Exercise HMU 3.4.2 (a)

It does not hold that $(R+S)^* = R^* + S^*$.

To see this choose R and S to be the regular expressions a and b, respectively. Then consider the string ab: we have $ab \in (a+b)^*$ but $ab \notin a^* + b^*$.

Exercise HMU 3.4.2 (b)

It does hold that $(RS+R)^* R = R(SR+R)^*$.

To see this we shall prove that for all $n \geq 0$:

$$(RS+R)^n R = R(SR+R)^n \qquad (\#)$$

We proceed by induction on n.

Base case: Then $n=0$ and we have

$$(RS+R)^0 R = \varepsilon R = R$$

$$R(SR+R)^0 = R\varepsilon = R$$

and clearly $(\#)$ holds.

Inductive case: We assume that $(\#)$ holds for n and have to prove it for $n+1$. We have

$$
\begin{aligned}
&(RS+R)^{n+1} R \\
&= (RS+R)(RS+R)^n R && (\text{defn. of } (..)^{n+1}) \\
&= (RS+R)\, R\, (SR+R)^n && (\text{ind. hyp.}) \\
&= (RSR+RR)(SR+R)^n && (\text{distr. law}) \\
&= R(SR+R)(SR+R)^n && (\text{distr. law}) \\
&= R(SR+R)^{n+1} && (\text{defn. of } (..)^{n+1})
\end{aligned}
$$

This completes the proof of $(\#)$.

The overall result follows since $w \in L((RS+R)^* R)$ implies $w \in L((RS+R)^n R)$ for some n and using (#) we the have $w \in L(R(SR+R)^n)$ which means $w \in L(R(SR+R)^*)$. In a similar way we can prove that if $w \in L(R(SR+R)^*)$ then $w \in L((RS+R)^*R)$.

An alternative – and smarter – proof goes as follows. First observe that

$$(LM)^* L = L(ML)^* \qquad (\#\#)$$

Then calculate as follows

$$\begin{aligned}
&(RS+R)^* R \\
&= (R(S+\varepsilon))^* R && \text{using } RS+R = R(S+\varepsilon) \\
&= R((S+\varepsilon)R)^* && \text{using } (\#\#) \\
&= R(SR+R)^*
\end{aligned}$$