] https://www.artofmathematics.org/books/truth-reasoning-certainty-and-proof

Chapter 2

Inductive and Deductive Reasoning

In order to more fully explore the deductive reasoning employed in mathematical explorations, we'll first contrast it with the inductive reasoning common in other areas of inquiry.

WRITE MORE HERE

2.1 Inductive Reasoning

Motivating Questions.

In this section, we will explore the following questions.

1. What is truth? What is fact? Is there a difference?
2. What is inductive reasoning? When is it helpful? What are its shortcomings?

We'll begin by attempting to define *truth* and *fact*, and compare them. We'll then explore *inductive reasoning* in depth, considering its benefits shortcomings.

Discussion 2.1.1 *What is truth? What is fact? How are the two concepts related? How are they different? Under which category does the sentence* $1 + 1 = 2$ *fall?*

Activity 2.1.2 What's my world? In the game *What's my World?*, one person thinks of a single law that defines a hypothetical world (e.g., "My world does not contain things that start with the letter 'C'."). The other players attempt to guess this governing law by taking turns asking questions of the creator, such as "Does your world contain dogs? Does it contain cats?" and so on. When one of the guessers believe they can identify the governing law, they ask the creator.

Play a few rounds of this game, taking turns in the role of creator. *Other than asking the creator directly*, how could you be *certain* that you had determined the law of the world?

Definition 2.1.3 Inductive reasoning is the process of drawing general conclusions from particular instances, generally known as **data**. ◊

The work of the guesser in Activity 2.1.2 is to employ inductive reasoning to determine the general law put in place by the creator. This is the scientific process at its best: looking at the world in its orderliness and using our curiosity and creativity to infer larger governing principles. If you have taken a statistics course, this is the type of reasoning employed there.

But how does this work in mathematics?

Exploration 2.1.4 Let's play a game with dots and lines. We'll start with at least two dots (though you'll probably want to increase this number pretty quickly). The rules are:

- Split your dots into two groups, group A and group B, and draw each group on its own line.
- Connect (some of) the dots from A to (some of) the dots in B with lines. The lines don't have to be straight—they can curve in any way you want!—but each line should connect precisely two dots: one from A and one from B.
- Each dot should be connected to at least one other dot—no lonely dots!

So, if I label four dots as X, Y, Z, W, *one possible* drawing is given in Figure 2.1.5.

Figure 2.1.5

However, there is a problem with this drawing: the lines cross! I know, this wasn't one of the rules above, but let's add it.

- The lines can be drawn so that none of them cross.

Now consider the following questions.

1. Redraw the picture in Figure 2.1.5 so that none of the lines cross.
2. Give a name to drawings of figures like Figure 2.1.5 which *can* be drawn so that none of the lines cross.
3. Which drawings are possible with two or three dots? Can they be drawn to have the property that none of the lines cross?
4. What other drawings are possible with four dots? Five?
5. Based on your work here, do you think it will always be possible to draw these pictures so that none of the lines cross? Explain your thinking.

Discussion 2.1.6 *Based on our work in this section, what are some strengths of inductive reasoning? What are some possible pitfalls? How can we minimize these potential downsides?*

Exercises

1. Do some internet research on the **Twin Prime Conjecture**. What is it? When was it first formulated? Is it true? Likely true?

2.2 Deductive Reasoning

Motivating Questions.

In this section, we will explore the following questions.

1. What is deductive reasoning? How does it differ from inductive reasoning?
2. How is deductive reasoning employed in mathematics?
3. What are some strengths and weaknesses of deductive reasoning?

Formal mathematical reasoning is deductive (defined momentarily), and begins with **axioms**, which are statements that should be self-evident and taken to be true. Note that while axioms are not always explicitly stated, they *can* be when necessary.

Investigation 2.2.1 The most famous set of axioms are Euclid's postulates for geometry, defined in *The Elements*[1], which not only shaped thousands of years of geometry, but solidified the deductive approach to doing and explaining mathematics that we will explore in this unit. At the beginning of Book I of *The Elements*, Euclid identified five postulates and five axioms.

Euclid's *postulates* are:

1. A straight line segment can be drawn joining any two points.
2. Any straight line segment can be extended indefinitely in a straight line.
3. Any circle can be described given a center and (radial) distance.
4. All right angles are equal to one another.
5. If a straight line intersecting two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on the side on which the angles are less than two right angles.

Euclid's *axioms* (or *common notions*) are:

1. Things which are equal to the same thing are also equal to one another.
2. If equals are added to equals, the wholes are equal.
3. If equals are subtracted from equals, the remainders are equal.
4. Things which coincide with one another are equal to one another.
5. The whole is greater than the part.

The desired qualities of a system of axioms are:

1. **consistency**: we cannot deduce contradictory propositions, such as "God exists" and "God does not exist" from the same set of axioms
2. **simplicity**: we have as few axioms as possible, and they are no more complicated than they need to be
3. **completeness**: the system can answer every question we can think to ask

In your groups, discuss Euclid's postulates and common notions, perhaps in view of the desired qualities of an axiomatic system. What strikes you as being interesting or noteworthy? Make a list of at least 2-3 observations. Then consider: on what axioms or assumptions do you make decisions (e.g., about how to spend your time, resources, etc)?

The process of **deductive reasoning** in mathematics begins from a set of generally agreed-upon axioms of set theory[2][3] and uses logic to make inevitable conclusions from those axioms. These conclusions are generally called **theorems**. They are usually given as conditional statements of the form "If P, then Q," where P and Q are sensible

statements. Moreover, since most deductive statements are presented in conditional form, their scope is generally limited. That is, the statement "if it is Monday, then we have math class" is only making a claim about what happens on Mondays; it says nothing whatsoever about any other day of the week. We will explore the consequences of this more in Chapter 3.

The author Lewis Carroll loved logic puzzles (he was actually a mathematics professor!), and wrote many of them. Here is one, axiomatized for easy reference.

Axiom 2.2.2 (Carroll). *Consider the axioms:*

1. *If a kitten loves fish, then it is teachable.*
2. *Every kitten without a tail will play with a gorilla.*
3. *All kittens with whiskers love fish.*
4. *If a kitten has green eyes, then it is not teachable.*
5. *All kittens have whiskers or do not have tails.*

Once you have a deductive argument that (generally) begins from your premises and reasons, step-by-step, to your conclusion, you can write out the argument in a short essay known as a *proof*. For our purposes, a proof is just a *convicing argument*. It should be written at a level appropriate to the reader and clearly lay out the steps necessary for a reader who accepts your hypotheses to believe the conclusion. As an example, consider the following.

Example 2.2.3 All kittens with whiskers are teachable.

Proof. Suppose we have a kitten with whiskers. Let's call him Arthur. By Axiom 2.2.2 #3, Arthur loves fish. Since Arthur loves fish, Axiom 2.2.2 #1 implies that Arthur is teachable.

Since Arthur was an arbitrary kitten, we conclude that all kittens with whiskers are teachable. □

There are several observations which are worth a moment of our time in the proof in Example 2.2.3.

- We first note that the proof is written using standard conventions of academic writing, including complete sentences, proper punctuation and capitalization, etc. This is important! In order to convince someone that your argument is valid, they need to be able to read it.
- While the statement to be proved is not written as "if P, then Q", it *can* be stated that way: "If A is a kitten with whiskers, then A is teachable". Thus, our proof begins by considering an arbitrary kitten with whiskers,

[1] https://en.wikipedia.org/wiki/Euclid's_Elements

[2] https://en.wikipedia.org/wiki/Zermelo\T1\textendashFraenkel_set_theory

[3] It should be noted that there is disagreement about the Axiom of Choice, mostly due to some of its surprising consequences.

who we name Arthur. We observe, however, that there is nothing special about Arthur that figures into our proof in a meaningful way, so the argument will apply just as well to any kitten with whiskers we may find.

- In each step we take throughout the proof, we refer to the specific axiom from Axiom 2.2.2 that allow us to take that step. It is valuable to be able to do this, but generally we do not specifically refer to the axioms by number. This is to improve the readability of the proof.

- Finally, note that our proof concludes with a conclusion: all kittens with whiskers are teachable. This is good practice and sends an unmistakable signal to the reader that you are done.

Now, prove the theorems that follow using Axiom 2.2.2.

Theorem 2.2.4 *If a kitten has green eyes, then it does not love fish.*

Theorem 2.2.5 *If a kitten has a tail, then it does not have green eyes.*

Theorem 2.2.6 *Every kitten with green eyes will play with a gorilla.*

Activity 2.2.7 Compare and contrast the structures of the proofs of the preceding theorems. Can you clarify the general reasoning patterns you used to prove them?

Discussion 2.2.8 *We have now explored both inductive and deductive reasoning. How are they similar? How are they different? How might you decide which type of reasoning to employ in a given situation? What are their strengths and weaknesses?*

Conclusion. In this section, we explored deductive reasoning, which begins from accepted axioms and premises and then reasons, step by logical step, toward a conclusion. This is the primary form of reasoning used in mathematics. We saw that while conclusions reached via deductive reasoning are generally tighter and more certain, there are still some drawbacks.

The main drawback of deductive reasoning involves scope. We must begin with axioms, so the axioms must be well-chosen and sensible. However, if one disagrees with the choice of a set of axioms, then one must be willing to set aside any results deduced from them (or, at least, deduced from the particular axioms with which one disagrees).

A second drawback having to do with scope concerns the premises of a conditional statement. In particular, if the premises of a statement are not satisfied, the statement makes no assertion whatsoever (though, as we will see in Chapter 3, there is still a *consistent* way to assign truth values to statements whose premises are not satisfied).

Exercises

1. Invent one or two additional theorems that can be deduced from Axiom 2.2.2. Prove them.

Chapter 3

Modern Mathematical Logic

In Chapter 2, we explored two types of reasoning: inductive reasoning, and deductive reasoning. The type of reasoning used most often in mathematics is deductive reasoning. In the late 19th/early 20th century, logic itself was mathematized by the likes of George Boole and Augustus De Morgan into a *propositional calculus*. This gave mathematicians a means of determining the truth value of a statement purely based on its inherent logical structure.

It is this method of calculating that we explore in this chapter.

3.1 Logical Connectives and Rules of Inference

Motivating Questions.

In this section, we will explore the following questions.

1. What is a proposition?
2. What are logical connectives? How can they be used to build new propositions?
3. What does it mean for two propositions to be logically equivalent?

In the late 19th and early 20th centuries, mathematicians began mathematize and formalize logic itself. Today we begin to explore these foundational issues. We'll start with some definitions.

Definition 3.1.1 A **proposition** is a declarative sentence which is either true or false, but not both. An **elementary proposition** is a sentence with a subject and a verb, but no connectives (such as *and, or, not, if-then,* or *if-and-only-if*). ♢

Activity 3.1.2 Determine which of the following are propositions (elementary or otherwise). If a given sentence is a proposition, determine its truth value. If it isn't, explain why not.

1. Erik Hoekstra is the president of Dordt University.
2. The square root of a whole number is always a whole number.
3. The Green Bay Packers are the worst football team.
4. Why is this class so much fun?
5. This sentence is false.

6. A group of crows is called a murder.

7. Everyone likes cats.

Now that we have a sense for what a proposition is, we'll take old propositions and make new ones using *logical connectives*. In order to describe how the connectives work, mathematicians define the truth values of the new propositions *formally*—that is, without regard to the content of the propositions themselves—in terms of the possible combinations of truth values from the constituent propositions. This gives us an abstract way of considering the truth values of propositions.

Definition 3.1.3 Suppose P and Q are statements (e.g., like those in Activity 3.1.2). The **negation** of P, denoted $\neg P$ and read "not P", has the opposite truth value of P and is defined by Table 3.1.4.

Table 3.1.4 The negation of P.

P	$\neg P$
T	F
F	T

◊

Definition 3.1.5 The **conjunction** of P and Q, denoted $P \wedge Q$ and read "P and Q", is true when both P and Q are true, and false otherwise. See Table 3.1.6.

Table 3.1.6 The conjunction of P and Q.

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

◊

Definition 3.1.7 The **disjunction** of P and Q, denoted $P \vee Q$ and read "P or Q", is true when P is true, Q is true, or both are true, and false otherwise. See Table 3.1.8.

Table 3.1.8 The disjunction of P and Q.

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

◊

Activity 3.1.9 Meaningfully negate the following propositions (without just saying "It is not the case that...").

1. e is a negative real number.

2. Iowa is the tenth largest state.

3. 17 is a prime number.

1. e is a nonnegative real number.

2. Iowa is not the tenth largest state.

3. 17 is a composite number.

Exploration 3.1.10 Determine the truth values of the following propositions.

1. Math 149 meets on Mondays and the capital of Iowa is Des Moines.

2. Math 149 meets on Mondays and the capital of Minnesota is Minneapolis.

3. Math 149 meets on Mondays or the capital of Minnesota is Minneapolis.

The last connective we'll consider (for now) is *implication*.

Definition 3.1.11 Let P and Q be statements. The **implication**, "P implies Q" (or "if P, then Q") is denoted $P \Rightarrow Q$, and is false only when P is true but Q is false. See Table 3.1.12.

Table 3.1.12 The implication $P \Rightarrow Q$.

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

◊

Exploration 3.1.13 Determine the truth values of the following statements. Identify which row of Table 3.1.12 you are in.

1. If x is a negative real number, then $-5x$ is a positive real number.

2. If we have Math 149 today, then it is Wednesday.

3. If $9 > 5$, then dogs do not have wings.

4. If $2 = 4$, then dogs do have wings.

The formalization of mathematical logic ramps up a bit when we consider conditional statements. It is important to remember that we define the truth value of the proposition $P \Rightarrow Q$ *purely formally* based on the structure of the conditional statement and the truth values of the constituents P and Q. There need not be a causal relationship between P and Q!

The last two rows of Table 3.1.12 are also worth a moment of our time. They state that if the statement P[1] is false, then the *implication* $P \Rightarrow Q$ is *true*. Note that this is different than saying that Q[2] is true. When the implication $P \Rightarrow Q$ is true because P is false, we usually say that $P \Rightarrow Q$ is **vacuously true**.

Activity 3.1.14 Suppose Dr. Janssen promises that, if everyone has solved their Rubik's Cube by Friday, then he will bring Defender sandwiches (on pretzel buns, with everything, as God intended) to class[3]. Unfortunately, a few students do not solve the Cube by Friday, so Dr. Janssen does not bring

[1] Often called the *antecedent*.
[2] Often called the *consequent*.

Defenders.

Decide the truth value of the implication "If everyone can solve their Cubes by Friday, Dr. Janssen will bring Defenders to class". How does this help you think about vacuous truth?

An important tool in our logical toolkit is one you likely employed in the theorems you deduced from Axiom 2.2.2.

Exploration 3.1.15 Let P and Q be statements. The **contrapositive** of the implication $P \Rightarrow Q$ is the implication $(\neg Q) \Rightarrow (\neg P)$. Complete Table 3.1.16. Is the contrapositive equivalent to anything we've looked at thus far?

Table 3.1.16 The contrapositive of $P \Rightarrow Q$.

P	Q	$\neg P$	$\neg Q$	$(\neg Q) \Rightarrow (\neg P)$
T	T			
T	F			
F	T			
F	F			

Activity 3.1.17 Write the contrapositives of the following statements. Be ready to explain why the contrapositives are equivalent to the original implications.

1. If a kitten loves fish, then it is teachable.
2. If a kitten does not have a tail, then it will play with a gorilla.
3. If a kitten has green eyes, then it is not teachable.
4. If today is Wednesday, then we have math class.

We have now explored several ways of combining existing propositions into larger propositions using logical connectives. When we use logic to write proofs, we also employ tools known as *rules of inference*. They clearly describe what steps we are allowed to take. There are many such rules; we will highlight two.

The way in which reasoning with implications is often done uses a rule of inference known as **modus ponens** which runs roughly:

- If P, then Q.
- P,
- Therefore, Q

A closely related rule of inference is known as **modus tollens**, and runs thusly:

- If P, then Q.
- Not Q,
- Therefore, not P.

Exercises

1.

2.

[3]This is purely hypothetical.

3.2 Formal Systems and Incompleteness

Motivating Questions.

In this section, we will explore the following questions.

1. What are formal systems?
2. What was Hilbert's goal? How was it resolved?
3. How did Cantor describe infinite sets?

3.2.1 Formal Systems

We have seen thus far a way of formalizing logic so that we can think about the truth of a statement purely syntactically (structurally) without regard for the semantic meaning of the statements under consideration. In the late 19th century, mathematicians developed what became known as **formal systems**, consisting of axioms, which were strings of symbols, such as

$$\exists B \forall C, (C \in B \Leftrightarrow C \subseteq A),$$

along with a **logical calculus**, which govern the rules of inference that can be used on the axioms to deduce new theorems.

Further, mathematicians had long assumed there were consistent foundational axioms for their discipline. Newly discovered paradoxes challenged this view.

Exploration 3.2.1 Russell's Paradox. In a certain town lives a barber who only cuts the hair of *all* people who do not cut their own hair. Who cuts the barber's hair?

Two main schools of mathematical philosophy sprung up in the wake of these discoveries. The **formalists** argued that statements of mathematics and logic are really just about the rules and consequences for manipulating symbols and strings of letters. That is, mathematics does not have a subject matter at all--just empty symbols, which may be given an interpretation in particular situations and thus have meaning.

The response came from the **intuitionists**. Intuitionism is an approach that considers mathematics to be purely the result of the constructive mental activity of humans rather than the discovery of any principles which we can reasonably claim exist in an objective reality. Thus, in some sense, mathematics is up to whoever is doing the mathematics. To the intuitionists, the formalists were reducing mathematics to a meaningless game with symbols.

Discussion 3.2.2 *Do you resonate more with a formalist approach to mathematics, or an intuitionist approach? Relatedly: do you believe mathematics is something humans invent, or something we discover in the world around us? Why?*

In 1900, at the second International Congress of Mathematicians in Paris, the esteemed mathematician David Hilbert posed 23 theretofore unanswered problems in mathematics that he thought were important to guide the development of mathematics in the 20th century. We've already seen one of them (the 8th): the Goldbach conjecture. Most of Hilbert's problems have been solved, but three are unresolved, two are thought to be too vague to ever get consensus on what a solution would look like, and one is the subject of much debate.

In an attempt to resolve the issues raised by paradoxes like Russell's Paradox, Hilbert posed this problem, the second on the list:

> But above all I wish to designate the following as the most important among the numerous questions which can be asked with regard to the axioms: To prove that they are not contradictory, that is, that a definite number of logical steps based upon them can never lead to contradictory results. . . .

That is, Hilbert wanted mathematicians to prove that the axioms on which mathematics was founded did not lead to a contradiction. The resolution to this problem is surprising, and to begin to explore it, we will turn to the infinite.

3.2.2 Infinity and Incompleteness

In Subsection 3.2.1, we learned about the push from mathematicians in the late 19th/early 20th century were trying to show that the axioms, the very foundations on which mathematics was built, were both *complete* (the truth of every sensible statement could be decided via deductions from the axioms) and *consistent* (one could never deduce contradictory statements from the axioms).

For reasons which are hopefully clear, we'll assume that the axioms are consistent, that is, no contradictions will arise from them. (If contradictions *can* arise, we are in trouble indeed.) But if the axioms are consistent, can it be shown that they're complete? To answer this question, we dive into the realm of infinity.

Definition 3.2.3 Let S and T be **sets**, which we may think of as collections of objects. We say that S and T have the same **cardinality** if there is a one-to-one correspondence between the objects of S and those of T, i.e., if each element of S can be paired up with one and only one unique element of T. In this case, we write $|S| = |T|$. $\Diamond$

Activity 3.2.4 Let $S = \{1, 2, 3, 4\}$, $T = \{\Box, \circ, \bigstar, \triangle\}$, and $U = \{\Diamond, \spadesuit, \clubsuit\}$.

1. Can you find a one-to-one correspondence between S and T? Describe it, or explain why none exist.

2. Can you find a one-to-one correspondence between S and U? Describe it, or explain why none exist.

3. Can you find a one-to-one correspondence between T and U? Describe it, or explain why none exist.

In order to explore our undecidable statement, we need to set some notation.

Definition 3.2.5 We define the following sets of numbers.

1. The **natural numbers** are given by $\mathbb{N} = \{1, 2, 3, \ldots\}$.

2. The **whole numbers** are given by $\mathbb{W} = \{0, 1, 2, 3, \ldots\}$.

3. The **integers** are given by $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.

4. The **even integers** are given by $2\mathbb{Z} = \{\ldots, -6, -4, -2, 0, 2, 4, 6, \ldots\}$.

5. The set of **rational numbers** is denoted by $\mathbb{Q}$ and consists of all fractions $\frac{a}{b}$, where a, b are integers and $b \neq 0$.

6. The set of **real numbers** is denoted by $\mathbb{R}$ and is given by all positive and negative infinite decimals (alternatively, every point on the number line).

$\Diamond$

Activity 3.2.6 For the numbers that follow, identify all sets described in Definition 3.2.5 they live in.

1. 7
2. -2
3. π
4. $\frac{4}{3}$
5. 0

We now look for some one-to-one correspondences.

Exploration 3.2.7 For the following pairs of sets, determine whether a one-to-one correspondence between the two sets exists. If it does, describe it. If it does not, give a justification.

1. $\mathbb{N}$ and $\mathbb{W}$
2. $\mathbb{W}$ and $\mathbb{Z}$
3. $\mathbb{Z}$ and $2\mathbb{Z}$
4. $\mathbb{N}$ and $2\mathbb{Z}$
5. (Challenge) $\mathbb{N}$ and $\mathbb{Q}$

Exploration 3.2.8 Let's explore the relationship between the cardinalities of $\mathbb{N}$ and $\mathbb{R}$ by considering the interval $[0,1]$ consisting of all real numbers (points on the number line) x satisfying $0 \leqslant x \leqslant 1$. We will use the notation $0.a_1a_2a_3\ldots$ to denote the infinite decimal with tenths place value a_1, hundredths a_2, and so on.

1. What is the relationship between $|\mathbb{R}|$ and $|[0,1]|$?
2. Suppose we have a one-to-one correspondence between $\mathbb{N}$ and $[0,1]$. Explain why this means that we can write

$$\begin{gathered} 1 \leftrightarrow 0.d_{11}d_{12}d_{13}\ldots \\ 2 \leftrightarrow 0.d_{21}d_{22}d_{23}\ldots \\ 3 \leftrightarrow 0.d_{31}d_{32}d_{33}\ldots \\ \vdots \end{gathered}$$

   where d_{ij} is the jth decimal digit of the ith number on the list.
3. Define a real number $M = 0.e_1e_2e_3\ldots$ where $e_j = 2$ if $d_{jj} \geqslant 5$ and $e_j = 7$ if $d_{jj} < 5$. Suppose the first three numbers on the list above are

$$\begin{gathered} 0.4548430426\ldots \\ 0.4607677961\ldots \\ 0.4702962689\ldots \end{gathered}$$

   What is M in this case?
4. In general, is it true that $0 \leqslant M \leqslant 1$?
5. Where is M on the list in Question 2?
6. What does your answer to the previous question suggest about the one-to-one correspondence we wrote down in Question 2?

7. What does your answer to the previous question suggest about the relationship between $|\mathbb{N}|$ and $|[0,1]|$? Between $|\mathbb{N}|$ and $|\mathbb{R}|$?

Discussion 3.2.9 *Given your responses to Exploration 3.2.7 and Exploration 3.2.8, do you think there is a set S such that $|S| > |\mathbb{N}|$ and $|S| < |\mathbb{R}|$? Why or why not?*

In the early 1930s, the Austrian mathematical logician Kurt Godel revolutionized mathematical logic with his two **incompleteness theorems**. Informally, first incompleteness theorem states that any sufficiently strong, consistent formal system contains undecidable statements. That is, there are sensible statements, such as the one raised in Discussion 3.2.9, which cannot be proved from within the system. In that case, we may choose to adopt either the statement *or its negation* as an additional axiom, and may do so without creating any contradiction.

The question in Discussion 3.2.9 leads to just such a proposition, namely that no such set S exists. This has been known as the **continuum hypothesis**. It was first suggested by Cantor in 1878, and was one of Hilbert's 23 problems. Godel himself proved in 1940 that its negation, i.e., that such a set S does exist, is independent of the usual axioms of set theory. The mathematician Paul Cohen proved in 1963 that the continuum hypothesis itself is independent of the usual axioms of set theory, thus verifying that the hypothesis is undecidable.

3.2.3 Exercises

1. How have mathematicians reacted to Godel's incompleteness theorems? What consequences, if any, have there been for the work of discovering new mathematics?

2. The existence of undecidable statements in mathematics, such as the continuum hypothesis, may seem like an esoteric quirk without any real consequences. However, there have been similar undecidable statements discovered in related disciplines such as computer science and physics. Find such a statement and, as best you can, describe it.

Part III

Power

In the Introduction to this book, we cited Francis Su's retiring MAA presidential address, *Mathematics for Human Flourishing*. In his address, Su argues that the practice of mathematics can cultivate virtues that help humans live lives of fullness and flourishing. We've explored the ways in which mathematics enables us to play and discern truth. We will next explore the ways in which the *power* of mathematics can be brought to bear to help us understand phenomena such as the current (as of this writing) coronavirus pandemic.

As Su writes in his follow-up book [1], *power* "often sounds like a bad word," especially when applied to humans. Yet Su means power in the sense in which Andy Crouch defines it in his book, *Playing God: Redeeming the Gift of Power*[1]:

> Power is the ability to make something of the world...the ability to participate in that stuff-making, sense-making process that is the most distinctive thing that human beings do.

In this unit, we will focus particularly on the *sense-making* component of power and ask: how do mathematical explorers make sense of the physical world via mathematics? We will explore some increasingly advanced discrete mathematical models, with the ultimate goal of understanding a basic model of how infectious disease spreads through a population. Let's get started.

[1] https://www.ivpress.com/playing-god

Chapter 4

Discrete Dynamical Systems

4.1 Sequences

Motivating Questions.

In this section, we will explore the following questions.

1. What is a sequence?
2. How can we use discrete sequences to describe real-world phenomena?
3. What is the difference between a recursive definition and an explicit formula for a discrete sequence?

We begin our study with an exploration of *sequences*.

Definition 4.1.1 A **sequence** is an ordered list of numbers, called **terms**. ◊

That's it! Typically, we explore sequences which continue forever; these are generally called **infinite sequences**. Such sequences are typically described using a mathematical rule of some sort. Let's look at an example.

Example 4.1.2 Joris is a collector of board games. His collection currently has 37 games, and each year he budgets enough to acquire 8 new games. We'll use the notation G_t to describe how many games Joris has in year t, where we consider G_0 (that is, in year $t = 0$) to be the number of games he currently has. Thus, $G_0 = 37$. We would also expect $G_1 = 37 + 8 = 45$, and $G_2 = 45 + 8 = 53$. We say that $37, 45, 53\ldots$ are the first three **terms** of the sequence.

This way of describing the number of games Joris has is known as a **recursive** description, where each term depends on previous terms. That is, the recursive description is given by the formula

$$G_0 = 37,\ G_{t+1} = G_t + 8, t \geqslant 1. \tag{4.1.1}$$

□

Since the sequence above only allows for whole number inputs, it cannot estimate the number of board games Joris will have in 1.3 years, or $\pi/4$ years; the system's prediction changes by 8 as the elapsed time changes from 1 to 2, 5 to 6, and so on. This makes the sequence **discrete**.

We also note the recursive nature of the definition of the sequence in Example 4.1.2. But there are other ways to describe sequences, and we turn to those now.

Activity 4.1.3 A recursive definition is nice because it is reasonably intuitive and fits with our usual understanding of how things change over time: they start from where they are now, and then change a little every so often. But perhaps we'd like to know how long it will take Joris to accumulate 500 games.

1. Clearly describe in 2-3 sentences *a process* for using the recursive definition in Example 4.1.2 that would allow you to determine how many years it would take Joris to accumulate 500 board games. Please do not use this process to answer the question!

2. In 2-3 sentences, describe the disadvantage in the process you used in Question 1 for finding how long it would take Joris to accumulate 500 games.

3. What we want is a **explicit formula** for G_t. That is, we want a formula that allows us to plug in a value for t that will give us the number of games G_t that Joris has in year t without having to know any of the previous terms in the sequence. Find such a formula, and compute the first three terms to convince your group that your formula produces the same sequence as (4.1.1).

The sequence described in Activity 4.1.3 is known as a **linear** sequence. This is because, like a line, it grows at a constant rate. Linear growth is extremely useful because it is straightforward to understand and apply. However, it has its shortcomings as well.

Activity 4.1.4 Lila is 6 years old, and is 43.5 inches tall. Her parents are told to expect her to grow at approximately 2.25 inches/year. Let $H_t, t \geqslant 0$ denote Lila's height t years from now.

1. Predict Lila's height when she turns 8.

2. Give a recursive description for Lila's height.

3. Give an explicit formula for Lila's height.

4. How tall will Lila be when she is 50? Give your answer in feet (remember: there are 12 inches in a foot).

5. In 2-3 sentences, clearly articulate at least one shortcoming of extrapolating using linear models.

Conclusion. In this section, we explored the idea of a (discrete) sequence. A sequence is just a list of numbers, and we considered sequences which are (in theory, at least) infinite. We saw two ways of describing a sequence other than just listing the numbers in order: recursively, and with an explicit formula. The recursive definition fits with our intuition about how quantities change over time, but it can be tedious to calculate terms that appear late in the sequence, as you need to calculate every term leading up to the one in which you're interested. Explicit formulas, on the other hand, give us shortcuts to calculating any term we want, but can often be hard to find and can obscure the actual behavior of the sequence.

Next time, we'll take a look at the ways in which we can combine multiple sequences to create *systems*. These systems can be used to describe the behavior of real-world interactions. In particular, we'll explore a basic predator-prey model (involving two populations, the predators and the prey), and then a system of three sequences that describes the basic dynamics of the spread of a disease.

Exercises

1. Find an explicit formula for each of the following.

(a) $2, 5, 8, 11, 14, \ldots$

(b) $50, 43, 36, 29, \ldots$

2. The first three terms in a sequence are listed below as numbers of dots. Determine a recursive description for the sequence. If possible, determine an explicit formula for the sequence.

Figure 4.1.5

3. The first four terms in a sequence are listed below as numbers of dots. Determine a recursive description for the sequence. If possible, determine an explicit formula for the sequence.

Figure 4.1.6

4.2 Discrete Dynamical Systems

Motivating Questions.

In this section, we will explore the following questions.

1. What is a discrete dynamical system? How do they relate to sequences?

2. How can we use discrete dynamical systems to describe real-world phenomena like predator-prey interactions, or the spread of a disease in a population?

In Section 4.1, we introduced the notion of a sequence. In this section, we will focus on situations in which our sequences represent a quantity changing over discrete, consistent periods of time. We will also consider *systems* of sequences: two or more interrelated sequences which describe the behavior of multiple changing quantities all at once. Before we dive into such systems, we consider another type of growth.

4.2.1 Exponential Growth

Populations of people and animals do not grow linearly. Instead, they usually grow by a percentage of the whatever the current population is. So, if that percentage is, say, 10%, and the current population of a group of fish in a pond is 1000, this model predicts a population of $1000 + 10\% \cdot 1000 = 1000 + 100 = 1100$ fish for next year.

In symbols, if P_t represents the number of fish t years from now, then $P_0 = 1000$, $P_1 = P_0 + 0.1 \cdot P_0 = (1 + 0.1)P_0 = 1.1P_0$. The number 0.1 is called the **growth rate**, r, and the number 1.1 is the **growth multiplier**.

In what follows, I will occasionally provide Sage cells in which you can do basic computations (and in which I may help get you started). To see how this works, click "Evaluate" in the Sage cell below to see what happens.

```
P_0 = 1000
P_1 = 1.1*P_0
P_1
```

Exploration 4.2.1 Consider a population of 1000 fish growing at 10% per year.

1. Give a formula for P_2 in terms of P_1, and then a formula for P_3 in terms of P_2.
2. Using your answer to the previous part as a guide, give a formula for P_t in terms of P_{t-1}.
3. Again, using your answer to Question 1 and the work in the paragraph preceding this activity, give a formula for P_2 in terms of P_0. Use this formula to find P_3 in terms of P_0. Why might these formulas be more useful than the ones you found in Question 1?
4. State your best guess for a formula for P_t in terms of P_0. Use your formula to estimate the number of fish in the pond after 10, 20, and 50 years. What is this model missing?
5. Press 'Evaluate' below to confirm your response to Question 4. What happens if you increase or decrease the growth rate, r? Try it, and reevaluate.

```
P_0 = 1000
r = 0.1
P(t) = (1+r)^t*P_0
tt = range(50)
PP = [P(t) for t in tt]
tP = list(zip(tt, PP))
P_dots = points(tP, color='red')
P_dots
```

The type of growth explored in Exploration 4.2.1 called **exponential**, as the growth comes from taking the growth multiplier to larger and larger powers (exponents). We saw the danger of extrapolation with linear models in Activity 4.1.4, and we note a similar danger in extrapolating with exponential models. If nothing else, it's likely that the pond cannot *hold* 100,000 fish. Thankfully, we can modify our exponential growth model by introducing an upper limit for what the habitat can hold.

Exploration 4.2.2 Suppose our pond can hold 5000 fish. This is known as the *carrying capacity*, and we'll denote it with the letter K

. We'll now let $r = 0.1$ be our *maximum* growth rate, but we'll let it slowly reduce as the population grows.

1. As P_t increases and gets closer and closer to K over time, what happens to the ratio P_t/K? What number does it get close to?
2. Thus, what happens to the expression $1 - \frac{P_t}{K}$ as P_t increases and gets closer and closer to K?
3. Ecologically speaking, what does it mean for P_t to get closer and closer to K?
4. As the phenomena described in Question 3 occurs, what do we expect the graph of P_t over time to look like?
5. Test your suspicion by evaluating the Sage code below.

```
P_0=1000
r = 0.1
K = 5000
maxterm = 100
P(t)= K/(1+(K-P_0)/P_0*e^(-r*t))
tt = range(maxterm)
PP = [P(t) for t in tt]
tP = list(zip(tt, PP))
P_dots = points(tP, color='red')
P_dots
```

4.2.2 A discrete predator-prey model

We now consider systems of discrete sequences, called **discrete dynamical systems**.

Definition 4.2.3 A **dynamical system** refers to any fixed mathematical rule which describes how a system changes over time. A **discrete** dynamical system changes at fixed intervals in time (e.g., each hour), and does not change between the fixed points in time (e.g., a system that changes each hour will view the changing quantity as static between the hours of 1:00pm and 2:00pm). ♢

There are two main types of dynamical systems: discrete and continuous. The study of continuous dynamical systems is the domain of calculus and its related disciplines (e.g., differential equations). Continuous dynamical systems treat time as infinitely divisible; discrete dynamical systems do not. Typically, a dynamical *system* involves multiple related quantities that change over time.

We begin with a classic predator-prey model adapted from the Feedback Systems Wiki[1] at Caltech. For historical reasons, we let L_t be the size of a population of Canadian lynxes in year t and H_t be the size of a population of snowshoe hares, the lynx's primary prey.

Definition 4.2.4 Let H_0 be the initial population of hares, and L_0 the initial population of lynxes. For $t \geqslant 1$, the **discrete Lotka-Volterra model** for the lynx/hare population is given by:

$$\begin{aligned} H_{t+1} &= H_t + b(u)H_t - aL_tH_t \\ L_{t+1} &= L_t + cL_tH_t - dL_t, \end{aligned}$$

where b is the hare birth rate per unit time as a function of the food supply u, d is the lynx mortality rate, and a and c are interaction coefficients. ♢

Note the interrelationship between the two equations: the formula for calculating H_{t+1} requires knowing not only H_t, but also L_t. This makes sense! These populations interact, so the presence (or absence) of lynx should reasonably affect the hare population. Let's further analyze this model.

Exploration 4.2.5 Consider the predator/prey model introduced in Definition 4.2.4.

1. What do the terms L_t and H_t represent?
2. Which term in the model represents an increase in the hare population? Which term represents a decrease in the hare population? Explain how you know.
3. Which term in the model represents an increase in the lynx population?

[1] https://www.cds.caltech.edu/~murray/amwiki/index.php/Predator_prey

Which term represents a decrease in the lynx population? Explain how you know.

4. What does the product $L_t H_t$ represent? (The multiplication principle[2] may be helpful here; also consider what a is described to be.)

5. What simplifying assumptions does this model make about how the populations increase and decrease?

Activity 4.2.6 Let's start by assuming that $H_0 = 20$ and $L_0 = 35$. That is, we begin with 20 hares and 35 lynxes. Let's further assume that $a = c = 0.001167$, $b = 0.05$, and $d = 0.0583$ (parameters scaled by 12 months).

1. Compute H_1, H_2, L_1, and L_2. What seems to be happening to the two populations? Confirm using the Sage cell below, which will display the first ten months' worth of predictions, or explore this spreadsheet[3]. Note that the hare population is given by the blue dots and the lynx population by the red.

```
H_0 = 10 # This is the initial hare population
L_0 = 10 # This is the initial lynx population
a = 0.014 # This represents the predation rate
b=0.6 # Growth rate of hares
c=a # This represents the predation rate
d=0.7 # death rate of lynx
nperiod = 12 # number of time periods in a year
def H(n):
        if n==0:
                return H_0
        else:
                return H(n-1) +
                    (b*H(n-1)-a*L(n-1)*H(n-1))/nperiod

def L(n):
        if n==0:
                return L_0
        else:
                return L(n-1) +
                    c*L(n-1)*H(n-1)-(d*L(n-1))/nperiod
time=10
nn=range(time)
HH = [H(n) for n in nn]
LL = [L(n) for n in nn]
nH = list(zip(nn, HH))
nL = list(zip(nn, LL))
H_dots = points(nH, color='blue')
L_dots = points(nL, color='red')
p = H_dots + L_dots
p
```

2. The interaction coefficients translate to a decrease in the lynx population and an increase in the hare population. What do you expect to happen if we increase it from 0.014 to 0.05?

3. Test your suspicion using the Sage cell below, or this spreadsheet[4].

[2] https://en.wikipedia.org/wiki/Rule_of_product

```
H_0 = 10 # This is the initial hare population
L_0 = 10 # This is the initial lynx population
a = 0.05 # This represents the predation rate
b=0.6 # Growth rate of hares
c=a # This represents the predation rate
d=0.7 # death rate of lynx
nperiod=12
def H(n):
        if n==0:
                return H_0
        else:
                return H(n-1) +
                    (b*H(n-1)-a*L(n-1)*H(n-1))/nperiod

def L(n):
        if n==0:
                return L_0
        else:
                return L(n-1) +
                    (c*L(n-1)*H(n-1)-d*L(n-1))/nperiod
time=10
nn=range(time)
HH = [H(n) for n in nn]
LL = [L(n) for n in nn]
nH = list(zip(nn, HH))
nL = list(zip(nn, LL))
H_dots = points(nH, color='blue')
L_dots = points(nL, color='red')
p = H_dots + L_dots
p
```

4. Qualitatively describe the dynamics displayed in the Sage output in the previous question.

One might reasonably wonder how such a simple model does in making predictions about the long-term dynamics of these populations. The answer is: surprisingly well! In Figure 4.2.7, observe the actual collected data on hare and lynx populations over 90 years, from 1845 to 1935. In Figure 4.2.8, we see dynamics predicted by the model. Note the same cyclical patterns of an increase in the hare population followed by an increase in the lynx population, which in turn causes a decrease in the hare population, etc.

[3] https://docs.google.com/spreadsheets/d/1sFOHJ-LNA39Ydj_GDg69XhtNwWVtpWNweBT5IBzjJGo/edit?usp=sharing

[4] https://docs.google.com/spreadsheets/d/1sFOHJ-LNA39Ydj_GDg69XhtNwWVtpWNweBT5IBzjJGo/edit?usp=sharing

https://www.cds.caltech.edu/~murray/amwiki/images/0/08/Predprey-graph.png

Figure 4.2.7 Data on hare and lynx populations over time. (Source[5])

https://www.cds.caltech.edu/~murray/amwiki/images/9/9a/Predprey-discrete.png

Figure 4.2.8 The predicted dynamics of hare and lynx populations over time. (Source[6])

To be clear, the purpose of the model is not to make absolutely certain predictions about the precise numbers of hares and lynxes present in the Canadian wilderness. Instead, we want to understand the broad dynamics of how the populations change relative to one another. Mathematical ecologists can then use these models to understand how small changes in the parameters (say, an increase in the rate of predation) affect the broader dynamics of the ecological system.

4.2.3 The discrete SIR epidemiological model

We now arrive at the main focus of this two-week unit: a basic mathematical model for the spread of an infectious disease. We'll first present the model itself and examine its features and assumptions. As was the case with the predator-prey model, we'll see that while it does make simplifying assumptions, it still allows us to analyze the broader dynamics of the disease transmission. We'll then look at ways of reducing the rate of infection, including the so-called "flattening the curve" method.

[5] https://www.cds.caltech.edu/~murray/amwiki/index.php/Predator_prey
[6] https://www.cds.caltech.edu/~murray/amwiki/index.php/Predator_prey

We begin with the model. It is known as a *compartmental model*, as it divides the population into compartments and assumes that every individual in the same compartment has the same characteristics (at least as far as the transmission of the disease is concerned). We'll look at the simplest such model, the discrete SIR model. Many more complex models are built on the SIR model.

Definition 4.2.9 Consider a population of N people through which a disease is spreading. For some discrete time $t \geqslant 0$, let S_t denote the number of individuals *susceptible* to the disease, I_t the number of individuals *infected* with the disease, and R_t the number of individuals who have *recovered* from the disease. We assume that that people move through the compartments as follows:

$$S \to I \to R. \tag{4.2.1}$$

For $t \geqslant 1$, the model is given by the following equations.

$$\begin{aligned} N &= S_t + I_t + R_t \\ S_{t+1} &= S_t - bS_tI_t \\ I_{t+1} &= I_t + bS_tI_t - aI_t \\ R_{t+1} &= R_t + aI_t. \end{aligned}$$

The constant a is known as the *recovery rate* parameter, which roughly describes how fast someone moves from the infected compartment to the recovered compartment. The constant b is known as the *infection rate*, and roughly describes how fast someone moves from the susceptible compartment to the infected compartment. ◊

Let's explore the equations.

Exploration 4.2.10

1. What does (4.2.1) mean? What assumptions does this make? How does it simplify our analysis?

2. What does the first equation in the set of four mean? What assumptions does it make about the population?

3. The second equation describes how the susceptible population changes over time. It contains the term $-bS_tI_t$. Based on our work above with a predator-prey model and the definition of b, what epidemiological event is this term describing?

4. Explain why it makes sense that we add bS_tI_t in the equation defining I_{t+1}.

5. What does the term $-aI_t$ mean in the equation for I_{t+1}? Why does it make sense to *add* aI_t in the equation for R_{t+1}?

6. Note that there are no terms being subtracted in the equation for R_{t+1}. What assumption does this tell us that the model is making?

Activity 4.2.11 Let's see what happens when we plug in some numbers. Assume that $N = 10,000$, $R_0 = 0$, and $I_0 = 50$.

1. Why does it make sense that we have $R_0 = 0$ (assuming we have a new disease entering a population).

2. What is S_0?

3. Let's assume that $a = 0.1$ and $b = 0.0001$. Compute S_1, I_1, and R_1.

4. Use your answer to the previous part to compute S_2, I_2, and R_2.

5. Now check your work using this spreadsheet[7] (download a copy to your device and edit it there).

Investigation 4.2.12 In this and the following activity, we'll use the spreadsheet found here[8]. Download the file and play around with the numbers at the top of the sheet to change some of the features; e.g., how does increasing/decreasing the number of initial infected individuals change the shape of the curves? What do you notice? What do you wonder? Give at least 2-3 observations or questions.

The COVID-19 pandemic introduced many people to a quantity called r_0.

This is known as the **basic reproduction number**, and is the expected number of new infections *directly generated* by a single case. So, if I were to get COVID-19, r_0 would be the expected number of people I would *directly* infect. We would then expect each of *them* to infect another r_0 people, and so on.

Generally, if $r_0 > 1$, we expect the disease to spread. If $r_0 < 1$, we expect it to die out.

The next activity explores the ways in which varying r_0 impacts new cases of the disease caused both directly and indirectly by a single person.

Activity 4.2.13 In this activity, we assume different values for r_0. However, we will make two assumptions that don't change.

First, assume that all direct infections are done within a 5-day period. Second, assume that that those infected don't infect others until the next five day period.

Let C_t be the number of cases I've caused after t five-day periods. Assume $C_0 = 1$ (me).

1. Explain why $C_1 = r_0 C_0 = r_0$.

2. Explain why $C_2 = C_1 + r_0 C_1 = r_0 + r_0^2$.

3. After 30 days, six five-day periods will have passed. Explain why

$$C_6 = r_0 + r_0^2 + r_0^3 + \cdots + r_0^6.$$

4. Our SIR model approximates r_0 by the formula

$$r_0 \approx \frac{bN}{a}. \tag{4.2.2}$$

Using that formula, what is the approximate r_0 for the situation described in Activity 4.2.11?

5. Given that value of r_0, use the Sage cell below (replacing the ? with the value you found) to estimate how many cases I am responsible for over a 30-day period.

6. The practice of social distancing[9] is intended to reduce r_0. Assume that strict social distancing is observed, and this reduces r_0 to approximately 1.25 (one direct infection fewer). Now how many cases are you responsible for over the course of a 30-day period?

---

[7] https://docs.google.com/spreadsheets/d/153L02021_TwEYyODq2Km90oRrilLEpfYmgb8w3AJSQo/edit?usp=sharing

[8] https://docs.google.com/spreadsheets/d/153L02021_TwEYyODq2Km90oRrilLEpfYmgb8w3AJSQo/edit?usp=sharing

7. As the COVID-19 situation is ongoing, estimates for r_0 vary significantly. One study from February 2020[10] found an average r_0 of 3.28. If that is the true number, approximately how many cases will a typical infected person be responsible for over the course of a month? Use the Sage cell to determine your answer.

8. Other studies suggest that, in the absence of any social interventions, the COVID-19 has $r_0 \approx 2.38$. In that case, how many cases would an infected person be responsible for over the course of a month?

9. The Delta variant of COVID-19 has a $r_0 \approx 5.1$. If I am infected with the Delta variant, approximately how many new cases will I cause within a month?

10. The Omicron variant of COVID-19 has an estimated $r_0 \approx 10$[11]. If I am infected with the Omicron variant, approximately how many new cases will I cause within a month?

```
r_0 = ?
r_0 + r_0^2 + r_0^3 + r_0^4 + r_0^5 + r_0^6
```

For our last activity, we'll explore how reducing r_0 "flattens" the curve of infected people at time t, I_t. There are two main advantages of a flattened infected curve. First, this often corresponds to fewer infected people overall. Second, the lack of a spike in infected persons makes it easier for the healthcare system to effectively treat those who *are* infected (not to mention anyone with other medical concerns).

---

[9] https://en.wikipedia.org/wiki/Social_distancing

[10] https://academic.oup.com/jtm/article/27/2/taaa021/5735319

[11] This is being written in February 2022, just as the first major Omicron wave is subsiding; it should therefore be treated as preliminary.

Exploration 4.2.14 One last time, consider the values for the variables we used in Activity 4.2.11; for reference, this was $S_0 = 10000$, $I_0 = 50$, $R_0 = 0$, $a = 0.1$, and $b = 0.0001$. We'll again use the Google sheet[12] to answer these questions.

1. What is r_0? (Recall (4.2.2) from Question 4 of Activity 4.2.13.)

2. This value is pretty high (though it is approximately the r_0 of diseases like measles and chicken pox!), but is convenient for our purposes. Nonetheless, we can still explore the ways in which changes in r_0 affect the shape of the curves; our qualitative observations will still apply to real-world situations like the current coronavirus pandemic.

   Recall that $r_0 \approx \frac{bN}{a}$ and assume our population size $N = 10000$ and recovery rate $a = 0.1$ are constant. Compute the three values of r_0 that result from infection rates of $b = 0.00005, 0.0001$, and 0.0002. In turn, plug these values into the Google sheet[13] and comment on the shape of the infection curve: how tall is the spike of infected individuals, and at what time t is it at its highest point?

3. Similarly, assuming $N = 10000$ and $b = 0.0001$ are constant, compute the values of r_0 that result from $a = 0.05, 0.1$, and 0.2. In turn, plug these values into the Google sheet[14] and comment on the shape of the (green) infection curve: how tall is the spike of infected individuals, and at what time t is it at its highest point?

[12] https://drive.google.com/file/d/1xSJ6KM8x9HVdo9-P4QoUOoSmmfpKmmIQ/view?usp=sharing
[13] https://drive.google.com/file/d/1xSJ6KM8x9HVdo9-P4QoUOoSmmfpKmmIQ/view?usp=sharing
[14] https://drive.google.com/file/d/1xSJ6KM8x9HVdo9-P4QoUOoSmmfpKmmIQ/view?usp=sharing

Chapter 5

Graphs and Networks

Chapter 6

Modular Arithmetic and Coding Theory

6.1 Adding in Circles

Motivating Questions.

In this section, we will explore the following questions.

1. What is congruence modulo m?
2. What are some real-world examples of congruence?
3. How can we do arithemtic modulo m?

This topic comes to us from the realm of mathematics known as *number theory*, which is all about the properties of the integers: positive whole numbers, negatives of whole numbers, and zero. Number theory is one of the oldest branches of mathematics; some of its hallmark theorems, still vital and taught today, have been known for thousands of years. And many of the deep structural features of the integers have found modern application in the hiding and transmitting of information. It is toward this last application that we now turn.

First, however, we need to be reminded of a suprisingly important result from school mathematics. We begin with some warmup questions.

Warmup 6.1.1 *For each pair of positive integers given below, perform long division to divide the first number by the second.*

1. *42, 6*
2. *42, 8*
3. *71, 9*
4. *0, 17*
5. *9, 71*
6. *8675309, 627*

The Division Algorithm.

Let a and m be integers, with $m > 0$. Then there exist unique integers q, r, $0 \leqslant r < m$, such that

$$a = m \cdot q + r.$$

We call a the **dividend**, m the **divisor**, q the **quotient**, and r the **remainder**.

Activity 6.1.2 For each long division problem from Warmup 6.1.1, identify the dividend, divisor, quotient, and remainder as described in the Division Algorithm. What do you notice? What do you wonder?

Discussion 6.1.3 *What is meant by the word "unique" in the Division Algorithm? Or, put slightly differently, when dividing 71 by 9, why do you think we do we not give a quotient of 6 and remainder of 17?*

Of primary importance for us will be a consideration of the remainders obtained by dividing by a fixed positive integer $m > 1$.

Activity 6.1.4 Throughout this activity, we will be dividing by $m = 5$. However, you should be thinking about how the answers might differ if we divide by a different integer m.

1. What remainders do you obtain when dividing the integers $12, 16, 20, 24, 30$ by 5?

2. What remainders do you obtain when dividing the integers $39, 52, 80, 107, 166$ by 5?

3. What remainders are *possible* upon division by 5? How do you know?

4. What remainders do you expect to be possible upon division by 103? How do you know?

5. For each of the five integers from the first question, find the integer from the second question whose remainder upon division by 5 is the same and write the pairs in a list.

6. For each pair in the list you made in the previous question, find the difference between the two integers and divide *that number* by $m = 5$. What do you notice? What do you wonder?

The work you did in Activity 6.1.4 provides motivation for the following definition.

Definition 6.1.5 Let a, b be integers and $m > 1$ an integer. We say that a and b are **congruent modulo** m if a and b have the same remainder upon division by m. In this case, we write $a \equiv b \mod m$ and call m the **modulus**.

Equivalently, we say a and b are congruent modulo m if m evenly divides $b - a$. ◊

Let's explore this definition a bit more.

Activity 6.1.6 For the given values below, determine whether $a \equiv b \mod m$.

1. $a = 71$, $b = 39$, $m = 16$

2. $a = 17$, $b = 19$, $m = 4$

3. $a = 832$, $b = 584$, $m = 31$

Congruence modulo m possesses several important properties. We highlight three of them in the following theorem.

Theorem 6.1.7 *Let $m > 1$ be an integer, and suppose a, b, c are integers.*

- *Congruence is* ***reflexive:*** $a \equiv a \mod m$
- *Congruence is* ***symmetric:*** *If* $a \equiv b \mod m$*, then* $b \equiv a \mod m$*.*
- *Congruence is* ***transitive:*** *If* $a \equiv b \mod m$ *and* $b \equiv c \mod m$*, then* $a \equiv c \mod m$*.*

Now that we are a bit more familiar with the definition, let's test its limits.

Exploration 6.1.8 Choose two different values of m, and for each value you choose, find two values of a and b so that $a \equiv b \mod m$.

1. Now choose a third integer c. For the integers you chose, is it true that $a + c \equiv b + c \mod m$? Is it true that $a - c \equiv b - c \mod m$? What about $a \cdot c \equiv b \cdot c \mod m$?
2. For each integer c you chose, find an integer d such that $c \equiv d \mod m$. Is it true that $a + c \equiv b + d \mod m$? Is it true that $a - c \equiv b - d \mod m$? What about $a \cdot c \equiv b \cdot d \mod m$?
3. Finally, note that $5 \cdot 2 \equiv 3 \cdot 2 \mod 4$. Does it follow that $5 \equiv 3 \mod 4$?

Our discoveries in Exploration 6.1.8 delineate the operations known as **modular arithmetic**: we may add, subtract, and multipy integers mod m, but we may not divide.

Activity 6.1.9 Find the smallest nonnegative integer x satisfying:

1. $x \equiv 9 \cdot 4 \mod 5$
2. $x \equiv 103 + 405 \mod 10$
3. $x \equiv 56 + 6 \mod 7$
4. $x \equiv 9 \cdot 99 \mod 5$

Questions.

Consider the following questions, and what they have to do with modular arithmetic.

1. What day of the week will it be 62 days from now?
2. What month will it be 40 months from now?
3. What time will it be 27 hours from now?

6.2 Coding Theory

Motivating Questions.

In this section, we will explore the following questions.

1. How can modular arithmetic be applied to the transmission of information?
2. How does the UPC check digit scheme work? What errors can it detect? What errors can it correct?

In Section 6.1, we learned about modular arithmetic, which works by considering the remainders obtained upon division by a fixed number m. In this section, we'll consider a relatively recent application of modular arithmetic to **coding theory**, which is the mathematical study of transmitting information.

The basic problem of coding theory is as follows: in order to send information from one entity to another, it needs to be encoded in some form by the *sender* (e.g., written, recorded, etc) and transmitted across a *channel* (e.g., mailed, emailed, uploaded/downloaded, etc) to the *receiver*. However, errors can creep into the process—written messages can be smudged, physical defects or packet loss can corrupt digital messages, and so on, resulting in information that either cannot be read at all, or can be misread. How can we be reasonably sure that common errors can be detected, and, if possible, corrected?

Activity 6.2.1 Alice wants to send a bit of information to her friend Bob; for simplicity's sake, let's assume she seeks to send a 0 or 1. To ensure that the digit is received correctly, she decides to send it three times in a row. Bob will then read the intended digit as the digit that appears most often.

1. How would Bob interpret Alice's message if he received 111? What if he received 010?

2. What strengths and weaknesses do you see in Alice's system? How could you improve upon it?

3. Perhaps you decided that Alice's system isn't sure enough, so you decide she should send the digit seven times. In what way(s) is this revised system better? Worse?

6.3 Opkpun Tlzzhnlz: Hu Vclycpld vm Jyfwavny-hwof

Part IV

Justice

One of the goals of this text is to expand the domain of mathematical questions to new areas. While we need to be careful to not overreach, mathematical ways of knowing have recently illuminated many interesting questions, and occasionally answered them.

An example of this involves recent mathematical explorations of democracy. In Chapter 7, we will consider mathematical analyses of various methods of choosing winner(s) of elections and discuss the question: is there a single best, fairest system? In Chapter 8, we will explore the systems of assigning representatives to geographical areas that have been employed throughout U.S. history. And in Chapter 9, we'll explore the process of drawing new districts based on shifting populations to ensure that those geographical areas are represented fairly.

As we will see, mathematics helps us explore these questions, but it cannot provide perfect answers, as human desires—for justice, fairness, and equity—are not mathematical constructs.

Chapter 7

Voting

7.1 Elections with Two Candidates

Motivating Questions.

In this section, we will explore the following questions.

1. What is a voting system?
2. What are some common means of choosing winners of elections?
3. What are some drawbacks for these common systems?

Our main task in this chapter is to explore the mathematics of **voting systems**, which refers to both the ways votes are cast in elections *and* the way those votes are used to determine a winner.

We will see in this section that while two-candidate systems are fairly straightforward, they provide a fertile ground for exploring the characteristics we desire in all systems.

We first want to determine how each system treats the voters and the candidates. A fair system should aim to treat each voter and each candidate equally.

Activity 7.1.1 It is time for the citizens of Sudden Valley to elect a new mayor, and they have two choices: Michael Bluth, and his sister, Lindsay Bluth. How should the winner of this election be determined?

Investigation 7.1.2 If Activity 7.1.1 seemed too easy, consider the following (likely alternative) suggestion.

The Bluths' father, George, has been a long-time player in Sudden Valley politics. It's important that all the citizens vote, but the winner will be whomever George votes for, regardless of how the other votes come in.

Of the 1001 citizens of Sudden Valley, suppose 1000 vote for Michael, and George votes for Lindsay. Who wins the election?

The method described in Investigation 7.1.2 is known as a **dictatorship**, with George as the dictator.

Exploration 7.1.3 Does a dictatorship treat all voters equally? Does it treat the candidates equally? Explain.

Investigation 7.1.4 If a dictatorship isn't your style, try this one: before the election runs and the citizens of Sudden Valley vote, it is decided that Lindsay will win. Does this method treat all of the voters equally? Does it treat the candidates equally? Explain.

The method described in Investigation 7.1.4 is known as **imposed rule**.

Let's try one more system on for size.

Investigation 7.1.5 Consider the following system. Each citizen of Sudden Valley votes, and the votes are counted. The winner is the candidate with the *smallest* number of votes. This is known as **minority rule**.

1. Suppose that the citizens of Sudden Valley vote, with 1000 voters voting for Michael and one (George, no longer a dictator) votes for Lindsay. Who wins under minority rule?

2. Now suppose that George convinces 500 of the prospective Michael voters to switch and vote for Lindsay. Who wins in this case under minority rule?

3. Does minority rule treat all voters equally? Does it treat all candidates equally? Explain.

4. Under minority rule, is it beneficial or detrimental for a candidate to receive additional votes? Explain.

In our quest for a mathematically just voting system, we will find it helpful to define certain desirable qualities for a fair and just system to satisfy. When we say that a voting system satisfies a certain criterion, we mean that it *always* satisfies it; that is, it can never violate that criterion. We will refer to these important criteria as **fairness criteria**. We have already observed a few important criteria that we now formalize.

Definition 7.1.6 A voting system for a two-candidate election is **anonymous** if it treats all voters equally. That is, if any two voters switched their votes, the outcome of the election should remain the same.

A voting system for a two-candidate election is **neutral** if it treats both candidates equally. That is, if *every* voter changes their vote, the outcome of the election should also change.

A voting system for a two-candidate election is **monotone** if is impossible for a winning candidate to become a losing candidate by gaining votes (and not losing any others), or for a losing candidate to become a winning candidate by losing votes (and not gaining any others). ◊

Investigation 7.1.7 Now suppose three members of the Bluth family are voting to determine who should take over the family business, The Bluth Company. The candidates are Michael and Lindsay. In order to vote, their friend, Barry, has devised a voting system. Three possible combinations of votes by Lucille, Buster, and Tobias are presented in Table 7.1.8.

Table 7.1.8 The results of Barry's voting system.

Lucille	Buster	Tobias	Winner
L	M	M	L
L	L	M	M
M	M	L	M

1. Which of the three properties described in Definition 7.1.6 are satisfied by Barry's voting system? Explain.

2. Is Barry's sytem equivalent to any of the other three systems we've

investigated thus far? Why or why not?

Investigation 7.1.9 Let's investigate the three criteria in Definition 7.1.6 in relation to the three voting systems we've examined.

1. Which of the three fairness criteria are satisfied by dictatorships? Explain clearly for each of your answers.

2. Which of the three fairness criteria are satisfied by imposed rule? Explain clearly for each of your answers.

3. Which of the three fairness criteria are satisfied by minority rule? Explain clearly for each of your answers.

The two-candidate voting system we haven't yet explored is probably the one that you actually suggested in Activity 7.1.1. Perhaps it is something like:

> Each voter should vote for the candidate they want to win the election. The votes are counted, and the candidate with the largest number of votes should be declared the winner.

This is known as **majority rule**.

Investigation 7.1.10 Which of the three fairness criteria does majority rule satisfy? Give a clear and convincing explanation of your answers.

In fact, for elections with two candidates, majority rule is the only system to satisfy all three fairness criteria.

Theorem 7.1.11 (K. May (1952)) *In a two-candidate election with an odd number of voters, majority rule is the only voting system that is anonymous, neutral, and monotone and avoids the possibility of a tie.*

Thus, for elections with two candidates, there is a clear just choice for a voting system subject to our adopted fairness criteria: majority rule. However, as we know, most elections don't have just two candidates, so we will next consider how to handle situations with more than two choices. We'll refine the existing fairness criteria to handle three or more candidates, and develop additional criteria due to surprising situations that can arise.

7.2 Elections with More than Two Candidates I

Motivating Questions.

In this section, we will explore the following questions.

1. How are elections with two candidates different from those with more than two?

2. What is a plurality, and how is it different than a majority?

3. How can we fairly evaluate voting systems with multiple candidates?

7.2.1 Describing the Problem

Let's begin with a warmup activity.

Activity 7.2.1 The popular vote totals from the state of Florida in the 2000 U.S. presidential election are given in Table 7.2.2.

Table 7.2.2 The Florida popular vote in 2000.

Candidate	Popular Votes
George W. Bush	2,912,790
Al Gore	2,912,254
Ralph Nader	97,488
Others	40,579

1. In this election, did any candidate receive a *majority* (more than half) of the popular votes cast in the state of Florida?

2. If George W. Bush and Al Gore had been the only candidates on the ballot in Florida in 2000, do you think that Gore might have possibly received more popular votes than Bush in Florida?

In fact, many political scientists believe that Ralph Nader was a *spoiler candidate* for Gore in 2000; that is, they believe that a large percentage of the Nader voters likely would have voted for Gore if Nader had not been on the ballot. Moreover, since Florida was the deciding factor in the election (we'll talk more about the electoral college later), the razor-thin margin of 537 votes (0.009% of the popular vote in Florida) swung the election from Gore to Bush. In this section, we'll explore the wrinkles introduced by additional candidates, and develop alternative systems for choosing winners of elections.

Definition 7.2.3 A candidate in an election who receives more votes than any of the other candidates is said to receive a **plurality** of the votes cast. ♢

Discussion 7.2.4

1. *For elections with two candidates, explain why the words* plurality *and* majority *mean exactly the same thing.*

2. *For elections with more than two candidates, explain why the words* plurality *and* majority *do not mean exactly the same thing.*

We adopt the following definitions.

Definition 7.2.5 Consider an election with more than two candidates.

- **Majority rule** is the voting system that elects the candidate who receives more than half the votes, if such a candidate exists. If no such candidate exists, the election is declared a tie with no winner.

- The **plurality method** is the voting system that elects the candidate that receives the largest number of votes. The plurality method only produces a tie when two candidates receive exactly the same number of votes, and this number is more than any other candidate.

♢

Discussion 7.2.6

1. *Which of the two methods described in Definition 7.2.5 is more likely to result in a tie?*

2. *If a candidate wins an election under majority rule, would that candidate also be guaranteed to win under the plurality method?*

3. *If a candidate wins an election under the plurality method, would that candidate also be guaranteed to win under majority rule?*

The plurality method is familiar, and so likely seems quite fair. But let's consider the next exploration, this time from the 2016 Republican presidential primary process.

Exploration 7.2.7 Twenty-one people filed paperwork with the U.S. Federal Election Commission as candidates for the 2016 Republication nomination for president. There were 31,183,841 votes cast in the Republican primaries nationwide in 2016.

1. Donald Trump received 14,015,993 of these votes. Did he receive a majority of the votes cast in the Republican primaries?

2. If the winner of the 2016 Republican nomination had been chosen by plurality from these 21 candidates, what is the smallest number of votes Trump could have received and still had a chance at winning the nomination? (Assume that the number of votes remains fixed at 31,183,841.)

3. Under the same assumptions as Question 2, what is the maximum number of voters who could have preferred Trump the *least* among the 21 candidates in order for him to still have had a chance of winning the nomination?

4. Using your answers to Questions 2 and 3, formulate a well-written criticism of the plurality method. You don't have to agree with your argument, but put yourself in the shoes of a critic and try to predict the type of argument that could be made.

7.2.2 A solution

As we saw in Exploration 7.2.7, when there are N candidates, it is possible for the vote to be split so thoroughly that one candidate can win with just over $1/N$-th of the vote. In this case, a majority of the voters will prefer someone other than the person who is ultimately elected. But the situation is even worse than that—it's possible for the candidate who wins the plurality of the vote to be the *last* choice of a *majority* of voters, but still win! The main reason that the plurality method is susceptible to this is that it only takes a voter's first choice into account; there is no penalty for being a voter's last choice, and no benefit to being the voter's second choice.

We will therefore explore methods of voting and fairness criteria that account for a full top-to-bottom ranking of candidates by voters, called a **preference order**. If there are three candidates in an election, say Michael, Lindsay, and Buster, and my preference order is that Michael is my top candidate, Buster my second choice, and Lindsay my third, we may write $M \succ B \succ L$, where the symbol $\succ$ means "is preferred to".

Activity 7.2.8

1. Consider a 3-candidate election for the presidency of the Bluth Company between Michael, Lindsay, and Buster. How many possible preference orders are there? In other words, in how many different ways could the voters rank them?

2. Suppose their brother George enters the race, bringing the total number of candidates to 4. Now how many preference orders are possible now?

Since our voters will be casting **preference ballots**, we need a different way of displaying the votes cast.

Exploration 7.2.9 Suppose Buster, George, Lindsay, and Michael are running for the presidency of The Bluth Company (TBC), the premier real estate company in Sudden Valley. The preference orders for each of the 27 shareholders in the company are displayed in Table 7.2.10, a visualization known as a **preference schedule**. The column headings indicate the number of voters with the preference order displayed in the column. For instance, the first column shows that 12 shareholders have the preference order $B \succ G \succ L \succ M$. Note that the preference orders displayed are the four that were cast as preference ballots in the election; there are many others that were not cast, and thus are not displayed.

Table 7.2.10 Preference schedule for the presidency of The Bluth Company.

Rank	12	7	5	3
1	B	G	L	M
2	G	L	M	L
3	L	M	B	G
4	M	B	G	B

1. Under majority rule, what would the outcome of the election be?
2. Under the plurality method, what would the outcome of the election be?
3. Rank the candidates based on the outcome produced by the plurality method. The final ranking of candidates by a voting system is known as the **societal preference order**.
4. Do you think the plurality winner best represents the will and preferences of the voters? If so, explain why. If not, give a convincing argument for why you think some other candidate would be better.

7.2.3 The Borda Count

One very popular method for choosing the winner of a multi-candidate election is the Borda count.

Definition 7.2.11 Consider an election with N candidates. The **Borda count** works as follows. Each voter submits a ballot that contains their entire preference order for all candidates in the election. For each ballot cast, points are awarded to each candidate according to the following rules:

- A last-place vote is worth 1 point.
- A second-to-last-place vote is worth 2 points.
- $\vdots$
- A third-place vote is worth $N - 2$ points.
- A second-place vote is worth $N - 1$ points.
- A first-place vote is worth N points.

The candidate who accumulates the largest number of total points from all of the ballots is declared the winner, and the societal preference order is determined by listing the candidates in order of the number of points they received, largest to smallest. In the event that two or more candidates are tied with the largest number of points, they are all declared winners (or some suitable prearranged tiebreaking procedure is used). ◊

Activity 7.2.12 Under the Borda count, what would the outcome of the Bluth Company presidential election in Exploration 7.2.9 be?

Consider the following interesting feature of the Borda count.

Activity 7.2.13 Controversy at the Bluth Company! Due to some shenanigans involving a major shareholder, the election displayed in Table 7.2.10 has to be rerun. The following preference schedule is produced.

Table 7.2.14 Preference schedule for the presidency of The Bluth Company.

Rank	10	3	5
1	B	G	L
2	L	L	G
3	G	M	M
4	M	B	B

1. Who wins under majority rule?
2. Who wins under the Borda count? Does this seem strange to you?

Lest you think this is a contrived example, be aware that things like this can happen in real life. A version of the Borda count is used by the Associated Press to rank the top 25 college football and basketball teams. In the 1971 AP preseason poll, my Nebraska Cornhuskers received 26 of 50 first-place votes, yet were ranked #2. The results of things like this AP polling anomaly or Activity 7.2.13 suggest a new fairness criterion.

Definition 7.2.15 A voting system satisfies the **majority criterion** if whenever a candidate is ranked first by a majority of voters, that candidate will be ranked first in the resulting societal preference order. ◊

Discussion 7.2.16 *Of all the voting systems we've explored thus far, which must always satisfy the majority criterion?*

Our definitions from Definition 7.1.6 can be modified to extend in a natural way to elections with three or more candidates.

Definition 7.2.17

- A voting system is **anonymous** if it treats all of the voters equally, meaning that if any two voters traded *preference orders*, the outcome of the election (and the resulting societal preference order) would remain the same.
- A voting system is **neutral** if it treats all of the candidates equally, meaning that if *every* voter switched the positions of two particular candidates in their individual preference orders, the positions of these two candidates would switch in the resulting societal preference order as well.
- A voting system is **monotone** if changes favorable only to a particular candidate in individual preference orders cannot cause that candidate to be ranked lower in the resulting societal preference order.

◊

Exploration 7.2.18

1. Which of the properties of anonymity, neutrality, and monotonicity are satisfied by plurality? Which are not satisfied? Give a convincing argument to justify each of your answers.

2. Which of the properties of anonymity, neutrality, and monotonicity are satisfied by the Borda count? Which of these three properties are not satisfied? Give a convincing argument to justify each of your answers.

3. Do either of your answers to Questions 1 or 2 contradict Theorem 7.1.11? Explain.

In this section, we have seen that things become more complicated when we consider more than two candidates. In an election with only two candidates, a vote for one implicitly ranks the other candidate second. With more than two candidates, not only can societal preferences be more diffuse, but they are also more complex than can be captured with simple plurality voting. It is possible for a candidate who is the least desirable choice of an *overwhelming majority* of the voters to win if there are enough other candidates.

In the next section, we will explore additional voting systems and fairness criteria that attempt to overcome the shortcomings of the Borda count.

7.3

7.3.1 Elections with More than Two Candidates II

Motivating Questions.

In this section, we will explore the following questions.

(a) How should head-to-head preferences figure into an election outcome?

(b) What is sequential pairwise voting? What are its strengths and weaknesses?

Consider the following situation[1].

Activity 7.3.1 Suppose Skip, Norm, and Jesse are all running for President of the 10,000 Lakes Club, with the preferences of the 100 members of the club as shown in Table 7.3.2.

Table 7.3.2 The preference schedule schedule for the 10,000 Lakes Club presidency.

Rank	35	28	20	17
1	N	S	J	J
2	S	N	N	S
3	J	J	S	N

(a) What would be the outcome of the election under majority rule?

(b) What would be the outcome of the election under plurality?

(c) What would be the outcome of the election under the Borda count?

(d) Which candidate is ranked first by the largest number of voters?

(e) Which candidate is ranked last by the largest number of voters?

(f) In a head-to-head contest between just Skip and Norm, who would win?

(g) In a head-to-head contest between just Skip and Jesse, who would win?

(h) In a head-to-head contest between just Norm and Jesse, who would win?

(i) Does anything about your answers to Questions 1-8 strike you as being strange or unusual?

In Activity 7.3.1, we saw that the plurality method can fail to elect a candidate who would win a head-to-head matchup against all other candidates. Perhaps worse, we also saw that plurality *can* elect a candidate who would *lose* a head-to-head matchup against all other candidates. This has struck voting theorists as unfair, and we make the following definition, named after Marie Jean Antoine Nicolas de Caritat, the Marquis de Condorcet.

Definition 7.3.3 A **Condorcet winner** is a candidate in an election who would win a head-to-head contest (with the winner decided by majority rule) against each of the other candidates.

A **Condorcet loser** is a candidate in an election who would lose a head-to-head contest (with the winner decided by majority rule) against each of the other candidates.

A voting system that will always elect a Condorcet winner, *whenever one exists*, is said to satisfy the **Condorcet winner criterion (CWC)**.

A voting system that will never elect a Condorcet loser is said to satisfy the **Condorcet loser criterion (CLC)**. ◊

Exploration 7.3.4 Consider the preference schedule in Table 7.3.5.

Table 7.3.5 A preference schedule.

Rank	1	1	1
1	A	B	C
2	B	C	A
3	C	A	B

(a) In a head-to-head contest between just candidates A and B, who would win?

(b) In a head-to-head contest between just candidates B and C, who would win?

[1]Borrowed, again, from Hodge and Klima's *The Mathematics of Voting and Elections*, 2nd ed.[2]

(c) In a head-to-head contest between just candidates A and C, who would win?

(d) Does anything about Questions 1-3 strike you as unusual?

(e) Is there a Condorcet winner and/or loser in this election? Explain.

Investigation 7.3.6

(a) Explain why, whenever majority rule does not result in a tie, the majority rule winner will be a Condorcet winner.

(b) Does your answer to Question 1 imply that majority rule satisfies the CWC? If so, explain why. Otherwise, give an example to show that majority rule can violate the CWC.

(c) Does your answer to Question 1 imply that majority rule satisfies the CLC? If so, explain why. Otherwise, give an example to show that majority rule can violate the CLC.

(d) Are there special types of elections for which majority rule does satisfy the CWC? Give a convincing argument to justify your answer.

(e) Use your answer to Question 1 to explain why any voting system that violates the majority criterion (Definition 7.2.15) must also violate the CWC.

(f) Use your answer to Question 5 to explain why the Borda count violates the CWC.

In order to find a voting system that satisfies the CWC and CLC, let's return to the Bluth Company presidential election in Exploration 7.2.9.

Exploration 7.3.7 Consider the following proposed voting system, using Table 7.2.10 as the preference schedule.

- Step 1: First, we'll ask voters to choose between George and Lindsay. Since this is a two-candidate election, we'll use majority rule to decide the winner.
- Step 2: Next, we'll ask the voters to choose between Buster and the winner from Step 1, again using majority rule to decide the winner.
- Step 3: Finally, we'll ask the voters to choose between Michael and the winner from Step 2. Whoever wins this third head-to-head contest will be declared the overall winner.

Consider the method described above.

(a) Under this method, who wins the presidency of the Bluth Company?

(b) Under this method, what societal preference order is produced?

(c) In light of the plurality and Borda count results for this election, does anything about your answers to Questions 1 and 2 strike you as being strange or unusual? Explain.

(d) Is there a Condorcet winner and/or loser in this election? Explain.

The method described above is known as **sequential pairwise voting**. What strikes you as being different or unusual about it, especially compared to the plurality and Borda count systems we've already explored?

We'll finish this section by exploring sequential pairwise voting and the CWC/CLC a bit more.

Investigation 7.3.8

(a) Could a Condorcet winner ever lose a head-to-head contest with another candidate? Why or why not?

(b) What does your answer to Question 1 allow you to conclude about sequential pairwise voting and the CWC?

(c) Does sequential pairwise voting satisfy the CLC? If so, explain why. Otherwise, give an example of a preference schedule for which sequential pairwise voting could elect a Condorcet loser.

As we have just seen, SPV satisfies the CWC. This means that it will always elect a Condorcet winner if one exists. However, when a Condorcet winner does not exist, as is the case in the Bluth Company presidential election, strange things can happen.

After the actual preference schedule, the biggest factor deciding the outcome of SPV is the order in which the candidates face off. This order is called the **agenda**, and is usually specified by listing the candidates in the order into which they are to be introduced to the comparisons. In Exploration 7.3.7, the agenda was G, L, B, M.

Investigation 7.3.9 Consider the familiar preference schedule given in Table 7.2.10.

(a) Who would win the Bluth Company presidency using SPV with the agenda B, G, L, M?

(b) Find a sequential pairwise voting agenda for which Buster would win the presidency.

(c) Find a sequential pairwise voting agenda for which George would win the presidency.

We thus see that when there is no Condorcet winner, the agenda can play an outsize role in the outcome of an election decided by sequential pairwise voting. Since the order in which the candidates are presented seems to matter, it's reasonable to guess that SPV violates the neutrality criterion, which we confirm in the next exploration.

Exploration 7.3.10 Suppose that all of the voters in the Bluth Company election switches the positions of Lindsay and Michael, yielding the preference schedule in Table 7.3.11.

Table 7.3.11 Revised Bluth Company preference schedule.

Rank	12	7	5	3
1	B	G	M	L
2	G	M	L	M
3	M	L	B	G
4	L	B	G	B

(a) Using sequential pairwise voting with the agenda G, L, B, M, what societal preference order would result from this new preference schedule?

(b) Explain why your answer to Exploration 7.3.7 and Question 1 show that SPV is not neutral.

Exploration 7.3.12 Is SPV anonymous? Monotone? Does it satisfy the majority criterion? Justify your answers.

Let's take stock of where we're at.

Activity 7.3.13 Fill in Table 7.3.14, indicating whether each voting system satisfies the given criteria.

Table 7.3.14 Voting systems and fairness criteria

	Anonymous	Neutral	Monotone	Majority	CWC	CLC
Majority rule						
Plurality						
Borda count						
Sequential pairwise voting						

7.4 Elections with More Than Two Candidates III

7.4.1 Worksheet

Motivating Questions.

In this section, we will explore the following questions.

(a) What is instant-runoff voting? What are its strengths and weaknesses?

(b) What is the independence of irrelevant alternatives, and why is it reasonable?

(c) What does Arrow's Theorem say, and what are its consequences for democracy?

In this section, we'll examine one last (new) voting system, called **instant runoff voting** (IRV) or **ranked choice voting** (RCV).

This system was proposed in the mid-1800s by Thomas Hare, and has slowly grown in popularity. Other than plurality, it is likely the most widely used system for choosing elected officials in the U.S. For example, the City of Minneapolis[1] uses IRV for its city-wide elections, and Maine uses it for all state-wide elections[2], including, for the first time in 2020, for the general election for president[3].

Definition 7.4.1 The instant runoff voting (IRV) system works as follows.

(a) Each voter in the election submits their entire preference order.

(b) The candidate (or candidates, in the case of a tie) with the fewest first-place votes is eliminated from each voter's preference order, and the remaining candidates are moved up, yielding a new preference schedule.

(c) Step 2 is repeated until a single candidate remains. That candidate is declared the winner.

◊

Activity 7.4.2 Consider the hypothetical preference schedule shown in Table 7.4.3.

Table 7.4.3 A hypothetical election

Rank	7	6	5	3
1	A	B	C	D
2	B	A	B	C
3	C	C	A	B
4	D	D	D	A

(a) Under IRV, which candidate is eliminated first?

(b) Under IRV, which candidate is eliminated second?

(c) Who would win the election under IRV? What would be the resulting societal preference order?

As has been our wont, let's explore which fairness criteria are satisfied by IRV.

Investigation 7.4.4 Use Definition 7.2.17 to write a thorough explanation of why IRV is both neutral and anonymous.

Investigation 7.4.5

(a) Explain why if, at any stage in the process of IRV, one candidate receives a majority of first-place votes, then that candidate can immediately be declared the winner of the election.

(b) Use your answer to Question 1 to explain why instant runoff voting satisfies the majority criterion.

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[1] `http://vote.minneapolismn.gov/rcv/index.htm`
[2] `http://www.rcvmaine.com`
[3] `https://www.huffpost.com/entry/maine-ranked-choice-voting-2020_n_5d72ca74e4b06451356df0f3`

Activity 7.4.6 Consider the election run in Activity 7.4.2. Suppose that the three voters in the rightmost column of Table 7.4.3 change their preferences to $A \succ D \succ C \succ B$. Note that this change is favorable only to candidate A.

(a) With these new preferences, who would win the election under IRV?

(b) Compare your answer to Question 1 of this activity to Question 3 of Activity 7.4.2. What conclusions can you draw?

(c) Do some research on FairVote.org[4] about this phenomenon. How do they respond to a potential critique?

Investigation 7.4.7 Consider an election between three candidates with the preference schedule shown in Table 7.4.8.

Table 7.4.8 A hypothetical election.

Rank	1	2	2
1	A	B	C
2	B	A	A
3	C	C	B

(a) Is there a Condorcet winner in this election?

(b) Who would win the election under IRV?

(c) Does IRV satisfy the Condorcet winner criterion? Use your answers to Questions 1 and 2, together with Definition 7.3.3 (and what we know about when implications are false! Recall Definition 3.1.11.).

Investigation 7.4.9

(a) What about the Bluths? Who wins the presidency of the Bluth Company with the preference schedule in Table 7.2.10 under IRV?

(b) Given all the investigations we've done into the Bluth Company president, write an argument to the company's board of directors arguing for a particular voting system to be used to choose the company's president. Your argument should make some allusion to the fairness criteria we've explored.

There is one more important fairness criterion we'll discuss. It's more subtle than some of the others, so we'll present a few versions of it. In short, what it says is that we want the societal preference between two candidates to depend *only on* the voters' preferences *between those two candidates.*

Definition 7.4.10 If a voting system has the property that the societal preference between any two candidates depends only on the voters' preferences between those two candidates, then the system is said to satisfy the **independence of irrelevant alternatives** criterion (IIA).

Put another way, a voting system satisfies IIA if some or all of the voters in an election change their preference ballots but no voter changes their preference between two candidates A and B, then the societal preference between A and B must also remain unchanged. ◊

IAA is often interpreted as saying that if a candidate (A) would win an election, and a new candidate (B) were added to the ballot, then either A or B should win the election. A further delightful illustration of a violation of IIA is attributed to Sidney Morgenbesser[5]:

> After finishing dinner, Sidney Morgenbesser decides to order dessert. The waitress tells him he has two choices: apple pie and blueberry pie. Sidney orders the apple pie. After a few minutes the waitress returns and says that they also have cherry pie at which point Morgenbesser says "In that case I'll have the blueberry pie."

Let's consider our current systems and IIA.

Investigation 7.4.11

(a) Does plurality satisfy IIA? Why or why not?

(b) Does the Borda count satisfy IIA? Why or why not?

(c) Does SPV satisfy IIA? Why or why not?

(d) Does IRV satisfy IIA? Why or why not?

At this point, you may be wondering which voting system is best. We saw a decisive answer if our election has only two candidates (the familiar *majority rule*), but when the election has more than two candidates, things have gotten complicated. Plurality, majority rule, the Borda count, SPV, and IRV all fail to satisfy at least one of our criteria.

[4] https://www.fairvote.org
[5] https://en.wikipedia.org/wiki/Sidney_Morgenbesser

In 1951, economist Kenneth Arrow[6] proved the following landmark result. It states that our quest is hopeless! There is no fairest voting system.

Arrow's Impossibility Theorem.

In an election with more than two candidates, it is impossible for a voting system to satisfy monotonicity, neutrality, anonymity, IIA, the majority criterion, and the CWC.

Discussion 7.4.12 *There is an ongoing debate about whether voting systems other than plurality are truly "democratic". For instance, lawmakers in Maine have repeatedly attempted to overturn the implementation of IRV for statewide elections, despite it passing on a statewide ballot initiative; see the "Timeline of Ranked Choice Voting in Maine" at the bottom of this page[7].*

What do you think? Regardless of which system you choose, what do you think of systems that take voter preferences beyond their first choice into account? What effect(s) do you think it might have on elections? On campaigning? Are there any other advantages or disadvantages you'd like to raise?

[6] https://en.wikipedia.org/wiki/Kenneth_Arrow
[7] http://www.rcvmaine.com

Chapter 8

Apportionment

8.1 Apportionment and Divisor Methods

8.1.1 Worksheet

Motivating Questions.

In this section, we will explore the following questions.

(a) What is apportionment?

(b) What is the mathematical question at the heart of the apportionment problem?

(c) What methods were proposed by Hamilton and Jefferson for apportionment of the U.S. house seats?

(d) What is the quota rule? Why is it reasonable?

Apportionment is the process by which the 435 seats in the U.S. House of Representatives are divided amongst the states. Every ten years, following a census, Congress must pass a new apportionment bill. A state's apportionment in Congress is also directly tied to its power in the electoral college, so the electoral votes after the 2024 presidential election will likely look different than it did following the 2020 election.

8.1.2 Apportionment

Warmup 8.1.1 *Round off the following fifteen numbers (i.e., turn them into whole numbers) so that the sum of the rounded numbers equals the sum of the unrounded numbers (which is exactly 105):*
Table 8.1.2

6.408,	*1.594,*	*2.226,*	*1.987,*	*8.622,*	*12.814,*	*3.826,*	*4.965,*
9.175,	*10.651,*	*1.864,*	*6.716,*	*2.301,*	*20.158,*	*11.693*	

What rounding method did you use? Describe in detail how you decided which numbers to round up to the next whole number, and which to round down to the previous whole number.

Article 1, Section 2 of the U.S. Constitution[1] states that representatives should be allocated to states based on population, but gives no method for doing so. The question of method is a thorny one, and has changed throughout history. Let's look at the first method for apportionment passed by Congress, due to Alexander Hamilton. To understand Hamilton's method, we need one bit of terminology: the **standard quota**.

Definition 8.1.3 The **standard quota** of a state is the exact number of seats it is entitled to, decimal and all. It is found by multiplying the state's proportion of the U.S. population by the number of seats in the legislature. ◊

The fundamental problem of apportionment, then, is as follows.

The Apportionment Problem.

Given a list of decimal numbers representing the standard quotas for each state, determine a fair method of allocating a fixed whole number of representatives to each state.

Alexander Hamilton proposed the following.

Hamilton's Apportionment Method.

(a) Find the standard quota for each state.

(b) Give each state a number of seats equal to its standard quota, rounded down (but at least 1).

(c) See how many seats are left to be allocated. Give those seats, one at a time, to the states whose standard quotas have the largest decimal parts.

[1] https://www.archives.gov/founding-docs/constitution-transcript

Table 8.1.4 Population totals by state, Census of 1790

State	Population
Connecticut	237,655
Delaware	59,096
Georgia	82,548
Kentucky	73,677
Maryland	319,728
Massachusetts	475,199
New Hampshire	141,899
New Jersey	184,139
New York	340,241
North Carolina	395,005
Pennsylvania	433,611
Rhode Island	69,112
South Carolina	249,073
Vermont	85,341
Virginia	747,550
Total:	3,893,874

Activity 8.1.5 Use Hamilton's method to apportion 105 seats to the fifteen states with population figures shown in Table 8.1.4. Which states were winners under Hamilton's method? Which were losers? What would it even mean to win/lose?

Then, for the standard quotas you determined for Maryland and Delaware, calculate the percentage that each state's decimal part is of its entire standard quota.

Exploration 8.1.6 For the standard quotas of each state in Activity 8.1.5, which state's decimal part makes up the largest percentage of its entire standard quota? The smallest?

In light of this, which state do you think was treated best in the apportionment from Activity 8.1.5? Worst?

Finally, articulate a critique of Hamilton's method.

Interestingly, Hamilton's method, passed by Congress, was the subject of the first presidential veto. No one knows exactly why President Washington vetoed it, but as we have seen, there are some problems with Hamiilton's method. For one, decimal parts are not really comparable. That is, similar decimal parts can represent very different numbers of people, depending on the populations of the states.

The method eventually passed by Congress and signed by Washington is due to Thomas Jefferson. We'll take a look at his method momentarily. But we're not done with Hamilton. Just you wait[2].

8.1.3 Divisors and the Quota Rule

The method that was ultimately approved by Congress and signed by Washington for the 1794 apportionment was due to Thomas Jefferson, and involved the calculation of the **standard divisor**.

Definition 8.1.7 If m seats are to be apportioned among a total population of N people, the **standard divisor** D is given by the formula $D = \frac{N}{m}$. For a state with population P, the state's **standard quota** is the number $\frac{P}{D}$. ◊

Exploration 8.1.8

(a) Recall the state population totals from the 1790 Census, presented again in Table 8.1.9. Calculate the standard divisor for the 1790 Census.

(b) Describe in words what the standard divisor *means*.

(c) Using the data below, calculate the standard quotas for each state.

[2] https://youtu.be/ZPrAKuOBWzw?t=544

Table 8.1.9 Population totals by state, Census of 1790

State	Population
Connecticut	237,655
Delaware	59,096
Georgia	82,548
Kentucky	73,677
Maryland	319,728
Massachusetts	475,199
New Hampshire	141,899
New Jersey	184,139
New York	340,241
North Carolina	395,005
Pennsylvania	433,611
Rhode Island	69,112
South Carolina	249,073
Vermont	85,341
Virginia	747,550
Total:	3,893,874

Jefferson's apportionment method is known as a *divisor method* because it works by modifying the standard divisor.

Jefferson's Method.

(a) Find the standard quota for each state.

(b) Round each standard quota down to the nearest whole number (but at least 1). If the sum of the rounded quotas equals the total number of seats being apportioned, we're done. Otherwise, continue to the next step.

(c) Choose a *modified* divisor different from the standard divisor, and use it to calculate modified quotas for each state by dividing the state's population by the modified divisor.

(d) Round each modified quota down to the nearest whole number (but at least 1). Sum the rounded modified quotas and check to see if the sum equals the number of seats to be apportioned. If so, we're done. Otherwise, repeat Step 3 with a different modified divisor.

Note that you may need to try Steps 3 and 4 a few times to get the sum of the quotas to equal the number of seats being apportioned. However, Jefferson's method has the advantage of rounding all the quotas the in the same way, potentially making it easier to understand and implement.

Activity 8.1.10 Use Jefferson's method to carry out the apportionment of 1794[3]. For each modified divisor that you try, write down the apportionment that results. Finally, describe how you knew whether to modfiy the divisor by making it larger or smaller based on the sum of the rounded modified quotas.

Question 8.1.11 Why will Step 2 of Jefferson's method always result in too few seats being apportioned? What will this cause you to always do to the standard quota to get to Step 3? What will the effect be for small states vs. large states? □

Exploration 8.1.12 In 1820, the Census recorded populations of 1,368,775 for New York and 8,969,878 for the entire U.S. Based on these numbers and the other states recorded, a total of 213 House seats were to be apportioned for the election of 1822.

(a) Using the 1820 Census data, calculate the standard divisor and New York's standard quota.

[3] You might find this Google sheet[4] helpful.
[4] https://docs.google.com/spreadsheets/d/1i5hEDQi1qVi9P0wY_CncNx06KOOyGE98RosP-e7cKhA/edit?usp=sharing

(b) In the apportionment of 1822, Jefferson's method with a modified divisor of 39,900 was used. Find New York's modified quota, and the final number of seats the state was given. Was this fair? Why or why not?

As one can imagine, this caused some consternation at the time. Unfortunately (though perhaps unsurprisingly), Congress did nothing. Perhaps they hoped the problem wouldn't happen again. But then it did, in the very next apportionment, in 1832. Further, it turns out that if Jefferson's method had continued to be used, something similar would have happened in every apportionment since 1852.

Discussion 8.1.13 *An alternative to Jefferson's method was proposed by John Adams, except rather than rounding the quotas down, he rounded up. Explain why this is no better than Jefferson's method.*

Definition 8.1.14 The **quota rule** states that, in an apportionment, each state should be given a number of seats equal to its standard quota, rounded up or down—no more, no less. An apportionment method is said to **violate quota** if it can produce apportionments which fail the quota rule

$\Diamond$

So, both Jefferson's method and Adams' method violate quota. The method used in 1842 was due to Daniel Webster, and modifies Jefferson and Adams only slightly. Rather than rounding the quotas always up or always down, Webster proposed that they should be rounded conventionally (i.e., a decimal part less than 0.5 gets rounded down, while a decimal part at least 0.5 gets rounded up)

It turns out that Webster's method can also violate quota, though this is much less likely. In fact, if Webster's method had been used in every apportionment from 1794 to 2012, it would not have violated quota once. As it happened, though, Webster's method was only used in 1842. Perhaps Congress was wary of continuing to use a method that could violate quota after 1822 and 1832.

In 1850, Samuel Vinton proposed what turned out to be Hamilton's method all over again, though no one remembered it as such[5] It happens that Hamilton's method cannot violate quota (why?). It was used for the apportionments of 1852 and 1862, though by 1872, Congress illegally and unconstitutionally apportioned seats based on no method at all. This had consequences for the presidential election of 1876, where Hayes defeated Tilden based on the electoral college, whereas if Hamilton's method had been used in 1872, Tilden would have easily won.

Congress went back to Hamilton's method in 1882, but not for long. In the next section, we'll look at some paradoxes that can arise from the apportionment process.

[5] Who tells your story, indeed.

8.2 Apportionment Paradoxes and Hills' Method

8.2.1 Worksheet

Motivating Questions.

In this section, we will explore the following questions.

(a) What surprising results can arise from the apportionment process?

(b) What does the Balinski-Young theorem say?

(c) What method of apportionment is used today?

We've already seen that apportionment is a trickier subject than it might seem at first. However, things are about to get downright weird as we study three apportionment *paradoxes*, by which we mean surprising or counterintuitive results. While these might seem farfetched, we'll see that they arose naturally in the history of apportionment in the U.S. and helped lead to the apportionment process we use today.

8.2.2 The Alabama Paradox

In the course of apportioning the House seats in 1882, the U.S. Census Bureau calculated apportionments using Hamilton's method for all sizes of House between 275 and 350 seats. Something strange was afoot for Alabama.

Exploration 8.2.1 The Apportionment of 1882. With a House size of 299 seats, Alabama's standard quota was 7.646, Illinois' was 18.640, and Texas's was 9.640. Ranked from largest to smallest by decimal part, Alabama was 20th on the list. With 299 seats, there were 20 surplus seats to dole out.

With 300 seats, the standard quotas for all states increased a bit. (Why?) Alabama's increased to 7.671, Illinois' to 18.702, and Texas' to 9.672. This moved Illinois into the 20th spot in line, with Texas next, and then Alabama. With a House of size 300, there were 21 surplus seats to be doled out, rather than 20.

(a) How many seats would Hamilton's method give to each of these states with 299 seats?

(b) How many seats would Hamilton's method give to each of these states with 300 seats?

(c) Does anything about these two potential apportionments strike you as strange? Unfair? Explain your thinking.

Definition 8.2.2 The phenomenon you observed in Exploration 8.2.1, in which a state loses a seat when the total number of seats increases, is known as the **Alabama paradox**. ◊

While it didn't present itself in the first few uses of Hamilton's method, the Alabama paradox can happen fairly regularly, and perhaps validates Washington's veto of Hamilton's method (though it is unlikely that Washington actually observed this phenomenon before deciding to veto Hamilton's method). Congress resolved this paradox by choosing a new House size of 325, a size at which the paradox did not occur.

Exploration 8.2.3 The Apportionment of 1902. For the 1902 apportionment, the Census Bureau gave Congress a table showing the apportionments under Hamilton's method for all House sizes from 350 to 400.

- When the number of seats in the House was 350-382, 386, 389, and 390, Maine would be given three seats, but for all other sizes Maine would be given four.
- When the number of House seats was 357, Colorado would be given two seats, but for all other House sizes Colorado would be given three seats.

Based only on these data, would the Alabama paradox have occurred for at least one House size between 350 and 400 seats (under Hamilton's method)? If so, for which House sizes and which states?

This was the end of Hamilton's method in apportioning House sizes. Congress used Webster's method on a final House size of 386 seats. However, Hamilton's method has been a fruitful source of additional paradoxes and so has remained a subject of study (in addition to its seemingly intuitive way of dealing with decimal parts).

8.2.3 The Population Paradox

Definition 8.2.4 The **population paradox** occurs when a small state with rapid population growth can lose a seat to a larger state with slower growth. ◊

Exploration 8.2.5 According to the Census Bureau, the population of Nevada grew from 1,998,257 in 2000 to 2,700,551 in 2010, and the population of Illinois grew from 12,419,293 in 2000 to 12,830,632 in 2010. In the 2012 apportionment, which was based on the these 2010 numbers, Nevada gained a seat in the House while Illinois lost a seat. Can you conclude that the population paradox occurred in 2012? Why or why not?

Exploration 8.2.6 According to the Census Bureau, the population of Nevada grew from 1,201,598 in 1990 to 1,998,257 in 2000, and the population of Illinois grew from 11,435,813 in 1990 to 12,419,293 in 2000. In the 2002 apportionment, which was based on the these 2000 numbers, Nevada gained a seat in the House while Illinois lost a seat. Can you conclude that the population paradox occurred in 2002? Why or why not? And why is this more interesting that Exploration 8.2.5?

8.2.4 The New-States Paradox and the Balinski-Young Theorem

Exploration 8.2.7 In 1907, Oklahoma was the 46th state to join the Union. Rather than completely reapportion the seats, Congress decided to give Oklahoma 5 seats, as this was in proportion to its share of the country's population. This increased the House size from 386 to 391. However, the following hypothetical apportionments were noted under Hamilton's method:

- With 45 states (pre-Oklahoma) and 386 seats, New York would have been awarded 38 seats and Maine 4 seats.
- With 46 seats (including Oklahoma) and 391 seats, New York would have been awarded 37 seats and Maine 5 seats.

Explain why this is yet another paradox (called the **new-states paradox**).

As it turns out, divisor methods (like Jefferson's, Adams', and Webster's) are not susceptible to these paradoxes. However, as we saw, they can violate quota. This is in contrast to Hamilton's method, which does not violate quota but is susceptible to these paradoxes. A reasonable person might wonder if there isn't some better method that both satisfies quota *and* cannot fall victim to these paradoxes. Fascinatingly, the answer is a resounding *no*.

Theorem 8.2.8 (Balinski-Young (1983)) *It is impossible for an apportionment method that divides representation among three or more states to satisfy quota and be incapable of producing paradoxes.*

8.2.5 Hill's Method

The last method we'll consider is the one used today. It was originally proposed by Joseph Hill, Chief Statistician of the U.S. Census Bureau, in the early 1900s and was eventually signed into law by FDR following the apportionment of 1942.

To understand Hill's method, a new perspective on conventional rounding is helpful. When we consider rounding a number like 11.494, we usually compare it to 11.5, which is halfway between the whole numbers 11 and 12. Another way to think of 11.5, though, is as the *mean* of 11 and 12: $\frac{11+12}{2} = 11.5$. But this mean is more properly known as the *arithmetic mean*. Another mean exists: the *geometric mean*.

Given whole numbers m, n, the geometric mean is the number $\sqrt{m \cdot n}$. In our example, $\sqrt{11 \cdot 12} = 11.4891$; this, then, is the number to which we'll compare our standard quotas in Hill's method. And thus, a standard quota of 11.494 would be rounded *up* to 12 rather than down to 11.

Hill's Method.

(a) Find the standard quota for each state.

(b) Round each standard quota based on the geometric mean of the two whole numbers between which the standard quota sits. If the sum of the rounded quotas equals the total number of seats being apportioned, we are done. Otherwise continue to the next step.

(c) Choose a modified divisor different from the standard divisor, and use it to calculate modified quotas for each state by dividing the state's population by the modified divisor.

(d) Round each modified quota based on the geometric mean of the two whole numbers between which the modified quota sits. If the sum of the rounded quotas equals the total number of seats being apportioned, we are done. Otherwise, repeat Step 3 with a different modified divisor.

Exploration 8.2.9 Recall the population table from 1790:

Table 8.2.10 Population totals by state, Census of 1790

State	Population
Connecticut	237,655
Delaware	59,096
Georgia	82,548
Kentucky	73,677
Maryland	319,728
Massachusetts	475,199
New Hampshire	141,899
New Jersey	184,139
New York	340,241
North Carolina	395,005
Pennsylvania	433,611
Rhode Island	69,112
South Carolina	249,073
Vermont	85,341
Virginia	747,550
Total:	3,893,874

(a) Use Hill's method to apportion 105 seats to the fifteen states with population figures shown in Table 8.2.10. Write down the apportionment that results from each modified divisor you try, including those that fail to give away exactly 105 seats.

(b) In your apportionment from question 1, which states were treated the best? Worst?

(c) How does this apportionment compare to what you found using Hamilton's, Jefferson's, Adams', and Webster's methods?

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