Exercise HMU 2.2.2.

We have to prove

$$\hat{\delta}(q, xy) = \hat{\delta}(\hat{\delta}(q, x), y) \qquad (*)$$

where $\hat{\delta}$ is defined by

$$\hat{\delta}(q, \varepsilon) = q \qquad (1)$$

$$\hat{\delta}(q, za) = \delta(\hat{\delta}(q, z), a) \qquad (2)$$

The proof is by induction on $|y|$.

Base case: Then $|y| = 0$ so $y = \varepsilon$. We calculate:

$$\hat{\delta}(q, x\varepsilon) = \hat{\delta}(q, x) \qquad \text{as } x\varepsilon = x$$

$$\hat{\delta}(\hat{\delta}(q, x), \varepsilon) = \hat{\delta}(q, x) \qquad \text{using (1)}$$

so $(*)$ follows.

Inductive case: The induction hypothesis gives

$$\hat{\delta}(q, xz) = \hat{\delta}(\hat{\delta}(q, x), z) \quad \text{if } |z| \leq k \qquad \text{(IH)}$$

Now assume $|y| = k+1$; then $y = za$ where $|z| = k$.

We calculate

$$\hat{\delta}(q, xza) = \delta(\hat{\delta}(q, xz), a) \qquad \text{using (2)}$$

$$= \delta(\hat{\delta}(\hat{\delta}(q, x), z), a) \qquad \text{using (IH)}$$

$$\hat{\delta}(\hat{\delta}(q, x), za) = \delta(\hat{\delta}(\hat{\delta}(q, x), z), z) \qquad \text{using (2)}$$

so $(*)$ follows.