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« وما أوتيتم من العلم إلا قليلا »

Il ne vous est donné en vérité
que fort peu de Science.

Le Saint Coran

A mes parents,
A ma femme,
A mes enfants,
A tous les miens.

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INTRODUCTION

Dans cette thèse nous étudions certains problèmes d'optimisation combinatoire, une des branches des mathématiques qui a connu un développement considérable durant les trentes dernières années.

Un problème d'optimisation combinatoire peut être défini comme étant celui de déterminer un plus petit (ou un plus grand) élément d'un ensemble fini valué.
Plusieurs problèmes concrets provenant de domaines divers, comme l'industrie, le transport, l'agriculture, l'économie, la gestion, peuvent être modélisés comme des problèmes d'optimisation combinatoire car ils mettent en jeu des variables de décision qui sont par nature entières et même souvent bivalentes.
La résolution de ces problèmes a suscité l'invention de nouvelles techniques théoriques comme l'optimisation dans les réseaux ou le développement de domaines plus anciens comme la théorie des graphes. Les recherches sur ces techniques ont permis à leur tour un élargissement du champ d'application de l'optimisation combinatoire dont on trouve aujourd'hui des exemples en médecine, biologie, physique, sociologie.

Complexité des problèmes

A première vue, il apparaît très simple de résoudre un problème d'optimisation combinatoire, vu le caractère fini de l'ensemble de ses solutions et on peut sans doute dire que ce type de problèmes n'existe pas pour la mathématique classique. On peut penser par exemple à une procédure qui consiste à énumérer toutes les solutions possibles et prendre la meilleure. Mais, en pratique, le nombre de ces solutions peut être très grand même pour des problèmes de petites tailles (ce nombre est généralement de l'ordre de 2^n où n caractérise la taille du problème). Or si n = 50, il faut au plus puissant ordinateur quelques centaines d'années pour éxtraire le plus petit élément parmi 2^{50} nombres et fournir une solution optimale. Vue cette efficacité désastreuse, il est clair que cette méthode énumérative triviale ne peut être envisagée que très rarement et que le problème central de l'optimisation combinatoire est celui de l'efficacité pratique des méthodes (algorithmes) qu'elle met en oeuvre.

Cette discipline ne peut donc se passer d'une théorie mathématique de la complexité des algorithmes qu'elle a puissamment contribuer à développer ces dernières années. Rappelons en brièvement les principes et résultats:

-la performance d'une méthode de résolution (algorithme) est en général mesuré par le "temps" d'exécution . Ce qui intéresse la théorie de la complexité c'est la manière dont varie ce temps en fonction de la taille (des données) du problème.

-En 1965 Edmonds a défini un algorithme "efficace" (ou polynomial) comme un algorithme tel que cette fonction puisse être borné par une fonction polynomiale . Un problème est dit polynomial (ou facile) s'il existe un algorithme polynomial qui le résoud. Parmi les problèmes polynomiaux, on trouve ceux du plus court-chemin,de l'arbre de poids minimum ou du couplage maximum.

-De la définition précédente il résulte que pour montrer qu'un problème est difficile il suffit de démontrer qu'il n'existe pas d'algorithme polynomial pour le résoudre. Malheureusement cette preuve semble hors de portée pour l'instant et on a à chercher plutôt à comparer la complexité de certains problèmes. C'est Cook [2] puis Karp [9] qui ont défini la classe des problèmes dits NP.Complets, pour lesquels on ne connaît pas d'algorithmes polynomiaux, mais qui possède la propriété que si un seul de ces problèmes était polynomial alors tous les problèmes NP (c'est à dire tous les problèmes de l'optimisation combinatoire) seraient polynomiaux.

Cook a trouvé, avec le problème dit de la satisfaisabilité, le premier représentant de cette classe, qui compte aujourd'hui plusieurs centaines de problèmes (qu'on peut consulter dans l'ouvrage de Garey et Johnson [3]), dont beaucoup de problèmes classiques comme celui de l'existence dans un graphe d'un cycle hamiltonien, ou d'un cocycle de cardinal donné.

Il est assez remarquable que cette séparation théorique entre problèmes faciles et difficiles qu'opère cette théorie de la complexité coïncide bien avec la distinction faite par les praticiens de l'optimisation combinatoire.

Optimisation combinatoire et Programmation Linéaire

C'est à G.B. Dantzig que revient le mérite d'avoir, le premier, proposé un algorithme, la méthode du Simplexe, pour résoudre les problèmes de Programmation Linéaire (c'est à dire les problèmes d'optimisation dont les variables sont continues, définis par des inéquations linéaires et une fonction objective linéaire). Non seulement cette méthode s'est révélée efficace en pratique mais elle a constitué un outil théorique très important de l' optimisation combinatoire.

Il est bien connu, en effet, qu'une solution optimale (de base) d'un programme linéaire donné, trouvée par l'algorithme du simplexe, correspond toujours à un point extrême du polyèdre défini par les inéquations du programme. Donc cet algorithme permet de résoudre un problème d'optimisation combinatoire dont l'ensemble des solutions correspond aux points extrêmes d'un polyèdre dont on connaît la caractérisation par un système d'inéquations linéaires.

Par conséquent, étant donné un problème d'optimisation combinatoire où toute solution peut être représentée par un point dans R^n (pour n donné), alors il est clair que si on peut caractériser le polyèdre, enveloppe convexe de ces points (voir Fig.1) par des inéquations linéaires, le problème se ramène à la résolution d'un simple programme linéaire.

(a) (b)

enveloppe convexe des points de la Fig.1 (a)

Fig. 1

Ce passage de l'optimisation sur un ensemble discret (qui peut avoir un nombre très grand de solutions possibles) à celui de l'optimisation sur un domaine convexe a été à l'origine d'un nouvel essor de l'optimisation combinatoire. En effet ce passage a permis d'envisager une nouvelle approche des problèmes d'optimisation combinatoire dite "polyèdrale". Cette approche consiste à ramener un tel problème à la résolution d'un programme combinatoire en décrivant l'enveloppe convexe de ses solutions par des inéquations linéaires. Cette approche a été véritablement initialisée par les travaux d'Edmonds sur le problème du couplage de poids maximum quand il a donné la caractérisation complète de l'enveloppe convexe des solutions de ce problème et par les travaux sur les coupes développés par Dantzig, Gomory et Balas. Par la suite, d'autres polyèdres ont été largement étudiés, en particulier celui associé au problème du voyageur de commerce [5] [6]. [7]

Malgré sa grande efficacité pratique l'algorithme du simplexe n'est pas polynomial. En effet il a été montré à partir d'un cas particulier que cet algorithme peut avoir une fonction de temps qui croît d'une manière exponentielle en fonction de la taille du problème. En 1979 le

Dans [8], J.F. MAURRAS et al ont décrit des algorithmes polynomiaux pour résoudre des classes particulières de programmes linéaires. En 1979 le mathématicien soviétique Kachian a proposé une nouvelle méthode dite "méthode des ellipsoides" qui permet de résoudre en temps polynomial tout programme linéaire donné. Cette méthode a joué par la suite un rôle central dans les derniers développements de l'optimisation combinatoire. Elle a permis d'engendrer des algorithmes polynomiaux pour des problèmes combinatoires en relation avec les polyèdres. En effet, en utilisant cette méthode, Grötschel, Lovàsz et Schrijver [4] ont montré qu'il existe un algorithme polynomial pour résoudre un problème d'optimisation sur un polyèdre donné si et seulement si il existe un algorithme polynomial pour le problème de séparation associé à ce polyèdre, c'est-à-dire un algorithme qui permet de décider pour un point x donné, si x appartient au polyèdre, et dans le cas contraire, de déterminer un hyperplan qui sépare x du polyèdre (une inéquation du polyèdre qui n'est pas vérifiée par x). La connaissance d'un algorithme polynomial pour le problème de séparation permet donc d'une part de montrer que le problème d'optimisation est polynomial et d'autre part de décrire un algorithme polynomial qui utilise la méthode des ellipsoides pour résoudre ce problème.

Par conséquent, étant donné un problème d'optimisation combinatoire, si ce problème est polynomial, alors il est généralement possible de donner une déscription linéaire complète à son polyèdre de solutions et de se ramener ainsi à l'étude d'un programme linéaire. Mais s'il est NP-complet, alors il y a un très faible espoir de pouvoir donner une telle caractérisation. Cependant, il est intéressant de montrer dans ce cas que certaines classes d'inéquations sont essentielles (facettes) pour décrire son polyèdre. D'après le résultat de Grötschel, Lovàsz et Schrijver, le système partiel décrit par ces inéquations peut être parfois suffisant pour donner en temps polynomial une solution optimale au problème. Ceci dépend de la complexité du problème de séparation associé à ce système.

Présentation de nos Travaux

C'est dans cette double filiation, de la théorie de la complexité et de l'utilisation de la Programmation Linéaire, que se situe l'essentiel des travaux présentés dans cette Thèse.

Dans le domaine du transport les techniques d'optimisation combinatoire sont largement appliquées pour déterminer des réseaux de transport optimaux. Le premier chapitre de cette thèse entre dans ce cadre. Nous étudions un problème de régulation de trafic qui nous a été soumis par l'Institut de Recherche sur le Transport à Paris. Nous donnons du problème un modèle mathématique qui conduit à la résolution d'un problème d'optimisation combinatoire.

Dans le deuxième chapitre nous nous intéressons à un des problèmes de la physique statistique qui relève de la combinatoire et de l'optimisation : la détermination du fondamental qui consiste à déterminer dans un système ayant un nombre très grand de configurations d'énergies une configuration qui minimise cette énergie.Dans le cas du modèle d'Ising des verres de spins, que nous allons étudier, ce problème peut se ramener à la recherche d'un cocycle de poids minimum dans un graphe donné . Dans le cas où le système est planaire (bidimensionnel), il a été montré que le problème est polynomial et il peut se résoudre d'une manière très éfficace en utilisant la théorie du couplage développée par J.Edmonds. Dans le cas où le système n'est pas planaire (tridimentionnel) il a été montré par contre que le problème est en général NP-complet.
Notre étude concerne en particulier l'approche polyédrale de ce problème. Nous étudions le polyèdre associé et nous caractérisons certaines classes de ses facettes. L'équivalence de ce problème avec celui du cocycle de poids minimum dans un graphe nous a ouvert la voie vers d'autres problèmes mathématiques intéressants que nous avons également étudiés.

Le troisième chapitre concerne deux généralisations du problème du verre de spins. La première sera donnée en terme de cocycle de poids maximum. Nous étudions le polyèdre associé à ce problème et nous donnons sa caractérisation dans la classe des graphes non contractibles à K_5. La deuxième sera donnée en terme d'indépendant de poids maximum. Nous étudions une composition particulière des systèmes d'indépendants et nous proposons une méthode qui permet de décrire les polyèdres correspondants. Nous étudions en particulier le cas des systèmes definis par les sous graphes sans circuit et nous décrivons le polyèdre associé dans une classe de graphes (orientés), celle des graphes symétriques non contractibles à $K_{3,3}$.

Dans le quatrième et dernier chapitre de cette thèse nous étudions deux autres problèmes d'optimisation combinatoire : l'absorbant et le K_i-recouvrement de poids minimum. Ces problèmes sont NP-complets dans le cas général. Nous avons essayé alors de décrire des systèmes de contraintes essentielles de leurs polyèdres associés, ainsi que des algorithmes efficaces pour les résoudre dans certains cas particuliers. Pour le premier problème nous donnons une déscription complète de son polyèdre dans la classe des graphes absorbants à seuil définie par Benzaken et Hammer [1]. Notre étude sur le problème du K_i-recouvrement nous a permis de donner des graphes dits "K_i-parfaits" une caractérisation très similaire à la conjecture forte des graphes parfaits.

Cette thèse est le résultat de plusieurs travaux écrits dans certains articles et rapports de recherche en cours de parution. Comme il a été présenté ci-dessus, elle comporte quatre parties qu'on va présenter ci-dessous avec les publications correspondantes.

I. Application de l' Optimisation Combinatoire à un Modèle de Transport

[I.1] "Improvement of traffic flow by the réduction of capacities"
avec P.H.Fargier, J.Fonlupt et J.P.Uhry.
RR n° 223. IMAG (soumis à Européen Journal of Operations Research).

[I.1]' "Résolution d'un problème de régulation de trafic"
Thèse de 3° cycle (partie I). Juin 81. IMAG.

II. Modèles de Verres de Spins et Optimisation Combinatoire

[II.1] "Composition of graphs and the bipartite subgraph polytope"
avec J. Fonlupt et J.P. Uhry. RR n°459. Laboratoire ARTEMIS (IMAG).
(soumis à Discrete Mathematics)

[II.2] "Facets of the bipartite subgraph polytope" avec F. Barahona et M. Grötschel.
Research Report n° 83264. Institut für Operations Research, Universität Bonn
(à paraître dans Mathematics of Operations Research).

III. Sur deux généralisations du probleme du Verre de Spins

[III.1] "On the cut polytope", avec F. Barahona.
Research Report n° 83271. Institut für Operations Research, Universität Bonn.
(à paraître dans Mathematical Programming).

[III.2] "Compositions in the acyclic subdigraph polytope"
avec F. Barahona. RR n°500. Laboratoire ARTEMIS (IMAG)
(soumis à Mathematics of Operations Research).

[III.3] "On a composition of Independence Systems by Circuit-Identification" avec R.Euler .RR n°501.Laboratoire ARTEMIS (IMAG) (soumis à Journal of Combinatoriol Theoty B).

IV. Etude algorithmique et polyédrale de deux problèmes d'optimisation combinatoire.

[IV.1] "Polytope des absorbants d'un graphe à seuil". Thèse de 3° cycle (partie II). (Annals of Discrete Maths, 17 (1983) 443-452)..

[IV.2] "K_i-covers I : complexity and polytope" avec M. Conforti et D.G. Corneil. RR n°444. Laboratoire ARTEMIS (IMAG). (à paraître dans Discrete Mathematics).

[IV.3] "K_i-cover II : K_i-perfect graphs" avec M. Conforti et D.G. Corneil. RR n°445. Laboratoire ARTEMIS (IMAG).(soumis à Journal of Combinatorial Theory B)

Chacune de ces parties qui constitue un chapitre de la thèse contient un résumé des résultats présentés dans les rapports correspondants. Ces documents sont tous mis en annexes . A la fin de chaque chapitre on donne la liste des références mentionnées dans ce chapitre. Si un auteur ou un résultat est cité sans référence alors celle-ci se trouve dans le travail décrit.

Notations et Définitions

Dans ce qui suit nous donnons quelques définitions et notations communes aux quatre chapitres.

Un graphe dont l'ensemble des sommets est V et l'ensemble des arêtes est E sera noté G = (V,E).

Les graphes considérés sont (sauf indication contraire), finis, non orientés, simples, sans arêtes multiples.

Une arête dont les extrémités sont v_i, v_j sera notée aussi $v_i v_j$. Un graphe H = (U, F) est un sous graphe de G si $U \subseteq V$ et $F \subseteq E$.

Le nombre de sommets |V| d'un graphe G est appelé l'ordre de G.

Pour un ensemble de sommets $U \subset V$, on note E(U) l'ensemble des arêtes dont les deux extémités sont dans U. De même on note V(F), pour $F \subset E$, l'ensemble des sommets adjacents aux arêtes de F.

Un cocycle de $G = (V, E)$ défini sur $U \subseteq V$, est l'ensemble d'arêtes de G dont une seule extrémité est dans U.

Pour un sommet $v \in V$, on note $\delta(v)$ l'ensemble d'arêtes adjacentes à v.

Un graphe $G = (V, E)$ est dit complet si $\forall v_i, v_j$, $v_i v_j \in E$. Un graphe complet d'ordre n sera noté K_n. Un sous graphe complet maximal d'un graphe G est dit une clique de G.

Un graphe est dit biparti si son ensemble de sommets peut être partitionné en deux sous ensembles V_1 et V_2 tels que toutes les arêtes du graphe aient un sommet dans V_1 et l'autre sommet dans V_2. Un graphe biparti G est dit maximal si pour toute arête ajoutée à G (entre deux sommets de G) le graphe obtenu n'est pas biparti.

Un graphe G est dit contractible à H, si H peut être obtenu à partir de G par une séquence de suppressions et de contractions d'arêtes. Chacune de ces dernières opérations consiste à identifier deux sommets adjacents de G. (Les arêtes multiples qui sont créées par cette identification sont remplacées par des arêtes simples).

Un polyèdre $P \subset R^n$ est une intersection finie d'hyperplans dans R^n. Il peut être défini aussi comme étant un ensemble de points de R^n vérifiant un nombre fini d'inéquations linéaires. un polytope est un polyèdre borné. On appelle enveloppe convexe d'un ensemble de points de R^n, le polyèdre défini par l'ensemble des combinaisons convexes de ces points. La dimension d'un polyèdre P donné est d-1 où d est le nombre maximum de points de P affinement indépendants.

Une inéquation $\sum a_i x_i = a^T x \leq a_0$ est dite valide pour P si elle est vérifiée par tout point $x \in P$. Un sous ensemble $Q \subseteq P$ est dit une face de P s'il existe une inéquation $a^T x \leq a_0$ valide pour P telle que $Q = \{x / a^T x = a_0\}$. Une face de P de dimension maximale (resp. zéro) est dite facette (resp. point extrême) de P. Etant donné un ensemble E (quelconque) et un sous ensemble F de E, on appelle vecteur d'incidence de F, le vecteur x^F défini par

$$x^F = \begin{cases} 1 & \text{si } e \in F \\ 0 & \text{sinon} \end{cases}$$

Ce vecteur est appelé aussi le vecteur représentatif ou caractéristique de F.

Pour $F \subseteq E$ on note $x(F) = \sum_{e \in F} x_e$. Enfin on note $\lceil x \rceil$ (resp. $\lfloor x \rfloor$) pour un réel x, le plus petit (resp. grand) entier $\geq x$ (resp $\leq x$).

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CHAPITRE I

APPLICATION DE L'OPTIMISATION COMBINATOIRE A UN MODELE DE TRANSPORT

1. Introduction

Dans ce chapitre nous étudions un problème de régulation de trafic par restrictions de capacités. Nous utilisons certaines techniques de l'optimisation combinatoire pour donner du problème un modèle mathématique et proposer une méthode qui permette de le résoudre.
Les phénomènes de congestion des réseaux routiers sont un des problèmes majeurs de la circulation urbaine et sont source de perte de temps à la fois pour l'usager et d'argent pour la collectivité. D'où la nécessité de trouver des politiques de régulation qui permettent d'éviter ces congestions ou de les déplacer de manière à améliorer au maximum l'écoulement du trafic dans le réseau.

Une congestion se manifeste souvent sur un tronçon par le fait que pendant un certain temps le trafic entrant sur ce tronçon est supérieur au trafic pouvant sortir de ce tronçon. Une approche possible pour modéliser ces situations de congestions est de considerer des réseaux avec capacités. C'est à dire de supposer que chaque tronçon e admet une capacité C(e) et que le trafic sur ce tronçon ne peut pas dépasser cette capacité. Une congestion se crée donc sur un tronçon e seulement si pendant une certaine période le trafic entrant dans e est supérieur à C(e). Dans ce cas une queue se crée sur le tronçon et le temps de passage est augmenté d'un temps dû à l'attente dans cette queue.
Une manière simple pour réguler un réseau est d'aggraver les conditions de circulation sur certains troncons. Cela peut sembler paradoxal mais il est facile de comprendre qu'en déplacant ainsi les congestions on peut améliorer le temps total passé par les usagers dans le réseau. Ceci est par exemple connu des automobilistes parisiens habitués aux restrictions d'accès sur les voies autoroutières.
Dans le modèle que nous utilisons cette régulation s'obtient simplement en abaissant les capacités

de certains tronçons, ce qui peut-être obtenu dans la pratique en supprimant une ou plusieurs voies ou bien en réglant un feu de signalisation d'une manière adéquate. Cette politique de régulation est l'objet de notre étude dans [I.1], [I,1]'.

Il existe (au moins) deux manières de modéliser le comportement des usagers dans le réseau.
Une première approche , qu'on peut qualifiée de dictatoriale, impose à chaque usager un itinéraire précis de manière à optimiser la satisfaction de l'ensemble, c'est à dire le temps total passé dans le système. Mais il est bien connu que cette régulation est purement théorique. Elle ne peut pas être appliquée dans la réalité puisqu'elle impose à certains usagers des itinéraires qui ne sont pas les plus intéressants pour eux, sauf à mettre un policier dans chaque voiture!

Il est beaucoup plus réaliste et intéressant d'étudier une politique de régulation qui suppose que les usagers s'affectent toujours selon leurs propres choix. Une telle affectation, dite optimale pour l'usager, vérifie le premier principe de Wardrop : "les coûts (temps de parcours) des itinéraires utilisés par le trafic entre une origine et une destination du réseau sont égaux et ne dépassent pas les coûts des itinéraires non utilisés". C'est cette approche que nous avons utilisée.

2. Le problème étudié

Considérons un réseau routier $R = (X, E)$ constitué de noeuds et de tronçons où chaque tronçon (arc) ayant une capacité maximum d'écoulement. Sur ce réseau circule un trafic entre plusieurs origines et plusieurs destinations.
Le problème que nous avons étudié consiste alors à trouver un modèle de régulation qui permette d'améliorer la circulation des usagers sur le réseau R, en abaissant éventuellement les capacités de certains arcs de ce réseau. On note $(\mathcal{P})$ ce problème.

Pour étudier le problème $(\mathcal{P})$ on s'est placé dans les hypothèses suivantes :

1) Toutes les caractéristiques du réseau ainsi que les données du trafic sont constantes (cas statique)
2) Le temps total de parcours $t(e)$ sur un tronçon e est linéaire si le trafic $f(e)$ vérifie $f(e) < C(e)$. Mais si $f(e) = C(e)$ alors ce temps peut être augmenté par un temps supplémentaire dû à la congestion qui peut être créée sur ce tronçon (voir Fig.2).

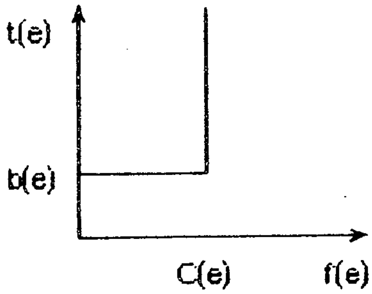


Fig. 2

où $b(e)$ est le temps de parcours à vide sur le tronçon e. Le temps de parcours réel $t(e)$ peut s'écrire donc

$$t(e) = b(e) + m(e)$$

où $m(e)$ est le temps supplémentaire de congestion.

La deuxième hypothèse qui est fondamentale dans notre étude a été utilisée par Fargier et Fonlupt (1981) dans leur étude sur le problème de l'affectation du trafic dans les réseaux avec capacités. Malgré le caractère très simplificateur de ces hypothèses les résultats obtenus dans cette étude restent significatifs dans des situations réalistes. (ex. régulation d'un corridor autoroutier).

3. Modélisation mathématique du problème

Notons $F_{ij}(e)$, le trafic circulant entre une origine x_i et une destination x_j du réseau $R = (X, E)$. Soit $f_{ij}(e)$ la portion de ce trafic qui circule sur l'arc e.

Une affectation optimale de trafic dans le reseau $R=(X,E)$ peut être définie comme une solution du programme linéaire

$$(P) \quad \left\{ \begin{array}{lll} \displaystyle\sum_{e/x_k=T(e)} f_{ij}(e) - \sum_{e/x_k=I(e)} f_{ij}(e) = \left\{ \begin{array}{l} -F_{ij} \text{ si } k=i \\ 0 \quad \text{si } k \neq i, j \\ F_{ij} \text{ si } k=j \end{array} \right. & & (1.1) \\ \displaystyle\sum_{x_i, x_j \in X} f_{ij}(e) = f(e) \leq c(e) & \forall\, e \in E & (1.2) \\ f_{ij}(e) \geq 0 & \forall\, e \in E & (1.3) \\ Z = \operatorname{Min} \displaystyle\sum_{e \in E} b(e)\, f(e) & & \end{array} \right.$$

D'après la dualité en programmation linéaire, une solution $f = (f_{ij}(e), e \in E)$ de (P) est optimale s'il existe une solution $(\Delta, m) = (\Delta(i,j,k); m(e), x_i, x_j, x_k \in X; e \in E)$ du dual de (P) telle que pour chaque $e = (x_k, x_l)$

$$\begin{cases} m(e) \geq 0 \qquad \forall\, e \in E & (1.4) \\ \Delta(i,j,l) - \Delta(i,j,k) \leq b(e) + m(e) & (1.5) \\ \text{si } f_{ij}(e) > 0,\ \Delta(i,j,l) - \Delta(i,j,k) = b(e) + m(e) & (1.6) \\ \text{si } m(e) > 0,\ f(e) = C(e) & (1.7) \\ \sum_{e \in E} b(e) f(e) = \sum_{x_i, x_j \in E} (\Delta(i,j,j) - \Delta(i,j,i))\, F_{ij} - \sum_{e \in E} m(e) C(e) & (1.8) \end{cases}$$

La valeur d'une variable $\Delta(i,j,k)$ peut être interprétée comme étant le temps nécessaire à un usager allant de x_i à x_j pour arriver à x_k. Ainsi on peut voir qu'une affectation optimale $(f_{ij}(e), e \in E)$ selon le premier principe de Wardrop, dans le réseau $R = (X, E)$ muni du système de capacité $C = (C(e), e \in E)$, peut être déterminée comme une solution du système linéaire $W(C)$, défini par les contraintes (1,1) - (1,8). Le temps total passé par les usagers dans cette affectation est donc

$$T = \sum_{e \in E} (b(e) + m(e)) f(e)$$

Nous avons alors

Théorème 1.1 :

Le problème (P) est équivalent au problème suivant

$$(P1) \begin{cases} (f, \Delta, m, C) \in W(C) \\ \text{tq } T \text{ soit minimun} \end{cases}$$

Le problème (P1) consiste donc à trouver un système de capacités $(C(e), e \in E)$ dans le réseau R qui minimise T. Ce problème, comme on peut le remarquer est un programme qui n'est pas en général convexe. Et ceci montre la difficulté du problème qui ne peut être résolu que par des méthodes adaptées (comme la méthode de Branch and Bound ou la méthode des coupes). Ce problème est difficile à résoudre même dans le cas d'un réseau simplifié avec une origine et une destination, où une telle régulation peut être intéressante. Nous avons donné dans [I.1] deux exemples significatifs qui montrent bien l'intérêt de la régulation dans ce cas. Notons de plus que si on suppose que la régulation ne modifie guère l'affectation, qui est une hypothèse couramment faite dans la pratique, alors, dans ce cas, le problème (P1) devient équivalent à un simple programme linéaire.

4. Résolution du problème

Comme nous l'avons mentionné, les méthodes de séparation et évaluation sont couramment utilisées pour la résolution effective de la plupart des problèmes d'optimisation combinatoire. Nous avons utilisé cette technique pour résoudre complètement le problème ($\mathcal{P}$) dans le cas où le réseau R comporte une origine et plusieurs destinations. Pour ce faire on a ramené le problème à la recherche d'un sous réseau R* approprié de R. La procédure d'évaluation utilisée est basée sur l'affectation de coût minimum (définie par le programme (P)), et la procédure de séparation consiste en chaque sommet de l'arborescence à interdir certains arcs de R et à obliger d'autres d'être dans R*. Nous avons testé par la suite cette méthode sur un réseau (ayant une origine et une destination) représentant une partie de la ville de Grenoble. Les résultats expérimentaux obtenus, bien qu'ils ne soient pas généraux, montrent en particulier que l'effet de la régulation étudiée, peut dépendre surtout de l'intensité du trafic qui circule dans le réseau et des caractéristiques de ce dernier. (Ces résultats sont présentés dans [I.1]).

Notons en conclusion que le modèle proposé, vues les hypothèses simplificatrices sur lesquelles il repose, ne peut pas être un outil de régulation en temps réel pour des cas concrets, mais il peut permettre de répondre à des questions intéressant les praticiens concernant par exemple les politiques de régulation qu'ils peuvent appliquer dans des cas types (ex. accidents sur autoroute), ou le gain du temps qu'on peut espérer par une régulation dans des réseaux complexes. Enfin ce modèle paraît constituer plus un outil " pédagogique" permettant aux praticiens de mieux cerner la réalité complexe d'un système routier plutôt qu'un outil directement opérationnel.

CHAPITRE II

MODELES DE VERRES DE SPINS ET OPTIMISATION COMBINATOIRE

1. Introduction

Un verre de spins est un système obtenu par une faible diluation (1%) d'un matériau magnétique (Fer) dans un matériau non magnétique (Or). L'intérêt des physiciens pour ce matériau vient de l'observation d'un pic dans la courbe de la susceptibilité magnétique X(T) en fonction de la température T (Fig. 3).

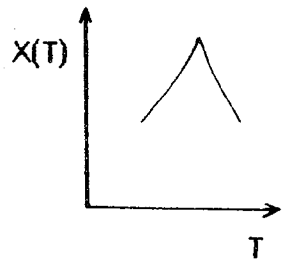


Fig.3

Un tel pic est généralement l'indice d'une transition de phase, d'un changement d'état du système. D'où la recherche de modèles susceptibles d'expliquer ce phénomène.
Dans un verre de spins les atomes magnétiques sont disposés aléatoirement dans l'espace. Entre deux atomes il existe une énergie d'interaction:

$$H_{12} = - J(R)\, S_1 . S_2$$

où S_i est le moment magnétique (spin) de l'atome i

$$J(R) = \frac{A \cos(2BR)}{(BR)^3}$$

R étant la distance entre les deux atomes, A et B étant des constantes (Fig. 4).

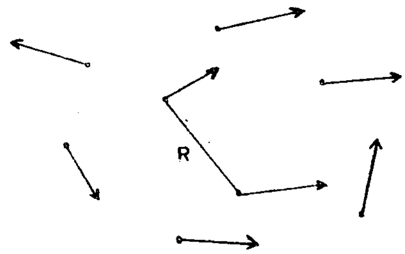
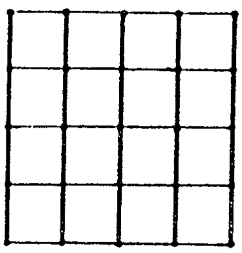
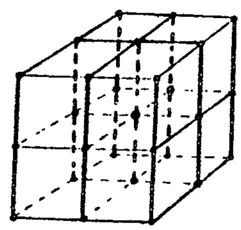


Fig.4

Pour bien modéliser ces systèmes, les physiciens ont construit un modèle très simplifié : ils supposent que les spins sont situés aux noeuds d'un maillage carré régulier (au lieu d'être répartis aléatoirement) et sont définis par des vecteurs unidimentionnels (au lieu d'être tridimentionnels) S_i prenant les valeurs +1 ou −1. Ces maillages sont généralement planaires carrés (Fig. 5 (a)) ou cubiques (Fig. 5 (b)).



(a)



(b)

Fig.5

Ils supposent en plus que les interactions entre les spins n'ont lieu qu'entre plus proches voisins (Elles sont représentées par les arêtes du maillage) et que leurs énergies sont des variables aléatoires indépendantes pouvant prendre des valeurs positives ou négatives. Une arête telle que l'énergie correspondante est positive (resp. négative) est dite liaison positive (resp. négative).

A une configuration des spins S (affectation des valeurs +1 ou -1 aux spins) correspond une énergie du système. Cette énergie est définie par

$$H(S) = - \sum J_{ij} S_i S_j \qquad (2.1)$$

où J_{ij} est l'interaction entre les spins i et j.

Les physiciens se posent depuis longtemps des questions sur les caractéristiques des états d'énergie minimums d'un verre de spins (à la température T= 0K) et sur le calcul de la fonction de partition qui est définie par

$$f(T) = \sum_{S \in \mathcal{S}} \exp(H(S)/KT)$$

où $\mathcal{S}$ est l'ensemble des configurations possibles des spins, K la constante de Boltzmann et T la température.

La fonction de partition a été obtenue par Onsager [8] en 1944 pour la grille carrée bidimentionnelle dans le cas férromagnétique où toutes les énergies d'interaction sont positives, ce qui lui a valu le prix Nobel. Quelques années plus tard Temperely [9] avait résolu la même question (dans le même cas ferromagnétique) en dénombrant les couplages parfaits d'un graphe planaire. Ce qui a montré la grande efficacité de l'optimisation combinatoire pour ce genre de modèles.

Pour déterminer un **état fondamental** d'un verre de spins (une configuration S qui minimise l'énergie H(S) du système), les physiciens utilisent traditionnellement des heuristiques de type Monte-Carlo (des méthodes qui consistent, à partir d'un état initial, de changer des configurations de spins de telle manière à diminuer l'énergie dans le système) pour obtenir des solutions approchées, même dans le cas où le maillage est planaire (travaux de Kirpatrick [7], Binder [4]).

Les recherches menées depuis certaines années dans l'équipe de Recherche Opérationnelle du Laboratoire ARTEMIS de l'Institut I.M.A.G., en collaboration avec MM. Meynard et Rammel du Laboratoire sur les basses températures du C.N.R.S. ont montré encore l'intérêt de l'optimisation combinatoire pour ce type de problèmes. En effet, dans leur thèse, E. Bièche [3] et F. Barahona [2] ont étudié le problème du fondamental, qu'on désignera dans la suite par ***PF***, dans le cas où le maillage décrivant le verre de spins est planaire (cas bidimentionnel). Ils ont montré que dans ce cas le problème est équivalent au problème du postier chinois . Ceci leur a permis d'étudier des approches efficaces de résolution, basées sur la théorie du couplage. L'approche étudiée par Bièche consiste à déterminer un couplage parfait de poids minimum dans un graphe complet approprié alors que celle étudiée par Barahona fait intervenir un algorithme de résolution directe du problème du postier chinois. Les deux approches nécessitent un temps de calcul de l'ordre de $O(n^3)$ où n est la taille de la maille. D'autre part Barahona a montré également que le calcul de l'entropie (dénombrement des états) et le calcul de la fonction de partition dans le cas planaire sont des problèmes qui peuvent être résolus aussi en temps polynomial. Il a montré par contre que dans le cas tridimentionnel le problème *PF* est NP-complet.

Puisque le problème *PF* est NP-complet dans le cas tridimentionnel, alors, comme on l'a déjà signalé, une approche possible de ce problème dans ce cas est d'étudier le polyèdre, enveloppe convexe de ses solutions. L'objet de ce chapitre est donc d'étudier ce polyèdre dans le cas général et de caractériser certaines classes de ses facettes, pour pouvoir simuler des échantillons tridimentionnels de tailles non triviales. Dans la suite nous donnons quelques résultats préliminaires utiles pour le reste du chapitre.

2. Préliminaires

2.1. Le problème *PF* en terme de graphes

A un système, verre de spins donné, on peut associer un graphe GS= (X, F) tel que X représente l'ensemble des spins et F soit partitionné en deux ensembles F^+ et F^- tel que F^+ (resp.F^-) represente les liaisons positives (resp. négatives). On associe à chaque liaison ij le poids $|J_{ij}|$. L'énergie totale du système donnée par (2.1) s'écrit donc

$$H(S) = -\sum_{ij\in F^+} |J_{ij}|\, S_i S_j + \sum_{ij\in F^-} |J_{ij}|\, S_i S_j \qquad (2.2)$$

Le problème *PF* se ramène donc à déterminer, dans le graphe GS, une affectation de l'ensemble des sommets X dans {-1, +1} qui minimise (2.2). Dans ce qui suit nous montrons qu'on peut étendre le graphe GS à un autre graphe dont toutes les liaisons sont négatives.

En effet, considérons le graphe G = (V, E), déduit de GS de la manière suivante : on remplace chaque liaison positive ij par une chaîne formée de deux liaisons négatives. A chacune de ces liaisons on associe le poids $-J_{ij}$. Le reste du graphe GS reste inchangé.

Lemme 2.1 :
A toute solution optimale dans GS on peut faire correspondre une solution optimale dans G et vice versa.

Démonstration :
Sans perte de généralité on suppose que $J_{ij} \neq 0 \;\; \forall\, ij$. Si S' est une affectation optimale dans G alors pour toute chaîne (i, k, j) correspondant à une liaison ij, on a : Si $S_i' = S_j' = +1$ (resp. -1) alors $S_k' = -1$ (resp. $+1$) (sinon S' ne peut pas être optimale). Considérons à partir de S', l'affectation S aux sommets de X définie par $S_i = S_i' \;\; \forall\, i \in X$. Il est simple de vérifier que :

$$H(S') = H(S) - \sum_{ij \in F} |J_{ij}|$$

Par conséquent S' est optimale dans G si et seulement si S est optimale dans GS.
■

Le lemme 1.1 montre que le problème de la frustration, associé à un verre de spins donné, peut toujours se ramener à déterminer, dans un graphe $G = (V, E)$ où toute arête e est munie d'un poids $w_e > 0$, une affectation $S : V \longmapsto \{-1, +1\}$ qui minimise :

$$H(S) = \sum_{e=ij \in E} w_e S_i S_j \tag{2.3}$$

Notons de plus que si le verre de spins est planaire (bidimentionnel) alors le graphe G l'est aussi.

2.2. Le problème *PF* en terme de sous graphe biparti de poids maximum

Pour une affectation S des valeurs +1 ou −1 aux sommets du graphe $G = (V, E)$, considérons la bipartition V_1, V_2 de V telle que $V_1 = \{i / S_i = +1\}$ et $V_2 = \{i / S_i = -1\}$. Nous avons :

$$H(S) = \sum_{ij \in V_1} w_{ij} + \sum_{ij \in V_2} w_{ij} - \sum_{i \in V_1, j \in V_2} w_{ij}$$

où w_{ij} est le poids de la liaison ij.
Donc

$$H(S) = \sum_{ij \in E} w_{ij} - 2 \sum_{i \in V_1, j \in V_2} w_{ij}.$$

Par conséquent, minimiser H(S) est équivalent à maximiser $\sum_{i \in V_1, j \in V_2} w_{ij}$, le poids du cocycle défini par la bipartition de V en V_1 et V_2.

Puisque les poids sur les arêtes sont tous (strictement) positifs alors un sous graphe biparti de poids maximum de G est toujours maximal et correspond par conséquent à un cocycle. Une solution optimale de *PF* dans G est caractérisée donc par un sous graphe biparti (maximal) de G de poids maximum.

Soit $P_B(G)$ l'enveloppe convexe des vecteurs X^F, où (V, F) est un sous graphe biparti de G. $P_B(G)$ sera appelé le polytope des sous graphes bipartis de G. Une solution optimale du problème *PF* peut donc être considérée comme une solution de base du programme

$$\text{Max}\ \{wx,\ x \in P_B(G)\}$$

Si ce programme peut être résolu en temps polynomial alors le problème *PF* peut l'être aussi. Notons de plus que si le polytope $P_B(G)$ est défini par le système d'inéquations linéaires

$$(2.4) \qquad \begin{cases} Ax \leq b \\ 0 \leq x \leq \mathbb{1} \end{cases}$$

où A et b sont données, et $\mathbb{1}^T = (1, 1, ..., 1)$ alors le polyèdre

$$(2.5) \qquad \begin{cases} Ax \geq b^* \\ x \geq 0 \end{cases}$$

déduit de (2.4) en changeant x par $\mathbb{1}$-x, a tous ses sommets en 0-1, et si x* est un de ses sommets alors $\mathbb{1}$-x* est un sommet de (2.4) et représente un sous graphe biparti maximal de G. Ceci implique que le polyèdre des solutions du problème *PF* est en effet le polyèdre (2.5) dont l'étude est équivalente à celle du polytope $P_B(G)$.

Il est bien connu qu'un graphe est biparti si et seulement si il ne contient pas de cycle impair.

Etant donné un graphe $G = (V, E)$, il est clair donc que les contraintes :

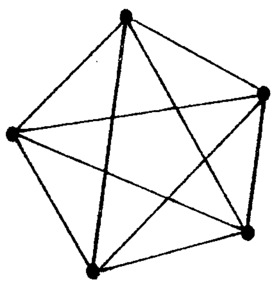
$$x(C) \leq |C| - 1 \qquad C \text{ cycle impair de } G \tag{2.6}$$
$$0 \leq x_e \leq 1 \qquad e \in E \tag{2.7}$$

sont valides pour le polytope $P_B(G)$.

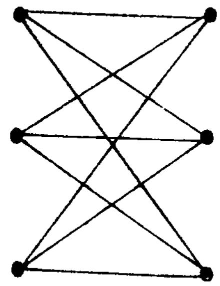
Dans [1], Barahona a montré que ces contraintes définissent des facettes de $P_B(G)$ et caractérisent ce polytope dans le cas où G est planaire.

En 1981, Grötschel et Pulleyblank ont proposé un algorithme polynomial basé sur la théorie du couplage qui permet de résoudre le problème de séparation pour les contraintes (2.6) et (2.7). Par conséquent, cet algorithme combiné avec la méthode des ellipsoides permet de résoudre en temps polynomial le problème *PF* dans les graphes G dont le polytope $P_B(G)$ est décrit complètement par les contraintes (2.6) et (2.7). Grötschel et Pulleyblank ont appelé ces graphes, les graphes faiblement bipartis. Les graphes planaires et les graphes bipartis font partie donc de cette classe de graphes. Dans un premier temps de notre étude nous avons essayé d'étendre la classe des graphes planaires à une classe plus générale de graphes faiblement bipartis.

On sait qu'un graphe est planaire si et seulement si il n'est pas contractible à K_5 ou à $K_{3,3}$ (Fig.6).



K_5



$K_{3,3}$

Fig.6

De plus on peut voir que les contraintes (2.6) et (2.7) ne suffisent pas pour caractériser le polytope $P_B(K_5)$. En effet la solution $x_e = 2/3$, pour $e \in K_5$ définit dans ce cas un point extrême du polyèdre décrit par (2.6) et (2.7). Pour séparer ce point, du polytope $P_B(K_5)$, on doit ajouter la contrainte

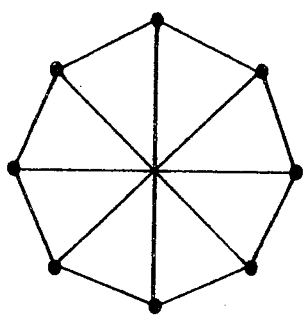
$$\sum_{e \in K_5} x_e \leq 6 \tag{2.8}$$

qui définit une facette de $P_B(K5)$. De même cette contrainte définit une facette de $P_B(G)$ pour tout graphe G contenant un K_5. Il est, dès lors, assez naturel de se demander si les graphes non contractibles (seulement) à K_5 sont faiblement bipartis. Cette question est le point de départ du travail que nous présentons dans le paragraphe suivant.

2. Composition of graphs and the bipartite subgraph polytope

(article [II.1] en collaboration avec J. Fonlupt et Jean-Pierre Uhry)

Etant donnés deux graphes G_1 et G_2, on dit que le graphe G est une k-somme de G_1 et G_2, noté $G = G_1 \boxed{k} G_2$ si G est obtenu à partir de G_1 et G_2 en identifiant deux graphes complets K_k contenus dans ces graphes (voir Fig. 9 (a) ,le cas où k=3). Wagner a montré que tout graphe non contractible à K_5 peut être obtenu par une suite finie de k-somme avec $1 \leq k \leq 3$ à partir des graphes planaires et des copies du graphe V_8 (Fig. 7). Dans cet article on s'est intéressé à l'étude du problème *PF* dans les graphes définis par des opérations de k-sommes, $1 \leq k \leq 3$, à partir d'autres graphes quelconques. Les graphes non contractibles à K_5 ne sont donc qu'une classe particulière de ces graphes.



Le graphe V_8

Fig. 7

Soit $G = (V, E)$ un graphe, k-somme de $G_1 = (V_1, E_1)$ et $G_2 = (V_2, E_2)$, tel que G_1 $G_2 = K_k$ et $1 \leq k \leq 3$. On peut alors remarquer qu'un cycle élémentaire C de G, qui intersecte $G_1 \backslash K_k$ et $G_2 \backslash K_k$, est impair si et seulement si il existe deux cycles C_1 et C_2 contenus respectivement dans G_1 et G_2 tels que si l'un est pair (resp. impair) l'autre est impair (resp. pair) et $C = C_1 \cup C_2 \backslash e_0$ où e_0 est une arête de K_k (voir Fig. 8)

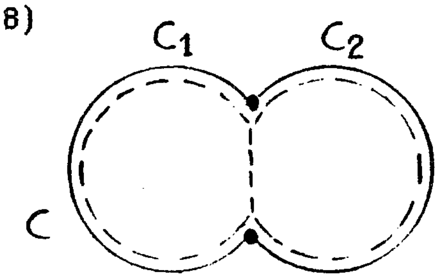


Fig.8

Maintenant si on considère les graphes $\bar{G}_1$ et $\bar{G}_2$ obtenus à partir de G_1 et G_2 en ajoutant entre chaque couple de sommets de K_k une chaîne de longueur deux (voir Fig. 9 (b) le cas où $k=3$) alors tout cycle pair de G_1 (resp. G_2) contenant une arête de K_k va correspondre à un cycle impair de $\bar{G}_1$ (resp. $\bar{G}_2$) contenant une chaîne supplémentaire et vice versa. Par conséquent tout cycle impair (élémentaire) dans G peut être défini à partir de cycles impairs dans $\bar{G}_1$ et $\bar{G}_2$. Ceci nous a permis d'étudier le problème ***PF***, sous les deux aspects algorithmique et polyédral, dans le graphe $G = G_1 \boxed{k} G_2$ à partir des graphes $\bar{G}_1$ et $\bar{G}_2$.

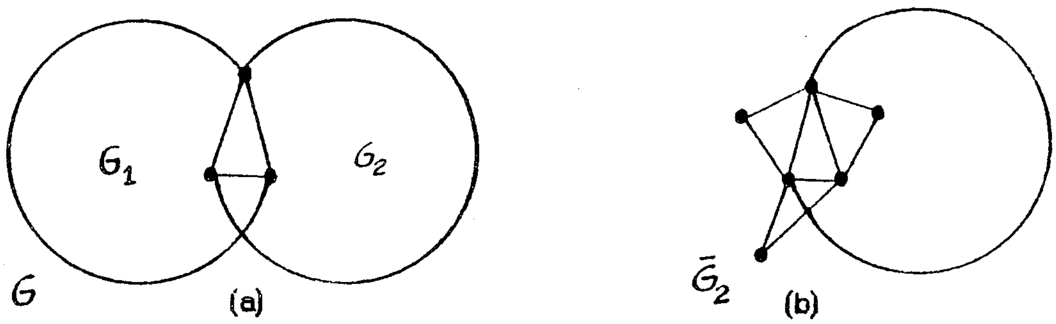


Fig. 9

En ce qui concerne l'aspect algorithmique nous avons proposé une méthode qui permet de calculer une solution optimale de ***PF*** dans G à partir des solutions optimales dans $\bar{G}_1$ et G_2. L'idée de la méthode consiste à ramener le calcul de la solution optimale dans G au calcul d'une solution optimale appropriée dans $\bar{G}_2$. Ceci permet par la suite une sorte de décomposition dans la recherche d'une solution optimale de ***PF*** dans un graphe G défini par une suite finie de k-somme, $1 \leq k \leq 3$ à partir d'autres graphes $G_1, ..., G_n$, deplus si l'on peut résoudre ***PF*** entemps polynomial dans les graphes $\bar{G}_1, ..., \bar{G}_n$ correspondants et si n est polynomial, alors on peut le résoudre également en temps polynomial dans G. Puisque le problème ***PF*** est polynomial dans les graphes planaires et dans V_8 (par énumération), alors d'après la décomposition de Wagner le problème ***PF*** reste polynomial dans les graphes non contractibles à K_5. Ce cas particulier est également démontré par Barahona.

Notre approche polyédrale a consisté par la suite à étudier le polytope $P_B(G)$ à partir des polytopes $P_B(\bar{G}_1)$ et $P_B(\bar{G}_2)$.

Notons d'abord que si un graphe H contient une chaîne sans corde alors les coefficients des variables associées aux arêtes de cette chaîne, dans toute inéquation définissant une facette de $P_B(H)$, sont égaux.

Considérons maintenant deux contraintes :

$$\sum_{e\in\bar{E}_1} a_e x_e + \sum_{e\in K_k} a_e x_e + \sum_{e\in K_k} \bar{a}_e (x_{e_1} + x_{e_2}) \le \partial \qquad (2.9)$$

$$\sum_{e\in\bar{E}_2} b_e x_e + \sum_{e\in K_k} b_e x_e + \sum_{e\in K_k} \bar{b}_e (x_{e_1} + x_{e_2}) \le \beta \qquad (2.10)$$

définies respectivement comme combinaisons linéaires des contraintes de $P_B(\bar{G}_1)$ et $P_B(\bar{G}_2)$, où $\bar{E}_i = E_i \setminus K_k$, $i=1, 2$ et e_1, e_2 définissent la chaîne ajoutée entre les sommets de e dans la construction de $\bar{G}_1$ et $\bar{G}_2$. Supposons de plus

$$\text{et} \quad \begin{cases} a_e \ge \bar{b}_e \\ b_e \ge \bar{a}_e \end{cases} \qquad \forall\, e \in K_k$$

alors on peut définir à partir de ces contraintes((2.9) et (2.10))une contrainte dite mixée valide pour le polytope $P_B(G)$ telle que

$$(2.11) \qquad \sum_{e\in\bar{E}_1} a_e x_e + \sum_{e\in\bar{E}_2} a_e x_e + \sum_{e\in K_k} ((a_e + b_e) - (\bar{a}_e + \bar{b}_e))\, x_e \le \partial + \beta - 2 \sum_{e\in K_k} (\bar{a}_e + \bar{b}_e)$$

Soit (P) l'ensemble convexe de $R^{|E|}$ défini par

i) les contraintes de $P_B(\bar{G}_1)$ et $P_B(\bar{G}_2)$ dont le support n'intersecte pas les chaînes supplémentaires

ii) les contraintes (2.11)

Le résultat fondamental de ce paragraphe est le :

<u>Théorème 2.1 :</u>

a) (P) est un polytope

b) (P) = $P_B(G)$

à partir de ce théorème nous avons déduit une caractérisation minimale (où toute contrainte étant une facette) du polytope $P_B(G)$ dans le cas où le graphe G est défini par une opération de 2-somme à partir de G_1 et G_2. Cette caractérisation est très simple à obtenir à partir de $P_B(\bar{G}_1)$ et $P_B(\bar{G}_2)$.

De même nous avons déduit le :

Corollaire 2.2 :
Soit G une k-somme de G_1 et G_2 avec $1 \leq k \leq 3$. Si G_1 et G_2 sont faiblement bipartis alors G est faiblement biparti.

A partir de ce corollaire et de la décomposition de Wagner des graphes non contractibles à K_5, on obtient :

Théorème 2.3 :
Les graphes non contractibles à K_5 sont faiblement bipartis.

Puisque les graphes non contractibles à K_5 sont faiblement bipartis, alors comme on l'a déjà signalé, le problème *PF* peut se résoudre aussi dans ces graphes en temps polynomial en utilisant la méthode des ellipsoides et l'algorithme de Grötschel et Pulleyblank déjà mentionné.
Par le corollaire 2.2 nous montrons aussi que les graphes définis comme des k-sommes des graphes non contractibles à K_5 et des graphes G tels que G\v soit biparti (où v est un sommet de G) sont également faiblement bipartis. La caractérisation complète des graphes faiblement bipartis est une question qui est encore ouverte.

Dans le paragraphe suivant nous étudions le polytope $P_B(G)$ dans le cas général et nous caractérisons certaines classes de ses facettes.

3. Facets of the bipartite subgraph polytope

(article [II.2] en collaboration avec F. Barahona et M. Grötschel)

Il a été mentionné au début de ce chapitre que si un graphe G = (V, E) contient un K_5 alors la contrainte (2.8) définit une facette de $P_B(G)$. Une première classe de facettes que nous avons obtenue généralise cette contrainte pour tout graphe complet d'ordre impair.

Théorème 2.4 :
Soit G = (V, E) un graphe et soit (W, E(W)) un sous graphe complet de G d'ordre $p \geq 3$. Alors

$$x(E(W)) \leq \lceil p/2 \rceil \lfloor p/2 \rfloor$$

est une inégalité valide de $P_B(G)$ si et seulement si p est impair.

A partir de ce théorème nous avons déduit que le polytope, enveloppe convexe des vecteurs représentatifs des cocycles de G (que nous étudions dans le prochain chapitre) est de pleine dimension.

Un graphe G = (V, E) est dit <u>p-roue de bicyclette</u> si G est défini par un cycle de longueur p et deux sommets adjacents à tout les sommets du cycle. Ces deux sommets sont dits universels. Une deuxième famille de facette obtenue est définie sur les 2k+1- roue de bicyclettes.

<u>Théorème 2.5 :</u>
Soit G = (V, E) un graphe et (W, F) une 2k+1-roue de bicyclette avec $k \geq 1$ contenue dans G. Alors l'inégalité

$$x(F) \leq 2(2k+1)$$

définit une facette de $P_B(G)$.

Nous avons décrit par la suite certaines procédures de construction de facettes du polytope $P_B(G)$ à partir d'autres facettes. En particulier nous avons montré, en utilisant le théorème 2.4, qu'étant donné un graphe complet (d'ordre quelconque) on peut associer des entiers ≥ 0 appropriés à ses sommets et obtenir sur ce graphe une nouvelle facette de $P_B(G)$. Les coefficients de ces facettes ne sont pas nécessairement en 0-1, ils peuvent être même de l'ordre de $O(n^2)$ où n est le nombre des sommets du graphe. Dans une deuxième procédure nous avons montré que l'opération qui consiste à remplacer dans un graphe une arête (resp. une chaîne impaire sans corde) par une chaîne impaire sans cordre (resp. une arête) permet elle aussi de définir de nouvelles facettes. De même celle qui consiste à diviser les arêtes d'un cocycle d'un graphe (mettre un nouveau sommet sur chacune de ces arêtes) conduit également à la construction de nouvelles facettes de $P_B(G)$.

En ce qui concerne l'aspect algorithmique de ces classes de facettes on peut d'abord remarquer qu'en utilisant des procédures énumératives, il est possible de résoudre polynomialement le problème de séparation pour chacune des classes de facettes associées aux K_{2k+1} et aux 2k+1- roue de bicyclettes avec $2k+1 \leq s$ où s est fixé. Par exemple pour K_5 cela nécessite un temps de l'ordre de $O(n^5)$. Pour les graphes complets d'ordre impair quelconque on ne sait pas encore si on peut résoudre en temps polynomial ou non le problème de séparation correspondant. Par contre pour les facettes associées aux 2k+1- roue de bicyclettes, Gérard [6] a montré dernièrement que le problème de séparation pour ces contraintes peut se résoudre en temps polynomial. Il a montré que ce problème peut se ramener à la recherche d'un cycle impair de poids minimum pour lequel on connaît déjà d'algorithme efficace. Notons de même que pour ce dernier problème Johnson et Gastou [5] ont proposé également un algorithme polynomial de résolution qui ramène le problème à la recherche de plus courtes chaînes dans un graphe.

Puisque K_5 est une 3-roue de bicyclette, il est maintenant très souhaitable de trouver un algorithme polynomial qui permet de résoudre en temps polynomial le problème de séparation pour les contraintes définies sur les graphes obtenus à partir de K_5 par des subdivisions impaires d'arêtes (deuxième procédure de construction de facettes mentionnée). Cette configuration simplifiée peut définir dans certains graphes, où le problème *PF* est NP- complet (par exemple les graphes G tel que G\v soit planaire), le seul type de contraintes essentielles du problème, définies à partir des graphes complets.

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CHAPITRE III

SUR DEUX GENERALISATIONS DU PROBLEME DU VERRE DE SPINS

L'objet de ce chapitre est l'étude de deux généralisations du problème du verre de spins. La première généralisation est donnée en terme de cocycle de poids maximum.

Considérons un graphe $G = (V, E)$ où chaque arête étant munie d'un poids quelconque w_e (au lieu d'être positif). Soit $\mathcal{P}'$ le problème qui consiste à déterminer une affectation $S : V \longrightarrow \{-1, +1\}$ qui minimise

$$\sum_{ij \in E} w_{ij} S_i S_j$$

Il est clair que ce problème généralise le problème $\mathcal{PF}$ du verre de spins (§II.1) et d'après ce qui est décrit dans (§II.2) toute solution de $\mathcal{P}'$ correspond à un cocycle de G et vice versa. Par conséquent le polyèdre des solutions de $\mathcal{P}'$ correspond au polyèdre, enveloppe convexe des vecteurs représentatifs des cocycles de G. On appelle ce polyèdre le polytope des cocycles de G et on le note $P_C(G)$. Dans le paragraphe suivant nous étudions le polyèdre $P_C(G)$ et nous donnons sa caractérisation dans les graphes non contractibles à K_5.

1. On the cut polytope

(article [III,1] en collaboration avec F. Barahona)

Etant donné un graphe $G = (V, E)$, il est évident que toute contrainte valide pour le polytope $P_B(G)$ est également valide pour $P_C(G)$, donc $P_C(G) \subseteq P_B(G)$. Mais en général $P_C(G) \neq P_B(G)$. En effet considérons un cycle C dans G, alors tout cocycle de G doit contenir un nombre pair d'arêtes de C. Ceci implique que les contraintes

$$x(F) - x(C \setminus F) \leq |F| - 1 \qquad \begin{array}{l} C \text{ cycle de } G, \\ F \subseteq C, |F| \text{ impair} \end{array} \qquad (3.1)$$

sont valides pour $P_C(G)$. Mais il est facile de voir que ces contraintes ne sont pas en général valides pour $P_B(G)$. Considérons de plus les contraintes dites triviales

$$0 \leq x_e \leq 1 \qquad \text{pour } e \in E \tag{3.2}$$

qui sont aussi valides pour $P_C(G)$.
Le résultat suivant caractérise parmi les contraintes (3.1) et (3.2) celles qui définissent des facettes de $P_C(G)$.

Théorème 3.1 :
(i) une contrainte parmi (3.1) définit une facette de $P_C(G)$ si et seulement si C est un cycle sans corde (une arete $e \notin C$ liant deux sommets du cycle).
(ii) une contrainte parmi (3.2) définit une facette de $P_C(G)$ si et seulement si e n'appartient pas à un triangle.

Nous avons étudié le problème de séparations associées aux contraintes (3.1) et (3.2) et nous avons proposé un algorithme qui permet de résoudre ce problème en temps polynomial. Pour ce faire l'algorithme ramène le problème à la recherche de plus courtes chaines dans un certain graphe. Cet algorithme combiné avec la méthode des ellipsoides permet donc de calculer en temps polynomial un cocycle de poids maximum dans les graphes G tels que le polytope $P_C(G)$ soit décrit complètement par les contraintes (3.1) et (3.2). Nous avons étudié ces graphes,ce qui nous a permis d'établir le :

Théorème 3.2 :
Un graphe G est non contractible à K_5 si et seulement si $P_C(G)$ est défini par les contraintes (3.1) et (3.2).

A notre connaissance, c'est la première fois qu'une caractérisation complète du polytope $P_C(G)$ est connue pour une classe de graphes non triviale.
Nous avons examiné également dans cet article certaines procédures génératrices de facettes du polytope $P_C(G)$. Nous avons montré en particulier que l'opération qui consiste à remplacer dans un graphe un sommet par un triangle, d'une manière appropriée, permet de construire de nouvelles facettes.
Enfin, dans cet article, nous avons étudié quelques propriétés liées à l'adjacence des points extrêmes du polytope $P_C(G)$. Par définition deux points extrêmes x_1 et x_2 d'un polyèdre (P) de R^n sont adjacents si et seulement si il existe un vecteur $C \in R^n$ tel que x_1 et x_2 soient les seuls points extrêmes qui maximisent Cx sur (P).

Nous avons alors le :

Théorème 3.3 :

Soit G = (V, E) un graphe et soient x^I et x^J deux points extrêmes de $P_C(G)$ correspondant aux cocycles I et J. Soit $F = E \setminus (I \triangle J)$ ($I \triangle J = (I \setminus J) \cup (J \setminus I)$). x^I et x^J sont adjacents dans $P_C(G)$ si et seulement si H = (V, F) admet deux composantes connexes.

A partir de ce théorème nous avons déduit que le polytope $P_C(G)$ possède la propriété de Hirsch et que quand le graphe G est complet, $P_C(G)$ est de diamètre 1.

La deuxième généralisation du problème du verre de spins que nous avons étudiée est donnée en terme d'indépendant de poids maximum. Rappelons d'abord en quelques mots certaines définitions de base liées à ce problème.

Un système d'indépendants est défini par un couple (E, $\mathcal{I}$) où E est un ensemble fini d'éléments et $\mathcal{I}$ une famille de sous ensembles de E qui vérifient

$$J \subseteq I \in \mathcal{I} \Rightarrow J \in \mathcal{I}$$

Les éléments de $\mathcal{I}$ sont appelés des ensembles indépendants et les sous ensembles de E non contenus dans $\mathcal{I}$ sont appelés des ensembles dépendants. Un ensemble indépendant maximal (resp. dépendant minimal) est dit une base (resp. circuit) de (E, $\mathcal{I}$). Un système d'indépendants peut être défini aussi par l'ensemble de ses bases ou de ses circuits.
Etant donné un système d'indépendants (E, $\mathcal{I}$) où E est un ensemble valué, le problème de l'indépendant de poids maximum consiste à déterminer un élément dans $\mathcal{I}$ de poids total maximum. (Remarquons ici que $\mathcal{I}$ peut avoir un nombre exponentiel d'éléments). Ce problème modélise une large classe de problèmes d'optimisation combinatoire, en particulier celui du sous graphe biparti de poids maximum (le problème $\mathcal{PF}$). En effet on peut voir que les ensembles d'arêtes des sous graphes bipartis d'un certain graphe définissent un système d'indépendants dont les circuits sont les cycles impairs de G. Un autre problème qui entre dans cette classe est celui du sous graphe sans circuit de poids maximum. Ce problème consiste à déterminer dans un graphe orienté donné où tout arc est muni d'un certain poids, un sous graphe sans circuit de poids total maximum. Le système d'indépendants correspondant est défini par les circuits de ce graphe. Ainsi on remarque que les ensembles de circuits de ce système et de celui associé aux sous graphes bipartis, sont définis les deux sur les mêmes structures, représentées par les

cycles d'un graphe. Ceci nous a permis d'utiliser des idées similaires à celles développées au paragraphe II.3 pour étudier ce dernier problème et de caractériser le polyèdre de ses solutions dans la classe des graphes symétriques non contractibles à $K_{3,3}$.

2. Composition in the acyclic subdigrapoh polytope

(article [III.2] en collaboration avec F. Barahona)

Dans ce paragraphe, on désigne par $D = (V, A)$ un graphe orienté dont V est l'ensemble des sommets et E celui des arcs. Un arc dont l'extrémité initiale est i et l'extrémité terminale est j est noté par (i, j). Etant donné un système de poids $(w(e), e \in A)$, le problème du sous graphe sans circuit dans G est donc de déterminer un sous ensemble $A' \subseteq A$ tel que le graphe $D' = (V, A')$ ne contienne pas de circuit et

$$w(A') = \sum_{e \in A'} w(e)$$

soit maximum.

Ce problème possède des applications dans différents domaines scientifiques (Kaas [1], Lenstra [2], Young [3]). Il est NP-complet dans le cas général mais polynomial dans la classe des graphes planaires.

On appelle polytope des sous graphes sans circuit de $D = (V, A)$ noté $P_{AC}(D)$, l'enveloppe convexe des vecteurs représentatifs x^F tel que (V, F) soit sans circuit. Il est clair que les contraintes

$x(C) \leq \lvert C\rvert - 1$	C circuit de D	(3.3)
$x_e \leq 1$	$e \in A$	(3.4)
$x_e \geq 0$	$e \in A$	(3.5)

sont valides pour $P_{AC}(D)$. Grötschel, Jünger et Reinelt (1982) ont montré que les contraintes (3.3) et (3.5) définissent des facettes de $P_{AC}(D)$ et une contrainte parmi (3.4) associée à un arc $e = (i, j)$ définit une facette de $P_{AC}(D)$ si et seulement si $(j, i) \notin A$. Ils ont montré que le problème de séparation pour ces contraintes peut se résoudre en temps polynomial. Par conséquent le problème du sous graphe sans circuit peut se résoudre en temps polynomial dans les graphes D où le polytope $P_{AC}(D)$ peut être décrit par les contraintes (3.3), (3.4) et (3.5). Ils ont appelé ces graphes, les graphes faiblement sans circuit. Ils ont montré à partir d'un résultat de Lucchesi (1978) que les graphes planaires orientés sont faiblement sans circuit.

Etant donné un graphe $G = (V, E)$ (non orienté), on appelle graphe symétrique de G, le graphe orienté $D(G) = (V, A)$ tel que $A = \{(i, j), (j, i) \ / \ ij \in E\}$. Dans cet article nous avons essayé en particulier de caractériser les graphes symétriques faiblement sans circuit, en utilisant des opérations de compositions du polytope des sous graphes sans circuit, similaires à celles décrites en (§ II.3) pour celui des sous graphes bipartis.

Soient deux graphes $D_1 = (V_1, A_1)$ et $D_2 = (V_2, A_2)$. Un graphe $D_3 = (V_3, A_3)$ est dit 2-somme de D_1 et D_2 s'il est obtenu à partir de D_1 et D_2 en identifiant deux circuits de D_1 et D_2 ayant chacun deux sommets (voir Fig. 10).

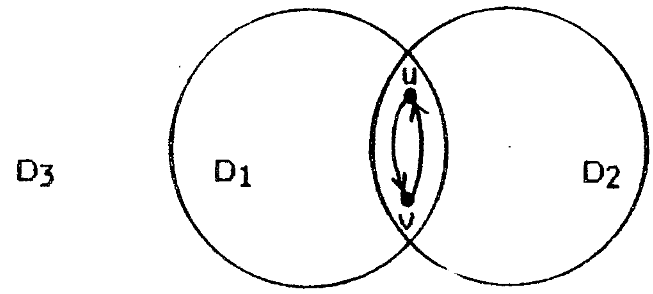

Fig. 10

Il est clair que la contrainte

$$x(u,v) + x(v,u) \le 1$$

définit une facette de $P_{AC}(D_1)$ et $P_{AC}(D_2)$. De plus on peut voir que cette contrainte définit la seule facette où les coefficients de $x(u, v)$ et $x(v, u)$ sont les deux non nuls. Supposons que les autres contraintes de $P_{AC}(D_1)$ (resp. $P_{AC}(D_2)$) dont le support intersecte le circuit $((u, v), (v, u))$ sont

$$\sum_{e \in \bar{A}_1} a_{ek} x_e + x(u,v) \le \alpha_k, \quad k \in K_1$$

$$\sum_{e \in \bar{A}_1} a_{ek} x_e + x(v,u) \le \alpha_k, \quad k \in K_2$$

$$\text{resp.} \left\{ \begin{array}{ll} \sum_{e \in \bar{A}_2} b_{le} x_e + x(u,v) \le \beta_l, & l \in L_1 \\ \sum_{e \in \bar{A}_2} b_{le} x_e + x(v,u) \le \beta_l, & l \in L_1 \end{array} \right.$$

où $\bar{A}_i = A_i \setminus \{(u,v), (v,u)\}$, $i=1, 2$

Et considérons les contraintes

$$\sum_{e\in A_1} a_{ke}\, x_e + \sum_{e\in A_2} b_{le}\, x_e \leq \alpha_k + \beta_l - 1 \qquad (3.6)$$

avec $(k, l) \in (K_1 \times L_2) \cup (K_2 \times L_1)$.

Nous avons montré que ces contraintes sont valides pour $P_{AC}(D)$ et définissent des facettes de ce polytope. Soit (P) le polyèdre de $R^{|A|}$ défini par les contraintes de $P_{AC}(D_1)$, $P_{AC}(D_2)$ et les contraintes (3.6). Nous avons le :

Théorèmle 3.4 :

$(P) = P_{AC}(D_3)$

A partir de ce théorème on peut déduire ainsi que si D_1 et D_2 sont faiblement sans circuit alors D_3 l'est aussi. De plus que le système (P) définissant $P_{AC}(D_3)$ est minimal (en supposant que les systèmes de $P_{AC}(D_1)$ et $P_{AC}(D_2)$ sont minimaux).

Un graphe G non contractible à $K_{3,3}$ est dit maximal si pour toute arête ajoutée à G, le graphe obtenu est contractible à $K_{3,3}$. Wagner a montré que tout graphe non contractible à $K_{3,3}$ maximal peut être obtenu par des opérations de 2-somme (§ II.3) à partir de graphes planaires et des copies de K_5.
On obtient le :

Théorème 3.5 :
$D(K_5)$ est faiblement sans circuit.

A partir des théorème 3.4 et 3.5 et du résultat de Wagner nous avons le :

Théorème 3.6 :
Un graphe D(G) est faiblement sans circuit si et seulement si G est non contractible à $K_{3,3}$.

Dans le paragraphe suivant nous généralisons l'opération de composition de polyèdres décrite ci-dessus, au cas des systèmes d'indépendants en général.

3. On a Composition of Independence Systems by Circuit-Identification

(article [III.3] en collaboration avec R. Euler.)

Etant donné un système d'indépendants $(E, \mathcal{I})$, on appelle polytope des indépendants de $(E, \mathcal{I})$, noté $P(\mathcal{I})$, l'enveloppe convexe des vecteurs x^F où $F \in \mathcal{I}$. Ce polytope a été étudié en particulier pour certains systèmes d'indépendants comme les ensembles stables, les couplages, les sous graphes bipartis (un ensemble S de sommets d' un graphe $G=(V,E)$ est un stable de G si $E(S)=\emptyset$. Et un ensemble $F \subseteq E$ est un couplage de G si les arêtes de F sont deux à deux disjointes). Dans ce paragraphe nous étudions ce polyèdre dans des systèmes d'indépendants quelconques définis à partir d'autres systèmes par une certaine opération de composition. Cette opération généralise celle de la 2-somme étudiée ci-dessus dans les cas des sous graphes bipartis et des sous graphes sans circuit.

Soit $(E, \mathcal{I})$ un système d'indépendants dont $\mathcal{C}$ est l'ensemble des circuits. Soit $C \in \mathcal{C}$ tel que $C = C^1 \cup C^2$ avec $C^1 \neq \emptyset \neq C^2$. On dit que $(E, \mathcal{I})$ vérifie pour $C = C^1 \cup C^2$ la propriété (**II**) si les deux conditions suivantes sont satisfaites

1) Pour tout $C' \in \mathcal{C}$ tel que $C' \cap C \neq \emptyset$, nous avons $C' \cap C = C^1$ ou $C' \cap C = C^2$

2) Pour tous circuits $C_1, C_2 \in \mathcal{C}$ tels que $C \cap C_1 = C^1$ et $C \cap C_2 = C^2$, il existe un circuit $C_3 \in \mathcal{C}$ tel que $C_3 \subseteq (C_1 \cup C_2) \setminus C$.

Considérons maintenant deux systèmes d'indépendants $(E_1, \mathcal{I}_1)$ et $(E_2, \mathcal{I}_2)$ ayant comme ensemble de circuits $\mathcal{C}_1$ et $\mathcal{C}_2$. Soient C_1 et C_2 deux circuits dans $\mathcal{C}_1$ et $\mathcal{C}_2$ ayant respectivement les partitions (C_1^1, C_1^2) et (C_2^1, C_2^2). Supposons que $(E_1, \mathcal{I}_1)$ (resp. $(E_2, \mathcal{I}_2)$) vérifie la propriété (**II**) pour $C_1 = C_1^1 \cup C_1^2$ (resp. $C_2 = C_2^1 \cup C_2^2$) et que

$$|C_1^1| = |C_2^1|,\ |C_1^2| = |C_2^2|$$

Soit $(E_3, \mathcal{I}_3)$ le système d'indépendants obtenu à partir de $(E_1, \mathcal{I}_1)$ et $(E_2, \mathcal{I}_2)$ en identifiant le circuit C_2 avec C_1 (C_2^1 avec C_1^1, et C_2^2 avec C_1^2) et dont l'ensemble des circuits est $\mathcal{C}_3$ tel que $\mathcal{C}_3 = \mathcal{C}_3^1 \cup \mathcal{C}_3^2$ avec

$$\mathcal{C}_3^1 = \mathcal{C}_1 \cup \bar{\mathcal{C}}_2$$

où $\bar{\mathcal{C}}_2$ est l'ensemble déduit de $\mathcal{C}_2$ en identifiant C_2 avec C_1

et

$$\mathcal{C}_3^2 = \{(C' \cup C'') \setminus C_1 \ / \ C' \in \mathcal{C}_1 \text{ et } C'' \in \mathcal{C}_2$$
$$\text{avec } C_1^1 \subseteq C' \text{ et } C_1^2 \subseteq C'' \text{ ou } C_1^1 \subseteq C'' \text{ et } C_1^2 \subseteq C' \}$$

Dans le cas où les systèmes $(E_1, \mathcal{I}_1)$, $(E_2, \mathcal{I}_2)$ sont définis par les sous graphes bipartis (resp. les sous graphes sans circuit) $\mathcal{C}_3^2$ correspond à l'ensemble des cycles impairs (resp. circuits) élémentaires créés par l'opération de la 2-somme.

Du fait que les systèmes $(E_1, \mathcal{I}_1)$ (resp. $(E_2, \mathcal{I}_2)$) vérifie la propriété (II) pour C_1 (resp. C_2), on peut voir facilement que toute base de $(E_1, \mathcal{I}_1)$ (resp. $E_2, \mathcal{I}_2)$) contient $(|C_1|-1)$ (resp. $|C_2|-1$) éléments de C_1 (resp. C_2). Ceci implique par conséquent que les contraintes

$$x(C_1) \leq |C_1|-1$$
$$\text{resp. } x(C_2) \leq |C_2|-1$$

définissent des facettes de $P(\mathcal{I}_1)$ (resp. $P(\mathcal{I}_2)$) et que les autres facettes de ce polytope dont le support intersecte C_1 (resp. C_2) sont de type

$$\sum_{e \in \bar{E}_1} a_{ek}\, x_e + x(C_1^1) \leq \alpha_k, \quad k \in K_1$$
$$\sum_{e \in \bar{E}_1} a_{ek}\, x_e + x(C_1^2) \leq \alpha_k, \quad k \in K_2$$

$$\text{resp. } \left\{ \begin{array}{ll} \sum_{e \in \bar{E}_2} b_{el}\, x_e + x(C_2^1) \leq \beta_l, & l \in L_1 \\ \sum_{e \in \bar{E}_2} b_{el}\, x_e + x(C_2^2) \leq \beta_l, & l \in L_2 \end{array} \right.$$

où $\bar{E}_i = E_i \setminus C_i$, $i=1, 2$

Considérons maintenant les contraintes

$$\sum_{e \in E_1} a_{ek}\, x_e + \sum_{e \in E_2} b_{el}\, x_e \leq \alpha_k + \beta_l - (|C_1|-1) \qquad (3.7)$$
$$\text{avec } (k, l) \in (K_1 \times L_2) \cup (K_2 \times L_1)$$

Les contraintes (3.7) sont valides pour le polytope $P(\mathcal{I}_3)$. De plus elles définissent des facettes de ce polytope. Soit (Q) le polèdre de $R^{|E|}$ défini par les contraintes des polytopes $P(\mathcal{I}_1)$, $P(\mathcal{I}_2)$ et les contraintes (3.7). Nous avons alors le :

<u>Théorème 3.7 :</u>

$$(Q) = P(\mathcal{I}_3)$$

Ce théorème nous permet donc de donner une description complète et minimale du polytope $P(\mathcal{I}_3)$ à partir des polytopes $P(\mathcal{I}_1)$ et $P(\mathcal{I}_2)$. Par exemple dans le cas où les systèmes $(E_1, \mathcal{I}_1)$ et $(E_2, \mathcal{I}_2)$ sont définis par les ensembles stables de deux graphes $G_1 = (V_1, E_1)$ et $G_2 = (V_2, E_2)$ parfaits, alors les polytopes $P(\mathcal{I}_1)$ et $P(\mathcal{I}_2)$ sont définis par les contraintes associées aux cliques de G_1 et G_2, ainsi les contraintes (3.7) sont définies sur des cliques du nouveau graphe G_3 déduit de G_1 et G_2 et qui représente le système $(E_3, \mathcal{I}_3)$. Le théorème 3.7 implique donc que le graphe G_3 est aussi parfait. Par conséquent cette opération de composition de système d'indépendants préserve la perfection dans le cas où ces systèmes sont définis par les ensembles stables dans les graphes.

Nous avons étudié également dans cet article la propriété "dual totalement entier" (total dual intégral) pour des systèmes d'inéquations en relation avec les polyèdres des indépendants. Par définition un système

$$Ax \le b$$

où A est une matrice (m, n) à coefficients entiers et $b \in R^m$, est dit" dual totalement entier" (**TDI**) si pour tout vecteur $C \in R^n$ à coefficients entiers, le dual du programme

$$\text{Max}\ \{ C^T x,\ Ax \le b \}$$

admet une solution optimale entière.
On obtient alors :

<u>Théorème 3.8 :</u>
Si les systèmes $P(\mathcal{I}_1)$ et $P(\mathcal{I}_2)$ sont **TDI** alors $P(\mathcal{I}_3)$ est **TDI**

Supposons que A est la matrice d'indicence dont les lignes représentent les circuits d'un système d'indépendants (E, $\mathcal{I}$). Considérons le système

$$(3.8) \qquad \begin{cases} Ax \leq b \\ 0 \leq x \leq 1 \end{cases}$$

où chaque coefficient de b est égal à $|C|-1$, avec C un circuit de (E, $\mathcal{I}$). Si (3.8) est **TDI** alors (3.8) = P($\mathcal{I}$). Nous avons également montré pour une classe particulière de matrice A que (3.8) est **TDI**

<u>Théorème 3.9 :</u>
Si A est équilibrée (ne contient pas de sous matrice carrée impaire ayant dans chaque ligne et chaque colonne deux 1) alors (3.8) est **TDI**.

De ce théorème on déduit que si (E, $\mathcal{I}$) est un matroide, alors P($\mathcal{I}$) = (3.8) si et seulement si les circuits de ce matroide sont deux à deux disjoints.

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CHAPITRE IV

ETUDE ALGORITHMIQUE ET POLYEDRALE DE DEUX PROBLEMES D'OPTIMISATION COMBINATOIRE

L'objet de ce dernier chapitre est l'étude de deux autres problèmes d'optimisation combinatoire : l'absorbant et le k_i-recouvrement de poids minimum. On s' intéresse en particulier à l'étude des polyèdres associés à ces problèmes.Nous étudions également une caractérisation des graphes dits "k_i-parfait" en la liant à la perfection de leur graphe d'intersection.

1. Polytope des absorbants dans une classe de graphes à seuil

(article [IV.1])

Etant donné un graphe $G = (V, E)$, un ensemble $A \subseteq V$ est dit absorbant (dominant) si tout-sommet dans $V \backslash A$ est adjacent à au moins un sommet de A. Le polytope des absorbants de G est l'enveloppe convexe des vecteurs représentatifs des absorbants de G.
Comme nous l'avons déjà signalé, il est souvent intéressant de caractériser des structures combinatoires par l'enveloppe convexe de leurs vecteurs représentatifs. Ceci à notre connaissance, n'a jamais été fait sur les absorbants d'une classe de graphes non triviale. Dans ce paragraphe nous donnons cette caractérisation dans la classe des graphes absorbants à seuil.

Définition 4.1 (Benzaken et Hammer)
Soit $G = (V, E)$ un graphe tel que $V = \{1, 2, ..., n\}$. G est dit absorbant à seuil s'il existe des poids $a_i \geq 0$ associés aux sommets de G et un seuil $b \geq 0$, tel que $A \subseteq V$ est un absorbant si et seulement si

$$\sum_{i \in A} a_i \geq b .$$

Parmi les caractérisations des graphes absorbants à seuil données par Benzaken et Hammer nous considérons la suivante :

Théorème 4.2

Les deux propriétés suivantes sont équivalentes

i) G est absorbant à seuil

ii) G peut se ramener à un graphe réduit à un sommet en enchaînant un nombre fini de fois les opérations suivantes :

$\Pi 1$: suppression d'un sommet isolé (sommet non adjacent à aucun autre sommet)

$\Pi 2$: suppression d'un sommet universel (sommet adjacent à tous les autres sommets)

$\Pi 3$: suppression d'une paire de pôles (deux sommets non liés entre eux mais adjacents à tous les autres sommets).

Considérons maintenant un graphe absorbant à seuil G. Soit n1 le nombre d'opérations de type $\Pi 1$ à faire pour ramener G à un sommet. Soit n1+1 ce sommet et soit i le sommet isolé définissant la $i^{\text{ème}}$ opération $\Pi 1$. Pour tout sommet j=n1+1 on désigne par V_j l'ensemble des sommets de G qui sont adjacents à j et par V_{n1+1} l'ensemble $X \setminus \{1, ..., n1\}$. On note $W_k = V_k - V_{k-1}$ pour $1 \leq k \leq n1+1$ et $V_0 = \emptyset$. Soient $(V_k)_u$ et $(W_k)_u$ (resp. $(V_k)_p$ et $(W_k)_p$) les ensembles de sommets universels (resp. paires de pôles) de V_k et W_k pour k=1,...,n1+1.

Le résultat central de cet article est :

Théorème 4.3

Le polytope des absorbants de G, P(G) est défini par les inégalités suivantes :

$$
Q \left\{
\begin{array}{ll}
\bullet \quad (1-\delta_{n1+1})x_i + \sum\limits_{j \in V_i} x_j \geq 1 & i=1,\dots,n1+1 \\
 & i \in (W_k)_p,\ k=1,\dots,n1+1 \\[2ex]
\bullet \quad \sum\limits_{i=k+1,\dots,n1} x_i + (j-k+2) \sum\limits_{i \in V_k \cup (W_{k+1})_u} x_i + \sum\limits_{t=k,\dots,j-1} (j+1-t) \sum\limits_{i \in (W_{t+1})_p \cup (W_{t+2})_u} x_i & \\[2ex]
\quad + \sum\limits_{i \in (V_{n1+1} - V_{j+1}) \cup (W_{j+1})_p} x_i \geq j-k+2 & 0 \leq k \leq j \leq n1 \\[2ex]
\bullet \quad 0 \leq x_i \leq 1 & \forall\ i \in V \\[2ex]
\quad \text{où } \delta_{n1+1,i} = \begin{cases} 1 & \text{si } n1+1=i \\ 0 & \text{sinon} \end{cases} &
\end{array}
\right.
$$

Il est facile de voir que toutes les contraintes dans Q sont valides pour P(G). Ceci implique que $P(G) \subseteq Q$. Pour démontrer le théorème il suffit donc de montrer que les points extrêmes de Q représentent tous des absorbants de G. Pour ce faire nous avons décrit d'abord un algorithme polynomial qui permet de construire un absorbant de poids minimum dans un graphe absorbant à seuil. Cet algorithme exige un nombre linéaire d'opérations. Par la suite nous avons utilisé cet algorithme pour montrer que pour tout $C \in R^n$ le programme linéaire

$$\mathrm{Min}\{Cx, x \in Q\},$$

admet une solution de base en 0-1. Ce qui montre que tous les points extrêmes de Q sont en 0-1 et représentent par conséquent des absorbants de G.

Notons que la description du polytope P(G) par le système linéaire Q n'est pas minimale. La description minimale de P(G) dans ce cas est une question qui est encore ouverte.

2. K_i-recouvrements : complexité, polytopes et graphes k_i-parfaits

Parmi les problèmes d'optimisation combinatoire qui ont été très étudiés ces dernières années, on trouve celui du transversal de poids minimum. Ce problème consiste à déterminer dans un graphe $G=(V,E)$, où tout sommet est muni d'un certain poids, un ensemble $T \subseteq V$ tel que $\forall$ $ij \in E$, $\{i,j\} \cap T \neq \emptyset$, et qui soit de poids total minimum. Ce problème est équivalent à celui du stable (indépendant) de poids maximum . Il est parmi les premiers problèmes montrés NP-complets. Le polyèdre associé à ce problème a été largement étudié. En particulier Padberg (1973) a étudié ce polytope dans le cas général et il a décrit certaines classes de ses facettes. Dans ce paragraphe nous étudions une généralisation de ce problème en terme de K_i-recouvrement.

Etant donné un graphe G = (V, E) et un entier $i \geq 2$ un ensemble $\mathcal{C}$ de K_{i-1} de G est dit K_i-recouvrement de G si tout K_i de G contient au moins un K_{i-1} de $\mathcal{C}$. Pour $i=2$, un K_2-recouvrement de G est un transversal. Si F est un ensemble de K_i de G, alors un K_i-recouvrement de F est défini de la même manière que dans G. Pour un système de poids $W = (w_j, K_{i-1}^j \subset G)$ associé aux K_{i-1} de G, le problème du K_i-recouvrement de poids minimum dans G consiste à déterminer un K_i-recouvrement $\mathcal{C}$ de G tel que $\sum_{K_{i-1} \in \mathcal{C}} w_j$ soit minimum.

Dans un premier temps nous avons étudié certains aspects algorithmiques et polyédraux de ce problème que nous présentons dans le paragraphe suivant.

2.1 K_i - Cover I : Complexity and polytopes

(article [IV.2] en collaboration avec M. Conforti et D.G. Corneil)

Pour i=2, le problème du K_2-recouvrement est équivalent à celui du transversal de poids minimum. Comme nous l'avons mentionné, ce problème est en général NP-complet. Le problème du K_3-recouvrement l'est également comme l'a montré Yannakakis. En utilisant une technique analogue on obtient le :

Théorème 4.2

Pour $i \geq 2$, le problème du K_i-recouvrement est NP-complet.

Un graphe est dit triangulé si tout cycle de ce graphe de longueur > 3 admet une corde. pour i=2 le problème du K_2-recouvrement est polynomial dans les graphes triangulés. Par contre pour i>2 nous avons le :

Théorème 4.3

Pour i>2, le problème du K_i-recouvrement est NP-complet dans les graphes triangulés.

Pour montrer ce théorème, on peut construire (en temps polynomial) à partir d'un graphe G donné un graphe G' triangulé de telle manière que les problèmes du K_i-recouvrement dans les deux graphes soient équivalents.

Cependant, dans un graphe triangulé où la taille maximale d'une clique est fixée, le problème du K_i-recouvrement est polynomial et nous avons donné dans cet article un algorithme basé sur la programmation dynamique qui permet de déterminer dans ce cas une solution optimale du problème en temps polynomial.

Etant donné un graphe G = (V, E), on désigne par $F_i(G)$ l'ensemble des K_i de G. Soit $\mathbf{{}^iA}$ la matrice d'incidence K_{i-1}-K_i dont les colonnes représentent les éléments de $F_i(G)$, les lignes ceux de $F_{i-1}(G)$, telle que

$$\mathbf{{}^iA}_{jk} = \begin{cases} 1 & \text{si } K_{i-1}^j \subset K_i^k \\ 0 & \text{sinon} \end{cases}$$

A la matrice $\mathbf{{}^iA}$ on associe le graphe $I_i(G)$ dit graphe K_i-intersection de G. Les sommets de $I_i(G)$

correspondent aux K_i dans $F_i(G)$, et deux sommets sont adjacents si et seulement si les K_i correspondants ont un K_{i-1} en commun. Notons que cette notion de graphe K_i-intersection est fondamentale dans cette étude.
On appelle polytope des K_i-recouvrements de G, noté $P_{Ki}(G)$, l'enveloppe convexe des vecteurs $x^{\mathcal{C}}$ où $\mathcal{C}$ est un K_i-recouvrement de G. Nous avons étudié ce polytope et nous avons examiné quelques classes de ses facettes.

Il est clair que les contraintes :

$$\mathbf{1}A^T x \geq \mathbb{1} \tag{4.1}$$

$$0 \leq x_j \leq 1 \qquad \text{pour } K_{i-1}{}^j \in F_{i-1}(G) \tag{4.2}$$

sont valides pour $P_{Ki}(G)$. Nous avons :

Théorème 4.4
Pour $i \geq 3$, les contraintes (4.1) et (4.2) définissent des facettes de $P_{Ki}(G)$.

Théorème 4.5
Soit $K_{i+1}{}^0$ un sous graphe complet de G de taille i+1. La contrainte

$$\sum_{K_{i-1}{}^j \subset K_{i+1}{}^0} x_j \geq \lceil i+1/2 \rceil$$

est valide pour $P_{Ki}(G)$. Elle définit une facette de $P_{Ki}(G)$ si et seulement si i est pair.

Un graphe H est dit K_i-p-trou si $I_i(H)$ est un trou de longueur p (un trou est un cycle sans corde). De même un graphe est dit p-roue d'ordre i s'il est défini par un trou C de longueur $p \geq 3$ et un K^*_i universel à C (tout sommet de K^*_i est adjacent à tout sommet de C). Notons qu'une p-roue d'ordre i-2 est un K_i-p-trou.

Théorème 4.6 :
Soit G un graphe et soit H un sous graphe de G, définissant un K_i-p-trou avec $i \geq 3$. Alors l'inégalité

$$\sum_{K_{i-1}{}^j \in F_{i-1}(H)} x_j \geq \lceil p/2 \rceil$$

est valide pour $P_{Ki}(G)$. Elle définit une facette de $P_{Ki}(G)$ si et seulement p est impair.

Théorème 4.7. :

Soit G un graphe et soit H un sous graphe de G définissant une $(2K+1)$-roue d'ordre $i-1$ où $k \geq 2$ et $i \geq 3$. Soit $\mathcal{C} = F_{i-1}(H) \setminus K^*_{i-1}$. Alors l'inégalité

$$\sum_{K_{i-1}^j \in \mathcal{C}} x_j \geq k(i-1) + \lceil i-1/2 \rceil$$

est valide pour $P_{K_i}(G)$. Elle définit une facette de $P_{K_i}(G)$ si et seulement i est pair.

Nous avons caractérisé également d'autres facettes du polytope $P_{K_i}(G)$ dans le cas où $i=3$. Ces facettes sont liées au polytopoe des sous graphes bipartis étudiés en (§, II.3). Par la suite nous avons étudié le problème de séparation pour les contraintes données par les théorèmes 4.6 et 4.7. En utilisant l'algorithme de Gérard (déjà mentionné) nous avons décrit un algorithme qui permet de résoudre ce problème en temps polynomial pour les contraintes associées aux $(2k+1)$-roues d'ordre $i-1$ et $i-2$ avec $i=2r$, où r est un entier. Pour les contraintes associées aux K_i-p-trous, autres que celles données par les p-roues d'ordre $i-2$, on ne connaît pas encore d'algorithme polynomial qui permet de résoudre le problème de séparation correspondant. L'existence d'un tel algorithme est un problème ouvert très intéressant, car si un tel algorithme existe alors il permet de résoudre en temps polynomial dans le graphe K_i- intersection le problème du trou de poids minimum.

Etant donné un graphe $G = (V, E)$, un ensemble de K_i de G est dit "K_i-packing" de G s'il ne contient pas deux K_i ayant un K_{i-1} en commun. Si $F \subseteq F_i(G)$, on définit un "K_i-packing" dans F de la même manière que dans G.

Soit $c_i(F)$ (resp. $p_i(F)$) la cardinalité d'un K_i-recouvrement minimum (resp. K_i-packing maximum) dans F. Il est clair que

$$c_i(F) \geq p_i(F) \qquad F \subseteq F_i(G) \tag{4.3}$$

Mais en général $c_i(F) \neq p_i(F)$. Dans le prochain paragraphe nous étudions les graphes dans lesquels l'inégalité (4.3) est toujours vérifiée avec égalité.

2.2. K_i-cover II : K_i-parfect graphs

(article [IV.3] en collaboration avec M. Conforti et D.G. Corneil)

Un graphe G est dit K_i-parfait si $c_i(F) = p_i(F)$ pour tout $F \subseteq F_i(G)$. On peut voir facilement que les graphes K_2-parfaits sont précisément les graphes bipartis. Le graphe K_i- intersection de G, $I_i(G)$ est dit conforme si pour toute clique de taille $k \geq 2$ dans $I_i(G)$, les K_i de G correspondants aux sommets de cette clique ont exactement i-1 sommets en commun. Par conséquent si $I_i(G)$ est conforme alors le nombre de cliques disjointes dans $I_i^F(G)$ est égal à $c_i(F)$ où $F \subseteq F_i(G)$ et $I_i^F(G)$ est le sous graphe induit de G dont les sommets correspondent aux K_i de F. Nous avons alors le :

Théorème 4.8 :

Le graphe $I_i(G)$ est conforme si et seulement si G ne contient pas un K_{i+1}.

A partir de ce théorème on peut déduire que si le graphe $I_i(G)$ est conforme, alors il ne contient pas un sous graphe induit K_4-e (le graphe K_4 moins une arête). Parthasarathy et Ravindra (1979) ont montré que les graphes sans K_4-e vérifient la conjecture forte des graphes parfaits : un graphe est parfait si et seulement si il ne contient pas de trou ou d'anti-trou (le complémentaire d'un trou) de longueur ≥ 5. Ceci nous a permis d'établir le :

Théorème 4.9 :

Un graphe G est K_i-parfait si et seulement si $I_i(G)$ est conforme et parfait.

Nous avons étudié par la suite une caractérisation des graphes K_i-parfaits en terme de matrices parfaites. Par définition une matrice A à coefficients en 0-1 est dite parfaite si le polyèdre

$$Ax \leq \mathbf{1}$$
$$x \geq 0$$

a tous ses sommets en 0-1. On obtient alors le :

Théorème 4.10 :

Un graphe G est K_i-parfait si et seulement si $\mathbf{iA}$ est une matrice parfaite.

Il est bien connu que toute matrice équilibrée (§ III.3) est parfaite. Donc le théorème 4.10

implique que si **iA** est équilibrée, alors le graphe correspondant est K_i-parfait. La réciproque de cette propriété n'est pas par contre vraie. Nous avons donné (dans [IV.3]) un graphe K_3-parfait pour lequel la matrice **3A** n'est pas équilibrée. Dans une dernière partie de cet article nous avons étudié les graphes pour lesquels la matrice **iA** est équilibrée et nous avons donné à ces graphes une caractérisation en terme de sous graphes partiels interdits dans le graphe K_i-intersection.

A la suite de cette étude plusieurs questions restent encore ouvertes. En particulier l'existence d'un algorithme polynomial de reconnaissance des graphes K_i-parfaits. Cette question est liée au problème de l'existence d'un trou impair dans le graphe K_i-intersection. De même il est intéressant d'étudier encore le polytope $P_{K_i}(G)$ et de caractériser d'autres familles de ses facettes pouvant avoir des coefficients différents de 0 et 1. Enfin on peut également étudier les graphes G pour lesquels $c_i(H) = p_i(H)$ pour tout sous graphe partiel (induit) H de G. Ces graphes peuvent contenir des graphes non K_i-parfaits.

ANNEXE 1

IMPROVEMENT OF TRAFFIC FLOW BY THE REDUCTION OF CAPACITIES

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ABSTRACT

In this paper we study the following traffic regulation problem :

Given a capacited transportation network with several origins and destinations, we assume that all the users follow the first Wardrop principle known as the user optimising assignment principle . It is well known that the time spent by the users in such an assignment equilibrium is not the best possible.

In order to improve the general traffic situation a central decision center may decide change locally the traffic parameters on some links of the network. In particular it may be interesting to reduce the value of the capacities of some links of the network and for many examples of networks, degrading the traffic conditions on some links may lead to a global improvement of traffic conditions on this network. The regulation problem consists in finding on which links it may be interesting to lower the initial capacities in order to improve the general traffic condition on this network. In this paper we study this problem from a theoretical point of view and also by providing some interesting examples which show the interest of this kind of study.

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I. INTRODUCTION

One of the major problems met in urban traffic theory is to find good regulation policies in order to solve congestion problems.

The best possible regulation method would consist in finding a traffic assignment which satisfies the second Wardrop principle (Wardrop, 1952) : "The average travel time is minimum".

However this assignment pattern is quite unrealistic in the sense that some drivers will have to use routes which are not optimal for them in order to reach the goal defined by this principle.

A more interesting approach is to assume that the users will behave according to the first Wardrop principle : (wardrop 1952), known also as the user optimising assignment principle :

> "The journey time on all routes actually used are equal, and less than those which whould be experienced by a single vehicle on an unused route".

An equilibrium situation which satisfies this principle is called a user optimising pattern. Of course this pattern will depend on the traffic conditions on each link of the network.

More precisely it is usually admitted that the travel time $\theta(e)$ on a link e is a non decreasing continuous function depending on the traffic flow f(e) observed on this link.
Moreover in capacited networks we assume that on each link e the traffic flow cannot exceed a given capacity c(e).
In our study we shall consider capacited networks rather than uncapacited networks. The reason is that the capacity of a link is a parameter which can be easily modified by a central decision center : for instance the capacity of a link can be reduced by the removal of one or several lanes of the link or by the use of an appropriate signal setting at the entrance of this link. On the other hand, it seems more difficult to act on the other parameters defining the travel time $\theta(e)$ of a link e.

We can now define the kind of regulation policies we want to study.

It is possible, by eventual reduction of the capacities of some links of the transportation network to improve the traffic conditions of this network (average time spent by the users), if we assume that the users will always behave according to the user optimising assignment principle ?

It may be surprising that degrading the traffic conditions on some links of the network may lead to a global improvement of traffic conditions, but this paradox has already been observed by FISK (1980), FISK and PALLOTINS (1979) MURCHLAND (1970) in the so-called paradoxes in the equilibrium assignment pattern or Braess's paradoxes.

In part II of this publication, we shall precise our notation and definitions and we shall set our main hypothesis concerning user assignment pattern in capacited networks.

In part III we shall give a mathematical formulation of our regulation model and we shall study this model.

In part IV we shall give some interesting examples showing what kind of improvement can be obtained by this regulation method.

In part V we shall propose some resolution method for the model defined in part III.

II. CAPACITED NETWORKS AND USER OPTIMISING PATTERNS

II.a. The main hypothesis

In this study we shall deal with linear capacited networks which are very simplified type of networks but all the results proved here could be easily extended to more classical types of capacited networks.

On a link e, the travel time is a constant b(e) defined as the initial travel time, provided that the traffic flow is strictly less than a given capacity c(e).

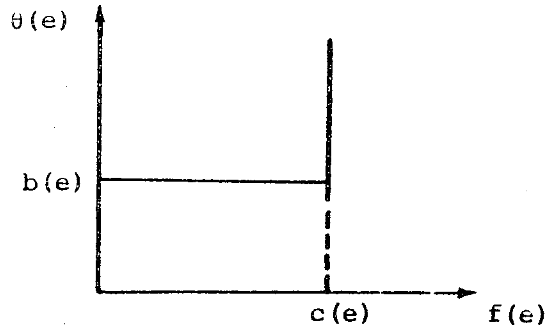


Fig. 1

Note that the travel time $\theta(e)$ is not defined when $f(e) = c(e)$ and this is one of the main difficulties incountered with capacited networks : it may be difficult to give an interpretation of the user optimising assignment principle in capacited networks. This is extensively discussed in the book of Potts and Oliver : "Flows in transportation networks".

However, in a recent publication, FARGIER and FONLUPT (1983) made the following assumption.

If the traffic flow on a link e is equal to the capacity of the link, $c(e)$ the travel time on this link is equal to

$$b(e) + m(e) \qquad \text{(Main hypothesis)}$$

This additional time $m(e) \geq 0$ can be interpreted as a congestion time due to the existence of a queue on the saturated link e. Moreover $m(e)$ is not given as a data but is itself the result of the traffic assignment pattern.

II.b. Notations and definitions

The transportation network will be denoted $R = (X,E,c,b)$ where :

- X is the set of nodes
- E is the set of links
- For any $e \in E$, $I(e)$ is the initial end of e and $T(e)$ is the final end of e ; $(I(e), T(e) \in X)$
- $c(e)$ is the capacity of link e. ($c(e)$ may be infinite).
- $b(e)$ is the initial travel time on link e.
- $\theta(e)$ is the average travel time on link e.
- The traffic demand is defined by a set $\{F_{ij} ; x_i, x_j \in X\}$ where F_{ij} is the amount of traffic which has to go from node x_i to node x_j. We note $f_{ij}(e)$ the flow going from x_i to x_j and using the link e.

We consider only static situations : the flows and travel times are independant of the time.

II.c. System and user optimising pattern

It is well known [see for instance Akcelik (1979) or Fargier, Fonlupt (1983)] that the system pattern which corresponds to the second Wardrop principle can be found as the solution generally unique of the following linear program P :

$$
(P)\begin{cases}
\forall x_i,\ \forall x_j \in X\ ;\ \forall e \in E \quad f_{ij}(e) \geq 0 & (1)\\[2ex]
\forall x_i,\ \forall x_j\ \forall x_k \in X : & \\
\displaystyle\sum_{\{e | x_k = T(e)\}} f_{ij}(e) - \sum_{\{e | x_k = I(e)\}} f_{ij}(e) = \begin{cases} -F_{ij} & \text{if } k = i \\ 0 & \text{if } k \neq i,j \\ F_{ij} & \text{if } k = j \end{cases} & (2)\\[2ex]
\forall e \in E \quad \displaystyle\sum_{x_i, x_j \in X} f_{ij}(e) = f(e) \leq c(e) & (3)\\[2ex]
Z = \min \displaystyle\sum_{e \in E} b(e)\ f(e) & (4)
\end{cases}
$$

Constraints (2) are flow constraints for each node x_k and each pair (x_i, x_j)
Constraints (3) are capacity constraints.
The objective function represents the time spent by all the users of the transportation network during a unit period of time.

Fargier and Fonlupt (1983) proved that the user optimised pattern can also be found by solving program (P). More precisely they proved the following result : A feasible solution $\{f_{ij}(e)\ ;\ x_i, x_j \in X,\ e \in E\}$ is a user optimised pattern if there exists a set of real variables $\pi(i,j,\ell)$ $(x_i, x_j, x_\ell \in X)$ defined on the nodes of the network and a set of real variables $m(e)$ defined on the links of the network such that :

for any couple of nodes (x_i, x_j) and any link $e = (x_k, x_\ell)$

$$\begin{cases} m(e) \geq 0 & (5) \\ \pi(i,j,\ell) - \pi(i,j,k) \leq b(e) + m(e) & (6) \\ \text{If } f_{ij}(e) > 0 \quad \pi(i,j,\ell) - \pi(i,j,k) = b(e) + m(e) & (7) \\ \text{If } m(e) > 0, \quad f(e) = c(e) & (8) \end{cases}$$

Interpretating $\pi(i,j,\ell)$ as the minimum time for a user going from x_i to x_j to reach node x_ℓ, these conditions appear clearly as equilibrium condition for defining a user optimising pattern. In particular condition (8) means that if there is a congestion on a link e, this link is saturated.

From a mathematical point of view note that constraints (5) and (6) are exactly the constraints of the dual program (D) of (P) [see for instance Dantzig (1963)].

Conditions (7) and (8) are the well known complementary slackness conditions of linear programming and a feasible solution of program (P) satisfying conditions (5) (6) (7) (8) is an optimal solution of program (P).

Therefore the user optimising pattern can also be found by solving linear program (P).

The relation between user optimised pattern and system optimised pattern can be stated in the following way.

a) The traffic observed in this two patterns at the same and this unique solution can be found by solving linear program (P).

b) There is an essential difference between the two patterns :
In the system optimised pattern the time spent by the users is

$$Z = \Sigma\, b(e)\, f(e).$$

In the user optimised pattern the time spent by the users is

$$T = \Sigma\, (b(e)+m(e))\, f(e).$$

where the variables m(e) are defined by optimality conditions (5) (6) (7) (8).

To illustrate this difference between the two patterns consider the following simple example, Fig. 2

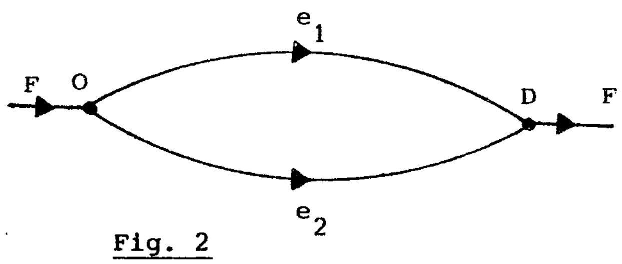


Fig. 2

The network is constituted by two nodes O and D and two disjoint links e_1 and e_2. Let us take an entering traffic F in O equal to 10.
Moreover let us suppose that the traffic is defined by the following parameter

$b(e_1) = 1 \qquad c(e_1) = 5$

$b(e_2) = 3 \qquad c(e_2) = 10.$

The traffic in the user and system assignment patterns is obviously :

$f(e_1) = 5 \qquad f(e_2) = 5.$

The time spent by the users in the system optimised pattern is :

$$Z = 1 \times 5 + 3 \times 5 = 20.$$

However in the user optimised pattern there is a congestion on link e_1 and the time spent by the users using link e_1 is the sum of the initial time $b(e_1)$ and an additional time $m(e_1)$ due to this queue. $m(e_1)$ is determined by equalling the time spent by users on links e_1 and e_2.

Therefore $b(e_1) + m(e_1) = b(e_2)$.

$$m(e_1) = 3-1 = 2.$$

The time spent by the users in the user optimised pattern is

$$w = (1+2) \times 5 + 3 \times 5 = 30.$$

III. MATHEMATICAL FORMULATION OF THE REGULATION PROBLEM

The dual (D) of the linear program (P) can be written :

For any couple of nodes (x_i, x_j) and any link $e = (x_k, x_\ell)$

$$(D) \begin{cases} m(e) \geq 0 & (5) \\ \pi(i,j,\ell) - \pi(i,j,k) \leq b(e) + m(e) & (6) \\ \max \sum_{x_i, x_j \in X} (\pi(i,j,j) - \pi(i,j,i)) F_{ij} - \sum_{e \in E} m(e)\, c(e) = w & (9) \end{cases}$$

Instead of using complementary slackness conditions for characterizing optimality in a linear program, we can also use the fact that a couple of feasible solution in the primal and the dual is optimal if and only if the value of the objective function in the primal is equal to the value of the objective function in the dual.

So in our situation optimality in (P) and (D) can be characterized by constraints (1) (2) (3) (5) (6) and the constraint (10)

$$\sum_{e \in E} b(e)\, f(e) = \sum_{x_i, x_j \in X} [\pi(i,j,j) - \pi(i,j,i)] F_{ij} - \sum_{e \in E} m(e)\, c(e) \qquad (10)$$

In conclusion, a user optimised pattern $\{f_{ij}(e) \; ; \; x_i, x_j \in X \; ; \; e \in E\}$ $\{m(e) \; ; \; e \in E\}$ can be found as the solution generally unique of the linear system of equations and inequations :(1) (2) (3) (5) (6) (10).

The global time spent by the users in such a pattern is :

$$T = \sum_{e \in E} [b(e) + m(e)]\, f(e)$$

Now let us assume we can modify the capacities of the links of the network : there are some unitial capacities $e \to \bar{c}(e)$ on each link of the network and the real capacity $c(e)$ of a link is now considered as a variable which satisfy the following constraints.

$$0 \leq c(e) \leq \bar{c}(e) \qquad \forall e \in E \qquad (11)$$

The regulation problem can be set under the following form : Find the value of the capacities $c(e)$ such that

$$T = \sum_{e \in E} [b(e) + m(e)]\, f(e) \text{ is minimum.}$$

In summary our problem has the following formulation :

The variables are - the traffic flows : $f_{ij}(e)$ $e \in E$; $x_i, x_j \in X$.

- the additional costs $m(e)$; $e \in E$.
- the capacities $c(e)$; $e \in E$.

These variables satisfy the linear constraints :

(1) (2) (3) (5) (6) (10)

The problem is to minimize the objective function

$$T = \sum_{e \in E} [b(e) + m(e)] \; f(e).$$

We shall call this mathematical program, program (R).

Remark : The preceding mathematical programming problem is among the most difficult problem encountered in the theory of optimisation. The main difficulty is that the objective function is a quadratic function neither convex or concave. Finding a local optimum is not a very hard task but there is no way at the present time to find the global optimum of this program (R) by an efficient algorithm.

Theoretically (R) can be solved by cutting planes methods [(see Zwart (1969)] or branch and bound methods [see for instance Hillier, Liberman (1974)].

We shall discuss some aspects of the resolution of (R) in the last section of this paper.

IV. SOME EXAMPLES OF REGULATION

a) Introduction. In this subsection we provide small but interesting examples showing that regulation may be interesting even in the case of simplified networks with one origin and one destination.

b) Example 1. Let R be the network depicted on Fig. 3.

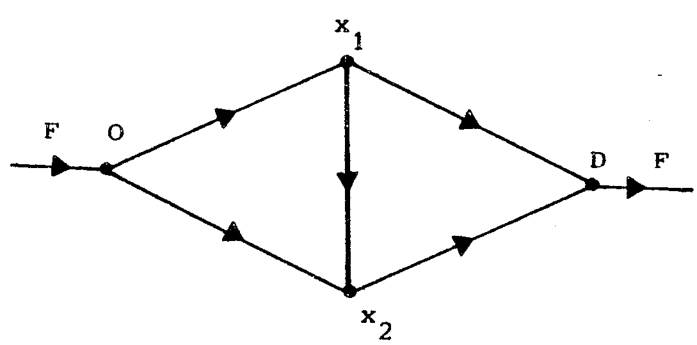


Fig. 3

The entering traffic in O is equal to 100 (vehicles per unit of time).

The initial costs are $b(O,x_1) = 2$; $b(O,x_2) = 9$; $b(x_1,x_2) = 2$

$b(x_1,D) = 8$; $b(x_2,D) = 3$.

The initial capacities of the links (O,x_1) and (x_2,D) are respectively 70 and 80. The other capacities are infinite .

Note that there are 3 possible routes in this network :

(O,x_1,D)	r_1
(O,x_1,x_2,D)	r_2
(O,x_2,D)	r_3

Without regulation the user optimised pattern is :

$$\begin{aligned} f(O,x_1) &= 70 \\ f(O,x_2) &= 30 \\ f(x_1,D) &= 20 \\ f(x_2,D) &= 80 \\ f(x_1,x_2) &= 50 \end{aligned} \qquad (12)$$

It is easy to check that the conditions (1) (2) (3) (5) (6) (7) (8) are satisfied by this assignment.

On the other hand there are additional costs on the saturated links (O,x_1) and (x_2,D)

$m(O,x_1)$ is determined by the condition that the cost of the route (O,x_1,x_2) is equal to the cost of the route (O,x_2).

$m(O,x_1) + b(O,x_1) + b(x_1,x_2) = b(O,x_2)$.

$m(O,x_1) = 5$

Similarly there is an additional cost on link (x_2,D), $m(x_2,D)$ equal to 3.

The global time spent by the users without regulation is equal to :

$$T_1 = \Sigma[b(e) + m(e)]\ f(e) = 15 \times 100 = 1500.$$

Note that the time spent by the drivers in the system assignment pattern is

$$T_3 = \sum_{e \in E} b(e)\ f(e) = 910.$$

Now lower the capacity of link (x_1,x_2) to 50.

It is easy to see that the flows in the preceding user optimised pattern (12) remain optimal with this new condition. However link (x_1,x_2) is now saturated and it is possible to imagine that congestion appears on this link.
The following solution satisfies also the first wardrop principle.

$$m(O,x_1) = 2$$

$$m(x_1,x_2) = 3$$

$$m(x_2,D) = 0.$$

The cost of this new solution is

$$T_2 = \Sigma[b(e) + m(e)]\ f(e) = 12 \times 100 = 1200 < T_1.$$

This situation illustrates the following paradox (studied in other context by KISK (1980) FISK and PALLOTINS (1979) or MURCHLAND (1970) see also M.J. SMITH (1979)).
If we deteriorate the traffic conditions on a link of the network [here link (x_1,x_2)] we globally improve the traffic conditions on the network.
Note that the explanation of this paradox is very simple.
The link (x_1,x_2) is now a saturated link. So the congestion which was on the link (x_2,D) has been replaced by a congestion on the link (x_1,x_2). As the traffic using link (x_2,D) is larger than the traffic using link (x_1,x_2), this implies an improvement in the global conditions of traffic.

An other interesting remark is that the traffic flow on each link is the same after the use of our regulation policy. To conclude with this example we assert (without proving it) that lowering the capacity of link (x_1,x_2) to 50 is the best possible regulation policy in this example (The proof is very easy).

c) Example 2. This example again corresponds to an other kind of paradoxes : the creation of a rapid road between two links of a transportation network may lead to an increase of the global time spent by the users in this network.
Consider the network depicted in Fig. 4.

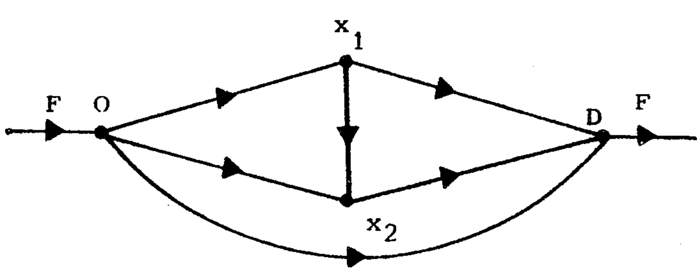


Fig. 4

For this example assume that : $F = 100$

$b(O,x_1) = 2$; $b(x_1,x_2) = 6$; $b(x_1,D) = 17$

$b(O,x_2) = 17$; $b(x_2,D) = 2$; $b(O,D) = 20.$

$c(O,x_1) = 90$ $c(x_2,D) = 90.$

The other capacities are infinite.

This network has 4 distinct routes :

(O,x_1,D) (1)
(O,x_2,D) (2)
(O,x_1,x_2,D) (3)
(O,D) (4)

Without regulation the traffic flows corresponding to the using assignment pattern is :

$f(O,x_1) = 90$; $f(x_1,x_2) = 90$; $f(x_2,x_3) = 90.$

$f(O,D) = 10$;

$f(O,x_2) = 0$ $f(x_1,D) = 0.$

[90 drivers use route (3) and 10 drivers use route (4)].
links (O,x_1) and (x_2,D) are saturated.
The congestion occurs on link (O,x_1) or link (x_2,D) and a possible solution is :

$m(O,x_1) = 8,$ $m(x_2,D) = 2$

(an other solution is $m(O,x_1) = 2$; $m(x_2,D) = 8$).
The total time spent by the users is :

$$T_1 = \sum_{e \in E} (b(e) + m(e))\, f(e) = 20 \times 90 + 20 \times 10 = 2000.$$

The time spent by the drivers in a system optimised pattern is

$$T_3 = \sum_{e \in E} b(e)\, f(e) = 1100.$$

It is easy to see in this example that if we want to improve the global situation of traffic flow it is necessary to change on some links the value of the trafics (this is the main difference with the first example).
If we take $c(x_1,x_2) = 0$ the rapid road (x_1,x_2) is closed to the traffic. The new user assignment pattern is given by the following solution :

50 drivers use route (1) and 50 drivers use route (2).

In this solution the links (O,x_1) and (x_2,D) are not saturated and

$m(O,x_1) = O$, $m(x_2,D) = O$ which means that there are no congestion on any link of the network.

The global time spent by the users is $T_2 = 19 \times 50 + 19 \times 50$

$$\underline{T_2 = 1900 < T_1}$$

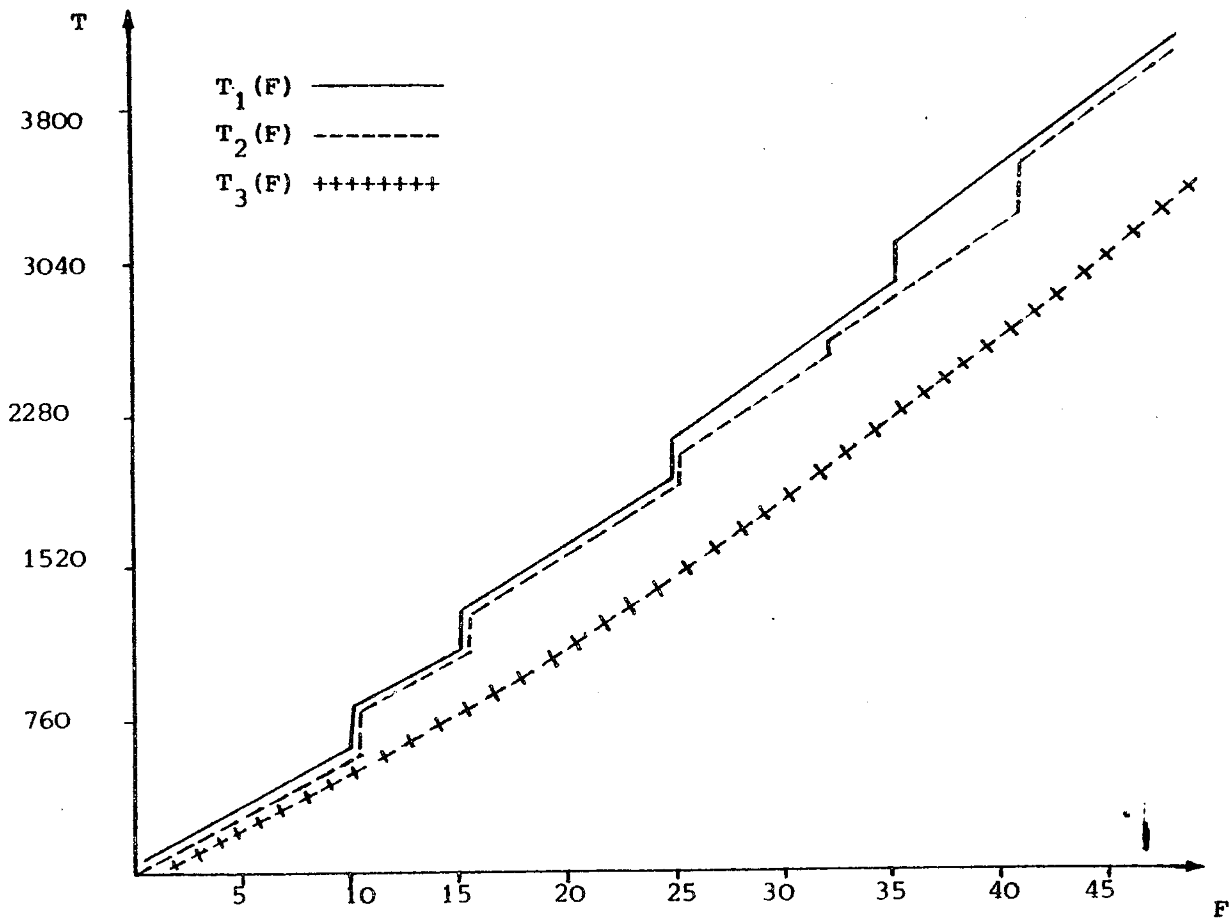
V. RESOLUTION OF PROGRAM (R) : AN OUTLINE

a) <u>Exact solution of (R)</u>

As it was previously noticed, program (R) is a very difficult to solve because of the non-convexity of the objective function.

A resolution method based on a branch and bound method has been proposed by FONLUPT, MAHJOUB and UHRY (1980). Thanks to this method it was possible to solve a real example corresponding to the city of Grenoble.

This network was with one origin and one final destination. If we call F the value of the flow going from O to D, it was possible to find for each value of F the best regulation policies. Results of this experimentation appear in Fig. (5).



<u>Fig. 5</u>

If F is the entering traffic flow T(F) is the time spent by the drivers.
$T_1(F)$ is the time spent by the drivers in the original network according to the first wardrop principle (user assignment pattern). $T_2(F)$ is the time spent by the driven with the use of the best regulation policy.
$T_3(F)$ is the time spent by the drivers according to the second wardrop principle (system assignment pattern).

Remark 1

Note that there are three kind of situations :

1°) If $F \leq 10$ (vehicles/second) $T_1(F) = T_3(F)$ and there is no need of studying a regulation policy (the charge of traffic is small and there are no saturated arcs).

2°) If $F \geq 45$ (vehicles/second) the network becomes completely saturated and all possible regulation policies are useless.

3°) $10 \leq F \leq 45$. This corresponds to an interesting situation where the network is not completely saturated and there may be advantage to remove congestion from certain links towards other links.

Remark 2

The largest improvement we could get was for F = 42. In this situation $T_1(F) = 3486$ and $T_2(F) = 3276$.
So one method provides an improvement of about 7 % in this situation.

Remark 3

Note that the function $F \to T_2(F)$ is an increasing function but the shape of the diagram $(F, T_2(F)$ is not very regular. This is not surprising, if we consider that the problem (R) is not on convex mathematical program.

b) Approximate methods for solving (R)

Program (R) is difficult to solve but it is easy to find a local optimum to (R).
If the variables $\{f_{ij}(e) \; ; \; x_i, x_j \in X \; ; \; e \in E\}$ are fixed, the program (R) becomes a linear program easy to solve. To fix the traffic variables means that we want to keep the actual values of the existing traffic before regulation. Then the problem that we have to solve is to find on which links it is interesting to locate the congestion, and this can be done by linear programming.
Note also that if we fix the variables $m(e), e \in E$ the program (R) becomes a

linear program (a multicommodety flow problem). Fixing alternatively the traffic flow variables $f_{ij}(e)$ and the congestion costs m(e) may lead to an interesting heuristique procedure to find of local optimum to (R).

VI. CONCLUSION

In this paper, we tried to implement a new approach to for studying regulation problems. Our main hypothesis was to assume that the drivers always behave according to the first Wardrop principle : regulation methods which do not on this principle seem not very realistic for general transportation networks. The only way for a central decision center to act on the decisions of drivers is to change locally the traffic conditions on the links of the network. A result of this study is that this problem is a very difficult one and that no exact solution may be found in the general case. However such regulation policies may give interesting results in some special cases and also enlightens from a new point of view the assignment problem. To continue this study it would be interesting to study some kinds of networks for which it is possible to solve program (R) : for instance we can think to the case of a corridor already studied by Payne and Thompson (1974) [see also Mahjoub (1981)].

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ANNEXE 2

COMPOSITION OF GRAPHS AND THE BIPARTITE SUBGRAPH POLYTOPE

by

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Abstract

Given a graph $G = (V, E)$ and a set of positive edge weights $(C_e,\ e \in E)$, the max-cut problem in G consists of determining a cut F in G such that $\sum_{e \in F} C_e$ is as large as possible. A graph G is said to be the k-sum of graphs G_i and G_j, if G is obtained from G_i and G_j by identifying a K_k contained in each of them, where K_k is the complete graph on k vertices, with $k \in N$. The bipartite subgraph polytope $P_B(G)$ of G, is the convex hull of the incidence vectors of all edge sets of bipartite subgraphs of G. For a graph G defined by the k-sum, $1 \leq k \leq 3$ of the graphs G_1 and G_2, we present in this paper, a method to calculate a max-weight cut in G from the max-weight cuts in $\bar{G}_1$ and $\bar{G}_2$ where $\bar{G}_1$ and $\bar{G}_2$ are two appropriate graphs defined from G_1 and G_2. Moreover we give a description of the polytope $P_B(G)$ when the polytopes $P_B(\bar{G}_1)$ and $P_B(\bar{G}_2)$ are known. We may then conclude that graphs noncontractible to K_5 are weakly bipartite. If G is the 2-sum of G_1 and G_2 then the polytope $P_B(G)$ will be characterized by a minimal system of constraints, where each one defines a facet of $P_B(G)$.

Résumé

Etant donné un graphe $G=(V,E)$ et un systeme de poids non négatifs sur les arêtes $(C_e,\ e \in E)$ le probleme du cocycle de poids maximum dans G, consiste à déterminer un cocycle F dans G tel que $\sum_{e \in F} C_e$ soit maximum.Ungraphe G est dit k-somme de deux graphes G_1 et G_2 si G est obtenu à partir ge G_1 et G_2 en identifiant un graphe complet K_k contenu dans G_1 et G_2 (K_k est le graphe complet ayant k sommets, avec $k \in N$). Le polytope des sous graphes bipartis de G noté $P_B(G)$ est l'enveloppe convexe des vecteurs d'incidence de tous les ensembles d'arètes des sous graphes bipartis de G.Pour un graphe G défini par la k-somme, $1 \leq k \leq 3$, de deux graphes G_1 et G_2, nous présentons dans ce papier une méthode qui permet de calculer un cocycle de poids maximum dans G à partir des cocycles maximums dans $\bar{G}_1$ et $\bar{G}_2$, où $\bar{G}_1$ et $\bar{G}_2$ sont deux graphes appropriés définis à partir de G_1 et G_2.De même, nous donnons une caractérisation du polytope $P_B(G)$ quand les polytopes $P_B(\bar{G}_1)$ et $P_B(\bar{G}_2)$ sont supposés connus. Nous montrons en particulier que si G_1 et G_2 sont faiblements bipartis alors G est également faiblement biparti.Ce résultat nous permet de déduire par la suite que les graphes non contractibles à K_5 sont faiblement bipartis.Dans le cas où G est la 2-somme de G_1 et G_2 le polytope $P_B(G)$ sera caractérisé par un système minimal de cotraintes où toute contrainte est une facette de $P_B(G)$.

1. Introduction and notation

In this paper we consider only finite, simple, undirected graphs, containing neither loops nor multiple edges. A graph is denoted by $G=(V,E)$, where V is the node set and E the edge set of G. An edge $e \in E$, whose vertices are v_i and v_j can also be denoted by v_iv_j. The graph $H=(U,F)$ is a subgraph of G if $U \subseteq V$ and $F \subseteq E$. We also say that H is contained in G and that G contains H.

Let $G=(V,E)$ be a graph, if $F \subseteq E$, then $V(F)$ denotes the set of nodes of V which occur at least once as an endnode of an edge in F, and $G(F)$ denotes the graph $(V(F),F)$. Similary, for $W \subseteq V$, $E(W)$ denotes the set of all edges of G with both endnodes in W.

A graph G is called bipartite if its node set may be partitioned into two nonempty disjoint sets V_1 and V_2 such that all edges have one node in V_1 and the other in V_2. The set of edges of a bipartite graph will be called a bipartite edge set. The family of bipartite edge sets of $G=(V,E)$ is denoted by $B(E)$. A bipartite edge set $B \in B(E)$ is said to be maximal if $\forall e \in E \setminus B$, $B \cup \{e\} \notin B(E)$.

A Cut of $G=(V,E)$ is a set $F \subseteq E$ such that there exists a set $U \subseteq V$ where $F=\{v_iv_j \in E \mid v_i \in U$ and $v_j \notin U\}$. A graph G is contractible to H, if H may be obtained from G by a sequence of elementary removals and contractions of edges. A contraction consists of identifying a pair of adjacent vertices and of preserving all other adjacencies between vertices (multiple edge arising from the identification are replaced by single edges).

Given a graph $G=(V,E)$ and a set of positive edge weights $(C_e, e \in E)$, the problem of finding a cut F such that its weight $\sum_{e \in F} C_e$ is as large as possible is called the max-cut problem and it is denoted by $\mathcal{P}$. Since the set of edges of any bipartite subgraph of G is contained in a cut, the optimal solution for $\mathcal{P}$ is a maximum weight bibartite subgraph of G. However if the weight function $b: E \longrightarrow R$ admits $b(e) < 0$ for some $e \in E$, then the max-weight cut may be different from the max-weight bipartite subgraph of G.

A polyhedron $P \subseteq R^m$ is the intersection of finitely many halfspaces in R^m. A polytope is a bounded polyhedron or equivalently the convex hull of finitely many points. The dimension of a polyhedron P, is the maximum number of affinely independent points in P minus one.

If $d > 0$ is the dimension of P, then a valid linear inequality of P,

$$\sum_{i=1,\ldots,m} a_i x_i \leq a_0$$

defines a facet of P if it is satisfied with equality by exactly d affinely independent points of P and $P \cap \{x \in P \mid \sum_{i=1,\ldots,m} a_i x_i = a_0\} \neq P$

If $G = (V,E)$ is a graph and $F \subseteq E$ an edge set then the 0/1-vector $x^F \in R^{|E|}$ with $x_e^F = 1$ if $e \in F$; and $x_e = 0$ if $e \notin F$ is called the <u>incidence vector</u> of F.

The convex hull $P_B(G)$ of the incidence vectors of all edge sets of bipartite subgraphs of G is called the bipartite subgraph polytope of G, ie

$$P_B(G) = \text{conv}\{x^B \in R^E \mid B \in \mathcal{B}(E)\}$$

Over the last few years, the study of this polytope has generated a great deal of interest (cf. [1], [3], [4], [10]). Note that an optimal solution of $\mathcal{P}$ is an optimal basic solution of the linear program

$$\text{Max } Cx,\ x \in P_B(G)$$

Since the problem $\mathcal{P}$ is NP-complete [18], we cannot expect to find a complete explicit characterization of $P_B(G)$ for all graphs G. This characterization is known only for certain classes of graphs such as planar graphs (cf, Barahona [1]) and weakly bipartite graphs (cf, Grötschel & Pulleyblank [10]). In [4], Barahona, Grötschel and Mahjoub have presented various classes of facets of this polytope in the general case. Barahona [1], showed that the constraints

$$0 \leq x_e \leq 1 \qquad \text{for } e \in E \qquad (1,1)$$

$$\sum_{e \in C} x_e \leq |C| - 1 \qquad \text{for all odd cycles } C \text{ in } G \qquad (1,2)$$

define facets of $P_B(G)$. (We call the inequalities (1,1) trivial and the inequalities (1,2) odd cycle constraints).

Grötschel & Pulleyblank [10], defined the class of weakly bipartite graphs as the set of graphs G for which the polytope $P_B(G)$ is defined by (1,1) and (1,2). They also presented a polynomial algorithm that solves the separation problem for the inequalities (1,1) and (1,2) : "Given $x \in R^{|E|}$, is x a point of $P_B(G)$?, if not, find the constraint among those in (1,1) and (1,2) violated by x". This algorithm, along with the ellipsoid method, solves $\mathcal{P}$ polynomially for this class of graphs. Two classes of graphs which satisfy the definition of weakly bipartite are bipartite graphs and planar graphs [1]. Furthermore Barahona showed in [3], that the graphs G satisfying the following property are weakly bipartite : there exist two vertices u, v of G such that $G \setminus \{u,v\}$ is bipartite.

Given two graphs G_i and G_j we say that graph G is the k-sum of G_i and G_j, denoted $G = G_i \boxed{k} G_j$, if G is obtained from G_i and G_j by identifying two K_k contained in both of them (K_k is a complete graph on k vertices, $k \in N$). It is easy to see that the 2-sum of two planar graphs is planar, but the 3-sum of two planar graphs may not necessarily be planar. Wagner [16], showed that any graph $G = (V,E)$ which is maximally noncontractible to K_5 (ie G is noncontractible to K_5 and $\forall\, v_i, v_j \in V \mid v_iv_j \notin E$, $G + v_iv_j$ is contractible to K_5) may be obtained by means of k-sums, $1 \le k \le 3$, of maximal planar graphs and copies of V_8 (V_8 is the graph of Fig.1). See also Ore [13], Seymour [14]). This characterization gives rise to a polynomial algorithms for recognizing whether a given graph is contractible to K_5.

If G is the k-sum obtained from G_1 and G_2 by identifying K_k, then we define $\bar{G}_i$, i=1, 2, to be the graphs derived from G_i, i=1, 2, by adding between each pair of vertices of K_k a path of size two (see Fig.2 and Fig.3).

From the work of Seymour on matroids and multicommodity flow [15], we can show that the graphs noncontractible to K_5 are weakly bipartite. In this paper we will present a generalization of this result. In section 3 we give a description of the polytope $P_B(G)$ when G is a k-sum of G_1 and G_2 with $1 \le k \le 3$, and the polytopes $P_B(\bar{G}_1)$ and $P_B(\bar{G}_2)$ are known. Moreover we show that if $\bar{G}_1$ and $\bar{G}_2$ are weakly bipartite then G is weakly bipartite. We can then conclude that graphs noncontractible to K_5 are weakly bipartite. In section 4, we describe a method of constructing facets of the polytope $P_B(G)$ from facets of $P_B(\bar{G}_1)$ and of $P_B(\bar{G}_2)$ where $G=G_1 \boxed{2} G_2$. In this case these constructions yield a minimal system of inequalities which characterize the polytope $P_B(G)$. In the following section, we present a method to calculate a maximum weight cut in a graph $G=G_1 \boxed{k} G_2$, $1 \le k \le 3$, with nonnegative weighted edges. This cut is expressed as a function of the optimum cuts in $\bar{G}_1$ and $\bar{G}_2$.

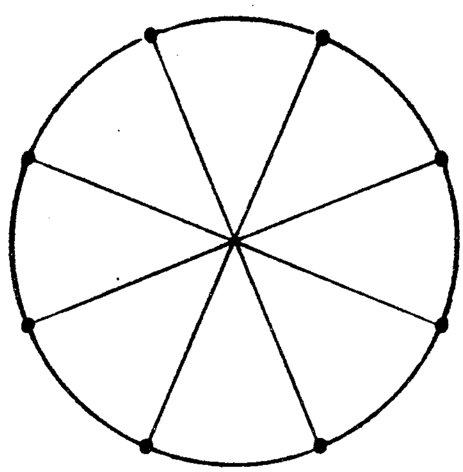


Fig.1 The graph V_8

2. k-sums and algorithm

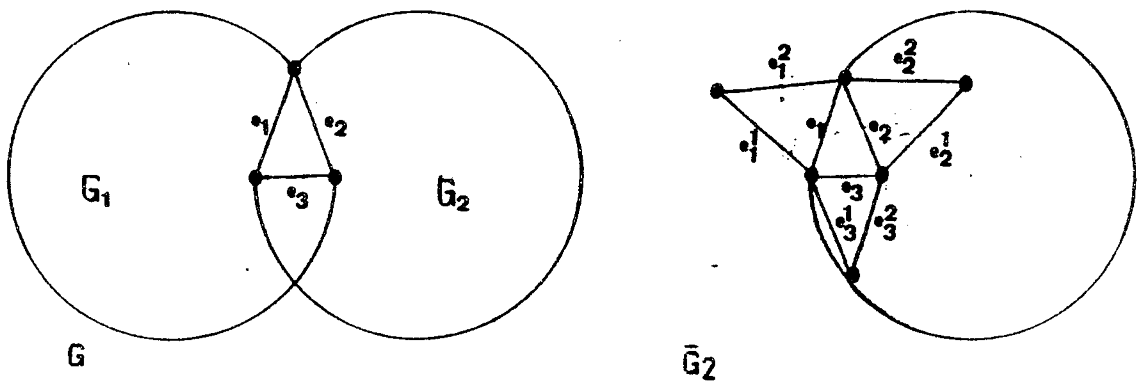
In this section we present an algorithm to calculate an optimum solution of the problem in a graph $G = G_1 \boxed{k} G_2$, $1 \le k \le 3$, where the optimum solutions are known for $\bar{G}_1$ and $\bar{G}_2$. We will extend this algorithm to the case where G is obtained by means of k-sums, $1 \le k \le 3$. Furthermore we assume that the edges of G have been assigned nonnegative weights. First note that if $G = G_1 \boxed{1} G_2$, then the optimum cut in G is the sum of two optimum cuts in G_1 and G_2. In the following we study the case of the 3-sum. The case of the 2-sum may be obtained as a variant of the first one.

2.1. 3-sum

Consider graph $G = (V, E)$, a 3-sum of $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ where

$$E_1 \cap E_2 = \{e_1, e_2, e_3\} \quad \text{(see Fig.2(a))}$$

Let (e_i^1, e_i^2), $i = 1, 2, 3$, be the paths added between the vertices of e_i, $i = 1, 2, 3$, for constructing the graphs $\bar{G}_1$ and $\bar{G}_2$ ($\bar{G}_2$ is given in Fig. 2.(b)). Let Z be edge set $\{e_i, e_i^1, e_i^2;\ i = 1, 2, 3\}$. Denote by $W^1_{\bar{e}_1}$; $W^1_{\bar{e}_2}$; $W^1_{\bar{e}_3}$; W^1_0 respectively the max-weight cut in G_i, for $i = 1, 2$, which contains e_2, e_3 but not e_1; e_1, e_3 but not e_2; e_1, e_2 but not e_3; not e_1, e_2, e_3, and denote by $w^1_{\bar{e}_1}$; $w^1_{\bar{e}_2}$; $w^1_{\bar{e}_3}$; w^1_0 the weights of these cuts. (In general upper case will signify a cut and lower case will signify the corresponding weight).



(a) (b)

Fig.2

it is obvious that a max-weight cut in $\bar{G}_1$ is among these cuts. We remark that these cuts may be calculated by considering appropriate weights on the edges in Z. Let:

$$w_1 = \frac{-w_0^1 - w_{\bar{e}_1}^1 + w_{\bar{e}_2}^1 + w_{\bar{e}_3}^1}{2}$$

$$w_2 = \frac{-w_0^1 + w_{\bar{e}_1}^1 - w_{\bar{e}_2}^1 + w_{\bar{e}_3}^1}{2}$$

$$w_3 = \frac{-w_0^1 + w_{\bar{e}_1}^1 + w_{\bar{e}_2}^1 - w_{\bar{e}_3}^1}{2}$$

and

$$\pi_j = \max(0, w_j) \quad \text{for } j = 1, 2, 3$$

$$\Delta_j = \pi_j - w_j \quad \text{for } j = 1, 2, 3$$

Let $H = (V(Z), Z)$. To H we assign the following system of edge weights denoted by (Δ, π): the edges e_i^1, e_i^2 receive the weights Δ_i and the edge e_i receives the weights π_i, for $i = 1, 2, 3$.

Let $w_{\bar{e}_1}^H$; $w_{\bar{e}_2}^H$; $w_{\bar{e}_3}^H$; w_0^H be the weights of the max-cuts in H which respectively contain e_2, e_3, but not e_1 ; e_1, e_3, but not e_2 ; e_1, e_2, but not e_3 ; not e_1, e_2, e_3. It is easy to see that there always exists a real Q which satisfies the following system :

$$(2,1)\begin{cases} w_{\bar{e}_1}^1 - Q = w_{\bar{e}_1}^H \\ w_{\bar{e}_2}^1 - Q = w_{\bar{e}_2}^H \\ w_{\bar{e}_3}^1 - Q = w_{\bar{e}_3}^H \\ w_0^1 - Q = w_0^H \end{cases}$$

Thus it becomes easy to determine a max-weight cut in G from G_2 there by establishing the following.

Theorem 2.1

Let $\bar{W}_2$ be an optimum cut in $\tilde{G}_2$ where the edges of H have the weight system (Δ, π) and the weights of the other edges of G remain unchanged. Set $\bar{\bar{W}}_2 = \bar{W}_2 \setminus Z$. Let W be a cut of G obtained as follow

(a) if $\{e_1, e_2\} \subset \bar{W}_2$ and $e_3 \notin \bar{W}_2$, $W = \bar{\bar{W}}_2 \cup W^1_{\bar{e}_3}$

(b) if $\{e_1, e_3\} \subset \bar{W}_2$ and $e_2 \notin \bar{W}_2$, $W = \bar{\bar{W}}_2 \cup W^1_{\bar{e}_2}$

(c) if $\{e_2, e_3\} \subset \bar{W}_2$ and $e_1 \notin \bar{W}_2$, $W = \bar{\bar{W}}_2 \cup W^1_{\bar{e}_1}$

(d) if $\{e_1, e_2, e_3\} \cap \bar{W}_2 = \emptyset$, $W = \bar{\bar{W}}_2 \cup W^1_0$

Then W is an optimum cut of G whose weight is $w = \bar{w}_2 + Q$.

Proof

On the one hand, the set W formulated by each operation (a), (b), (c), (d), defines a cut of G whose weight is $w = \bar{w}_2 + Q$. On the other hand, w^* the maximum weight of a cut in G must satisfy

$$w^* = \max\,(w^1_{\bar{e}_1} + w^2_{\bar{e}_1}, w^1_{\bar{e}_2} + w^2_{\bar{e}_2}, w^1_{\bar{e}_3} + w^2_{\bar{e}_3}, w^1_0 + w^2_0)$$

From system (2, 1), it follows that

$$w^* = \bar{w}_2 + Q$$

Thus the cut is optimum.

■

2.2 2-sum

As mentioned above, this case may be studied as a variant of the preceding case. In fact consider a graph $G = (V, E)$, the 2-sum of $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ where

$E_1 \cap E_2 = \{e_1\}$ (see Fig.3 (a))

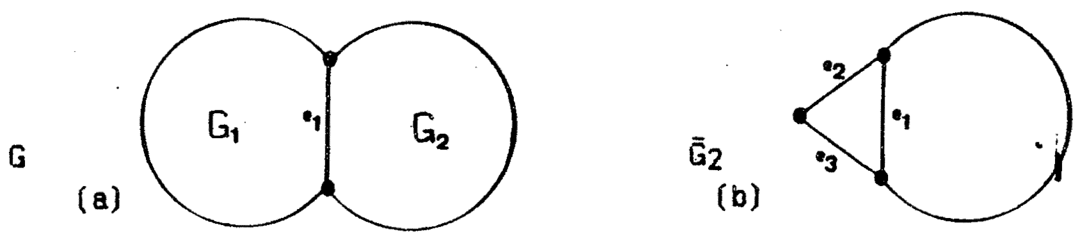


Fig.3

Let $\bar{G}$, be the 3-sum of the graphs $\bar{G}_1$ and $\bar{G}_2$ obtained by identifying the triangle (e_1, e_2, e_3) where (e_2, e_3) is the path added to both G_1 and G_2 for constructing the graphs $\bar{G}_1$ and $\bar{G}_2$ (see Fig.3(b)). ($\bar{G}$ is the graph G with the path (e_2, e_3) added between the vertices of e_1). If we consider null weights on the edges e_2, e_3, it is clear that a cut W is optimum in $\bar{G}$ if and only if the cut $W - \{e_2, e_3\}$ is optimum in G. As a consequence, if $W^1_{e_1}$(resp. $W^1_{\bar{e}_1}$) denotes the max-weight cut in G which contains (resp does not contain) the edge e_1, and if we set

$$w_1^1 = w^1_{e_1} - w^1_{\bar{e}_1}$$

$$\pi = \max(0, w_1^1)$$

$$\Delta = \pi - w_1^1$$

Then a max-weight cut in G may be determined by the following corollary.

<u>**Corollary 2.2**</u>

Let $\bar{W}_2$ be an optimum cut of $\bar{G}_2$ where e_2, e_3 have the same weight Δ, and e_1 has weight π. The other weights remain unchanged. Let W be the cut obtained as follow

(a) if $e_1 \in \bar{W}_2$, $W = \bar{W}_2 \setminus \{e_2, e_3\} \cup W^1_{e_1}$

(b) if $e_1 \notin \bar{W}_2$, $W = \bar{W}_2 \setminus \{e_2, e_3\} \cup W^1_{\bar{e}_1}$

Then W is an optimum cut in G whose weight is $w = \bar{w}_2 + Q$.

Theorem 2.1 and corollary 2.2 allow a kind of decomposition in the determination of an optimum cut in graphs formed by the k-sum operation, $1 \leq k \leq 3$. In fact consider a graph G obtained by means, of k-sums, $1 \leq k \leq 3$, of the graphs $G_1, G_2, \ldots, G_n$. G may be decomposed serially in the following way :

$$(2,2)\begin{cases} H_1 = G_1 \\ H_{i+1} = H_i \; \boxed{k_i} \; G_{i+1} \qquad \text{with } 1 \leq k_i \leq 3 \qquad \forall i \in \{1, 2, \ldots, n-1\} \\ G = H_n \end{cases}$$

Note that if G is a graph which has several plarallel paths of size two between two vertices v_i, v_j, then an optimum cut in G may be calculated by replacing these parallel paths by a single one with appropriate weights on its edges.

Let $\widetilde{G}_i$, $i = 1, 2, \ldots, n$ be the graph obtained from G_i by adding between the vertices of every edge $e \in G_i \cap G_j$ for all $j \neq i$, a path containing two edges.

We consider the sequence of graphs G^i, $i = n, n-1, \ldots, 1$
defined as follow

$$\begin{cases} G^n = G, & T_n = H_{n-1} \\ G^i = T_i \boxed{k_i} G'_i & \\ G^{i-1} = \bar{T}_i & \end{cases}$$

with $\bar{G}'_i = \widetilde{G}_i$ and $\bar{T}_i$ is defined from T_i by adding a path of size two between the vertices of an edge $e \in K_{k_i}$, only if a such path does not already exist. From the above remarks, an optimum cut in G^i may be calculated from G^{i-1} with an appropriate system of weights deduced from the optimum cuts in $\widetilde{G}_i$. An optimum solution in G may thus be calculated recursively from the graphs G^i, $i = n, n-1, \ldots, 1$. To do this we need to calculate in each graph $\widetilde{G}_i$, for $i = 1, \ldots, n$, k_{i+1} max-weight cuts. Thus we have the following.

<u>**Corollary 2.3**</u>

Let G be a graph obtained by means of k-sums, $1 \leq k \leq 3$, from the graphs $G_1, \ldots, G_n$. (as in (2,2)). If the problem $\mathcal{P}$ is polynomial for the graphs $\widetilde{G}_1, \ldots, \widetilde{G}_n$ then $\mathcal{P}$ is polynomial for G.

Since the problem $\mathcal{P}$ is polynomial in planar graphs and in V_8 (by enumeration of solutions) then using Wagner's decompositon and corollary 2.3 we can conclude that problem $\mathcal{P}$ is polynomially solvable for graphs noncontractible to K_5. This has also been shown by Barahona[2].

3. k-sums and the bipartite subgraph polytope

As mentioned above, a complete description of the bipartite subgraph polytope $P_B(G)$ by a finite system of inequalities is not known in the general case. Moreover for a graph G defined by a k-sum, $1 \le k \le 3$, of two graphs G_1 and G_2 we can give such a description where the polytopes $P_B(\bar{G}_1)$ and $P_B(\bar{G}_2)$ are known. If G is the 1-sum of G_1 and G_2 then $P_B(G)$ is given by the juxtaposition of $P_B(G_1)$ and $P_B(G_2)$. In the following paragraph we give this description in the 3-sum case. The case of the 2-sum may be obtained as a particular case of the first one.

3.1. 3-sum

Let $G = (V, E)$ be a 3-sum of $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ (see Fig.2). Let the triangle
$T_0 = E_1 \cap E_2 = \{e_1, e_2, e_3\}$.

Denote T_i the triangle (e_i, e_i^1, e_i^2) for $i = 1, 2, 3$. We keep the notation of the previous section). Moreover, denote $E^i = E_i \setminus T_0$ for $i = 1, 2$. We have then

$$\bar{E}_1 = E^1 \cup T_1 \cup T_2 \cup T_3$$

$$\bar{E}_2 = E^2 \cup T_1 \cup T_2 \cup T_3$$

Notation : We introduce the following notations which will be used in the remainder of the paper. Given a graph $G = G_1 \boxed{k} G_2$, $1 \le k \le 3$, where $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, we associate with each edge $e \in \bar{E}_1 \cup \bar{E}_2$ a variable x_e. For all subset $F \subseteq \bar{E}_1 \cup \bar{E}_2$ we set

$$x_F = (x_e,\ e \in F)$$

Let $a_F = (a_e,\ e \in F)$ be a vector of $R^{|F|}$. We denote

$$\langle a_F, x_F \rangle = \sum_{e \in F} a_e x_e$$

Suppose, polytopes $P_B(\bar{G}_1)$ and $P_B(\bar{G}_2)$ are defined by the following systems of inequalities

$$P_B(\bar{G}_1) = \begin{cases} 0 \le x_e \le 1 & \text{for } e \in E_1 \\ \langle a^k_{E1}, x_{E1}\rangle + \sum\limits_{i=1,\dots,3} \langle a^k_{T_i}, x_{T_i}\rangle \le \alpha^K & k=1,\dots,K \qquad (3,1) \end{cases}$$

and

$$P_B(\bar{G}_2) = \begin{cases} 0 \le x_e \le 1 & \text{for } e \in E_2 \\ \langle b^l_{E2}, x_{E2}\rangle + \sum\limits_{i=1,\dots,3} \langle b^l_{T_i}, x_{T_i}\rangle \le \beta^1 & 1=1,\dots,L \qquad (3,2) \end{cases}$$

The constraints described in (3,1) and (3,2) will be called proper constraints. We suppose that each of these constraints is nonredundant (i.e. each defines a facet of the polytope). A proper constraint which corresponds to $k \in \{1, 2, \dots, k\}$ (resp $1 \in \{1, 2, \dots, L\}$) will be denoted by (k) (resp (1)).

Recall, that for all $B \in \mathcal{B}(\bar{E}_1)$ (resp $B \in \mathcal{B}(\bar{E}_2)$), x^B is a vertex of $P_B(\bar{G}_1)$ (resp $P_B(\bar{G}_2)$ (and vice versa) and, moreover, the coefficents in a proper constraint are nonnegative (this is a consequence of the fact that $\mathcal{B}(E)$ defines a family of independents). In the following we introduce some preliminary lemmas and definitions.

Lemma 3.1

For a constraint (k) (resp(1)) which is different than the constraintes associated with triangles T_i, $i=1,2,3$, at least one of the coefficients $a^k_{e_i}$, $a^k_{e_i^1}$ and $a^k_{e_i^2}$ (resp $b^l_{e_i}$, $b^l_{e_i^1}$ and $b^l_{e_i^2}$) is equal to zero for $i=1,2,3$.

Proof

we prove this lemma for $P_B(\bar{G}_1)$ (the proof for $P_B(\bar{G}_2)$ is similar). Suppose that there is $i_0 \in \{1,2,3\}$ and a facet defining inequality (k) of $P_B(G_1)$ such that

$$a^k_{e_{i_0}} > 0,\ a^k_{e^1_{i_0}} > 0,\ a^k_{e^2_{i_0}} > 0$$

Thus every bibartite edge set $B \subseteq \bar{E}_1$ whose incidence vector x^B satisfies (k) with equality, also satisfies

$$x^B_{e_{i_0}} + x^B_{e^1_{i_0}} + x^B_{e^2_{i_0}} = 2$$

This implies that facet (k) and the facet associated with triangle T define the same hyperplane which is impossible.

■

Lemma 3.2

$$a^k_{e^1_i} = a^k_{e^2_i}, \qquad k = 1, 2, \ldots, K; \quad i = 1, 2, 3$$

$$b^l_{e^1_i} = b^l_{e^2_i}, \qquad l = 1, 2, \ldots, L; \quad i = 1, 2, 3$$

Proof :

We prove this lemma for $P_B(\bar{G}_1)$ (the proof for $P_B(\bar{G}_2)$ is similar). Suppose not, and there exist i and k where, $1 \leq i \leq 3$; $1 \leq k \leq K$, such that we have, for example, $0 \leq a^k_{e^1_i} < a^k_{e^2_i}$. Since, constraint (k) is nontrivial, there exists $B \in \mathcal{B}(E_1)$ such that B is maximal and its incidence vector x^B satisfies (k) with equality and $x^B_{e^2_i} = 0$ (then $x^B_{e^1_i} = 1$).
Let $B' = B \setminus \{e^1_i\} \cup \{e^2_i\}$. It's obvious that B' defines a bipartite edge set of $\bar{G}_1$. But the vector $x^{B'}$ violates the constraint (k) and, thus, we have a contradiction.

■

3.1.1. Mixture of proper constraints

For every constraint (k)(resp(l)), we associate a real coefficient $\pi_k \geq 0$ (resp μ_l) and we set

$$\pi = (\pi_k,\ k = 1, \ldots, K)$$

and $\mu = (\mu_l,\ l = 1, \ldots, L)$

We may, then, define constraint (π) obtained by a linear combination of the K proper constraints such that

$$(\pi) := \langle a_{\bar{E}_1}(\pi),\ x_{\bar{E}_1} \rangle \leq \alpha(\pi)$$

with $a_{\bar{E}_1}(\pi) = \sum_{k=1,\ldots,K} \pi_k\, a^k_{\bar{E}_1}$ and $\alpha(\pi) = \sum_{k=1,\ldots,K} \alpha^k \pi_k$

In the same manner, we may define the constraint

$$(\mu) := \langle b_{\bar{E}_2}(\mu),\ x_{\bar{E}_2} \rangle \leq \beta(\mu)$$

with $b_{\bar{E}_2}(\mu) = \sum_{l=1,\ldots,L} \mu_l\, b^l_{\bar{E}_2}$ and $\beta(\mu) = \sum_{l=1,\ldots,L} \beta^l(\mu_l)$.

Note that the constraints (π) and (μ) are in general redundant and in order for them to be nonzero, it is necessary that $\pi \neq 0$ and $\mu \neq 0$. Moreover note that by the precedent lemma we have

$$a_{e_i}1(\pi) = a_{e_i}2(\pi), \quad i = 1, 2, 3$$

and

$$b_{e_i}1(\mu) = b_{e_i}2(\mu), \quad i = 1, 2, 3$$

Now, consider two constraints (π) and (μ) expressed as

$$(\pi) := \langle a_E1(\pi), \; x_E1\rangle + \sum_{i=1,\dots,3} \left(a_{e_i}(\pi)x_{e_i} + a_{e_i}1(\pi)(x_{e_i}1 + x_{e_i}2)\right) \leq \alpha(\pi)$$

$$(\mu) := \langle b_E2(\mu), \; x_E2\rangle + \sum_{i=1,\dots,3} \left(b_{e_i}(\mu)x_{e_i} + b_{e_i}1(\mu)(x_{e_i}1 + x_{e_i}2)\right) \leq \beta(\mu)$$

Suppose, moreover,

$$(3,3) \begin{cases} a_{e_i}(\pi) \geq b_{e_i}1(\mu), & i = 1, 2, 3 \\ b_{e_i}(\mu) \geq a_{e_i}1(\pi), & i = 1, 2, 3 \end{cases}$$

and

$$(3,4) \begin{cases} \sum_{k=1,\dots,K} \pi_k + \sum_{l=1,\dots,L} \mu_l = 1 \\ \pi_k \geq 0, & k = 1, \dots, K \\ \mu_l \geq 0, & l = 1, \dots, L \end{cases}$$

(the first constraint of (3,4) means just that, at least one of the constraints (π) and (μ) is nonzero).

We call mixed constraint of (π) and (μ) denoted by $[\pi, \mu]$, the constraint defined on the set E by

$$[\pi, \mu] := \langle a_E1(\pi), x_E1\rangle + \langle b_E2(\mu), x_E2\rangle + \sum_{i=1,\dots,3} \left(a_{e_i}(\pi) + b_{e_i}(\mu)\right.$$

$$\left. - a_{e_i}1(\pi) - b_{e_i}1(\mu)\right)x_{e_i} \leq \alpha(\pi) + \beta(\mu) - 2\sum_{i=1,\dots,3} \left(a_{e_i}1(\pi) + b_{e_i}1(\mu)\right)$$

In the following we denote by (R), the system of linear inequalities defined by conditions (3,3) and (3,4). Let (P) be the convex subset of $R^{|E|}$ defined by all mixed constraints $[\pi, \mu]$ satisfying (R), and the trivial constraints.

$$0 \le x_e \le 1 \quad \text{for } e \in E \tag{3,5}$$

Note that there exists an uncountable infinity of mixed constraints. In the next lemma, we show that only a finite number of constraints suffices to define (P).

Lemma 3.3

(P) is a polytope

Proof

Let $[\pi, \mu]$ be a mixed constraint satisfying the system (R) which is formed by seven linear equalities and inequalities whose variables are π_k, $k = 1, \ldots, K$ and μ_l, $l = 1, \ldots, L$. If (π, μ) isn't a basic solution of (R) then (π, μ) may be written as a half sum of other solutions of (R), (π_1, μ_1) and (π_2, μ_2). But then, the constraint $[\pi, \mu]$ may be written as a half sum of mixed constraints $[\pi_1, \mu_1]$ and $[\pi_2, \mu_2]$, which shows that $[\pi, \mu]$ is redundant. Therefore, the nontrivial constraints which may be nonredundant for (P) are the mixed constraints $[\pi, \mu]$ where (π, μ) is a basic solution of (R). This shows that the number of nonredundant constraints of (P) is finite and therefore (P) is a polytope.

■

Remark 3.4

All constraints defining facets of $P_B(G_1)$ or of $P_B(G_2)$ (which define also facets of $P_B(\bar{G}_1)$ or of $P_B(\bar{G}_2)$) are among the constraints of (P).

3.1.2 Extension of a solution

Definition 3.5 :

Let $x_E = (x_{E1}, x_{E2}, x_{T_0})$ be a vector of $R^{|E|}$.

We call extension of x_E, the vector $x_{\bar{E}}$ of $R^{|\bar{E}_1 \cup \bar{E}_2|}$ defined by

$$x_{\bar{E}} = (x_{E1}, x_{E2}, x_{T_1}, x_{T_2}, x_{T_3})$$

with
$$x_{e_i 1} = 1 \quad , \ i = 1, 2, 3$$
$$x_{e_i 2} = 1 - x_{e_i}, \ i = 1, 2, 3$$

Definition 3.6 :

Given a feasible solution x_E of (P), then x_E is said to be nondominated if there does not exist a feasible solution x'_E of (P) such that

$$x'_E \neq x_E \text{ and } x'_e \geq x_e \text{ for } e \in E.$$

Definition 3.7 :

Given an inequality C, and a real-valued vector x, we say that x saturates C if x satisfies C at equality.

We have the following fundamental result.

Lemma 3.8

Let $x_E = (x_{E1}, x_{E2}, x_{T_0})$ be a feasible, nondominated solution of (P), and let

$$x_{\bar{E}} = (x_{E1}, x_{E2}, x_{T_1}, x_{T_2}, x_{T_3}) \text{ be its extension.}$$

Then the vectors

$$x_{\bar{E}_1} = (x_{E1}, x_{T_1}, x_{T_2}, x_{T_3})$$

$$x_{\bar{E}_2} = (x_{E2}, x_{T_1}, x_{T_2}, x_{T_3})$$

are both feasible solutions for $P_B(\bar{G}_1)$ and $P_B(\bar{G}_2)$ respectively. Furthermore, a mixed constraint $[\pi, \mu]$ is saturated by $x_{\bar{E}}$ if and only if (π) is saturated by $x_{\bar{E}_1}$ and (μ) is saturated by $x_{\bar{E}_2}$.

Proof

We will show that $x_{\bar{E}_1} \in P_B(\bar{G}_1)$ (the proof for $x_{\bar{E}_2} \in P_B(\bar{G}_2)$ is similar). Consider a constraint (π) written as

$$(\pi) := \langle a_{E1}(\pi), x_{E1}\rangle + \langle a_{T_0}(\pi), x_{T_0}\rangle + \sum_{i\in I} a_{e_i^1}(\pi)\,(x_{e_i^1} + x_{e_i^2}) \leq \partial(\pi)$$

with $- I \subseteq \{1, 2, 3\}$

- $a_{e_i^1} > 0$, for $i \in I$ ($a_{e_i^1} = 0$ for $i \in \{1, 2, 3\} \setminus I$)
- $\pi = \{\pi_k, k=1, \ldots, K\} \geq 0$

1°/ To prove that $x_{\bar{E}_1}$ satisfies (π), we may assume that for $i \in I$, $x_{e_i} < 1$. In fact, if for example, $x_{e_i} = 1$ for $i \in I' \subseteq I$ then we'll have to show that the constraint

$$(\pi') := \langle a_{E}1(\pi), x_{E}1\rangle + \langle a_{T_0}(\pi), x_{T_0}\rangle + \sum_{i \in I \setminus I'} a_{e_i}1(\pi)(x_{e_i}1 + x_{e_i}2) \leq \partial(\pi) - \sum_{i \in I'} a_{e_i}1(\pi)$$

is satisfied by $x_{\bar{E}_1}$. But it is easy to see that if (π) is valid for $P_B(\bar{G}_1)$, so is (π') and therefore we may consider instead of (π), the constraint (π').

2°/ Since solution x_E is assumed to be nondominated, for all $i \in I$ there exists at least one constraint in (P) whose support (the edge set whose the corresponding coefficients are nonzero) contains e_i and which is saturated by x_E. Hence we may assume that there exists a mixed constraint in (P) such that

$$[\bar{\pi}, \bar{\mu}] := \langle a_{E}1(\bar{\pi}), x_{E}1\rangle + \langle b_{E}2(\bar{\mu}), x_{E}2\rangle + \sum_{i=1,\dots,3} C_{e_i} x_{e_i} \leq \bar{\partial}$$

with $C_{e_i} > 0$ for $i \in I$, which is saturated by x_E and obtained from the constraints

$$(\bar{\pi}) := \langle a_{E}1(\bar{\pi}), x_{E}1\rangle + \sum_{i=1,\dots,3} (a_{e_i}(\bar{\pi}) x_{e_i} + a_{e_i}1(\bar{\pi})\,(x_{e_i}1 + x_{e_i}2)) \leq \partial(\bar{\pi})$$

$$(\bar{\mu}) := \langle b_{E}2(\bar{\mu}), x_{E}2\rangle + \sum_{i=1,\dots,3} ((b_{e_i}(\bar{\mu}) x_{e_i} + b_{e_i}1(\bar{\mu})\,(x_{e_i}1 + x_{e_i}2)) \leq \beta(\bar{\mu})$$

we have then

$$C_{e_i} = (a_{e_i}(\bar{\pi}) - b_{e_i}1(\bar{\mu})) + (b_{e_i}(\bar{\mu}) - a_{e_i}1(\bar{\pi})), \quad i = 1, 2, 3$$

$$\bar{\partial} = \partial(\bar{\pi}) + \beta(\bar{\mu}) - \sum_{i=1,\dots,3} 2(a_{e_i}1(\bar{\pi}) + b_{e_i}1(\bar{\mu}))$$

$$a_{e_i}(\bar{\pi}) \geq b_{e_i}1(\bar{\mu}) \text{ and } b_{e_i}(\bar{\mu}) \geq a_{e_i}1(\bar{\pi}), \qquad i = 1, 2, 3$$

Let

$$I_1 = \{i \in I \mid b_{e_i}(\bar{\mu}) = a_{e_i}1(\bar{\pi})$$

$$I_2 = I \setminus I_1$$

without loss of generality, we may assume that

$$a_{e_i}(\bar{\pi}) - b_{e_i}1(\bar{\mu}) \geq a_{e_i}1(\pi) \qquad \text{for } i \in I_1$$

$$b_{e_i}(\bar{\mu}) - a_{e_i}1(\bar{\pi}) \geq a_{e_i}1(\pi) \qquad \text{for } i \in I_2$$

(If not, we may multiply $(\bar{\pi})$ and $(\bar{\mu})$ by sufficiently large values for the coefficients).

Moreover consider the following constraints of $P_B(\bar{G}_2)$

$$a_{e_i^1}(x_{e_i} + x_{e_i^1} + x_{e_i^2}) \leq 2\, a_{e_i^1} \qquad \text{for } i \in I_1 \qquad (3,6)$$

Now if we sum up term by term, on the one hand, the constraints (π) and $(\bar{\pi})$ and, on the other hand, the constraints (3, 6) and $(\bar{\mu})$, and if we mix the newly found constraints, we obtain the constraint

$$\langle a_{E^1}(\pi) + a_{E^1}(\bar{\pi}), x_{E^1}\rangle + \langle b_{E^2}(\bar{\mu}), x_{E^2}\rangle + \langle a_{T_0}(\dot{\pi}), x_{T_0}\rangle$$

$$+ \sum_{i\in I} (C_{e_i} - a_{e_i^1}(\pi))x_{e_i} + \sum_{i\in\{1,2,3\}\setminus I} C_{e_i} x_{e_i} \leq \partial + \partial(\pi) - 2\sum_{i\in I} a_{e_i^1}$$

Since, this constraint is satisfied by x_E, and the constraint $[\bar{\pi}, \bar{\mu}]$ is saturated by x_E, we may deduce that constraint (π) is satisfied by $x_{\bar{E}_1}$. Therefore, by the definition of an extension it is obvious that a mixed constraint $[\pi, \mu]$ is saturated by x_E if and only if (π) and (μ) are saturated respectively by $x_{\bar{E}_1}$ and $x_{\bar{E}_2}$ and this completes the proof.

■

Lemma 3.9

The constraints of $P_B(G)$ associated with odd cycles, are among the constraints of (P).

Proof

Let C be an odd cycle of G. If $C \subset G_1$ or $C \subset G_2$, then, from remark 3.4 the constraint associated with C belong to the set of constraints of (P). If not, it is easy to see that there may be exist two odd cycles $C_1 \subset \bar{G}_1$ and $C_2 \subset \bar{G}_2$ and a mixed constraint defined from the constraints associated with C_1 and C_2 which is the constraint associated with C in $P_B(G)$.

■

Now we can introduce this major result

Theorem 3.10

$P_B(G) = (P)$

Proof

First we prove that $P_B(G) \subseteq (P)$. Let B, be a bipartite edge set of G.

Let $x^B = (x^B_{E1}, x^B_{E2}, x^B_{T_0})$ be its incidence vector. Then $(x^B_{E1}, x^B_{T_0})$ and $(x^B_{E2}, x^B_{T_0})$ are the incidence vectors of two bipartite edge sets B_1 and B_2 respectively of G_1 and G_2. Moreover these sets may be extended to two bipartite edge sets $\bar{B}_1$, $\bar{B}_2$ of $\bar{G}_1$, $\bar{G}_2$ such that

$|\bar{B}_1 \cap T_i| = |\bar{B}_2 \cap T_i| = 2 \quad$ for $i = 1, 2, 3$

Since $x^{\bar{B}_1} \in P_B(\bar{G}_1)$ and $x^{\bar{B}_2} \in P_B(\bar{G}_2)$ then all mixed constraints are satisfied by x^B. Therefore $x^B \in (P)$.

On the other hand, by lemma 3.9, each 0.1 feasible solution for (P) is the incidence vector of a bipartite edge set of G. Then to prove the theorem, it suffices to schow that all extreme points of (P) are in 0.1. Suppose not, then there exists non 0-1 extreme point of (P), $x_E = (x_{E1}, x_{E2}, x_{T_0})$. x_E may be assumed to be nondominated (if not, x_E is dominated by another non 0-1 extreme point of (P)). Let $x_{\bar{E}} = (x_{E1}, x_{E2}, x_{T_1}, x_{T_2}, x_{T_3})$ be the extension of x_E. From lemma 3.8, $x_{\bar{E}_1} = (x_{E1}, x_{T_1}, x_{T_2}, x_{T_3})$ and $x_{\bar{E}_2} = (x_{E2}, x_{T_1}, x_{T_2}, x_{T_3})$ are feasible solutions for $P_B(\bar{G}_1)$ and $P_B(\bar{G}_2)$ respectively. Then they may be written as

$$x_{\bar{E}_1} = \sum_{j=1,\dots,s} \lambda_j x^{B^1_j} \text{ with } B^1_j \in B(\bar{E}_1)\,;\ \lambda_j > 0 \text{ for } j=1,\dots,s \text{ and } \sum_{j=1,\dots,s} \lambda_j = 1$$

and

$$x_{\bar{E}_2} = \sum_{k=1,\dots,t} Q_k x^{B^2_k} \text{ with } B^2_k \in B(\bar{E}_2)\,;\ Q_k > 0 \text{ for } k=1,\dots,t \text{ and } \sum_{k=1,\dots,t} Q_k = 1$$

Note that all constraints of $P_B(\bar{G}_1)$ (resp $P_B(\bar{G}_2)$) saturated by $x_{\bar{E}_1}$ (resp $x_{\bar{E}_2}$) are also saturated by vectors $x^{B^1_j}$, $j=1,\dots,s$ (resp $x^{B^2_k}$, $k=1,\dots,t$). We claim that for each set B^1_j, $j=1,\dots,s$ (resp B^2_k, $k=1,\dots,t$) the cardinality of its intersection with T_0 is zero or two. In fact, if there exists $j_0 \in \{1, 2, \dots, s\}$ such that $B^1_{j_0} \cap T_0 = \{e_{i_0}\}$ where $i_0 \in \{1, 2, 3\}$ then, since $\sum_{e \in T_i} x_e^{B_{j_0}} = 2$ for $i = 1, 2, 3$ (because $\sum_{e \in T_i} x_e = 2$ for $i = 1, 2, 3$), we have

$$x^{B_{j_0}}_{e^1_i} + x^{B_{j_0}}_{e^2_i} = 2 \qquad \text{for } i \in \{1, 2, 3\} \setminus \{i_0\}$$

Therefore the constraint in $P_B(\bar{G}_1)$ associated with the odd cycle

$$(e_i, e^1_i, e^2_i\,;\ i \in \{1, 2, 3\} \setminus \{i_0\})$$

will be violated by $x^{B_{j_0}}$ and this is impossible. Thus our claim is proved. Now we'll consider two cases

1°/ $x_{e_1} + x_{e_2} + x_{e_3} < 2$

Then, there exist $B^1_{j_0}$, $j_0 \in \{1, 2, \dots, s\}$ and $B^2_{k_0}$, $k_0 \in \{1, 2, \dots, t\}$ such that $B^1_{j_0} \cap T_0 = \emptyset$

and $B^2_{k_0} \cap T_0 = \emptyset$. Let $B_0 = (B^1_{j_0} \cup B^2_{k_0}) \setminus \{e^1_i, e^2_i ; i = 1, 2, 3\}$. It is clear that B_0 is a bipartite edge set of G. Furthermore, every constraint of (P) saturated by x_E is also saturated by x^{B_0}. In fact, for a trivial constraint, the result is obvious. Consider a mixed constraint $[\pi, \mu]$, then from Lemma 3.8, $x_{\bar{E}_1}$, (resp $x_{\bar{E}_2}$) saturates the constraint (π) (resp (μ)). On the other hand, $x^{B^1_{j_0}}$ (resp $x^{B^2_{k_0}}$) saturates (π) (resp (μ)).

Since

$$\sum_{e \in T_i} x_e^{B^1_{j_0}} = 2 \qquad ; \quad i = 1, 2, 3$$

and

$$\sum_{e \in T_i} x_e^{B^2_{k_0}} = 2 \qquad ; \quad i = 1, 2, 3$$

then x^{B_0} saturates the constraint $[\pi, \mu]$. But x_E is assumed to be an extreme point of (P). Since $x^{B_0} \neq x_E$, we have a contradiction.

2°/ $x_{e_1} + x_{e_2} + x_{e_3} = 2$

We may assume that $x_{e_1} \geq x_{e_2} \geq x_{e_3}$. Then it is easy to see that there exist in this case $B^1_{j_0}$, $j_0 \in \{1, \ldots, s\}$ and $B^2_{k_0}$, $k_0 \in \{1, 2, \ldots, t\}$ such that $B^1_{j_0} \cap T_0 = \{e_1, e_2\}$ and $B^2_{k_0} \cap T_0 = \{e_1, e_2\}$. If we consider $B_0 = (B^1_{j_0} \cup B^2_{k_0}) \setminus \{e^1_i, e^2_i ; i = 1, 2, 3\}$ we have the same result as in the previous case. We have thus proved our theorem.

■

3.2. 2-sum

Consider graph $G = (V, E)$, a 2-sum of $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. As mentioned above, the characterization of the polytope $P_B(G)$ in this case may be presented as a particular case of the 3-sum. In fact suppose $E_1 \cap E_2 = \{e_1\}$ and consider the graphs $\bar{G}_1$, $\bar{G}_2$ obtained from G_1, G_2 by adding the path (e_2, e_3) between the nodes of e_1.

Let G' be the 3-sum of $\bar{G}_1$ and $\bar{G}_2$ obtained by identifying the triangle $\{e_1, e_2, e_3\}$. (G' is the graph G in which we add the path (e_2, e_3) between the nodes of e_1 (see Fig.4). The polytope P_B (G) may thus be determined from the polytope $P_B(G')$ by considering only the constraints where the support does not intersect the path (e_2, e_3). From lemmas 3.1 and 3.2, each polytope $P_B(\bar{G}_1)$ and $P_B(\bar{G}_2)$ may be described by a minimal system of inequalities as follow (by E^1(resp E^2) we denote the set $E_1-\{e_1\}$ (resp $E_2 \setminus \{e_2\}$).

$$
P_B(\bar{G}_1) \begin{cases}
0 \le x_e \le 1 & \text{for } e \in \bar{E}_1 & \\
x_{e_1} + x_{e_2} + x_{e_3} \le 2 & & (3,7) \\
\langle a^k_{E1}, x_{E1} \rangle \le a^k & k = 1, 2, \ldots, K_1 & (3,8) \\
\langle a^k_{E1}, x_{E1} \rangle + x_{e_1} \le a^k & k = K_1+1, \ldots, K_2 & (3,9) \\
\langle a^k_{E1}, x_{E1} \rangle + x_{e_2} + x_{e_3} \le a^k & k = K_2+1, \ldots, K_3 & (3,10)
\end{cases}
$$

and

$$
P_B(\bar{G}_2) \begin{cases}
0 \le x_e \le 1 & \text{for } e \in \bar{E}_2 & \\
x_{e_1} + x_{e_2} + x_{e_3} \le 2 & & (3,11) \\
\langle b^l_{E2}, x_{E2} \rangle \le \beta^l & l = 1, 2, \ldots, L_1 & (3,12) \\
\langle b^l_{E2}, x_{E2} \rangle + x_{e_1} \le \beta^l & l = L_1+1, \ldots, L_2 & (3,13) \\
\langle b^l_{E2}, x_{E2} \rangle + x_{e_2} + x_{e_3} \le \beta^l & l = L_2+1, \ldots, L_3 & (3,14)
\end{cases}
$$

(Note that by lemma 3.1 if graph G contains triangle T such that one of its nodes is of degree two, then $\sum_{e \in T} x_e \le 2$ is the single facet of $P_B(G)$ whose the support contains T).

Lemma 3.11

The nontrivial facets of the polytope $P_B(\bar{\bar{G}}_1)$ (recall that $\bar{\bar{G}}_1$ (resp $\bar{\bar{G}}_2$) is the graph obtained from $\bar{G}_1$ (resp $\bar{G}_2$) by adding the path (e_i^1, e_i^2), between the nodes of each edge e_i, for $i = 1, 2, 3$) whose support intersects E^1 and the paths added between the nodes of e_2 and e_3 are of three types

a) $\langle a^k_{E1}, x_{E1}\rangle + x_{e_2} + x_{e_3^1} + x_{e_3^2} \le \alpha^k$ (3, 15)

b) $\langle a^k_{E1}, x_{E1}\rangle + x_{e_3} + x_{e_2^1} + x_{e_2^2} \le \alpha^k$ (3, 16)

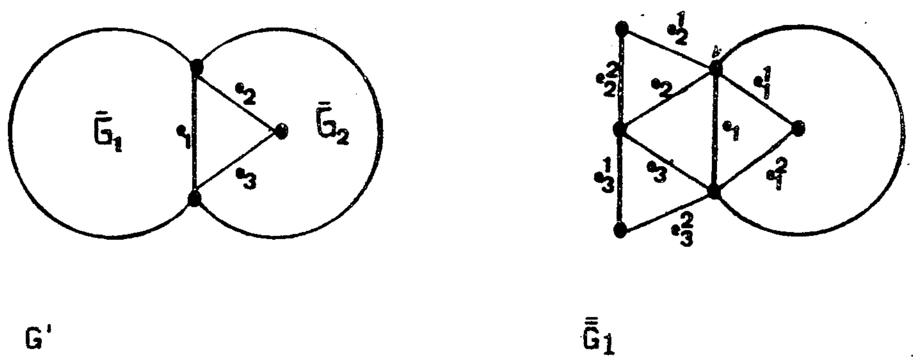
c) $\langle a^k_{E1}, x_{E1}\rangle + x_{e_2^1} + x_{e_2^2} + x_{e_3^1} + x_{e_3^2} \le \alpha^k$ (3, 17)

(a similar lemma may be established for the polytope $P_B(\bar{\bar{G}}_2)$).

Proof

On the one hand, the support of such a facet may not contain neither e_1, e_2 simultaneously, nor e_1, e_3 simultaneously, nor the six edges of the added paths, since in each of these cases, two even or odd chordless paths will be created between the nodes of e_1 and this is not possible for a facet. On the other hand, this support may not contain a cycle of length five contained in $\{e_i, e_i^1, e_i^2 \; ; \; i=1,2,3\}$ (if not, the constraint associated with this cycle defines the same hyperplane as that of the facet). Thus we have only the three types of facets given by the lemma.

■



G' (a) $\bar{\bar{G}}_1$ (b)

Fig.4

In [4], the following has been shown.

Theorem 3.12

Let $G = (V, E)$ be a graph and $\langle a_E, x_E \rangle \leq \partial$ be a facet of $P_B(G)$ whose support contains a shordless path S of odd length $k+1 \geq 3$ with $a_e = \partial^*$ for all $e \in S$. Let $G^* = (V^*, E^*)$ be the graph obtained from G by replacing the path S by an edge e_0. Let $a^*{}_{E^*} \in R^{|E^*|}$ be defined as follows

$$a^*{}_e = a_e \quad \text{if } e \neq e_0$$

$$a^*{}_{e_0} = \partial^*$$

Then

$$\langle a^*{}_{E^*}, x_{E^*} \rangle \leq \partial - k\,\partial^*$$

defines a facet of $P_B(G^*)$.

From theorem 3.12 it is easy to see that :

- For a constraint described in (3, 15) or (3, 16) (resp (3, 17)) which defines a facet of $P_B(\bar{\bar{G}}_1)$), the constraint $\langle a^k_{E^1}, x_{E^1} \rangle + x_{e_1} \leq \partial^k - 2$ (resp $\langle a^k_{E^1}, x_{E^1} \rangle + x_{e_2} + x_{e_3} \leq \partial^k - 2$) defines also a facet of $P_B(\bar{\bar{G}}_1)$.

- Each mixed constraint $[\pi, \mu]$ associated with a constraint (π) of $P_B(\bar{\bar{G}}_1)$, described in (3, 17) and a constraint (μ) of $P_B(\bar{\bar{G}}_2)$ such that

$$(\mu) := \langle b^l_{E^2}, x_{E^2} \rangle + x_{e_2} + x_{e_3} \leq \beta^l$$

may not define a facet of $P_B(G')$. ($[\pi, \mu]$ may be written in this case as a sum of two valid constraints of $P_B(G')$).

From the previous remarks and from theorem 3.10 we have

Corollary 3.13

The polytope $P_B(G)$ is defined by the following system of inequalities (Q)

- $0 \leq x_e \leq 1$ for $e \in E$
- (k) , $k = 1, \ldots, K_2$
- (l) , $l = 1, \ldots, L_2$

$$- \langle a^k_{E1}, x_{E1} \rangle + \langle b^l_{E2}, x_{E2} \rangle \le \alpha^k + \beta^l - 2 \qquad (3,18)$$

with $(k,l) \in \{\{(k_1+1, \ldots, k_2) \times (L_2+1, \ldots, L_3)\} \cup \{(L_1+1, \ldots, L_2) \times (k_2+1, \ldots, k_3)\}\}$.

We can also deduce the following corollaries

Corollary 3.14

Let G be a k-sum, $1 \le k \le 3$, of G_1 and G_2. If $\bar{G}_1$ and $\bar{G}_2$ are weakly bipartite then G is weakly bipartite.

Corollary 3.15

Let G be a graph obtained by means of k-sums, $1 \le k \le 3$, starting from $G_1, G_2, \ldots, G_n$ (cf 2.2). If $\tilde{G}_1, \tilde{G}_2, \ldots, \tilde{G}_n$ are weakly bipartite then G is weakly bipartite.

Corollary 3.16

Let G be a 2-sum of G_1 and G_2 such that G_1 is weakly bipartite and G_2 is bipartite, then G is weakly bipartite.

Using Corollary 3.15, we will show in section 5 that the graphs noncontractible to K_5 are weakly bipartite. In the following section we introduce a method for constructing other facets of $P_B(G)$ from the known ones. We show that the constraints described in (3,18) define facets of $P_B(G)$.

4. Construction of facets

We may ask whether the "mixed" inequalifies of (P) and (Q) not originating in $P_B(G_1)$ and $P_B(G_2)$ define facets of the polytope $P_B(G)$, (where G is the 3-sum or the 2-sum of G_1 and G_2). In this section we show that each such inequality in (Q) (described in (3, 18)) defines a facet of $P_B(G)$ but this is note true for (P).

Theorem 4.1

Given a graph G, a 2-sum of G_1 and G_2 (as defined in cf. 3.2), constraints (3, 18) define facets of $P_B(G)$.

Proof :

Consider such a constraint

$$[\pi, \mu] := \langle a_E^{k_0}1, x_E1\rangle + \langle b_E^{l_0}2, x_E2\rangle \leq \partial^{k_0} + \beta^{l_0} - 2$$

and suppose for example that $k_0 \in \{k_1+1, \ldots, K_2\}$ and $l_0 \in \{L_2+1, \ldots, L_3\}$. From theorem 3.10 this constraint is valid for $P_B(G)$. To prove that $[\pi, \mu]$ is a facet of $P_B(G)$ it suffises to exhibit $|E|$ bipartite edge sets of G whose incidence vectors are linearly independent and satisfy $[\pi, \mu]$ with equality.

Since the constraints

$$(\pi) := \langle a_E^{k_0}1, x_E1\rangle + x_{e_1} \leq \partial^{k_0}$$

and $$(\mu) := \langle b_E^{l_0}2, x_E2\rangle + x_{e_2} + x_{e_3} \leq \beta^{l_0}$$

are nontrivial facets respectively of $P_B(\bar{G}_1)$ and $P_B(\bar{G}_2)$, (π) is a facet of $P_B(G_1)$. Moreover if we let $|E_1| = m_1+1$ and $|E_2| = m_2$ then there are m_1+1 (resp m_2+2) bipartite edge sets $B_1^1, \ldots, B_{m+1}^1 \in E_1$ (resp $B_1^2, \ldots, B_{m_2+2}^2 \in E_2$) whose incidence vectors

$$x^{B_1^1}, \ldots, x^{B^1_{m_1+1}} \text{ (resp } x^{B_1^2}, \ldots, x^{B^2_{m_2+2}})$$

are linearly independent and satisfy the constraint (π) (resp(μ)) with equality.

Let M_1(resp M_2) be the (m_1+1, m_1+1) (resp(m_2+2, m_2+2)) matrix whose columns are the incidence vectors $x^{B_1^1}, \ldots, x^{B^1_{m_1+1}}$(resp $x^{B_1^2}, \ldots, x^{B^2_{m_2+2}}$). We assume that the last two rows of M_2 correspond to edges e_2 and e_3. Moreover we assume that the sets $B_1^2, \ldots, B^2_{m_2+2}$ are ordered in such way that the sets, $B_1^2, \ldots, B_r^2$, contain both edges e_2, e_3 ; $B^2_{r+1}, \ldots, B_s^2$

contain e_2 but not e_3; $B^2_{s+1}, \ldots, B^2_{m+2}$ contain e_3 but not e_2. Note if $x^{B^2_k}$, for $r+1 \le k \le s$ satisfies (μ) with equality, then $x^{B^2_k{}'}$ where $B^2_k{}' = B^2_k \setminus \{e_2\} \cup \{e_3\}$ also satisfies (μ) with equality. Hence we may assume that $B^2_{r+1} = B^2_{s+1} \setminus \{e_2\} \cup \{e_3\}$. Finally we assume that the sets $B^1_1, \ldots, B^1_{m+1}$ are such that only $B^1_1, \ldots, B^1_k$ contain e_1 with $k < m_1+1$ (recall that (π) is nontrivial). Thus our matrices M_1 and M_2 look as follows.

$$
M_1 = \begin{array}{cc|c|c|}
 & & \overbrace{\qquad}^{k} & \overbrace{\qquad}^{m_1+1-k} \\
\cline{3-4}
e_1 & & 1\ldots\ldots1 & 0\ldots\ldots0 \\
\cline{3-4}
m_1 & \Big\{ & A_1 & A_2 \\
\cline{3-4}
\end{array}
\qquad
M_2 = \begin{array}{cc|c|c|c|}
 & & \overbrace{\qquad}^{r} & \overbrace{\qquad}^{s-r} & \overbrace{\qquad}^{m_2+2-s} \\
\cline{3-5}
m_2 & \Big\{ & D_1 & D_2 & D_3 \\
\cline{3-5}
e_2 & & 1\ldots1 & 1\ldots1 & 0\ldots0 \\
\cline{3-5}
e_3 & & 1\ldots1 & 0\ldots0 & 1\ldots1 \\
\cline{3-5}
\end{array}
$$

where A_1, A_2, D_1, D_2, D_3 are some $0-1$ matrices.

Now we transforme the edge sets B^1_j, $j=1, \ldots, m_1+1$ and B^2_j, $j=1, \ldots, m_2+2$ into bipartite edge sets in G as follows

$$
B_j = \begin{cases}
B^1_{k+1} \cup \bar{B}^2_j & , \quad 1 \le j \le r \\
B^1_1 \setminus \{e_1\} \cup \bar{B}^2_j & , \quad r+1 \le j \le m_2+2 \\
B^1_{j-m_2-2} \setminus \{e_1\} \cup \bar{B}^2_{r+1} & , \quad m_2+3 \le j \le m_2+2+k \\
B^1_{j-m_2-2} \cup \bar{B}^2_1 & , \quad m_2+3+k \le j \le m_2+m_1+3
\end{cases}
$$

where $\bar{B}^2_j = B^2_j \setminus \{e_2, e_3\}$

Thus we have a family of $m_1 + m_2 + 3$ bipartite edge sets B_j, $j=1, \ldots, m_1+m_2+3$ of G whose incidence vectors satisfy the constraint (π, μ) with equality.

Let M be the (m_1+m_2, m_1+m_2) matrix whose columns are the incidence vectors x^{B_j} such that $1 \le j \le m_1+m_2+3$ with j different than $1, r+1, s+1$. We denote by a_1, a_2, d_1, d_2, d_3 respectively the first column of A_1, A_2, D_1, D_2 and D_3; thus $d_2 = d_3$. Furthermore let D^*_i for

$i=1, 2, 3,$ be the matrix obtained from D_i by removing d_i, then M looks as follows

$$M = \left.\begin{array}{c} \\ m_1+m_2 \\ \\ \end{array}\right\{ \begin{array}{c|c|c|c|c} \overbrace{}^{r-1} & \overbrace{}^{s-r-1} & \overbrace{}^{m_2+1-s} & \overbrace{}^{k} & \overbrace{}^{m_1+1-k} \\ \hline D_1^* & D_2^* & D_3^* & d_2, \ldots, d_2 & d_1, \ldots, d_1 \\ \hline a_2, \ldots, a_2 & a_1, \ldots, a_1 & a_1, \ldots, a_1 & A_1 & A_2 \end{array}$$

To prove the theorem,it remains to show that M is nonsingular. Consider matrices M_{1c} and M_{2c} obtained from M_1 and M_2 as follows

$$M_{1c} = \begin{array}{|c|c|c|} \hline 1\ldots\ldots1 & 0\ldots\ldots0 & 1 \\ \hline A_1 & A_2 & \begin{matrix}0\\ \vdots\\ 0\end{matrix} \\ \hline 1\ldots\ldots1 & 1\ldots\ldots1 & c \\ \hline \end{array} \qquad M_{2c} = \begin{array}{|c|c|c|c|} \hline D_1 & D_2 & D_3 & \begin{matrix}0\\ \vdots\\ 0\end{matrix} \\ \hline 1\ldots\ldots1 & 1\ldots\ldots1 & 0\ldots\ldots0 & 1 \\ 1\ldots\ldots1 & 0\ldots\ldots0 & 1\ldots\ldots1 & 1 \\ \hline 1\ldots\ldots1 & 1\ldots\ldots1 & 1\ldots\ldots1 & c \\ \hline \end{array}$$

where c is a real constant.

We claim that M_{1c} and M_{2c} are nonsingular for $c > 1$ and $c \leq 0$ (we show the result for M_{2c}; the proof for M_{1c} is similar). Since the matrix M_2 is nonsingular, the equation system

$M_2^T Q = \mathbb{1}$ (where $\mathbb{1}^T$ is the m_2+2 vector $(1, \ldots, 1)$ and M_2^T is the transpose of M_2) has a unique solution given by

$$Q_e = \frac{b_e^{l_0}}{\beta^{l_0}} \qquad \text{for } e \in E^2$$

If the last row of M_{2c} is a dependent of the first ones then

$$\frac{b_e^{l_0} + b_e^{l_0}}{\beta^{l_0}} = \frac{2}{\beta^{l_0}} = c$$

must hold. But $\beta^{l_0} \geq 2$ thus implying that $0 < c \leq 1$ and our claim is shown.

Let $\bar{M}_{1c}$ (resp $\bar{M}_{2c}$) be the matrix obtained from M_{1c} (resp M_{2c}) by subtracting the column which contains a_i (resp d_i) from each column which intersects the matrix A_i (resp D_i) for

$i = 1, 2$ (resp $i = 1, 2, 3$).

$\bar{M}_{1c}$ and $\bar{M}_{2c}$ look as follows

$$\bar{M}_{1c} = \left[\begin{array}{c|c|c|c|c} \bar{A}_1 & \bar{A}_2 & a_1 & a_2 & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline \bar{A}_3 & \bar{A}_4 & \begin{matrix} 1 \\ 1 \end{matrix} & \begin{matrix} 0 \\ 1 \end{matrix} & \begin{matrix} 1 \\ c \end{matrix} \end{array}\right] \qquad \bar{M}_{2c} = \left[\begin{array}{ccc|c|c|c|c} \bar{D}_1 & \bar{D}_2 & \bar{D}_3 & d_1 & d_2 & d_3 & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline & & & 1 & 1 & 0 & 1 \\ \bar{D}_4 & \bar{D}_5 & \bar{D}_6 & 1 & 0 & 1 & 1 \\ & & & 1 & 1 & 1 & c \end{array}\right]$$

where $\bar{A}_3$, $\bar{A}_4$, $\bar{D}_4$, $\bar{D}_5$, $\bar{D}_6$ are all zeros matrices and $\bar{A}_i$, $i=1,2$ (resp $\bar{D}_i$, $i=1,2,3$) is the matrix obtained from A_i (resp D_i) by deleting column a_i (resp d_i) after subtracting it from each column of A_i (resp D_i).

Since the matrices $\bar{M}_{1c}$ and $\bar{M}_{2c}$ are nonsingular for $c > 1$ and $c \leq 0$, and $d_2 = d_3$ it is easy to show by elementary operations of subtraction and addition on the three last rows and columns of these matrices that the following square matrices $\bar{\bar{M}}_{1c}$ and $\bar{\bar{M}}_{2c}$ are nonsingular for $c > 1$ and $c \leq 0$.

$$\bar{\bar{M}}_{1c} = \left[\begin{array}{c|c|c} \bar{A}_1 & \bar{A}_2 & a_1 - (1-c)a_2 \end{array}\right] \qquad \bar{\bar{M}}_{2c} = \left[\begin{array}{c|c|c|c} \bar{D}_1 & \bar{D}_2 & \bar{D}_3 & td_1 - d_2 \end{array}\right]$$

where $t = 1/2 + 1/2(1-c)$.

Now, from matrix M construct the matrix $\bar{M}$ by subtracting column m_2 (resp $m_2 + k$) from all columns i such that

$$r \leq i \leq m_2+k-1 \text{ and } i \neq m_2 \text{ (resp } 1 \leq i < r-1 \text{ and } m_2 + k + 1 \leq i \leq m_1 + m_2)$$

$\bar{M}$ may be written as

$$\bar{M} = \left[\begin{array}{c|c|c|c|c|c|c} \bar{D}_1 & \bar{D}_2 & \bar{D}_3 & d_2 & \bar{A}_3 & \bar{A}_4 & d_1 \\ \hline \bar{D}_4 & \bar{D}_5 & \bar{D}_6 & a_1 & \bar{A}_1 & \bar{A}_2 & a_2 \end{array}\right]$$

where $\bar{D}_4$, $\bar{D}_5$, $\bar{D}_6$, $\bar{A}_3$, and $\bar{A}_4$ are all zero matrices.

Moreover, since the matrix $\bar{\bar{M}}_{2c}$ is non singular for $c = 2$, the following (m_2, m_2) submatrix $\bar{M}_2$ of $\bar{M}$ is nonsingular

$\bar{M}_2 =$

$\bar{D}_1$	$\bar{D}_2$	$\bar{D}_3$	d_2

Hence, the vector d_1 may be written as

$$d_1 = Q\, d_2 + \sum_{i=1}^{m_2-1} \Delta_i \bar{d}_i$$

where Q and Δ_i, $i=1, \ldots, m_2-1$ are reals with $Q \neq 0$ and $\bar{d}_i$ is column i of $\bar{M}_2$.

Then

$$1/Q\; d_1 - d_2 = \sum_{i=1}^{m_2-1} \Delta'_i \bar{d}_i \qquad \text{where} \qquad \Delta'_i = \Delta_i / Q$$

Since the matrix $\bar{\bar{M}}_{2c}$ is nonsingular for $c > 1$ and $c \leq 0$ (ie for $t \in]-\infty, 1]$) it follows that $1/Q \notin]-\infty, 1]$.

We now construct the matrix $\bar{\bar{M}}$ from the matrix $\bar{M}$ by subtracting the vector

$$Q\, d_2 + \sum_{i=1}^{m_2-1} \Delta'_i \bar{d}_i$$

from the column which contains d_1. $\bar{\bar{M}}$ may be written as

$\bar{\bar{M}} =$

$\bar{D}_1$	$\bar{D}_2$	$\bar{D}_3$	d_2	$\bar{A}_3$	$\bar{A}_4$	$0 \; \vdots \; 0$
$\bar{D}_4$	$\bar{D}_5$	$\bar{D}_6$	a_1	$\bar{A}_1$	$\bar{A}_2$	$Qa_1 - a_2$

Since the matrix $\bar{\bar{M}}_{1c}$ is nonsingular, for $c > 1$ and $c \leq 0$, and $1/Q \in]1, +\infty[$, the (m_1, m_1) submatrix

$\bar{A}_1$	$\bar{A}_2$	$Qa_1 - a_2$

is nonsingular. Moreover since $\bar{M}_2$ is nonsingular and $\bar{A}_3$, $\bar{A}_4$ are two null matrices, we may conclude that $\bar{\bar{M}}$ is nonsingular. Consequently the matrices M and $\bar{M}$ are nonsingular and the proof is complete. ∎

Theorem 4.1 shows that the polyhedron (Q) (given in the preceding section) defines a minimal system of inequalities which describe the polytope $P_B(G)$. But the system of inequalities which defines the polytope (P) need not be minimal. In this case suppose there exists a constraint (π) of $P_B(\bar{G}_1)$ associated with an odd cycle of $\bar{G}_1$ such that

$$(\pi) := \sum_{e \in F_1} x_e + x_{e_1} + x_{e_2} \leq |F_1| + 1$$

and a constraint (μ) of $P_B(\bar{G}_2)$ asspciated with an odd cycle of $\bar{G}_2$ suoh that

$$(\mu) := \sum_{e \in F_2} x_e + x_{e_1^1} + x_{e_1^2} + x_{e_2^1} + x_{e_2^2} \leq |F_2| + 3$$

Then the constraint

$$[\pi, \mu] := \sum_{e \in F_1} x_e + \sum_{e \in F_2} x_e \leq |F_1| + |F_2|$$

will be a constraint of (P) which does not define a facet of $P_B(G)$ because the set $F_1 \cup F_2$ defines an even cycle of G.

5. Final remarks and examples

In [4] the following has been shown

Theorem 5.1

Let $G = (V, E)$ be a graph and $\langle a_E, x_E \rangle \leq \alpha$ a nontrivial facet of $P_B(G)$. Let $v \in V$ be any node and suppose the star F of $v (F = \{v v_i \in E \mid v_i \neq v\})$ consists of the edges $v v_1, \ldots, v v_k$. construct a new graph $G' = (V', E')$ from G as follows. Subdivide each edge of the star F, i.e. add k new nodes $u_1, \ldots, u_k$ and replace the edges $v v_1, \ldots, v v_k$ by the edges $v u_1$, $u_1 v_1$, $v u_2$, $u_2 v_2$, ..., $v u_k$, $u_k v_k$. Keep all other nodes and edges, and moreover optionally add any further edge linking a new node u_s to any other node in V', or any edge $v v_s$, $s=1, \ldots, k$.

Let $\bar{a}_{E'} \in R^{|E'|}$ be defined as follows

$$\bar{a}_e = a_e \qquad \text{for } e \in E \cap E'$$

$$\bar{a}_{v u_s} = a_{u_s v_s} = a_{v v_s} \qquad \text{for } s = 1, \ldots, k$$

$$\bar{a}_e = 0 \qquad \text{else}$$

and let

$$\bar{\alpha} = \alpha + \sum_{s=1}^{k} a_{v v_s}$$

then $\langle \bar{a}_E', x_E' \rangle \leq \bar{\delta}$ defines a facet of $P_B(G')$.

Using Wagner's decomposition of graphs noncontractible to K_5, and the above results we can show the following.

Corollary 5.2

The graphs noncontractible to K_5 are weakly bipartite.

Proof

Since planar graphs are weakly bipartite, to prove this result it suffices to show by corollary 3.13 and Wagner's decomposition that the graph obtained from V_8 by adding a path of length two between the nodes of each edge is weakly bipartite. Let $\bar{V}_8$ be this graph. Note that graph V_8 is weakly bipartite [3]. If $\bar{V}_8$ is not weakly bipartite then there exists a subgraph H of $\bar{V}_8$ which is not an odd cycle and which produces a nontrivial facet of $P_B(\bar{V}_8)$.

By lemma 3.1 H may be obtained from V_8 by dividing some edges and removing others. Thus by theorems 3.12 and 5.1 we may assume that in H, only edges of the exterior cycle of V_8 (cycle of length eight) are divided. Now it is easy to see that in this case, H is such that $H - \{u, v\}$ is bipartite where u and v are two nodes of H. From Barahona's result [3], H is weakly bipartite and we have a contradiction.

■

We can easily deduce from this corollary the following equivalent result which has been conjectured by Johnson and Gaston in [9].

Corollary 5.3

Given a graph $G = (V, E)$ noncontractible to k_5, the polyhedron

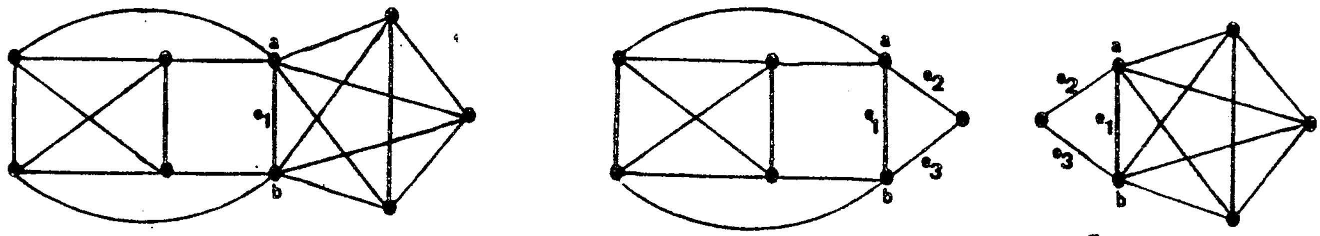
$$\begin{cases} \sum_{e \in C} x_e \geq 1 & \text{for all odd cycles } C \text{ in } G \\ x_e \geq 0 & \text{for } e \in E \end{cases}$$

has all vertices in 0.1.

Consider now a graph G such $G \setminus v_0$ is bipartite where v_0 is a node of G. If we divide the edges of any triangle of G, the resulting graph is also weakly bipartite [3]. Thus the 2-sum and the 3-sum of this graph and a graph noncontractible to k_5 gives also a weakly bipartite graph.

In the following we give an example which illustrates the method introduced above for describing the polytope $P_B(G)$ in the case of 2-sum.

Consider the graph G given in Fig.5 which is the 2-sum of two graphs G_1 and G_2 whose graphs $\bar{G}_1$ and $\bar{G}_2$ are presented respectively in Fig. 5 (b) and Fig. 5 (c).



G (a) $\bar{G}_1$ (b) $\bar{G}_2$ (c)

Fig.5

It is easy to see from the results in [3], [4], [10], that the polytopes $P_B(\bar{G}_1)$ and $P_B(\bar{G}_2)$ are defined by the following two minimal systems

$$P_B(G_1) = \begin{cases} \sum_{e\in C} x_e < |C|-1 & \text{for } C \text{ odd cycle in } \bar{G}_1 \\ \sum_{e\in E^1} x_e + 2\,x_{e_2} + 2\,x_{e_3} \leq 10 & \\ 0 \leq x_e \leq 1 & \text{for } e \in E_1 \end{cases}$$

$$P_B(G_2) = \begin{cases} \sum_{e\in C} x_e \leq |C|-1 & \text{for } C \text{ odd cycle in } G_2 \\ \sum_{e\in E_2} x_e \leq 6 & \\ 0 \leq x_e \leq 1 & \text{for } e \in E_2 \end{cases}$$

From corollary3.11 we deduce that the polytope $P_B(G)$ is defined by the following constraints (where each constraints is a facet of $P_B(G)$)

$$P_B(G) = \begin{cases} \sum_{e\in C} x_e \le |C|-1 & \text{for } C \text{ odd cycle in } G \\ \sum_{e\in E^2} x_e + \sum_{e\in L} x_e \le |L|+5 & \text{for } L \subset G_1, \text{ odd path between a and b} \\ 1/2 \sum_{e\in E^1} x_e + \sum_{e\in T} x_e \le |T|+3 & \text{for } T \subset G_2, \text{ even path between a and b} \\ 1/2 \sum_{e\in E^1} x_e + \sum_{e\in E^2} x_e \le 9 & \\ 0 \le x_e \le 1 & \text{for } e\in E \end{cases}$$

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ANNEXE 3

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FACETS OF THE BIPARTITE SUBGRAPH POLYTOPE

by

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Abstract

The bipartite subgraph polytope $P_B(G)$ of a graph $G=[V,E]$ is the convex hull of the incidence vectors of all edge sets of bipartite subgraphs of G. We show that all complete subgraphs of G of odd order and all so-called odd bicycle wheels contained in G induce facets of $P_B(G)$. Moreover, we describe several methods with which new facet defining inequalities of $P_B(G)$ can be constructed from known ones. Examples of these methods are contraction of node sets in odd complete subgraphs, odd subdivision of edges, certain splittings of nodes, and subdivision of all edges of a cut. Using these methods we can construct facet defining inequalities of $P_B(G)$ having coefficients of order $|V|^2$.

Keywords: Bipartite Subgraphs, Max-Cut Problem, Polyhedral Combinatorics, Facets of Combinatorial Polytopes, Cutting Planes.

1. Introduction and Notation

In this paper we study the polytope associated with the bipartite subgraphs of a graph. We shall exhibit various new classes of facets of this polytope, in particular, we describe several methods with which new classes of facets can be derived from known ones. Using these constructions we obtain facet defining inequalities with integral coefficients in the interval $[0, n^2/4]$ where n is the order of the given graph.

Results about the bipartite subgraph polytope have been used by GRÖTSCHEL and PULLEYBLANK (1981) for the design of polynomial time algorithms for special cases of the max-cut problem. These ideas have been successfully implemented by BARAHONA and MACCIONI (1982) for the solution of medium sized real-world problems of this type. We hope that our polyhedral results may lead to the design of new efficient LP-based cutting plane algorithms for the max-cut problem.

The graphs we consider are finite and undirected. Loops and multiple edges do not play a role in what follows, so we assume that our graphs have no multiple edges and no loops. We denote a graph by $G = [V,E]$, where V is the *node set* and E the *edge set* of G. The cardinality of V is called the *order* of G. If $e \in E$ is an edge with endnodes i and j we also write ij to to denote the edge e. If $H = [W,F]$ is a graph with $W \subseteq V$ and $F \subseteq E$ then H is called a *subgraph* of G. We also say that H is contained in G and that G contains H.

If $G = [V,E]$ is a graph and $F \subseteq E$, then $V(F)$ denotes the set of nodes of V which occur at least once as an endnode of an edge in F. Similarly, for $W \subseteq V$, $E(W)$ denotes the set of all edges of G with both endnodes in W.

A graph G is called *complete* if every two different nodes of G are linked by an edge. The complete graph with n nodes is denoted by K_n.
A graph is called *bipartite* if its node set can be partitioned into two nonempty, disjoint sets V_1 and V_2 such that no two nodes in V_1 and no two nodes in V_2 are linked by an edge. We call V_1, V_2 a *bipartition* of V.

If $W \subseteq V$ then $\delta(W)$ is the set of edges with one endnode in W and the other in $V \setminus W$. The edge set $\delta(W)$ is called a *cut*. Clearly, $[V,\delta(W)]$ is a bipartite subgraph of G. We write $\delta(v)$ instead of $\delta(\{v\})$ for $v \in V$ and call $\delta(v)$ the *star* of v. For a node $v \in V$ the cardinality of $\delta(v)$ is called the *degree* of v. For $W \subseteq V$ the set of nodes in $V \setminus W$ which are adjacent to a node in W is denoted by $\Gamma(W)$.

If U,W are disjoint subsets of V, then $[U:W]$ denotes the set of edges of G which have one endnode in U and the other endnode in W, for instance $[\{v\}: V \setminus \{v\}] = \delta(v)$.

A *path* P in $G = [V,E]$ is a sequence of edges $e_1, e_2, \ldots, e_k$ such that $e_1 = v_0v_1$, $e_2 = v_1v_2, \ldots, e_k = v_{k-1}v_k$ and such that $v_i \neq v_j$ for $i \neq j$. The nodes v_0 and v_k are the endnodes of P and we say that P links v_0 and v_k or goes from v_0 to v_k. The number k of edges of P is called the *length* of P. If $P = e_1, e_2, \ldots, e_k$ is a path linking v_0 and v_k and $e_{k+1} = v_0v_k \in E$, then the sequence $e_1, e_2, \ldots, e_k, e_{k+1}$ is called a *cycle* of length $k + 1$. A cycle (path) is called *odd* if its length is odd, otherwise it is called *even*. If P is a cycle or a path and uv an edge of $E \setminus P$ with $u, v \in V(P)$, then uv is called a *chord* of P.

If v is a node in a graph G, then *G−v* denotes the subgraph of G obtained by removing node v and all edges incident to v from G.

A *polyhedron* $P \subseteq \mathbb{R}^m$ is the intersection of finitely many halfspaces in $\mathbb{R}^m$. A *polytope* is a bounded polyhedron or equivalently the convex hull of finitely many points. The dimension of a polyhedron P, denoted by *dim P*, is the maximum number of affinely independent points in P minus one.

If $a \in \mathbb{R}^m \setminus \{0\}$, $a_0 \in \mathbb{R}$, then the inequality $a^T x \leq a_0$ is said to be *valid* with respect to a polyhedron $P \subseteq \mathbb{R}^m$ if $P \subseteq \{x \in \mathbb{R}^m \mid a^T x \leq a_0\}$. We say that a valid inequality $a^T x \leq a_0$ *supports* P or defines a *face* of P if $\emptyset \neq P \cap \{x \mid a^T x = a_0\} \neq P$. A valid inequality $a^T x \leq a_0$ defines a *facet* of P if it defines a face of P and if there exist dim P affinely independent points in $P \cap \{x \mid a^T x = a_0\}$. Two face-defining inequalities $a^T x \leq a_0$, $b^T x \leq b_0$ are called *equivalent* if
$P \cap \{x \mid a^T x = a_0\} = P \cap \{x \mid b^T x = b_0\}$.

If $P = \{x \mid Ax \leq b\}$, then $Ax \leq b$ is called *nonredundant* if $P \neq \{x \mid A'x \leq b'\}$ for all inequality systems $A'x \leq b'$ where $A'x \leq b'$ arises from $Ax \leq b$ by deleting one inequality.

For every facet F of a *full-dimensional* polyhedron $P \subseteq \mathbb{R}^m$, i. e. dim P = m, there exists a unique (up to multiplication by a positive constant) valid inequality $a^T x \leq a_0$ such that $F = \{x \in P \mid a^T x = a_0\}$. This implies that (up to multiplication by a positive constant) there is a unique complete and nonredundant inequality system $Ax \leq b$ such that

$$P = \{x \in \mathbb{R}^m \mid Ax \leq b\}$$

We shall present a quite large, but partial, nonredundant system of inequalities for the polyhedron associated with the bipartite subgraphs of a graph.

If x is a real number, then $\lceil x \rceil$ resp. $\lfloor x \rfloor$ denotes the smallest resp. largest integer not smaller resp. larger than x.

2. The Bipartite Subgraph Polytope and the Max-Cut Problem

Given a graph $G = [V,E]$ with edge weights $c_e > 0$ for all $e \in E$, then the *max-cut problem* can be stated as follows: Find a cut $\delta(W)$ in G such that $c(\delta(W)) := \sum_{e\in\delta(W)} c_e$ is as large as possible. This problem can be approached from a polyhedral point of view in the following way.

If $G = [V,E]$ is a graph and $F \subseteq E$ an edge set then the 0/1-vector $x^F \in \mathbb{R}^E$ with $x_e^F = 1$ if $e \in F$, and $x_e^F = 0$ if $e \notin F$ is called the *incidence vector* of F. The convex hull $P_B(G)$ of the incidence vectors of all edge sets of bipartite subgraphs of G is called the *bipartite subgraph polytope* of G, i. e.

$$P_B(G) = \text{conv}\,\{x^F \in \mathbb{R}^E \mid [V,F] \text{ is a bipartite subgraph of } G\}. \tag{2.1}$$

Obviously, every edge set of a bipartite subgraph of G is contained in a cut of G. This implies that for positive edge weights c_e, $e \in E$, every optimum basic solution of the linear program

$$\max c^T x, \quad x \in P_B(G) \tag{2.2}$$

is the incidence vector of a cut of G. Hence, whenever the linear program (2.2) can be solved in polynomial time, the max-cut problem can be solved in polynomial time (this implication also holds the other way around).

Since the max-cut problem is NP-complete we cannot expect to find a complete explicit characterization of $P_B(G)$ for all graphs G. It may however be that for certain classes of graphs G, $P_B(G)$ can be described by means of a few classes of linear inequalities and that for these classes of inequalities polynomial time separation algorithms can be designed, so that the max-cut problem for these graphs is solvable in polynomial time. Examples of such classes of graphs are planar graphs, cf. BARAHONA (1980), and weakly bipartite graphs, cf. GRÖTSCHEL & PULLEYBLANK (1981).

It is trivial to see that $P_B(G)$ is full-dimensional, which implies that $P_B(G)$ has a unique (up to positive scaling) nonredundant defining linear inequality system. BARAHONA (1980) showed that the following inequalities

$$0 \le x_e \le 1 \qquad \text{for all } e \in E \tag{2.3}$$

$$x(C) \le |C|-1 \qquad \text{for all odd cycles } C \text{ in } G \tag{2.4}$$

define facets of $P_B(G)$. (We call the inequalities (2.3) *trivial* and the inequalities (2.4) *odd cycle constraints.*) Using matching techniques GRÖTSCHEL & PULLEYBLANK (1981) devised a polynomial time separation algorithm for the system (2.3), (2.4) (i. e. an algorithm which decides in polynomial time whether a given vector $y \in \mathbb{Q}^E$ satisfies (2.3) and (2.4), and if not, finds a violated inequality). Thus, the ellipsoid method implies that there is a polynomial time algorithm for the solution of (2.2) whenever $P_B(G)$ is completely determined by the trivial inequalities (2.3) and the odd cycle constraints (2.4). Such graphs are called *weakly bipartite*. It is for instance known that all planar graphs are weakly bipartite, cf. BARAHONA (1980).
In the next two paragraphs we shall introduce various new classes of inequalities defining facets of $P_B(G)$.

To simplify technical details in subsequent proofs we now state a lemma which we shall use frequently in the following sections.

(2.5) <u>Lemma.</u> *Let $b^Tx \le \beta$ be a valid inequality with respect to $P_B(G)$. Let Y, Y', Z and Z' be mutually disjoint subsets of V and let $v \in V \setminus (Y \cup Y' \cup Z \cup Z')$. suppose that the incidence vectors of the following edge sets of bipartite subgraphs*

$$B_1 := [Y' \cup Y \cup \{v\} : Z' \cup Z]$$

$$B_2 := [Y' \cup Y : Z' \cup Z \cup \{v\}]$$

$$B_3 := [Y' \cup Z \cup \{v\} : Z' \cup Y]$$

$$B_4 := [Y' \cup Z : Z' \cup Y \cup \{v\}]$$

satisfy $b^Tx \le \beta$ with equality. Then

$$\sum_{z \in Z} b_{zv} = \sum_{y \in Y} b_{yv}$$

<u>Proof.</u> Clearly,

$$0 = \beta - \beta = b^T{}_x B_1 - b^T{}_x B_3$$

$$= \sum_{z\in Z} (b_{zv} + \sum_{y'\in Y'} b_{y'z}) + \sum_{y\in Y} \sum_{z'\in Z'} b_{yz'}$$

$$- \sum_{z\in Z} \sum_{z'\in Z'} b_{zz'} - \sum_{y\in Y} (b_{yv} + \sum_{y'\in Y'} b_{yy'}).$$

$$0 = \beta - \beta = b^T{}_x B_2 - b^T{}_x B_4$$

$$= \sum_{z\in Z} \sum_{y'\in Y'} b_{y'z} + \sum_{y\in Y} \left(\sum_{z'\in Z'} b_{yz'} + b_{yv}\right)$$

$$- \sum_{z\in Z} \left(\sum_{z'\in Z'} b_{zz'} + b_{zv}\right) - \sum_{y\in Y} \sum_{y'\in Y'} b_{yy'}.$$

Thus, subtracting the two equations from each other we obtain

$0 = 2 \sum_{z\in Z} b_{zv} - 2 \sum_{y\in Y} b_{yv}$ which proves our claim.

$\blacksquare$

3. Further Rank Facets of $P_B(G)$

For a graph $G = [V,E]$ let

$$\mathcal{B}(G) := \{B \subseteq E \mid [V,B] \text{ is bipartite}\} \tag{3.1}$$

denote the set of edge sets of bipartite subgraphs of G. Clearly, $\mathcal{B}(G)$ is an independence system on E, i. e. $\mathcal{B}(G)$ satisfies $\emptyset \in \mathcal{B}(G)$ and $B' \subseteq B \in \mathcal{B}(G) \Rightarrow B' \in \mathcal{B}(G)$. For every set $F \subseteq E$ the number

$$r(F) := \max \{|B| : B \in \mathcal{B}(G),\ B \subseteq F\}$$

is called the *rank* of F (with respect to $\mathcal{B}(G)$). If $F \subseteq E$ then the 0/1-inequality

$$x(F) := \sum_{e \in F} x_e \leq r(F) \tag{3.2}$$

is by definition a supporting inequality of $P_B(G)$. The inequalities of type (3.2) are called *rank inequalities*, e. g. the odd cycle constraints (2.4) are rank inequalities.

Obviously, every facet of $P_B(G)$ defined by a 0/1-inequality is a rank inequality, and it is a very interesting open problem to characterize those rank inequalities of $P_B(G)$ that define facets of $P_B(G)$.

In this section we introduce two new "basic" classes of facet defining rank inequalities, further such inequalities will be derived in the next section.

(3.3) Theorem. *Let $G = [V,E]$ be a graph and let $[W,E(W)]$ be a complete subgraph of order $p \geq 3$ of G. Then*

$$x(E(W)) \leq \lceil \tfrac{p}{2} \rceil \lfloor \tfrac{p}{2} \rfloor \tag{3.4}$$

is a valid inequality with respect to $P_B(G)$. (3.4) defines a facet of $P_B(G)$ if and only if p is odd.

Proof. First of all, it is trivial to see that the right hand side of (3.4) gives the rank of a complete subgraph with respect to $\mathcal{B}(G)$.

We now show that for p even, inequality (3.4) is the sum of p (positively scaled) inequalities (3.4) for which $|W|$ is odd. Namely if $[W,E(W)]$ is a complete subgraph of G with $p = |W|$ even, then

$$x(E(W)) = \frac{1}{p-2} \sum_{v \in W} x(E(W \setminus \{v\})) \leq \frac{1}{p-2}\, p \left\lceil \frac{p-1}{2} \right\rceil \left\lfloor \frac{p-1}{2} \right\rfloor = \frac{p^2}{4} = \left\lceil \frac{p}{2} \right\rceil \left\lfloor \frac{p}{2} \right\rfloor$$

This implies that (3.4) does not define a facet of $P_B(G)$ in case p is even.

Now let us assume that $[W,E(W)]$ is a complete subgraph of G with $|W| = 2k+1$, $k \geq 1$, and let us denote the corresponding inequality $x(E(W)) \leq k(k+1)$ by $a^Tx \leq \alpha$. Assume, moreover, that $b^Tx \leq \beta$ is a facet defining inequalitiy for $P_B(G)$ which has the following property. If a vertex $x \in P_B(G)$ satisfies $a^Tx = \alpha$ then x also satisfies $b^Tx = \beta$. If we can prove that $b = \rho a$ for some $\rho > 0$ then we can conclude that $a^Tx \leq \alpha$ is equivalent to $b^Tx \leq \beta$, i. e. defines a facet of $P_B(G)$, and we are done.

To prove the claim above we first show that for any three different nodes $r, s, t \in W$ we have $b_{rt} = b_{st}$. To prove this, we take two disjoint node sets $U, U' \subseteq W \setminus \{r,s,t\}$ of cardinality $k-1$, and apply Lemma (2.5) with $Y' = U$, $Z' = U'$, $Y = \{r\}$, $Z = \{s\}$, and $v = t$.

Since r, s, t were arbitrarily chosen and since $[W,E(W)]$ is connected we can conclude that for some $\rho \in \mathbb{R}$

$$b_{ij} = \rho \qquad \text{for all } ij \in E(W).$$

It is a trivial matter to prove that

$$b_{ij} = 0 \qquad \text{for all } ij \in E \setminus E(W).$$

Moreover, it is easy to see that for every edge e in G there is a set $B \in \mathcal{B}(G)$ with $e \in B$ and $a^Tx^B = \alpha$. This implies that the face defined by $a^Tx \leq \alpha$ is not contained in the trivial facet $\{x \in P_B(G) \mid x_e = 0\}$ for some $e \in E$. Hence $b^Tx \leq \beta$ defines a nontrivial facet of $P_B(G)$ which by the monotonicity (i. e. $0 \leq x \leq y \in P_B(G)$ implies $x \in P_B(G)$) of $P_B(G)$ implies that all coefficients of b are nonnegative and at least one is positive. Hence we can conclude that $\rho > 0$ and, thus, $b = \rho a$ holds which proves our claim.

$\square$

In the following we shall refer to the facet defining inequalities introduced in (3.3) as *odd complete subgraph inequalities* or as the K_{2k+1}-*inequalities*.

Note that Theorem (3.3) can be used to prove the following

Remark: *For every graph $G = [V,E]$ the cut polytope*

$$conv\ \{x^{\delta(W)} \in \mathbb{R}^E \mid W \subseteq V\}$$

has full dimension $|E|$.

Namely, we embed G into a complete graph K_{2k+1}. Then we know that the K_{2k+1}-inequality defines a facet of $P_B(K_{2k+1})$. We therefore can make up a nonsingular $\left(\binom{2k+1}{2},\binom{2k+1}{2}\right)$-matrix M whose rows are incidence vectors of bipartite edge sets in K_{2k+1} satisfying the K_{2k+1}-inequality with equality. Note that each of these bipartite edge sets necessarily is a cut of K_{2k+1}. Now we remove all columns from M corresponding to edges of K_{2k+1} which are not in G. The resulting matrix M' has full rank $|E|$ and the rows of M' correspond to cuts of G. Thus, there are $|E|$ cuts of G whose incidence vectors are linearly independent. Adding the zero vector (which is the incidence vector of $\delta(V)$) we obtain a set of $|E| + 1$ affinely independent points in the cut polytope of G.

A graph $G = [V,E]$ is called a *bicycle p-wheel* if G consists of a cycle of length p and two (so-called *universal*) nodes which are adjacent to each other and to every node in the cycle. A bicycle p-wheel is called *odd* if p is odd. Figure 3.1 shows a bicycle 7-wheel.

(3.5) Theorem. *Let $G = [V,E]$ be a graph and let $[W,F]$ be a bicycle $(2k+1)$-wheel, $k \geq 1$, contained in G. Then the inequality*

$$x(F) \leq 2\ (2k+1) \qquad (3.6)$$

defines a facet of $P_B(G)$.

Proof. Let v,w be the two universal nodes of the bicycle (2k+1)-wheel $[W,F]$ in G, and let the cycle C of $[W,F]$ be formed by the edges $u_1u_2, u_2u_3, u_3u_4, \ldots, u_{2k}u_{2k+1}, u_{2k+1}u_1$. $[W,F]$ contains the following triangles

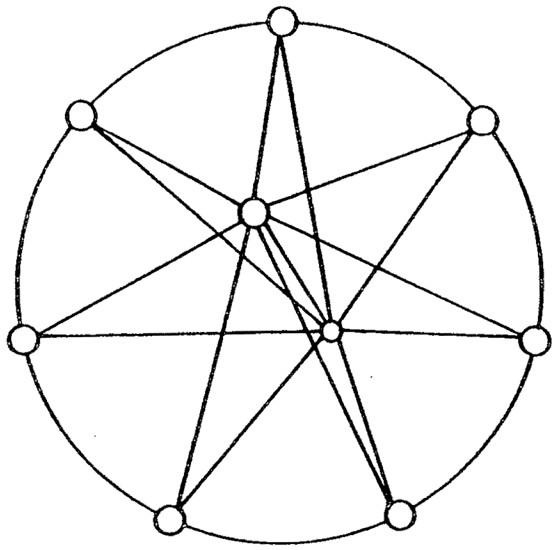


Figure 3.1

$T_i := \{vw,\ vu_i,\ wu_i\}$, $\quad i = 1,\dots,2k+1$,

$T_i^v := \{vu_i,\ vu_{i+1},\ u_iu_{i+1}\}$, $\ i = 1,\dots,2k$, $\ T_{2k+1}^v := \{vu_1,\ vu_{2k+1},\ u_1u_{2k+1}\}$

$T_i^w := \{wu_i,\ wu_{i+1},\ u_iu_{i+1}\}$, $\ i = 1,\dots,2k$, $\ T_{2k+1}^w := \{wu_1,\ wu_{2k+1},\ u_1u_{2k+1}\}$

By summing up appropriately we get

$$(2k+1)x(F) = \sum_{i=1}^{2k+1} \left(x(T_i)+kx(T_i^v)+kx(T_i^w)\right) + x(C) \le (2k+1)(2+2k+2k) + 2k =$$

$$= (2k+1)2(2k+1) + 2k.$$

Dividing through 2k+1 we obtain

$$x(F) \le 2\ (2k+1) + 2k/(2k+1).$$

Since for every vertex of $P_B(G)$ the left hand side of the above inequality is an integer which, by the validity of the odd cycle inequalities, does not exceed the right hand side, we can round the right hand side down to the next integer and obtain that (3.6) is valid for $P_B(G)$.

Let us denote our inequality $x(F) \leq 2(2k+1)$ by $a^Tx \leq \alpha$ and let us assume (as in the proof before) that $b^Tx \leq \beta$ is a facet defining inequality such that $\{x \in P_B(G) \mid a^Tx = \alpha\} \subseteq \{x \in P_B(G) \mid b^Tx = \beta\}$. We are done if we can prove that there is $\rho > 0$ such that $b = \rho a$.

First we show that for every node u_i, $1 \leq i \leq 2k+1$, $b_{vu_i} = b_{vw}$ holds. We simply pick an arbitrary node of the cycle C, say u_1, and apply Lemma (2.5) with

$$Y' := \{u_3, u_5, \ldots, u_{2k+1}\}, \quad Z' := \{u_2, u_4, \ldots, u_{2k}\},$$

$$Y := \{u_1\}, \quad Z := \{w\} \quad \text{and } v := v.$$

By symmetry we obtain $b_{u_iw} = b_{vw}$ for $i = 1,\ldots,2k+1$. Thus, if we set

$$\rho := b_{vw}$$

we can conclude that

$$b_{vu_i} = b_{wu_i} = \rho, \qquad i = 1,\ldots,2k+1.$$

In the next step we show that the b-values of the cycle edges are equal. Set $B_1 := [Y' \cup Z : Z' \cup \{u_1,v\}]$

$$B_2 := [Y' \cup \{w,u_1\} : Z' \cup \{v\}]$$

then again $B_j \in \mathcal{B}(G)$ and $a^Tx^{B_j} = \alpha$, $j = 1,2$. So

$$0 = b^Tx^{B_2} - b^Tx^{B_1} = b_{u_1u_2} + b_{u_1v} - b_{u_1u_{2k+1}} - b_{u_1w} = b_{u_1u_2} - b_{u_1u_{2k+1}}$$

which implies

$$b_{u_1u_{2k+1}} = b_{u_iu_{i+1}} = \rho' \qquad i = 1,\ldots,2k$$

for some $\rho' \in \mathbb{R}$.

To prove that $\rho = \rho'$ consider the sets

$$B_3 := [Y' \cup \{u_1,u_2\} : (Z' \setminus \{u_2\}) \cup \{v,w\}]$$

$$B_4 := [Y' \cup \{u_2\} : (Z' \setminus \{u_2\}) \cup \{v,w,u_1\}]$$

which are in $\mathcal{B}(G)$ and for which $a^T x^{B_3} = a^T x^{B_4} = \alpha$. We get

$0 = b^T x^{B_4} - b^T x^{B_3} = b_{u_1u_2} + b_{u_1u_{2k+1}} - b_{u_1v} - b_{u_1w} = 2(\rho'-\rho)$. Hence we have shown

$$b_{ij} = \rho \qquad \text{for all } ij \in F.$$

For every pair of nodes u_i, u_j which are not neighbours on the cycle C, it is easy to construct a bipartite subgraph $[W,B]$ of $[W,F]$ with bipartition W_1, W_2 such that $u_i \in W_1$, $u_j \in W_2$ and such that $a^T x^B = \alpha$. If $u_iu_j \in E$, then of course $B' = B \cup \{u_iu_j\} \in \mathcal{B}(G)$ and $a^T x^{B'} = \alpha$. Hence $0 = b^T x^{B'} - b^T x^B = b_{u_iu_j}$.

If ij is an edge of E with one endnode not in W, then it is trivial to construct $B \in \mathcal{B}(G)$ such that $B \cup \{ij\} \in \mathcal{B}(G)$ and $a^T x^B = \alpha$, which implies $b_{ij} = 0$. Altogether we have now shown

$$b_{ij} = \rho \qquad \text{for all } ij \in F,$$

$$b_{ij} = 0 \qquad \text{for all } ij \in E \setminus F.$$

As in the proof of (3.3) it follows that $\rho > 0$ and thus that $b = \rho a$ which finishes the proof of (3.5).

$\square$

In the following we shall call the inequalities (3.6) *odd bicycle wheel inequalities*. Of course, we could have considered even bicycle wheel inequalities as well, but as in (3.3) these "even inequalities" do not define facets of $P_B(G)$.

Note that the bicycle 3-inequalities (3.6) are the same as the K_5-inequalities (3.4). But this is the only overlapping of these two classes of inequalities.

Let us denote the polytope defined by the trivial inequalities (2.3) and the odd cycle inequalities (2.4) by $P_C(G)$. BARAHONA (1980) has shown that the max-cut problem for the class of graphs G which have the following property "there exists a node v such the G - v is planar" is NP-complete. Since linear programs over $P_C(G)$ are solvable in polynomial time, we can conclude (unless P = NP) that $P_C(G)$ does not equal $P_B(G)$. This observation lead us to the discovery of the odd bicycle wheel inequalities. Namely, if one of the universal nodes of a bicycle wheel is removed then the remaining graph is planar.

In fact, BARAHONA (1982) has shown that the max-cut problem is NP-complete for the following class of graphs G : " G contains a node v such that G - v is planar and all nodes in G - v have degree at most three". The odd bicycle wheels (except p = 3) do not belong to this class. We shall, however, give a construction (node-splitting) with which facets of $P_B(G)$ for graphs G of this type can sometimes be constructed from odd bicycle wheel inequalities. These inequalities however are not rank inequalities, since they have coefficients larger than one (see section G for an example).

It is obvious that

$$P_B(G) = \text{conv}\{x \in P_C(G) \mid x \text{ integer}\},$$

so one may ask what is the Chvátal rank, cf. CHVATAL (1973), for a class of facets of $P_B(G)$ with respect to the trivial inequalities (2.3) and the odd cycle constraints (2.4). The validity proof for the odd bicycle wheel inequalities given in (3.5) shows that these inequalities are of Chvátal rank one. This is, however, not the case for the K_{2k+1}-inequalities. For instance, one can easily show that the K_7-inequalities are of Chvátal rank 2, and we conjecture the Chvátal rank (with respect to $P_C(G)$) of K_{2k+1}-inequalities grows linearly with k.

4. Contraction and Splitting of Nodes

In this section we shall describe three methods for constructing "facets from facets". The first one called "node set contraction" can only be applied to the odd complete subgraph inequalities introduced in section 3, while the other two, named "odd subdivision of edges" and "node splitting", are universal and can be applied to any facet defining inequality. The latter two operations can also be reversed.

(4.1) Theorem. (Node Set Contraction) *Let $[W,E(W)]$ be a complete subgraph of order $2k+1$, $k \geq 3$, of a graph $G = [V,E]$ and let $W_1,W_2,\dots,W_s$ be a partition of W into nonempty mutually disjoints subsets such that* $\sum_{i=1,|W_i|>1}^{s} |W_i| \leq k-1$.

Let $G' = [V',E']$ denote the graph obtained from G by shrinking the nodes in W_i into a single node w_i, $i = 1,\dots,s$ (i. e. all new nodes w_i are adjacent in G', and a new node w_i is adjacent in G' to an old node $v \in V \setminus W$ iff W_i contains a node adjacent to v in G). Set

$$a_{w_iw_j} := |W_i||W_j| \qquad \text{for all } 1 \leq i < j \leq s$$

$$a_{vw} := 0 \qquad \text{for all other } vw \in E'$$

then

$$a^Tx \leq \alpha := k(k+1) \tag{4.2}$$

defines a facet of $P_B(G')$.

Proof. Denoting by $\bar{a}^Tx := x(E(W)) \leq k(k+1) =: \alpha$ the odd complete subgraph inequality of $[W,E(W)]$ with respect to $P_B(G)$ we immediately see that a nonzero coefficient $a_{w_iw_j}$ of the new inequality (4.2) is equal to $\sum_{v\in W_i} \sum_{w\in W_j} \bar{a}_{vw}$. If we have a bipartition V_1', V_2' of V' we can expand the nodes of V' into nodes of V to obtain the corresponding bipartition V_1, V_2 of V. Then it immediately follows that for the edge sets $B' = [V_1':V_2']$ and $B = [V_1:V_2]$ we have $\bar{a}^Tx^B = a^Tx^{B'}$ which implies that $a^Tx \leq \alpha$ is valid with respect to $P_B(G')$.

Assume as in the previous proofs of this type that $b^T x \leq \beta$ is a facet defining inequality for $P_B(G')$ with $\{x \in P_B(G') | a^T x = \alpha\} \subseteq \{x \in P_B(G') | b^T x = \beta\}$ To prove our theorem we have to show that $b = \rho a$ holds for some $\rho > 0$.

Let W' denote the subset of V' obtained by shrinking the node sets $W_1,\dots,W_s$. We may assume that $|W_i| > 1$ for $i = 1,\dots,r$ and $|W_i| = 1$ for $i = r+1,\dots,s$. (If no set W_i has cardinality more than one, there is nothing to prove.) Note t $s-r \geq k+2$ because of the assumption $q := |W_1| + \dots + |W_r| \leq k-1$. Set $S := \{w_1,\dots,w_r\}$, $T := W' \setminus S$.

Let, u, v, w be three different nodes in T. Let U be any subset of $T\setminus\{u,v,w\}$ of cardinality $k-1$ and $S' := T\setminus(U \cup \{u,v,w\})$. Set $Y := \{u\}$, $Z := \{w\}$, $Y' := U$, $Z' := S \cup S'$, then the incidence vectors of $B_1,\dots,B_4$ as defined in (2.5) satisfy $a^T x \leq \alpha$ with equality, and hence $b^T x \leq \beta$ also with equality. Therefore Lemma (2.5) implies $b_{uv} = b_{wv}$ and we can conclude that there is $\rho \in \mathbb{R}$ with

$$b_{uv} = \rho \qquad \text{for all } u,v \in T.$$

Now we choose any $w_i \in S$ and any $v \in T$. Set $Y := \{w_i\}$, let Z be any subset of $T\setminus\{v\}$ of cardinality $|W_i|$, Z' be any subset of $T\setminus(Z \cup \{v\})$ of cardinality $k - |W_i|$ and $Y' := (S \cup T)\setminus(Z \cup Z' \cup \{v\} \cup \{w_i\})$. Again, the incidence vectors c $B_1,\dots,B_4$ as defined in (2.5) satisfy $a^T x \leq \alpha$ and hence $b^T x \leq \beta$ with equality, so we can conclude from (2.5) that $b_{vw_i} = \sum_{z \in Z} b_{zv} = \rho|W_i|$ thus

$$b_{vw_i} = \rho|W_i| \quad \text{for all } w_i \in S \text{ and all } v \in T.$$

Let S_1 be any subset of T of cardinality $k+1-q$ and set $S_2 := T\setminus S_1$. Then $B = [S \cup S_1 : S_2]$ is an edge set in $\mathcal{B}(G)$ satisfying $a^T x = \alpha$. Hence

$$\beta = b^T x^B = \sum_{w_i \in S} \sum_{j \in S_2} b_{jw_i} + \sum_{i \in S_1} \sum_{j \in S_2} b_{ij} = \rho\, k(k+1).$$

To determine the coefficients $b_{w_i w_j}$ for $w_i, w_j \in S$ (in case $|S| > 1$) we use the following argument. Let S_1, S_2 be any bipartition of S, then by our assumptions we have that $k_1 := \sum_{w_i \in S_1} |W_i| \leq k-3$ and $k_2 := \sum_{w_j \in S_2} |W_j| \leq k-3$.

Choosing any subset S_1' of T of cardinality $k+1-k_1$ and setting $S_2' := T\setminus S_1'$ we see that $B := [S_1 \cup S_1' : S_2 \cup S_2']$ is the edge set of a bipartite subgraph of G' such that $a^Tx^B = \alpha$ and therefore $b^Tx^B = \beta$ holds. Thus, for every bipartition S_1, S_2 of S we can find an edge set $B \in \mathcal{B}(G')$ with $B \subseteq E(W')$ such that $b^Tx^B = \beta$.

Observe that for all edges $ij \in E' \setminus E(W')$ the components x^B_{ij} of the incidence vectors x^B of all these edge sets B are zero by construction. Let $\bar{y} \in \mathbb{R}^{E(S)}$ $\left(\text{resp } \bar{\bar{y}} \in \mathbb{R}^{E(W') \setminus E(S)}\right)$ denote the vector obtained from a vector $y \in \mathbb{R}^{E'}$ by dropping the components corresponding to edges $ij \in E' \setminus E(S)$ (resp. edges $ij \in E' \setminus (E(W') \setminus E(S))$). Thus for every edge set $B \in \mathcal{B}(G)$ as constructed above we have $\beta = b^Tx^B = \bar{b}^T\bar{x}^B + \bar{\bar{b}}^T\bar{\bar{x}}^B$. Since we know already all coefficients b_{ij} for $ij \in E(W') \setminus E(S)$, we get

$$\begin{aligned}
\bar{\bar{b}}^T\bar{\bar{x}}^B &= \sum_{w_i \in S_1} \sum_{j \in S_2'} b_{jw_i} + \sum_{w_j \in S_2} \sum_{i \in S_1'} b_{iw_j} + \sum_{i \in S_1'} \sum_{j \in S_2'} b_{ij} \\
&= \sum_{w_i \in S_1} \rho |S_2'||W_i| + \sum_{w_j \in S_2} \rho |S_1'||W_j| + \rho |S_1'||S_2'| \\
&= \rho(k_1|S_2'| + k_2|S_1'| + |S_1'||S_2'|) \\
&= \rho(k_1(k-k_2) + k_2(k+1-k_1) + (k-k_2)(k+1-k_1)) \\
&= \rho(k(k+1) - k_1k_2)
\end{aligned}$$

which implies that for every set $B \in \mathcal{B}(G)$ constructed from a bipartition S_1, S_2 of S as above we obtain an equation

$$\begin{aligned}
\bar{b}^T\bar{x}^B &= \beta - \rho(k(k+1) - k_1k_2 \\
&= \rho\, k_1k_2
\end{aligned}$$

Let us write this equation system in the form

$$A\bar{b} = \bar{\beta}.$$

I. e. we have $|E(S)|$ unknowns b_{ij}, $ij \in E(S)$, and the matrix A is an $(2^{|S|-1} - 1, |E(S)|)$-matrix whose rows are the incidence vectors of the cuts $\delta(S_1)$ of $[S,E(S)]$, $\emptyset \neq S_1 \neq S$, $S_1 \subseteq S$, and the right hand side $\bar{\beta}$ is given as calculated above. This matrix A has full rank $|E(S)|$ (see Remark behind Theorem (3.3)), so, if the system $A\bar{b} = \bar{\beta}$ has a solution, it has a unique solution. Now we simply set

$$b_{w_iw_j} = \rho |W_i||W_j| \qquad \text{for all } w_i, w_j \in S$$

and can easily see that this stipulation in fact gives a solution of our system. Hence we also know the coefficients of b corresponding to edges in E(S).

It is trivial to show that

$$b_{ij} = 0 \qquad \text{for all } ij \in E' \setminus E(W'),$$

thus we have proved that $b = \rho\alpha$ holds.

∎

Theorem (4.1) shows that for any K_{2k+1}-inequality, $k \geq 3$, one can shrink one or more subsets of nodes (subject to some cardinality conditions) to obtain new facet defining inequalities with coefficients which are not only zero and one. Moreover the *supports* of these inequalities $a^T x \leq \alpha$ (i. e. the edge sets $F_a := \{ij \in E' \mid a_{ij} > 0\}$) may also be the edge sets of complete subgraphs of G' of even order.

(4.3) <u>Corollary.</u> *Let $G = [V,E]$ be a graph containing a complete subgraph $[W,E(W)]$ of order $k+4$, $k \geq 2$. Then $P_B(G)$ has a facet defining inequality $a^T x \leq k(k+1)$ whose support is $E(W)$ and whose coefficients are $0,\ 1,\ \frac{1}{2}(k-1),\ \frac{1}{4}(k-1)^2$ in case k is odd, resp. $0,\ 1,\ \frac{k-2}{2},\ \frac{k}{2},\ \frac{1}{4}k(k-2)$ in case k is even.*

∎

In particular, if G itself is the complete graph of order $n = k+4$, Corollary (4.3) shows that the coefficients of facets of $P_B(G)$ may be of order n^2.

We now introduce a facet construction method which will turn out to be a special case of another such method but which is of particular interest and therefore gets its own name.

(4.4) <u>Theorem.</u>

(a) (Odd Edge Subdivision). *Let $G = [V,E]$ be a graph and $a^T x \leq \alpha$ be a nontrivial facet defining inequality for $P_B(G)$. Let $ij \in E$ be an edge with $a_{ij} > 0$. Let $G' = [V',E']$ be a graph obtained from G in the following way. An even number of new nodes $i_1, i_2, \ldots, i_k$ is added. The edge set $P = ii_1, i_1i_2, \ldots, i_{k-1}i_k, i_kj$ (i. e. a path of odd length) is added. Any further edge having at least one of the* *new nodes* $i_1,\dots,i_k$ *as an endnode may be added. The edge* ij *may be removed. Let* $\bar{a} \in \mathbb{R}^{E'}$ *be defined as follows*

$$\bar{a}_{uv} := a_{uv} \qquad \text{for all } uv \in (E \cap E') \setminus \{ij\},$$

$$\bar{a}_{uv} := a_{ij} \qquad \text{for all } uv \in P,$$

$$\bar{a}_{uv} := 0 \qquad \text{otherwise.}$$

Then

$$\bar{a}^T x \leq \alpha + k a_{ij}$$

defines a facet of $P_B(G')$.

(b) (Replacing an odd path by an edge). *Let* $G' = [V',E']$ *be a graph and* $\bar{a}^T x \leq \bar{\alpha}$ *be a facet defining inequality for* $P_B(G')$. *Suppose the support of* $\bar{a}$ *contains a path* $P = \{ii_1, i_1i_2, \dots, i_{k-1}i_k, i_kj\}$ *of odd length* $k+1 \geq 3$ *with* $a_{uv} = \alpha'$ *for all* $uv \in P$ *and where the degree of all nodes* $i_1,\dots,i_k$ *in the support of* $\bar{a}$ *is 2 and* ij *is not in the support of* $\bar{a}$. *Let* $G = [V,E]$ *be the graph obtained from* G' *by removing the nodes* $i_1,\dots,i_k$ *and adding the edge* ij *(if* ij *is not already contained in* G'*). Let* $a \in \mathbb{R}^E$ *be defined as follows:*

$$a_{uv} := \bar{a}_{uv} \qquad \text{for all } uv \in E \cap E'$$

$$a_{ij} := \alpha'$$

Then

$$a^T x \leq \bar{\alpha} - k\alpha'$$

defines a facet of $P_B(G)$.

$\square$

Thus Theorem (4.4) (a) shows that we can subdivide "positive edges" of a facet defining inequality an odd number of times and that we will get another facet defining inequality this way. Theorem (4.4) (b) tells us that this operation can also be reversed without losing the desired properties. Theorem (4.4) is a corollary of the following more general result (set $F = \{ij\}$ and apply (4.4) repeatedly).

(4.5) <u>Theorem.</u>

(a) (Node Splitting). *Let* $G = [V,E]$ *be a graph and* $a^T x \leq \alpha$ *be a nontrivial facet defining inequality for* $P_B(G)$ ...

support of a, and let v be a node in $V(E_a)$. *Let* V_1, V_2 *be any bipartition of V such that* $a^T x^{\delta(V_1)} = \alpha$ *and assume that* $v \in V_1$. *Choose any nonempty subset* $F \subseteq \delta(v) \cap E_a \cap E(V_1)$ *and construct a new graph* $G' = [V',E']$ *from G as follows. Split node v into two nodes* v_1, v_2 *such that* v_1 *is incident to all edges contained in F and* v_2 *is incident to all edges in* $\delta(v) \setminus F$. *The edge* v_1v_2 *may be added, as well as any edge* v_2u *with* $vu \in F$. *Moreover, add a new node* v_3 *adjacent to* v_1 *and* v_2; *in addition any further edge having* v_3 *or* v_1 *as one endnode may be added. The other parts of G remain unchanged. Set*

$$\bar{a}_{ij} := a_{ij} \qquad \text{for all } ij \in E \setminus \delta(v)$$

$$\bar{a}_{v_1u} := a_{vu} \qquad \text{for all } v_1u \in E' \text{ with } vu \in F,$$

$$\bar{a}_{v_2u} := a_{vu} \qquad \text{for all } v_2u \in E' \text{ with } vu \in (\delta(v) \cap E_a) \setminus F,$$

$$\bar{a}_{v_3v_1} := \bar{a}_{v_3v_2} := \gamma := \sum_{ij \in F} a_{ij},$$

$$\bar{a}_{ij} := 0 \qquad \text{otherwise}$$

$$\bar{\alpha} := \alpha + 2\gamma,$$

then $\bar{a}^T x \leq \bar{\alpha}$ *defines a facet of* $P_B(G')$.

(b) (Contraction of a path of length two). *Let* $G' = [V',E']$ *be a graph and* $\bar{a}^T x \leq \bar{\alpha}$ *be an inequality defining a facet of* $P_B(G')$. *Suppose G' contains a node* v_3 *contained in exactly two edges, say* v_1v_3 *and* v_2v_3, *of the support* $F_{\bar{a}}$ *of* $\bar{a}$, *that* $v_1v_2 \notin F_{\bar{a}}$, *that* v_1 *and* v_2 *have no common neighbour in* $[V',F_{\bar{a}}]$ *except* v_3, *and that* $\bar{a}_{v_1v_3} = \bar{a}_{v_2v_3} = \bar{a}(\delta(v_1) \setminus \{v_1v_3\}) \leq \bar{a}(\delta(v_2) \setminus \{v_2v_3\})$. *Let* $G = [V,E]$ *be the graph obtained from G' by removing the nodes* v_1, v_2, v_3 *adding a new node v and the edges* $\{vj \mid \bar{a}_{v_1j} > 0\} \cup \{vj \mid v_2j \in E'\}$.

Set

$$a_{ij} := \bar{a}_{ij} \qquad \text{for all } ij \in E \cap E'$$

$$a_{vj} := \bar{a}_{v_1j} \qquad \text{if } \bar{a}_{v_1j} > 0$$

$$a_{vj} := \bar{a}_{v_2 j} \qquad \text{if } v_2 j \in E' \text{ unless } \bar{a}_{v_1 j} > 0$$

$$\alpha := \bar{\alpha} - 2\bar{a}_{v_1 v_3},$$

then $a^T x \leq \alpha$ *defines a facet of* $P_B(G)$.

Proof. The validity of the new inequalities defined in (a) and (b) follows by elementary construction. It remains to prove that the faces defined by the inequalities contain sufficiently many affinely independent points.

(a) Since $a^T x \leq \alpha$ is nontrivial, there are $m = |E|$ bipartite edge sets $B_1, \ldots, B_m \subseteq E$ whose incidence vectors are linearly independent and satisfy $a^T x^{B_i} = \alpha$, $i = 1, \ldots, m$. Set
$B_i' := (B_i \setminus \delta_G(v)) \cup \{v_1 v_3, v_2 v_3\} \cup \{v_1 u \mid vu \in B_i \cap F\} \cup \{v_2 u \mid vu \in B_i \setminus F\}$
$i = 1, \ldots m$, then B_i' is a bipartite edge set in G' and $\bar{a}^T x^{B_i'} = \alpha + 2\gamma$. Moreover, the incidence vectors $x^{B_i'} \in \mathbb{R}^{E'}$, $i = 1, \ldots m$ are linearly independent.

Let $F_{\bar{a}} \subseteq E'$ denote the support of $\bar{a}$. Using the assumption that $a^T x^{\delta(V_1)} = \alpha$ and $v \in V_1$ we immediately see that the incidence vectors of the bipartite edge sets $B_{m+1} := \delta_{G'}((V_1 \setminus \{v\}) \cup \{v_2, v_3\}) \cap F_{\bar{a}}$ and $B_{m+2} := \delta_{G'}((V_1 \setminus \{v\}) \cup \{v_2\}) \cap F_{\bar{a}}$ in G' satisfy $\bar{a}^T x^{B_i} = \alpha + 2\gamma$, $i = m+1, m+2$. It is obvious , that each of the vectors $x^{B_{m+1}}$ and $x^{B_{m+2}}$ is linearly independent from the vectors x^{B_i}, $i = 1, \ldots, m$, and it is an easy exercise in linear algebra to prove that in fact all the vectors x^{B_i} , $i = 1, \ldots, m+2$ are linearly independent.

Moreover, consider the incidence vectors of the following bipartite edge sets. If $v_1 v_2 \in E'$ set $B_{v_1 v_2} = B_{m+2} \cup \{v_1 v_2\}$. If $v_3 u \in E'$, $v_1 \neq u \neq v_2$, then, if $u \in V_1$ set $B_{v_3 u} = B_{m+2} \cup \{v_3 u\}$, if $u \in V \setminus V_1$ set $B_{v_3 u} = B_{m+1} \cup \{v_3 u\}$. If $v_1 u \in E'$ such that $vu \notin F$, then, if $u \in V_1$ set $B_{v_1 u} := B_{m+2} \cup \{v_1 u\}$, if $u \notin V_1$ then set $B_{v_1 u} := (B_{m+2} \setminus \delta(v_1)) \cup \{v_1 v_3, v_1 u\}$. If $v_2 u \in E'$ with $vu \in F$, then because $a^T x \leq \alpha$ defines a nontrivial facet of $P_B(G)$ there exists a bipartite edge set $B_u \subseteq E$ with $vu \in B_u$ and $a^T x^{B_u} = \alpha$, and we set
$B_{v_2 u} = (B_u \setminus \delta_G(v)) \cup \{v_1 v_3, v_2 v_3\} \cup \{v_1 j \mid vj \in B_u \cap F\} \cup \{v_2 j \mid vj \in B_u \setminus F\} \cup \{v_2 u\}$.
Clearly, these incidence vectors satisfy $\bar{a}^T x \leq \alpha + 2\gamma$ with equality and the set of $|E'|$ vectors constructed above is linearly independent.

Thus $P_B(G') \cap \{x \mid \bar{a}^T x = \alpha + 2\gamma\}$ contains $|E'|$ linearly independent points which proves that $a^T x \leq \alpha + 2\gamma$ defines a facet of $P_B(G')$.

(b) Let $Q = \{v_3u \in E' \mid v_1 \neq u \neq v_2\} \cup \{v_1u \in E' \mid \bar{a}_{v_1u} = 0\} \cup \{v_2u \in E' \mid \bar{a}_{v_1u} > 0\}$. Then by assumption $\bar{a}_{ij} = 0$ for all $ij \in Q$. If $\bar{a}^T x \leq \bar{\alpha}$ defines a facet of $P_B(G')$ then the inequality $\sum_{ij \in E' \setminus Q} \bar{a}_{ij} x_{ij} \leq \bar{\alpha}$ clearly defines a facet of $P_B(G'-Q)$. Thus we may assume without loss of generality that $Q = \emptyset$, i. e. that in G' no node in $V' \setminus \{v_1, v_2, v_3\}$ is adjacent to v_3 and that no such node is adjacent to both v_1 and v_2.

By assumption $\bar{a}^T x \leq \bar{\alpha}$ defines a nontrivial facet of $P_B(G')$, so there are $m := |E'|$ bipartite edge sets $B_1', B_2', \ldots, B_m'$ whose incidence vectors are linearly independent and satisfy $\bar{a}^T x^{B_i'} = \bar{\alpha}$, $i = 1, \ldots, m$. We may assume that $N \subseteq V'$ is the set of neighbours of v_1 in $[V', F_{\bar{a}}]$ different from v_3, and that $k := |N|$.

Now let M be the (m,m)-matrix whose rows are the incidence vectors $x^{B_1'}, \ldots, x^{B_m'}$. We may assume that the last $k+2$ columns correspond to the edges v_1i, $i \in N$, and v_1v_3, v_2v_3. Moreover, we assume that the sets $B_1', \ldots, B_m'$ are ordered in such a way that the sets $B_1', \ldots B_r'$ contain both edges v_1v_3, v_2v_3, the sets $B_{r+1}', \ldots, B_s'$ contain v_1v_3 and not v_2v_3 and the sets $B_{s+1}', \ldots, B_m'$ contain v_2v_3 but not v_1v_3.

Note that the assumptions of the theorem imply that if $\bar{a}^T x^{B'} = \bar{\alpha}$, then B' contains at least one of the edges v_1v_3, v_2v_3, and moreover, if B' contains only one of the edges v_1v_3, v_2v_3 then B' necessarily contains all edges v_1i, $i \in N$, otherwise $B'' = (B' \setminus \{v_1i \mid i \in N\}) \cup \{v_1v_3, v_3v_2\}$ would be a bipartite edge set with $\bar{a}^T x^{B''} > \bar{\alpha}$. Thus, our matrix M looks as follows

$$\begin{array}{cc} & \begin{array}{cccc} \overbrace{\qquad\quad}^{m-k-2} & \overbrace{\qquad}^{k} & \overbrace{\quad}^{2} & \end{array} \\ \begin{array}{r} r \left\{ \right. \\ s-r \left\{ \right. \\ m-s \left\{ \right. \end{array} & \begin{array}{|c|c|c|c|} \hline M_1 & M_2 & \mathbb{1} & \mathbb{1} \\ \hline M_3 & M_5 & \mathbb{1} & 0 \\ \hline M_4 & M_6 & 0 & \mathbb{1} \\ \hline \end{array} \end{array} = M$$

where M_1, M_2, M_3, M_4 are some 0/1-matrices, the matrices M_5 and M_6 contain ones only and the last two columns are as indicated, where $\mathbb{1}$ denotes a column of ones.

Now we transform the edge sets $B'_i \subseteq E'$ into bipartite edge sets $B_i \subseteq E$, $i = 1,\ldots,m$ as follows

$$B_i := \left(B'_i \setminus (\delta_{G'}(v_1) \cup \delta_{G'}(v_2))\right) \cup \{vj \mid v_2 j \in B'_i,\ j \neq v_3\}$$
$$\cup \{vj \mid v_1 j \in B'_i,\ j \neq v_3\}$$

Note that this transformation corresponds to a shrinking of the nodes v_1, v_2, v_3 into the node v. (By our assumption $Q = \emptyset$, no multiple edges will occur.) It follows from our remarks above, that $a^T x^{B_i} = \alpha$ for $i = 1,\ldots,m$. If A is the $(m,m-2)$-matrix whose rows are the incidence vectors x^{B_i}, $i = 1,\ldots,m$, then A looks as follows

$$\begin{array}{r|c|c|}
 & \overbrace{\qquad\qquad}^{m-k-2} & \overbrace{\qquad}^{k} \\
\hline
r \left\{ \right. & A_1 & A_2 \\
\hline
s-r \left\{ \right. & A_3 & A_5 \\
\hline
m-s \left\{ \right. & A_4 & A_6 \\
\hline
\end{array} \quad = \quad A$$

where by construction the following holds

$$A_i = M_i \qquad i = 1,\ldots,4$$

and A_5, A_6 are matrices containing zeros only.

To prove the theorem we now show that the rank of A equals m-2.

First we add to M a (m+1)st column which contains twos only and then we add to this matrix a (m+1)st row having zeros everywhere exept in positions m-k-1,...,m-2 and m+1 which contain ones. Let us denote this matrix by M'

$$\left[\begin{array}{c|c|c|c|c} M_1 & M_2 & \mathbb{1} & \mathbb{1} & 2 \\ & & & & \vdots \\ \hline M_3 & M_5 & \mathbb{1} & 0 & \\ \hline M_4 & M_6 & 0 & \mathbb{1} & \vdots \\ & & & & 2 \\ \hline & 1\ldots1 & 0 & 0 & 1 \end{array}\right] = M'$$

Since M is nonsingular the equation system $M\lambda = \underline{2}$ has a unique solution which, in fact, is nothing but a multiple of the normal vector $\bar{a}$ of the given facet, i. e. we obtain

$$\lambda_{ij} = \frac{2\bar{a}_{ij}}{\bar{\alpha}} \qquad \text{for all } ij \in E'$$

Suppose the last column of M' is dependent of the first m columns of M', then

$$\sum_{i\in N} \lambda_{v_1 i} = \sum_{i\in N} \frac{2\bar{a}_{v_1 i}}{\bar{\alpha}} = \frac{2\bar{a}_{v_1 v_3}}{\bar{\alpha}} = 1$$

has to hold. But this implies that $2\bar{a}_{v_1 v_3} = \bar{\alpha}$, hence the incidence vector of the bipartite edge set $\{v_1v_3, v_2v_3\}$ satisfies $\bar{a}^T x \leq \bar{\alpha}$ with equality which is impossible by our assumptions. Therefore we can conclude that M' is a nonsingular matrix.

Now we perform the following operations on M'. To each of the columns m-k-1,...,m-2, i. e. the columns corresponding to the edges $v_1 i$, $i \in N$, we add the column m-1 and the column m and subtract the column m+1. It is easy to see that the matrix M" we obtain this way looks as follows:

$$\left[\begin{array}{c|c|c|c|c} A_1 & A_2 & \mathbb{1} & \mathbb{1} & 2 \\ & & & & \vdots \\ \hline A_3 & A_5 & 1 & 0 & \\ \hline A_4 & A_6 & 0 & 1 & \vdots \\ & & & & 2 \\ \hline & 0 & & & 1 \end{array}\right] = M''$$

M'' is nonsingular, since M' is nonsingular. Deleting the last column and the last row from M'' we again obtain a nonsingular matrix M'''. Deleting the last two columns from M''' we obtain a matrix of full column rank m-2. But this matrix is the matrix A, and our proof is complete.

$\square$

To illustrate the methods of facet construction introduced before we now give an example. Let $[V,E_7]$ be the complete graph K_7, then $x(E_7) \leq 12$ defines a facet of $P_B(K_7)$ by Theorem (3.3). We pick the nodes 6 and 7 of K_7 and shrink them by node set contraction into a new node 6' to obtain the complete graph K_6. By Theorem (4.1) the inequality

$$2x(\delta_{K_6}(6')) + x(E(\{1,2,...,5\})) \leq 12$$

defines a facet of $P_B(K_6)$. $\delta_{K_6}(V_1)$ with $V_1 = \{4,5,6'\}$ is a bipartite edge set whose incidence vector satisfies the above inequality with equation. Choosing $F = \{46',56'\}$ we construct the following new graph G' (with 8 nodes) by splitting node 6' into v_1 and v_2 and adding a new node v_3 with the method described in Theorem (4.5) (a), cf. Figure 4.1.

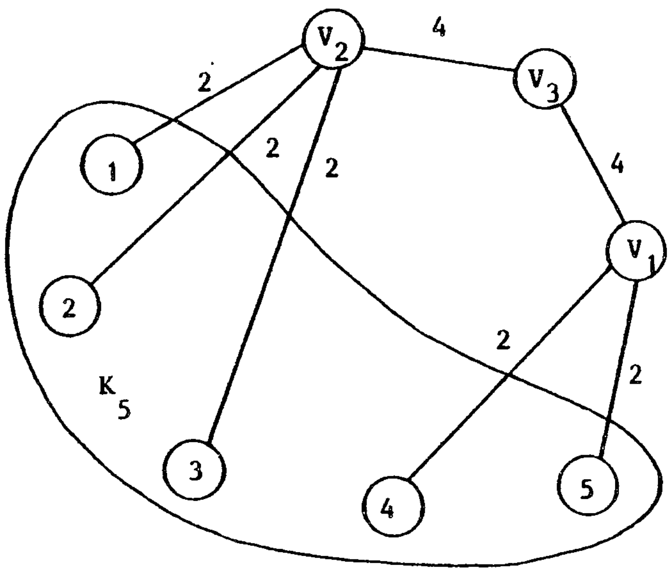


Figure 4.1

The inequality $\bar{a}^T x \leq 20$ defines a facet of $P_B(G')$ where the coefficients $\bar{a}_{ij}$ are equal to one for ij with $1 \leq i < j \leq 5$, and the other coefficients have the value indicated on the corresponding edges in Figure 4.1. Moreover, this inequality defines a facet of $P_B(H)$ for all graphs H containing G' as a subgraph.

5. Cut Subdivisions

We now describe how a cut of the support of a facet defining inequality can be subdivided to get a new facet. We start with a special case.

(5.1) Theorem. (Subdivision of a Star) *Let $G = [V,E]$ be a graph and $a^T x \le \alpha$ be a nontrivial facet defining inequality for $P_B(G)$. Let $v \in V$ be any node and suppose the star $\delta(v)$ of v consists of the edges $vi_1,\dots,vi_k$. Construct a new graph $G' = [V',E']$ from G as follows. Subdivide each edge of the star $\delta(v)$, i. e. add k new nodes $j_1,\dots,j_k$ and replace the edges $vi_1,vi_2,\dots,vi_k$ by the edges $vj_1,j_1i_1;vj_2,j_2i_2;\dots;vj_k,j_ki_k$. Keep all other nodes and edges, and moreover, any further, edge linking a new node j_s to any other node in V', and any edge vi_s, $s = 1,\dots,k$ may be added.*
Set

$$\bar{a}_{ij} := a_{ij} \qquad \text{for all } ij \in E \cap E'$$

$$\bar{a}_{vj_s} := \bar{a}_{j_s i_s} := a_{vi_s} \qquad \text{for } s = 1,\dots,k,$$

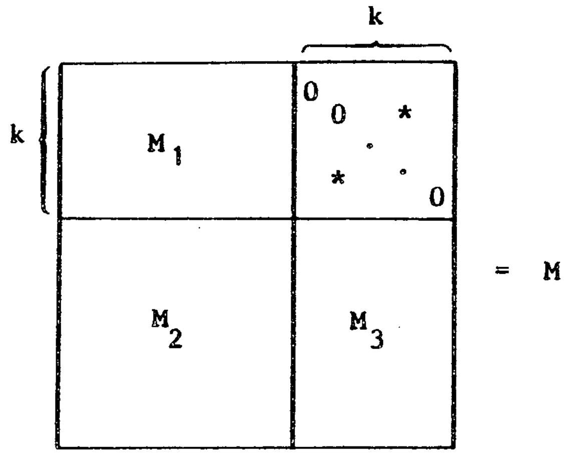
$$\bar{a}_{ij} := 0 \qquad \text{else}$$

$$\bar{\alpha} := \alpha + \sum_{s=1}^{k} a_{vi_s},$$

then $\bar{a}^T x \le \bar{\alpha}$ defines a facet of $P_B(G')$.

Proof. It should be clear that $\bar{a}^T x \le \bar{\alpha}$ is valid with respect to $P_B(G')$. Since $a^T x \le \alpha$ is a nontrivial facet of $P_B(G)$ there are bipartite edge sets $B_1,\dots,B_m \subseteq E$, $m := |E|$, whose incidence vectors are linearly independent and satisfy the inequality with equality.

We claim that there is an injective mapping $\varphi: \{1,\dots,k\} \to \{1,\dots,m\}$ such that $B_{\varphi(s)}$ does not contain the edge vi_s, $1 \le s \le k$, i. e. that we may assume that B_s does not contain vi_s, $1 \le s \le k$. Namely, first of all for each edge vi_s there is a set B_j not containing vi_s, otherwise $a^T x \le \alpha$ would define the (trivial) facet $\{x \in P_B(G) \mid x_{vi_s} = 1\}$. Secondly, if there were two edges vi_s, vi_r such that each set B_j not containing vi_s does not contain vi_r and vice versa, then the nonsingular (m,m)-Matrix M whose rows are the incidence vectors of the sets B_j would contain two identical columns, a contradiction. We may also assume that the last k columns of M correspond to the edges vi_s, $1 \leq s \leq k$, i. e. that M has the following form



where M_1, M_2, M_3 are some 0/1-matrices and the (k,k)-submatrix in the upper right hand corner is a 0/1-matrix M_4 with zeros on the main diagonal.

Let $m' := |E'|$. We now construct edge sets B'_t in G' as follows:

$$B'_t := (B_t \setminus \delta_G(v)) \cup \{vj_s \mid s=1,\dots,k\} \cup \{j_s i_s \mid vi_s \in B_t\} \quad t = 1,\dots,m,$$

$$B'_{m+s} := (B'_s \setminus \{vj_s\}) \cup \{j_s i_s\} \qquad s = 1,\dots,k,$$

and for the k', say, edges $e_{m+k+1},\dots,e_{m+k+k'}$ in G' for which one of the endnodes is one of the nodes j_s, $s \in \{1,\dots,k\}$, and the other endnode is a node in $V' \setminus \{j_s, i_s, v\}$, say $e_{m+k+t} = j_s u$, we define B'_{m+k+t}, $t = 1,\dots,k'$ to be the bipartite edge set among the two sets

$$B'_{m+s} \cup \{j_s u\} \quad \text{or} \quad B'_s \cup \{j_s u\}.$$

Moreover, for the (possibly existing) further $m'-(m+k+k')$ edges $e_{m+k+k'+1},\dots,e_{m'}$ in G' for which one of the endnodes is v and the other is one of the endnodes i_s, $s \in \{1,\dots k\}$, say $e_{m+k+k'+t} = vi_r$, we set

$$B'_{m+k+k'+t} := (B_{i_r} \setminus \delta_G(v)) \cup \{vj_s \mid s=1,\dots,k\} \cup \{j_s i_s \mid vi_s \in B_{i_r}\} \cup \{vi_r\}$$

where B_{i_r} is a bipartite edge set in G such that v and i_r are in the same set of the corresponding bipartition of V and such that $a^T x^{B_{i_r}} = \alpha$. Note that such a set B_{i_r} exists because $a^T x \leq \alpha$ is nontrivial.

It is clear from the construction that each of the sets B'_t, $t = 1,\ldots,m'$, is the edge set of a bipartite graph and that its incidence vector satisfies $\bar{a}^T x \le \bar{\alpha}$ with equality.

Consider now the (m',m')-matrix M' whose rows are the incidence vectors of the sets B'_t, $t = 1,\ldots,m'$. We may assume that the columns of M' corresponding to the edges in $E \cap E'$ are ordered in the same way as in M, that the k columns $m-k+1,\ldots,m$ of M' correspond to the edges $j_s i_s$, $s = 1,\ldots,k$, that the next k columns $m+1,\ldots,m+k$ correspond to the edges $vj_1,\ldots,vj_k$ and that the last $m'-(m+k)$ columns correspond to the further edges $e_{m+k+1},\ldots,e_{m'}$ of G'. Then M' looks as follows

$$\begin{array}{c|c|c|c|c|}
\multicolumn{1}{c}{} & \multicolumn{2}{c}{\overbrace{\hphantom{xxxxxxxxxxxxxxxx}}^{m}} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{2-5}
 & M_1 & \begin{smallmatrix}0 & & *\\ & \ddots & \\ * & & 0\end{smallmatrix} & & \\ \cline{2-3}
m & M_2 & M_3 & \mathbb{1} & 0 \\ \cline{2-5}
 & M_5 & \begin{smallmatrix}1 & & *\\ & \ddots & \\ * & & 1\end{smallmatrix} & \begin{smallmatrix}0 & & *\\ & \ddots & \\ * & & 0\end{smallmatrix} & 0 \\ \cline{2-5}
 & M_6 & M_7 & M_8 & I \\ \cline{2-5}
\end{array} \quad = \quad M'$$

where the principal (m,m)-submatrix is equal to M, M_5, M_6, M_7, M_8 are 0/1-matrices and the other matrices are of the indicated type. It is not too difficult to prove that M' is nonsingular. This finishes our proof.

$\square$

Using the above construction and Theorem (4.4) we get a more general method.

(5.2) <u>Theorem.</u>

(a) (Subdivision of a Cut) *Let $G = [V,E]$ be a graph and $a^T x \le \alpha$ be a nontrivial facet defining inequality for $P_B(G)$. Let $W \subseteq V$ and* *suppose that* $\delta(W) = \{w_1u_1,\ldots,w_ku_k\}$, *with* $w_i \in W$, $i = 1,\ldots,k$. *Construct a new graph* $G' = [V',E']$ *from* G *as follows. Subdivide each edge of* $\delta(W)$, *i. e. add* k *new nodes* $j_1,\ldots,j_k$ *and replace the edges* $w_1u_1, w_2u_2,\ldots,w_ku_k$ *by the edges* $w_1j_1, j_1u_1; w_2j_2, j_2u_2; \ldots; w_kj_k, j_ku_k$. *Keep all other nodes and edges, and moreover, any further edge linking a new node* j_s *to any other node* $v \in V' \setminus \{j_s,u_s,w_s\}$ *may be added as well as any of the edges* $w_1u_1,\ldots,w_ku_k$. *Set*

$$\bar{a}_{ij} := a_{ij} \qquad \text{for all } ij \in E \cap E',$$

$$\bar{a}_{w_sj_s} := \bar{a}_{u_sj_s} := a_{w_su_s} \qquad s = 1,\ldots,k,$$

$$\bar{a}_{ij} := 0 \qquad \text{otherwise},$$

$$\bar{\alpha} := \alpha + \sum_{s=1}^{k} a_{w_su_s},$$

then $\bar{a}^T x \leq \bar{\alpha}$ *defines a facet of* $P_B(G')$.

(b) (Contraction of a Cut) *Let* $G' = [V',E']$ *be a graph and* $\bar{a}^T x \leq \bar{\alpha}$ *be a facet of* $P_B(G')$. *Let* $W \subseteq V'$ *be a node set, and let* $W' \subseteq V' \setminus W$ *be the set of nodes which are linked to a node in* W *by an edge contained in the support* $F_{\bar{a}}$ *of* $\bar{a}$. *Suppose that* $E'(W') \cap F_{\bar{a}} = \emptyset$, *that* $(\delta_{G'}(W') \setminus \delta(W)) \cap F_{\bar{a}}$ *is a matching and that every node* $j \in W'$ *is contained in exactly two edges, say* w_jj, u_jj *where* $w_j \in W$ *and* $u_j \in V \setminus (W \cup W')$, *of the support of* $\bar{a}$ *which in addition satisfy* $\bar{a}_{u_jj} = \bar{a}_{w_jj}$. *Construct a new graph* $G = [V,E]$ *from* G' *as follows. Remove the node set* W'. *Remove all edges of* $\delta_{G'}(W)$. *Keep all other nodes and edges and add the edges* u_jw_j *for all* $j \in W'$. *Set*

$$a_{ij} := \bar{a}_{ij} \qquad \text{for all } ij \in E \cap E'$$

$$a_{u_jw_j} := \bar{a}_{u_jj} \qquad \text{for all } j \in W'$$

$$\alpha := \bar{\alpha} - \bar{a}(\delta(W))$$

then the inequality $a^T x \leq \alpha$ *defines a facet of* $P_B(G)$.

<u>Proof.</u> It is clear that the new inequalities defined in (a) and (b) are valid.

(a) For every node $w \in W$ we perform the subdivision of its cut $\delta(w)$ as described in Theorem (5.1) (and add the additional edges we want to have). This gives a facet defining inequality where each edge of the cut $\delta(w)$ is subdivided as described in the theorem and each edge $vw \in E(W)$ is replaced by a path of length three. Using Theorem (4.4) (b) we replace all these paths by the (original) edges vw finishing the proof.

(b) First we remove all edges ij from G' which have (at least) one endnode in W' and for which $\bar{a}_{ij} = 0$ to obtain a graph G'' where all nodes of W' have degree two. Clearly, the projected inequality $\bar{\bar{a}}^T x \leq \bar{\alpha}$ (leaving out the deleted coefficients) defines a facet of $P_B(G'')$. Now we apply part (a) of the Theorem to subdivide the cut $\delta(W)$. This way we obtain paths of length three where in each path all edges have the same positive $\bar{\bar{a}}$-coefficient. We replace these paths by a single edge using Theorem (4.4) (b) to obtain a facet defining inequality for $P_B(G)$.

$\square$

6. Examples and Final Remarks

The constructions defined in sections 4 and 5 can be used to obtain an incredibly large number of quite "wild" facets from the facets described in section 3. We shall discuss some examples.

Taking the K_{2k+1}-inequalities (3.3) and node splitting (4.5) we can generate facet defining inequalities for $P_B(G)$ where G is a graph of order 4k such that all integers 0,1,...,k occur as coefficients. For example, consider the graph K_7, i. e. k = 3. The inequality $x(E_7) \leq \alpha = 12$ defines a facet of $P_B(K_7)$ by (3.3). Now we pick a node, say 7, take any set of three edges incident to 7, say 47, 57, 67 as set F and apply the node splitting described in (4.5). We obtain the graph shown in Figure 6.1 (coefficients of $\bar{a}$ larger than one are written on the corresponding edges).

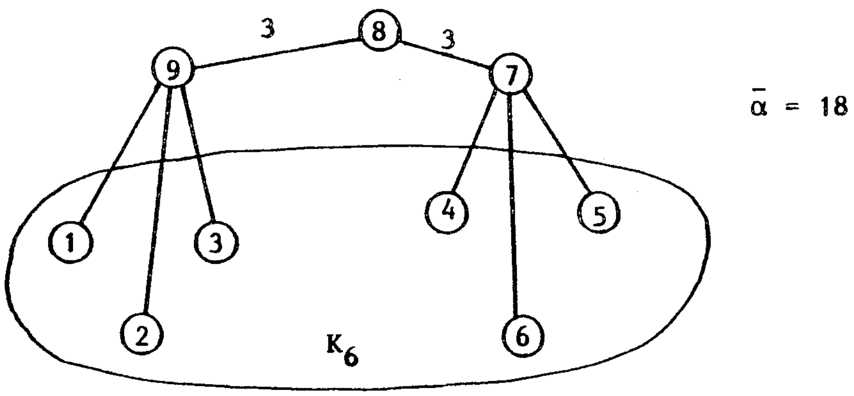


Figure 6.1

Now we split on node 9 taking 19, 29 as F to get the graph of Figure 6.2.

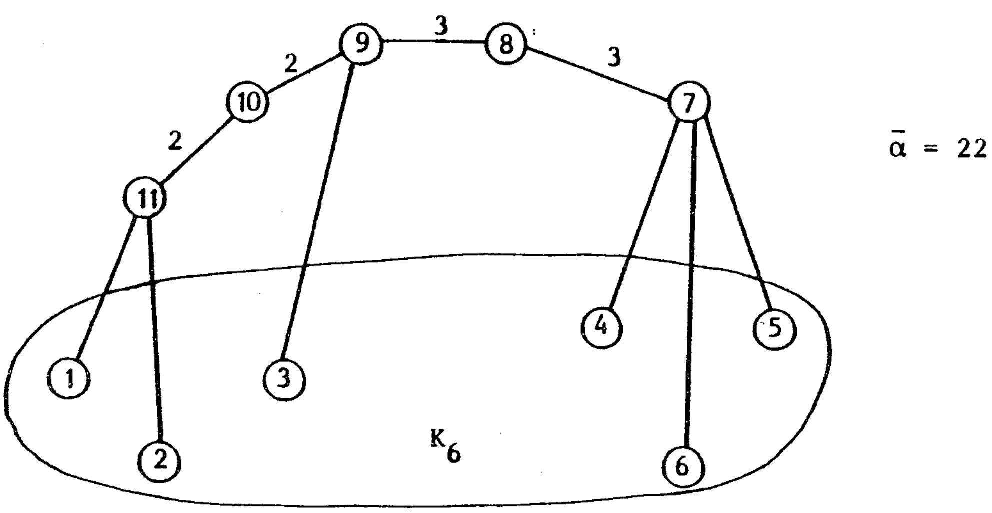


Figure 6.2

Weighting the edges of the graph G shown in Figure 6.2 as indicated we get a facet of $P_B(G)$ with coefficients 1,2,3 (zero coefficients can be obtained by adding further edges).

A graph - of interest in all parts of graph theory - is the Petersen graph P. We display a facet of $P_B(P)$ with coefficients 1 and 2 and right hand side 16 which is obtained by a metamorphosis of K_5 (showing this way that P is not weakly bipartite). We start from $x(E_5) \leq 6$ and then apply various constructions. We indicate the constructions on the arrows of Figure 6.3, a double line states that the coefficient of the edge in the corresponding inequality is two. The right hand sides, denoted by RS, of the inequalities corresponding to the graphs shown in Figure 6.3 are also indicated.

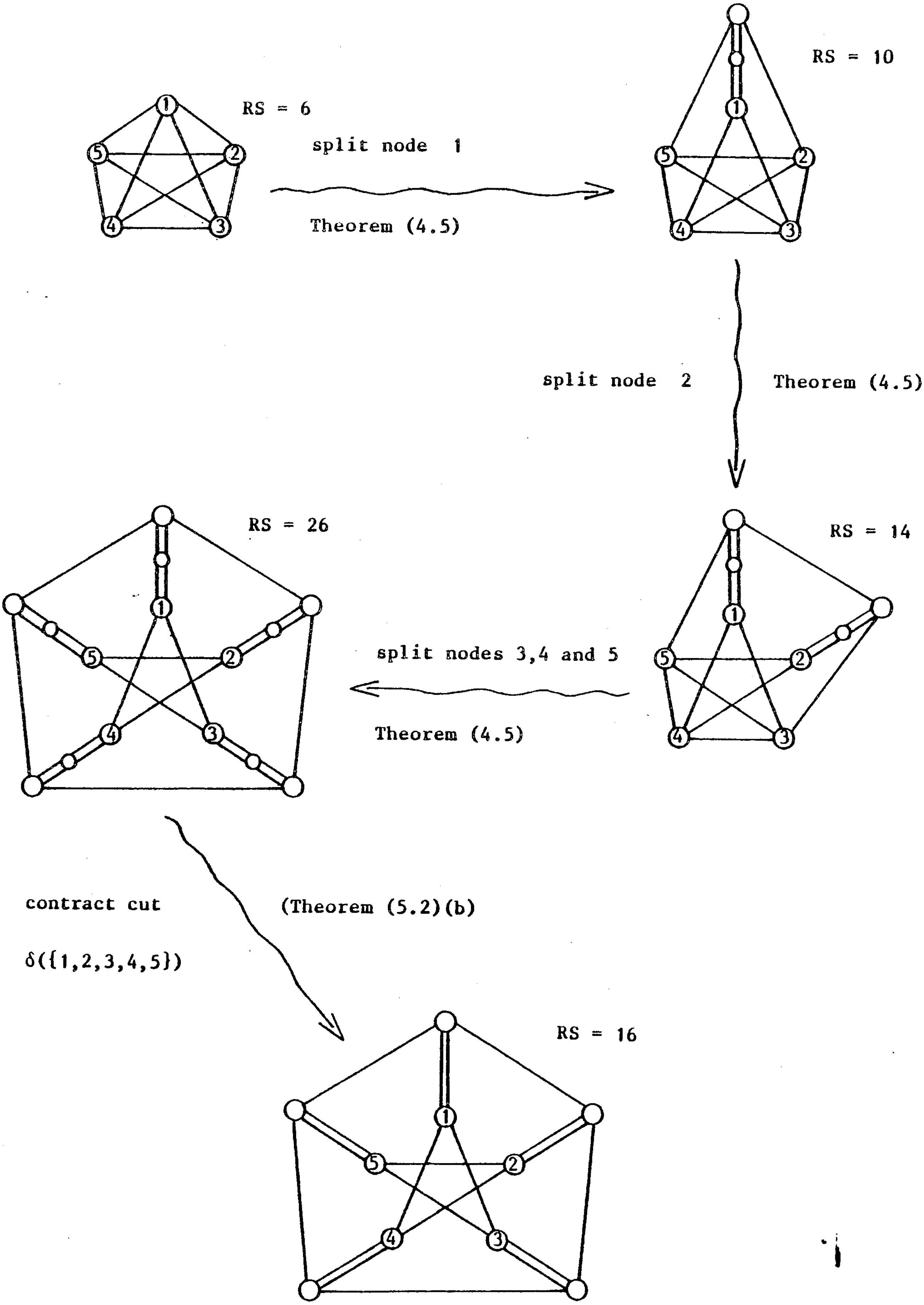

RS = 6
split node 1
Theorem (4.5)
RS = 10
split node 2
Theorem (4.5)
RS = 14
split nodes 3,4 and 5
Theorem (4.5)
RS = 26
contract cut
δ({1,2,3,4,5})
(Theorem (5.2)(b)
RS = 16

Figure 6.3

By repeated splitting of one of the universal nodes of a bicycle p-wheel we can construct graphs which have the property that they contain a node v (namely the remaining universal node) whose removal results in a planar graph where all nodes have degree at most three. Figure 6.4 shows such a construction for a bicycle 5-wheel. (The method, however, does not work for all bicycle wheels.)

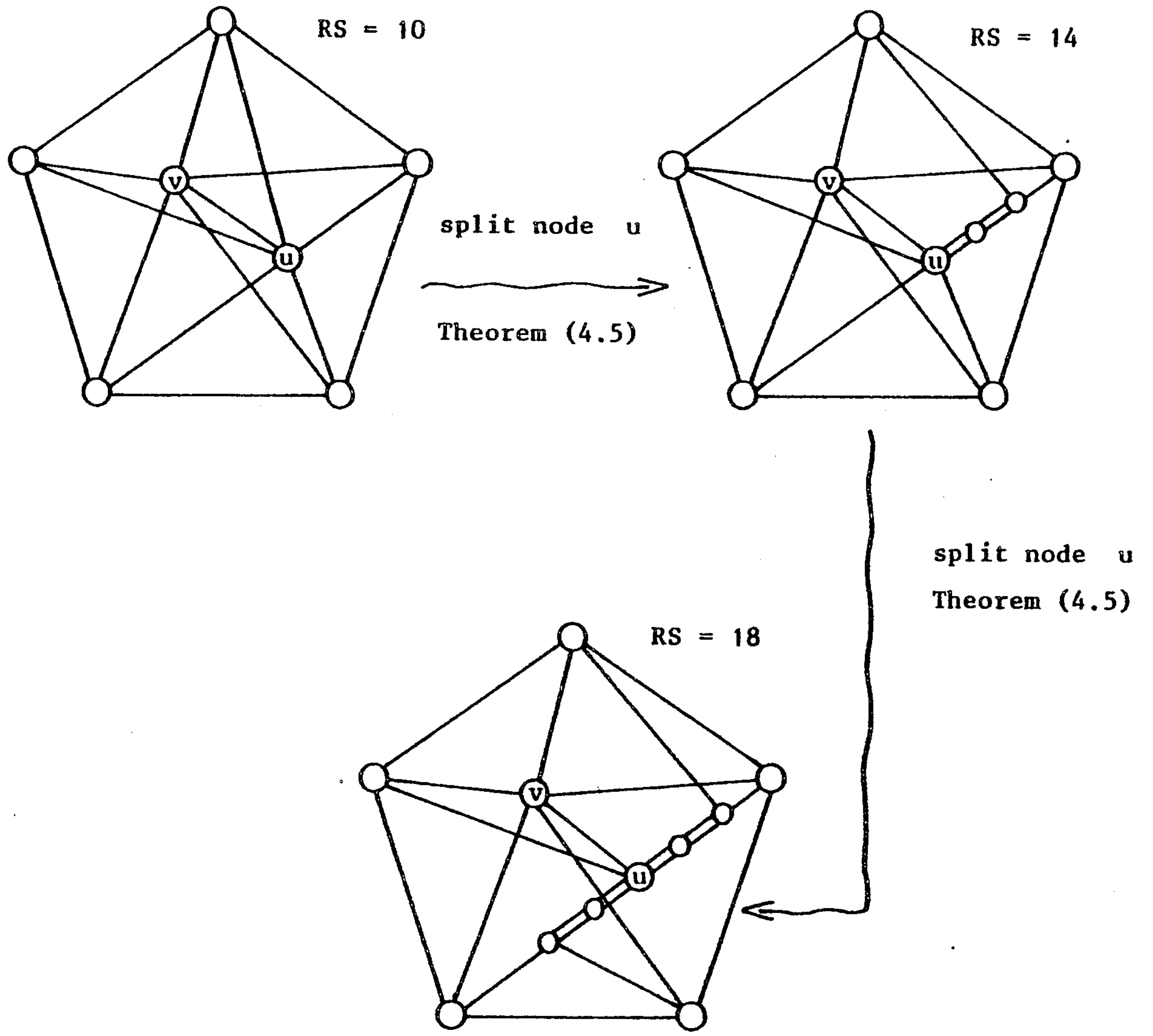


Figure 6.4

The final graph G of the sequence of splittings shown in Figure 6.4 has the property that G-v is planar and all nodes have degree at most three. The inequality derived from this graph (edges drawn with a double line get coefficient 2, all other coefficient 1) with right hand side 18 defines a facet of $P_B(G)$.

The algorithmic aspects of our theoretical analysis are still unexplored. We would like to know whether there are polynomial time algorithms for the separation problems involved with the classes of facets of $P_B(G)$ described here.

For instance, for s fixed, one can check whether all K_{2k+1}-inequalities and all bicycle (2k+1)-wheel inequalities, $2k+1 \leq s$, are satisfied by brute force enumeration in polynomial time. But even for K_5 this takes $O(n^5)$ time. We do not know whether there is a polynomial time algorithm for checking all odd complete subgraph inequalities or all odd bicycle wheel inequalities. (Note added in proof: See the paper of B. Gerards in this issue of MOR on this subject.)

It seems quite hopeless to find reasonable algorithms that solve the separation problems for the classes of facets obtained from the odd complete subgraph inequalities or odd bicycle wheel inequalities by any of the constructions described in sections 4 and 5.

Even the simplest case is open, the graphs obtained from K_5 by odd subdivision of edges (Theorem (4.5) (a)). Let us call these graphs *odd homeomorphs* of K_5. What one has to solve here is the following. Given a graph $G = [V,E]$ and a point $y \in \mathbb{Q}^E$, say $0 \leq y_e \leq 1$ for all $e \in E$. Check whether there is a subgraph $H = [W,F]$ of G obtained from K_5 by odd subdivision of edges such that the corresponding facet defining inequality of $P_B(G)$ is violated.

For each odd homeomorph $H = [W,F]$ of K_5 the corresponding inequality can be written as

$$x(F) \leq |F| - 4 \qquad (6.1)$$

see (4.4) (a) and (3.3). If y is given as above then we set $w_e := 1-y_e$ for all $e \in E$. If we can solve in polynomial time

$$\min \{w(F) \mid [V(F),F] \text{ odd homeomorph of } K_5\} \qquad (6.2)$$

then we can solve the separation problem for the inequalities (6.1) in polynomial time. Namely, if the minimum in (6.2) is 4 or larger then y satisfies all inequalities (6.1), otherwise for the minimum F*

$$y(F^*) > |F^*| - 4$$

holds and a violated inequality is found.

Of course, this question can be asked for the odd homeomorphs of K_7, K_9 etc. as well. We have the feeling that at least for the odd homeomorphs of K_5 a polynomial separation algorithms can be found.

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ANNEXE 4

Report No. 83271 - OR

ON THE CUT POLYTOPE

by

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Abstract

The Cut Polytope $P_C(G)$ of a graph $G = (V, E)$ is the convex hull of the incidence vectors of all edge sets of cuts of G. We show some classes of facet defining inequalities of $P_C(G)$. We describe three methods with which new facet defining inequalities of $P_C(G)$ can be constructed from known ones. In particular, we show that inequalities associated with chordless cycles define facets of this polytope, moreover, for these inequalities a polynomial algorithm to solve the separation problem is presented. We characterize the facet defining inequalities of $P_C(G)$ if G is not contractible to K_5. We give a simple characterization of adjacency in $P_C(G)$, we prove that for complete graphs this polytope has diameter one, and that $P_C(G)$ has the Hirsch property. A relationship between $P_C(G)$ and the convex hull of incidence vectors of balancing edge sets of a signed graph is studied.

Introduction and Notation

The graphs we consider are finite and undirected. We denote a graph by $G=(V,E)$, where V is the *node set* and E the *edge set* of G. The cardinality of V is called the *order* of G. If $e \in E$ is an edge with endnodes i and j we also write ij to denote the edge e. If $H=(W,F)$ is a graph with $W \subseteq V$ and $F \subseteq E$ then H is called a *subgraph* of G.

If $G=(V,E)$ is a graph and $F \subseteq E$, then $V(F)$ denotes the set of nodes of V which occur at least once as an endnode of an edge in F. Similarly, for $W \subseteq V$, $E(W)$ denotes the set of all edges of G with both endnodes in W.

A graph G is called *complete* if every two different nodes of G are linked by an edge. The complete graph with n nodes is denoted by K_n. A graph is called *bipartite* if its node set can be partitioned into two nonempty, disjoint sets V_1 and V_2 such that no two nodes in V_1 and no two nodes in V_2 are linked by an edge. We call V_1, V_2 a *bipartition* of V. If $|V_1|=p$, $|V_2|=q$ and G is a maximal bipartite graph, it is denoted by $K_{p,q}$. If $W \subseteq V$ then $\delta(W)$ is the set of edges with one endnode in W and the other in $V \setminus W$. The edge set $\delta(W)$ is called a *cut*. We write $\delta(v)$ instead of $\delta(\{v\})$ for $v \in V$ and call $\delta(v)$ the *star* of v.

If U, W are disjoint subsets of V, then $[U : W]$ denotes the set of edges of G which have one endnode in U and the other endnode in W. We write $[u : W]$ instead of $[\{u\} : W]$ for $u \in V$.

A *path* P in $G=(V,E)$ is a sequence of edges $e_1, e_2, \ldots, e_k$ such that $e_1 = v_0v_1$, $e_2 = v_1v_2, \ldots, e_k = v_{k-1}v_k$ and such that $v_i \neq v_j$ for $i \neq j$. The nodes v_0 and v_k are the endnodes of P and we say that P links v_0 and v_k or goes from v_0 to v_k. If $P = e_1, e_2, \ldots, e_k$ is a path linking v_0 and v_k and $e_{k+1} = v_0v_k \in E$ then the sequence $e_1, e_2, \ldots, e_k, e_{k+1}$ is called a *cycle*.
If P is a cycle an uv and edge of $E \setminus P$ with $u, v \in V(P)$, the uv is called a *chord* of P. A cycle with three edges is called a *triangle*.

If v is a node of a graph G, then $G \setminus v$ denotes the subgraph of G obtained by removing node v and all edges incident to v from G.

A graph G is *contractible* to G' if G' can be obtained from G by a sequence of elementary contractions, in which a pair of adjacent vertices are identified and all other adjacencies between vertices are preserved (multiple edges arising from the identification being replaced by single edges).

A *polyhedron* $P \subseteq \mathbb{R}^m$ is the intersection of finitely many halfspaces in $\mathbb{R}^m$. A *polytope* is a bounded polyhedron or equivalently the convex hull of finitely many points. The dimension of a polyhedron P, denoted by dim P, is the maximum number of affinely independent points in P minus one.

If $a \in \mathbb{R}^m \setminus \{0\}$, $a_0 \in \mathbb{R}$, then the inequality $a^Tx \leq a_0$ is said to be *valid* with respect to a polyhedron $P \subseteq \mathbb{R}^m$ if $P \subseteq \{x \in \mathbb{R}^m \mid a^T x \leq a_0\}$. We say that a valid inequality $a^T x \leq a_0$ *supports* P or defines a *face* of P if $\emptyset \neq P \cap \{x \mid a^T x = a_0\} \neq P$. A valid inequality $a^T x \leq a_0$ defines a *facet* of P if it defines a face of P and if there exist dim P affinely independent points in $P \cap \{a^T x = a_0\}$.

If $P \subseteq \mathbb{R}^m$ is a full dimensional polyhedron, i.e. dim $P = m$, a linear system $Ax \leq b$ that defines P is minimal if and only if there is a bijection between the inequalities of the system and the facets of P. Moreover, these facet inducing inequalities are unique up to positive multiples.

Given $F \subseteq E$ the incidence vector of F, x^F is defined by

$$x^F(e) = \begin{cases} 1, & \text{if } e \in F \\ 0, & \text{if } e \in E \setminus F \end{cases}$$

Given $b : E \to \mathbb{R}$, and $F \subseteq E, b(F)$ will denote $\sum_{e \in F} b(e)$. The support of b, E_b will be $E_b = \{e \mid b(e) \neq 0\}$.

The cut polytope $P_C(G)$ of a graph $G = (V, E)$ is the convex hull of incidence vectors of all edge sets of cuts of G. The bipartite subgraph polytope $P_B(G)$ is defined in an analogous way. It is clear that $P_C(G) \subseteq P_B(G)$, but in general $P_C(G) \neq P_B(G)$. Moreover, given a weighting function $w : E \to \mathbb{R}$, a maximum weighted cut could be different from a maximum weighted bipartite subgraph if $w(e) < 0$ for some edges $e \in E$.

In Barahona, Grötschel and Mahjoub [2] it has been shown that $P_C(G)$ is full dimensional, moreover some of the facets defining inequalities of $P_B(G)$ there studied, are also facet defining inequalities of $P_C(G)$.

We will present methods to derive new facet defining inequalities of $P_C(G)$ from known ones. In particular, we are looking for facet defining inequalities of $P_C(G)$ that are not valid for $P_B(G)$. We shall present a polynomial algortihm to solve the separation problem for inequalities associated with cycles. Moreover, we prove that these inequalities and inequalities associated with edges suffice to define $P_C(G)$ if and only if G is not contractible to K_5. We give a simple answer to the question when are two extreme points adjacent in $P_C(G)$. This characterization permits us to derive upper bounds for the diameter of this polytope,in particular, we prove that if G is complete then $P_C(G)$ has diameter one. We will use our results to study the polytope of balancing edge sets of a signed graph.

2. Construction of Facets.

First, we will state three theorems that characterize some facet defining inequalities of $P_C(G)$.we will omit the proofs because they are analogous to those that appear in [2].

Theorem 2.1 *Let $G = (V, E)$ be a graph and let (W, F) be a complete subgraph of order $p \geq 3$ of G. Then*

$$x(E(W)) \leq \lceil \frac{p}{2} \rceil \lfloor \frac{p}{2} \rfloor \tag{2.2}$$

is a valid inequality with respect to $P_C(G)$. (2.2) defines a facet of $P_C(G)$ if and only if p is odd.

□

A graph is called a bicycle p-wheel if G consists of a cycle of length p and two nodes which are adjacent to each other and to every node in the cycle.

Theorem 2.3

Let $G = (V, E)$ be a graph and let (W, F) be a bicycle $(2k+1)$-wheel, $k \geq 1$, contained in G. Then the inequality

$$x(F) \leq 2(2k+1)$$

defines a facet of $P_C(G)$.

□

Theorem 2.4 *(Node Set Contraction) Let $(W, E(W))$ be a complete subgraph of order $2k+1$, $k \geq 3$, of a graph $G = (V, E)$ and let $W_1, W_2, \ldots, W_s$ be a partition of W into nonempty mutually disjoint subsets such that*

$$\sum_{\substack{i=1 \\ |W_i|>1}}^{s} |W_i| \leq k-1.$$

Let $G' = (V', E')$ denote the graph obtained from G by shrinking the nodes in W_i into a single node $w_i, i = 1, \ldots, s$ (i.e. all new nodes w_i are adjacent in G', and a new node w_i is adjacent in G' to an old node $v \in V \setminus W$ iff W_i contains a node adjacent to v in G). Set

$$\begin{aligned} a_{w_i w_j} &:= |W_i||W_j| && \text{for all } 1 \leq i < j \leq s \\ a_{vw} &:= 0 && \text{for all other } vw \in E' \end{aligned}$$

then

$$a^T x \leq \alpha := k(k+1)$$

defines a facet of $P_C(G')$.

To simplify technical details in subsequent proofs we first state a lemma.

Lemma 2.5 *Let $b^T x \leq \beta$ be a valid inequality with respect to $P_C(G)$. Given* adjacent *nodes p and q let S be a subset of $V \setminus \{p,q\}$ and $T = V \setminus (S \cup \{p,q\})$. Suppose that the incidence vectors of the edge sets $\delta(S), \delta(T), \delta(S \cup \{p\}), \delta(S \cup \{q\})$ satisfy $b^T x \leq \beta$ with equality. Then*

$$b_{pq} = 0.$$

Proof. $0 = \beta - \beta = b^T x^{\delta(T)} - b^T x^{\delta(S \cup \{p\})} = b([q:T]) - b([q:S]) - b_{pq}$, and $0 = \beta - \beta = b^T x^{\delta(S)} - b^T x^{\delta(S \cup \{q\})} = b([q:S]) - b([q:T]) - b_{pq}$. Thus, summing the two equations we obtain

$$-2b_{pq} = 0.$$

$\square$

Now we shall describe three methods to construct "facets" from facets".

Theorem 2.6

(a) (Node Splitting) Let $G = (V,E)$ be graph and $a^T x \leq \alpha$ be a facet defining inequality for $P_C(G)$. Let E_a be the support of a, and let v be a node in $V(E_a)$. Let W be a subset of $V(E_a)$ such that $a^T x^{\delta(W)} = \alpha$ and assume that $v \in W$. Choose any nonempty subset $F \subseteq \delta(v) \cap E(W)$ such that $a_e > 0$ for $e \in F$, and construct a new graph $G' = (V',E')$ from G as follows. Split node v into two nodes v_1, v_2 such that v_1 is incident to all edges contained in F and v_2 is incident to all edges in $\delta(v) \setminus F$. The edge $v_1 v_2$ is added, in addition any further edges $v_1 u$ with $u \notin V(E_a)$ may be added. The other parts of G remain unchanged. Set

$$\begin{aligned}
\bar{a}_{ij} &:= a_{ij} && \text{for all } ij \in E \setminus \delta(v)\\
\bar{a}_{v_1 u} &:= a_{vu} && \text{for all } v_1 u \in E' \text{ with } vu \in F,\\
\bar{a}_{v_2 u} &:= a_{vu} && \text{for all } v_2 u \in E' \text{ with } vu \in (\delta(v) \cap E_a) \setminus F,\\
\bar{a}_{v_1 v_2} &:= -a(F)\\
\bar{a}_{ij} &:= 0 && \text{otherwise}\\
\bar{\alpha} &:= \alpha
\end{aligned}$$

then $\bar{a}^T x \leq \bar{\alpha}$ defines a facet of $P_C(G')$.

(b) (Contraction of an edge) Let $G' = (V',E')$ be a graph and $\bar{a}^T x \leq \bar{\alpha}$ be an inequality defining a facet of $P_C(G')$. Suppose that $v_1 v_2 \in E'_{\bar{a}}$, that v_1 and v_2 have no common neighbour in $(V', E_{\bar{a}})$, that $a_{v_1 u} \geq 0$ for $v_1 u \in \delta(v_1) \setminus \{v_1 v_2\}$ and that $-\bar{a}_{v_1 v_2} = \bar{a}(\delta(v_1) \setminus \{v_1 v_2\}) \geq \bar{a}(\delta(v_2) \setminus \{v_1 v_2\})$. Let $G = (V,E)$ be the graph obtained from G' by removing the nodes v_1, v_2, adding a new node v and the edges $\{vj \mid \bar{a}_{v_1 j} > 0\} \cup \{vj \mid v_2 j \in E'\}$. Set

$$\begin{aligned}
a_{ij} &:= \bar{a}_{ij} && \text{for all } ij \in E \cap E'\\
a_{vj} &:= \bar{a}_{v_1 j} && \text{if } \bar{a}_{v_1 j} > 0\\
a_{vj} &:= \bar{a}_{v_2 j} && \text{if } v_2 j \in E' \text{ unless } \bar{a}_{v_1 j} > 0\\
\alpha &:= \bar{\alpha}
\end{aligned}$$

then $a^T x \leq \alpha$ defines a facet of $P_C(G)$.

<u>Proof.</u> The validity of the new inequalities defined in (a) and (b) follows by elementary construction. Then, let us assume that there is a facet defining inequality $b^T x \leq \beta$ for $P_C(G)$ which has the following property. If a vector $x \in P_C(G)$ satisfies $\bar{a}^T x = \alpha$ then x also satisfies $b^T x = \beta$. If we can prove that $\bar{a} = \rho b$ for some $\rho > 0$, then we can conclude that $\bar{a}^T x \leq \alpha$ is equivalent to $b^T x \leq \beta$, i.e. $a^T x \leq \alpha$ defines a facet of $P_C(G)$.

(a) First, we will show that for $u \notin V(E_{\bar{a}})$ $b_{uv_1} = 0$. To prove this apply lemma (2.5) with $S := (W \setminus \{v\}) \cup \{v_2\}$, $p := v_1$, $q := u$.

Since $a^T x \leq \alpha$ defines a facet of $P_C(G)$ there are $m = |E|$ cuts $\delta(U_1), \ldots, \delta(U_m) \subseteq E$ whose incidence vectors are affinely independent and satisfy $a^T x^{\delta(U_i)} = \alpha$, $i = 1, \ldots, m$. Set

$$U'_i := \begin{cases} U_i, & \text{if } v \notin U_i \\ (U_i \setminus \{v\}) \cup \{v_1, v_2\}, & \text{if } v \in U_i. \end{cases}$$

Since $\bar{a}^T x^{\delta(U_i')} = \alpha$, then $b^T x^{\delta(U_i')} = \beta$, $i = 1, \ldots, m$. The vectors $x^{\delta(U_i')}$, $i = 1, \ldots, m$ are affinely independent, then we can conclude that $b_{uw} = \rho \bar{a}_{uw}$ for $uw \neq v_1 v_2$. Let W' be $(W \setminus \{v\}) \cup \{v_1, v_2\}$, and $W'' = W' \setminus \{v_1\}$. Since $\bar{a}^T x^{\delta(W')} = \bar{a}^T x^{\delta(W'')} = \alpha$, we have that

$$\begin{aligned} b^T x^{\delta(W')} &= b^T x^{\delta(W'')} = \beta, \quad \text{then} \\ 0 = b^T x^{\delta(W')} - b^T x^{\delta(W'')} &= -b_{v_1 v_2} - b(F), \quad \text{hence} \\ b_{v_1 v_2} &= -b(F) = -\rho \bar{a}(F) = \rho \bar{a}_{v_1 v_2}. \end{aligned}$$

$b^T x \leq \beta$ is valid, thus $\rho > 0$.

(b) By assumption $\bar{a}^T x \leq \bar{a}$ defines a facet of $P_C(G)$, so there are $m := |E'|$ edge sets $\delta(U_1'), \ldots, \delta(U_m')$ whose incidence vectors are affinely independent and satisfy $\bar{a}^T x^{\delta(U_i')} = \bar{a}$, $i = 1, \ldots, m$. We may assume that $N \subseteq V'$ is the set of neighbours of v_1 in $(V', E_{\bar{a}})$ different from v_2, and that $k := |N|$.

Now let M be the (m,m)-matrix whose rows are the incidence vectors $x^{\delta(U_1')}, \ldots, x^{\delta(U_m')}$. We may assume that the last $k+1$ columns correspond to the edges $v_1 i$, $i \in N$ and $v_1 v_2$. Moreover, we assume that the set $\delta(U_1'), \ldots, \delta(U_m')$ are ordered in such a way that only the sets $\delta(U_1'), \ldots, \delta(U_r')$ contain the edge $v_1 v_2$, and that $v_2 \in U_i'$, $i = 1, \ldots, m$.

Note that the assumptions of the theorem imply that if U_i' does not contain v_1 then it necessarily contains all nodes $i \in N$, otherwise $U'' = U_i' \cup \{v_1\}$ would define a cut $\delta(U'')$ such that $\bar{a}^T x^{\delta(U'')} > \alpha$. Thus, our matrix M looks as follows

$$M = \begin{array}{cccc|c} m-t-k & t-1 & k & 1 & \\ \left[\begin{array}{c|c|c|c} M_1 & M_2 & M_3 & \mathbb{1} \\ \hline M_4 & M_5 & M_6 & 0 \end{array}\right] & & & & \begin{array}{l} \}\, r \\ \}\, m-r \end{array} \end{array}$$

Where M_3 contains only ones and columns s, $m-k-t < s < m-k$, correspond to edges in $\delta(v_1) \setminus E'_{\bar{a}}$ and $v_2 j \in E'$ for which $\bar{a}_{v_{1j}} > 0$.

Now we transform the sets $U_i' \subseteq V'$ into sets $U_i \subseteq V$, $i = 1,\ldots,m$ as follows.

$$U_i = (U_i' \setminus \{v_1, v_2\}) \cup \{v\}.$$

This transformation corresponds to contracting the edge $v_1 v_2$. It follows from our remarks above that $a^T x^{\delta(U_i)} = \alpha$ for $i = 1,\ldots,m$. If A is the $(m, m-1)$-matrix whose rows are the incidence vectors $x^{\delta(U_i)}$ $i = 1,\ldots,m$ then A looks as follows

$$A = \begin{array}{cc|c} m-k-t & k & \\ \left[\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \end{array}\right] & & \begin{array}{l} \}\, r \\ \}\, m-r \end{array} \end{array}$$

where

$$A_1 = M_1$$
$$A_3 = M_4$$
$$A_4 = M_6$$

and A_2 contains zeros only.

To obtain A we can perform the following operations on M.
Subtract the last column from the columns $m-k, \ldots, m-1$. Then, delete the last column and columns s such that $m-k-t < s < m-k$. It is clear that the rows of this matrix have affine rank $m-t$ and our proof is complete.

$\square$

Theorem 2.7 *(Replacing a node by a triangle) Let $G = (V, E)$ be a graph and $a^T x \leq \alpha$ be a facet defining inequality for $P_C(G)$. Let v be a node in $V(E_a)$ such that $a_e \geq 0$ for each $e \in \delta(v)$. Let F_1, F_2, F_3 be a partition of $\delta(v)$ and assume that there exist $W_1, W_2, W_3 \subseteq V$ such that $a^T x^{\delta(W_i)} = \alpha$ and $F_i \subseteq E(W_i)$ for $i = 1,2,3$. (W_i may coincide with W_j for $i \neq j$). Construct a new graph $G' = (V', E')$ from G as follows. Replace v by v_1, v_2, v_3 such that v_i is incident to all edges contained in F_i, $i = 1,2,3$. Add edges v_1v_2, v_1v_3 and v_2v_3, in addition any further edge v_iu, $i = 1,2,3$, with $u \notin V(E_a)$ may be added. The other parts of G remain unchanged. Set*

$$
\begin{aligned}
\bar{a}_{ij} &:= a_{ij} && \text{for all } ij \in E_a \setminus \delta(v) \\
\bar{a}_{v_iu} &:= a_{vu} && \text{for all } v_iu \in E' \\
& && \quad \text{with } v_iu \in F_i, i = 1,2,3 \\
\bar{a}_{v_1v_2} &:= \frac{-a(F_1) - a(F_2) + a(F_3)}{2} \\
\bar{a}_{v_1v_3} &:= \frac{-a(F_1) - a(F_3) + a(F_2)}{2} \\
\bar{a}_{v_2v_3} &:= \frac{-a(F_2) - a(F_3) + a(F_1)}{2} \\
\bar{a}_{ij} &:= 0 && \text{otherwise} \\
\bar{\alpha} &:= \alpha
\end{aligned}
$$

then $\bar{a}^T x \leq \bar{\alpha}$ defines a facet of $P_C(G')$.

□

The proof of this Theorem is analogous to the proof of Theorem 2.6(a).We leave it to the reader.

To illustrate these constructions we will give some examples of facet defining inequalities of $P_C(G)$ that are not valid for $P_B(G)$.
Let $G = (V, E)$ be the complete graph K_7, then $X(E) \leq 12$ defines a facet of $P_C(G)$, by Theorem 2.1. Choosing $F = \{71, 72, 73\}$ we split the node 7 into v_1 and v_2 and we obtain a graph $G' = (V', E')$, cf. Figure 2.1 .

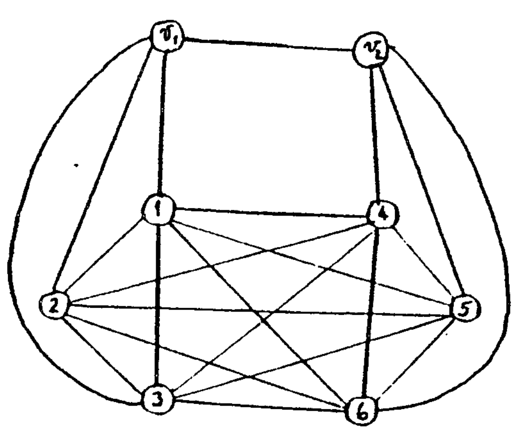

Figure 2.1

The inequality

$$x(E' \setminus \{v_1 v_2\}) - 3x(v_1 v_2) \leq 12$$

defines a facet of $P_C(G)$.

By splitting some nodes of a bicycle 5-wheel as it is shown in Figure 2.2 we obtain the graph of Figure 2.3 .

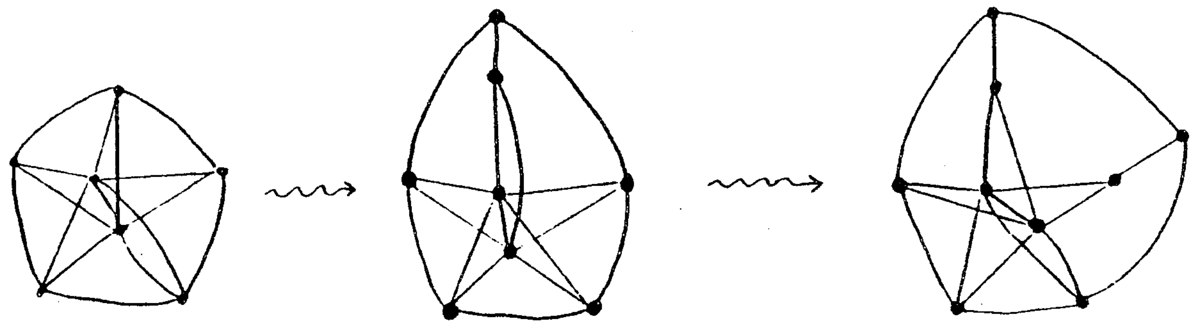

Figure 2.2

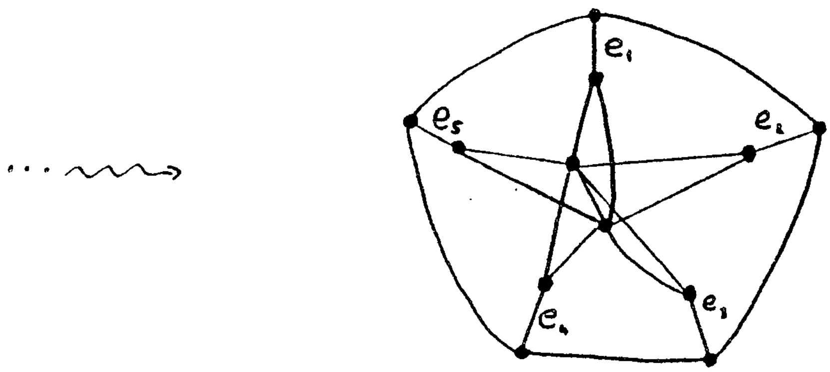

Figure 2.3

Let $G = (V, E)$ be this graph and $E' = \{e_1, \ldots, e_5\}$, the inequality

$$x(E \setminus E') - 2x(E') \leq 10$$

defines a facet of $P_C(G)$.

Let $G = (V, E)$ be the complete graph K_7. Replacing the node 7 by a triangle we can obtain the graph $G' = (V', E')$ in Figure 2.4

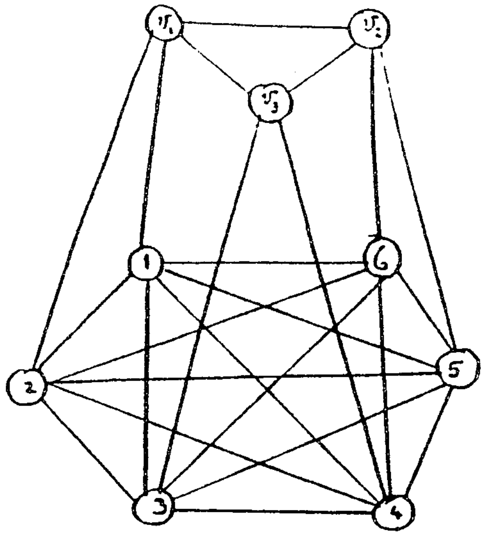

Figure 2.4

Since $x(E) \leq 12$ defines a facet of $P_C(G)$, by Theorem 2.7, the inequality

$$x(E \setminus \{v_1v_2, v_2v_3, v_1v_3\}) - x(\{v_1v_2, v_2v_3, v_1v_3\}) \leq 12$$

defines a facet of $P_C(G')$.

The third method for obtaining "facets from facets" is the following

Theorem 2.8 *(Changing the signs of a Star) Let $G = (V, E)$ be a graph and $a^Tx \leq \alpha$ be a facet defining inequality for $P_C(G)$. For any $v \in V$ the inequality*

$$-\sum_{e \in \delta(v)} a_e x(e) + \sum_{e \notin \delta(v)} a_e x(e) \leq \alpha - a(\delta(v)) \tag{2.8}$$

defines a facet for $P_C(G)$.

Proof. Let us denote $\bar{a}^Tx \leq \bar{\alpha}$ inequality (2.8) . $\bar{a}^Tx \leq \bar{\alpha}$ is valid for $P_C(G)$, otherwise it exists $U \subseteq V$ with $v \in U$ such that

$$\bar{a}^T x^{\delta(U)} > \bar{\alpha} .$$

But this implies that for $U' = U \setminus \{v\}$

$$a^T x^{\delta(U')} > \alpha .$$

Since $a^Tx \leq \alpha$ defines a facet, there are $m = |E|$ sets U_i, $i = 1, \ldots, m$ such that $a^Tx^{\delta(U_i)} = \alpha$, $v \in U_i$, $i = 1, \ldots, m$ and the vectors $x^{\delta(U_1)}, \ldots, x^{\delta(U_m)}$ are affinely independent. Set $U_i' = U_i \setminus \{v\}$, then

$$x^{\delta(U_i')}(e) = \begin{cases} x^{\delta(U_i)}(e), & \text{if } e \notin \delta(v) \\ 1 - x^{\delta(U_i)}(e), & \text{if } e \in \delta(v) \end{cases}$$

Thus the vectors $x^{\delta(U_1')},\ldots,x^{\delta(U_m')}$ are affinely independent and satisfy $\bar{a}x \leq \bar{\alpha}$ with equality.

□

Theorems 2.6 and 2.8 can be combined to give new procedures of construction of facet defining inequalities. The following corollaries illustrate this fact.

Corollary 2.9 *Let $G = (V,E)$ be a graph and $a^T x \leq \alpha$ be a facet defining inequality for $P_C(G)$. Let $W \subseteq V$, and set*

$$\begin{aligned} \bar{a}_{ij} &:= a_{ij} && \text{for } ij \in E \setminus \delta(W), \\ \bar{a}_{ij} &:= -a_{ij} && \text{for } ij \in \delta(W), \\ \bar{\alpha} &= \alpha - a(\delta(W)), \end{aligned}$$

then $\bar{a}^T x \leq \bar{\alpha}$ defines a facet of $P_C(G)$.

Proof. Using Theorem 2.8 we change the signs of the coefficients associated with $\delta(v)$ for each $v \in W$.

□

Corollary 2.10

(a) (Subdivision of an edge.). Let $G = (V,E)$ be a graph and $a^T x \leq \alpha$ be a facet defining inequality for $P_C(G)$. Let $ij \in E$ be an edge with $a_{ij} \neq 0$. Let $G' = (V',E')$ be a graph obtained from G in the following way. Nodes $i = i_1,\ldots,i_k$ are added. The edge set $P = ii_1, i_1i_2,\ldots,i_{k-1}i_k, i_kj$ is added. Any further edge $i_\ell u$ with $u \notin V(E_a)$ may be added. The edge ij is removed. Let P^+, P^- define a partition of P with $|P^+|$ odd if $a_{ij} > 0$, and $|P^-|$ even if $a_{ij} < 0$. Let $\bar{a} \in \mathbb{R}^{|E'|}$ be defined as follows

$$\begin{aligned} \bar{a}_{uv} &= a_{uv} && \text{for all } uv \in E \cap E' \\ \bar{a}_{uv} &= a_{ij} && \text{for all } uv \in P^+ \\ \bar{a}_{uv} &= -a_{ij} && \text{for all } uv \in P^- \\ \bar{a}_{uv} &= 0 \text{ otherwise} \end{aligned}$$

Then

$$\begin{aligned} \bar{a}^T x &\leq \alpha + (|P^+| - 1)\, a_{ij} && \text{if } a_{ij} > 0 \\ \bar{a}^T x &\leq \alpha - |P^-| a_{ij} && \text{if } a_{ij} < 0 \end{aligned}$$

defines a facet of $P_C(G')$.

(b) (Replacing a path by an edge). Let $G' = (V',E')$ be a graph and $\bar{a}^T x \leq \bar{\alpha}$ be a facet defining inequality for $P_C(G')$. Let $E_{\bar{a}}$ be the support of $\bar{a}^T x \leq \bar{\alpha}$. Suppose that $E_{\bar{a}}$ contains a path $P = \{ii_1, i_2i_3,\ldots,i_kj\}$ such that

(i) i_l *has degree two in* $E_{\bar{a}}$, *for* $l = 1,\ldots,k$.

(ii) $ij \notin E_{\bar{a}}$.

(iii) $\bar{a}_{uv} = \alpha'$ *for* $uv \in P^+ \subseteq P$, $\bar{a}_{uv} = -\alpha'$ *for* $uv \in P^- = P \setminus P^+$, *with* $|P^+|$ *odd if* $\alpha' > 0$ *and* $|P^-|$ *even if* $\alpha' < 0$.

Let $G = (V, E)$ be the graph obtained from G' be removing the nodes $i_1, \ldots, i_k$ and adding the edge ij. Let $a \in \mathbb{R}^{|E|}$ be defined as follows:

$$a_{uv} = \bar{a}_{uv} \qquad \text{for all } uv \in E \cap E'$$
$$a_{ij} = \alpha'$$

Then

$$a^T x \leq \alpha - (|P^+| - 1)\,\alpha' \qquad \text{if } \alpha' > 0$$
$$a^T x \leq \alpha + |P^-|\,\alpha' \qquad \text{if } \alpha' < 0$$

defines a facet of $P_C(G)$.

□

The proof of Corollary 2.10 follows by applying Theorems 2.6 and 2.8 repeatedly.

Corollary 2.11 *For any pair of cuts, C and D, there is a one-to-one correspondence between the facets adjacent to x^C and to x^D.*

<u>Proof.</u> Let $ax \leq \alpha$ be a facet defining inequality such that $ax^C = \alpha$, If we apply Corollary 2.9 with $\delta(W) = C \triangle D$, we obtain a facet defining inequality $bx \leq \beta$, such that $bx^D = \beta$.

□

Let us define

$$CONE(P_C(G)) = \{y \mid y = \lambda x\,,\ x \in P_C(G), \lambda \in \mathbb{R}_+\}.$$

If $P_C(G) = \{x \mid Ax \geq b\,,\ Ex \geq 0\}$, $b < 0$, then $CONE(P_C(G)) = \{x \mid Ex \geq 0\}$.

Corollary 2.11 shows that a set of inequalities defining $P_C(G)$ can be obtained from a set of inequalities defining the facets adjacent to one extreme point of $P_C(G)$. Hence, getting a characterization of $CONE(P_C(G))$ is as hard as getting a characterization of $P_C(G)$.

3. Facets associated with edges and cycles.

Given a graph $G = (V, E)$, an incidence vector x must verify the inequalities

$$0 \leq x(e) \leq 1 \qquad \text{for } e \in E. \tag{3.1}$$

Moreover, if x is an incidence vector of a cut then for each cycle C , $x(C)$ is an even number. These conditions imply the inequalities

$$x(F) - x(C \setminus F) \leq |F| - 1 \qquad \text{for each cycle } C, \quad F \subseteq C, |F| \text{ odd}. \tag{3.2}$$

In what follows we shall study when the above inequalities define facets of $P_C(G)$.

Theorem 3.3 *(3.2) defines a facet of $P_C(G)$ if and only if C is a chordless cycle.*

Proof. If C is chordless cycle, inequality (3.2) can be obtained from an inequality associated with a triangle (Theorem 2.1), by applying Theorems 2.6 and 2.8 repeatedly as in Corollary 2.10.

If $C = \{v_1v_2, v_2v_3, \ldots, v_kv_1\}$ has a chord , say v_1v_l. It is easy to see that an inequality (3.2) associated with C can be obtained by summing inequalities (3.2) associated to $C_1 = \{v_1v_2, v_2v_3, \ldots, v_{l-1}v_l, v_lv_1\}$ and $C_2 = \{v_1v_l, \ldots, v_{k-1}v_k, v_kv_1\}$. This finishes our proof.

□

Theorem 3.4 *(3.1) define facets if and only if e does not belong to a triangle.*

Proof. Let us suppose that e does not belong to a triangle and let us study the inequality

$$x(e) \geq 0. \tag{3.5}$$

Let us assume that there is a facet defining inequality $b^Tx \geq \beta$ such that

$$S = \{x \in P_C(G) \mid x(e) = 0\} \subseteq \{x \in P_C(G) \mid b^Tx = \beta\}$$

S is the cut polytope of the graph G' obtained by contracting e. Since $P_C(G)$ is full dimensional we can conclude that $b_f = 0$ for $f \in E \setminus \{e\}$.

$b^Tx \geq \beta$ is valid, hence $b_e > 0$.

Now we can conclude that

$$x(e) \leq 1$$

defines a facet by applying Theorem 2.8 to inequality (3.5).

That finishes the first part of the proof.

Let us suppose that e belong to a triangle say $\{e, f, g\}$. By Theorem 3.3 we have the following facet defining inequalities.

$$x(e) + x(f) + x(g) \leq 2 \tag{3.6}$$

$$x(e) - x(f) - x(g) \leq 0 \tag{3.7}$$

$$-x(e) + x(f) - x(g) \leq 0 \tag{3.8}$$

$$-x(e) - x(f) + x(g) \leq 0 \tag{3.9}$$

Summing (3.6) and (3.7) we obtain

$$2x(e) \leq 2,$$

and (3.8) plus (3.9) gives

$$2x(e) \geq 0\,.$$

Hence, in this case (3.1) do not define facets.

□

Grötschel, Lovász and Schrijver [3] have shown that there exists a polynomially bounded algorithm for a linear optimization problem over a polyhedron if and only if there is a polynomially bounded algorithm for the associated separation problem : given a point x, either verify that it belongs to the polyhedron or else find a hyperplane that separates it from the polyhedron.

The knowledge of an efficient method to solve the separation problem gives an answer to some theoretical and practical questions. It proves that a problem is polynomially solvable, and it permits the design of Linear Programming based cutting plane algorithms.

In what follows we shall give a polynomial algorithm to solve the separation problem for (3.2) inequalities. Let us write these inequalities as

$$\sum_{e \in C \setminus F} x(e) + \sum_{e \in F} (1 - x(e)) \geq 1 \quad \text{for a cycle } C,$$

$$\text{and } F \subseteq C, |F| \text{ odd.}$$

Given x, we are looking for a minimum weighted cycle, where some edges have the weight $x(\cdot)$, and an odd number of edges have weight $1 - x(\cdot)$.

From the given graph G we form a new graph G' with two nodes i' and i'', for every node i of G. For every edge ij of G we put edges $i'j'$ and $i''j''$ with weight $x(ij)$, and edges $i'j''$ and $i''j'$ with weight $1 - x(ij)$. Now, for node i of G we find a shortest path f m i' to i''. The minimum over the vertices of the lenghts of the corresponding shortest path is the weight of the required cycle. As the computation of a shortest path takes $0(n^2)$ calculations, this procedure has a time complexity of $0(n^3)$.

The existence of a good algorithm for the separation problem associated to inequalities (3.1) and (3.2) leads us to ask which are the graphs such that these inequalities suffice to define $P_C(G)$. In what follows we shall see that those are the graphs not contractible to K_5.

Corollary 2.11 implies that $P_C(G)$ is defined by the inequalities associated to cycles and to edges if and only if $CONE(P_C(G))$ is defined by :

$$\begin{array}{ll} x(f) \leq x(C \setminus \{f\}), & C \text{ a cycle }, f \in C, \\ x(e) \geq 0 & e \in E \end{array}$$

This is called the "sum of circuits" property by Seymour [5]. Actually, he proved that the "sum of circuits" property holds if and only if G is not contractible to K_5.

We can state the following:

Corollary 3.12 *A graph G is not contractible to K_5 if and only if $P_C(G)$ is defined by:*

$$0 \leq x(e) \leq 1 \text{, for each edge } e \text{ that does not belong to a triangle,}$$
$$x(F) - x(C \setminus F) \leq |F| - 1 \text{, for each chordless cycle } C, F \subseteq C, |F| \text{ odd.}$$

□

4. Adjacency in $P_C(G)$.

In this section we will give a simple characterization of adjacency in $P_C(G)$, and we will derive a bound for the diameter of $P_C(G)$.

Let $I \triangle J$ denote $(I \setminus J) \cup (J \setminus I)$. If I and J are cuts then $I \triangle J$ is a cut.

Theorem 4.1 *Let $G = (V, E)$, be a graph. Let x^I, x^J be extreme points of $P_C(G)$; let I, J be the corresponding cuts. Let $F = E \setminus (I \triangle J)$. Then x^I and x^J are adjacent in $P_C(G)$ if and only if $H = (V, F)$ has two connected components.*

Proof. By definition, x^I and x^J are adjacent in $P_C(G)$ if and only if there is a vector $c = (c_e : e \in E)$ such that x^I and x^J are the only two extreme points which maximize cx over $P_C(G)$.

(i) "If" part. Let $G_i = (V_i, E_i)$, $i = 1, 2$ be the connected components of H. Let T_i be a spanning tree of G_i, $i = 1, 2$. Set

$$c_e = \begin{cases} 1 & \text{if } e \in T_i \cap (I \cap J),\ i = 1, 2, \\ -1 & e \in T_i \setminus (I \cap J),\ i = 1, 2, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$cx \leq |T_1 \cap (I \cap J)| + |T_2 \cap (I \cap J)|$$

for all $x \in P_C(G)$. The equality holds for an extreme point x if and only if $x = x^I$ or $x = x^J$.

(ii) "Only if" part. Since H can not be connected, let us suppose that $G_i = (V_i, E_i)$, $i = 1, \ldots, k$, $k \geq 3$, are the connected components of H. Assume that there is a vector c with the desired properties. Setting

$$a(e) = \begin{cases} 1 - x(e)^I, & \text{if } e \in \delta(V_1), \\ x(e)^I, & \text{otherwise,} \end{cases}$$

$$b(e) = \begin{cases} 1 - x(e)^J, & \text{if } e \in \delta(V_1), \\ x(e)^J, & \text{otherwise,} \end{cases}$$

a and b belong to $P_C(G)$, and $a + b = x^I + x^J$, so $cx^I + cx^J = ca + cb$. Consequently, $max\{ca, cb\} \geq cx^I = cx^J$ which is a contradiction as both a and b are extreme points of $P_C(G)$. That finishes the proof.

□

The graph of a polyhedron is the graph whose nodes correspond to the extreme points of this polyhedron and which has an edge joining each pair of nodes for which the corresponding extreme points are adjacent.

Given two cuts I and J, let i and j be the corresponding nodes of the graph of $P_C(G)$. The distance from I to J, $d(I, J)$ is the number of edges in the shortest path from i to j. The diameter of $P_C(G)$ is

$$max\{d(I,J) : I, J \text{ are cuts of } G\}.$$

Theorem 4.1 implies the following:

Corollary 4.2 *If G is a complete graph then $P_C(G)$ has diameter one.*

□

If G is connected and C is a cut let us define $T(C)$ as the graph obtained by contracting edges not in C, and replacing multiple edges by single edges. Let $m(C)$ be the number of edges of $T(C)$.

Theorem 4.3 *If I and J are cuts of G, then*

$$d(I,J) \leq m(I \Delta J).$$

Proof. Let P be a minimal cut included in $I \Delta J$.

Set $L = I \Delta P$. L is a cut and x^L is adjacent to x^I. Furthermore, $m(L \Delta J) < m(I \Delta J)$, and the theorem follows by induction on $m(I \Delta J)$.

□

The bound in Theorem 4.3 may be realized. For instance, if $G = (V, E)$ is the graph $K_{1,p}$, $I = \emptyset$, and $J = E$, then $d(I,J) = p$.

Corollary 4.4 *The diameter of $P_C(G)$ is at most*

$$max\{m(C) : C \text{ is a cut}\}.$$

□

Again, the graph $K_{1,p}$ shows that the bound of the corollary may be realized.

A d-dimensional polyhedron P with k facets has the Hirsch property if the diameter of P is at most $k - d$.

Theorem 4.5 *$P_C(G)$ has the Hirsch property.*

Proof. $P_C(G)$ has dimension $|E|$. Let us partition E into E_1 and E_2, where E_1 contains the edges that belong to a triangle. Let $T_1, \ldots, T_r$ be the triangles of G. By Theorems 3.3 and 3.4 $P_C(G)$ has at least $4r + 2|E_2|$ facets. Since $r \geq |E_1|/3$, $4r + 2|E_2| - |E| \geq |E_1|/3 + |E_2|$.
On the other hand, if C is a cut

$$m(C) \leq |E_1|/3 + |E_2|,$$

and our proof is complete.

□

5. Signed Graphs

A signed graph is a graph $G = (V, E)$ where E is partioned into E_- and E_+. Elements of E_- (E_+) are called negative (positive). For instance, if this structure represents a social group, we can think of positive and negative relationships between the members of the group.

A signed graph is called balanced if there exists $U \subseteq V$ such that $E_- = \delta(U)$. A graph is balanced if and only if each of its cycles includes an even number of negative edges, c.f. Harary [4].

A balancing set is an edge set $S \subseteq E$ such that when the signs of the elements of S are changed the resulting graph is balanced.

It is easy to see that S is a balancing set if and only if there exists $U \subseteq V$, such that $S \cap E_+ = S \cap \delta(U) = E_+ \cap \delta(U)$ and $S \cap E_- = S \cap (E \setminus \delta(U)) = E_- \cap (E \setminus \delta(U))$. Then y is the incidence vector of a balancing set if and only if there exists an incidence vector of a cut x, such that

$$y(e) = \begin{cases} x(e) & \text{if } e \in E_+ \\ 1 - x(e) & \text{if } e \in E_- \end{cases} \tag{5.1}$$

Thus, a minimum weighted balancing set can be founded by solving a minimum (maximum) cut problem.

Let us call $P_{BS}(G)$ the convex hull of incidence vectors of balancing sets of a signed graph G. Facet defining inequalities of $P_{BS}(G)$ can be obtained from facet defining inequalities of $P_C(G)$ by using relation (5.1). In particular, Theorem 3.5 can be stated as

Remark 5.2 *A signed graph G is not contractible to K_5 if and only if $P_{BS}(G)$ is defined by*

$$x(C \setminus F) - x(F) \geq 1 - |F|, \quad \text{for each chordless cycle } C,$$
$$F \subseteq C,\ |C \cap E_-| + |F| \text{ odd},$$

$$0 \leq x(e) \leq 1 \qquad \text{for each edge } e \text{ that does not belong to a triangle.}$$

Again relation (5.1) enables us to translate adjacency results in $P_C(G)$ into adjacency results in $P_{BS}(G)$. For instance Corollary 4.2 gives us

Remark 5.3 *If G is a complete signed graph then $P_{BS}(G)$ has diameter one.*

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ANNEXE 5

Compositions in the Acyclic Subdigraph Polytope

by

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ABSTRACT

Given a digraph D that is a 2-sum of two digraphs D_1 and D_2, we characterise a system of linear inequalities that define the acyclic subdigraph polytope of D, provided that such systems are known for D_1 and D_2. This fact is used to characterise this polytope for symmetric digraphs not contractible to $K_{3,3}$.

RESUME

Etant donné un graphe orienté D défini comme une 2-somme de deux graphes D_1 et D_2, on caractérise un système d'inéquations linéaires qui définit le polytope des sous graphes sans circuits de D quand les systèmes correspondants à D_1 et D_2 sont supposés connus. Nous utilisons par la suite ce résultat pour caractériser ce polytope dans les graphes orientés symétriques, non contractibles à $K_{3,3}$.

1. INTRODUCTION

A digraph $D = (V,A)$ consists of a finite nonempty set V of nodes and a set A of arcs which are ordered pairs of different elements of V. If $D' = (V, A')$ is a digraph with $A' \subseteq A$, D' is called a subdigraph of D.

Given $w : A \longrightarrow \mathbb{R}$, the acyclic subdigraph problem consists in finding a set $A' \subseteq A$ such that A' does not contain directed cycles, and

$$w(A') = \sum_{e \in A'} w(e)$$

is as large as possible.

If $S \subseteq A$, the incidence vector x^S of S is defined as

$$x^S(e) = \begin{cases} 1 & \text{if } e \in S \\ 0 & \text{if } e \in A \setminus S \end{cases}$$

The acyclic subdigraph polytope is

$$P_{AC}(D) = \operatorname{conv}\{x^S \mid S \subseteq A \text{ , } S \text{ has no directed cycle}\}$$

The acyclic subdigraph problem can be stated as

$$\begin{aligned} &\text{Maximize } wx \\ &\text{s.t. } x \in P_{AC}(D) \end{aligned}$$

This problem is NP-Hard [3], and it is known to be solvable in polynomial time for planar digraphs [9].

If D is a planar digraph $P_{AC}(D)$ is defined by the so-called dicycle inequalities [9], [4].

Grötschel, Jünger & Reinelt [4], characterised several facet defining inequalities of this polytope. They used them to solve several real-world problems [6], [7].

We shall prove that if a digraph D is a 2-sum of D_1 and D_2 then a system of inequalities defining $P_{AC}(D)$ may be obtained from the systems of inequalities that define $P_{AC}(D_1)$ and $P_{AC}(D_2)$, the ideas we use are similar to those of [1]. This fact and a theorem of Wagner will enable us to characterise $P_{AC}(D)$ for symmetric digraphs not contractible to $K_{3,3}$.

A graph G = (V,E) consists of a finite nonempty node set V, and a set E of edges which are unordered pairs of different nodes. If G = (V,E) is a graph then an orientation of G is a digraph which contains an arc (i,j) or (j,i) but not both whenever $ij \in E$. The symmetric digraph D(G) is a digraph (V,A) where (i,j) and (j,i) belong to A whenever $ij \in E$. If G is not contractible to $K_{3,3}$ we say that D(G) is not contractible to $K_{3,3}$.

2. Sums

In this Section we shall study some composition of facets defining inequalities of $P_{AC}(D)$.

If $(u,v) \in A$ and $(v,u) \in A$ it has been proved that

$$x(u,v) + x(v,u) \leq 1$$

defines a facet [4].

Since $P_{AC}(D)$ is full-dimensional there is no other facet defined by an inequality with nonzero coefficients for $x(u,v)$ and $x(v,u)$ simultaneously.

Moreover, the inequalities that define facets have non-negative coefficients.

A digraph $D = (V,A)$ is a 2-sum of the digraphs $D_1 = (V_1,A_1)$ and $D_2 = (V_2, A_1)$ if:

i) $V_1 \cup V_2 = V$

ii) $V_1 \cap V_2 = \{u,v\}$

iii) $A = A_1 \cup A_2$

iv) $A_1 \cap A_2 = \{(u,v),(v,u)\}$.

See Figure 2.1.

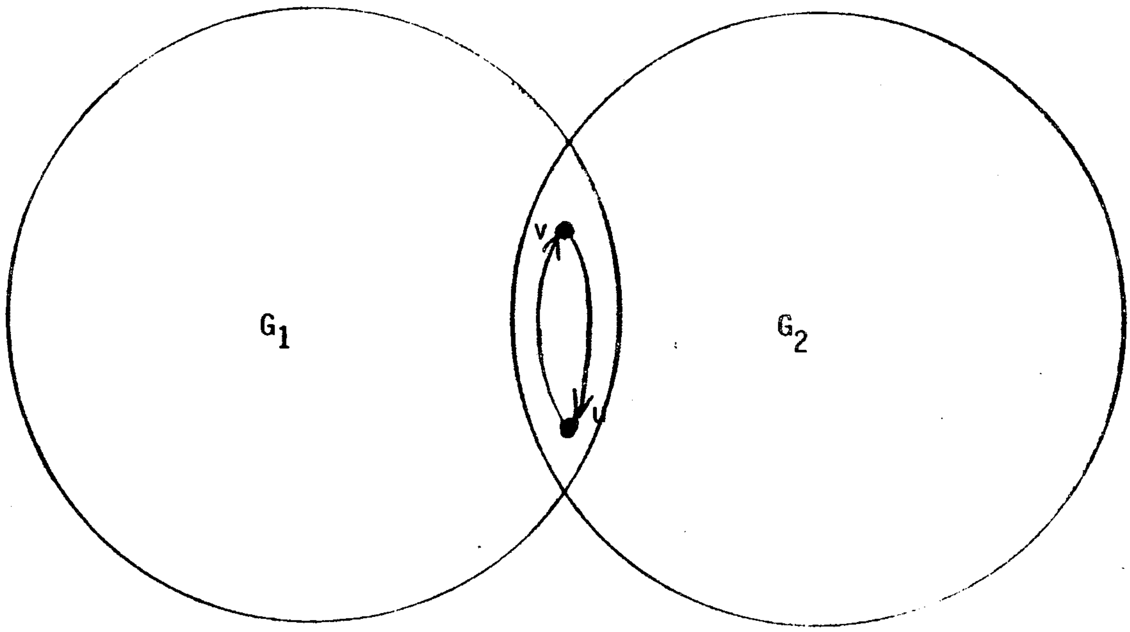


Figure 2.1

Futhermore, let us assume that the facet defining inequalities of $P_{AC}(D_1)$ are

$$(2.1)\quad \begin{cases} (a) & 0 \leq x(e) \quad e\in A_1 \\ (b) & \sum_{e\in\bar{A}_1} a_{ke}x(e) + x(u,v) \leq \alpha_k, \quad k\in K_1, \\ (c) & \sum_{e\in\bar{A}_1} a_{ke}x(e) + x(v,u) \leq \alpha_k, \quad k\in K_2, \\ (d) & x(u,v)+x(v,u) \leq 1 \\ (e) & x(i,j) \leq 1 \quad (i,j)\in\bar{A}_1 \ ,(j,i)\notin \bar{A}_1 \\ (f) & \sum_{e\in\bar{A}_1} a_{ke}x(e) \leq \alpha_k, \quad k\in K_3, \end{cases}$$

and that those of $P_{AC}(D_2)$ are

$$(2.2)\quad \begin{cases} (a) & 0 \leq x(e) \quad , \quad e\in A_1, \\ (b) & \sum_{e\in\bar{A}_2} b_{\ell e}x(e) + x(u,v) \leq \beta_\ell \ , \quad \ell\in L_1, \\ (c) & \sum_{e\in\bar{A}_2} b_{\ell e}x(e) + x(v,u) \leq \beta_\ell \ , \quad \ell\in L_2, \\ (d) & x(u,v) + x(v,u) \leq 1, \\ (e) & x(i,j) \leq 1, \quad (i,j)\in\bar{A}_2, \quad (j,i)\notin\bar{A}_2 \\ (f) & \sum_{e\in\bar{A}_2} b_{\ell e}x(e) \leq \beta_\ell \ , \quad \ell\in L_3 \end{cases}$$

where $\bar{A}_i = A_i \setminus \{(u,v), (v,u)\}$.

In what follows we shall study the following inequalities.

$$(2.3)\qquad \sum_{e\in\bar{A}_1} a_{ke}x(e) + \sum_{e\in\bar{A}_2} b_{\ell e}x(e) \leq \alpha_k + \beta_\ell -1$$

for $k\in K_1$ and $\ell\in L_2$ or $k\in K_2$ and $\ell\in L_1$.

<u>Lemma 2.4</u> Inequalities (2.3) are valid for $P_{AC}(D)$.

Proof.

Let B an acyclic edge set of D. Let $B_1(B_2)$ be the restriction of B to $D_1(D_2)$. We can extend B_1 and B_2 to two acyclic edge sets $\overline{B}_1 \subseteq A_1$ and $\overline{B}_2 \subseteq A_2$ such that

$$|\overline{B}_1 \cap \{(u,v)(v,u)\}| = |\overline{B}_2 \cap \{(u,v),(v,u)\}| = 1$$

Since $x^{\overline{B}_1}$ and $x^{\overline{B}_2}$ satisfy (2.1) and (2.2) respectively it is clear that x^B satisfies (2.3)

□

Let P be the polytope defined by (2.1), (2.2) and (2.3). Before proving that $P = P_{AC}(D)$ we shall introduce a technical Lemma.

Lemma 2.5 Let $\overline{x} \in P$, such that at least one of the inequalities (2.1)(b), (2.2)(b), $x(u,v) \leq 1$, holds with equality. Let y_1 and y_2 be defined as follows,

$$y_1(e) = \begin{cases} \overline{x}(e) \ , & \text{for } e \in A_1 \ , \quad e \neq (v,u) \ , \\ 1 - \overline{x}(u,v) \ , & \text{if } e = (v,u) \ , \end{cases}$$

$$y_2(e) = \begin{cases} \overline{x}(e) \ , & \text{for } e \in A_2 \ , \quad e \neq (v,u) \ , \\ 1 - \overline{x}(u,v) \ , & \text{if } e = (v,u) \ , \end{cases}$$

then $y_1 \in P_{AC}(D_1)$, $y_2 \in P_{AC}(D_2)$.

Moreover, an inequality of (2.3) holds with equality if and only if the corresponding inequalities of (2.1) and (2.2) hold with equality for y_1 and y_2 respectively.

Proof.

If $\bar{x}(u,v)=1$ it is clear that $y_1 \in P_{AC}(D_1)$.

Let us assume that $\bar{x}(u,v)<1$.

Let us consider the inequality

$$(2.6) \qquad \sum_{e\in\bar{A}_1} a_{ke}x(e) + x(v,u) \leq \alpha_k, \quad \text{for } k\in K_2$$

Case 1. Suppose that it exists $r\in K_1$ such that

$$\sum_{e\in\bar{A}_1} a_{re}x(e) + x(u,v) = \alpha_r.$$

Applying Lemma 2.4 to the 2-sum of D_1 with itself we obtain that

$$\sum_{e\in\bar{A}_1} a_{ke}x(e) + \sum_{e\in\bar{A}_1} a_{re}x(e) \leq \alpha_k + \alpha_r - 1$$

is valid for $P_{AC}(D_1)$. Moreover, it is satisfied by y_1.

Hence if y_1 does not satisfy (2.6) we would have a contradiction.

Case 2. Suppose that it exists $r\in L_1$ such that

$$\sum_{e\in\bar{A}_2} a_{re}x(e) + x(u,v) = \beta_r,$$

the constraint

$$\sum_{e\in\bar{A}_1} a_{ke}x(e) + \sum_{e\in\bar{A}_2} a_{re}x(e) \leq \alpha_k + \beta_r - 1$$

must be satisfied by $\bar{x}$.

Again, if y_1 does not satisfy (2.6) we would have a contradiction.

The proof that $y_2 \in P_{AC}(D_2)$ is analogous.

Furthermore, the way that y_1 and y_2 are defined implies that an inequality (2.3) holds with equality for $\bar{x}$ if and only if the associated inequalities (2.1) and (2.2) hold with equality for y_1 and y_2 respectively.

□

At this point we are able to prove the main result of this Section.

<u>Theorem 2.7</u> $P = P_{AC}(D)$.

Proof.

Since the dicycle inequalities are included in (2.1), (2.2), (2.3) it is clear that a point of P is integral if and only if it is the incidence vector of an acyclic subdigraph.

Lemma 2.4 implies that $P_{AC}(D) \subseteq P$, then we have to show that all the extreme points of P are integral.

Let us suppose that $\bar{x}$ is an extreme point that has fractional components.

Case 1. $\bar{x}$ satisfies the hypotesis of Lemma 2.5. Let y_1 anf y_2 be defined in this Lemma.
as

$$y_1 = \sum \Delta_i x^{S_i}, \qquad \Delta_i \geq 0, \sum \Delta_i = 1,$$
$$y_2 = \sum \Pi_j x^{T_j}, \qquad \Pi_j \geq 0, \sum \Pi_j = 1.$$

where $x^{S_i}(x^{T_j})$ are incidence vector of a cyclic subdigraphs of $D_1(D_2)$.

Note that all the constraints of $P_{AC}(D_1)$ $(P_{AC}(D_2))$ that hold with equality for $y_1(y_2)$ also hold with equality for $x^{S_i}(x^{T_j})$.

Subcase 1(a) $\bar{x}(u,v) < 1$.

We have that $y_1(v,u) > 0$ and $y_2 \neq (v,u) > 0$.
There exists S_r and T_s such that $(v,u) \in S_r \cap T_s$.
If $\bar{x}(u,v) + \bar{x}(v,u) < 1$ then no inequality of type (2.1)(c) or (2.2)(c) could hold as equation for $\bar{x}$.
In fact, let us assume that

$$\sum_{e \in \bar{A}_1} a_{k_0 e} \bar{x}(e) + \bar{x}(v,u) = \alpha_{k_0}, \qquad \text{for } k_0 \in K_2.$$

Since $\bar{x}$ satisfies the hypotesis of Lemma 2.5 we have that

$$\sum_{e \in \bar{A}_1} a_{k_1 e} \bar{x}(e) + \bar{x}(u,v) = \alpha_{k_1}, \qquad \text{for some } k_1 \in K_1,$$

or

$$\sum_{e \in \bar{A}_2} b_{l_1 e} \bar{x}(e) + \bar{x}(u,v) = \beta_{l_1}, \qquad \text{for some } l_1 \in L_1$$

This implies

$$\sum_{e \in \bar{A}_1} a_{k_0 e} \bar{x}(e) + \sum_{e \in \bar{A}_1} a_{k_1 e} \bar{x}(e) > \alpha_{k_0} + \alpha_{k_1} - 1,$$

or

$$\sum_{e \in \bar{A}_1} a_{k_0 e} \bar{x}(e) + \sum_{e \in \bar{A}_2} b_{l_1 e} \bar{x}(e) > \alpha_{k_0} + \beta_{l_1} - 1.$$

Then set $W = S_r \cup T_s \setminus \{(v, u)\}$. W definies an acyclic subdigraph and any constraint that holds as equation for $\bar{x}$ also holds as equation for x^W. This is clear for (2.1) and (2.2), for (2.3) it follows from Lemma 2.5.

This contradicts the fact that $\bar{x}$ is an extreme point.

If $\bar{x}(u, v) + \bar{x}(v, u) = 1$ set $W = S_r \cup T_s$. x^W has the same properties mentioned above.

Subcase 1(b) $\bar{x}(u, v) = 1$

There exists S_r and T_s such that $(u, v) \in S_r \cap T_s$. Set $W = S_r \cup T_s$.

The proof follows as in the subcase 1.(a).

Case 2

If none of the inequalities (2.1) (b), (2.2) (b), $\bar{x}(u, v) \leq 1$, holds with equality, we must have $\bar{x}(u, v) = 0$. Hence we can increase the value of $\bar{x}(u, v)$ and get another fractional extreme point that can be treated in case 1. Our proof is complete.

□

3. Facets

In this Section we shall show that inequalities (2.3) define facets of $P_{AC}(D)$. Since inequalities (2.1) and (2.2) also define facets of $P_{AC}(D)$, we have a minimal system that defines this polytope.

For $k \in K_1$ and $\ell \in L_2$ let us assume that

$$(3.1) \qquad \sum_{e \in \bar{A}_1} a_{ke} x(e) + x(u,v) \leqslant \alpha_k \text{ , and}$$

$$(3.2) \qquad \sum_{e \in \bar{A}_2} b_{\ell e} x(e) + x(v,u) \leqslant \beta_\ell$$

define facets of $P_{AC}(D_1)$ and $P_{AC}(D_2)$ respectively.

Let us remark that (3.1) and (3.2) also define facets of $P_{AC}(D_1')$ and $P_{AC}(D_2')$, where

$$D_1' = (V_1, A_1 \setminus \{(v,u)\}) \text{ and } D_2' = (V_2, A_2 \setminus \{(u,v)\}).$$

Hence there are $p = |A_1| - 1$ (respectively $q = |A_2| - 1$), acyclic subdigraphs $S_1,\ldots,S_p$ of D_1' ($T_1,\ldots,T_q$ of D_2') whose incidence vectors are linearly independent and satisfy (3.1)((3.2)) with equality.

We assume that

$$x(uv)^{S_i} = \begin{cases} 1, & \text{for } \quad 1 \leqslant i \leqslant r \text{ ,} \\ 0, & \text{for } \quad r+1 \leqslant i \leqslant p, \end{cases}$$

and

$$x(vu)^{T_i} = \begin{cases} 1 & , \text{ for } 1 \leqslant i \leqslant t , \\ 0 & , \text{ for } t+1 \leqslant i \leqslant q . \end{cases}$$

Set $S_{p+1} = S_{r+1} \cup \{(v,u)\}$ and

$$T_{q+1} = T_{t+1} \cup \{(u,v)\} .$$

Let $M_1 (M_2)$ be the $(p+1)x(p+1)$ $((q+1)x(q+1))$ matrix whose columns are the vectors x^{S_i} (x^{T_i}).

M_1 and M_2 look as follows

$$M_1 = \begin{array}{|c|c|c|c|c|l} \hline c & N & d & P & d & \\ \hline 1 & 1 \ldots 1 & 0 & 0 \ldots\ldots 0 & 0 & \leftarrow x(u,v) \\ \hline 0 & 0 \ldots 0 & 0 & 0 \ldots\ldots 0 & 1 & \leftarrow x(v,u) \\ \hline \multicolumn{2}{c}{\underbrace{\hspace{3cm}}_{r}} & \multicolumn{3}{c}{\underbrace{\hspace{4cm}}_{p+1-r}} & \end{array}$$

$$M_2 = \begin{array}{|c|c|c|c|c|l} \hline 0 & 0 \ldots 0 & 0 & 0 \ldots 0 & 1 & \leftarrow x(u,v) \\ \hline 1 & 1 \ldots 1 & 0 & 0 \ldots 0 & 0 & \leftarrow x(v,u) \\ \hline e & Q & f & R & f & \\ \hline \multicolumn{2}{c}{\underbrace{\hspace{3cm}}_{t}} & \multicolumn{3}{c}{\underbrace{\hspace{4cm}}_{q+1-t}} & \end{array}$$

Before proving the main Theorem of this Section we will introduce two technical Lemmas.

Let N' be the matrix obtained from N by subtracting c from each column.

Let P' be the matrix obtained from P by subtracting d from each column.

Let Q' and R' be the matrices obtained from Q and R by doing the same with e and f respectively.

The matrices

$$\left[\begin{array}{c|c|c} N' & d & P' \end{array}\right]$$

$$\left[\begin{array}{c|c|c} Q' & f & R' \end{array}\right]$$

are nonsingular.

<u>Lemma 3.3.</u> Let M_{1y} be

$$M_{1y} = \left[\begin{array}{c|c} \begin{array}{c} \\ M_1 \\ \\ \\ \end{array} & \begin{array}{c} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \end{array} \\ \hline 1 \;\; \ldots \;\; 1 & y \end{array}\right] ,$$

and M_{2y} be

$$M_{2y} = \left[\begin{array}{ccc|c} & & & 0 \\ & & & 1 \\ & M_2 & & 0 \\ & & & \vdots \\ & & & 0 \\ \hline 1 & \cdots & 1 & y \end{array}\right]$$

if $y \notin (0,1)$ then M_{1y} and M_{2y} are nonsingular.

Proof.

Let $\mathbf{1}$ be a row of ones, since M_1 is nonsingular, the solution of

$$\lambda M_1 = \mathbf{1} \quad \text{is}$$

$$\lambda(e) = \frac{a_{ke}}{\alpha_k} \quad .$$

Since $\lambda(uv) = \dfrac{1}{\alpha_k}$, and $0 < \alpha_k > 1$, $\lambda(uv) \in (0,1)$. If $y \notin (0,1)$ then M_{1y} is nonsingular.

The proof for M_{2y} is analogous.

$\square$

<u>Lemma 3.4.</u> Let M'_{1y} be

$$M'_{1y} = \left[\begin{array}{c|c|c} c-yd & N' & P' \end{array}\right]$$

and M'_{2y} be

$$M'_{2y} = \begin{array}{|c|c|c|}\hline e-yf & Q' & R' \\ \hline\end{array}\ ,$$

if $y \notin (0,1)$ then M'_{1y} and M'_{2y} are nonsingular

Proof.

$$\begin{array}{|c|ccc|c|ccc|c|c|}\hline c & & N' & & d & & P' & & \begin{matrix}0\\ \vdots\\ 0\end{matrix} & \begin{matrix}0\\ \vdots\\ 0\end{matrix} \\ \hline 1 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 & 1 \\ \hline 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 1 & 0 \\ \hline 1 & 0 & \ldots & 0 & 1 & 0 & \ldots & 0 & 0 & y \\ \hline\end{array}$$

is nonsingular, by Lemma 3.3.

Thus

$$\begin{array}{|c|c|c|c|c|}\hline c-(1-y)d & N' & d & P' & \begin{matrix}0\\ \vdots\\ 0\end{matrix} \\ \hline 0 & 0 \ldots 0 & -1/y & 0 \ldots 0 & 1 \\ \hline 0 & 0 \ldots 0 & 0 & 0 \ldots 0 & y \\ \hline\end{array}$$

is nonsingular, and $(1-y) \notin (0,1) \iff y \notin (0,1)$.

The proof for M'_{2y} is analogous.

$\square$

<u>Theorem 3.5</u> The inequality

$$(3.6) \qquad \sum_{e \in \bar{A}_1} a_{ke} x(e) + \sum_{e \in \bar{A}_2} b_{\ell e} x(e) \leq \alpha_k + \beta_\ell - 1$$

defines a facet of $P_{AC}(D)$.

Proof.

It is easy to see that the columns of

M =

c ... c	d ... d	N	P	c	d	c	d
0 ... 0	0 ... 0	0 ... 0	0 ... 0	0	0	1	0
0 ... 0	0 ... 0	0 ... 0	0 ... 0	0	0	0	1
R	Q	f ... f	e ... e	f	e	f	e

satisfy (3.6) with equality and belong to $P_{AC}(D)$. We shall prove that M is nonsingular. It is enough to prove that

M' =

0	0	N'	P'	c	d	c	d
0	0	0	0	0	0	1	0
0	0	0	0	0	0	0	1
R'	Q'	0	0	f	e	f	e

is nonsingular.

We only need to prove that

$$M'' = \left[\begin{array}{c|c|c|c|c|c} 0 & 0 & N' & P' & c & d \\ \hline R' & Q' & 0 & 0 & f & e \end{array}\right]$$

is nonsingular.

Since

$$\left[\begin{array}{c|c|c} N' & P' & d \end{array}\right]$$

is nonsingular, we have that

$$c = \left[\begin{array}{c|c|c} N' & P' & d \end{array}\right] \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$$

Then by columns operations we get

$$M'' = \left[\begin{array}{c|c|c|c|c|c} 0 & 0 & N' & P' & 0 & d \\ \hline R' & Q' & 0 & 0 & f-\gamma e & e \end{array}\right]$$

We have to prove that

$$\left[\begin{array}{c|c|c} R' & Q' & f-\gamma e \end{array}\right]$$

is nonsingular.

Since

N'	P'	c-γd

is singular, by Lemma 3.4 we have that $\gamma\epsilon(0,1)$.

This implies that $1/\gamma>1$, and that

R'	Q'	-γ(e-1/γf)

is nonsingular.

The proof is complete.

□

Now, we shall see an example. Let D = (V,A) be the digraph of Figure 3.1

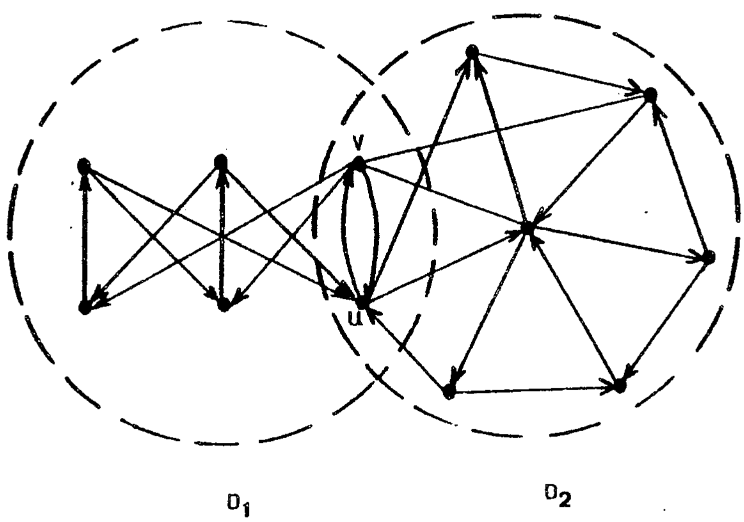


Figure 3.1

D is a 2-sum of $D_1 = (V_1, A_1)$ and $D_2 = (V_2, A_2)$. Let A_1' be $A_1 \setminus \{(v,u)\}$ and $A_2' = A_2 \setminus \{(u,v)\}$ Grötschel et al [4] proved that

$$\sum_{e \in A_1'} x(e) \leq 7, \qquad \text{and}$$

$$\sum_{e \in A_2'} x(e) \leq 11$$

define facets of $P_{AC}(D_1)$ and of $P_{AC}(D_2)$ respectively.

Theorem 3.5 implies that

$$\sum_{e \in A'} x(e) \leq 17$$

defines a facet of $P_{AC}(D)$, where

$$A' = A \setminus \{(u,v),(v,u)\}.$$

If we continue with this procedure we can generate complicated facet defining inequalities.

We can state the following Corollaries.

<u>Corollary 3.7</u>. The characterisation of $P_{AC}(D)$ given by Theorem 2.7 (i.e. by (2.1), (2.2) and (2.3)) is minimal.

□

<u>Corollary 3.8</u> If $P_{AC}(D_1)$ and $P_{AC}(D_2)$ are defined by rank inequalities then $P_{AC}(D)$ is also defined by rank inequalities.

□

Rank inequalities are inequalities with 0 and 1 coefficients.

4. Weakly Acyclic Digraphs

A digraph D = (V,A) is called weakly acyclic (w.a.) [4] if $P_{AC}(D)$ is defined by

$$0 \leq x(e) \leq 1, \quad \text{for} \quad e \in A,$$

$$x(C) \leq |C| -1, \quad C \text{ a dicycle of } D.$$

$x(S)$ denotes $\sum_{e \in S} x(e)$.

Since the separation problem [8] can be solved in polynomial time for these inequalities we can optimize a linear function over this polyhedron in polynomial time.

A characterisation or a polynomial algorithm to recognize w.a. digraphs is not known. Planar digraphs are w.a. [9], [4]. Theorem 2.7 shows that a 2-sum of w.a. digraphs gives a w.a. digraph.

Let us remark $D(K_{3,3})$ and $D(G)$ for any graph G contractible to $K_{3,3}$ are not w.a. [4]. Let us denote by $\mathcal{C}$ the class of graphs not contractible to $K_{3,3}$. We shall prove that $G \in \mathcal{C}$ if and only if $D(G)$ is w.a.

A graph G in $\mathcal{C}$ is said to be maximal if for any edge that we add to G we obtain a graph not in $\mathcal{C}$. Let us denote by $\hat{\mathcal{C}}$ the class of maximal graphs in $\mathcal{C}$.

Wagner [11] proved that any nonplanar graph in $\hat{C}$ can be obtained by doing 2-sums of planar triangulated graphs and copies of K_5.

We just need to prove that $D(K_5)$ is w.a.

Set $D(K_5) = (V,A)$

We have to prove that for any w the problem

$$(4.1) \quad \begin{aligned} &\text{Max } wx \\ &\text{s.t.} \\ &\qquad x(C) \leq |C| - 1, \quad C \text{ a dicycle of } D(K_5), \\ &\qquad x \geq 0 \quad , \end{aligned}$$

has an integer optimal solution.

The dual problem is

$$(4.2) \quad \begin{aligned} &\text{Min } \sum(|C| - 1)y_C \\ &\text{s.t.} \\ &\qquad \sum\{y_C : e \in C\} \geq w(e) \\ &\qquad y \geq 0. \end{aligned}$$

Let us define $D' = (V,A')$ as follows:

a) If $w(i,j) > w(j,i) \geq 0$ then $(i,j) \in A'$ and $w'(i,j) = w(i,j) - w(j,i)$

b) If $w(i,j) = w(j,i) \geq 0$ then $(i,j) \in A'$ or $(j,i) \in A'$ but not both, say $(i,j) \in A'$; in this case $w'(i,j) = 0$.

c) If $w(i,j) \geq 0 > w(j,i)$ then $(i,j) \in A'$ and $w'(i,j) = w(i,j)$.

d) If $w(i,j) < 0$ and $w(j,i) < 0$ we do not put any arc between i and j.

Let us suppose that

(4.3) Max $w'x$

s.t.

$$x(C) \leq |C| - 1, \quad C \text{ a dicycle of } D'.$$

$$0 \leq x(e) \leq 1, \quad e \in A'$$

has an integer optimal solution $\bar{x}$ and $(\bar{z}, \bar{\gamma})$ is an optimal solution of

$$\text{Min } \sum (|C|-1) z_C + \sum \gamma(e)$$

s.t.

$$\sum \{z_C : e \in C\} + \gamma(e) \geq w'(e)$$

$$z \geq 0, \quad \gamma \geq 0$$

Set

$\bar{\bar{x}}(i,j) = \bar{x}(i,j), \bar{\bar{x}}(j,i) = 1 - \bar{\bar{x}}(i,j)$ if a) or b) hold,

$\bar{\bar{x}}(i,j) = \bar{x}(i,j), \bar{\bar{x}}(j,i) = 0,$ if c) holds,

$\bar{\bar{x}}(i,j) = \bar{\bar{x}}(j,i) = 0$ if d) holds,

$\bar{y}_C = \bar{z}_C$ if C is a dicycle of D',

$\bar{y}_C = \gamma(i,j) + w(j,i)$ if $C = \{(i,j),(j,i)\}$ and a) or b) hold,

$\bar{y}_C = \gamma(i,j)$ if $C = \{(i,j),(j,i)\}$ and c) holds,

$\bar{y}_C = 0$ otherwise.

It is easy to see that $\bar{\bar{x}}$ and $\bar{y}$ are optimal solutions of (4.1) and (4.2) respectively.

Thus, we need to prove that for any $w' \geq 0$ (4.3) has an integer optimal solution.

If case d) holds then D' is planar and we are done.

We shall prove that if D' is any orientation of K_5 then the polyhedron defined in (4.3) has integer vertices.

Let $\bar{x}$ be an extreme point.

Lemma 4.4. If $\bar{x}(e) = 0$ for some arc e then $\bar{x}$ is integer valued.

Proof.

Set $D'' = D' \setminus e$, and let x'' be the vector $\bar{x}$ without the component $\bar{x}(e)$.

$x'' \in P_{AC}(D'')$, because D" is planar.

$$x'' = \sum \lambda_i x_i \quad , \quad \sum \lambda_i = 1, \quad \lambda_i \geq 0 \qquad \forall i$$

where x_i is an extreme point of $P_{AC}(D'')$, i.e. x_i is a representative vector of an acyclic subdigraph of D".

Let $\bar{x}_i$ be the vector obtained by adding to x_i a 0 in the component $\bar{x}_i(e)$

$$\bar{x} = \sum \lambda_i \bar{x}_i \ .$$

For any inequality $ax \leq \alpha$ of (4.3) we have that

$$a\bar{x} = \alpha \Longrightarrow a\,\bar{x}_i = \alpha \quad \forall\, i.$$

This implies that $\bar{x} = \bar{x}_i\ \forall i$.

□

<u>Lemma 4.5</u> If there exists an arc e that does not belong to a dicycle then $\bar{x}$ is integer valued.

Proof.

If e does not belong to a dicycle the nontrivial inequalities of (4.3) are associated to $D' \setminus e$. Since this digraph is planar we are done.

□

<u>Lemma 4.6</u> If $\bar{x}(e) > 0$ for any $e \in A'$ then for any nonempty set $S \subset V$ there is at least one arc $e \in \delta(S)$ with $\bar{x}(e) = 1$.

Proof.

Let S be such a set, let $\delta^+(S)(\delta^-(S))$ be the set of arcs that enter (leave) S. If $\delta^+(S) = \emptyset$ (or $\delta^-(S) = \emptyset$) there are some arcs that do not belong to any dicycle, this implies that $\bar{x}$ is integer valued.

Otherwise if $\bar{x}(e) < 1$ for any arc $e \in \delta(S)$ we can define x' as follows:

$$x'(e) = \begin{cases} \bar{x}(e)+\varepsilon & \text{if } e \in \delta^+(S), \\ \bar{x}(e)-\varepsilon & \text{if } e \in \delta^-(S), \\ \bar{x}(e) & \text{otherwise.} \end{cases}$$

For ε sufficiently small x' belongs to the polyhedron (4.3), and if $\bar{x}$ satisfy one of the inequalities as equation then x' also does.

This contradicts the fact that $\bar{x}$ is an extreme point.

$\square$

<u>Lemma 4.7</u> If there is a vertex v that covers every dicycle then $\bar{x}$ is integer valued.

Proof.

Let $D'' = (V'', A'')$ be defined as follows:

$$V'' = V \setminus \{v\} \cup \{s,t\}.$$

$$(i,j) \in A' \Longrightarrow (i,j) \in A'' \quad \text{for} \quad i \neq v,\ j \neq v,$$

$$(v,i) \in A' \Longrightarrow (s,i) \in A'' ,$$

$$(i,v) \in A' \Longrightarrow (i,t) \in A'' .$$

Let A be the incidence matrix of all directed paths from s to t in D".

It is well known that for any $w' \geqslant 0$ the problem

$$\begin{aligned} &\text{Min } w'x \\ &\text{s.t.} \quad Ax \geqslant 1 \\ &\qquad\quad\ x \geqslant 0 \end{aligned}$$

has an integer valued optimal solution $\tilde{x}$.

But A is also the incidence matrix of the dicycles of D'. Then the vector $\tilde{x}'$ defined as

$$\tilde{x}'(e) = 1-\tilde{x}(e) \quad \text{for } e \in A' ,$$

is an optimal solution of (4.3).

$\square$

In what follows we shall assume that $\bar{x}(e) > 0$ for any arc $e \in A'$. Lemma 4.6 implies that there is a spanning tree T of arcs e with $\bar{x}(e) = 1$.

We will study three cases.

Case 1. T contains a directed path P of lenght three. This implies the orientation of Figure 4.1, where the double arcs are those of P.

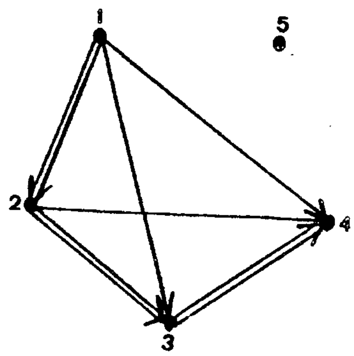
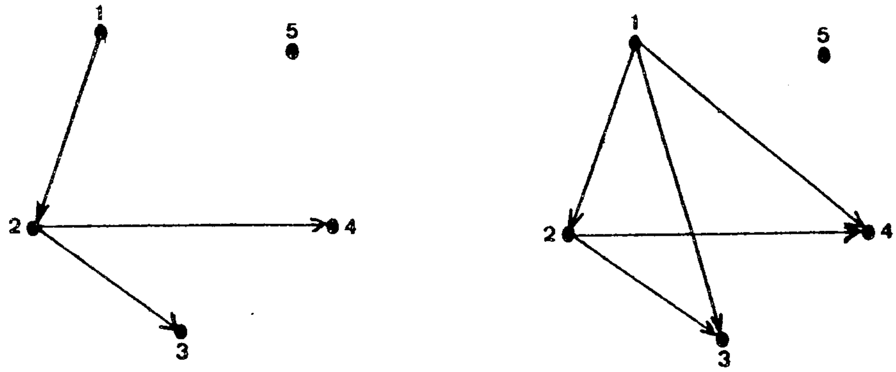


Figure 4.1

Since 5 covers every dicycle we have that $\bar{x}$ is integer valued.

Case 2. T contains a directed path of lenght two, and no directed path of lenght three.

Subcase 2.1 Suppose that T contains the arcs of Figure 4.2(a), this implies the orientations of Figure 4.2(b).

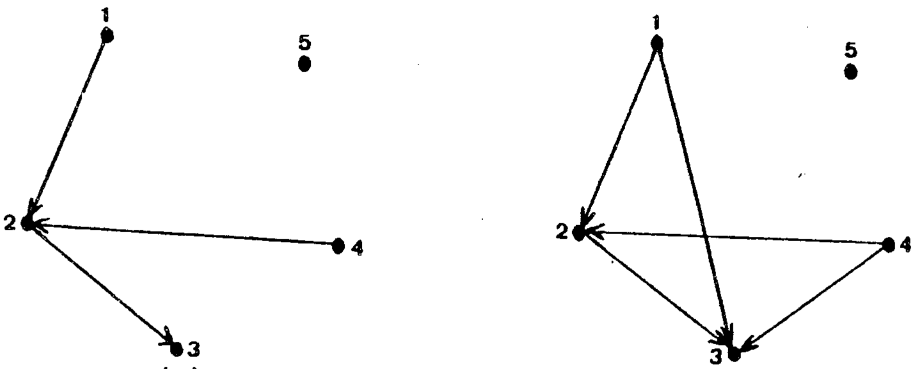


(a) (b)

Figure 4.2

Since 5 covers every dicycle this implies that $\bar{x}$ is integer valued.

<u>Subcase 2.2</u> Suppose that T contains the arcs of Figure 4.3(a), it implies the orientations of Figure 4.3(b).

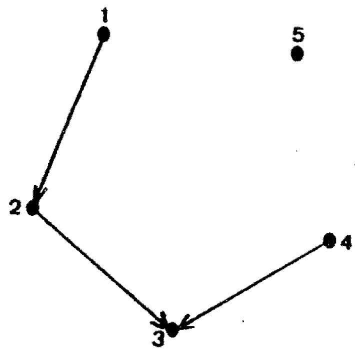


(a) (b)

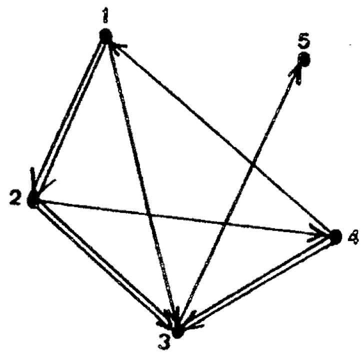
Figure 4.3

Again we have that 5 covers every dicycle.

Subcase 2.3 T contains the arcs of Figure 4.4 (a). It implies that $(1,3)\epsilon A'$.



(a)



(b)

Figure 4.4

If the subdigraph induced by $\{1,2,3,4\}$ has no dicycle we have that 5 covers every dicycle, then we should assume that (2,4) and (4,1) belong to A'. If 3 has only entering arcs we are done, then we should assume that $(3,5)\epsilon A'$.

If the arcs (5,2) or (2,5) belongs to T we would have the orientations of Figure 4.3(a) or Figure 4.2(a), if $(5,4)\epsilon T$ we would have a contradiction with the constarint

$$x(4,3) + x(3,5) + x(5,4) \leqslant 2,$$

hence we should assume that $(1,5)\epsilon T$ or $(4,5)\epsilon T$.

Subcase 2.3.1 $(1,5)\epsilon T$.

We should suppose that $(5,4)\epsilon A'$, otherwise the digraph induced by $\{1,3,4,5\}$ would have no dicycle,and that $(5,2)\epsilon A'$, otherwi se the subgraph induced by $\{1,2,3,5\}$ would have no dicycle. We have then the orientation of Figure 4.5.

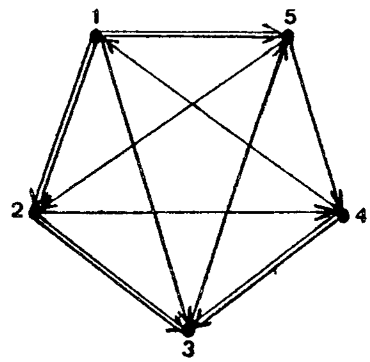


Figure 4.5

$\bar{x}$ satisfies

(4.8)			$\bar{x}(2,4)$ +	$\bar{x}(4,1)$				≤ 1
(4.9)		$\bar{x}(5,2)$ +			+ $\bar{x}(3,5)$			≤ 1
(4.10)					$\bar{x}(3,5)$ +	$\bar{x}(5,4)$		≤ 1
(4.11)				$\bar{x}(4,1)$		+ $\bar{x}(5,4)$		≤ 1
(4.12)	$\bar{x}(1,3)$ +	$\bar{x}(5,2)$ +	$\bar{x}(2,4)$ +	$\bar{x}(4,1)$ +	$\bar{x}(3,5)$			≤ 4
(4.13)	$\bar{x}(1,3)$ +			$\bar{x}(4,1)$ +	$\bar{x}(3,5)$+	$\bar{x}(5,4)$		≤ 3

(4.8) and (4.9) imply that (4.12) holds with strict inequality.

If $\bar{x}(4,1) = 1$ then $\bar{x}(5,4) = 0$. Thus we may assume that $\bar{x}(4,1) < 1$. This and (4.10) imply that (4.13) holds with strict inequality.

Thus we have six variables and four inequalities that could hold as equation, this implies that some variable should be equal to zero or one. In the former case we are done, and in the last case we would have one of the preceding cases.

Subcase 2.3.2 $(4,5) \in T$.

We should assume that $(5,1) \in A'$, otherwise the subdigraph induced by $\{1,3,4,5\}$ would have no dicycle. We assume also that $(5,2) \in A'$, otherwise the subdigraph induced by $\{2,3,4,5\}$ would have no dicycle. We have then the orientation of Figure 4.6

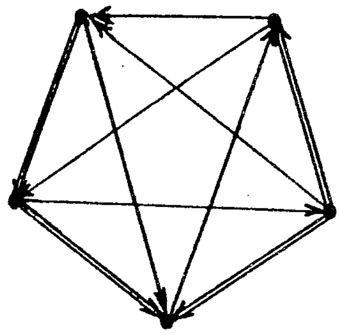


Figure 4.6

$\bar{x}$ satisfies

$$
\begin{array}{lllllll}
(4.14) & & \bar{x}(2,4) & + \bar{x}(4,1) & & & \leq 1 \\
(4.15) & & & & \bar{x}(3,5) + \bar{x}(5,2) & & \leq 1 \\
(4.16) & & \bar{x}(2,4) & & \quad\quad\quad\;\; + \bar{x}(5,2) & & \leq 1 \\
(4.17) & & & & \bar{x}(3,5) & + \bar{x}(5,1) & \leq 1 \\
(4.18) & & \bar{x}(2,4) & & & + \bar{x}(5,1) & \leq 1 \\
(4.19) & \bar{x}(1,3) & & & + \bar{x}(3,5) & + \bar{x}(5,1) & \leq 2 \\
(4.20) & \bar{x}(1,3) & + \bar{x}(2,4) & + \bar{x}(4,1) & + \bar{x}(3,5) + \bar{x}(5,2) & & \leq 4
\end{array}
$$

(4.17) implies that (4.19) can hold with equality only if (4.17) does and $\bar{x}(1,3) = 1$.

(4.14) and (4.17) imply that (4.20) cannot hold as equation.

Thus we should assume that $\bar{x}(1,3) = 1$, and restrict our attention to (4.14),...,(4.18).

Let us define x' as follows:

$$x'(2,4) = \bar{x}(2,4) + \varepsilon,$$
$$x'(4,1) = \bar{x}(4,1) - \varepsilon,$$
$$x'(3,5) = \bar{x}(3,5) + \varepsilon,$$
$$x'(5,2) = \bar{x}(5,2) - \varepsilon,$$
$$x'(5,1) = \bar{x}(5,1) - \varepsilon,$$

$x'(i,j) = \bar{x}(i,j)$ for all the other arcs $(i,j)\epsilon A'$.

For ε sufficiently small x' belongs to the polytope of (4.3), and if $\bar{x}$ satisfy with equation one inequality of (4.14),..., (4.18) then x' also does, this contradicts the fact that $\bar{x}$ is an extreme point.

<u>Subcase 2.4</u> T contains the edges of Figure 4.7

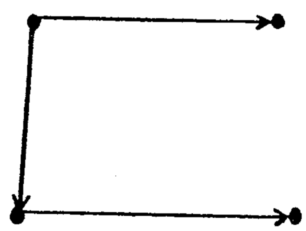


Figure 4.7

This is analogous to the subcase 2.3.

<u>Case 3</u>. T does not contain a directed path of lenght 2.

<u>Subcase 3.1</u> T is one of the trees of Figure 4.8

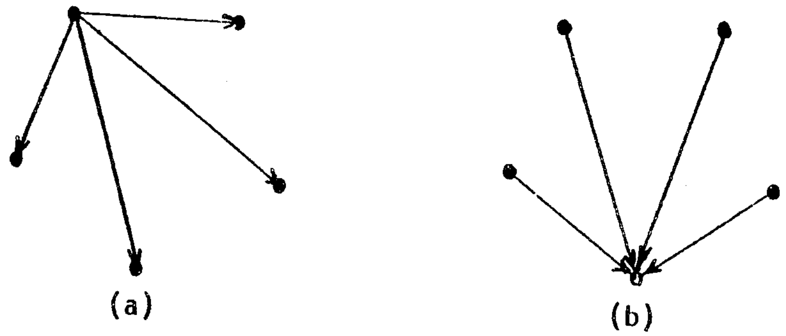


Figure 4.8

Since the arcs of T do not belong to a dicycle we are done.

Subcase 3.2 T is one of the trees of Figure 4.9. That of Figure 4.9(a) say.

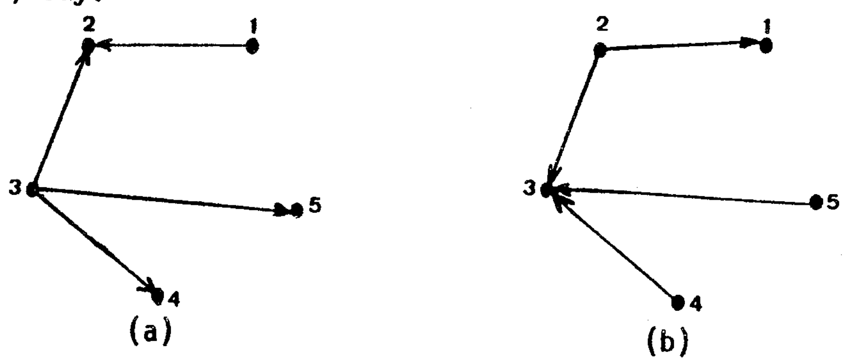


Figure 4.9

Without loss of generality we can assume that $(4,5)\epsilon A'$. We should assume that $(1,3)\epsilon A'$, otherwise arcs adjacent to 3 would not belong to a dicycle. Let us suppose that $(5,1)\epsilon A'$, otherwisse the subdigraph induced by $\{1,2,3,5\}$ would have no dicycle. By the same reasons we suppose that $(4,1)\epsilon A'$. We assume that $(5,2)$ and $(2,4)$ belong to A', otherwise the subdigraph induced by $\{2,3,4,5\}$ would have no dicycle. Thus we obtain the orientation of Figure 4.10

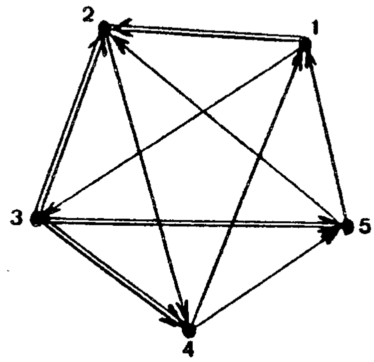


Figure 4.10

(4.21)	$\bar{x}(2,4) + \bar{x}(4,1)$	≤ 1
(4.22)	$\bar{x}(4,1) + \bar{x}(1,3)$	≤ 1
(4.23)	$\bar{x}(1,3) + \bar{x}(5,1)$	≤ 1
(4.24)	$\bar{x}(2,4) + \bar{x}(4,5) + \bar{x}(5,2)$	≤ 2
(4.25)	$\bar{x}(2,4) + \bar{x}(5,1) + \bar{x}(4,5)$	≤ 2
(4.26)	$\bar{x}(1,3) + \bar{x}(5,1) + \bar{x}(4,5)$	≤ 2
(4.27)	$\bar{x}(2,4) + \bar{x}(4,1) + \bar{x}(1,3)$	≤ 2
(4.28)	$\bar{x}(2,4) + \bar{x}(4,1) + \bar{x}(1,3) + \bar{x}(5,2)$	≤ 3
(4.29)	$\bar{x}(2,4) + \bar{x}(1,3) + \bar{x}(5,1) + \bar{x}(4.5)$	≤ 3

(4.23) implies that (4.26) holds with equality only if $\bar{x}(4,5) = 1$, (4.22) implies that (4.27) holds with equality only if $\bar{x}(2,4) = 1$, (4.27) implies that (4.28) holds with equality only if $\bar{x}(5,2) = 1$, (4.26) implies that (4.29) holds as equation only if $\bar{x}(2,4) = 1$. In any of these cases we would have a path of lenght two of arcs (i,j) with $\bar{x}(i,j) = 1$.

Thus we will assume that, (4,26),...,(4.29) holds with strict inequality.

Again we have six variables and five constraints, hence one of these variables should be equal to zero or one.

<u>Subcase 3.3</u> T is one of the trees of Figure 4.11. That of Figure 4.11(a) say.

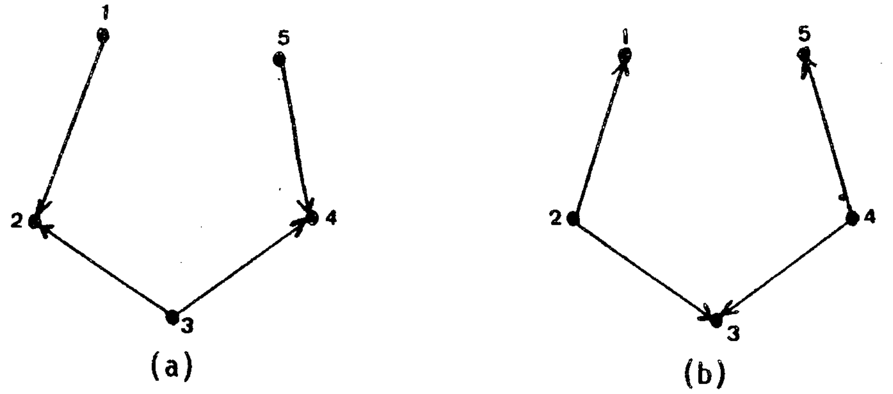


Figure 4.11

We do no loose generality if we suppose tat $(1,5)\epsilon A'$. We should assume that $(4,1)\epsilon A'$, otherwise the subdigraph induced by $\{1,2,3,4\}$ would have no dicycle. By the same reasons we assume that $(2,5)\epsilon A'$ and $(5,3)\epsilon A'$.

If (4,2) and (3,1) belong to A' the subdigraph induced by $\{1,2,3,4\}$ would have no dicycle.

There are three more cases to study.

<u>Subcase 3.3.1</u>. (4,2) and (1,3) belong to A', we have then the orientation of Figure 4.12.

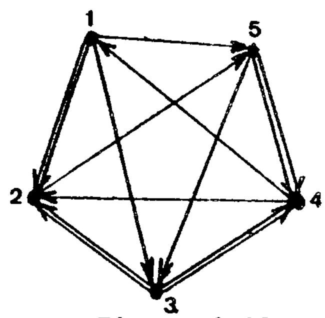


Figure 4.12

$\bar{x}$ satisfies

$$
\begin{array}{lllllll}
(4.30) & \bar{x}(1,3) + \bar{x}(4,1) & & & & & \leqslant 1 \\
(4.31) & & \bar{x}(4,1) + \bar{x}(1,5) & & & & \leqslant 1 \\
(4.32) & & & \bar{x}(2,5) + \bar{x}(5,3) & & & \leqslant 1 \\
(4.33) & & & \bar{x}(2,5) & & + \bar{x}(4,2) & \leqslant 1 \\
(4.34) & & & \bar{x}(2,5) + \bar{x}(5,3) & & + \bar{x}(4,2) & \leqslant 2 \\
(4.35) & & \bar{x}(4,1) & + \bar{x}(2,5) & & & \leqslant 1 \\
(4.36) & & \bar{x}(4,1) + \bar{x}(1,5) & & + \bar{x}(5,3) & & \leqslant 2 \\
(4.37) & & \bar{x}(4,1) & + \bar{x}(5,3) & & & \leqslant 2 \\
(4.38) & \bar{x}(1,3) + \bar{x}(4,1) & & + \bar{x}(2,5) & & & \leqslant 2
\end{array}
$$

We assume that (4.34), (4.36), (4.37), (4,38) hold with strict inequality, otherwise we would have that one of these six variables would be equal to one.

In this case we have only five inequalities.

Subcase 3.3.2 (2,4) and (3,1) belong to A', we have then orientation of Figure 4.13.

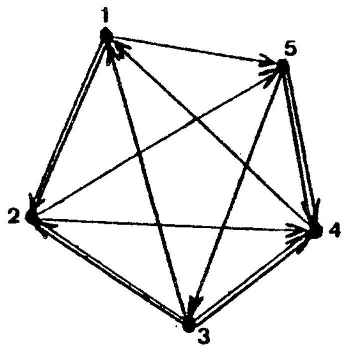


Figure 4.13

$\bar{x}$ satisfies

$$\begin{array}{llr}
(4.39) & \bar{x}(2,4) + \bar{x}(4,1) & \leq 1 \\
(4.40) & \bar{x}(5,3) + \bar{x}(3,1) + \bar{x}(1,5) & \leq 2 \\
(4.41) & \bar{x}(5,3) + \bar{x}(2,5) & \leq 1 \\
(4.42) & \bar{x}(4,1) + \bar{x}(1,5) & \leq 1 \\
(4.43) & \bar{x}(4,1) + \bar{x}(5,3) + \bar{x}(1,5) & \leq 2 \\
(4.44) & \bar{x}(4,1) + \bar{x}(2,5) & \leq 1
\end{array}$$

(4.42) implies that (4.43) holds as equation only if $\bar{x}(5,3) = 1$, but in this case we would have a path of lenght two of arcs (i,j) with $\bar{x}(i,j) = 1$.

We suppose then that (4.43) holds with strict inequality, in this case we have five constraints.

Subcase 3.3.3 (2,4) and (1,3) belong to A', we have the orientation of Figure 4.14.

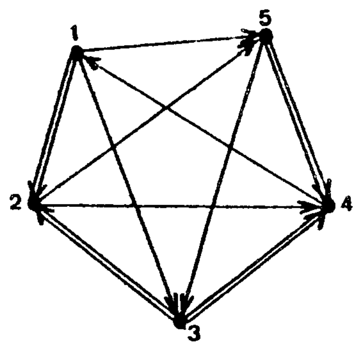


Figure 4.14

$\bar{x}$ satisfies

(4.45)	$\bar{x}(4,1)$	$+ \bar{x}(2,4)$					≤ 1
(4.46)			$\bar{x}(2,5)$	$+ \bar{x}(5,3)$			≤ 1
(4.47)	$\bar{x}(4,1)$				$+ \bar{x}(1,3)$		≤ 1
(4.48)	$\bar{x}(4,1)$					$+ \bar{x}(1,5)$	≤ 1
(4.49)	$\bar{x}(4,1)$			$+ \bar{x}(5,3)$		$+ \bar{x}(1,5)$	≤ 2
(4.50)	$\bar{x}(4,1)$	$+ \bar{x}(2,4)$			$+ \bar{x}(1,3)$		≤ 2
(4.51)	$\bar{x}(4,1)$		$+ \bar{x}(2,5)$				≤ 1
(4.52)	$\bar{x}(4,1)$	$+ \bar{x}(2,4)$		$+ \bar{x}(5,3)$		$+ \bar{x}(1,5)$	≤ 3
(4.53)	$\bar{x}(4,1)$		$+ \bar{x}(2,5)$	$+ \bar{x}(5,3)$			≤ 2
(4.54)	$\bar{x}(4,1)$		$+ \bar{x}(2,5)$		$+ \bar{x}(1,3)$		≤ 2

If any of the inequalities (4.49), (4.50), (4.52), (4.53), (4.54) holds as equation then one of these variables has the value one.

The remaining inequalities are only five.

We have proved the following Lemma.

<u>Lemma 4.55</u> $D(K_5)$ is weakly acyclic.

□

Then we can state the main Theorem of this Section.

<u>Theorem 4.56</u> $D(G)$ is weakly acyclic if and only if G is not contractible to $K_{3,3}$.

□

Let A be a matrix with nonnegative entries. Let $P=\{x|Ax\geq 1, x\geq 0\}$, and let us suppose that this system of inequalities is not redundant. If B is the matrix whose rows are the extreme points of P, the pair (A,B) is called a blocking pair [2] . Actually the extreme points of $Q = \{x|B\geq 1, x\geq 0\}$ are the rows of A.

Theorem 4.56 can be stated as follows.

<u>Theorem 4.56'</u>. Given an undirected graph G, let A be the incidence matrix of the dicycles of D(G), and B the matrix whose rows are the incidence vectors of minimal sets that cover every dicycle. (A,B) is a blocking pair if and only if G is not contractible to $K_{3,3}$.

□

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ANNEXE 6

ON A COMPOSITION OF INDEPENDENCE SYSTEMS BY CIRCUIT-IDENTIFICATION

by

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Abstract

Call an independence system $(E, \mathcal{I})$ linearly relaxable, if the circuit-inequalities $x(C) \leq |C|-1$ together with the constraints $0 \leq x_e \leq 1$ for all $e \in E$ are sufficient to define $P(\mathcal{I})$, the convex hull of the incidence vectors of members of $\mathcal{I}$. Examples are given by independence systems associated with bipartite subgraphs of graphs not contractible to K_5 (cf. Fonlupt et al. [6]) or with acyclic subdigraphs of digraphs obtainable from graphs not contractible to $K_{3,3}$ (cf. Barahona and Mahjoub [1]). Further classes are stable set independence systems of bipartite graphs and, as we are going to show here, a generalization of these within the framework of balanced matrices.

The composition of general bipartite subgraph resp. acyclic subdigraph independence systems and in particular of their associated polyhedra by the identification of a pair of 3-cycles resp. 2-dicycles together with its implications for an algorithmic treatment has been the central subject of the above mentioned two references. We generalize this kind of composition within the framework of independence systems satisfying a special circuit-exchange property and discuss its implications for associated polyhedra, totally dual integral linear systems describing these as well as related optimization problems. Special attention is given to linearly relaxable such independence systems.

Key Words

Independence System, Circuit-Exchange Property, Composition, Polyhedral Description, Total Dual Integrality, Algorithm.

Resumé

Un système d'independants $(E, \mathcal{I})$ est dit linéairement relaxable si les contraintes associés aux circuits $x(C) \leq |C|-1$ et les contraintes triviales $0 \leq x_e \leq 1$ pour $e \in E$ decrivent complètement le polyèdre $P(\mathcal{I})$, enveloppe convexe des vecteurs réprésentatifs des éléments de $\mathcal{I}$. Dans un prémier temps nous montrons que les systèmes d'indépendants dont la matrice d'incidence des circuits est équilibrée, sont linéairement relaxables. Par la suite nous étudions une opération de composition de systèmes d'indépendants liée à une propriété d'échange de circuits. Si $(E_3, \mathcal{I}_3)$ est le système defini par cette opération à partir de $(E_1, \mathcal{I}_1)$ et $(E_2, \mathcal{I}_2)$ alors nous donnons une déscription complète du polyèdre $P(\mathcal{I}_3)$ à partir des polyèdres $P(\mathcal{I}_1)$ et $P(\mathcal{I}_2)$. Nous décrivons également une méthode qui permet de calculer un indépendant de poids maximum dans $(E_3, \mathcal{I}_3)$ à partir de telles éléments dans $(E_1, \mathcal{I}_1)$ et $(E_2, \mathcal{I}_2)$. Ceci nous permettra d'étudier la propriété "totalement dual entier" pour le système linéaire décrivant le polytope $P(\mathcal{I}_3)$.

1) Introduction

The concept of an independence system has turned out to be suitable in embracing a large variety of combinatorial structures. It is defined to be an ordered pair $(E, \mathcal{I})$ consisting of a finite ground set E and a nonempty system $\mathcal{I}$ of subsets of E, which fulfills the following condition of "subclusiveness":

$$J \subseteq I \in \mathcal{I} \rightarrow J \in \mathcal{I}$$

We call the members of $\mathcal{I}$ independent sets and those of $2^E \setminus \mathcal{I}$, 2^E denoting the power set of E, dependent sets. A maximal independent set is a base and a minimal dependent set a circuit of $(E, \mathcal{I})$. The corresponding set systems are denoted by $\mathcal{B}$ and $\mathcal{C}$. Throughout the following we restrict ourselves to normal independence systems, i.e. those for which $\{e\} \in \mathcal{I}$ for all $e \in E$. The restriction $(X, \mathcal{I}_X)$ of $(E, \mathcal{I})$ to $X \in E$ is that independence system with $\mathcal{I}_X = \mathcal{I} \cap 2^X$. With any independence system we may associate an optimization problem, the independent set problem (ISP) as follows:

> Given real weights w_e on the elements of E find an independent set I which has maximum total weight $w(I) = \sum_{e \in I} w_e$.

In general, this problem is NP-Hard but if we restrict ourselves to special classes of independence systems, as for instance given by matroids, 2-matroid-intersections or matchings in a graph, then the resulting problem is polynomially solvable provided one can test in polynomial time whether a given set I is independent or not. Very often the validity of these algorithms could be shown by using linear programming tools such as complementary slackness or the duality theorem. Necessary for such an approach is an explicit knowledge of the associated polytope $P(\mathcal{I})$, defined to be the convex hull of incidence vectors of all members of $\mathcal{I}$. Whereas the "pioneering" results have been obtained by establishing linear systems sufficient to define $P(\mathcal{I})$ for selected classes of independence systems, recent efforts have been made to investigate independence systems, whose associated polytopes can be described in a straightforward way. One such class is given by those independence systems $(E, \mathcal{I})$, for which the circuit-inequalities $x(C) \leq |C|-1, C \in \mathcal{C}$, and the trivial inequalities $0 \leq x_e \leq 1$, $e \in E$, are sufficient to define $P(\mathcal{I})$. In such a case we will speak of a linearly relaxable independence system. By definition examples are given by bipartite subgraph systems in weakly bipartite graphs (cf. Grötschel and Pulleyblank [8]) or acyclic subdigraphs systems in weakly acyclic digraphs (cf. Grötschel, Jünger and Reinelt [7]). Both of these are nontrivial examples since the first class contains as special case bipartite subgraph systems over graphs not contractible to K_5 and the second class contains acyclic subdigraph systems over planar digraphs, for instance. An important characteristic of both classes is the fact, that a minimum weighted odd cycle resp. a minimum weighted dicycle can be found in polynomial time so that together with the ellipsoid method the associated independent set problem can be solved in polynomial time, too. An interesting challenge therefore consists in finding such algorithms for further classes of linearly relaxable independence systems or even for the whole class and, in particular, to develop purely combinatorial such algorithms. A further well known independence system of such a kind is that given by the stable sets in a bipartite graph, the independent set problem of which is known to be polynomially solvable by network-flow techniques. As we will show in section 2), this class can be generalized within the framework of balanced matrices as introduced by Berge [3] thereby providing a characterization of those matroids, which are linearly relaxable.

Very recently, the composition of independence systems arising from bipartite subgraphs of graphs (cf. Fonlupt et al. [6]) or from acyclic subdigraphs of digraphs (cf. Barahona and Mahjoub [1]) has been studied. For the first class a method has been presented, which calculates a bipartite subgraph of maximum weight in a graph G obtained by composing two given graphs G_1 and G_2 in a specific way. Moreover, a system of linear inequalities (in short : a <u>linear system</u>) sufficient to describe the associated polytope $P(\mathcal{I})$ has been shown to be obtainable from those describing $P(\mathcal{I}_1)$ and $P(\mathcal{I}_2)$. For the special case of a "2-sum" $P(\mathcal{I})$ could even be characterized by a minimal such system. Finally, the first paper contains a discussion of the implications for weakly bipartite graphs. A similar investigation has been accomplished for the second class of independence systems. We should point to the fact that in both cases a certain "circuit-exchange property" is satisfied, which we are going to study in full detail in section 3). It was especially this property that gave us the motivation for writing this paper. Its organisation is much along the line of the two aforementioned ones : in section 2) we present linearly relaxable independence systems together with some examples and extensions within the framework of balanced matrices. Section 3) is devoted to a discussion of the "circuit-exchange property" as already mentioned. The composition of independence systems having this property is the subject of the following part. Section 5) contains our results on a composition of associated polyhedra, followed by a study of minimal linear systems sufficient to describe these polyhedra. Algorithmic aspects are treated in section 7) providing at the same time basic tools for a conclusive treatment of linear systems having the property of total dual integrality.

2) <u>Linearly Relaxable Independence Systems</u>

The class of independence systems, the associated polytope of which can be fully described by means of the circuit- and the trivial inequalities and probably known for the longest time is that given by the stable sets in a bipartite graph G. Since these graphs are perfect, the linear system

$$Ax \le \mathbb{1}, \quad x \ge 0,$$

where A is the edge-node incidence matrix of G, has the property of <u>total dual integrality</u>

(TDI), which means that for any integer vector c there is an integral optimum solution to the problem

$$\text{Minimise} \quad \mathbb{1}y \quad \text{subject to} \quad yA \geq c, \quad y \geq 0.$$

This concept has been introduced by Hoffman [9] and Edmonds and Giles [5] and proved to be very useful in formulating a large number of max-min results in a purely combinatorial way. In particular Edmonds and Giles [5] have shown that if a linear system is TDI, then every face of the associated polyhedron contains an integer point. Sufficient for a linear system to fully describe the polytope associated with an independence system is therefore to show that it is TDI and this is exactly the way we will obtain a more general class of linearly relaxable independence systems than that given by bipartite graphs in the above mentioned sense. Berge has introduced the notion of a balanced matrix, defined to be a 0-1 matrix which does not contain a submatrix of odd order having all row and column sums equal to 2. With any 0-1 matrix A, whose rows represent the incidence vectors of a clutter on E, a set of elements numbered by the column indices, we can associate an independence system $(E_A, \mathcal{I}_A)$ in a straightforward way, namely by just taking the rows of A as the incidence vectors of a system $\mathcal{C}_A$ of circuits. In the sequel we will show that the linear system

$$Ax \leq b, \quad 0 \leq x \leq \mathbb{1}$$

where A is balanced, $b^T = (\mathbb{1}a_1 - 1, \dots, \mathbb{1}a_m - 1)$ and a_i is the i-th row vector of A, is TDI. For this let the duplication of an element $e \in E$, say e_1, transform A into a 0-1 matrix A' as shown in Figure 1 :

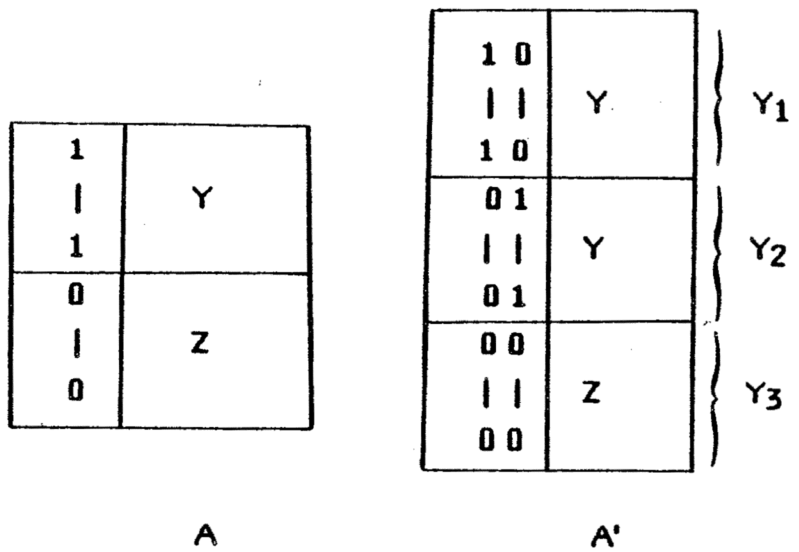


Figure 1

Lemma (2.1)

If A is balanced, then A' as shown in Fig.1 is also balanced.

<u>**Proof**</u>:

Suppose that A' is not balanced, i.e. there is a submatrix C of odd order having row-and column-sums all equal 2. Then since A is balanced C must "use" rows of Y_1 as well as of Y_2. If the first resp. second column of A' is not used by C, then all rows from C within Y_1 resp. Y_2 can be exchanged against their counterparts in Y_2 resp. Y_1, since these in turn cannot be used by C at the same time due to the structure of C. This, however, yields a contradiction to A being balanced and therefore C must use columns 1 <u>and</u> 2 of A'.

<u>**Case 1**</u>

The two lines used by C in Y_1 resp. Y_2 correspond to each other. But then C contains a submatrix of order 4 with row- and column-sums equal to 2, a contradiction to the structure of C.

<u>**Case 2**</u>

Two of the 4 rows (indexed by i_1, i_2 in Y_1 and j_1, j_2 in Y_2) say lines i_1 and j_1, correspond to each other. Then by using only lines j_2 and the counterpart of i_2 in Y_2 yields a forbidden submatrix in A.

<u>**Case 3**</u>

None of the 4 rows i_1, i_2, j_1, j_2 correspond to another. Then w.l.o.g. we can start with line j_2 in Y_2 and proceed line by line "along the odd cycle" (i.e. two consecutive lines have a nonzero element in common) until we reach for instance line i_1 in C. If the number of lines met this way (including lines j_2 and i_1) is odd, the exchange of line i_1 against its counterpart in Y_2 and the deletion of lines i_1, i_2, j_1 from C yields a forbidden submatrix in A. If the number of lines met is even, then the remaining lines within C form an odd number of lines and the exchange of line i_2 against its counterpart in Y_2 and the deletion of rows i_1, i_2, j_2 from C yields also a forbidden submatrix in A, a contradiction. ■

From the theory of balanced hypergraphs (cf. Berge and Las Vergnas [3]) it is known that the maximum cardinality ν of pairwise disjoint edges in such a hypergraph equals the minimum cardinality τ of a transversal. This implies for $(E_A, \mathcal{C}_A)$ that the maximum cardinality $r(E)$ of an independent set within E, also called the <u>rank</u> of E, equals $|E| - \nu$. Hence, if M represents a collection of circuits within $\mathcal{C}_A$ forming a matching of cardinality ν in the associated hypergraph we get

$$|E| - \nu = \sum_{C \in M} (|C|-1) + |E \setminus \bigcup_{C \in M} C| = r(E)$$

This in turn gives us a 0–1 solution y to the dual program of

$$(P) \quad \begin{array}{ll} \text{Maximise} & \sum_{e \in E} x_e \\ \text{s.t.} & \sum_{e \in C} x_e \leq |C|-1 \qquad \forall C \in \mathcal{C}_A \\ & 0 \leq x_e \leq 1 \qquad \forall e \in E, \end{array}$$

more precisely,

$$Y_F = 1 \iff F \in M \text{ or } F = \{e\} \text{ with } e \in (E \setminus \bigcup_{C \in M} C).$$

Thus for $a \equiv \mathit{1}$ we have an integral optimum solution to (D), the dual of (P). It remains to show the existence of such a solution for every other integral vector a, which we can suppose to have strictly positive components without loss of generality. If then $\bar{A}$ denotes the incidence matrix of all those circuits, which define the corresponding restriction of $(E_A, \mathcal{I}_A)$, we can "multiply" $\bar{A}$ by a by duplicating consecutively column i of $\bar{A}$ a_i times just as described in Figure 1. Due to Lemma (2.1) the resulting matrix $\bar{A}_a$ is again a balanced matrix. As just demonstrated for $a \equiv 1$ we can now find an optimum 0-1 solution to the corresponding dual program over $\bar{A}_a$, which can be transformed into an integral optimum solution to the dual of the problem

$$\begin{array}{ll} \text{Maximize} & a^T x \\ \text{s.t.} & Ax \leq b, \\ & 0 \leq x \leq \mathit{1}. \end{array}$$

Summing up we have thus shown

<u>Theorem (2.2)</u>

Let A be the incidence matrix of a clutter over E. Then if A is balanced the linear system

$$Ax \leq b, \quad 0 \leq x \leq \mathit{1}$$

is TDI (implying that $(E_A, \mathcal{I}_A)$ is linearly relaxable).

Using this result we are now able to characterize those matroids, which are at the same time linearly relaxable :

<u>Corollary (2.3)</u>

If $(E, \mathcal{I})$ is a matroid then $(E, \mathcal{I})$ is linearly relaxable iff the members of $\mathcal{C}$ are pairwise disjoint.

<u>Proof:</u>

"->"

Suppose there are two circuits $C_1, C_2 \in \mathcal{C}$ having non empty intersection. Then we may consider the restriction of $(E, \mathcal{C})$ to $C_1 \cup C_2$. Clearly, the inequality $\sum_{e \in C_1 \cup C_2} x_e \leq |C_1 \cup C_2| - 1$ defines a facet of the polytope $P(\mathcal{I}_{C_1 \cup C_2})$, which may be lifted to a facet of $P(\mathcal{I})$. This, however, yields a contradiction to the property of $(E, \mathcal{I})$ to be linearly relaxable.

"<-"

If the members of $\mathcal{C}$ are pairwise disjoint, the incidence matrix A of $\mathcal{C}$ is clearly balanced. By Theorem (2.2) $(E, \mathcal{I})$ is linearly relaxable.

■

We mention at this point, that there is still no polynomial algorithm for the solution of the associated independent set problem.

A further example is given by <u>bipartite subgraph independence systems</u> over so called "weakly bipartite" graphs. For a precise explanation let $G = [V,E]$ be a finite, undirected graph and $\mathcal{I}$ be the collection of all those edge sets in G, which induce a bipartite subgraph. Clearly, the corresponding system $\mathcal{C}$ of circuits consists of all those sets of edges, which induce an elementary odd cycle. Finally, a graph $G = [V,E]$ is called <u>weakly bipartite</u>, if its bipartite subgraph independence system is linearly relaxable. Due to a result of Grötschel and Pulleyblank [8] a minimum weighted odd cycle in an arbitrary graph G can be found in polynomial time so that together with the ellipsoid method a polynomial algorithm to solve the so-called "max-cut"- problem over weakly bipartite graphs is at hand. Also note that this class contains as a special case those graphs which are not contractible to K_5 (cf. Fonlupt et al. [6]).

Similar results can be derived for "weakly acyclic" digraphs respectively acyclic subdigraph independence systems over digraphs. The interested reader is referred to the papers by Grötschel, Jünger and Reinelt [7] as well as Barahona and Mahjoub [1] for the details.

We close this section by stating that both the circuits of bipartite subgraph and acyclic subdigraph independence systems share a certain circuit-exchange property, which we will discuss in more detail in the next section. We feel that this abstract property deserves attention for its own, in particular in connection with the problem of finding a minimum weighted circuit among a collection C of such circuits.

3) A Certain Circuit-Exchange Property

As already announced previously we are now going to introduce independence systems, whose systems of circuits share a certain "exchange-property". We will see later that it is this property which is to be kept when composing independence systems by circuit-identification.

For a precise presentation let C be a circuit of an independence system $(E, \mathcal{I})$ and (C^1, C^2) denote a partition of C into two nonempty subsets of C. We say that the system $\mathcal{C}$ of circuits has the exchange-property, if the following condition holds:

$(\mathcal{C}1)$ For all $C, C_1, C_2 \in \mathcal{C}$ and all partitions (C^1, C^2) of C such that $C \cap C_1 = C^1$, $C \cap C_2 = C^2$ there exists a further circuit $C_3 \in \mathcal{C}$ which is contained in $(C_1 \cup C_2) \setminus C$.

It is not difficult to see that this property is especially shared by the circuits of a binary matroid. Using a simple counting argument it can be shown for elementary cycles of odd cardinality in a graph as well. A similary easy argument based on the orientations of edges finally may be used to verify this property for elementary dicycles in a digraph, so that the "prototypes" of both bipartite subgraph and acyclic subdigraph independence systems are recognized to share property $(\mathcal{C}1)$. For general stable set independence systems over graphs we can derive

Proposition 3.1

Let $G = [V,E]$ be a finite, undirected graph, let $K_i = [V_i, E_i]$, $i = 1,\ldots,m$, denote its connected components and $(V, \mathcal{I})$ be the independence system given by all stable sets of G having E as system of circuits. Then E satisfies condition $(\mathcal{C}1)$ if and only if the components K_i induce either complete bipartite graphs or complete subgraphs, i.e. cliques.

Proof:

If K_i does not contain an odd elementary cycle let (V_i^1, V_i^2) be the partition of V_i such that every edge from E_i connects an element from V_i^1 to one from V_i^2. If the nodes $v_1, v_2 \in V_i^1$, V_i^2, respectively, are not connected by such an edge, by connectedness of K_i and the circuit-exchange-property we obtain a contradiction.

If K_i does contain an odd elementary cycle C, then again by the circuit-exchange property applied to that "halfcycle" of C connecting two nodes of C and using an even number of nodes, the subgraph of K_i induced by C must be complete. Moreover, any other node connected somehow to C must be adjacent to every other node of C and, therefore, K_i must be a complete subgraph of G. ∎

So far the examples for this special class of independence systems. As we will see in the following sections, the composition of independence systems by "circuit-identification" will be defined in such a way as to maintain exactly property ($\mathcal{C}1$) for the resulting independence system. An appropriate treatment of linear inequalities (the "mixing") will enable us to derive similar results for the composition of associated polyhedra, for an algorithmic approach and, at the end, within the framework of total dual integrality.

4) Composing Independence Systems by Circuit-Identification

This section is devoted to the presentation of those ideas which underly the operation of composing independence systems having property ($\mathcal{C}1$) by circuit-identification. To enable this identification each of the two independence systems must contain a circuit which fulfills certain properties. The first of these can be stated as follows. If $(E,\mathcal{I})$ is given together with its system $\mathcal{C}$ of circuits there is a $C \in \mathcal{C}$ and a partition (C^1, C^2) of C such that

($\mathcal{C}2$) For all $C' \in \mathcal{C}$, $C' \cap C \neq \emptyset$ implies either that $C' \cap C = C^1$ or that $C' \cap C = C^2$.

Now let $(E_1, \mathcal{I}_1)$, $(E_2, \mathcal{I}_2)$ be two independence systems and C_1, C_2 members of $\mathcal{C}_1, \mathcal{C}_2$ respectively, together with partitions $(C_1{}^1, C_1{}^2)$, $(C_2{}^1, C_2{}^2)$ so that ($\mathcal{C}2$) is fulfilled for both with respect to C_1 and C_2 as well as their partitions. In order to be able to identify C_1 with C_2 we will have to assure that the partitions given are "compatible", which means that

($\mathcal{C}3$) $|C_1{}^1| = |C_2{}^1|$ and $|C_1{}^2| = |C_2{}^2|$.

This, however, can always be achieved by "contracting" $(E_1, \mathcal{I}_1)$ resp. $(E_2, \mathcal{I}_2)$ "appropriately", an operation, which can be defined as follows : given $(E, \mathcal{I})$ and an element $e \in E$ the independence system obtained by <u>contracting e</u> is $(E \setminus \{e\}, \mathcal{I}^e)$ with $\mathcal{I}^e = \{I \subseteq E \setminus \{e\} : I \cup \{e\} \in \mathcal{I}\}$ or equivalently, $(E \setminus \{e\}, \mathcal{C}^e)$ where

$$\mathcal{C}^e = \{C \in \mathcal{C} : (e \notin C) \text{ and } (C' \setminus \{e\}) \not\subseteq C \text{ for all } C' \in \mathcal{C} \text{ containing } e\}$$
$$\cup \{C \setminus \{e\} : C \in \mathcal{C} \text{ and } e \in C\}.$$

As pointed out in Cunningham [4] it is not only easy to get polyhedral descriptions for $(E \setminus \{e\}, \mathcal{I}^e)$ from one for $(E, \mathcal{I})$; one can also solve the independent set problem for $(E \setminus \{e\}, \mathcal{I}^e)$ efficiently, whenever this is the case for $(E, \mathcal{I})$.

We are now ready to define the composition $(E_3, \mathcal{I}_3)$ of two independence systems $(E_i, \mathcal{I}_i)$, $i=1,2$, sharing property $(\mathcal{C}1)$ and possessing at the same time two distinguished circuits C_1 and C_2 together with partitions $(C_1^1, C_1^2), (C_2^1, C_2^2)$ such that $(\mathcal{C}2)$ and $(\mathcal{C}3)$ are satisfied with respect to these.

The new ground set is given by

$$E_3 := E_1 \cup (E_2 \setminus C_2).$$

The new system $\mathcal{C}_3$ of circuits consists of two systems of subsets of E:

- $\mathcal{C}_3^1 := \mathcal{C}_1 \cup \bar{\mathcal{C}}_2$, where $\mathcal{C}_1$ is the system of circuits of $(E_1, \mathcal{I}_1)$ and $\bar{\mathcal{C}}_2$ is obtained from $\mathcal{C}_2$ by identifying the elements in C_2^1 resp. C_2^2 with those in C_1^1 resp. C_1^2 in some injective way;
- $\mathcal{C}_3^2 := \{(C' \cup C'') \setminus C_1 : (C_1^1 \subseteq C'$ and $C_1^2 \subseteq C'')$

  or $(C_1^2 \subseteq C'$ and $C_1^1 \subseteq C'')$

  and $C' \in \mathcal{C}_1, C'' \in \bar{\mathcal{C}}_2\}$

Clearly, the structure of $\mathcal{C}_3^2$ ensures that $(E_3, \mathcal{I}_3)$ again satisfies the exchange-property $(\mathcal{C}1)$ as introduced in section 3).

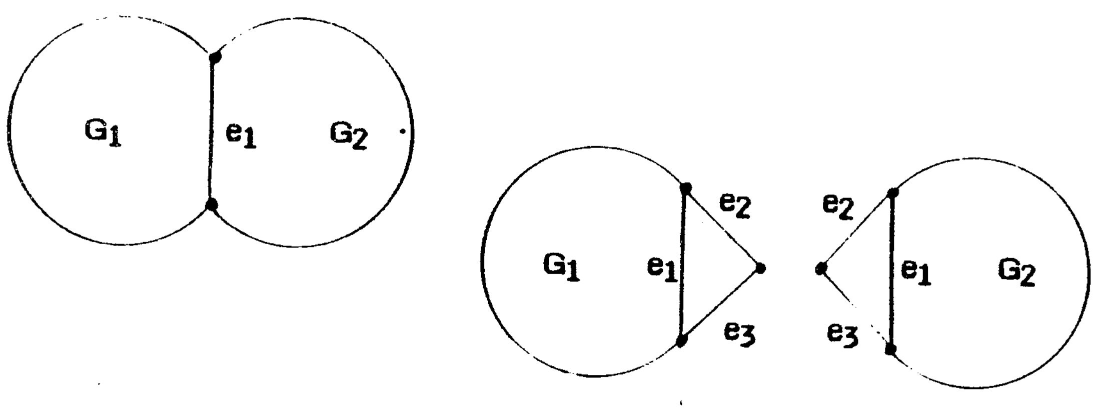


Figure 2

As an example just consider a so-called 3-sum of two graphs $G_1 = [V_1, E_1]$, $G_2 = [V_2, E_2]$ as investigated by Fonlupt et al. in [6] in connection with the max-cut problem over the 2-sum of 2 given graphs (see Fig. 2). One easily verifies that this operation corresponds exactly to the more general composition as introduced above when specialised to bipartite subgraphs and distinguished 3-element circuits. The approach undertaken by Barahona and Mahjoub in [1] for the acyclic subdigraph independence systems is exactly along the same line.

To conclude this section we would like to formulate a result, which is of more technical a nature and which will be useful for many of the proofs to follow. For this let $(E, \mathcal{I})$ be an independence system having $\mathcal{C}$ as system of circuits. Moreover, let $C \in \mathcal{C}$ and (C^1, C^2) be a partition of C such that $(\mathcal{C}1)$, $(\mathcal{C}2)$ hold with respect to $(E, \mathcal{I})$, C and (C^1, C^2). Then we have

<u>Lemma (4.1)</u>

Any base of $(E, \mathcal{I})$ contains $|C|-1$ many elements from C.

<u>Proof</u>:

Suppose there is a base $B \in \mathcal{B}$ containing less than $|C|-1$ many elements from C.

<u>Case 1</u> $\quad C^1 \subsetneq B$ or $C^2 \subsetneq B$

Suppose $C^1 \subseteq B$, i.e. $|B \cap C^2| < |C^2|-1$ by supposition. Since B is a base, there is a circuit in $B \cup \{e\}$ for every $e \in E \setminus B$. So let $e_0 \in C^2 \setminus B$ and C' be a circuit contained in $B \cup \{e_0\}$. Then C' contains at most $|C^2|-1$ many elements from C, a contradiction to property $(\mathcal{C}2)$ with respect to C and (C^1, C^2).

<u>Case 2</u> $\quad C^1 \not\subseteq B$ and $C^2 \not\subseteq B$

Let $e_1 \in C^1 \setminus B$ and $e_2 \in C^2 \setminus B$. If now $(B \cap C^1) \cup \{e_1\} \neq C^1$ or $(B \cap C^2) \cup \{e_2\} \neq C^2$, the situation reduces to Case 1. Therefore, we may suppose that

$$(B \cap C^1) \cup \{e_1\} = C^1 \text{ and } (B \cap C^2) \cup \{e_2\} = C^2.$$

Since B is a base, there are two circuits C' and C" such that

$$C' \subsetneq B \cup \{e_1\} \text{ (and, moreover, } C^1 = C' \cap C\text{)},$$
$$C'' \subsetneq B \cup \{e_2\} \text{ (and, moreover, } C^2 = C'' \cap C\text{)}.$$

By $(\mathcal{C}1)$ there exists a circuit C_3 in $(C' \cup C'') \setminus C$, a contradiction to the membership of B in $\mathcal{I}$

■

5) Composing Associated Polyhedra

The main subject of this section will be the discussion of the composition of polyhedra associated with independence systems as they have been studied in the last section. It will turn out that in particular the operation of "mixing" circuits has a straightforward counterpart in terms of linear inequalities. First of all let us introduce some basic notions from polyhedral theory. A polyhedron $P \subseteq R^m$ is the intersection of finitely many halfspaces in R^m, i.e. sets of points of the form $\{x \in R^m : a^T x \le a_0\}$, where $a \in R^m \setminus \{0\}$, $a_0 \in R$. If P is bounded or, equivalently, the convex hull of finitely many points we speak of a polytope. A linear inequality $a^T x \le a_0$ with $a \in R^m \setminus \{0\}$, $a_0 \in R$, is said to be valid for P, if $P \subseteq \{x \in R^m : a^T x \le a_0\}$. A subset F of P is called a face of P, if there exists an inequality $a^T x \le a_0$ valid for P such that $F = \{x \in P : a^T x = a_0\}$. We also say that $a^T x \le a_0$ defines F. A face F is called proper, if $F \neq P$. A proper, nonempty face maximal (minimal) with respect to set-inclusion is called a facet (extreme point or vertex) of P. Since we only deal with full-dimensional polytopes, any such P has a representation of the form $P = \{x \in R^m : Ax \le b\}$. We will particularly consider those polytopes, which represent the convex hull of incidence vectors x^I of independent sets in a given independence system $(E, \mathcal{I})$.

So let $(E, \mathcal{I})$ be an independence system, $\mathcal{C}$ its system of circuits and $C \in \mathcal{C}$ together with a partition (C^1, C^2) such that $(\mathcal{C}1)$, $(\mathcal{C}2)$, $(\mathcal{C}3)$ hold. The following is the first of a sequence of lemmata, which will lead us to the main result of this section. It concerns the support $S(a)$ of certain inequalities $a^T x \le a_0$, which is defined to consist of all those elements $e \in E$, for which $a_e \neq 0$.

Lemma (5.1)

The inequality $\sum_{e \in C} x_e \le |C|-1$ for the distinguished circuit C defines a facet of $P(\mathcal{I})$.

Moreover, for the support S of any other facet-defining inequality the following holds:

$S \cap C \neq \emptyset \Rightarrow$ either $S \cap C = C^1$ or $S \cap C = C^2$.

Proof:

Clearly, $x(C) := \sum_{e \in C} x_e \le |C|-1$ defines a facet of $P(\mathcal{I}_C)$, the polytope associated with the restriction $(C, \mathcal{I}_C)$. Now any $e \in E \setminus C$ can be enlarged to a base B of E containing $|C|-1$ many elements from C due to Lemma (4.1). Therefore, the coefficients for the remaining elements

can all be lifted to zero, which proves the first part of our assertion.
For the second part suppose w.l.o.g. that there is a facet-defining inequality $a^T x \le a_0$, whose support $S(a)$ has nonempty intersection with C^1, but $S(a) \cap C^1 \neq C^1$, where $|C^1| \ge 2$ can be assumed. Let $I \in \mathcal{I}$ with $a^T x^I = a_0$ and let e_0 be an element of $(C^1 \cap S(a)) \setminus I$. Then we can enlarge I to a base B of E (only by elements e having $a_e = 0$), which by Lemma (4.1) has $|C|-1$ many elements in common with C. Since $a_e > 0$, B does not contain e_0. Moreover, $B \cup \{e_0\}$ contains C and possibly a number of those circuits, which contain exactly C^1. Therefore, $B \cup \{e_0\} \setminus \{e_1\} \in \mathcal{I}$ where $e_1 \in C^1 \setminus (S(a) \cup \{e_0\})$ with $a_{e_1} = 0$. This implies that the corresponding incidence vector does not satisfy the inequality $a^T x \le a_0$, a contradiction. Consequently, any such I must contain $S(a) \cap C^1$ entirely. But then the inequality $a^T x \le a_0$ cannot define a facet of $P(\mathcal{I})$. By a similar reasoning we can derive that the coefficients a_e, $e \in S(a) \cap C^1$, must all be equal.

Finally, suppose that, in addition to $x(C) \le |C| - 1$, there is a facet-defining inequality $a^T x \le a_0$ for $P(\mathcal{I})$, whose support contains C. Then every incidence vector x^I of an independent set I, which satisfies this inequality with equality, must contain $|C|-1$ many elements of C. Otherwise, by Lemma (4.1) $a^T x \le a_0$ would not be valid for $P(\mathcal{I})$. But then $a^T x \le a_0$ and $x(C) \le |C|-1$ define the same hyperplane, a contradiction.

■

Using Lemma (5.1) the two polytopes $P(\mathcal{I}_1)$ and $P(\mathcal{I}_2)$ can be described by linear systems having the following form ($(C_1{}^1, C_1{}^2)$, $(C_2{}^1, C_2{}^2)$ are the partitions of the distinguished circuits):

$$P(\mathcal{I}_1)\left\{\begin{array}{llll}
\sum\limits_{e \in \bar{E}_1} a_e^i x_e & \le a_0^i & \text{for } i \in K_1 & (5.2) \\[2ex]
\sum\limits_{e \in \bar{E}_1} a_e^i x_e + \sum\limits_{e \in C_1{}^1} x_e & \le a_0^i & \text{for } i \in K_2 & (5.3) \\[2ex]
\sum\limits_{e \in \bar{E}_1} a_e^i x_e + \sum\limits_{e \in C_1{}^2} x_e & \le a_0^i & \text{for } i \in K_3 & (5.4) \\[2ex]
\sum\limits_{e \in C_1} x_e & \le |C_1| - 1 & & (5.5) \\[2ex]
0 \le x_e \le 1 & & \text{for all } e \in E_1 & (5.6)
\end{array}\right.$$

$$
P(\mathcal{I}_2)\begin{cases}
\sum_{e\in\bar{E}_2} b_e^j x_e \le b_0^j & \text{for } j\in L_1 & (5.7)\\[2ex]
\sum_{e\in\bar{E}_2} b_e^j x_e + \sum_{e\in C_2^1} x_e \le b_0^j & \text{for } j\in L_2 & (5.8)\\[2ex]
\sum_{e\in\bar{E}_2} b_e^j x_e + \sum_{e\in C_2^2} x_e \le b_0^j & \text{for } j\in L_3 & (5.9)\\[2ex]
\sum_{e\in C_2} x_e \le 1 \le |C_2|-1 & & (5.10)\\[2ex]
0\le x_e\le 1 & \text{for all } e\in E_2 & (5.11)
\end{cases}
$$

where $\bar{E}_i = E_i \setminus C_i$ for $i = 1,2$.

Let us now proceed to define "mixed constraints", in some sense the geometric analogue of the members of $\mathcal{C}_3^2$.

<u>Definition (5.12)</u>

Given a constraint of the type (5.3) (resp. (5.4)) and one of type (5.9) (resp. (5.8)) the associated <u>mixed constraint</u> is given by

$$(5.13)\qquad \sum_{e\in\bar{E}_1} a_e^i x_e + \sum_{e\in\bar{E}_2} b_e^j x_e \le a_0^i + b_0^j - (|C_1|-1).$$

An immediate consequence is

<u>Lemma (5.14)</u>

Any mixed constraint (5.13) is valid for $P(\mathcal{I}_3)$.

<u>Proof:</u>

Let I be a member of $\mathcal{I}_3$, $I_1 := I\cap E_1$, $I_2 := I\cap(\bar{E}_2\cup C_1)$ and $\bar{I}_2$ the corresponding member of $\mathcal{I}_2$. Clearly, I_1, and $\bar{I}_2$ can be enlarged to bases B_1, B_2 of E_1, E_2, containing $|C_1|-1$, $|C_2|-1$ many elements of C_1, C_2, respectively, by Lemma (4.1). Since x^{B_1}, x^{B_2} both satisfy the constraints for $P(\mathcal{I}_1)$, $P(\mathcal{I}_2)$ used to form the mixed constraint, x^I satisfies this latter constraint.

■

In order to establish a linear system sufficient to describe $P(\mathcal{I}_3)$ and based upon those for $P(\mathcal{I}_1)$, $P(\mathcal{I}_2)$ we will introduce an "auxiliary" independence system $(E_3, \mathcal{C}_1^*)$, which is induced by the following system of circuits:

$$\mathcal{C}_1^* = \{ C \in \mathcal{C}_3 : C_1^2 \not\subseteq C \}$$

It is a polyhedral description of this independence system that we are now going to present. For this let $\underline{P}$ be the polyhedron defined by all constraints of type (5.2), (5.3), (5.6), (5.7), (5.8), (5.11) and the collection of all possible constraints (5.13).

<u>Lemma (5.15)</u>

Any inequality $x(C) \leq |C| - 1$ for $C \in \mathcal{C}_1^*$ is redundant with respect to the inequalities defining $\underline{P}$.

<u>Proof:</u>

Let C be a circuit from $\mathcal{C}_1^*$. If $C \subseteq E_1 \setminus C_1^2$ or $C \subseteq \bar{E}_2 \cup C_1^1$ then the associated inequality is redundant with respect to (5.2), (5.3), (5.7), (5.8). Moreover, if there is a circuit $C \in \mathcal{C}_1^*$ satisfying $C \cap E_1 \neq \emptyset$ and $C \cap \bar{E}_2 \neq \emptyset$ then by definition of $(E_3, \mathcal{I}_3)$ this circuit is obtained by mixing two circuits $C_1 \in \mathcal{C}_1$ and $C_2 \in \mathcal{C}_2$. Since the corresponding inequalities can be lifted to facet-defining ones for $P(\mathcal{I}_1)$, $P(\mathcal{I}_2)$, respectively, the circuit-inequality $x(C) \leq |C| - 1$ can be obtained from a mixed constraint by putting the remaining coefficients to zero.

■

<u>Lemma (5.16)</u>

Given two inequalities of type (5.3) and (5.4), i.e.

$$\sum_{e \in \bar{E}_1} a_e^i x_e + \sum_{e \in C_1^1} x_e \leq a_0^i, \ i \in K_2 \quad \text{and} \quad \sum_{e \in \bar{E}_1} a_e^j x_e + \sum_{e \in C_1^2} x_e \leq a_0^j, \ j \in K_3,$$

both valid for $P(\mathcal{I}_1)$, the mixed inequality

$$\sum_{e \in \bar{E}_1} (a_e^i + a_e^j) x_e \leq a_0^i + a_0^j - (|C_1| - 1)$$

is also valid for $P(\mathcal{I}_1)$.

Proof:

In analogy to the proof of Lemma (5.14) any $I \in \mathcal{I}_1$ can be enlarged to a base B_1 of E_1 having $|C_1| - 1$ many elements in common with C_1, the distinguished circuit. The assertion follows immediately.

■

Lemma (5.17)

Let $\bar{x}$ be a feasible solution of the linear system defining P and let at least be one of the inequalities (5.3), (5.8) or $x(C_1{}^1) \le |C_1{}^1|$ be satisfied with equality by $\bar{x}$. Moreover, let $\delta := |C_1{}^1| - \bar{x}(C_1{}^1)$ (≥ 0), e_0 be a distinguished element from $C_1{}^2$ resp. $C_2{}^2$ and

$$x_e{}^1 := \begin{cases} \bar{x}_e & \text{for } e \in E_1 \setminus C_1{}^2, \\ 1 & \text{for } e \in C_1{}^2 \setminus \{e_0\}, \\ \delta & \text{for } e = e_0 \end{cases}$$

$$\bar{x}_e{}^2 := \begin{cases} \bar{x}_e & \text{for } e \in E_2 \setminus C_2{}^2, \\ 1 & \text{for } e \in C_2{}^2 \setminus \{e_0\}, \\ \delta & \text{for } e = e_0 \end{cases}$$

Then $\bar{x}^1, \bar{x}^2$ are both feasible for $P(\mathcal{I}_1)$, $P(\mathcal{I}_2)$. Furthermore, a mixed constraint (5.13) is satisfied by $\bar{x}$ with equality iff the associated inequalities of type (5.3) (resp. (5.4)) and (5.9) (resp. (5.8)) are satisfied with equality by $\bar{x}^1, \bar{x}^2$.

Proof:

First of all we claim that $\bar{x}$ has to satisfy the inequality $\bar{x}(C_1{}^1) \ge |C_1{}^1| - 1$. Otherwise, the constraint assumed to be satisfied with equality by $\bar{x}$ would be of type (5.3) or (5.8). But then

$$\sum_{e \in \bar{E}_1} a_e{}^1 \bar{x}_e > a_0{}^1 - (|C_1{}^1| - 1) \quad \text{resp.} \quad \sum_{e \in \bar{E}_2} b_e{}^j \bar{x}_e > b_0{}^j - (|C_2{}^1| - 1).$$

This, however, contradicts the validity of these inequalities with ">" replaced by "≤", which must hold due to a similar reasoning as has been used for the proofs of Lemmata (5.14) and (5.16).

Consequently, $0 \le \delta \le 1$ and thus the trivial inequalities are fulfilled. We will now proceed to showing that $\bar{x}^1 \in P(I_1)$ (the proof for $x^2 \in P(I_2)$ is almost identical).

Clearly, the constraints (5.2), (5.3) are fulfilled by $\bar{x}^1$. If now $x(C_1^1) = |C_1^1|$ we know from the definition of P that an inequality of type (5.4), i.e.

$$\sum_{e \in \bar{E}_1} a_e^i x_e + \sum_{e \in C_1^2} x_e \le a_0^i \quad \text{for} \quad i \in K_3$$

is satisfied by $\bar{x}^1$, since the inequality

$$\sum_{e \in \bar{E}_1} a_e^i x_e \le a_0^i + |C_1^1| - (|C_1|-1) \; (= a_0^i - (|C_1^2|-1)),$$

obtained by mixing

$$\sum_{e \in \bar{E}_1} a_e^i x_e + \sum_{e \in C_1^2} x_e \le a_0^i \text{ with } x(C_1^1) \le |C_1^1|,$$

is satisfied by $\bar{x}^1$ and $\bar{x}^1(C_1^2) = |C_1^2|-1$.

If, however, $x(C_1^1) < |C_1^1|$, there is an inequality of type (5.3) or (5.8) satisfied by $\bar{x}$ with equality.
If we have

$$(+) \; \sum_{e \in \bar{E}_1} a_e^{i_0} \bar{x}_e + \sum_{e \in C_1^1} \bar{x}_e = a^{i_0},$$

then for an inequality of type (5.4) we get together with Lemma (5.16) that

$$(++) \; \sum_{e \in \bar{E}_1} (a_e^{i_0} + a_e^i) \bar{x}_e \le a_0^{i_0} + a_0^i - (|C_1|-1).$$

Subtracting (+) from (++) and taking account of the fact that $\bar{x}(C_1) \le |C_1|-1$ we obtain the validity of all inequalities (5.4) for $\bar{x}^1$.

If in turn

$$\sum_{e\in\bar{E}_2} b_e^{j_0}\bar{x}_e + \sum_{e\in C_2^1} \bar{x}_e = b_0^{j_0}, \text{ then}$$

$$\sum_{e\in\bar{E}_1} a^i \bar{x}_e + \sum_{e\in E_2} b_e^{j_0}x_e \le a_0^i + b_0^{j_0} - (|C_1|-1)$$

and by subtraction we obtain as above that

$$\sum_{e\in\bar{E}_1} a_e^i \bar{x}_e + \sum_{e\in C_1^2} \bar{x}_e \le a_0^i.$$

Note that by identification of C_1 and C_2 we can replace $\bar{x}(C_1^2)$ by $\bar{x}(C_2^2)$ etc.

Finally, it is obvious that by definition of $\bar{x}^1, \bar{x}^2$ a mixed constraint is satisfied with equality iff both inequalities used for this are satisfied with equality by $\bar{x}^1, \bar{x}^2$, respectively.

■

We are now able to state the key result of this section:

__Theorem (5.18)__

$P(\mathcal{I}_1^*) = P$.

__Proof :__

By Lemma (5.15) a point $x\in P$ is integral iff it is the incidence vector of a set I independent in $\mathcal{I}_1^*$. Since $P(\mathcal{I}_1^*)$ is contained in P, we only have to show that every extreme point of P is integral.
For this suppose there is an extreme point of P having at least one nonintegral component.

__Case 1__

There is an element $e^*\in C_1^1$ such that $0 < \bar{x}_{e^*} < 1$; in this case $\bar{x}$ must satisfy at least one of the constraints (5.3), (5.8) with equality, for otherwise $\bar{x}$ would not define an extreme point of P. Let

$\bar{x}^1$, $\bar{x}^2$ be defined as in Lemma (5.17), by which we have seen that $\bar{x}^1 \in P(\mathcal{I}_1)$ and $\bar{x}^2 \in P(\mathcal{I}_2)$. Consequently, there is a representation of the form

$$\bar{x}^1 = \sum_{i=1}^{z} \nu_i x^{S_i^1}, \quad \bar{x}^2 = \sum_{j=1}^{s} \mu_j x^{S_j^2}$$

with $\sum_{i=1}^{z} \nu_i = 1$, $\sum_{j=1}^{s} \mu_j = 1$, $\nu_i, \mu_j \geq 0$ and $S_i^1 \in \mathcal{I}_1$, $S_j^2 \in \mathcal{I}_2$ for all i, j.

We note at this place that any constraint satisfied by $\bar{x}^1$ resp. $\bar{x}^2$ with equality is at the same time satisfied with equality by $x^{S_i^1}$, $i = 1,\dots, z$ and $x^{S_j^2}$, $j = 1,\dots, s$. respectively. Now since $\bar{x}_{e^*} < 1$, there must exist indices i_0 and j_0 such that $e^* \notin S_{i_0}^1$ and $e^* \notin S_{j_0}^2$. By definition of $\bar{x}^1, \bar{x}^2$ it follows that $C_1^2 \subseteq S_{i_0}^1 \cap S_{j_0}^2$. Let $S := (S_{i_0}^1 \cup S_{j_0}^2) \setminus C_1^2$.
S is a member of $\mathcal{I}_1^*$ as is clear from the independence of $S_{i_0}^1$, $S_{j_0}^2$ and the structure of the circuits of $(E_3, \mathcal{I}_1^*)$.

We now claim that any constraint satisfied by $\bar{x}$ with equality is at the same time satisfied with equality by x^S. This is immediately clear for constraints of type (5.2), (5.3), (5.7), (5.8) as well as the trivial inequalities. So let us consider a mixed constraint (5.13). By Lemma (5.17) the two constraints used for getting the mixed one are satisfied with equality by $\bar{x}^1$ and $\bar{x}^2$. In addition, the incidence vectors of $S_{i_0}^1$ and $S_{j_0}^2$ satisfy the inequality $x(C_1) \leq |C_1|-1$ with equality, since this is the case for $\bar{x}^1$ and $\bar{x}^2$ by definition and the result stated in Lemma (5.17). Therefore, x^S satisfies the mixed constraint with equality, too. Since $x^S \neq \bar{x}$, $\bar{x}$ cannot define an extreme point, a contradiction.

<u>Case 2</u>

For all $e \in C_1^1$ we have $\bar{x}_e = 1$;
In this case there are $S_{i_0}^1$ and $S_{j_0}^2$ such that $C_1^1 \subseteq S_{i_0}^1 \cap S_{j_0}^2$. Putting $S := (S_{i_0}^1 \cup S_{j_0}^2) \setminus C_1^1$, we again obtain a member of $\mathcal{I}_1^*$, whose incidence vector satisfies the same constraints with equality as $\bar{x}$, a contradiction.

Case 3

There is an element $e^* \in C_1^1$ with $\bar{x}_{e^*} = 0$; if there is at least one inequality of type (5.3) or (5.8), which is satisfied with equality by $\bar{x}$, case 1 applies.
Otherwise, we can augment $\bar{x}_{e^*}$ properly to obtain another extreme point of P, which satisfies one of the constraints (5.3) or (5.8) or $x(C_1^1) \leq |C_1^1|-1$ with equality. But then again case 1 applies.

■

Corollary (5.19)

If $\mathcal{I}_2^*$ is obtained by deleting from $\mathcal{C}_3$ all those circuits which contain C_1^1, the linear system given by the inequalities (5.2), (5.4), (5.7), (5.9), the trivial inequalities and all mixed inequalities (5.13) fully describe the polytope $P(\mathcal{I}_2^*)$.

Corollary (5.20)

The polytope $P(\mathcal{I}_3)$ is fully described by the constraints (5.2), (5.3), (5.4), (5.7), (5.8), (5.9), (5.13), the inequality $x(C_1) \leq |C_1|-1$ as well as all trivial inequalities $0 \leq x_e \leq 1$ for $e \in E_3$.

Proof :

First, we show that the support of a facet-defining inequality of $P(\mathcal{I}_3)$ does not contain C_1 (except if it is the circuit-inequality itself). For suppose such a further inequality $a^T x \leq a_0$ exists. By Lemma (4.1) and our assumption any independent set I, whose incidence vector x^I satisfies this inequality with equality must already contain $|C_1|-1$ many elements from C_1. But then $a^T x \leq a_0$ defines the same hyperplane as $x(C_1) \leq |C_1|-1$, a contradiction.
So if the constraints proposed do not suffice to describe $P(\mathcal{I}_3)$ an additional facet-defining inequality must exist, whose support has either C_1^1 or C_1^2 as common intersection with C_1. But then it defines at the same time a facet for $P(\mathcal{I}_1^*)$ or $P(\mathcal{I}_2^*)$, which contradicts the fact that all such inequalities are among those just presented.

■

The conclusion with particular reference to the class of linearly relaxable independence systems is as follows :

Corollary (5.21)

If (E_1, l_1) and (E_2, l_2) as given above are both linearly relaxable, then (E_3, l_3) obtained by composition is also linearly relaxable.

6. A Minimal Linear System Sufficient to Define $P(\mathcal{I}_3)$

There are two main reasons for the interest in deriving minimal linear systems sufficient to define $P(\mathcal{I})$ for a general independence system $(E, \mathcal{I})$. The first is that the number of variables for the dual program to that associated with (ISP) is reduced to its minimum. The second, algorithmically perhaps more important, is that the inequalities of such a minimal system all define facets of $P(\mathcal{I})$, i.e. they are best possible supporting hyperplanes. We will show in this section that once we are given minimal linear systems sufficient to define $P(\mathcal{I}_1)$ and $P(\mathcal{I}_2)$ the linear system as derived in section 5) for describing $P(\mathcal{I}_3)$ is again a minimal such system. First observe that an inequality of type (5.2), (5.3), (5.4) or (5.7), (5.8), (5.9) is easily shown to be facet-defining for $P(\mathcal{I}_3)$ provided it has this property for $P(\mathcal{I}_1)$ or $P(\mathcal{I}_2)$, respectively. So let us consider the mixed inequalities and start with

(6.1) $$\sum_{e\in\bar{E}_1} a_e x_e + x(C_1^1) \le a^i, \quad i\in K_2,$$

(6.2) $$\sum_{e\in\bar{E}_2} b_e x_e + x(C_2^2) \le b^j, \quad j\in L_3.$$

The constraint obtained from these by mixing them is

(6.3) $$\sum_{e\in\bar{E}_1} a^i x_e + \sum_{e\in\bar{E}_2} b^j x_e \le a^i + b^j - (|C_1|-1).$$

If now (6.1) and (6.2) define facets of $P(\mathcal{I}_1)$ and $P(\mathcal{I}_2)$, respectively, there are members $S_1, \ldots, S_{|E_1|}$ in $\mathcal{I}_1$ as well as $T_1, \ldots, T_{|E_2|}$ in $\mathcal{I}_2$ such that their incidence vectors are linearly independent and all satisfy (6.1) respectively (6.2) with equality. For convenience let us introduce some notations:

i) $|C_1^1| =: p, \ |C_1^2| =: q, \ p+q =: \gamma;$

ii) $C_1^1 = \{e_1, \ldots, e_p\}, \ C_1^2 = \{e_1', \ldots, e_q'\};$

iii) $|E_1| - \gamma =: n, \ |E_2| - \gamma =: m.$

Moreover, let us index the sets $S_1, \ldots, S_{n+\gamma}$ such that

$S_1, \ldots, S_{r_1}$ contain the elements $e_2, \ldots, e_p$ but not e_1,

$S_{r_1+1}, \ldots, S_{r_2}$ " " " $e_1, e_3, \ldots, e_p$ but not e_2,

$S_{r_{p-1}+1}, \ldots, S_{r_p}$ " " " $e_1, e_2, \ldots, e_{p-1}$ but not e_p,

and

$S_{r_p+1}, \ldots, S_{n+\gamma}$ " " " $e_1, e_2, \ldots, e_p$.

and suppose that the indices $s_1, s_2, \ldots, s_q$ induce a similar indexing of the sets $T_1, \ldots, T_{m+\gamma}$. Now let $M1$ ($M2$) be the $(n+\gamma) \times (n+\gamma)$ ($(m+\gamma) \times (m+\gamma)$) -matrix having the vectors x^{S_i} (x^{T_j}) as columns for $i=1,\ldots,n+\gamma$ ($j=1,\ldots,m+\gamma$). M_1 for instance can be represented as follows:

	$\underbrace{\quad}_{}$ r_1	r_2-r_1		r_p-r_{p-1}	$n+\gamma-r_p$	columns
	N_1	N_2		N_p	N_{p+1}	
line $\hat{=}$ e_1	0---0	1---1		1---1	1---1	
line $\hat{=}$ e_2	1---1	0---0		1---1	1---1	
line $\hat{=}$ e_p	1---1	1---1		0---0	1---1	

Figure (6.4)

A similar representation can be given for the incidence vectors of $T_1, \ldots, T_{m+\gamma}$, where the submatrices N_i, $i=1, \ldots, p+1$, are replaced by matrices R_j, $j=1, \ldots, q+1$, having the appropriate sizes with respect to $s_1, \ldots, s_q$.

The following lemma will be of use for the proof of our main result:

<u>Lemma (6.5):</u>

For $c \notin [0,1]$ the matrices M_1^c and M_2^c given as follows are nonsingular :

$$M_1^c = \left[\begin{array}{ccc|c} & & & 0 \\ & & & \vdots \\ & M_1 & & 0 \\ & & & 1 \\ & & & \vdots \\ & & & 1 \\ \hline 1 & \cdots & 1 & c \end{array}\right] \begin{array}{l} \\ \\ \\ \hat{=}\, e_1 \\ \\ \hat{=}\, e_p \\ \\ \end{array} \qquad M_2^c = \left[\begin{array}{ccc|c} & & & 0 \\ & & & \vdots \\ & M_2 & & 0 \\ & & & 1 \\ & & & \vdots \\ & & & 1 \\ \hline 1 & \cdots & 1 & c \end{array}\right] \begin{array}{l} \\ \\ \\ \hat{=}\, e_1 \\ \\ \hat{=}\, e_q \\ \\ \end{array}$$

<u>Proof :</u>

Since M_1 is nonsingular, the system of equations

$$\lambda M_1 = 1$$

allows one unique solution namely that given by

$$\lambda_e = \begin{cases} a_e^1 / a_0^1 & \text{for } e \ \bar{E}_1 , \\ 1 / a_0^1 & \text{for } e \ C_1^1, \\ 0 & \text{otherwise.} \end{cases}$$

If the last line of M_1^c can be written as a linear combination of the remaining lines we will get :

$$\sum \lambda_{e_i} = p / a_0^1 = c$$

implying by independence of the set C_1^1 that $p \leq a_0^1$ and therefore $0 \leq c \leq 1$, in contradiction to our assumption.

The proof for M_2^c is along the same line.

■

Let us point out that if S is a member of $\mathcal{I}_1$, containing an element $e_j \in C_1^1$ but not containing $e_i \in C_1^1$, then $S' := (S \setminus \{e_j\}) \cup \{e_i\}$ is again a member of $\mathcal{I}_1$. Moreover, if the incidence vector x^S of S satisfies (6.1) with equality, the same holds for the incidence vector of S'. Clearly, this observation is also valid for all $T \in \mathcal{I}_2$, appropriate elements from C_1^2 and inequality (6.2). We may therefore assume that the first columns $d_1, \ldots, d_p$ of each of $N_1, \ldots, N_p$ and $f_1, \ldots, f_q$ of $R_1, \ldots, R_q$, respectively, are identical. Moreover, let d_{p+1} (f_{q+1}) denote the first column of the matrix N_{p+1} (R_{q+1}).

We remark at this point that all these columns do exist in the matrices M_1 and M_2.

Now let matrices $\hat{M}_1^c$ ($\hat{M}_2^c$) be obtained by subtracting column d_i from all remaining ones in N_i (f_j from those in R_j) for $i=1, \ldots, p+1$ ($j=1, \ldots, q+1$), the result of which is shown in Figure (6.6) for the first matrix:

$$\hat{M}_1^c = \begin{array}{|cccc|c|ccc|c|} \hline d_1 & d_2 & \cdots & d_p & d_{p+1} & N'_1 & \cdots & N'_{p+1} & 0 \\ \hline 0 & & & 1 & 1 & & & & 1 \\ & \ddots & & & \vdots & & 0 & & \vdots \\ 1 & & & 0 & 1 & & & & 1 \\ \hline 1 & \cdots & \cdots & 1 & 1 & 0 & \cdots & 0 & c \\ \hline \end{array}$$

Figure (6.6)

<u>Lemma (6.7)</u>

If t is not in $]+1, +\infty[$, the two matrices

$$\bar{M}_1^c = \begin{array}{|c|c|c|c|c|c|} \hline N'_1 & N'_2 & \cdots & N'_p & N'_{p+1} & t\,d_{p+1} - d_1 \\ \hline \end{array}$$

$$\bar{M}_2^c = \begin{array}{|c|c|c|c|c|c|} \hline R'_1 & R'_2 & \cdots & R'_q & R'_{q+1} & t\,f_{q+1} - f_1 \\ \hline \end{array}$$

where R'_j are obtained from R_j exactly as the N'_i from N_i, $i=1, \ldots, p+1$, $j=1, \ldots, q+1$, are nonsingular.

Proof:

By performing additions and subtractions on rows and columns of $\bar{M}_1^c$, $\bar{M}_2^c$, we find out that

$$t = \frac{(p-1)(1-c)+1}{p(1-c)} = 1 + \frac{c}{p(1-c)} .$$

Since we know that M_1^c is nonsingular for $c \notin [0, 1]$, our assumption implies that $\bar{M}_1^c$ is nonsingular, exactly as is shown for $\bar{M}_2^c$.

■

From this lemma we can conclude that the matrix

$$\bar{M}_1 = \boxed{N'_1 \mid N'_2 \mid \ldots \mid N'_p \mid N'_{p+1} \mid d_1}$$

is nonsingular. If, namely, $p \neq 1$ then t becomes 0 for $c = p/(p-1)$; and if $p = 1$, we can derive this result from M_1.

We will now proceed to construct independent sets within I_3 from S_i, T_j, $i=1,\ldots,n+\gamma$, $j=1,\ldots,m+\gamma$. For this let

$$W_i := \begin{cases} (S_i \setminus C_1^1) \cup (T_{s_q+1} \setminus C_1^2), & i = 1,\ldots,r_p , \\ (S_i \setminus C_1^1) \cup (T_1 \setminus C_1^2), & i = r_{p+1},\ldots, n+\gamma; \end{cases}$$

$$W'_j := \begin{cases} (T_j \setminus C_1^2) \cup (S_{r_p+1} \setminus C_1^1), & j = 1,\ldots,s_q , \\ (T_j \setminus C_1^2) \cup (S_1 \setminus C_1^1), & j = s_{q+1},\ldots,m+\gamma \end{cases}$$

The sets W_i, W'_j, for all such i, j are in I_3 and their incidence vectors satisfy the constraint (6.3) with equality. Since the columns $d_1,\ldots,d_p$ $(f_1,\ldots,f_q)$ are identical we have that

$$S_1 \setminus C_1^1 = S_{r_i+1} \setminus C_1^1 \quad \text{for } i=1,\ldots,p-1$$

and

$$T_1 \setminus C_1^2 = T_{s_i+1} \setminus C_1^2 \quad \text{for } i=1,\ldots,q-1 .$$

This implies that

$$W_i = W_{r_i+1} \quad \text{for } i=1,\dots,p-1$$

and

$$W'_i = W_{s_i+1} \quad \text{for } i=1,\dots,q-1 .$$

Identically, $W_1 = W'_{s_q+1}$ and $W_{r_p+1} = W'_1$. We now consider all those sets W_i, W'_j such that $i \neq 1, r_i+1$ for $i=1,\dots,p-1$ and $j \neq 1, s_i+1$ for $i=1,\dots,q-1$.

Let M^* be the matrix, whose columns are just the incidence vectors of these sets. It has the following form:

$$M^* = \left[\begin{array}{c|c|c|c|c|c|c|c} N_1^* & \cdots\cdots & N_p^* & N_{p+1} & d_{p+1}\dots d_{p+1} & \cdots & d_{p+1}\dots d_{p+1} & d_1\dots d_1 \\ \hline f_{q+1}\dots f_{q+1} & \cdots\cdots & f_{q+1}\dots f_{q+1} & f_1\dots f_1 & R_1^* & \cdots & R_q^* & R_{q+1} \end{array}\right]$$

where N_i^* (R_j^*) are derived from N_i (R_j) by deleting the first column, $i=1,\dots,p$, $j=1,\dots,q$.

To show that (6.3) defines a facet of $P(I_3)$ it remains to verify that M^* has full rank. For this we consider the matrix M^{**}, derived from M^* by subtracting the columns $\begin{pmatrix} d_1 \\ f_{q+1} \end{pmatrix}$ and $\begin{pmatrix} d_{p+1} \\ f_1 \end{pmatrix}$ from every column occurring in one of N_i^*, R_{q+1} and R_j^*, N_{p+1}, respectively, $i \in \{1,\dots,p\}$, $j \in \{1,\dots,q\}$. M^{**} can thus be written as follows:

$$M^{**} = \left[\begin{array}{c|c|c|c|c|c|c|c} N'_1 & \cdots & N'_{p+1} & & 0 & & d_{p+1} & d_1 \\ \hline & 0 & & R'_1 & \cdots & R'_{q+1} & f_1 & f_{q+1} \end{array}\right]$$

We know that the matrix $\bar{M}_1$ as described above is nonsingular. Therefore, there are vectors $\rho_1,\dots,\rho_{p+1}$ and a scaler $\rho_0 \neq 0$ such that

$$(6.8) \qquad d_{p+1} = \bar{M}_1 \cdot \begin{pmatrix} \rho_1 \\ \vdots \\ \rho_{p+1} \\ \rho_0 \end{pmatrix}$$

Now since $\bar{M}_1^C$ is nonsingular for all $t \notin]1,\infty[$, (6.8) implies that

$$\frac{1}{\rho_0} \in]1,\infty[\,.$$

Therefore, it remains to show that the matrix

$$\begin{array}{|c|c|c|c|c|}\hline R'_1 & \cdots & R'_q & R'_{q+1} & f_1 - \rho_0 f_{q+1} \\ \hline \end{array}$$

is nonsingular. This, however, follows from the fact that $1/\rho_0 \in]1,\infty[$, and thus $\rho_0 \in]0,1[$, and the application of Lemma (6.7).
Summing up we have shown:

<u>Theorem (6.9):</u>

Any mixed constraint defines a facet of $P(I_3)$ provided it is obtained from two such inequalities for $P(I_1)$ and $P(I_2)$.

An immediate consequence is

<u>Corollary (6.10):</u>

If the linear systems sufficient to define $P(I_1)$ and $P(I_2)$ contain only 0–1 inequalities then this is also the case for the derived system sufficient to define $P(I_3)$.

Finally, Theorem (6.9) together with the observation concerning the property of inequalities (5.2)–(5.4) and (5.7)–(5.9) of being facet-defining yields the main result of this section:

<u>Theorem (6.11):</u>

If we are given minimal linear systems sufficient to define the polytopes $P(I_1)$ and $P(I_2)$, then the linear system derived in section 5) to describe $P(I_3)$ is also a minimal such system.

7. Solving the independent set problem for (E_3, I_3)

After having discussed how to compose an independence system (E_3, I_3) out of (E_1, I_1), (E_2, I_2) and, in particular, how to obtain a linear system sufficient to define $P(I_3)$ once we dispose of such systems for $P(I_1)$ and $P(I_2)$ we are now going to elaborate on algorithmic aspects of our composition. As above and throughout this section let (E_1, I_1), (E_2, I_2) satisfy conditions $(\mathcal{C}1)$ and $(\mathcal{C}2)$, $(\mathcal{C}3)$ with respect to distinguished circuits $C_1 \in \mathcal{C}_1$, $C_2 \in \mathcal{C}_2$ and partitions (C_1^1, C_1^2), (C_2^1, C_2^2), and let (E_3, I_3) be the composition of these two as defined above. For notational convenience we will write $I_3 = I_1 \Delta I_2$. Moreover, let $w : E_3 \mapsto R$ be a weight-function on E_3, which can be assumed to be nonnegative without loss of generality. In the sequel we will show how a maximum-weight independent set (with respect to w) within I_3 can be determined from such a set within I_1 (w.r.t. the restriction of w to E_1) and one within I_2 with respect to a slightly different objective function $\bar{w}_2 : E_2 \mapsto R_+$ than the restriction of w to E_2. As will turn out the method proposed is polynomial in $|E_3|$ provided polynomial algorithms for solving problem (ISP) over (E_1, I_1) and (E_2, I_2) are available. Of specific importance is the fact that this method can be generalized to polynomially solve problem (ISP) over an independence system (E, I), which is obtained by sequentially composing $(E_1, I_1), \ldots, (E_m, I_m)$, m being polynomial in $|E|$.

First observe that by Lemma (4.1) any base of (E_3, I_3) contains $|C_1|-1$ many elements from C_1, the circuit from $\mathcal{C}_1$ identified with $C_2 \in \mathcal{C}_2$. For the given partition (C_1^1, C_1^2) of C_1 let $e^* \in C_1^1$ and $e^{**} \in C_1^2$ be chosen such that

$$w(e^*) = \min\{w(e) : e \in C_1^1\} \quad \text{and} \quad w(e^{**}) = \min\{w(e) : e \in C_1^2\}.$$

As a consequence any maximum-weight base of (E_3, I_3) can be transformed into one containing the whole set $C_1 \setminus \{e^*, e^{**}\}$ and either e^* or e^{**}, but not both. This can be achieved by appropriate element interchangements within C_1^1 or C_1^2 and due to the special structure of the members of $\mathcal{C}_3$.

Let us now consider the restrictions w^1 and w^2 of w to E_1 and E_2, respectively, i.e. the weight-functions $w^i : E_i \mapsto R_+$ given by $w^i(e) = w(e) =: w_e$ for all $e \in E_i$, $i = 1, 2$. Then let B_1^* and B_1^{**} (B_2^* and B_2^{**}) be bases within I_1 (I_2) having maximum total weight ω_1^* and ω_1^{**} (ω_2^* and ω_2^{**}) with respect to w^1 (w^2) and containing e^* and e^{**}, but not both, respectively.

If then ω_3 is the maximum total weight of a member I_3 from $\mathcal{I}_3$ (with respect to w) we have

$$\omega_3 = \max\{\omega_{1^*} + \omega_{2^*} - w(C_1 \setminus \{e^{**}\}), \omega_{1^{**}} + \omega_{2^{**}} - w(C_1 \setminus \{e^*\})\},$$

since any such I_3 decomposes into optimal bases $B_1 \in \mathcal{I}_1$ and $B_2 \in \mathcal{I}_2$. We should remark at this point that a straightforward way to determine such an I_3 would consist just in determining optimal bases B_1^*, B_2^* and B_1^{**}, B_2^{**} and comparing the total weights of $B_1^* \cup B_2^*$ and $B_1^{**} \cup B_2^{**}$, respectively. This procedure, however, would provide us with an exponential algorithm when generalized to the composition of more than 2, say m, independence systems, even when m is polynomial in $|E|$.

As already mentioned the method presented here will require a maximum-weight base within $\mathcal{I}_2$ and with respect to $\bar{w}^2$, a modification of w^2. The details of this modification are now being developed. First of all by putting $\gamma := w(C_1 \setminus \{e^*, e^{**}\})$ and considering the system of 2 equations

$$\text{(7.1)} \quad \begin{cases} x + \gamma = \omega_{1^*} - \sigma \\ y + \gamma = \omega_{1^{**}} - \sigma \end{cases},$$

in which x, y, σ are parameters, we are able to give a slightly different expression for ω_3 in terms of these parameters. Namely, if (7.1) admits a solution having x, y both positive, we get

$$\text{(7.2)} \quad \omega_3 = \sigma + \max\{x + \omega_{2^*} - w_{e^*}, y + \omega_{2^{**}} - w_{e^{**}}\}.$$

So if we define

$$\alpha_0 := \omega_{1^*} - \omega_{1^{**}}, \quad x := \max\{0, \alpha_0\}, \quad y := x - \alpha_0,$$

one easily verifies that together with arbitrary σ, x and y satisfy the system (7.1). As a modification of w^2 we now assign the weights x and y to e^* and e^{**}, respectively, and a weight $w_e + M$, M chosen to be sufficiently large (for instance $M = 2(x+y)$) to the remaining elements of C_1. Together with w_e^2 as weights for all $e \in E_2 \setminus C_2$ we hereby obtain a new weight-function $\bar{w}^2$ over E_2, assuring by the choice of M that any maximum weight base of $(E_2, \mathcal{I}_2)$ with respect to $\bar{w}^2$ will either not contain e^* or e^{**}. The total weight of such a base is thus given by

$$\bar{\omega}_2 = \max\{\omega_{2^*} - w_{e^*} + x + M(|C_1|-2), \omega_{2^{**}} - w_{e^{**}} + y + M(|C_1|-2)\}$$

$$= M(|C_1|-2) + \max\{x + \omega_{2^*} - w_{e^*}, y + \omega_{2^{**}} - w_{e^{**}}\}$$

implying by (7.2)

(7.3) $\quad \omega_3 = \sigma + \bar{\omega}_2 - M(|C_1|-2).$

This allows us to derive the following result:

Theorem (7.4)

Let $\bar{w}^2$ be given as just introduced and $\bar{B}_2$ be a maximum-weight base within $(E_2, \mathcal{I}_2)$ with respect to $\bar{w}^2$. Moreover, let B_3 be the subset of E_3 defined as follows:

i) If $e^* \in B_2$ then $B_3 := \bar{B}_2 \cup B_1^*$,

ii) If $e^{**} \in B_2$ then $B_3 := \bar{B}_2 \cup B_1^{**}$.

Then B_3 is a maximum-weight independent set with respect to w within $(E_3, \mathcal{I}_3)$.

Proof:

Clearly, B_3 is a member of $\mathcal{I}_3$ by definition of $\mathcal{C}_3$. Moreover, for $w(B_3)$ we have in case

i) $\bar{\omega}_2 + \omega_1^* - \gamma - x - M(|C_1|-2) = \bar{\omega}_2 + \sigma - M(|C_1|-2)$

and in case

ii) $\bar{\omega}_2 + \omega_1^{**} - \gamma - y - M(|C_1|-2) = \bar{\omega}_2 + \sigma - M(|C_1|-2)$,

which in both cases equals ω_3 by (7.3).

Let us now reflect on the situation that an independence system $(E, \mathcal{I})$ is the composition of more than two appropriate $(E_i, \mathcal{I}_i)$. For convenience let

$$H_n := \mathcal{I}_n, \ H_{n-i} := H_{n-i+1} \Delta \mathcal{I}_{n-i}, \ i = 1, \dots, n-1, \text{ and } \mathcal{I} := H_1.$$

By Theorem (7.4) a maximum-weight independent set with respect to w and $(E, \mathcal{I})$ can be found by determining B_1^*, B_1^{**} as already considered above and by solving (ISP) over H_2 with respect to a modified weight-function $\bar{w}^2$. This latter problem can now be "decomposed" identically and so on until we reach the point to determine an optimal independent set within I_n with respect to a weight-function $\bar{w}^n$. Together with B_{n-1}^*, B_{n-1}^{**} this yields by application of Theorem (7.4) an optimal solution for (ISP) over H_n and so on until we obtain such a solution for H_1, which equals I. Figure (7.5) illustrates this procedure schematically, the B_i representing maximum-weight independent sets within H_i and with respect to weight-functions $\bar{w}^i$ for $i = n, n-1, \dots, 1$.

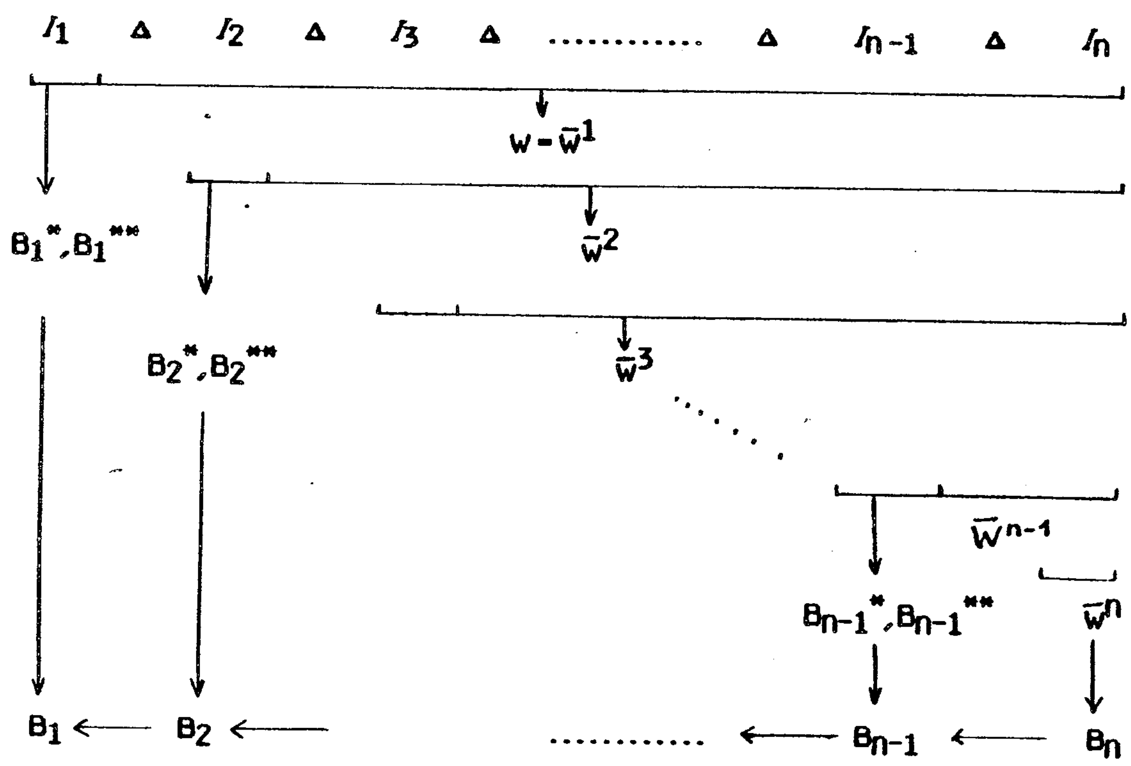


Figure (7.5)

8. On Linear Systems Defining $P(\mathcal{I}_1), P(\mathcal{I}_2), P(\mathcal{I}_3)$ and Total Dual Integrality

The results we are going to present in this section strongly refer to the property of total dual integrality of linear systems as introduced in section 2) and the particular descriptions of $P(\mathcal{I}_1), P(\mathcal{I}_2), P(\mathcal{I}_3)$ as considered in section 5). We assume $(E_i, \mathcal{I}_i)$ $i = 1,2,3$, to be given as above and, for the sake of brevity, we denote the linear system (5.2) - (5.6) defining $P(\mathcal{I}_1)$ in section 5) by (L_1), the corresponding system for $P(\mathcal{I}_2)$ by (L_2) and the system defining $P(\mathcal{I}_3)$ given by all linear inequalities of type (5.2) - (5.4), (5.7) - (5.9), (5.13) as well as the circuit inequality $x(C_1) \leq |C_1|-1$ and the trivial inequalities by (L_3). The main result of this section will consist in proving (L_3) to be TDI, whenever (L_1) and (L_2) are so. For this we assign to any inequality of (L_1) a dual variable y_i^1, $i \in K_1 \cup K_2 \cup K_3$, a variable y_C^1 for the circuit-inequality and variables z_e^1 to all inequalities $x_e \leq 1$, $e \in E_1$. The same assignment is done for (L_2) so that being faced with the linear programs

$$(P_1)\begin{cases} \text{Maximize } w^1x \\ \\ \text{subject to } (L_1) \end{cases} \qquad \text{and} \qquad (P_2)\begin{cases} \text{Maximize } w^2x \\ \\ \text{subject to } (L_2) \end{cases}$$

we can easily formulate the associated dual programs as follows (we restrict ourselves to explicitly describing it for (P_1)) :

$$(D_1)\begin{cases} \text{Minimize} \quad \sum\limits_{i \in K_1 \cup K_2 \cup K_3} a_0^i y_i^1 + y_C^1(|C_1|-1) + \sum\limits_{e \in E_1} z_e^1 \\ \\ \text{subject to} \\ \\ \sum\limits_{i \in K_1 \cup K_2 \cup K_3} a_e^i y_i^1 + z_e^1 \geq w_e^1 \quad \text{for all } e \in \bar{E}^1, \\ \\ \sum\limits_{i \in K_2} y_i^1 + y_C^1 + z_e^1 \geq w_e^1 \quad \text{for all } e \in C_1^1, \\ \\ \sum\limits_{i \in K_3} y_i^1 + y_C^1 + z_e^1 \geq w_e^1 \quad \text{for all } e \in C_1^2, \\ \\ y_i^1 \geq 0 \text{ for all } i \in K_1 \cup K_2 \cup K_3, \; y_C^1 \geq 0, \; z_e^1 \geq 0 \text{ for all } e \in E_1. \end{cases}$$

For the related problem over (L_3), i.e.

$$(P_3)\quad \begin{cases} \text{Maximize } wx \\ \\ \text{subject to } (L_3) \end{cases}$$

we obtain as its dual the following program:

$$(D_3)\quad \begin{cases} \text{Minimize } \sum\limits_{i\in K_1\cup K_2\cup K_3} y_i^1 a_0^i + \sum\limits_{i\in L_1\cup L_2\cup L_3} y_i^2 b_0^i + y_C(|C_1|-1) \\ \quad + \sum\limits_{e\in E_3} z_e + \sum\limits_{\substack{ij\in K_2\times L_3 \\ \text{or } L_2\times K_3}} \varepsilon_{ij}(a_0^i + b_0^j - |C_1|+1) \\ \\ \text{subject to} \\ \\ \sum\limits_{i\in K_1\cup K_2\cup K_3} a_e^i y_i^1 + \sum\limits_{ij} a_e^i \varepsilon_{ij} + z_e \geq w_e \text{ for all } e\in \bar{E}_1, \\ \\ \sum\limits_{i\in L_1\cup L_2\cup L_3} b_e^i y_i^2 + \sum\limits_{ij} b_e^j \varepsilon_{ij} + z_e \geq w_e \text{ for all } e\in \bar{E}_2, \\ \\ \sum\limits_{i\in K_2} y_i^1 + \sum\limits_{i\in L_2} y_i^2 + y_C + z_e \geq w_e \text{ for all } e\in C_1^1, \\ \\ \sum\limits_{i\in K_3} y_i^1 + \sum\limits_{i\in L_3} y_i^2 + y_C + z_e \geq w_e \text{ for all } e\in C_1^2, \\ \\ y_i^1 \geq 0 \ \forall\, i\in K_1\cup K_2\cup K_3,\ y_i^2 \geq 0 \ \forall\, i\in L_1\cup L_2\cup L_3, \\ \\ \varepsilon_{ij} \geq 0 \ \forall\, ij\in K_2\times L_3 \text{ and } K_3\times L_2,\ y_C \geq 0 \text{ and } z_e \geq 0 \ \forall e\in E_3. \end{cases}$$

Assuming now that the linear systems (L_1) and (L_2) are TDI, we will construct for an integral weight-function $w : E_3 \longrightarrow Z_+$ an integer-valued optimal solution to (D_3) from integral optimal solutions to (D_1) and (D_2) for the restrictions of w to w^1 and w to w^2. Two main cases are to be distinguished for α_0, which was defined to be equal to $\omega_{1^*} - \omega_{1^{**}}$:

<u>Case 1:</u> $\alpha_0 \geq 0$

Let us change w^2 to $\bar{w}^2$ as follows:

- to each element $e \in C_1^1$ we assign the weight α_0,
- to e^{**} we assign the weight 0,
- the remaining weights w_e^2 are left unchanged.

Then together with

$$\bar{\gamma} := \alpha_0(|C_1^1|-1) + w(C_1^2 \setminus \{e^{**}\}),$$
$$\gamma = w(C_1 \setminus \{e^{**}\}) \text{ as above and}$$
$$\sigma := \omega_{1^*} - \alpha_0 - \bar{\gamma} \text{ we obtain for } \omega_3:$$

$$\begin{aligned}\omega_3 &= \max\{\omega_{1^*} + \omega_{2^*} - \gamma - w_{e^*}, \omega_{1^{**}} + \omega_{2^{**}} - \gamma - w_{e^{**}}\}\\ &= \max\{\sigma + \alpha_0 + \bar{\gamma} + \omega_{2^*} - \gamma - w_{e^*}, \sigma + \bar{\gamma} + \omega_{2^{**}} - \gamma - w_{e^{**}}\}\\ &= \sigma + \max\{\omega_{2^*} + \alpha_0 - w_{e^*} + \sum_{e \in C_1^1 \setminus \{e^*\}} (\alpha_0 - w_e),\\ &\qquad \omega_{2^{**}} - w_{e^{**}} + \sum_{e \in C_1^1 \setminus \{e^*\}} (\alpha_0 - w_e)\}\\ &= \sigma + \bar{\omega}_2,\end{aligned}$$

where $\bar{\omega}_2$ is the maximum $\bar{w}^2$-weight of an independent set $\bar{B}_2$ within I_2.

<u>Subcase 1.1:</u> $\bar{B}_2$ does not contain e^{**}.

Clearly, $\bar{B}_2$ contains $C_1 \setminus \{e^{**}\}$ and can therefore be written as

$$\bar{B}_2 = I \cup (C_1^2 \setminus \{e^{**}\}).$$

The incidence vector x^I of I is therefore an optimal solution for (P_2) with respect to the weight-function $\hat{w}^2$ given by

$$\hat{w}_e^2 = \begin{cases} \bar{w}_e^2 & \text{for all } e \in E_2 \setminus C_1^2, \\ 0 & \text{for all } e \in C_1^2. \end{cases}$$

The objective function value for x^1 and $\hat{w}^2$ is equal to $\bar{\omega}_2 - w(C_1{}^2 \setminus \{e^{**}\})$. Since the linear system (L_2) is TDI, there is an integral optimal solution $(y^2, y_C{}^2, z^2)$ to (D_2) with respect to $\hat{w}^2$. By complementary slackness we get

$$\sum_{i \in L_2} y_i{}^2 + y_C{}^2 + z_e{}^2 = \alpha_0 \quad \text{for all } e \in C_1{}^1,$$

implying $z_e{}^2 = z_{e'}{}^2$ for all $e, e' \in C_1{}^1$. Identically,

$$\sum_{i \in L_3} y_i{}^2 + y_C{}^2 + z_e{}^2 = 0 \quad \text{for all } e \in C_1{}^2,$$

implying $y_i{}^2 = 0$ for all $i \in L_3$, $y_C{}^2 = 0$ and $z_e{}^2 = 0$ for all $e \in C_1{}^2$.

Now we put

$$t := \alpha_0 - z_{e^*}^2 .$$

If then $\hat{w}^1$ is the weight-function obtained from w^1 by changing the weights w_e for $e \in C_1{}^2$ to $w_e + t$ and leaving the others as they are, there is $\hat{I}_1 \in \mathcal{I}_1$ having maximum $\hat{w}^1$-weight and not containing e^{**}, as follows from the definition of α_0 and the assumption that $\alpha_0 \geq 0$. Consequently, there is an integral optimal solution $(y^1, y_C{}^1, z^1)$ for (D_1) with respect to $\hat{w}_1$ satisfying $z_{e^{**}}^1 = 0$. By feasibility we get

$$\sum_{i \in K_3} y_i{}^1 + y_C{}^1 + z_e{}^1 \geq w_e + t \quad \text{for all } e \in C_1{}^2,$$

implying by $z_{e^{**}}^1 = 0$ that

$$\text{(6.1)} \qquad \sum_{i \in K_3} y_i{}^1 + y_C{}^1 \geq t .$$

To simplify our presentation we remark at this point that any constraint of type (5.3) and (5.4) ((5.8) and (5.9)) of the linear system (L_3) describing $P(\mathcal{I}_3)$ can be considered as being obtained from mixing itself with the circuit inequality $x(C_2) \leq |C_2|-1$ $(x(C_1) \leq |C_1|-1)$. Since we will have to index all inequalities of (5.13), we assign to the aforementioned inequalities the indices (i, γ_2) $((\gamma_1, j))$, where $i \in K_1 \cup K_2 \cup K_3$, $j \in L_1 \cup L_2 \cup L_3$ and γ_1, γ_2 are indices for the circuit-inequalities in (L_1), (L_2), respectively. As a consequence, the inequality $x(C_1) \leq |C_1|-1$ in (L_3) will get the index (γ_1, γ_2).

The following lemma of more technical a nature is easily derived from the total unimodularity of matrices describing the transportation polytope:

<u>**Lemma (8.2):**</u>

Consider p+1 constraints from (L_1) indexed by $i_1, \dots, i_p$ either being elements from K_2 or from K_3 and γ_1 as well as q+1 constraints from (L_2) indexed by $j_1, \dots, j_q$ either from (L_3) or (L_2), respectively, and γ_2. Suppose in addition that there are nonnegative integers $\lambda(i_s)$, $s = 1, \dots, p$, and $\lambda(C_1)$ as well as such numbers $\mu(j_r)$, $r = 1, \dots, q$ and $\mu(C_2)$ fulfilling the condition

$$\lambda(C_1) + \sum_{s=1}^{p} \lambda(i_s) = \mu(C_2) + \sum_{r=1}^{q} \mu(j_r) = \Omega.$$

Then we can associate with any mixed constraint indexed by (ρ, δ), $\rho \in M_1 := \{i_1, \dots, i_p, \gamma_1\}$, $\delta \in M_2 := \{j_1, \dots, j_q, \gamma_2\}$ an integral, nonnegative number $\varepsilon(\rho, \delta)$ such that the following holds:

$$\sum_{\rho \in M_1} \varepsilon(\rho, \delta) = \mu(\delta) \quad \text{for all } \delta \in M_2,$$

$$\sum_{\delta \in M_2} \varepsilon(\rho, \delta) = \lambda(\rho) \quad \text{for all } \rho \in M_1$$

(implying that $\sum_{\rho} \sum_{\delta} \varepsilon(\rho, \delta) = \Omega$).

By (8.1) there are indices $i_1, \dots, i_p \in K_3$ such that

$$\sum_{s=1}^{p} y_i^1 < t \quad \text{and} \quad \sum_{s=1}^{p} y_i^1 + y_C^1 \geq t$$

(if $t = 0$ or $y_C^1 \geq t$ the set $\{i_1, \dots, i_p\}$ is empty). We now put

$$\xi := t - \sum_{s=1}^{p} y_{i_s}^1$$

Since $\sum_{i \in L_2} y_i^2 + y_C^2 = t$ by definition of t we can apply Lemma (8.2) and assign nonnegative values $\varepsilon(\rho, \delta)$ to every mixed constraint indexed by (ρ, δ), $\rho \in M_1$, $\delta \in M_2$ such that

$$(8.3)\quad \begin{cases} \sum_{\rho} \varepsilon(\rho, i) = y_i^2 \text{ for all } i \in L_2, \\ \sum_{\rho} \varepsilon(\rho, \gamma_2) = y_C^2, \\ \sum_{\delta} \varepsilon(i_s, \delta) = y_{i_s}^1 \text{ for all } s = 1, \dots, p, \\ \sum_{\delta} \varepsilon(\gamma_1, \delta) = \xi, \end{cases}$$

holds; moreover these $\varepsilon(\rho, \delta)$ can be chosen to have integral-valued components only, since the right-hand sides are all integral-valued.

From this we now define the nonnegative, integral-valued vector $(\bar{y}^1, \bar{y}^2, y_C, \varepsilon, z)$ as follows:

$$\bar{y}_i^1 = \begin{cases} y_i^1, \text{ whenever } i \neq i_1, \dots, i_p, \\ \varepsilon(i, \gamma_2) \text{ for } i = i_1, \dots, i_p, \end{cases}$$

$$\bar{y}_i^2 = \begin{cases} y_i^2, \text{ whenever } i \notin L_2, \\ \varepsilon(\gamma_1, i) \text{ for } i \in L_2, \end{cases}$$

$$\bar{y}_C = (y_C^1 - \xi) + \varepsilon(\gamma_1, \gamma_2),$$

$$\varepsilon_{ij} = \varepsilon(i, j) \text{ for } i = i_1, \dots, i_p,\ j \in L_2,$$

$$z_e = \begin{cases} z_e^1 \text{ for } e \in E_1 \\ z_e^2 \text{ for } e \in \bar{E}_2. \end{cases}$$

By observing that $\varepsilon(\rho, \gamma_2) = 0$ for all $\rho \in M_1$, due to $y_C^2 = 0$, and using (8.3) it takes a bit of purely technical effort to verify that indeed $(\bar{y}^1, \bar{y}^2, y_C, \varepsilon, z)$ is a feasible solution for (D_3) and, in particular, its objective function value is exactly equal to $\bar{\omega}_2 + \sigma$, which equals ω_3. We have hereby found an integral-valued optimal solution to (D_3).

<u>Subcase 1.2:</u> $\bar{B}_2$ contains e^{**}.

Then by complementary slackness the optimal solution $(y^2, y_C{}^2, z^2)$ to (D_2) with respect to $\hat{w}^2$ has the property:

$$- z_e{}^2 = 0 \quad \text{for all } e \in C_1{}^1 ;$$

$$- \sum_{l \in L_3} y_l{}^2 + y_C{}^2 + z_e{}^2 = \begin{cases} w_e & \text{for all } e \in C_1{}^2 \setminus \{e^{**}\} \\ 0 & \text{for } e = e^{**} \end{cases}$$

which implies that

$$z_e{}^2 = w_e \quad \text{for all } e \in C_1{}^2, \text{ since } y_C{}^2 = 0 = \sum_{l \in L_3} y_l{}^2 .$$

In addition, we have

$$\sum_{l \in L_2} y_l{}^2 + y_C{}^2 + z_e{}^2 \geq \alpha_0 \quad \text{for all } e \in C_1{}^1 \text{, implying}$$

(8.4)
$$\sum_{l \in L_2} y_l{}^2 \geq \alpha_0 \quad .$$

Now let $\hat{w}^1$ be the weight-function obtained from w^1 by changing just the w_e to $w_e + \alpha_0$ for all $e \in C_1{}^2$. Then we can show:

<u>Lemma (8.5)</u>

There is an optimal solution to (P_1) with weight-function $\hat{w}^1$, which contains e^* and one such solution, which contains e^{**}.

Proof:

If an optimal solution to (P_1) with $\hat{w}^1$ contains e^* (e^{**}), its total weight is

$$\omega_{1^*} + \alpha_0(|C_1^2|-1), \quad (\omega_{1^{**}} + \alpha_0|C_1^2|),$$

and, consequently, the set B_1^{**} (B_1^{*}) considered in section 7) has total weight

$$\begin{aligned}\omega_{1^{**}} + \alpha_0|C_1^2| &= \omega_{1^{**}} + \alpha_0 + \alpha_0(|C_1^2|-1)\\ &= \omega_{1^*} + \alpha_0(|C_1^2|-1).\end{aligned}$$

■

From this lemma we can conclude that there is a solution to (D_1) which satisfies

$$z^1_{e^{**}} = 0 \text{ and}$$

(8.6) $$\sum_{i\in K_3} y_i^1 + y_C^1 = \omega_{e^{**}} + \alpha_0$$

Therefore, there are elements $i_1,\dots,i_q$ in K_3 such that

$$\sum_{s=1}^{q} y^1_{i_s} < \alpha_0 \quad\text{and}\quad \sum_{s=1}^{q} y^1_{i_s} + y_C^1 \geq \alpha_0 .$$

$$(\text{ if } \alpha_0 = 0 \text{ or } y_C^1 \geq \alpha_0,\ \{i_1,\dots,i_q\} = \emptyset)$$

Now we define

$$\xi_1 := \alpha_0 - \sum_{s=1}^{q} y^1_{i_s} .$$

By (8.4) there are $j_1,\dots,j_{p+1} \in L_2$ satisfying

$$\sum_{r=1}^{p} y^2_{j_r} < \alpha_0 \quad\text{and}\quad \sum_{r=1}^{p+1} y^2_{j_r} \geq 0 .$$

Here we put

$$\xi_2 := \alpha_0 - \sum_{r=1}^{p} y_{j_r}^2$$

By Lemma (8.2) we can assign to the mixed constraints indexed by (ρ,δ), $\delta \in M_1 := \{i_1,\dots,i_q,\gamma_1\}$, $\rho \in M_2 := \{j_1,\dots,j_{p+1}\}$ values $\varepsilon(\rho,\delta) \geq 0$ such that

$$\sum_{\rho \in M_1} \varepsilon(\rho, j_r) = \begin{cases} y_{j_r}^2, & r=1,\dots,p, \\ \xi_2, & r=p+1, \end{cases}$$

$$\sum_{\delta \in M_2} \varepsilon(i_s,\delta) = y_{i_s}^1, \; s=1,\dots,q,$$

$$\sum_{\delta \in M_2} \varepsilon(\gamma_1,\delta) = \xi_1$$

$$\text{implying } \sum_{\rho,\delta} \varepsilon(\rho,\delta) = \alpha_0 .$$

If we now consider the solution

$$\bar{y}_i^1 := \begin{cases} y_i^1 & \text{if } i \neq i_1,\dots,i_q, \\ 0 & \text{else}, \end{cases}$$

$$\bar{y}_j^2 := \begin{cases} y_j^2 & \text{for } j \neq j_1 \dots j_{p+1}, \\ \varepsilon(\gamma_1, j) & \text{for } j = j_1,\dots,j_p, \\ \varepsilon(\gamma_1, j) + (y_{j_{p+1}}^2 - \xi_2) & \text{for } j = j_{p+1}, \end{cases}$$

$$y_C := y_C^1 - \xi_1 ,$$

$$\varepsilon_{ij} := \varepsilon(i,j),\; i \in M_1 \setminus \{\gamma_1\},\; j \in M_2 ,$$

$$z_e := \begin{cases} z_e^1 & \text{for } e \in E_1 \\ z_e^2 & \text{for } e \in \bar{E}_2 \end{cases}$$

we can easily verify its feasibility for (D_3). Moreover, its objective function value can be shown to be equal to $\bar{\omega}_2 + \bar{\sigma}$, which equals ω_3.

The case $\alpha_0 \leq 0$ can be treated very similarly to that given by $\alpha_0 > 0$ just by considering appropriately defined weight-functions $\bar{w}^1, \bar{w}^2$ over $(E_1, \mathcal{I}_1)$ and $(E_2, \mathcal{I}_2)$.

9. Final Remarks

It appears that most of the results presented in this paper are also valid for the situation that $(E_1, \mathcal{I}_1)$, $(E_2, \mathcal{I}_2)$ do not necessarily share property ($\mathcal{C}1$). We have made some investigations in this direction for the special case that $(E_i, \mathcal{I}_i)$ arise from stable sets in graphs G_i, $i=1,2$. Let us discuss some of these aspects in more detail. Let G_3 be the graph obtained from G_1 and G_2 by the identification of an edge $\{a,b\}$. Figure (9.1) gives an illustration of such a graph G_3, where S_a, S_b (T_a, T_b) correspond to the sets of neighbours of a, b in G_1 (G_2). The edges appearing in Figure (9.1) are exactly those which correspond to the mixed circuits. Also note that by ($\mathcal{C}2$), ($\mathcal{C}3$) S_a and S_b (T_a and T_b) have empty intersection.

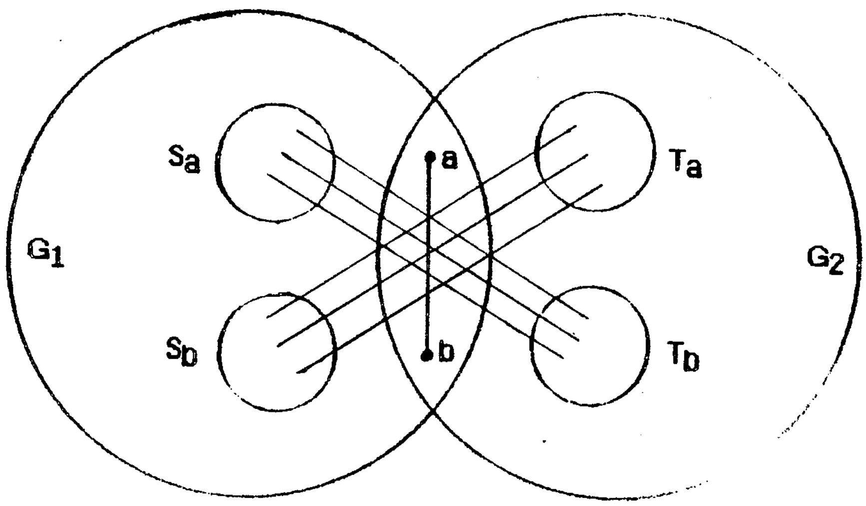
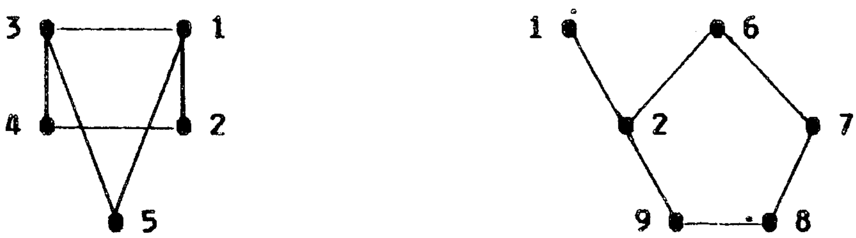


Figure (9.1)

Let us now suppose that G_1 and G_2 are perfect. $P(\mathcal{I}_1)$, $P(\mathcal{I}_2)$ are then describable by their clique- and the trivial inequalities. Obviously, $P(\mathcal{I}_3)$ is describable by the linear systems obtained from $P(\mathcal{I}_1)$ and $P(\mathcal{I}_2)$ augmented by all clique inequalities, which are induced by $K \setminus \{a\}$, $K' \setminus \{b\}$, K, K' being maximal cliques in G_1, G_2 and containing a, b, respectively. It follows that the graph G_3 is perfect again, i.e. our composition preserves perfection.
If, however, G_1 and G_2 are h-perfect, i.e. the clique-, odd cycle- and the trivial inequalities are sufficient to describe $P(\mathcal{I}_1)$, $P(\mathcal{I}_2)$, G_3 is not necessarily h-perfect. To see this just consider

the two graphs of Figure (9.2):



G_1 G_2

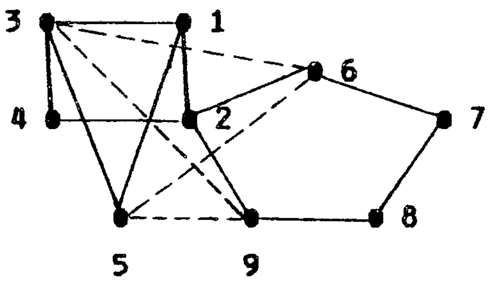
Figure (9.2)

G_1 is perfect, G_2 is series-parallel. The associated polyhedra can be described as follows:

$$P(I_1) = \{ x \in R^5 : x_3 + x_4 + x_5 \leq 1 \ , x_1 + x_3 + x_5 \leq 1 \ , \\ x_1 + x_2 \leq 1 \ , x_2 + x_4 \leq 1, \ x_i \geq 0, i=1,\dots,5 \ \}$$

$$P(I_2) = \{ x \in R^6 : \begin{array}{l} x_1 + x_2 \leq 1 \ , x_2 + x_6 + x_7 + x_8 + x_9 \leq 2, \\ x_6 + x_7 \leq 1 \ , x_7 + x_8 \leq 1, x_2 + x_6 \leq 1, \\ x_2 + x_9 \leq 1, x_8 + x_9 \leq 1, x_i \geq 0, i-1,\dots,9 \end{array} \}$$

G_3 as the composition of G_1 and G_2 is as follows:



G_3

Figure (9.3)

The associated polytope $P(\mathcal{I}_3)$ can be described as follows :

$$(L_3)\quad \begin{cases} (L_1), & \\ (L_2), & \\ x_3 + x_5 + x_6 \leq 1, & (9.4) \\ x_3 + x_5 + x_9 \leq 1, & (9.5) \\ x_3 + x_5 + x_6 + x_7 + x_8 + x_9 \leq 2 & (9.6)\ , \end{cases}$$

where (9.4) and (9.5) are clique inequalities and (9.6), however, does not correspond to such neither to an odd cycle inequality, implying that G_3 is <u>not</u> h-perfect.

To conclude we would like to mention some problems which to our mind deserve particular attention. The first is the problem of designing a polynomial algorithm for the solution of (ISP) over those independence systems, which arise from balanced matrices in the way presented in section 2). A second line of further research could consist in a thorough study of those independence systems , whose system $\mathcal{C}$ of circuits fulfills condition ($\mathcal{C}1$), with particular respect to finding a minimum weighted circuit in $\mathcal{C}$ for a given weight function w. Finally, a characterization of linearly relaxable independence systems much along the line as it has been done in [1] and [6] for the two "prototypes" could be of special interest.

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ANNEXE 7

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POLYTOPE DES ABSORBANTS DANS UNE CLASSE DE GRAPHE A SEUIL

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A graph G is defined to be *domishold* (Benzaken and Hammer (1978)) if there exist real positive numbers associated to their vertices so that a set of vertices is *dominating* if and only if the sum of the corresponding weights exceeds a certain threshold b.

In this paper, we characterize the polytope of the dominants in this class of graphs, using a polynomial algorithm to find a minimum weight dominating set.

1. Introduction

Soit $G=(X,E)$ un graphe à n sommets, simple, non orienté, sans boucle, dont X représente l'ensemble des sommets, E celui des arêtes. Un ensemble $A \subseteq X$ est dit absorbant si tout sommet de $X-A$ est adjacent à au moins un sommet de A.

On peut associer à chaque absorbant A de G son vecteur représentatif x^A dans $\{0,1\}^n$ ($x_i^A=1$ si $i \in A$, 0 sinon). En considérant chacun de ces vecteurs comme un point de R^n, on peut définir l'enveloppe convexe $P(G)$ de ces absorbants qu'on appellera *polytope des absorbants de G*.

Il est en effet souvent intéressant de caractériser des structures combinatoires par l'enveloppe convexe de ses vecteurs représentatifs (cf. les travaux d'Edmonds sur les couplages, de Fulkerson sur les graphes parfaits, de Chvatal, Uhry et Boulala sur les stables dans un graphe série parallèle). Ceci, à notre connaissance, n'a jamais été fait sur les absorbants d'une classe de graphes non triviale. De plus, cette caractérisation est souvent liée à des algorithmes de reconnaissance et d'optimisation polynomiaux.

Dans cet article, nous caractérisons le polytope $P(G)$ dans la classe des graphes absorbants à seuils définie par Benzaken et Hammer [1] et, pour ce faire, nous donnons un algorithme polynomial en $O(n)$ de construction d'un absorbant de poids minimum dans un tel graphe, alors que ce problème est NP-complet dans le cas général.

Définition 1 (Benzaken et Hammer [1]). Soit $G=(X,E)$ un graphe fini, sans boucle, simple, non orienté, $X=\{1,2,\dots,n\}$, G est dit *absorbant à seuil* s'il existe des poids $a_i \geq 0$ associés aux sommets de G et un seuil $b \geq 0$ tel que $A \subseteq X$ est un absorbant (ou dominant) de G si et seulement si $\sum_{i \in A} a_i \geq b$.

Parmi les caractérisations des graphes absorbants à seuil ([1]), nous utiliserons essentiellement la suivante.

Théorème 1. *Les deux propriétés suivantes sont équivalentes:*

(1) *G est absorbant à seuil.*

(2) *G peut se ramener à un graphe réduit à un sommet en enchaînant un nombre fini de fois les opérations suivantes* (cf. Fig. 1):

ϕ_1 : *Supprimer un sommet isolé.*

ϕ_2 : *Supprimer un sommet universel (sommet adjacent à tous les autres sommets).*

ϕ_3 : *Supprimer une paire de pôles (deux sommets non liés entre eux, mais adjacents à tous les autres sommets).*

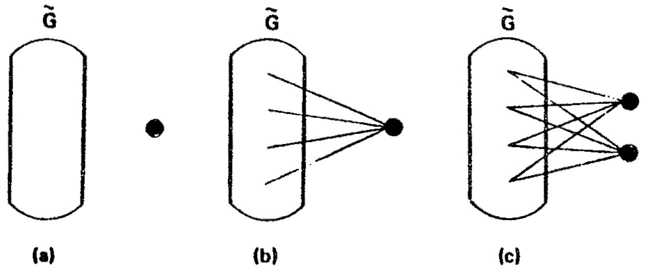


Fig. 1. $\tilde{G}$ est le graphe restant après avoir fait l'une des opérations ϕ_1, ϕ_2 ou ϕ_3.

Remarque 1. L'operation ϕ_1 va définir des étapes dans le 'démontage' de G. Chaque étape consiste à supprimer des sommets universels et des paires de pôle adjacents à un sommet qui, par la suite, devient un sommet isolé. Autrement dit, il s'agit dans chaque étape d'enchaîner des opérations de type ϕ_2 ou ϕ_3 suivies d'une opération de type ϕ_1.

Soit n_1 le nombre d'opérations de type ϕ_1, à faire, pour ramener G à un sommet. On numérotera $n_1 + 1$ ce sommet, et i le sommet isolé définissant la $i$ème opération ϕ_1. Pour tout sommet $j \neq n_1 + 1$, on notera V_j l'ensemble des sommets de G qui sont adjacents à j et $V_{n_1+1} = X - \{1, \ldots, n_1\}$.

D'après ce qui précède, on peut représenter G par le schéma symbolique de la Fig. 2.

On posera dans la suite $W_k = V_k - V_{k-1}$, $1 \leq k \leq n_1$.

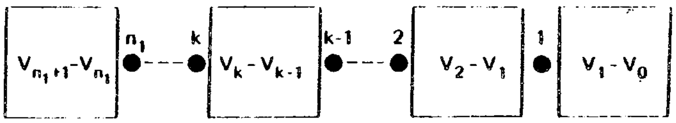


Fig. 2. $V_k - V_{k-1} = \{i \mid i \in V_k \text{ et } i \notin V_{k-1}\}$ et $V_0 = \emptyset$.

Remarque 2. Dans la suite on considère le sommet $n_1 + 1$ comme un sommet *universel* dans W_{n_1+1}.

Soit $C = (C_1, \ldots, C_n)$ un vecteur de R^n. On désignera par $P(G, C)$ le problème de la recherche d'un absorbant A_0 de G tel que $\sum_{i \in A_0} C_i$ soit minimum (on dit que A_0 est de poids minimum dans G). Dans ce qui suit, nous allons donner un algorithme de résolution de $P(G, C)$ qui exige seulement un nombre d'opérations élémentaires proportionnel à n. Pour ce faire on peut supposer que $C_i \geq 0$ pour $i \in X$, car les sommets ayant des poids négatifs appartiennent forcément à A_0.

2. Algorithme de résolution de $P(G, C)$

2.1. Algorithme

On notera $(V_k)_u$ et $(W_k)_u$ (resp. $(V_k)_p$ et $(W_k)_p$) les ensembles de sommets universels (resp. paires de pôles) de V_k et W_k. Pour $k = 1, \ldots, n_1 + 1$, soit p_k (resp. u_k) un sommet de $(W_k)_p$ (resp. $(W_k)_u$) tel que $C_{p_k} = \min\{C_i \mid i \in (W_k)_p\}$ (resp. $C_{u_k} = \min\{C_i \mid i \in (W_k)_u\}$. Soient i_k^* et j^* respectivement les sommets tels que

$$C_{i_k^*} = \min\{C_i \mid i = k, \ldots, n_1\}$$

et

$$C_{j^*} = \min\{C_i \mid i \in \{p_1, \ldots, p_{n_1+1}, u_1, \ldots, u_{n_1+1}\}\}$$

et soit k_0 tel que $j^* \in W_{k_0+1}$.

D'après les notations précédentes, l'absorbant A_0 de poids minimum parmi les absorbants suivants est solution de $P(G, C)$:

(a) $\{1, 2, \ldots, r-1, P_r, j^*\}$, $r = 1, \ldots, k_0$,
(b) $\{1, 2, \ldots, r-1, u_r\}$, $r = 1, \ldots, k_0 + 1$,
(c) $\{1, 2, \ldots, r-1, P_r, i_r^*\}$, $r = 1, \ldots, k_0$,
(d) $\{1, 2, \ldots, k_0, P_{k_0+1}, s\}$,

où s est un sommet tel que

$$C_s = \min\{C_i \mid i \in \{(W_{k_0+1})_p - \{P_{k_0+1}\} \cup (V_{n_t+1} - V_{k_0+1}) \cup \{i^*_{k_0+1}\}\}\}$$

2.2. *Justification*

Soit A_0 une solution de $P(G, C)$, et soit r le plus petit entier compris entre 1 et k_0 (s'il existe) tel que $A_0 \cap W_r \neq \emptyset$.

(1) r existe.

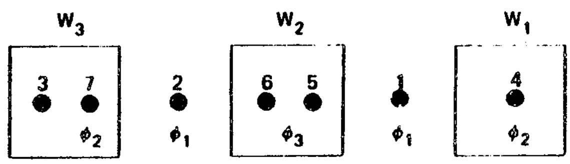
Donc $\{1, 2, \ldots, r-1\} \subset A_0$. Si $A_0 \cap (W_r)_u \neq \emptyset$ alors $A_0 = \{1, 2, \ldots, r-1, u_r\}$, sinon, il existe $i \neq p_r$ tel que $A_0 = \{1, 2, \ldots, r-1, p_r, i\}$, donc $i \in \{i^*_r, j\}$.

(2) r n'existe pas.

Donc $A_0 \cap W_{k_0} = \emptyset$ et par suite $\{1, 2, \ldots, k_0\} \subset A_0$. D'après la définition de j^*, il existe dans ce cas, un absorbant A_0 de poids minimum tel que $A_0 \cap W_{k_0+1} \neq \emptyset$, donc $A_0 = \{1, 2, \ldots, k_0, u_{k_0+1}\}$ ou bien $A_0 = \{1, 2, \ldots, k_0, p_{k_0+1}, s\}$.

2.3. *Exemple*

Soit G le graphe ci-dessous (Fig. 3), défini par l'enchaînement des opérations ϕ_2, ϕ_1, ϕ_3, ϕ_1, ϕ_2.



Représentation schématique de G.

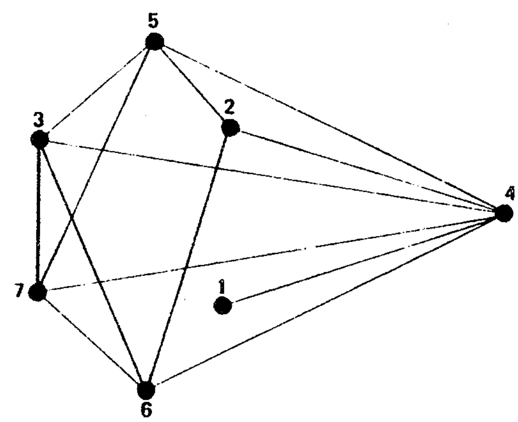


Fig. 3.

Polytope des absorbants

On affecte aux sommets de G les poids $C_1 = 2$, $C_2 = 3$, $C_3 = 12$, $C_4 = 20$, $C_5 = 8$, $C_6 = 16$, $C_7 = 4$. D'après l'algorithme, un absorbant A_0 de poids minimum dans G est parmi les absorbants $\{4\}$, $\{1,5,7\}$, $\{1,5,2\}$, $\{1,2,7\}$ donc $A_0 = \{1,2,7\}$ de poids 9.

3. Polytope des absorbants d'un graphe absorbant a seuil

3.1. Une famille de facette de $P(G)$

Lemme 1. *Si G n'a pas de sommets isolés (i.e., $V_1 \neq \emptyset$), alors $P(G)$ est de pleine dimension.*

Démonstration. Si $V_1 \neq \emptyset$, les ensembles $X - \{i\}$, $i = 1, \ldots, n$ et l'ensemble X lui-même forment une famille de $n + 1$ absorbants affinement indépendants.

Remarque 3. Pour caractériser $P(G)$, on supposera, sans perte de généralité, que G n'a pas de sommets isolés.

Définition 2. L'inéquation $\sum_{i=1}^{n} \alpha_i x_i \geqslant \alpha_0$, $\alpha_i \in R$, $\forall i$, est dite facette (face de dimension $n - 1$) de $P(G)$ si et seulement si

(1) elle est valide pour tout absorbant de G;

(2) il existe n absorbants affinement indépendants vérifiant cette inéquation avec égalité.

Lemme 2. *Pour tout sommet $i \in X$, la contrainte $x_i \leqslant 1$ définit une facette de $P(G)$.*

Démonstration. D'une part, la contrainte $x_i \leqslant 1$ est satisfaite par tous les absorbants, d'autre part les n absorbants suivants $X, X - \{1\}, \ldots, X - \{i - 1\}$, $X - \{i + 1\}, \ldots, X - \{n\}$ sont affinement indépendants et vérifient la contrainte avec égalité.

3.2. Le polytope $P(G)$

3.2.1

Théorème 2. *Le polytope $P(G)$ est défini par les inégalités suivantes:*

$$Q \begin{cases} (1-\delta_{n_1+1,i})x_i + \sum_{t \in V_i} x_t \geq 1, \quad i = 1, \ldots, n_1+1, & (1) \\ \qquad\qquad\qquad\qquad i \in (W_k)_p, \; k = 1, \ldots, n_1+1, & \\ \sum_{i=k+1}^{n_1} x_i + (j-k+2) \sum_{i \in V_k \cup (W_{k+1})_u} x_i + \sum_{t=k}^{j-1} (j+1-t) \sum_{i \in (W_{t+1})_p \cup (W_{t+1})_u} x_i & (2) \\ \quad + \sum_{i \in (V_{n_1+1} \setminus V_{j+1}) \cup (W_{j+1})_p} x_i \geq j-k+2, \quad 0 \leq k \leq j \leq n_1, & \\ x_i \leq 1 \quad \forall i \in X, & (3) \\ x_i \geq 0 \quad \forall i \in X, & (4) \end{cases}$$

où

$$\delta_{n_1+1,i} = \begin{cases} 1 & si \; n_1+1 = i, \\ 0 & sinon. \end{cases}$$

Il est clair que les contraintes (1), (3) et (4) sont satisfaites pour tout absorbant du graphe G. Pour une contrainte (C) de type (2), définie par j, k tels que $0 \leq k \leq j \leq n_1$, on considère un absorbant A de G.

Si $A \cap (V_k \cup (W_{k+1})_u) \neq \emptyset$, alors (C) est vérifiée, sinon soit t_0, $k \leq t_0 \leq n_1$, le plus petit entier tel que $A \cap V_{t_0} = \emptyset$ et $A \cap V_{t_0+1} \neq \emptyset$, donc $\{1, 2, \ldots, t_0\} \subset A$. Suivant que $t_0 \leq j$ ou $t_0 > j$ on peut voir facilement que la contrainte (C) est bien vérifiée.

Remarque 4. Les contraintes de type (1) et (2) correspondantes respectivement à $i = n_1 + 1$, et (k, j) tel que $k = j = n_1$, sont les mêmes.

3.2.2. Démonstration du Théorème 2

Pour démontrer le théorème, il suffit de montrer que les points extrêmes du polytope Q représentent tous des absorbants de G. Pour ce faire, considérons le programme linéaire suivant:

$$P_G = \begin{cases} X \in Q, \\ \min CX, \quad C \in R^n. \end{cases}$$

On sait que pour tout point extrême de Q, il existe un vecteur C tel que ce point soit solution optimale unique de P_G. Par conséquent, il suffit de montrer que pour tout système de poids $C = (C_1, \ldots, C_n) \in R^n$, sur les sommets de G, il existe une solution optimale de P_G qui est le vecteur représentatif d'un absorbant de G. Pour cela, nous montrons que pour tout absorbant A_0 de poids minimum dans G et de vecteur représentatif x^{A_0}, donné par l'algorithme précédent, il existe une solution y^{A_0} du dual D_G de P_G qui vérifie avec x^{A_0} les conditions du théorème des écarts complémentaires.

Puisque on a supposé $C_i \geqslant 0$, $\forall i \in X$, on peut omettre les contraintes (3).

A une contrainte de type (1), correspondant à un sommet i (isolé ou appartenant à une paire de pôle), on associe la variable duale y_i, et à une contrainte de type (2) définie par les entiers k, j on associe la variable duale $\varepsilon_{k,j}$.

Le dual D_G s'écrit:

$$D_G \quad \begin{cases} y_i + \sum\limits_{j \in (V_i)_p} y_j + \sum\limits_{\substack{0 \leqslant k \leqslant i-1 \\ k \leqslant j \leqslant n_1}}\sum \varepsilon_{k,j} \leqslant C_i, \quad \text{pour } i = 1, \ldots, n_1, & (5) \\[2ex] \sum\limits_{i=h}^{n_1} y_i + \sum\limits_{j \in (V_{n_1+1})_p} y_j + \sum\limits_{0 \leqslant k \leqslant j \leqslant h-2}\sum \varepsilon_{k,j} + \sum\limits_{\substack{0 \leqslant k \leqslant h-2 \\ h-1 \leqslant j \leqslant n_1}}\sum (j-h+3)\varepsilon_{k,j} & \\ \qquad + \sum\limits_{h \leqslant k \leqslant j \leqslant n_1}\sum (j-k+2)\varepsilon_{k,j} \leqslant C_u, \quad \text{pour } u \in (W_h)_u & (6) \\[2ex] \sum\limits_{i=h}^{n_1+1} y_i + \sum\limits_{j \neq p^*} y_j + \sum\limits_{0 \leqslant k \leqslant j \leqslant h-1}\sum \varepsilon_{k,j} + \sum\limits_{\substack{0 \leqslant k \leqslant h-1 \\ h \leqslant j \leqslant n_1}}\sum (j-h+2)\varepsilon_{k,j} & \\ \qquad + \sum\limits_{h \leqslant k \leqslant j \leqslant n_1}\sum (j-k+2)\varepsilon_{k,j} \leqslant C_p, \quad \text{pour } p \text{ tq } (p, p^*) \text{ est une paire de pôle de } (W_h), & (7) \\[2ex] y_i \geqslant 0, \qquad i \in \{1, \ldots, n_1\} \cup (V_{n_1+1})_p, & \\ \varepsilon_{k,j} \geqslant 0, \quad 0 \leqslant k \leqslant j \leqslant n_1, & (8) \\ \max \left(\sum\limits_{i=1}^{n_1+1} y_i + \sum\limits_{j \in (V_{n_1+1})_p} y_j + \sum\limits_{0 \leqslant k \leqslant j \leqslant n_1}\sum (j-k+2)\varepsilon_{k,j} \right). & \end{cases}$$

Soit $P(A_0)$ le poids d'un absorbant A_0, solution de $P(G, C)$, alors d'après l'algorithme précédent, il existe r, $1 \leqslant r \leqslant k_0 + 1$, tel que:

$$P(A_0) \geqslant C_{j^*} + C_1 + C_2 + \cdots + C_{r-1},$$

et

$$P(A_0) \leqslant C_{j^*} + C_1 + C_2 + \cdots + C_{k_0} + C_r.$$

Donc, il existe t, $r - 1 \leqslant t \leqslant k_0$, tel que

$$P(A_0) = C_{j^*} + C_1 + C_2 + \cdots + C_t + \delta \quad \text{avec } 0 \leqslant \delta \leqslant C_{t+1}.$$

D'où le lemme suivant.

Lemme 3. *Si A_0 est solution de $P(G, C)$, alors il existe t, $r - 1 \leqslant t \leqslant k_0$, et un réel δ, $0 \leqslant \delta \leqslant C_{t+1}$, tel que*

$$P(A_0) = C_{j^*} + C_1 + C_2 + \cdots + C_t + \delta. \tag{9}$$

Soit

$$C_i^* = \min\{C_i, \ldots, C_{k_0}, C_0\}, \quad i = 1, \ldots, k_0, \tag{10}$$

avec $C_0^* = 0$, $C_{k_0+1}^* = C_0$, *où* $C_0 = \min\{C_{j^*}, C_{k_0+1}, \ldots, C_{n_1}\}$.

Nous avons $C_i^* \leq C_{i+1}^*$ pour $i = 1, \ldots, k_0$. Alors, si A_0 est solution de $P(G, C)$ et de poids donné par la relation (9), il en résulte:

$$C_h + C_{h+1} + \cdots + C_t + C_{j^*} + \delta \leq C_{p_h} + C_h^*, \tag{11}$$

$$C_h + C_{h+1} + \cdots + C_t + C_{j^*} + \delta \leq C_{u_h}, \quad \text{pour } h \ tq. \ 1 \leq h \leq t,$$

et si

$$t = k_0, \qquad \delta \leq C_*. \tag{12}$$

p_h et u_h sont déjà définis dans l'algorithme au Section 2.1. Soit i_0 tel que

$$\begin{cases} 0 \leq i_0 \leq t, & \text{si } 0 \leq \delta \leq C_{t+1}^* \text{ et } C_{i_0}^* \leq \delta \leq C_{i_0+1}^*, \\ i_0 = t+1, & \text{sinon.} \end{cases}$$

Lemme 4. *Si* $P(A_0) = C_{j^*} + C_1 + C_2 + \cdots + C_t + \delta$ *avec* $r - 1 \leq t \leq k_0$ *et* $0 \leq \delta \leq C_{t+1}$, *alors*

$$y^{A_0} = \begin{cases} \varepsilon_{k,t} = C_{k+1}^* - C_k^*, & k = 0, 1, \ldots, i_0 - 1, \\ \varepsilon_{i_0,t} = \delta - C_{i_0}^*, & \\ \varepsilon_{i_0,t+1} = C_{i_0+1}^* - \delta, & \\ \varepsilon_{k,t+1} = C_{k+1}^* - C_k^*, & k = i_0 + 1, \ldots, t - 1, \\ y_i = C_i - C_i^*, & i = 1, \ldots, t, \\ (1 - \delta_{t,k_0}) y_{t+1} + \delta_{t,k_0} y_{p_{k_0+1}} = \lambda(\delta - C_{t+1}^*), & \\ y_{n_1+1} = C_{j^*} - \varepsilon, & \\ y_i = \varepsilon_{k,j} = 0, & \textit{ailleurs} \end{cases}$$

où

$$\delta_{t,k_0} = \begin{cases} 1 & \textit{si } t = k_0, \\ 0 & \textit{sinon,} \end{cases} \quad \textit{et} \quad \lambda = \begin{cases} 1 & \textit{si } \delta > C_{t+1}^*, \\ 0 & \textit{sinon,} \end{cases}$$

$$\varepsilon = \sum_{0 < k < j < n_1} \sum \varepsilon_{k,j}$$

est une solution réalisable du dual D_G.

Démonstration. Simple à vérifier à partir de (10), (11) et (12).

D'après les Lemmes 3 et 4, on déduit que pour tout absorbant A_0 solution de $P(G, C)$, on peut associer une solution réalisable y^{A_0} du dual D_G.

Lemme 5. *Pour tout absorbant A_0, solution de $P(G, C)$, déterminé par l'algorithme, x^{A_0} est une solution optimale de P_G.*

Démonstration. Il est facile de vérifier que pour tout absorbant A_0, de poids minimum proposé par l'algorithme, x^{A_0} vérifie avec y^{A_0} les conditions du théorème des écarts complémentaires.

Le Lemme 5 achève la démonstration du Théorème 2.

Exemple. On reprend le graphe G donné par la Fig. 3, le polytope $P(G)$ sera dans ce cas:

$$P(G)\left\{\begin{array}{rrrrrrrl} x_1 & & & +\, x_4 & & & & \geq 1, \\ & x_2 & & +\, x_4 + & x_5 + & x_6 & & \geq 1, \\ & & x_3 + & x_4 + & x_5 + & x_6 + & x_7 & \geq 1, \\ & x_2 + & x_3 + & x_4 + & x_5 + & x_7 & & \geq 1, \\ & x_2 + & x_3 + & x_4 & & +\, x_6 + & x_7 & \geq 1, \\ x_1 + & x_2 + & x_3 + & 2x_4 + & x_5 + & x_6 + & x_7 & \geq 2, \; * \\ x_1 + & x_2 + & x_3 + & 3x_4 + & x_5 + & x_6 + & x_7 & \geq 3, \; * \\ x_1 + & x_2 + & 2x_3 + & 4x_4 + & 2x_5 + & 2x_6 + & 2x_7 & \geq 4, \; * \\ & x_2 + & x_3 + & 2x_4 + & x_5 + & x_6 + & x_7 & \geq 2, \\ & x_2 + & 2x_3 + & 3x_4 + & 2x_5 + & 2x_6 + & 2x_7 & \geq 3. \; * \end{array}\right.$$

Les quatres contraintes marquées d'une astérisque sont redondantes, ce qui montre que les contraintes de type (2) ne sont pas toutes des facettes.

La solution optimale du dual D_G associée à l'absorbant de poids minimum A_0 est:

$$y^{A_0} = \begin{cases} y_1 = 1, \\ \varepsilon_{0,1} = 2, \\ \varepsilon_{1,1} = 1, \\ y_i = \varepsilon_{j,k} = 0, \quad \text{ailleurs.} \end{cases}$$

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ANNEXE 8

K_i - covers I: Complexity and Polytopes

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ABSTRACT

A K_i in a graph is a complete subgraph of size i. A K_i-cover of a graph G(V,E) is a set C of K_{i-1}'s of G such that every K_i in G contains at least one K_{i-1} in C. Thus a K_2-cover is a vertex cover. The problem of determining whether a graph has a K_i-cover ($i \geq 2$) of cardinality $\leq k$ is shown to be NP-complete for graphs in general. For chordal graphs with fixed maximum clique size, the problem is polynomial; however, it is NP-complete for arbitrary chordal graphs when $i \geq 3$. The NP-completeness results motivate the examination of some facets of the corresponding polytope. In particular we show that various induced subgraphs of G define facets of the K_i-cover polytope. Further results of this type are also produced for the K_3-cover polytope. We conclude by describing polynomial algorithms for solving the separation problem for some classes of facets of the K_i-cover polytope.

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1. Introduction

In this paper we begin our study of various aspects of K_i-covers in graphs, a generalization of the notion of vertex cover. The concept of covering is well established in combinatorics and includes set covering and edge covering as well as vertex covering, clique covering and coloring. Other types of covering and applications are discussed in [1]. The paradigm for combinatorial covering may be stated as follows:

Given a set X and a family $\mathcal{S}$ of subsets of X, a set $\mathcal{C}$ of elements from 2^X having a prescribed property is a <u>cover</u> of $\mathcal{S}$ if, for each $S_i \in \mathcal{S}$, there is a $C_j \in \mathcal{C}$ such that C_j is contained in or "covers" S_i.

In many applications one is interested in finding a cover of minimum cardinality or more generally to find a cover with minimum weight where the elements in $\mathcal{C}$ have each been assigned a weight. To illustrate the general definition of cover, we examine each of the five particular coverings mentioned above. For the set covering problem, $\mathcal{C} \subseteq X$ and thus we require a subset of X such that each element in $\mathcal{S}$ contains at least one of these elements. For the graph theoretic coverings X = V where the given graph has vertex set V and edge set E. In a vertex cover, $\mathcal{S}$ = E and $\mathcal{C} \subseteq$ V; in an edge cover $\mathcal{S}$ = V and $\mathcal{C} \subseteq$ E (note here that edge e "covers" vertex v if e is incident with v). In the clique covering, X = V, $\mathcal{S}$ = V and $\mathcal{C}$ is the family of completely connected nodesets; in the coloring X and $\mathcal{S}$ do not change, but $\mathcal{C}$ is the family of stable subsets of nodes of G. Incidentally, the only well solved covering problem is the edge cover, by matching theory.

It should be noted that some confusion arises in the naming of various covering problems. For vertex and edge covers the adjectives "vertex" and "edge" indicate the nature of the covering set C, whereas for set covers, the adjective "set" describes the contents of S, the set to be covered. It is this latter convention that we follow in our definition of K_i-cover. Given a graph $G(V,E)$ we let $F_i(G)$ denote $\{K_i | K_i \subseteq G\}$ (i.e., $F_i(G)$ is the set of complete subgraphs on i vertices of G). For the K_i-cover problem, $X = V$, $S = F_i(G)$ and $C \subseteq F_{i-1}(G)$, $i \geq 2$. In other words, a K_i-cover of G is a set C of K_{i-1}'s such that every K_i in G contains at least one K_{i-1} in C. Note that this definition of K_2-cover is equivalent to that of vertex cover and a K_3-cover is a set of edges meeting all the triangles of G. For a graph G, the K_i-cover number $c_i(G)$ is the cardinality of a smallest K_i-cover of G. If we associate a weight to each K_{i-1} of G, then $c_i(G)$ for the weighted version of the problem is defined as the minimum total weight of any K_i-cover, where the weight of a K_i-cover C_j is the sum of the weights of the K_{i-1}'s in C_j.

In this paper we address various computational aspects of K_i-covers. First we examine the complexity of the K_i-cover problem on graphs in general and then on the restricted class of chordal graphs. The decision version of the K_i-cover problem is "given a graph G and integers $i \geq 2$ and $k \geq 1$, is $c_i(G) \leq k$?". Having seen that the general problem is NP-complete, we study some families of facets of the K_i-cover polytope and show that various induced subgraphs of G define facets of this polytope.

In [6] we continue the study of K_i-covers by examining the relationship between $c_i(G)$ and $p_i(G)$, the K_i-packing number defined to be the largest cardinality of any set of K_i's in G not having $i - 1$ nodes in common. For any $F \subseteq \mathcal{F}_i(G)$ we may define $c_i(F)$ and $p_i(F)$ in a manner similar to their definitions for graphs. Clearly for all such F, $c_i(F) \geq p_i(F)$. We define a graph to be <u>K_i-perfect</u> if $\forall F \subseteq \mathcal{F}_i(G)$, $c_i(F) = p_i(F)$. In [6] we provide a characterization of K_i-perfect graphs in terms of a class of graphs which satisfies Berge's Strong Perfect Graph Conjecture [10]. Furthermore, we study the associated $\mathcal{F}_{i-1}(G) \times \mathcal{F}_i(G)$ intersection matrices.

Before presenting our material on the computational aspects of K_i-covers, we introduce the definitions and terminology used in this paper. Given a graph $G(V,E)$ with $X \subseteq V$, $G[X]$ denotes the induced subgraph of G restricted to X. For $v \in V$, $\Gamma(v) = \{u | u \in V, (u,v) \in E\}$. A vertex v is <u>universal</u> to $X \subseteq V \setminus \{v\}$ if $X \subseteq \Gamma(v)$. Similarly, a set of vertices may be universal to X. A <u>clique</u> in $G(V,E)$ is a maximal complete subgraph and $\omega(G)$ is the <u>clique number of G</u>, namely, the size of the largest clique in G.

A graph is <u>chordal</u> (or <u>triangulated</u>) [8], [10] if every cycle of length greater than three has a chord. A <u>simplicial vertex</u> v is one for which $\Gamma(v)$ is complete. A graph has a <u>perfect elimination scheme</u> if there exists an order of eliminating the vertices such that each vertex is simplicial at the time of its elimination. A graph is chordal iff it has a perfect elimination scheme (see [10]).

The relationship between the K_i's and the K_{i-1}'s in G may be represented by the <u>K_i-intersection graph</u>, $I_i(G)$. The vertices of $I_i(G)$ are the K_i's in $\mathcal{F}_i(G)$; two such vertices are adjacent iff they have a K_{i-1} in common. Note that $I_2(G)$ is the line graph $L(G)$. Throughout the paper $n = |V|$, and $k_i(G) = |\mathcal{F}_i(G)|$, the number of K_i's in G.

2. Complexity Results

In this section we examine the complexity of the K_i-cover problem for different values of i and for restricted inputs. This examination consists of showing the problem to be NP-complete for graphs in general and also for chordal graphs. It is then shown that the problem is polynomial for chordal graphs with fixed clique size.

2.1. NP-completeness

As mentioned in the introduction, the K_2-cover problem is the well-known vertex cover problem, one of the first problems shown to be NP-complete [15]. The K_3-cover problem was shown to be NP-complete by Yannakakis [19] using a reduction from the vertex cover problem. We now use Yannakakis' proof technique to show that the K_i-cover problem is NP-complete for all $i \geq 2$.

Theorem 2.1. For any $i \geq 2$, the K_i-cover problem is NP-complete.

Proof: The reduction is from the vertex cover problem. Let G(V,E) be the input graph to the vertex cover problem. As in [19] we may assume that G has no triangles by replacing each edge of G with a path on four vertices. It is clear that for this new graph G_1, $c_2(G_1) = c_2(G) + |E|$. We now form the graph G' by adding a universal K_{i-2}, C to G_1. We claim that $c_i(G') = c_2(G_1)$.

$c_i(G') \geq c_2(G_1)$: Let C be a K_i-cover of G' where $|C| = \beta$. Examine all K_{i-1}'s in C; if any K_{i-1}, say X, contains an edge (u,v) in G_1, then replace X with a new complete subgraph $X' = C \cup \{u\}$ or $X' = C \cup \{v\}$. Since G_1 is Δ-free, X covers only the K_i, $C \cup \{u,v\}$, whereas X' covers this K_i

and possibly others. Thus the set C' resulting from the replacement of all such K_{i-1}'s in C is also a K_i-cover of G' with cardinality β. Now let A be the set of all vertices in G_1 such that the vertex belongs to a K_{i-1} in C'. It is clear that A is a vertex-cover of G_1.

$\underline{c_i(G') \leq c_2(G_1)}$: Assume A is an optimum vertex cover of G_1. Set $C = \{C \cup \{x\} | x \in A\}$. C is a K_i-cover of G' since any K_i not covered by C would imply the existence of an edge in G_1 which is not covered by A. ■

Using the above construction we immediately have

<u>Corollary 2.2</u>. For any $i \geq 2$, the K_i-cover problem is NP-complete when restricted to graphs with $\omega(G) = i$.

Although the vertex-cover problem is polynomial for chordal graphs [8], we now show that for any fixed $i \geq 3$, the K_i-cover problem is NP-complete for chordal graphs.

<u>Corollary 2.3</u>. For any fixed $i \geq 3$, the K_i-cover problem on chordal graphs is NP-complete.

<u>Proof</u>: The reduction will be from the general K_i-cover problem. Given a graph $G(V,E)$ we construct a chordal graph $G'(V',E')$ as follows:

(i) Set $C = K_n$, $(n = |V|)$ where v_j' in C represents $v_j \in V$.

(ii) Examine all $\binom{n}{i-1}$ subsets of $i - 1$ vertices in V. If such a subset S does not form a K_{i-1}, then add a new vertex v_s to V' where v_s is adjacent to all vertices in C corresponding to the vertices in S.

Clearly, G' is chordal and since i is fixed, G' may be constructed in polynomial time. We claim that $c_i(G') = c_i(G) + \binom{n}{i-1} - k_{i-1}(G)$. (Recall that $K_{i-1}(G) = |F_{i-1}(G)|$.)

$\underline{c_i(G') \geq c_i(G) + \binom{n}{i-1} - k_{i-1}(G)}$: Let C be a K_i-cover of G' where $|C| = c_i(G')$. Examine each K_{i-1} in C; if any such K_{i-1}, say X, contains a v_s, then replace X with $X' = S$. If S is already in C, then choose any other K_{i-1} in G. Thus X' covers the K_i, $\{v_s\} \cup S$ and, possibly others. Thus the set C', resulting from the replacement of all such K_{i-1}'s in C is also a K_i-cover of G' with cardinality $c_i(G')$. Form C^* from C' by removing all $\binom{n}{i-1} - k_{i-1}(G)$ elements of C' which correspond to subsets of G which do not form a K_{i-1}. None of the K_i's in G is covered by any K_{i-1} in $C' \backslash C^*$ and thus these K_i's are covered by the K_{i-1}'s in C^*. Therefore C^* is a K_i-cover of G with cardinality $\leq c_i(G') - \binom{n}{i-1} + k_{i-1}(G)$.

$\underline{c_i(G') \leq c_i(G) + \binom{n}{i-1} - k_{i-1}(G)}$: Assume C^* is an optimal K_i-cover of G. Set $C = C^* \bigcup_{v_s \in V'} \Gamma(v_s)$.

Clearly, $|C| = c_i(G) + \binom{n}{i-1} - k_{i-1}(G)$. C is a K_i-cover of G' since any uncovered K_i must be in G and furthermore must be uncovered in G, which contradicts the assumption that C^* is a K_i-cover of G. ■

2.2. Polynomial Algorithm

In light of Corollaries 2.2 and 2.3, it is somewhat surprising to note that for any fixed j, the K_i-cover problem is polynomial on chordal graphs with maximum clique size j. Before presenting this algorithm we note some facts about chordal graphs. As mentioned in §1, a graph is chordal iff it has a perfect elimination scheme thereby indicating that the cliques of the chordal graph G interlock in a very tree-like way. Given a chordal graph G and an associated perfect elimination scheme, we may construct a rooted clique tree T [8] where the nodes of T are the cliques of G. Furthermore, this tree may be constructed in linear time [17]. Given G with a rooted clique tree T and a clique C of G (note $|C| \le j$) we let G(C) denote the subgraph of G induced by the cliques in the subtree of T rooted at C. Thus if R is the root of T then $G(R) \equiv G$. We let $C_1, C_2, \ldots, C_\ell$ denote the children of C in T ($\ell = 0$ iff C is a leaf of T). If C_k is a child of C, then we let x_k denote the vertex $C_k \setminus C$. See Figure 2.1, where the vertex number is its order in a perfect elimination scheme. Clique $C_k = \{3,5,8\}$ is a child of $\{5,6,7,8\}$ and the node x_k is 3. $G(\{3,5,8\})$ is the subgraph induced by the nodeset $\{1,3,5,8\}$.

Given a clique C of G and $\mathcal{C}$ any set of K_{i-1}'s which cover all K_i's in C, we define:

$COVi(G(C),\mathcal{C})$ to be a minimum cardinality set of K_{i-1}'s in G(C) which is a K_i-cover of G(C) and $COVi(G(C),\mathcal{C}) \cap F_{i-1}(C) = \mathcal{C}$ (i.e., $\mathcal{C} \subset COVi(G(C),\mathcal{C})$; however, no other K_{i-1}'s in $\mathcal{F}_{i-1}(C) \setminus \mathcal{C}$ belong to $COVi(G(C),\mathcal{C})$).

The algorithm will find $COVi(G(C),\mathcal{C})$ for all cliques C in G and for all covering sets $\mathcal{C}$ of C. The general form of this dynamic programming is described in [7].

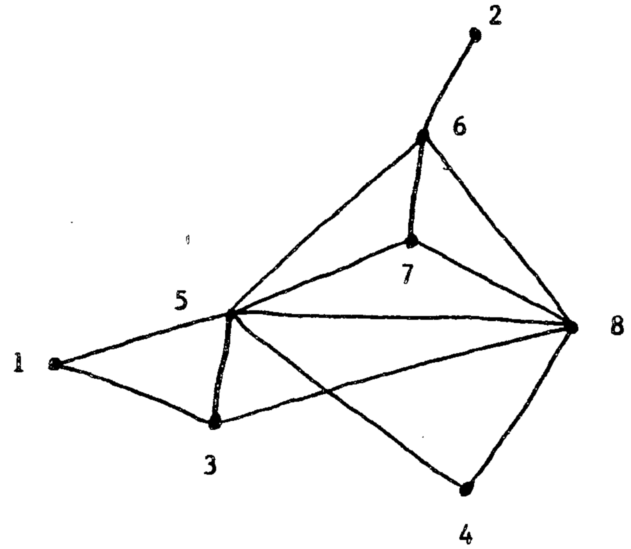
<u>Algorithm 2.1</u>. K_i-cover of chordal graphs with fixed clique size j.

<u>Input</u>: Chordal graph G with $\omega(G) \le j$, a constant; integer i.

<u>Output</u>: A set C^* of K_{i-1}'s such that C^* covers $\mathcal{F}_i(G)$ and $|C^*| = c_i(G)$.

1. If $i > j$, then set $C^* = \emptyset$ and stop.

G:



T:

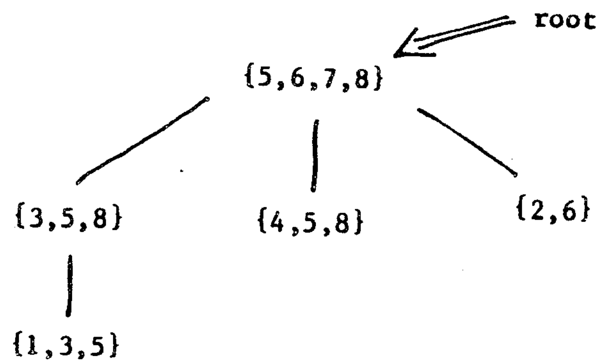


Figure 2.1. Chordal graph G with clique tree T.

2. Construct a clique tree T rooted at R.

3. Do a bottom-up scan of T calculating the COVi(G(C),$\mathcal{C}$)'s for each node C in T in the following way: Assume node C has children $C_1, C_2, \ldots, C_\ell$ ($\ell = 0$ iff C is a leaf of T).

   (i) If C is a leaf then for all $\mathcal{C} \subset F_{i-1}(C)$ such that $\mathcal{C}$ covers C set

   $$\text{COVi}(G(C),\mathcal{C}) = \mathcal{C}.$$

   (ii) If C is not a leaf, then for all $\mathcal{C}$ covering C do the following: For each clique C_k ($1 \leq k \leq \ell$) examine all sets $\mathcal{D}_k$ of K_{i-1}'s such that for each $\mathcal{D}_k$, $\mathcal{D}_k$ covers all K_i's in C_k and $\mathcal{D}_k$ does not include any new K_{i-1}'s in C that are not in $\mathcal{C}$. Let B_k be a set with minimum cardinality in $\{\mathcal{C} \cup \text{COVi}(G(C_k),\mathcal{D}_k)\}$. Set

   $$\text{COVi}(G(C),\mathcal{C}) = \bigcup_{k=1}^{\ell} B_k.$$

4. If C is the root R, then set $\mathcal{C}^*$ to be any set of minimum cardinality in $\{\text{COVi}(G(R),\mathcal{C})\}$ where $\mathcal{C}$ is a set of K_{i-1}'s which cover all K_i's in R. ∎

As an example of this algorithm consider the graph G in Figure 2.1 where $i = 3$. For cliques $\{1,3,5\}$ and $\{4,5,8\}$ the COV3 sets are calculated immediately by step 3(i). Now examine clique $\{3,5,8\}$. Its different $\mathcal{C}$ sets and the corresponding COV3 sets are listed in Table 2.1.
For the root $R = \{5,6,7,8\}$, all of its $\mathcal{C}$ sets of cardinality 2 or 3 and the corresponding COV3 sets are shown in Table 2.2. From this table we see that $\mathcal{C}^* = \{(5,8)(6,7)(1,5)\}$ is a K_3-cover of G with minimum cardinality.

Table 2.1. COV3 sets for C = {3,5,8}.

C	COV3($G(C)$,C)
{(3,5)}	{(3,5)}
{(3,8)}	{(3,8), (1,3)}
{(5,8)}	{(5,8), (1,5)}
{(3,5)(3,8)}	{(3,5), (3,8)}
{(3,5)(5,8)}	{(3,5), (5,8)}
{(3,8)(5,8)}	{(3,8), (5,8), (1,3)}
{(3,5)(3,8)(5,8)}	{(3,5), (3,8), (5,8)}

Table 2.2. Some COV3 sets for C = {5,6,7,8}

C	COV3(G,C)	
{(5,6)(7,8)}	{(5,6)(7,8)(3,5)(4,8)}	
{(5,8)(6,7)}	{(5,8)(6,7)(1,5)}	*
{(6,8)(5,7)}	{(6,8)(5,7)(3,5)(4,5)}	
{(5,6)(7,8)(5,7)}	{(5,6)(7,8)(5,7)(3,5)(4,5)}	
{(5,6)(7,8)(5,8)}	{(5,6)(7,8)(5,8)(1,5)}	
{(5,6)(7,8)(6,7)}	{(5,6)(7,8)(6,7)(3,5)(4,5)}	
{(5,6)(7,8)(6,8)}	{(5,6)(7,8)(6,8)(3,5)(4,5)}	
{(5,8)(6,7)(5,6)}	{(5,8)(6,7)(5,6)(3,5)}	
{(5,8)(6,7)(5,7)}	{(5,8)(6,7)(5,7)(3,5)}	
{(5,8)(6,7)(6,8)}	{(5,8)(6,7)(6,8)(3,5)}	
{(5,8)(6,7)(7,8)}	{(5,8)(6,7)(7,8)(3,5)}	
{(6,8)(5,7)(5,6)}	{(6,8)(5,7)(5,6)(3,5)(4,5)}	
{(6,8)(5,7)(5,8)}	{(6,8)(5,7)(5,8)(3,5)}	
{(6,8)(5,7)(6,7)}	{(6,8)(5,7)(6,7)(3,5)(4,5)}	
{(6,8)(5,7)(7,8)}	{(6,8)(5,7)(7,8)(3,5)(4,5)}	
{(5,6)(5,7)(6,7)}	{(5,6)(5,7)(6,7)(3,5)(4,5)}	
{(5,7)(5,8)(7,8)}	{(5,7)(5,8)(7,8)(3,5)}	
{(5,6)(5,8)(6,8)}	{(5,6)(5,8)(6,8)(3,5)}	
{(6,7)(6,8)(7,8)}	{(6,7)(6,8)(7,8)(3,5)(4,5)}	

We now establish the correctness of Algorithm 2.1 and then discuss its efficiency. First we state some straightforward lemmas.

<u>Lemma 2.4</u>. Let C be a K_i-cover of $G(V,E)$. For any $X \subseteq V$, the restriction of C to X covers all K_i's in $G[X]$.

<u>Lemma 2.5</u>. Let $G(V,E)$ be a chordal graph with clique tree T and C a node of T with children $C_1,\ldots,C_\ell$. Then,

$$\text{for any } j,k,\ 1 \le j < k \le \ell,\ G(C_j) \cap G(C_k) = C_j \cap C_k \subset C.$$

<u>Corollary 2.6</u>. Under the conditions of lemma 2.5, $x \in G(C_j)\backslash C$, $y \in G(C_k)\backslash C \Rightarrow (x,y) \notin E$.

<u>Corollary 2.7</u>. Under the conditions of lemma 2.5, let $\mathcal{C}_j$ be a K_i-cover of $G(C_j)$ and let $\mathcal{C}_k$ be a K_i-cover of $G(C_k)$. Then $\mathcal{C}_j \cap \mathcal{C}_k \subseteq F_{i-1}(C)$.

<u>Theorem 2.8</u>. Algorithm 2.1 determines a minimum cardinality K_i-cover for the chordal graph with $\omega(G) = j$.

<u>Proof</u>: We only need to show that in step 3 the algorithm correctly determines a $COVi(G(C),\mathcal{C})$ for each possible $\mathcal{C}$ of C. Under this assumption, step 4 will obviously find a minimum cardinality K_i-cover of G.

The proof of the correct calculation of the COVi sets proceeds by induction on the height of the clique tree T. If C is a leaf, then obviously the COVi sets are calculated correctly. Assume now that C is not a leaf and for all children $C_1,C_2,\ldots,C_\ell$ of C, the COVi sets are determined accurately. We now show that $C' = \mathrm{COVi}(G(C),C)$ is calculated correctly for $C \subseteq F_{i-1}(C)$ such that C covers all K_i's in C.

Clearly, $C \subseteq C'$. First we show that C' covers G(C) and then that it is of minimum cardinality. From the inductive assumption $G(C_k)$ is covered $\forall$ k, $1 \leq k \leq \ell$. From Corollary 2.6 we see that all K_i's in G(C) are of one of the following two types: (i) entirely within $G(C_k)$ for some k, in which case it is covered, or (ii) entirely within C, in which case it is covered by C. Furthermore, it is clear that $C' \cap F_{i-1}(C) = C$ as required.

To establish the minimum cardinality of C' we assume to the contrary that there exists X which satisfies the various conditions for C' and $|X| < |C'|$. By Lemma 2.4 the restriction of C' to $G(C_k)$ covers $G(C_k)$ $\forall$ k, $1 \leq k \leq \ell$. Let X_k denote the restriction of X to $G(C_k)$. By Corollary 2.7, $X_j \cap X_k \subseteq F_{i-1}(C)$, $1 \leq j < k \leq \ell$. In fact, since $X \cap F_{i-1}(C) = C$, $X_j \cap X_k \subseteq C$. Thus $|X| = \sum_{j=1}^{\ell} |X_j \backslash (C \cap F_{i-1}(C_j))| + |C|$. Similarly, $|C'| = \sum_{j=1}^{\ell} |C'_j \backslash (C \cap F_{i-1}(C_j))| + |C|$. Since $|X| < |C'|$ there exists k such that $|X_k \backslash (C \cap F_{i-1}(C_k))| < |C'_k \backslash (C \cap F_{i-1}(C_k))|$. Since both X and C' must contain all K_i's in $C \cap F_{i-1}(C_k)$ this means that $|X_k| < |C'_k|$. But $X \cap F_{i-1}(C_k)$ covers C_k and $X \cap F_{i-1}(C) = C$, and thus $X \cap F_{i-1}(C_k)$ is one of the $\mathcal{D}_k$'s considered by the algorithm. However, by the inductive assumption $\mathrm{COVi}(G(C_k),\mathcal{D}_k)$ is of minimum cardinality thereby contradicting the assumption that $|X| < |C'|$. ∎

Since for any clique in G, the number of subsets of K_{i-1}'s to be examined is bounded by $2^{\binom{j}{i-1}}$ which is a constant (albeit quite large!)

we see that Algorithm 2.1's running time and storage requirements are bounded by polynomials in the size of the input graph. In light of Corollary 2.3, this exponential growth with j is hardly surprising. For particular values of i and j it is expected that algorithms which are more efficient than Algorithm 2.1 can be developed; as an example for $i = 2$, $j = 3$ a straightforward greedy algorithm suffices. We also note that Algorithm 2.1 shows that the K_i-cover problem is polynomial for chordal graphs whose largest degree is bounded by a constant.

We now turn our attention to examining the facets of the K_i-cover polytope.

3. Facets of the K_i-cover Polytope

As before, let $F_i(G)$ and $F_{i-1}(G)$ be the families of K_i's and K_{i-1}'s in G. Throughout this section the following terminology and notation will be used. The matrix ${}_iA$ will denote the $K_{i-1}(G)$ versus $K_i(G)$ incidence matrix. The $(j,k)^{th}$ entry of ${}_iA$ will equal 1 iff the j^{th} K_{i-1} in $F_{i-1}(G)$ is contained in the k^{th} K_i in $F_i(G)$, and equal 0 otherwise. $|F_{i-1}(G)|$ will be denoted by h. To any $C \subseteq F_{i-1}(G)$ we may associate the incidence vector $x^C \in (0,1)^h$ where

$$x_j^C = \begin{cases} 1 & \text{if } K_{i-1}^j \in C \\ 0 & \text{otherwise} \end{cases}$$

The all ones vector will be denoted by $\mathbf{1}$, and ${}_iA_j$ refers to the j^{th} row of ${}_iA$. Very often we will wish to refer to a specific complete graph in a given graph. In the notation ${}^{\alpha}K_i^j$, i gives the size of the complete graph, j is an index and α is a list of nodes included in or excluded from the complete graph. For example, $\alpha = x,\bar{y},\bar{z}$ indicates that node x is included, whereas nodes y and z are excluded.

We now examine the minimum weight K_i-cover problem by presenting a partial characterization of the corresponding polytope. The problem of finding a minimum weight K_i-cover of G is equivalent to solving the following integer linear programming problem.

$$(P) = \begin{cases} {}_iA^T x \geq 1 \\ x_j \in \{0,1\} \quad j = 1,\ldots,h \\ \text{MIN } Wx \end{cases}$$

where W is the system of weights associated with $F_{i-1}(G)$. By relaxing the integrality constraint on the x_j's in (P) we get the following linear program:

$$(P') = \begin{cases} {}_iA^T x \geq 1 & (1) \\ 0 \leq x_j \leq 1 \quad j = 1,\ldots,h & (2) \\ \text{MIN } Wx \end{cases}$$

If the polyhedron defined by (1) and (2) has integer valued vertices, then the problems (P) and (P') are equivalent. If this is not the case, it becomes necessary to determine the hyperplanes (facets) which in addition to constraints (1) and (2) define the convex hull of the integer solutions of (P'). This convex hull will be called the <u>K_i-cover polytope of G</u> and will be denoted by $P_{K_i}(G)$. The minimum weight K_i-cover problem may be stated as the following linear program: MIN Wx, $x \in P_{K_i}(G)$. When the constraint matrix in (1) is a general 0 - 1 matrix, then (P) is known as the set covering problem. Hence our K_i-cover polytope is a special case of the set covering polytope.

An independent set is a set of nonadjacent vertices. If S is an independent set, then V - S is a K_2-cover. Thus, for i = 2, the K_2-cover polytope of G is equivalent to the independence polytope (each vertex of the polytope is the incidence vector of an independent set of G). A great deal of work has been done on this polytope [16,5,18,4]; in particular Padberg [16] has studied it for arbitrary graphs and has described some classes of its hyperplanes.

Since the K_i-cover problem is NP-complete, in light of the implications brought by the ellipsoid method [12], there is very little hope of completely characterizing $P_{K_i}(G)$ for an arbitrary graph G. Nevertheless, it is interesting to produce a partial characterization of the polytope corresponding to such an NP-complete problem. In particular, one often focuses such attention on facets for which the separation problem is polynomial. The separation problem is to decide whether a point x belongs to the polytope and, if not, to find a hyperplane which separates x from the polytope. For examples of this approach see [13,11] on the travelling salesman problem and [3,14] on the max cut problem.

3.1. Facets of $P_{K_i}(G)$

We now present a partial non-redundant system of inequalities defining some hyperplanes of $P_{K_i}(G)$. These inequalities are essential inequalities or facets of $P_{K_i}(G)$. For two of these families we present polynomial algorithms for solving the separation problem (see §3.3). First we prove the following

Lemma 3.1: The polytope $P_{K_i}(G)$ is of full dimension (i.e.,

$\dim(P_{K_i}(G)) = h$).

<u>Proof</u>: The sets $F_{i-1}(G)\backslash K_{i-1}^{j}$, $j = 1,\ldots,h$ and the set $F_{i-1}(G)$ form a family of $h+1$ K_i-covers of G whose incidence vectors are affinely independent. ∎

Thus a valid inequality $a^T x \geq a_0$ (satisfied by all points of $P_{K_i}(G)$) defines a facet of $P_{K_i}(G)$ iff $\emptyset \neq \{P_{K_i}(G) \cap \{x \mid a^T x = a_0\}\} \neq P_{K_i}(G)$, and there exists h affinely independent points in $P_{K_i}(G) \cap \{x \mid a^T x = a_0\}$. It is easy to see that the trivial constraints $x_j \leq 1$, $\forall\, j$, define facets for $P_{K_i}(G)$ for all $i \geq 2$, and the constraints $x_j \geq 0$, $\forall\, j$ define facets for $P_{K_i}(G)$ only if $i \geq 3$. We now present three different families of facets. The first is defined on the K_i's and K_{i+1}'s.

<u>3.1.1. Complete Subgraphs K_i and K_{i+1} and Facets</u>

<u>Theorem 3.2</u>: For $i \geq 3$, the constraints ${}_iA^T x \geq 1$ define facets of $P_{K_i}(G)$.

<u>Proof</u>: It is clear that these constraints are valid for $P_{K_i}(G)$. Given

$$K_i^j \in F_i(G), \text{ let } \{K_{i-1}^1, K_{i-1}^2, \ldots, K_{i-1}^i\} \text{ be the } K_{i-1}\text{'s in } K_i^j.$$

We now examine the following K_i-covers of G:

$$C_\ell = \{K_{i-1}^\ell, K_{i-1}^{i+1}, \ldots, K_{i-1}^h\} \qquad \ell = 1,2,\ldots,i$$

$$C_r = \{K_{i-1}^1\} \cup \{K_{i-1}^j,\ i+1 \leq j \leq h,\ j \neq r\} \qquad r = i+1,\ldots,h.$$

The vectors $x^{C_1},\ldots,x^{C_h}$ all verify $\sum_{k=1}^{i} x_k = {}_iA_j^T x = 1$ and they are linearly independent. Thus ${}_iA_j^T x \geq 1$ is a facet of $P_{K_i}(G)$. ∎

<u>Lemma 3.3</u>: Every facet defining inequality of $P_{K_i}(G)$ except those given by $x_j \leq 1$, $\forall j$, is of the form $\sum_{j=1,\ldots,h} a_j x_j \geq a_0$, with $a_j \geq 0$, $\forall j = 0,1,\ldots,h$.

<u>Proof</u>: Suppose that $a_{j_0} < 0$ for $j_0 \in \{1,\ldots,h\}$. Since $\sum_{j=1,\ldots,h} a_j x_j \geq a_0$ is different from $x_{j_0} \leq 1$, there exists a K_i-cover C_1 with incidence vector x^{C_1} such that $K_{i-1}^{j_0} \notin C_1$ and $\sum_{j=1,\ldots,h} a_j x_j^{C_1} = a_0$. Let $C_2 = C_1 \cup \{K_{i-1}^{j_0}\}$. It is obvious that C_2 is a K_i-cover of G, but $\sum_{j=1,\ldots,h} a_j x_j^{C_2} < a_0$, where x^{C_2} is the incidence vector of C_2. This is a contradiction. ∎

<u>Theorem 3.4</u>: Let K_{i+1}^0 be a complete subgraph of size $i + 1$ in G; then

$$\sum_{K_{i-1}^j \subset K_{i+1}^0} x_j \geq \left\lceil \frac{i+1}{2} \right\rceil \tag{3}$$

is a valid inequality for $P_{K_i}(G)$. Furthermore, (3) defines a facet of $P_{K_i}(G)$ iff i is even.

<u>Proof</u>: First we show that (3) is valid for $P_{K_i}(G)$. For any $K_i^j \subset K_{i+1}^0$, $\sum_{K_{i-1}^k \subset K_i^j} x_k \geq 1$ is valid for $P_{K_i}(G)$. By summing all of these inequalities for $j = 1,2,\ldots,i + 1$ we get $2 \sum_{K_{i-1}^j \subset K_{i+1}^0} x_j \geq i + 1$. Thus $\sum_{K_{i-1}^j \subset K_{i+1}^0} x_j \geq \frac{i+1}{2}$. Since the sum of the x_j's is an integer, (3) is valid for $P_{K_i}(G)$. This also implies that (3) does not define a facet of $P_{K_i}(G)$ when i is odd since (3) could then be written as the linear combination of constraints defined on the K_i's contained in K_{i+1}^0.

We now assume that $i = 2r$, $(r \geq 1)$ and denote the corresponding constraint $\sum_{K^j_{i-1} \subset K^0_{i+1}} x_j \geq r + 1$ by $a^T x \geq a_0$. Furthermore, assume that there exists an inequality $b^T x \geq b_0$ which defines a facet of $P_{K_i}(G)$ and which also obeys the following: if $x \in P_{K_i}(G)$ verifies $a^T x = a_0$ then x also verifies $b^T x = b_0$. If we are able to show the existence of $\rho > 0$ such that $b = \rho a$, then we may conclude that $a^T x \geq a_0$ is a facet of $P_{K_i}(G)$. To do this, we first show that for two complete graphs K^j_{i-1} and K^k_{i-1} in K^0_{i+1}, $b_j = b_k$.

Let $V_0 = \{v_0, v_1, \ldots, v_{2r}\}$ denote the set of vertices of K^0_{i+1}. For $v_\alpha, v_\beta \in V_0$, ${}^{\bar\alpha,\bar\beta}K_{i-1}$ denotes the K_{i-1} defined on $V_0 \setminus \{v_\alpha, v_\beta\}$. Let $C_0 = \{K^k_{i-1} \mid K^k_{i-1} \in F_{i-1}(G),\ K^k_{i-1} \not\subset K^0_{i+1}\}$, and let $j \in \{0,1,\ldots,2r\}$. We now examine the following sets of K_{i-1}'s where the indices are modulo $2r + 1$.

$$\tilde C_j = \{{}^{\bar\alpha,\overline{\alpha+1}}K_{i-1} \mid \alpha = j, j+2, \ldots, j+2r\} \cup C_0$$

$$\tilde C^k_j = (\tilde C_j \setminus \{{}^{\overline{j-1},\bar j}K_{i-1}\}) \cup \{{}^{\overline{j-1},\bar k}K_{i-1}\} \quad k \in \{0,1,\ldots,2r\} \setminus \{j-1, j\}.$$

It is clear that for any j, the sets $\tilde C_j$ and $\tilde C^k_j$ are each K_i-covers of G and that their vector $x^{\tilde C_j}$ and $x^{\tilde C^k_j}$ satisfy $a^T x^{\tilde C_j} = a^T x^{\tilde C^k_j} = a_0$. Thus $b^T x^{\tilde C_j} = b^T x^{\tilde C^k_j} = b_0$, which implies that $b_{j,j-1} = b_{j-1,k}$ for $k \neq j$, $j - 1$ and $0 \leq k \leq 2r$, where $b_{j,j-1}$ and $b_{j-1,k}$ are respectively the coefficients of b associated with ${}^{\bar j,\overline{j-1}}K_{i-1}$ and ${}^{\overline{j-1},\bar k}K_{i-1}$.

Since j is chosen arbitrarily, we conclude that there exists $\rho \in \mathbb{R}$ such that $b_k = \rho$ for all $K^k_{i-1} \subset K^0_{i+1}$. For any $K^k_{i-1} \in C_0$ note that the set $\mathcal{C}_k = \tilde{\mathcal{C}}_j \setminus \{K_{i-1}^k\}$ is a K_i-cover of G and that $x^{\mathcal{C}_k}$ verifies $a^T x = a_0$. Thus $0 = b^T x^{\tilde{\mathcal{C}}_j} - b^T x^{\mathcal{C}_k} = b_k$ and $b_k = 0 \ \forall K_{i-1}^k \in \mathcal{C}_0$. Furthermore, it is easy to see that $\forall K_{i-1}^j \in \mathcal{F}_{i-1}(G)$, there exists $\mathcal{C}$ a K_i-cover of G such that $K_{i-1}^j \notin \mathcal{C}$ and $a^T x^{\mathcal{C}} = a_0$. This implies that the face $a^T x \geq a_0$ is not contained in a trivial facet $\{x \in P_{K_i}(G) \mid x_j = 1\}$ for a $K_{i-1}^j \in \mathcal{F}_{i-1}(G)$. Therefore $b^T x \geq b_0$ defines a non-trivial facet of $P_{K_i}(G)$. By lemma 3.3 it follows that $\rho > 0$ and thus $b = \rho a$. ∎

3.1.2. Chordless Cycles in $I_i(G)$ and Facets

Our second family of facets of $P_{K_i}(G)$ is defined by graphs called K_i-p-holes, defined as follows. Let H be a graph where $|\mathcal{F}_i(H)| = p$ and each K_{i-1} in H is contained in at least one of these p K_i's. H is a K_i-p-hole (recall that a hole is a chordless cycle) if the K_i-intersection graph $I_i(H)$ is a hole of size p. Three nonisomorphic K_3-9-holes are presented in Figure 3.1.

Remark 3.1. If H is a K_i-p-hole with $\mathcal{F}_i(H) = \{K_i^0, K_i^1, \ldots, K_i^{p-1}\}$ then $\mathcal{F}_{i-1}(H)$ may be partitioned into p + 1 pairwise disjoint sets $\tilde{\mathcal{C}}, \mathcal{C}_0, \ldots, \mathcal{C}_{p-1}$ such that $\tilde{\mathcal{C}} = \{\tilde{K}_{i-1}^0, \ldots, \tilde{K}_{i-1}^{p-1}\}$ is formed by a bijection with the edges of $I_i(H)$ where $\tilde{K}_{i-1}^k = K_i^k \cap K_i^{k+1}$ (superscripts modulo p) and $\mathcal{C}_j = \{K_{i-1}^\ell \mid K_{i-1}^\ell \subset K_i^j$ and $K_{i-1}^\ell \notin \tilde{\mathcal{C}}\}$ for $j = 0, \ldots, p-1$. This notation will be used throughout this subsection.

G:

H:

J:

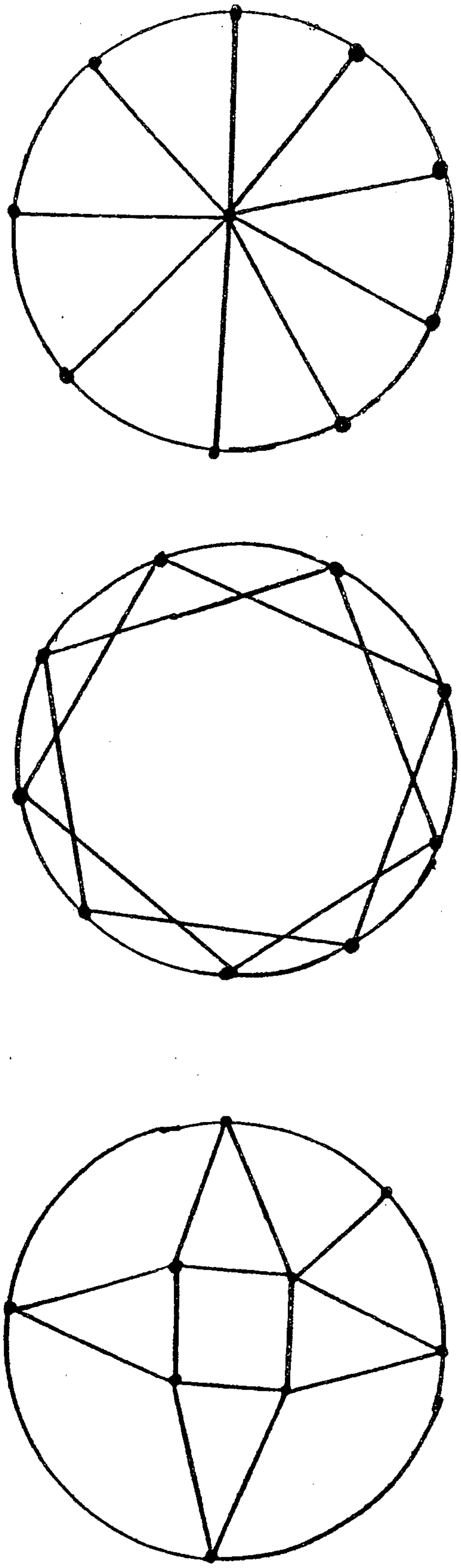


Figure 3.1. K_3-9-holes.

<u>Theorem 3.5</u>: Let G be a graph with an induced subgraph H, which is a is a K_i-p-hole, $i \geq 3$. Then the inequality

$$\sum_{K_{i-1}^j \in F_{i-1}(H)} x_j \geq \left\lceil \frac{p}{2} \right\rceil \qquad (4)$$

is valid for $P_{K_i}(G)$, and is a facet of $P_{K_i}(G)$ iff p is odd.

<u>Proof</u>: From Remark 3.1 we note that $\sum_{K_{i-1}^j \subset K_i^\ell} x_j \geq 1$

for all $K_i^\ell \in F_i(H)$ yields (by summing these inequalities)

$2 \sum_{\tilde{K}_{i-1}^j \in \tilde{C}} x_j + \sum_{K_{i-1}^k \in C_0 \cup C_1 \cup \ldots \cup C_{p-1}} x_k \geq p$, which implies $\sum_{K_{i-1}^j \in F_{i-1}(H)} x_j \geq \frac{p}{2}$.

Since the sum is an integer, inequality (4) is valid for $P_{K_i}(G)$.

Furthermore, when p is even (4) may be written as a linear combination of the constraints defined on the K_i's of H, and then (4) does not define a facet of $P_{K_i}(G)$.

Now assume that $p = 2r + 1$ and denote the constraint $\sum_{K_{i-1}^j \in F_{i-1}(H)} x_j \geq \left\lceil \frac{p}{2} \right\rceil$ by $a^T x \geq a_0$. As in the proof of Theorem 3.3 we assume that $b^T x \geq b_0$ is a facet of $P_{K_i}(G)$ such that $\{x \in P_{K_i}(G) \mid a^T x = a_0\} \subseteq \{x \in P_{K_i}(G) \mid b^T x = b_0\}$. It suffices to show that there exists $\rho > 0$ such that $b = \rho a$. To do this we will show that for any $K_i^j \in F_i(H)$ with $K_{i-1}^{j_1}, \ldots, K_{i-1}^{j_i} \subseteq K_i^j$ we have $b_{j_k} = b_{j_\ell}$, $1 \leq k < \ell \leq i$. Let $C = F_{i-1}(G) \setminus F_{i-1}(H)$ and let $j \in \{0,1,\ldots,2r\}$. We now examine the following sets, where indices are modulo $2r + 1$:

$$C_{j_\ell} = \{\tilde{K}_{i-1}^{j+1}, \tilde{K}_{i-1}^{j+3}, \ldots, \tilde{K}_{i-1}^{j+2r-1}\} \cup \{K_{i-1}^{j_\ell}\} \cup C, \quad \ell = 1, \ldots, i$$

All C_{j_ℓ}'s are K_i-covers of G and the incidence vectors satisfy $a^T x^{C_{j1}} = a^T x^{C_{j2}} = \ldots = a^T x^{C_{ji}} = a_0$. Thus $b^T x^{C_{j1}} = b^T x^{C_{j2}} = \ldots = b^T x^{C_{ji}} = b_0$, which implies that $b_{j_\ell} = b_{j_k}$ for $1 \le \ell < k \le i$.

Since each K_i^k in H intersects K_i^{k+1} in $\tilde{K}_{i-1}^k$ we may conclude that there exists $\rho \in \mathbb{R}$ such that $b_j = \rho$ for $K_{i-1}^j \in F_i(H)$. It is easy to see that $b_j = 0$ for $K_{i-1}^j \notin F_i(H)$. As in the proof of Theorem 3.3, $\rho > 0$ and thus $b = \rho a$. ∎

If H is one of the K_3-9-holes presented in Figure 3.1, the inequality $\sum_{K_2^k} x_k \ge 5$ is a facet of $P_{K_3}(G)$. Furthermore, note that if $i = 2$, (4) does not always define a facet of $P_{K_2}(G)$. In this case Padberg [16] has presented a general procedure to generate the facets.

Remark 3.2. The facets associated with the K_i-p-holes as given in theorem 3.5 may be generalized to facets of the set covering polytope, as defined in section 3. In fact, consider the intersection graph Γ associated with A, the constraint matrix of our set covering problem denoted by (P) which the nodes correspond to the rows of A and two nodes i,j are adjacent if and only if the corresponding rows a_i, a_j verify $a_i^T a_j > 0$. Let H be a hole of Γ of size p (where p is odd). Let $r_0, \ldots, r_{p-1}$ be the rows of A that correspond to nodes $i_0, \ldots, i_{p-1}$ of H. Suppose that $i_0 i_1, i_1 i_2, \ldots, i_{p-2} i_{p-1}, i_0 i_{p-1}$ are the edges of H and let $c_0, \ldots, c_{p-1}$ be the columns of A such that each c_k has a 1 in rows r_k, r_{k+1} for

$k = 0,\ldots,p-1$ (where the indices are modulo p). (Note that the submatrix of A where the rows are $r_0,\ldots,r_{p-1}$ and columns are $c_0,\ldots,c_{p-1}$ is an odd cycle submatrix of A.) Let $r_p,\ldots,r_{m-1}$; $c_p,\ldots,c_{n-1}$ be the other rows and columns of A. Suppose that each column c_j, $j \geq p$, has a 1 in at most one of the rows $r_0,\ldots,r_{p-1}$ and each row r_i, $i \geq p$, has a 1 in at least two of the columns $c_p,\ldots,c_{n-1}$. Furthermore, let $T = \{j/\text{column } c_j \text{ has a 1 in one of the rows } r_0,\ldots,r_{p-1}\}$. Then

$$\sum_{j \in T} x_j \geq \left\lceil \frac{p}{2} \right\rceil$$

defines a facet of the polyhedron associated with (P).

The proof is similar to the proof of theorem 3.5.

3.1.3. p-wheels and Facets

Our third and final family of facets of $P_{K_i}(G)$ involves a subfamily of K_i-p-holes, the p-wheels of order i defined as follows: Graph H is a <u>p-wheel of order i</u> if H consists of a hole C of length $p \geq 3$ and a K_i^* universal to C. See Figure 3.2 for the 5-wheel of order 3. Note that a p-wheel of order i - 2 is a K_i-p-hole; for example, Graph G in Figure 3.1 is a 9-wheel of order 1.

We now show that if p and i are odd, then the p-wheels of order i define facets of $P_{K_i}(G)$.

<u>Theorem 3.6</u>: Let G(V,E) be a graph with an induced subgraph H, which is a (2k+1)-wheel of order i - 1 where $k \geq 2$ and $i \geq 3$. Let $C = F_{i-1}(H)\backslash\{K_{i-1}^*\}$ (i.e., the set of K_{i-1}'s in H excluding the universal K_{i-1}); then the inequality

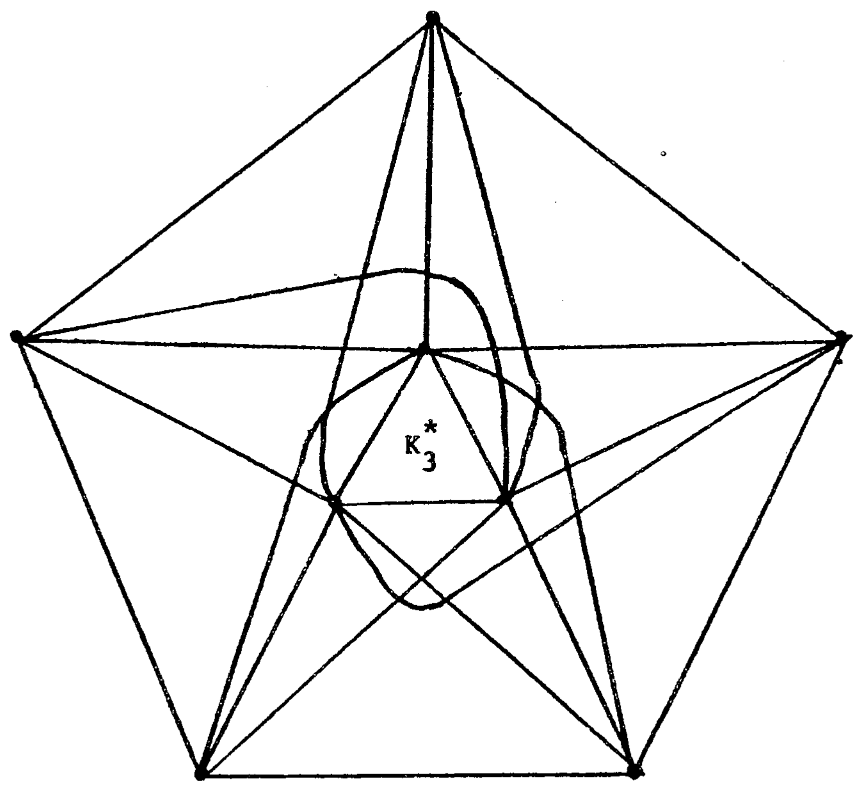


Figure 3.2. 5-wheel of order 3.

$$\sum_{K_{i-1}^j \in C} x_j \geq k(i-1) + \left\lceil \frac{i-1}{2} \right\rceil \tag{5}$$

is valid for $P_{K_i}(G)$. Furthermore (5) defines a facet of $P_{K_i}(G)$ iff i is even.

<u>Proof:</u> First we show that (5) is valid. In H we let $V^* = \{v_0, v_1, \ldots, v_{i-2}\}$ denote the vertices of K_{i-1}^* and $U = \{u_0, u_1, \ldots, u_{2k}\}$ denote the vertices of the hole C where u_j is adjacent to u_{j-1} and u_{j+1} (subscripts modulo $2k+1$). Each edge (u_j, u_{j+1}) is contained in $i-1$ K_i's of the form ${}^{\bar{\ell}}K_i^j = \{u_j, u_{j+1}, V^* \backslash \{v_\ell\}\}$, $\ell = 0, 1, \ldots, i-2$. Summing all the constraints (1) defined by these K_i's for all edges of C we get:

$$2 \sum_{K_{i-1}^j \in C} x_j \geq (2k+1)(i-1), \text{ thus } \sum_{K_{i-1}^j \in C} x_j \geq k(i-1) + \frac{(i-1)}{2},$$

which implies that (5) is valid for $P_{K_i}(G)$. Furthermore, if i is odd, (5) does not define a facet.

Now we consider the case where $i = 2r + 2$ and denote the inequality $\sum_{K_{i-1}^j \in C} x_j \geq r(2k+1) + k + 1$ by $a^T x \geq a_0$. We assume that $b^T x \geq b_0$ is a facet of $P_{K_i}(G)$ such that $\{x \in P_{K_i}(G) \mid a^T x = a_0\} \subseteq \{x \in P_{K_i}(G) \mid b^T x = b_0\}$. To prove this it suffices to show that $b = \rho a$ with $\rho > 0$. Any vertex u_j in C belongs to two different types of K_{i-1}'s:

$${}^{\bar{f},\bar{\ell}}K_{i-1}^j = \{u_j, u_{j+1}, V^* \backslash \{v_f, v_\ell\}\} \quad 0 \leq f \neq \ell \leq 2r$$

$${}^{\bar{f}}K_{i-1}^j = \{u_j, V^* \backslash \{v_f\}\} \qquad 0 \leq f \leq 2r.$$

Note that $C = \{{}^{\bar f,\bar \ell}K^j_{i-1}, {}^{\bar f}K^j_{i-1},\ j = 0,\dots,2k,\ 0 \le f \ne \ell \le 2r\}$ and the subscripts are modulo $2k + 1$ and $2r + 1$ respectively.

Let u_q and v_s be two arbitrary vertices in U and V^* respectively. We now define the following three sets of K_{i-1}'s:

$$C_0 = F_{i-1}(G)\backslash F_{i-1}(H)$$

$$C^s = \{{}^{\bar f,\overline{f+1}}K^j_{i-1} \mid j = 0,1,\dots,2k;\ f = s+1, s+3,\dots,s+2r-1\}$$

$$C^q_s = \{{}^{\bar s}K^j_{i-1} \mid j = q+2, q+4,\dots,q+2k\}$$

Given t and t' (different from s) where $0 \le t < t' \le 2r$ define

$$C_1 = C_0 \cup C^s \cup C^q_s \cup \{K^*_{i-1}, {}^{\bar s,\bar t}K^q_{i-1}\}$$

$$C_2 = (C_1\backslash\{{}^{\bar s,\bar t}K^q_{i-1}\}) \cup \{{}^{\bar s,\bar t'}K^q_{i-1}\}$$

It is easily seen that C_1 and C_2 are both K_i-covers of G and that their incidence vectors x^{C_1} and x^{C_2} verify $a^Tx = a_0$. Thus x^{C_1} and x^{C_2} verify $b^Tx = b_0$ and

$$b_{q,\bar s,\bar t} = b_{q,\bar s,\bar t'} \tag{6}$$

where $b_{q,\bar s,\bar t}$ and $b_{q,\bar s,\bar t'}$ are the coefficients of b associated with ${}^{\bar s,\bar t}K^q_{i-1}$ and ${}^{\bar s,\bar t'}K^q_{i-1}$ respectively. Now look at the K_i-covers

$$C_3 = (C_1\backslash\{{}^{\bar s,\bar t}K^q_{i-1}\}) \cup \{{}^{\bar s}K^q_{i-1}\}$$

$$C_4 = (C_1\backslash\{{}^{\bar s,\bar t}K^q_{i-1}\}) \cup \{{}^{\bar s}K^{q+1}_{i-1}\}$$

Again we see that $a^Tx^{C_3} = a^Tx^{C_4} = a_0$ and thus $b^Tx^{C_3} = b^Tx^{C_4} = b_0$. Thus

$$b_{q,\bar{s},\bar{t}} = b_{q,\bar{s}} = b_{q+1,\bar{s}} \tag{7}$$

Since s, q, t and t' are chosen arbitrarily, we may conclude from (6) and (7) that $\exists \rho \in \mathbb{R}$ such that $b_j = \rho \; \forall \; K^j_{i-1} \in C$. For $K^j_{i-1} \in C_0$, the set $C'_j = C_1 \setminus \{K^j_{i-1}\}$ is a K_i-cover of G and $x^{C'_j}$ verifies $a^Tx = a_0$ and thus $b_j = 0$ for $K^j_{i-1} \in C_0$.

Now we must show that $b_* = 0$ (b_* corresponds to K^*_{i-1}). Let

$$\bar{C} = \begin{cases} \{^{\overline{\ell},\overline{\ell+1}}K^j_{i-1} \mid j = 0,1,\ldots,2k;\ \ell = 4,6,\ldots,2r\} & \text{if } r \geq 2 \\ \phi & \text{if } r < 2 \end{cases}$$

Using $\bar{C}$ we have the following two K_i-covers of G:

$$\bar{C}_0 = \bar{C} \cup C^0_1 \cup C^1_2 \cup C^1_3 \cup C_0 \cup \{^{\bar{2},\bar{3}}K^1_{i-1},\ ^{\bar{1}}K^1_{i-1}\}$$

$$\bar{C}'_0 = \bar{C}_0 \cup \{K^*_{i-1}\}$$

where $a^Tx^{\bar{C}_0} = a^Tx^{\bar{C}'_0} = a_0$. Thus $0 = b^Tx^{\bar{C}_0} - b^Tx^{\bar{C}'_0} = b_*$. We have thus shown that

$$b_j = \begin{cases} \rho & \text{for } K^j_{i-1} \in C \\ 0 & \text{for } K^j_{i-1} \notin C \end{cases}$$

To complete the proof we note that, as before, $b^Tx \geq b_0$ defines a nontrivial facet of $P_{k_i}(G)$. Then, by lemma 3.3, $\rho > 0$ and thus $b = \rho a$. ∎

Note that if $k = 1$, (5) is still valid but does not always define a facet. If, however, we set C' to be C without the K_{i-1}'s containing the three vertices of the exterior cycle, then a proof similar to that used for Theorem 3.5 shows that

$$\sum_{K^j_{i-1} \in C'} x_j \geq k(i-1) + \left\lceil \frac{i-1}{2} \right\rceil \text{ defines a facet for } P_{K_i}(G)$$

iff i is even. As an example of Theorem 3.5 consider the 5-wheel of order 3 (Figure 4.2). For this graph, the inequality

$$\sum_{K^j_3 \neq K^*_3} x_j \geq 8 \quad \text{is a facet of } P_{K_4}(G).$$

<u>Remark 3.3</u>. For G a $(2k+1)$-wheel of order $i - 1 = 2r$ $(r > 1)$, it is easy to see that $c_i(G) = r(2k+1)$; however, for $r = 1$, then $i = 3$ and $c_3(G) = 2k + 2$ (see §3.2.1). Then from the previous theorem, the constraint $\sum_{K^j_{i-1} \in F_{i-1}(G)} x_j \geq c_i(G)$ does not define a facet of $P_{K_i}(G)$ for $i = 2r + 1 > 3$; however, if $i = 3$, the corresponding constraint does define a facet of $P_{K_3}(G)$. In the next section we show this result and examine other facets of $P_{K_3}(G)$ related to the polytope of the bipartite subgraphs.

3.2. Facets of $P_{K_3}(G)$

3.2.1. The Polytope of the Bipartite Subgraphs and Facets of $P_{K_3}(G)$

$P_B(G)$, the polytope of the bipartite subgraphs of a graph $G(V,E)$, is the convex hull of the incidence vectors of the bipartite subgraphs of G. In [2] Barahona, Grötschel and Mahjoub have presented a large family of facets of this polytope. It is clear that if (V,F) is a bipartite subgraph of G, then E\F is a K_3-cover of G. Thus if a constraint $a^T\bar{x} \leq \bar{a}_0$ is a facet of $P_B(G)$ and $a^Tx \geq a_0$ (the inequality resulting by changing $\bar{x}$ to $1 - x$ in $a^T\bar{x} \leq \bar{a}_0$ is valid for $P_{K_3}(G)$, then $a^Tx \geq a_0$ is a facet of $P_{K_3}(G)$.

In the following we describe three families of this type of facet of $P_{K_3}(G)$. The first family arises from the $(2k+1)$-wheels of order 2. The notation developed in §3.1.3 will be used here

<u>Theorem 3.7</u>. Let $H(W,F)$, a $(2k + 1)$-wheel $(k \geq 1)$ of order 2 be an induced subgraph of G. The inequality

$$\sum_{e\in F} x_e \geq 2(k + 1) \tag{8}$$

defines a facet of $P_{K_3}(G)$.

<u>Proof</u>: In [2] it was shown that

$$\sum_{e\in F} \bar{x}_e \leq 2(2k + 1) \tag{9}$$

is a facet of $P_B(G)$. We now show that (8) is valid for $P_{K_3}(G)$. Set $K_2^* = \{v_1,v_2\}$ and let H_1 $(W\backslash\{v_2\},F_1)$ be the $(2k+1)$-wheel of order 1 defined by $H\backslash\{v_2\}$. Similarly, define $H_2(W\backslash\{v_1\},F_2)$. Given the triangles defined by $K_3^j = \{u_j,v_1,v_2\}$, $j = 0,1,\ldots,2k$ the following constraints are valid for $P_{K_3}(G)$:

$$\sum_{e\in F_i} x_e \geq k + 1 \qquad i = 1,2$$

$$\sum_{e\in K_3^j} x_e \geq 1 \qquad j = 0,1,\ldots,2k$$

Summing these constraints yields:

$$2 \sum_{e\in F\setminus\{(v_1,v_2)\}} x_e + (2k + 1)x_{(v_1,v_2)} \geq 2(k + 1) + 2k + 1 \tag{10}$$

From Theorem 3.5 we know that

$$\sum_{e\in F\setminus\{(v_1,v_2)\}} x_e \geq 2k + 1 \tag{11}$$

is valid for $P_{K_3}(G)$. Therefore, $(2k + 1) \sum_{e\in F} x_e \geq (2k + 1)(2k + 1) + 1$ which implies $\sum_{e\in F} x_e \geq (2k + 1) + \frac{1}{2k + 1}$. Since the sum is an integer, we conclude that (8) is valid for $P_{K_3}(G)$ and thus is a facet. ■

The second family of facets of $P_{K_3}(G)$ consists of complete subgraphs of odd order.

<u>Theorem 3.8</u>: Let $H(W,F)$ be a complete subgraph of G, where $|W| = 2k + 1$.

Then
$$\sum_{e\in F} x_e \geq k^2 \tag{12}$$

is a facet of $P_{K_3}(G)$.

<u>Proof</u>: Since $\sum_{e\in F} \bar{x}_e \leq k(k + 1)$ is a facet of $P_B(G)$ [2] we only need to show that (12) is valid for $P_{K_3}(G)$. This may be shown by induction on k and by examining the subgraphs of size $2k - 1$ of H. ■

We now present the third family of facets of $P_{K_3}(G)$.

<u>Theorem 3.8</u>: Let $H(W,F)$ be a complete subgraph of $G(V,E)$ where $W = \{1,2,\ldots,q\}$. Let positive integers $t_i (1 \leq i \leq q)$ satisfy $\sum_{i=1}^{q} t_i = 2k+1$, $k \geq 3$ and $\sum_{t_i > 1} t_i \leq k-1$. Set

$$a_{ij} = \begin{cases} t_i t_j & 1 \leq i < j \leq q \\ 0 & (i,j) \in E \setminus F \end{cases}$$

Then

$$a^T x \geq k^2 - \sum_{i=1}^{q} \frac{t_i(t_i - 1)}{2} \tag{13}$$

is a facet of $P_{K_3}(G)$.

<u>Proof</u>: In [2] the inequality $a^T \bar{x} \leq k(k+1)$ was shown to be a facet of $P_B(G)$. To prove that (13) is valid for $P_{K_3}(G)$ we let $C_1 \subseteq E$ be a K_3-cover of G. Now form the graph $G'(V',E')$ from G by replacing node i of W with V^i a K_{t_i} (if $t_i \geq 1$). All vertices in V^i are completely connected to any vertex in G adjacent to i. Thus H has been replaced by a K_{2k+1}. We now construct C_1' a K_3-cover of G' as follows:

(i) If $(i,j) \in C_1 \cap F$, then put all edges between V^i and V^j into C_1'.

(ii) If $(v,j) \in C_1$, $v \in V \setminus W$, $j \in W$, then add all edges between v and V^j into C_1'.

(iii) For each $1 \leq i \leq q$, where $t_i > 1$, add to C_1' all edges (u,v), such that $u,v \in V^i$.

Since C_1 is a K_3-cover of G, it is clear that C_1' is a K_3-cover of G'. Furthermore, $a^T x^{C_1'} = a^T x^{C_1} + \sum_{i=1}^{q} \frac{t_i(t_i - 1)}{2} \geq k^2$; therefore (13) is valid and thus is a facet of $P_{K_3}(G)$. ∎

As an example let H(W,F) be a complete subgraph K_9 then for all edges (u,v) F, the inequality

$$4x_{u,v} + 2 \sum_{i \neq v} x_{u,i} + 2 \sum_{j \neq u} x_{v,j} + \sum_{(i,j) \neq (u,v)} x_{i,j} \geq 23 \tag{14}$$

is a facet of $P_{K_3}(G)$. (14) may be obtained from Theorem 3.8 by setting $t_u = t_v = 2$, $t_i = 1 \quad u \neq i \neq v$.

3.2.2. Construction of Facets

Let G' be obtained from G by the addition of an edge. We now show how to construct the facets of $P_{K_3}(G')$ from the facets of $P_{K_3}(G)$.

Theorem 3.10: Let $a^T x \geq a_0$ be a facet of $P_{K_3}(G)$ for a graph G(V,E) and denote by G' the graph obtained from G by the addition of an edge e_0. Let a be a system of weights of E and let γ be the minimum weight of $C_0 \subset E$ where C_0 is a K_3-cover of both G and G' ($\gamma \geq a_0$). Set

$$\bar{a}_{e_0} = \gamma - a_0$$

$$\bar{a}_e = a_e \qquad \text{for } e \neq e_0$$

$$\bar{a}_0 = \gamma$$

Then

$$\bar{a}^T x \geq \bar{a}_0 \tag{15}$$

defines a facet of $P_{K_3}(G')$.

<u>Proof</u>: (15) is valid for $P_{K_3}(G')$ since if C' is a K_3-cover of G' which contains e_0, then $x^{C'}$ verifies (15). If C' does not contain e_0, then $a^T x^{C'} \geq a_1$ and again (15) is verified.

Since $a^T x \geq a_0$ is a facet of $P_{K_3}(G)$, there exists $|E|$ K_3-covers of G, $C_1, C_2, \ldots, C_{|E|}$ such that $x^{C_1}, \ldots, x^{C_{|E|}}$ verify $a^T x = a_0$ and are affinely independent. Let $C'_i = C_i \cup \{e_0\}$, $i = 1,2,\ldots|E|$ and $C'_{|E|+1} = C_0$. The sets C'_i, $i = 1,\ldots,|E|+1$ are all K_3-covers of G' and their incidence vectors $x^{C'_i}$ verify $\bar{a}^T x = \bar{a}_0$ and are affinely independent. Thus (15) is a facet of $P_{K_3}(G')$. ∎

To illustrate this method consider the graph $H(W,F)$ in Figure 3.3. H is obtained from the K_3-9-hole J presented in Figure 3.1 by the addition of edge e_0. From Theorem 3.4 we know that

$$\sum_{e \in F \setminus \{e_0\}} x_e \geq 5 \tag{16}$$

is a facet of $P_{K_3}(J)$. It is easy to see that $c_3(H) = 6$ and thus by Theorem 3.9, $\sum_{e \in F} x_e \geq 6$ is a facet of $P_{K_3}(H)$.

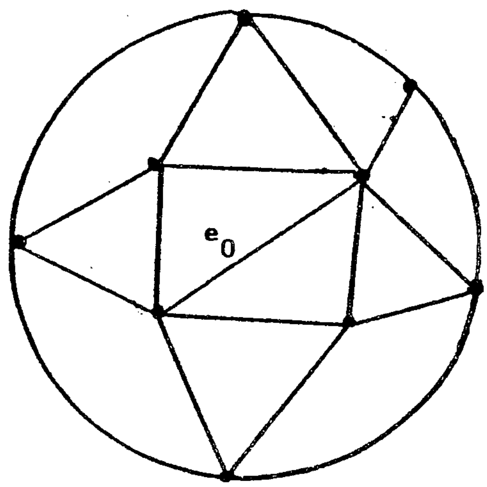


Figure 3.3. H(W,F).

In this example we note that although $H\setminus\{e_0\}$ is a partial subgraph of H, constraint (16) does not define a facet of $P_{K_3}(H)$. From Theorem 3.9 we see that $a^T x \geq a_0$ is also a facet of $P_{K_3}(G')$ iff $a_1 = a_0$. This gives the following corollary:

<u>Corollary 3.10</u>: Let H(W,F) be a partial subgraph of G(V,E) where $\sum_{e\in F} x_e \geq a_0$ is a facet of $P_{K_3}(H)$. If for all edges $e_0 \in E\setminus F \cap (W \times W)$ there exists a K_3-cover of $H + \{e_0\}$ of cardinality a_0, then $\sum_{e\in F} x_e \geq a_0$ is a facet of $P_{K_3}(G)$.

It is easy to see that the $(2k+1)$-wheels of order 1 satisfy the condition of Corollary 3.10. Therefore, if a graph contains a partial subgraph H(W,F) of this type, then $\sum_{e\in F} x_e \geq k+1$ is a facet of $P_{K_3}(G)$.

<u>3.3. Polynomial Algorithms to Test the Facets Defined by the (2k+1)-wheels</u>

Using the ellipsoid method, Grötschel, Lovász and Schrijver [12] have shown that there exists a polynomial time algorithm for a linear optimization problem on a polyhedron iff for each constraint of the polyhedron, there exists a polynomial time algorithm for the separation problem. Knowledge of an efficient method solving the separation problem not only shows that the problem is polynomial but also allows the use of these facets as cutting planes in a "dual" algorithm for the solution of our problem.

We now present polynomial algorithms to solve the separation problem associated with some facets of $P_{K_1}(G)$. First of all, it is easy to test the facets defined by the inequalities (1), (2) and (3). Gerards

[9] has presented a polynomial algorithm to test the facets of $P_B(G)$ defined by the $(2k+1)$-wheels of order 2 (see inequality (9)). The same algorithm may be used to test (8). Gerards' algorithm reduces the problem to finding all cycles of minimum length in a graph with positive edge weights. Grötschel and Pulleyblank [14] have shown that this problem has a polynomial solution. We also use this idea to develop a polynomial algorithm to test the facets of the polytope $P_{K_i}(G)$ defined by the $(2k+1)$-wheels H of order $2r-2$ or $2r-1$ (where $i = 2r$), whose inequalities are:

$$\sum_{K^j_{i-1} \in F_{i-1}(H)} x_j \geq k+1 \tag{1}$$

and

$$\sum_{K^j_{i-1} \in (F_{i-1}(H) \setminus K^*_{i-1})} x_j \geq (r-1)(2k+1)+k+1 \tag{18}$$

(See (4) and (5) respectively.)

<u>Algorithm 3.1</u>: Testing facets of $P_{K_i}(G)$ defined by $(2k+1)$-wheel subgraphs.

Given an element $x \in \mathbb{R}^h$, without loss of generality we may assume that the constraints (1), (2) and (3) are verified by x. To test (17) (respectively (18)), do the following:

For each K^j_{i-2} (resp. K^j_{i-1}) in G set:

$$V_j = \{w \in V \mid (w,v) \in E,\ \forall\ v \in K^j_{i-2} \text{ (resp. } K^j_{i-1})\}$$

$$E_j = \{(w,w') \mid w,w' \in V_j\}$$

For $(w,w')\in E_j$, let ${}^{ww'}C_0^j$ denote the set of K_{i-1}'s in G which contain nodes w and w' and i - 3 nodes of K_{i-2}^j (resp. K_{i-1}^j) and let ${}^{w}C_1^j$ denote the set of K_{i-1}'s in G which contain node w and i - 2 nodes of K_{i-2}^j (resp. K_{i-1}^j). Set

$$\alpha_{ww'} = \sum_{K_{i-1}^{\ell}\in {}^{ww'}C_0^j} x_\ell$$

$$\beta_w = \sum_{K_{i-1}^{\ell}\in {}^{w}C_1^j} x_\ell$$

For $(w,w')\in E_j$ we set $y_{ww'}^j = -\frac{1}{2} + (\alpha_{ww'} + \frac{1}{2}(\beta_w + \beta_{w'}))$ (resp. $y_{ww'}^j = -r + \frac{1}{2} + \alpha_{ww'} + \frac{1}{2}(\beta_w + \beta_{w'}))$.

It is straightforward to show that there is a $(2k+1)$-wheel of order i - 2 where $K_{i-2}^* = K_{i-2}^j$ (resp. of order i - 1 where $K_{i-1}^* = K_{i-1}^j$) for which (17) (resp. (18)) is not verified by x iff the minimum weight of an odd cycle in (V_j,E_j) is less than $\frac{1}{2}$. Furthermore, by summing all the constraints (1) corresponding to K_i's which contain nodes w and w' and i - 2 nodes of K_{i-2}^j (resp. K_{i-1}^j) we get

$$2\,\alpha_{ww'} + \beta_w + \beta_{w'} \geq 1 \quad .$$

$$(\text{resp. } 2\alpha_{ww'} + \beta_w + \beta_{w'} \geq 2r - 1).$$

Since the constraints are assumed to be verified by x, $y_{ww'}^j \geq 0$. We now apply the Grötschel-Pulleyblank minimum weight odd cycles algorithm [14] on the graphs (V_j,E_j) weighted by y^j to test the constraints of type (17) and (18). ∎

For constraints defined by the K_i-p-holes, other than those given by the p-wheels of order i - 2, no polynomial testing algorithm is known. The existence of such an algorithm seems to be an interesting open problem.

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ANNEXE 9

K_i-covers II: K_i-perfect graphs

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Abstract

A K_i is a complete subgraph of size i. A K_i-cover of a graph G(V,E) is a set C of K_{i-1}'s of G such that every K_i in G contains at least one K_{i-1} in C. $c_i(G)$ is the cardinality of a smallest K_i-cover of G. A K_i-packing of G is a set of K_i's such that no two K_i's have $i - 1$ nodes in common. $p_i(G)$ is the cardinality of a largest K_i-packing of G. Let $F_i(G)$ denote the set of K_i's in G and define $c_i(F)$ and $p_i(F)$ analogously for $F \subseteq F_i(G)$. G is K_i-perfect if $\forall\ F \subseteq F_i(G)$, $c_i(F) = p_i(F)$. The K_2-perfect graphs are precisely the bipartite graphs. We present a characterization of K_i-perfect graphs which is related to the Strong Perfect Graph Conjecture. Furthermore, if ${}^iA$ denotes the 0 - 1 matrix of G where the rows are the elements of $F_{i-1}(G)$ which belong to at least one K_i and the columns are the elements of $F_i(G)$, then we show that ${}^iA$ is perfect iff G is a K_i-perfect graph. Furthermore, we characterize the K_i-perfect graphs for which ${}^iA$ is balanced.

*This research was done while the authors visited IMAG, Université de Grenoble.

1. Introduction

In this paper we continue our study of K_i-covers of graphs (see [4]). Given a graph $G(V,E)$ we let $\mathcal{F}_i(G)$ denote $\{K_i | K_i \subseteq G\}$ (i.e., $\mathcal{F}_i(G)$ is the set of complete graphs on i vertices in G). A <u>K_i-cover of G</u> is a set $\mathcal{C}$ of K_{i-1}'s such that every K_i in G contains at least one K_{i-1} in $\mathcal{C}$. Note that the definition of K_2-cover is equivalent to that of vertex cover. For a graph G, the <u>K_i-cover number</u> $c_i(G)$ is the cardinality of a smallest K_i-cover of G. In [4] we showed that the problem of determining whether $c_i(G) \leq k$ for a given graph G and integers $i \geq 2$ and $k \geq 1$ is NP-complete. We also studied the complexity of the problem on the restricted family of chordal graphs. Having seen that the general problem is NP-complete, we examined some families of facets of the K_i-cover polytope and showed that various induced subgraphs of G define facets of this polytope.

A <u>K_i-packing</u> is a set of K_i's such that no two K_i's have $i - 1$ nodes in common. For a graph G, the <u>K_i-packing number</u> $p_i(G)$ is the cardinality of a largest K_i-packing of G. For any $F \subseteq \mathcal{F}_i(G)$ we may define $c_i(F)$ and $p_i(F)$ in a similar manner to their definitions for graphs. We shall see that for all such F, $c_i(F) \geq p_i(F)$. We define a graph to be <u>K_i-perfect</u> if $\forall\, F \subseteq \mathcal{F}_i(G)$, $c_i(F) = p_i(F)$. It is clear from the definitions that for $i = 2$, the K_2-perfect graphs are precisely the bipartite graphs and thus our characterizations of the K_i-perfect graphs may be viewed as generalizations of bipartite graphs.

Given two graphs $G(V,E)$ and $G'(V',E')$ we say that G' is an <u>induced subgraph of G</u>, denoted $G' \triangleleft G$ if $V' \subseteq V$ and $E' = (V' \times V') \cap E$. G' is a partial subgraph of G, denoted $G' < G$ if $V' \subseteq V$ and $E' \subseteq (V' \times V') \cap E$. A hole of a graph G is an induced subgraph which is a cycle. An anti-hole is an induced subgraph which is the complement of a cycle. The stability number of a graph, $\alpha(G)$ is the size of a largest maximal induced set of nonadjacent vertices. A clique is a maximal complete subgraph. $k(G)$ the clique cover number of G is the minimum number of cliques of G whose union is G. Equivalently $k(G)$ may be defined to be the minimum number of disjoint complete subgraphs of G whose union is G. Berge [1] has defined a graph G to be perfect if $\forall\, H \triangleleft G \;\; \alpha(H) = k(H)$. One of the outstanding open problems in graph theory is Berge's Strong Perfect Graph Conjecture (SPGC), which states that a graph is perfect iff G contains no odd holes or anti-holes of size ≥ 5. It is seen that our definition of K_i-perfect graphs has the same flavor as the definition of perfect graphs and in fact in section 2 we prove a characterization of K_i-perfect graphs which is very similar to the SPGC. This characterization uses the Parathasarathy-Ravindra [7] proof that $(K_4 - e)$-free graphs are valid for the SPGC (i.e., the SPGC is true for this family of graphs). Throughout the paper n will denote $|V|$ and $K_{i,j}$ will refer to the complete bipartite graph with cell sizes i and j.

Given a graph G and integer $i \geq 2$ we let iA denote the $0-1$ matrix where the rows represent K_{i-1}'s in G which belong to at least one K_i and the columns represent the K_i's. Clearly each column has i 1's and iA contains no all zeros rows. In the following $\mathbf{1}$ will denote an all 1's vector and a cycle matrix is a square matrix where each row and column contain exactly two 1's. A triangle matrix is a cycle matrix of size three. A $0-1$ matrix A is perfect if the associated set packing polytope $\{x \mid Ax \leq 1, x \geq 0\}$ has all integral extreme points [6]. A 0 - 1 matrix A is <u>balanced</u> if A does not contain any square submatrix of odd order with two 1's per row and per column (see [2],[5]). A matrix is <u>totally unimodular</u> iff every square submatrix has determinant 0, 1 or -1. Any balanced matrix is perfect, and any totally unimodular matrix is balanced.

In section 3 we will show that the matrix iA is perfect iff G is K_i-perfect. Furthermore, for $i \geq 3$ we will characterize those K_i-perfect graphs for which iA is balanced.

2. Characterization of K_i-Perfect Graphs

We now show that the K_i-perfect graphs have a characterization which is very similar to Berge's Strong Perfect Graph Conjecture for perfect graphs. As mentioned in §1, the diamond-free (or $(K_4 - e)$-free) graphs are valid for this conjecture [7]. This result may be stated as follows:

<u>Lemma 2.1</u>: [7] A diamond-free graph is perfect iff it does not contain C_{2k+1}, $k > 1$ as an induced subgraph.

<u>Proof</u>: This follows immediately from the Parathasarathy and Ravindra result since a diamond-free graph cannot contain an odd anti-hole of size greater than 5. ∎

In order to use this lemma to characterize the K_i-perfect graphs we need to establish a relationship between the parameters of K_i-perfectness. namely $c_i(S)$ and $p_i(S)$, and the parameters of perfectness, namely, $\alpha(G)$ and $k(G)$. First we note the following relationship between $c_i(S)$ and $p_i(S)$.

<u>Lemma 2.2</u>: For any $i \geq 2$ and $S \subseteq \mathcal{F}_i(G)$ we have $p_i(S) \leq c_i(S) \leq i \cdot p_i(S)$.

<u>Proof</u>: By definition, no two K_i's in P, any largest packing of S, have a K_{i-1} in common; therefore, each K_i in P must be covered by a distinct K_{i-1} and thus $c_i(S) \geq p_i(S)$.

To show that $c_i(S) \leq i \cdot p_i(S)$ we first note that all of the K_i's in $S \setminus P$ have at least $i - 1$ nodes in common with a $K_i \in P$, otherwise P is not a maximal packing. Therefore, the family of K_{i-1}'s contained in K_i's in P is a K_i-cover of S. Since there are i K_{i-1}'s in a K_i, the number of K_{i-1}'s in P equals $i \cdot p_i(S)$. ■

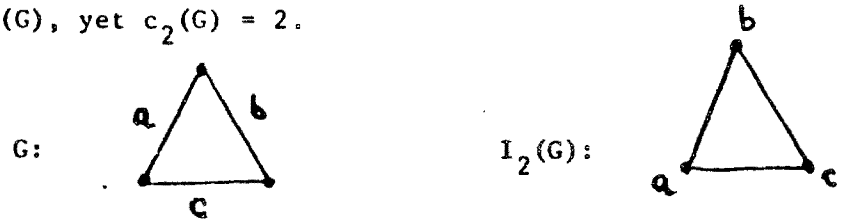
Note that the upper bound on $c_i(S)$ may be strengthened to $c_i(S) \leq i \cdot p_i^*(S)$ where $p_i^*(S)$ is the cardinality of the smallest maximal packing. The proof is exactly the same.

To establish the previously mentioned relationship between the parameters of K_i-perfectness and the parameters of perfectness we define the <u>K_i-intersection graph</u> of G denoted $I_i(G)$ as follows: The nodes of $I_i(G)$ are the K_i's in $\mathcal{F}_i(G)$. Two nodes of $I_i(G)$ are adjacent iff the corresponding K_i's in G have $i - 1$ nodes in common. If $S \subseteq \mathcal{F}_i(G)$ we let $I_i^S(G)$ denote the corresponding induced subgraph of $I_i(G)$. For $i = 2$ it is clear that $I_2(G)$ is the line graph $L(G)$. An important result by Whitney [8] states that if $n > 4$ then $G_1 \cong G_2 \iff L(G_1) \cong L(G_2)$ ($\cong$ means isomorphic). It is clear that if $i \geq 3$ we have $G_1 \cong G_2 \Rightarrow I_i(G_1) \cong I_i(G_2)$; however, the converse does not hold. The next two lemmas follow immediately from the definitions.

<u>Lemma 2.3</u>: $I_i(G)$ does not contain an induced $K_{1,i+1}$.

<u>Lemma 2.4</u>: If $S \subseteq F_i(G)$ then $\alpha(I_i^S(G)) = p_i(S)$.

This lemma leads us to hope that there is a relationship between $c_i(S)$ and $k(I_i^S(G))$. Although a K_i-cover of S does correspond to a clique cover of $I_i^S(G)$ of the same size, the converse does not always hold, as illustrated by the graph in Figure 1.2 where $i = 2$, $\{a,b,c\}$ is a clique cover of $I_2(G)$, yet $c_2(G) = 2$.



<u>Figure 2.1</u>: $c_2(G) \neq k(I_2(G))$

Although $c_i(S) \neq k(I_i^S(G))$ in general, we are able to characterize the $I_i(G)$'s where equality does hold. $I_i(G)$ is <u>conformal</u> if for any clique of size h, $h \geq 2$ in $I_i(G)$, the corresponding family of K_i's in G have exactly $i - 1$ nodes in common. In this case we have equality between $k(I_i^S(G))$ and $c_i(S)$ for any $S \subseteq F_i(G)$, namely,

<u>Lemma 2.5</u>: $I_i(G)$ is conformal if and only if, for all $S \subseteq F_i(G)$, we have $k(I_i^S(G)) = c_i(S)$.

We now characterize conformal K_i-intersection graphs:

<u>Theorem 2.6</u>: $I_i(G)$ is conformal iff G does not contain a K_{i+1}.

<u>Proof</u>: (=>): Suppose there exists a K_{i+1} in G; let S denote the set of $i + 1$ K_i's in this K_{i+1}. The graph $I_i^S(G)$ is complete; however, it is clear that, in G, the intersection of the K_i's in S is the null set since every vertex in the K_{i+1} is avoided by exactly one K_i in S. Thus $I_i(G)$ is not conformal.

(<=): Suppose G does not contain a K_{i+1} and that $I_i(G)$ is not conformal. Thus $I_i(G)$ contains a complete subgraph $I_i^S(G)$ but the corresponding K_i's in G do not have a common set of $i - 1$ nodes. Clearly this can only happen if $|S| \geq 3$. Thus there are three K_i's in S, A, B, C, such that $|A \cap B \cap C| < i - 1$. Assume that $A \cap B$ and $B \cap C$ represent different K_{i-1}'s. In G let node $\{a\} = A \backslash B$ and $\{b\} = B \backslash A$. Similarly, let node $\{c\} = C \backslash B$. If $c \neq a$ then $|A \cap C| < i - 1$, contradicting the existence of edge (A,C) in $I_i(G)$. If $c = a$, then we have a K_{i+1} in G against assumption. ■

We now establish the relationship between conformal intersection graphs and diamond-free graphs.

<u>Lemma 2.7</u>: If $I_i(G)$ is conformal then it is diamond-free.

<u>Proof</u>: Suppose $I_i(G)$ has a diamond on the vertices A,B,C,D where the edge (A,D) is missing. Since $I_i(G)$ is conformal, the K_i's A, B, and C share a K_{i-1} in G and the K_i's B, C, and D also share a different K_{i-1} in G. Therefore B and C have two different K_{i-1}'s in common which is impossible. ■

We are now ready to characterize K_i-perfect graphs:

<u>Theorem 2.8</u>: G is K_i-perfect iff $I_i(G)$ is conformal and does not contain an odd hole of size ≥ 5.

<u>Proof</u>: (=>): Suppose $I_i(G)$ is not conformal and let S be a complete subgraph in $I_i(G)$ which violates conformality. Since $\alpha(I_i^S(G)) = 1$ we know that $p_i(S) = 1$ by Lemma 2.4. However, since the K_i's in S do not have a K_{i-1} in common, $c_i(S) \geq 2$, thereby contradicting the K_i-perfectness of G. We now assume that $I_i(G)$ is conformal and contains $I_i^S(G)$, an odd hole of size $2k + 1$, $(k \geq 2)$. For this odd hole $k(I_i^S(G)) = k + 1$, whereas $\alpha(I_i^S(G)) = k$. Since $I_i(G)$ is conformal, this implies that $c_i(S) > p_i(S)$ (by Lemmas 2.4 and 2.5), again contradicting the K_i-perfectness of G.

(<=): Since $I_i(G)$ is conformal, it is diamond-free by Lemma 2.7. The non-existence of an odd hole allows Lemma 2.1 to be applied and we conclude that $I_i(G)$ is perfect and thus that $\alpha(I_i^S(G)) = k(I_i^S(G))$ for all induced subgraphs $I_i^S(G)$ of $I_i(G)$. Since $I_i(G)$ is conformal, we may use Lemmas 2.4 and 2.5 to conclude that for any set $S \subseteq F_i(G)$, $c_i(S) = p_i(S)$, thus implying that G is K_i-perfect. ■

An equivalent form of Theorem 2.8 is the following:

<u>Theorem 2.9</u>: G is K_i-perfect iff G contains neither a K_{i+1} nor a subgraph whose K_i-intersection graph contains an odd hole of size ≥ 5.

As an immediate corollary to Theorem 2.8 we have:

<u>Corollary 2.10</u>: G is K_i-perfect iff $I_i(G)$ is conformal and perfect.

Theorems 2.8 and 2.9, when restricted to $i = 2$, state that a graph is K_2-perfect (i.e., bipartite) iff G does not contain a triangle and L(G) does not contain an odd hole of size ≥ 5. Since L(G) has such an odd hole iff G does, our result is equivalent to the standard characterization of bipartite graphs.

We now examine the matrix iA and show that iA is perfect iff G is K_i-perfect.

3. iA and K_i-perfect Graphs

We first introduce some definitions. Given a 0 - 1 matrix A, the intersection graph I(A) is constructed by associating one node to each column of A and joining two nodes by an edge if the corresponding columns have at least one 1 in the same position (i.e., the columns are not orthogonal). A is a clique matrix if the undominated rows of A form the list of the representative vectors of all the cliques of I(A). In [6] Padberg has shown the following:

Lemma 3.1: A matrix A is perfect iff the undominated rows of A constitute the clique matrix of I(A) and I(A) is a perfect graph.

We now study the matrix iA for a graph G and see immediately that $I(iA) \equiv I_i(G)$. Furthermore, we have:

Lemma 3.2: $I_i(G)$ is conformal iff iA is a clique matrix.

Proof: We have the following identities: iA is a clique matrix <=> all cliques of $I_i(G)$ are represented by rows of iA <=> any $S \subseteq F_i(G)$ such that any two K_i's in S have $i - 1$ points in common has the property that $|\bigcap_{K_i^j \in S} K_i^j| = i - 1 \Leftrightarrow I_i(G)$ is conformal. (This last identity uses Theorem 2.6). ∎

<u>Theorem 3.3</u>: iA is a clique matrix iff iA does not have a triangle submatrix.

<u>Proof</u>: From Lemma 3.2 and Theorem 2.6 we know that iA is a clique matrix iff $G \not\supset K_{i+1}$.

(=>): Assume that there is a triangle submatrix in iA. It is straightforward to see that the three K_i's whose columns are in this triangle submatrix, must form a K_{i+1} in G.

(<=): If a $K_{i+1} \subset G$, then for any $i \geq 2$, there must exist three distinct K_i's in the K_{i+1} such that any two of them have $i - 1$ points in common; however, the intersection of all three contains fewer than $i - 1$ points. Therefore, iA contains a triangle submatrix. ∎

We may now characterize K_i-perfect graphs in terms of perfect matrices:

<u>Theorem 3.4</u>: G is K_i-perfect iff iA is a perfect matrix.

<u>Proof</u>: Follows immediately from Corollary 2.10 and Lemma 3.2. ∎

Chvátal [3] has shown that a graph G is perfect iff the polytope

$$P_1(G) \begin{cases} Ax \leq 1 \\ x \geq 0 \end{cases}$$

where A is the clique-node incidence matrix, has all its extreme points in 0,1. This polytope defines the convex hull of the incidence vectors of the independent sets of G. (The incidence vector of $A \subseteq B$ is the vector x^A in $(0,1)^{|B|}$ defined by $x_i^A = 1$ if $i \in A$, 0 otherwise.) As a result of Lemma 2.1 and Theorem 3.4 we conclude that a graph G is K_i-perfect iff the polytope

$$P_2(G) \begin{cases} iAx \leq 1 \\ \quad x \geq 0 \end{cases}$$

has all its vertices in 0,1. Thus since $I_i(G)$ is conformal, the polyhedra $P_1(I_i(G))$ and $P_2(G)$ are the same.

As a summary, we have the following theorem:

<u>Theorem 3.5</u>: Given a graph G, the following seven conditions are equivalent:

(i) G is K_i-perfect.

(ii) $I_i(G)$ is conformal and does not contain an odd hole of size ≥ 5.

(iii) $I_i(G)$ is conformal and perfect.

(iv) iA is a perfect matrix.

(v) iA is a clique matrix and $I_i(G)$ is perfect.

(vi) $I_i(G)$ is conformal and $P_1(I_i(G))$ has all its points in 0,1.

(vii) $P_2(G)$ has all its points in 0,1.

It is well known that for K_2-perfect graphs (bipartite graphs) 2A (the vertex-edge incidence matrix) is totally unimodular. One might hope that the structure of iA might allow a stronger statement than that made in Theorem 3.4, in particular that G is K_i-perfect iff iA is a balanced matrix. However, already for $i = 3$ one may construct K_3-perfect graphs for which 3A is not balanced. The smallest such graph is shown in Figure 3.1. $I_3(G)$ is presented in Figure 3.2 and 3A is in Table 3.1. The first 13 rows and the 13 columns of 3A form a cycle submatrix.

We have seen that G is K_i-perfect iff iA is a perfect matrix and that for $i \geq 3$ there exist K_i-perfect graphs for which iA is not balanced. We now characterize the K_i-perfect graphs for which iA is balanced. Clearly, any K_i-perfect graph with unbalanced iA must have a partial odd hole C in $I_i(G)$ where the edges of this hole correspond to different K_{i-1}'s in G. Since $I_i(G)$ is perfect the induced subgraph on the nodes in C must contain chords. Furthermore, $I_i(G)$ is conformal and thus the edges of $I_i(G)$ may be partitioned into edge-disjoint cliques such that each vertex of $I_i(G)$ belongs to at most i such cliques. In the following we will characterize the number, size and structure of the complete subgraphs formed by the chords of a smallest such partial odd hole.

Throughout this discussion, given C a partial odd hole of size $2k + 1$ ($k \geq 2$) in $I_i(G)$ we let $\{K_i^0, \ldots, K_i^{2k}\}$ denote the K_i's of G corresponding to the vertices of C and let $\{K_{i-1}^0, \ldots, K_{i-1}^{2k}\}$ denote the K_{i-1}'s of G where K_{i-1}^j represents the edge (K_i^j, K_i^{j+1}) in $I_i(G)$ (addition modulo 2k). The K_{i-1}'s are all distinct, otherwise the submatrix of iA representing C would not be a cycle matrix. A subgraph of $I_i(G)$ which has the above properties together with at least one "chording" complete subgraph which corresponds to $K_{i-1}^* \notin \{K_{i-1}^0, \ldots, K_{i-1}^{2k}\}$ is called a <u>clique-chorded hole</u>. We now have

<u>Lemma 3.6</u>: iA is balanced iff G is K_i-perfect and $I_i(G)$ does not contain an odd clique-chorded hole.

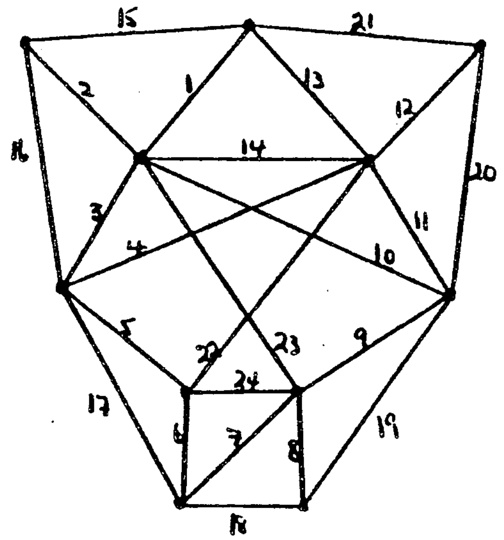


Figure 3.1: G, K_3-perfect, 3A not balanced.

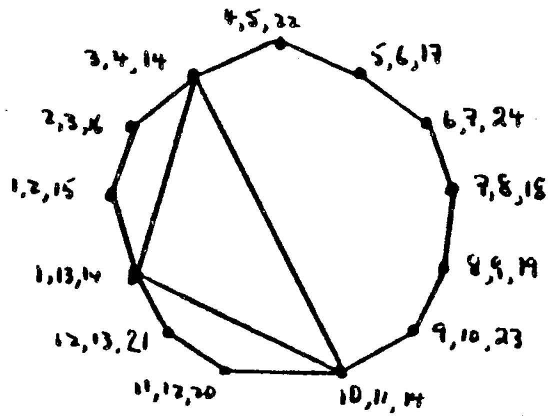


Figure 3.2: $I_3(G)$.

	1	1	2	3	4	5	6	7	8	9	10	11	12
	13	2	3	4	5	6	7	8	9	10	11	12	13
	14	15	16	14	22	17	24	18	19	23	14	20	21
1	1	1											
2		1	1										
3			1	1									
4				1	1								
5					1	1							
6						1	1						
7							1	1					
8								1	1				
9									1	1			
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11											1	1	
12												1	1
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21													1
22					1								
23										1			
24							1						

Table 3.1: 3A

Proof: (=>): Assume one of the following holds:

(i) $I_i(G)$ is not conformal, thus G contains a K_{i+1}. From Lemma 3.2 and Theorem 3.3 this implies that iA contains a triangle submatrix and thus is not balanced.

(ii) $I_i(G)$ is conformal but contains an induced odd hole. The submatrix of iA corresponding to this hole is an odd cycle matrix and thus iA is not balanced.

(iii) G is K_i-perfect but $I_i(G)$ contains an odd clique-chorded hole. The submatrix of iA corresponding to the $\{K_{i-1}\}$ and $\{K_i\}$ comprising the cycle is an odd cycle matrix and thus iA is not balanced.

(<=): If iA is not perfect then from Theorem 3.4, G is not K_i-perfect. Assume iA is perfect but not balanced. From the previous discussion this implies the $I_i(G)$ contains an odd clique-chorded hole. ■

The definition of clique-chorded hole allows the chording complete subgraphs to have an unrestricted interaction. We will show in fact that this interaction must be greatly constrained. To this end we let X_C denote the set of chords of C in $I_i(G)$ and partition X_C into ℓ edge disjoint complete subgraphs, X_j, $1 \leq j \leq \ell$. The vertices in each X_j may be considered using the ordering of the K_i's in C. Each X_j represents a K_{i-1} in G which does not belong to $\{K_{i-1}^0, \ldots, K_{i-1}^{2k}\}$. In the following lemmas, a minimum C refers to any partial odd hole in $I_i(G)$, G K_i-perfect, whose corresponding submatrix in iA is a cycle matrix and there does not exist any other such C' where $|C'| < |C|$. From Theorem 3.3 we know that $|C| > 3$.

Lemma 3.7: For each $X_j \in X_C$ where C is minimum, the number of vertices of C between consecutive vertices in X_j is even.

Proof: Since K^*_{i-1}, the $K_{i-1} \in F_{i-1}(G)$ which represents X_j does not belong to $\{K^0_{i-1},\ldots,K^{2k}_{i-1}\}$, consecutive vertices in C cannot be vertices in X_j. Now assume that there exist vertices K^α_i, K^β_i, $\alpha < \beta$ consecutive vertices in X_j such that $\beta - \alpha - 1$, the number of C vertices trapped between K^α_i and K^β_i is odd. We immediately see that the submatrix of iA defined on $\{K^\alpha_i, K^{\alpha+1}_i, \ldots, K^\beta_i\}$ is a cycle matrix which contradicts the minimality of C. ∎

Corollary 3.8: Each X_j of X_C for a minimum C, is of odd order.

Proof: Follows immediately from Lemma 3.7 since $|C|$ is odd. ∎

We now examine the situation where X_C contains at least two complete subgraphs. An exterior edge of such a complete subgraph is the chord of C between two consecutive vertices of the complete subgraph. Given two complete subgraphs X_1 and X_2 where (K^α_i, K^β_i) is a chord of X_1, we say that this chord traps vertex K^γ_i of X_2 if $\alpha \leq \gamma \leq \beta$. Note that X_1 and X_2 may have at most one vertex in common. In the proof of the next lemma, an informal diagrammatic type of proof will be used to show the existence of a smaller cycle submatrix of iA, thereby contradicting the minimum cardinality of C. This informal proof can easily be replaced by a verbose formal argument. Given two vertices A, B on C, $A \sim B$ denotes the set of C vertices in the range from A to B inclusive. $|A \sim B|$ denotes the number of such vertices. If $A = B$ then $|A \sim B| = 0$.

<u>Lemma 3.9</u>: Given a minimum C, there cannot exist two complete subgraphs $X_1, X_2 \in X_C$ where an exterior edge of X_1 traps an odd number of vertices of X_2.

<u>Proof</u>: In the following, solid lines will denote the edges of X_1 and dotted lines will denote the edges of X_2. The proof is divided into four cases. In the first three, the exterior edge of X_1 is vertex disjoint with X_2.

<u>Case 1</u>: Assume an exterior edge of X_1 traps an odd number > 1 of vertices of X_2 (see Fig. 3.3). Note that all X_2 vertices in $A - D$ are in $B - C$. From Lemma 3.7 $|A - D|$ is odd. Since $|B - C|$ is odd, $|A - B| + |C - D|$ is odd. Thus the cycle defined on the vertices in $A - B$ and $C - D$ forms an odd hole and is of smaller cardinality than $|C|$.

<u>Case 2</u>: Assume an exterior edge of X_1 traps 1 vertex of X_2, namely B, and that an exterior edge of X_2 from B traps an even number of vertices of X_1 (see Fig. 3.4). Since $|A - C|$ is even, $|A - B|$ and $|B - C|$ are of opposite parity. Since $|C - D|$ and $|B - E|$ are even, $|B - C| + |D - E|$ is even and thus $|A - B| + |D - E|$ is odd. Thus the cycle defined on the vertices in $A - B$ and $D - E$ forms an odd hole and is of smaller cardinality than $|C|$.

<u>Case 3</u>: Assume an exterior edge of X_1 traps 1 vertex of X_2, namely B, and that both exterior edges of B trap one vertex of X_1, namely the endpoints of the exterior edge of X_1 (see Fig. 3.5 where (A,C) is the exterior edge of X_1). Since $|E - B|$ is even, $|E - A| + |A - B|$ is odd.

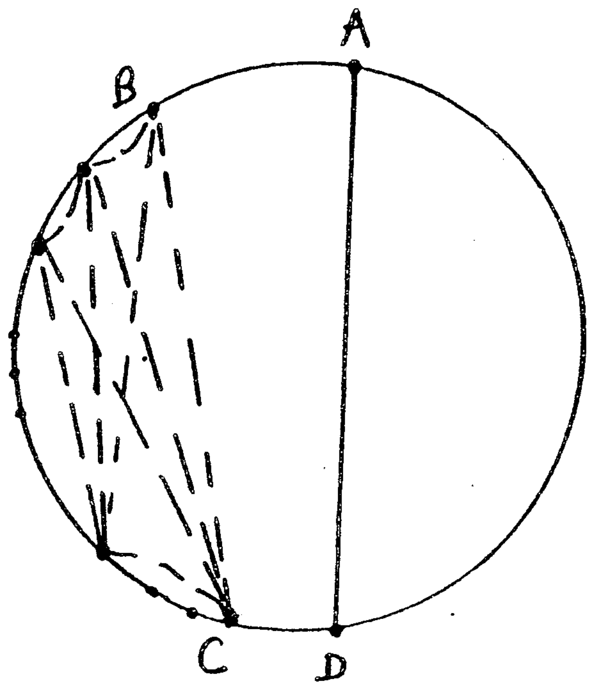

Figure 3.3

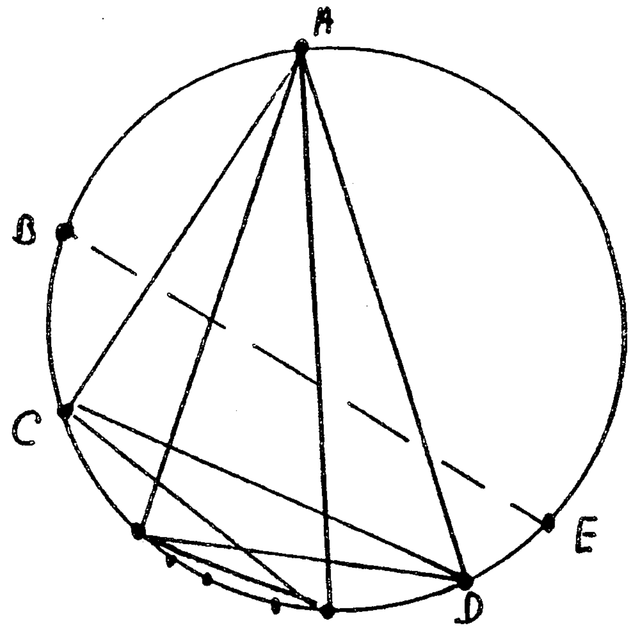

Figure 3.4

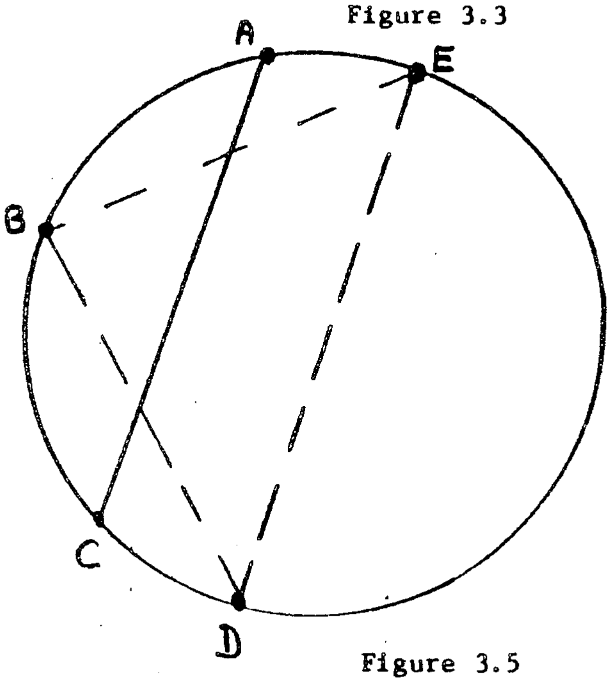

Figure 3.5

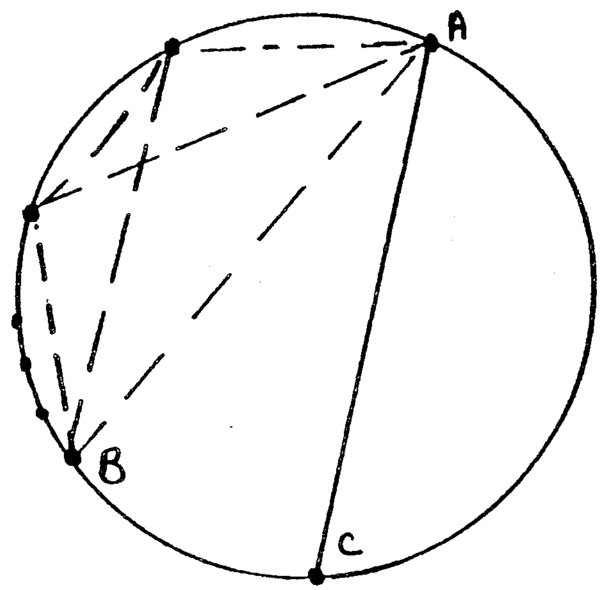

Figure 3.6

Since $|A - C|$ is even, $|A - B| + |B - C|$ is odd. Since $|B - D|$ is even, $|B - C| + |C - D|$ is odd and thus $|E - A| + |C - D|$ is odd. Thus the cycle defined on the vertices in $E - A$ and $C - D$ forms an odd hole of smaller cardinality than $|C|$.

Case 4: Assume an exterior edge of X_1 which shares a vertex with X_2, traps an odd number of vertices of X_2 (see Fig. 3.6). The proof is similar to that of Case 1. ■

From this lemma we have the following corollary.

Corollary 3.10: Given a minimum C, there cannot exist two disjoint X_j's in X_C.

Proof: If X_1 and X_2 are two disjoint elements of X_C then from Lemma 3.9, every exterior edge of X_1 must trap an even number of vertices of X_2. This implies that X_2 has even cardinality contradicting Corollary 3.8. ■

Thus each pair of X_j's in X_C must have a common vertex. We now show that all such X_j's must have the same common vertex.

Lemma 3.11: In a minimum C, if $|X_C| > 1$, then there exists a single vertex which belongs to all $X_j \in X_C$.

Proof: Assume there exist three complete subgraphs in X_C, X_1, X_2, X_3 such that $X_1 \cap X_2 = x$, $X_2 \cap X_3 = y$ and $X_1 \cap X_3 = z$ where x, y and z are all different. In $I_i(G)$ the K_i's x, y and z are the vertices of a triangle. Conformality of $I_i(G)$ requires that these K_i's all have a K_{i-1} in common which contradicts the assumption that X_1, X_2 and X_3 are different. ■

We can now make a statement about the maximum number of elements in X_C. All such elements have node $\bar{x}$ in common.

<u>Lemma 3.12</u>: In a minimum C, the maximum number of complete graphs in X_C is $i - 2$.

<u>Proof</u>: From Lemma 2.3, there is no $K_{1,i+1}$ in $I_i(G)$. Since $\bar{x}$ is on C it already has two independent cycle edges emanating from it and thus since each $X_i \in X_C$ adds another independent edge, the maximum is $i - 2$. ∎

For $i = 3$, we thus see that there is exactly one element in X_C for C minimum. We now examine the case where $i > 3$ and show that a certain nesting property must hold among the complete graphs in X_C. Given two $X_1, X_2 \in X_C$ we say that <u>X_1 is nested in X_2</u>, if the two exterior edges of X_2 from $\bar{x}$ trap no vertices of X_1. A set of complete graphs is called <u>nested</u> if for any pair, one is nested in the other. We now show that the set X_C is nested in this sense.

<u>Lemma 3.13</u>: If $X_1, X_2 \in X_C$ where C is minimum then either X_1 is nested in X_2 or X_2 is nested in X_1.

<u>Proof</u>: If not, then from Lemma 3.9 we see that both X_1 and X_2 have an even number of vertices, which contradicts Corollary 3.8. ∎

We are now ready to characterize the K_i-perfect graphs for which iA is balanced. A graph $G(V,E)$ is called an <u>i-nested clique hole</u> if E may be partitioned into a Hamiltonian cycle C and a set of at most $i - 2$ nested odd cliques such that the intersection of any two cliques is $\bar{x} \in V$. For any clique each exterior edge traps an even (> 2) number of vertices of C and either 0 or an even number of vertices of any other clique.

Theorem 3.14: The following conditions are equivalent:

(i) iA is balanced.

(ii) G is K_i-perfect and $I_i(G)$ does not contain an odd induced i-nested clique hole.

(iii) G is K_i-perfect and $I_i(G)$ does not contain an odd clique-chorded hole.

Proof: (i) <=> (iii) follows from Lemma 3.6.

(i) => (ii): This proof is the same as the (=>) proof in Lemma 3.6 except we must show that the K_{i-1}'s corresponding to the edges of the cycle are distinct and different from the K_{i-1}'s corresponding to the interior cliques. This follows immediately from the definition of i-nested clique hole since each exterior edge of an interior clique must trap at least four vertices of C. Thus if G is K_i-perfect and contains an odd induced i-nested clique hole, iA is not balanced.

(i) <= (ii): Assume iA is perfect but not balanced. Let C be a minimum size odd cycle matrix of iA. The fact that the subgraph of $I_i(G)$ induced on the K_i's in C is an odd i-nested clique hole follows from the definition of i-nested clique hole and Lemmas 3.7, 3.9, 3.11, 3.12, 3.13 and Corollary 3.8. ∎

4. Concluding Remarks

Our definition of K_i-perfect states that $\forall\, F \subseteq F_i(G)$, $c_i(F) = p_i(F)$. Although this definition allows an elegant characterization of K_i-perfect graphs and of perfect iA matrices, it does not capture the same flavor of perfection as does Berge's definition of perfect graphs. In particular our definition has the universal quantification over the set $F_i(G)$ rather than over the partial or induced subgraphs of G. We now examine each of these two alternative definitions and show by counter-example that they are not equivalent to the definition of K_i-perfect used in this paper.

We define a graph to be <u>K_i-partial perfect</u> if $\forall H < G$, $c_i(H) = p_i(H)$ and a graph to be <u>K_i-induced perfect</u> if $\forall H \triangleleft G$, $c_i(H) = p_i(H)$. Figure 4.1 shows G_1 a K_3-partial perfect graph which is not K_3-perfect. This graph's K_3 intersection graph is presented in Figure 4.2. For this graph $c_3(G_1) = p_3(G_1) = 11$. It is easy to verify that all $H < G_1$ satisfy $c_3(H) = p_3(H)$. The odd hole in $I_3(G_1)$ shows that G_1 is not K_3-perfect.

G_2 presented in Figure 4.3 is a K_3-induced perfect graph which is not K_3-perfect. $I_3(G_2)$ is drawn in Figure 4.3. It is seen that $p_3(G_2) = c_3(G_2) = 8$ and that $p_3(H) = c_3(H)\ \forall H \triangleleft G_2$. Again the odd hole in $I_3(G_2)$ shows that G_2 is not K_3-perfect.

The characterization of the K_i-partial perfect and the K_i-induced perfect graphs is left as an open problem.

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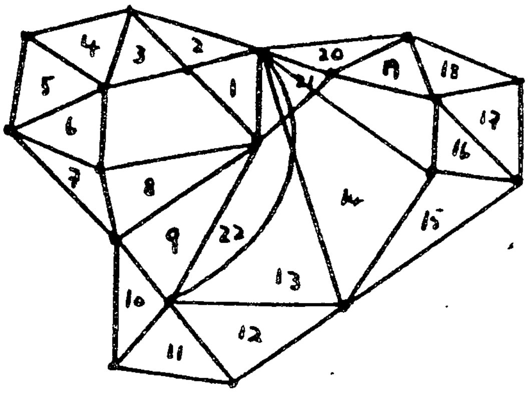

Figure 4.1: G_1

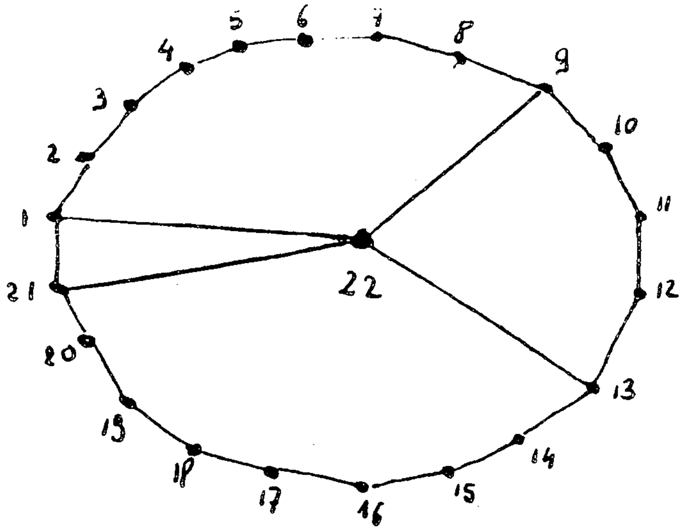

Figure 4.2: $I_3(G_1)$

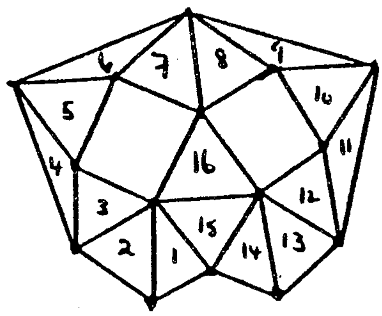

Figure 4.3: G_2

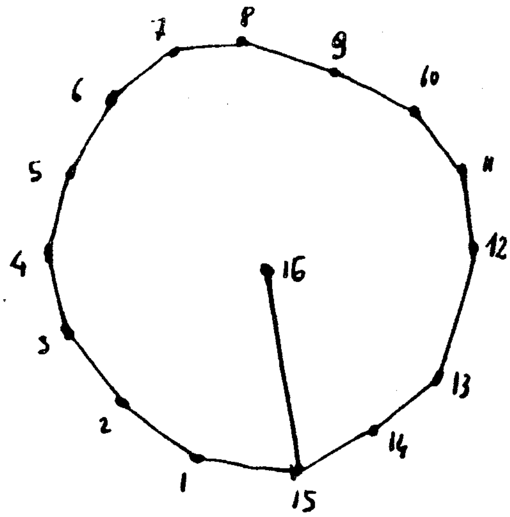

Figure 4.4: $I_3(G_2)$

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DOCTORAT D'ETAT

Vu les dispositions de l'article 5 de l'arrêté du 16 avril 1974,

Vu les rapports de M. Jean FONLUPT.............
M. Michel LAS VERGNAS.......
M. Th. M. LIEBLING..............

M. Ali Ridha MAHJOUB.......................... est autorisé à présenter une thèse en vue de l'obtention du grade de DOCTEUR D'ETAT ES SCIENCES.

Fait à Grenoble, le - 6 JUIN 1985

Le Président de l'U.S.M.G.

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