



# The $k$ edge-disjoint 3-hop-constrained paths polytope

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## ABSTRACT

Given a graph  $G$  with two distinguished nodes  $s$  and  $t$ , a cost on each edge of  $G$  and two fixed integers  $k \geq 2, L \geq 2$ , the  $k$  edge-disjoint  $L$ -hop-constrained paths problem is to find a minimum cost subgraph of  $G$  such that between  $s$  and  $t$  there are at least  $k$  edge-disjoint paths of length at most  $L$ . In this paper we consider this problem from a polyhedral point of view. We give an integer programming formulation for the problem and discuss the associated polytope. In particular, we show that when  $L = 3$  and  $k \geq 2$ , the linear relaxation of the associated polytope, given by the trivial, the  $st$ -cut and the so-called  $L$ -path-cut inequalities, is integral. As a consequence, we obtain a polynomial time cutting plane algorithm for the problem when  $L = 2, 3$  and  $k \geq 1$ . This generalizes the results of Huygens et al. (2004) [1] for  $k = 2$  and  $L = 2, 3$  and those of Dahl et al. (2006) [2] for  $L = 2$  and  $k \geq 2$ . This also proves a conjecture in [1].

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## 1. Introduction

Given a graph  $G = (V, E)$  with two distinguished nodes  $s, t \in V$  and a positive integer  $L \geq 2$ , an  $L$ - $st$ -path in  $G$  is a path between  $s$  and  $t$  of length at most  $L$ , where the length is the number of its edges. Given a function  $c : E \rightarrow \mathbb{R}$  which associates a cost  $c(e)$  to each edge  $e \in E$  and an integer  $k \geq 2$ , the  $k$  edge-disjoint  $L$ -hop-constrained paths problem (kHPP) is to find a minimum cost subgraph such that between  $s$  and  $t$  there exist at least  $k$  edge-disjoint  $L$ - $st$ -paths.

Given an edge subset  $F \subseteq E$ , the 0–1 vector  $x^F \in \mathbb{R}^E$ , such that  $x^F(e) = 1$  if  $e \in F$  and  $x^F(e) = 0$  otherwise, is called the incidence vector of  $F$ . The kHPP polytope, denoted by  $kHPP(G)$ , is the convex hull of the incidence vectors of the solutions of the kHPP on  $G$ , i.e.,

$$kHPP(G) = \text{conv}\{x^F \in \mathbb{R}^E \mid F \subseteq E \text{ and } (V, F) \text{ contains } k \text{ edge-disjoint } st\text{-paths of length at most } L\}.$$

In [1] Huygens et al. conjectured that when  $L = 2, 3$  and  $k \geq 3$ , the kHPP polytope is given by the so-called  $st$ -cut and  $L$ -path-cut inequalities together with the trivial inequalities. This has been proved by Dahl et al. [2] for  $L = 2$  and  $k \geq 2$ . In this paper, we show the conjecture for  $L = 3$  and  $k \geq 2$ . This has been an open question for a few years. We give an integer programming formulation for the problem. We also describe necessary and sufficient conditions for these inequalities to be facet defining. Using this, we give a complete description of the kHPP polytope. These results can be seen as a generalization of those obtained by Huygens et al. [1] for  $k = 2$  and  $L = 2, 3$  and by Dahl et al. [2] for  $k \geq 2$  and  $L = 2$ . We also show that the kHPP reduces to a minimum cost flow problem in an appropriate graph.

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One of the main concerns when designing telecommunication networks is to compute network topologies that provide a sufficient degree of survivability. Survivable networks – networks that are still functional after the failures of certain links – must satisfy some connectivity requirements. As pointed out in [3] (see also [4]), the topology that seems to be very efficient (and needed in practice) is the uniform topology, that is to say that corresponding to networks that survive after the failure of  $k - 1$  or fewer edges, for some  $k \geq 2$ . The 2-connected topology ( $k = 2$ ) provides an adequate level of survivability since most failure usually can be repaired relatively quickly. However, for some types of networks (like ATM and IP networks), it may be necessary to provide a higher level of connectivity.

However, this requirement is often insufficient regarding the reliability of a telecommunication network. In fact, the alternative paths could be too long to guarantee an effective routing. In data networks, such as Internet, the elongation of the route of the information could cause a strong loss in the transfer speed. For other networks, the signal itself could be degraded by a longer routing. In such cases, the  $L$ -path requirement guarantees exactly the needed quality of the alternative routes.

The  $k$ HPP can be seen as a special case of the more general problem when more than one pair of terminals are considered. This is the case, for instance, when several commodities have to be routed in the network. Thus, an efficient algorithm for solving the  $k$ HPP would be useful to solve (or produce upper bounds for) this more general problem.

The  $k$ HPP has been studied in some special cases. Huygens et al. [1] have investigated the case where  $k = 2$  and  $L = 2, 3$ . They give an integer programming formulation for the problem and show that the linear relaxation of this formulation completely describes the  $k$ HPP polytope in this case. They also show that this formulation is no longer valid when  $L \geq 4$ . In [2], Dahl et al. study the  $k$ HPP for  $L = 2$  and  $k \geq 2$ . They give a complete description of the associated polytope. There has been however a considerable amount of research on many related problems. In [5], Dahl considers the  $k$ HPP for  $k = 1$  and  $L \geq 3$ . He gives a complete description of the dominant of the associated polytope when  $L \leq 3$ . Dahl and Gouveia [6] consider the directed hop-constrained path problem. They describe valid inequalities and characterize the associated polytope when  $L \leq 3$ .

In [7], Coullard et al. investigate the structure of the polyhedron associated with the  $st$ -walks of length  $K$  of a graph, where a walk is a path that may go through the same node more than once. They present an extended formulation of the problem, and, using projection, they give a linear description of the associated polyhedron. They also discuss classes of facets of that polyhedron. Dahl et al. [8] also consider the hop-constrained walk polytope in directed graphs when  $L = 4$ . By introducing extended variables in addition to the design variables the authors characterize the associated polytope. They also introduce a large class of facet defining inequality for the dominant of that polytope. This is the first work that addresses a polyhedral analysis for a hop-constrained network design problem with  $L = 4$ . Huygens and Mahjoub [9] study the  $k$ HPP when  $L \geq 4$ . They also study the variant of the  $k$ HPP where  $k$  node-disjoint paths of length at most  $L$  are required between two terminals. They give integer programming formulations for these two problems in the case  $k = 2$  and  $L = 4$ .

The more general version of the problem where several pairs  $(s, t)$  of terminals have to be linked by  $L$ -hop-constrained paths has also been studied in the literature. In [10], Dahl and Johannessen consider the 2-path network design problem which consists in finding a minimum cost subgraph connecting each pair of terminal nodes by at least one path of length at most 2. Huygens et al. [11] consider the problem of finding a minimum cost subgraph with at least two edge-disjoint  $L$ -hop-constrained paths between each pair of terminal nodes. They give an integer programming formulation of that problem for  $L = 2, 3$  and present several classes of valid inequalities. They also devise a Branch-and-Cut algorithm, and discuss some computational results. In [12–14], Gouveia et al. consider the node case of that general problem within the context of an MPLS (Multi-Protocol Label Switching) network design model. In [12], the authors study a network design problem involving hop-constraints for MPLS over optical networks. They discuss a hop-indexed formulation and devise a heuristic which shows to be very efficient for large instances. In [13,14], they consider a hop-constrained node survivable network design problem. Here survivability requires that between every pair of demand nodes, there are at least  $k$  node-disjoint paths with at most  $L$  edges, where  $k$  and  $L$  are two given integers. In [13], they discuss the problem of designing one layer of the network (the MPLS layer) with  $k = 2$ . They propose two compact formulations for the problem, a hop-indexed formulation and a multicommodity flow formulation. They show, in particular, that the LP bound of the former is as good as the LP bound given by the latter. They also present computational results for  $L = 4, 5, 6$ . In [14], they consider the problem in multilayer MPLS over WDM networks. They give an integer linear programming model based on the results in [12,13], and discuss computational results for  $k = 2, 3, 4$  and  $L = 4$ .

The problem of finding a minimum cost spanning tree with hop-constraints is also considered in [15–17]. Here, the hop-constraints limit to  $L$  the number of links between the root and any terminal in the network. In [15] Gouveia gives a multicommodity flow formulation for that problem and devises a Lagrangian relaxation. In [16] he proposes a hop-indexed reformulation of a multicommodity flow formulation which is based on an extended description of the  $L$ -walk polyhedron. In [17] Gouveia and Requejo propose a Lagrangian relaxation for the problem that dualizes the hop-indexed flow conservation constraints. Dahl [18] also studies the minimum cost spanning tree with hop-constraints problem where  $L = 2$  from a polyhedral point of view. He gives a complete description of the associated polytope when the graph is a wheel.

Another reliability condition used in telecommunication networks in order to limit the length of the routing, requires that each link of the network belongs to a ring (cycle) of bounded length. In [19], Fortz et al. consider the 2-node connected subgraph problem with bounded rings. This problem consists in finding a minimum cost 2-node connected subgraph  $(V, F)$  such that each edge of  $F$  belongs to a cycle of length at most  $L$ . They describe several classes of facet defining inequalities for the associated polytope and devise a Branch-and-Cut algorithm for the problem. In [20], Fortz et al. study the edge version

of that problem. They give an integer programming formulation for the problem in the space of the natural design variables and describe different classes of valid inequalities. They study the separation problem of these inequalities and discuss Branch-and-Cut algorithm.

The related  $k$  edge-connected subgraph problem and its associated polytope have also been the subject of extensive research in the past years. Grötschel and Monma [21] and Grötschel et al. [22,23] study the  $k$ -edge connected subgraph problem within the framework of a general survivable model. They discuss polyhedral aspects and devise cutting plane algorithms. Didi Biha and Mahjoub [24] study that problem and give a complete description of the associated polytope when the graph is series-parallel. In [25], Didi Biha and Mahjoub study the Steiner version of that problem and characterize the polytope when  $k$  is even. Chopra [26] studies the dominant of that problem and introduces a class of valid inequalities for its polyhedron. Barahona and Mahjoub [27] characterize the polytope for the class of Halin graphs. In [28], Fonlupt and Mahjoub study the fractional extreme points of the linear relaxation of the 2-edge connected subgraph polytope. They introduce an ordering on these extreme points and characterize the minimal extreme points with respect to that ordering. As a consequence, they obtain a characterization of the graphs for which the linear relaxation of that problem is integral. Didi Biha and Mahjoub [29], extend the results of Fonlupt and Mahjoub [28] to the case  $k \geq 3$  and introduce some graph reduction operations. Kerivin et al. [30] study that problem in the more general case where each node of the graph has a specific connectivity requirement. They present different classes of facets of the associated polytope when the connectivity requirement of each node is at most 2, and devise a Branch-and-Cut algorithm for the problem in this case. In [31], Bendali et al. study the  $k$ -edge connected subgraph problem for the case  $k \geq 3$ . They introduce several classes of valid inequalities and discuss the separation algorithm for these inequalities. They also devise a Branch-and-Cut algorithm and give some computational results for  $k = 3, 4, 5$ . A complete survey of the  $k$ -edge connected subgraph problem can be found in [4].

The paper is organized as follows. In next section, we give some notations and preliminary results we will use along this paper. In Section 3, we describe necessary and sufficient conditions for the  $st$ -cut and  $L$ -path-cut inequalities to be facet defining. Our main result, which gives the complete description of the  $k$ HPP polytope for  $L = 3$ , is presented in Section 4, and concluding remarks are given in Section 5.

## 2. Notations and preliminary results

### 2.1. Notations

We denote a graph by  $G = (V, E)$  where  $V$  is the node set and  $E$  the edge set. Given two node subsets  $W$  and  $W'$ , we denote by  $[W, W']$  the set of edges having one endnode in  $W$  and the other in  $W'$ . If  $W = \{u\}$ , we then write  $[u, W']$  for  $[\{u\}, W']$ . We also denote by  $\bar{W}$  the node set  $V \setminus W$ . If  $[u, v] \neq \emptyset$  for two nodes  $u, v \in V$ , we let  $uv$  be an edge of  $[u, v]$ . The set of edges having only one node in  $W$  is called a *cut* and denoted by  $\delta(W)$ . We will write  $\delta(u)$  for  $\delta(\{u\})$ . Given two nodes  $s, t \in V$ , a cut  $\delta(W)$  such that  $s \in W$  and  $t \in \bar{W}$  is called an *st-cut*.

Throughout the paper, we consider graphs without loops and which may have multiple edges.

If  $\pi = (V_0, \dots, V_p)$ ,  $p \geq 1$ , is an ordered partition of  $V$ , the graph  $G_\pi = (V_\pi, E_\pi)$  denotes the subgraph of  $G$  obtained by contracting the sets  $V_i$ ,  $i = 1, \dots, p$ . The edge set  $E_\pi$  may be identified with the set of edges of  $G$  having their endnodes in different elements of  $\pi$ .

For an edge subset  $F \subseteq E$ , we will denote by  $G \setminus F$  the subgraph of  $G$  obtained by removing all the edges of  $F$ .

Given a weight vector  $w \in \mathbb{R}^E$  and an edge subset  $F \subseteq E$ , we let  $w(F) = \sum_{e \in F} w(e)$ .

Given a solution  $\bar{x} \in \mathbb{R}^E$ , an inequality  $a^T x \geq \alpha$  is said to be *tight* for  $\bar{x}$  if and only if  $a^T \bar{x} = \alpha$ .

We will also denote by  $D = (V, A)$  a directed graph. An arc  $a$  with its tail node  $u$  and head node  $v$  will be denoted by  $(u, v)$ . Given two node subsets  $W$  and  $W'$  of  $V$ , we will denote by  $[W, W']$  the set of arcs whose tail nodes are in  $W$  and head nodes are in  $W'$ . As before, we will write  $[u, W']$  for  $[\{u\}, W']$  and  $\bar{W}'$  will denote the node set  $V \setminus W'$ . The set of arcs having their tail in  $W$  and their head in  $\bar{W}$  is called a *cut* in  $D$  and is denoted by  $\delta^{out}(W)$ . We will also write  $\delta^{out}(u)$  for  $\delta^{out}(\{u\})$  with  $u \in V$ . If  $s$  and  $t$  are two nodes of  $H$  such that  $s \in W$  and  $t \in \bar{W}$ , then  $\delta^{out}(W)$  will be called a *st-dicut* in  $D$ .

### 2.2. Valid inequalities for the $k$ HPP polytope

Let  $G = (V, E)$  be an undirected graph, two nodes  $s, t$  of  $V$  and a positive integer  $k \geq 2$ . If  $F$  is a solution of the  $k$ HPP, then clearly  $x^F$  satisfies the following inequalities:

$$x(\delta(W)) \geq k, \quad \text{for all } st\text{-cut } \delta(W), \quad (2.1)$$

$$0 \leq x(e) \leq 1, \quad \text{for all } e \in E. \quad (2.2)$$

Inequalities (2.1) will be called *st-cut inequalities* and inequalities (2.2) *trivial inequalities*.

In [5], Dahl considers the problem of finding a minimum cost path between two given terminal nodes  $s$  and  $t$  of length at most  $L$ . He describes a class of valid inequalities for the problem and gives a complete characterization of the associated  $L$ -path polyhedron when  $L \leq 3$ . In particular he introduces a class of valid inequalities as follows.

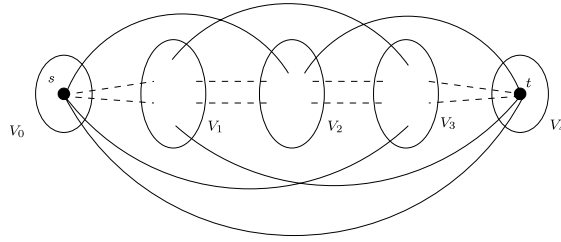


Fig. 1. Support graph of a 3-path-cut inequality.

Let  $V_0, V_1, \dots, V_{L+1}$  be a partition of  $V$  such that  $s \in V_0$  and  $t \in V_{L+1}$ , and  $V_i \neq \emptyset$  for all  $i = 1, \dots, L$ . Let  $T$  be the set of edges  $e = uv$ , where  $u \in V_i$ ,  $v \in V_j$ , and  $|i - j| > 1$ . Then the inequality

$$x(T) \geq 1$$

is valid for the  $L$ -path polyhedron.

Using the same partition, this inequality can be generalized in a straightforward way to the  $k$ HPP polytope as

$$x(T) \geq k. \quad (2.3)$$

The set  $T$  is called an  $L$ -path-cut, and a constraint of type (2.3) is called an  $L$ -path-cut inequality. See Fig. 1 for an example of a 3-path-cut inequality with  $V_0 = \{s\}$  and  $V_4 = \{t\}$ . Note that  $T$  intersects every 3- $st$ -path in at least one edge and each  $st$ -cut  $\delta(W)$  intersects every 3- $st$ -path. We denote by  $P_k(G)$  the polytope given by inequalities (2.1)–(2.3).

The separation problem associated with inequalities (2.1) can be solved in polynomial time by computing a maximum flow between  $s$  and  $t$  in the graph  $G$ . Fortz et al. [20] shows that the separation of inequalities (2.3) can also be performed in polynomial time by computing a maximum flow in an auxiliary graph.

### 2.3. Formulation

In this subsection, we give an integer programming formulation for the  $k$ HPP. We will show that the  $st$ -cut, 3-path-cut and trivial inequalities, together with the integrality constraints suffice to formulate the  $k$ HPP as a 0–1 linear program. To this end, we first give a lemma. Its proof can be found in [1].

**Lemma 2.1** ([1]). *Let  $G = (V, E)$  be an undirected graph and  $s$  and  $t$  two nodes of  $V$ . Suppose that there do not exist  $k$  edge-disjoint 3- $st$ -paths in  $G$ , with  $k \geq 2$ . Then there exists a set of at most  $k - 1$  edges that intersects every 3- $st$ -path.*

**Theorem 2.1.** *Let  $G = (V, E)$  be a graph and  $k \geq 2$ . Then the  $k$ HPP is equivalent to the integer program*

$$\text{Min } \{cx; x \in P_k(G), x \in \{0, 1\}^E\}.$$

**Proof.** To prove the theorem, it is sufficient to show that every 0–1 solution  $x$  of  $P_k(G)$  induces a solution of the  $k$ HPP. Let us assume the contrary and suppose that  $x$  does not induce a solution of the  $k$ HPP but satisfies the  $st$ -cut and trivial inequalities. We will show that  $x$  necessarily violates at least one 3-path-cut inequality. Let  $G(x)$  be the subgraph of  $G$  induced by  $x$ , that is the graph obtained from  $G$  by deleting every edge  $e \in E$  such that  $x(e) = 0$ . As  $x$  is not a solution of the problem,  $G(x)$  does not contain  $k$  edge-disjoint 3- $st$ -paths. By Lemma 2.1, it follows that there exist at most  $k - 1$  edges in  $G(x)$  that intersect every 3- $st$ -path. Consider the graph  $G'(x)$  obtained from  $G(x)$  by deleting these edges. Obviously,  $G'(x)$  does not contain any 3- $st$ -path.

We claim that  $G'(x)$  contains at least one  $st$ -path of length at least 4. In fact, as  $x$  is a 0–1 solution and satisfies the  $st$ -cut inequalities,  $G(x)$  contains at least  $k$  edge-disjoint  $st$ -paths. Since at most  $k - 1$  edges were removed from  $G(x)$ , at least one path remains between  $s$  and  $t$ . However, since  $G'(x)$  does not contain a 3- $st$ -path, that  $st$ -path must be of length at least 4.

Now consider the partition  $(V_0, \dots, V_4)$  of  $V$  with  $V_0 = \{s\}$ ,  $V_i$  the set of nodes at distance  $i$  from  $s$  in  $G'(x)$  for  $i = 1, 2, 3$ , and  $V_4 = V \setminus (\bigcup_{i=0}^3 V_i)$ , where the distance between two nodes is the length of a shortest path between these nodes. Since there does not exist a 3- $st$ -path in  $G'(x)$ , it is clear that  $t \in V_4$ . Moreover, as by the claim above,  $G'(x)$  contains an  $st$ -path of length at least 4, the sets  $V_1, V_2$  and  $V_3$  are nonempty. Furthermore, no edge of  $G'(x)$  is a chord of the partition (that is an edge between two sets  $V_i$  and  $V_j$  where  $|i - j| > 1$ ). In fact, if there exists an edge  $e = v_i v_j \in [V_i, V_j]$  with  $|i - j| > 1$  and  $i < j$ , then  $v_j$  is at distance  $i + 1 < j$ , from  $s$ , a contradiction.

Thus, the edges deleted from  $G'(x)$  are the only edges that may be chords of the partition  $G(x)$ . In consequence, if  $T$  is the set of chords of the partition in  $G$ , then  $x(T) \leq k - 1$ . But this implies that the corresponding 3-path-cut inequality is violated by  $x$ .  $\square$

### 3. Facets of $k$ HPP( $G$ )

In this section, we give necessary and sufficient conditions for inequalities (2.1)–(2.3) to define facets. These will be useful in what follows.

Let  $G = (V, E)$  be an undirected graph,  $s$  and  $t$  two nodes of  $G$  and  $k$  a positive integer  $\geq 2$ . An edge  $e \in E$  is said to be 3- $st$ -essential if  $e$  belongs to an  $st$ -cut or a 3-path-cut of cardinality  $k$ . Let  $E^*$  be the set of the 3- $st$ -essential edges. We have the following results that can be easily seen to be true.

**Theorem 3.1.**  $\dim(kHPP(G)) = |E| - |E^*|$ .

An immediate consequence of Theorem 3.1 is the following.

**Corollary 3.1.** If  $G = (V, E)$  is a complete graph such that  $|V| \geq k + 2$ , then  $kHPP(G)$  is full dimensional.

In the rest of the paper, we will consider that  $G = (V, E)$  is a complete graph with  $|V| \geq k + 2$ , and which may contain multiple edges. Thus, by Corollary 3.1,  $kHPP(G)$  is full dimensional.

**Lemma 3.1.** Let  $a^T x \geq \alpha$  be an inequality which defines a facet of  $kHPP(G)$ , different from (2.2). Then  $a(e) \geq 0$  for all  $e \in E$ .

The following theorems show when inequalities (2.1)–(2.3) define facets for  $kHPP(G)$ .

**Theorem 3.2.** (1) Inequality  $x(e) \leq 1$  defines a facet of  $kHPP(G)$  for all  $e \in E$ .

(2) Inequality  $x(e) \geq 0$  defines a facet of  $kHPP(G)$  if and only if either  $|V| \geq k + 3$  or  $|V| = k + 2$  and  $e$  does not belong neither to an  $st$ -cut nor to a 3-path-cut containing exactly  $k + 1$  edges.

**Proof.** (1) As  $|V| \geq k + 2$  and  $G$  is complete, the edge set  $E_f = E \setminus \{f\}$  is a solution of  $kHPP$ , for all  $f \in E \setminus \{e\}$ . Hence, the sets  $E$  and  $E_f$ , for all  $f \in E \setminus \{e\}$ , constitute a set of  $|E|$  solutions of the  $kHPP$ . Moreover, their incidence vectors satisfy  $x(e) = 1$  and are affinely independent.

(2) Suppose that  $|V| \geq k + 3$ . Then  $G$  contains  $k + 2$  node-disjoint  $st$ -paths (an edge of  $[s, t]$  and  $k + 1$  paths of the form  $(s, u, t), u \in V \setminus \{s, t\}$ ). Hence any edge set  $E \setminus \{f, g\}, f, g \in E$ , contains  $k$  edge-disjoint 3- $st$ -paths among these 3- $st$ -paths.

Consider the  $|E|$  edge sets  $E \setminus \{e\}$  and  $E_f = E \setminus \{e, f\}$  for all  $f \in E \setminus \{e\}$ . Therefore, these sets induce solutions of the  $kHPP$ . Moreover the incidence vectors of these solutions satisfy  $x(e) = 0$  and are affinely independent.

Now suppose that  $|V| = k + 2$ . If  $e$  belongs to an  $st$ -cut  $\delta(W)$  (resp. a 3-path-cut  $T$ ) with  $k + 1$  edges, then  $x(e) \geq 0$  is redundant with respect to the inequalities

$$\begin{aligned} x(\delta(W)) &\geq k \quad (\text{resp. } x(T) \geq k), \\ -x(f) &\geq -1 \quad \text{for all } f \in \delta(W) \setminus \{e\} (\text{resp. } f \in T \setminus \{e\}), \end{aligned}$$

and cannot hence be facet defining. If  $e$  does not belong neither to an  $st$ -cut nor to a 3-path-cut with  $k + 1$  edges, then the edge sets  $E \setminus \{e\}$  and  $E_f, f \in E \setminus \{e\}$ , introduced above, are still solutions of  $kHPP$ . Moreover, their incidence vectors satisfy  $x(e) = 0$  and are affinely independent.  $\square$

**Theorem 3.3.** Every  $st$ -cut inequality defines a facet of  $kHPP(G)$ .

**Proof.** Let  $W \subseteq V$  such that  $s \in W$  and  $t \in \overline{W}$ . Observe that  $[s, t] \subseteq \delta(W)$ . Let us denote by  $a^T x \geq \alpha$  the  $st$ -cut inequality induced by  $W$  and let  $\mathcal{F}_a = \{x \in kHPP(G) \mid a^T x = \alpha\}$ . We first show that  $\mathcal{F}_a$  is a proper face of  $kHPP(G)$ . As  $|V| \geq k + 2$ , there exist  $W_1 \subseteq W \setminus \{s\}$  and  $W_2 \subseteq \overline{W} \setminus \{t\}$  such that  $|W_1| + |W_2| = k$ . Note that  $W_1$  and  $W_2$  may be empty but not both. Let  $F_1 = \{sv, v \in W_2\} \cup \{ut, u \in W_1\}$  and  $E_1 = F_1 \cup E_0$  where  $E_0 = E(W) \cup E(\overline{W})$ . It is not hard to see that  $E_1$  is a solution of the  $kHPP$  whose incidence vector satisfies  $a^T x \geq \alpha$  with equality. Hence,  $\mathcal{F}_a \neq \emptyset$  and, therefore, is a proper face of  $kHPP(G)$ .

Now suppose that there exists a facet defining inequality  $b^T x \geq \beta$  such that  $\mathcal{F}_a \subseteq \{x \in kHPP(G) \mid b^T x = \beta\}$ . We will show that there exists a scalar  $\rho$  such that  $b = \rho a$ .

Consider an edge  $e \in F_1$ . Clearly, the edge set  $E_2 = (E_1 \setminus \{e\}) \cup \{st\}$  is a solution of the  $kHPP$  and its incidence vector satisfies  $a^T x \geq \alpha$  with equality. It then follows that  $b^T x^{E_2} = b^T x^{E_1} - b(e) + b(st)$ . Since  $x^{E_1} \in \mathcal{F}_a$ , we obtain that  $b(e) = b(st)$ . As  $e$  is arbitrary in  $F_1$ , this implies that

$$b(e) = b(st) = \rho \quad \text{for all } e \in F_1. \quad (3.1)$$

Now let  $f = uv \in \delta(W) \setminus F_1$ , with  $u \in W \setminus \{s\}$  and  $v \in \overline{W} \setminus \{t\}$ . If  $u \in W_1$  and  $v \in W_2$ , then let  $E_3 = (E_1 \setminus \{sv, ut\}) \cup \{f, st\}$ . Clearly,  $E_3$  is a solution of the  $kHPP$  and its incidence vector satisfies  $a^T x \geq \alpha$  with equality. Hence, we have that  $b^T x^{E_3} = b^T x^{E_1}$ . This implies that  $b(sv) + b(ut) = b(f) + b(st)$ . From (3.1), it follows that  $b(f) = \rho$ .

If  $u \in W_1 \cup \{s\}$  (resp.  $u \in W \setminus (W_1 \cup \{s\})$ ) and  $v \in \overline{W} \setminus (W_2 \cup \{t\})$  (resp.  $v \in W_2 \cup \{t\}$ ), by considering the edge set  $E_4 = (E_1 \setminus \{ut\}) \cup \{f\}$  (resp.  $E_4 = (E_1 \setminus \{sv\}) \cup \{f\}$ ), we similarly obtain that  $b(f) = \rho$ .

If  $u \notin W_1$  and  $v \notin W_2$ , then one can consider the solution  $E_5 = (E_1 \setminus \{e\}) \cup \{f\}$ , where  $e$  is an edge of  $F_1$ , and obtain along the same lines that  $b(f) = \rho$ .

Thus, together with (3.1), this yields

$$b(e) = \rho \quad \text{for all } e \in \delta(W).$$

Now let  $e \in E_0$ , and suppose, w.l.o.g., that  $e \in E(W)$ . If  $e$  does not belong to a 3- $st$ -path of  $E_1$ , then the edge set  $E_6 = E_1 \setminus \{e\}$  also induces a solution of the  $kHPP$  and satisfies  $a^T x \geq \alpha$  with equality. We then have that  $b^T x^{E_6} = b^T x^{E_1}$  implying  $b(e) = 0$ .



If  $e$  belongs to a 3- $st$ -path of  $E_1$ , say  $(su, ut)$ , then the edge set  $E_7 = (E_1 \setminus \{su, ut\}) \cup \{st\}$  induces a solution of the  $k$ HPP and its incidence vector satisfies  $a^T x \geq \alpha$  with equality. It then follows that  $b^T x^{E_7} = b^T x^{E_1}$  and hence  $b(st) = b(su) + b(ut)$ . As by (3.1),  $b(ut) = b(st)$ , we get  $b(e) = 0$ .

Consequently, we have that

$$b(e) = \begin{cases} \rho & \text{for all } e \in \delta(W), \\ 0 & \text{if not.} \end{cases}$$

Thus,  $b = \rho a$  with  $\rho \in \mathbb{R}$ , and the result follows.  $\square$

**Theorem 3.4.** An inequality (2.3), induced by a partition  $\pi = (V_0, \dots, V_4)$  with  $s \in V_0$  and  $t \in V_4$ , defines a facet of  $k$ HPP( $G$ ), different from a trivial inequality, if and only if

- (1)  $|V_0| = |V_4| = 1$ ;
- (2)  $||[s, V_1]| + |[V_3, t]| + |[s, t]| \geq k + 1$ .

**Proof.** Let  $T$  be the 3-path-cut induced by  $\pi$ . Let  $a^T x \geq \alpha$  denote the 3-path-cut inequality produced by  $T$  and  $\mathcal{F} = \{x \in k$ HPP( $G$ ) |  $a^T x = \alpha\}$ .

*Necessity.* (1) We will show that if  $|V_0| \geq 2$ , inequality  $x(T) \geq k$  does not define a facet. The case where  $|V_4| \geq 2$  follows by symmetry. Suppose that  $|V_0| \geq 2$  and consider the partition  $\pi' = (V'_0, \dots, V'_4)$  given by

$$\begin{aligned} V'_0 &= \{s\}, \\ V'_1 &= V_1 \cup (V_0 \setminus \{s\}), \\ V'_i &= V_i, \quad i = 2, 3, 4. \end{aligned}$$

The partition  $\pi'$  produces a 3-path-cut inequality  $x(T') \geq k$ , where  $T' = T \setminus [V_0 \setminus \{s\}, V_2]$ . Since  $G$  is complete,  $[V_0 \setminus \{s\}, V_2] \neq \emptyset$  and  $T'$  is strictly contained in  $T$ . Thus,  $x(T) \geq k$  is redundant with respect to the inequalities

$$\begin{aligned} x(T') &\geq k, \\ x(e) &\geq 0 \quad \text{for all } e \in [V_0 \setminus \{s\}, V_2], \end{aligned}$$

and hence cannot define a facet of  $k$ HPP( $G$ ).

(2) Suppose that condition (1) holds. Let  $A = [s, V_1] \cup [V_3, t] \cup [s, t]$  and let  $u_i$  be a fixed node of  $V_i$ ,  $i = 1, 2, 3$ . Let us suppose that  $\mathcal{F}$  is a facet of  $k$ HPP( $G$ ) different from a trivial inequality. Thus there exists a solution  $F$  of the  $k$ HPP such that  $x^F \in \mathcal{F}$  and  $F \cap [V_1, V_3] \neq \emptyset$ . If this is not the case, then  $\mathcal{F}$  would be equivalent to a facet defined by any of the inequalities  $x(e) \geq 0$ ,  $e \in [V_1, V_3]$ . Note that, since each 3- $st$ -path of  $F$  intersects  $T$  at least once and  $|F \cap T| = k$ ,  $F$  necessarily contains exactly  $k$  edge-disjoint 3- $st$ -paths. Moreover, each of these paths intersects  $T$  only once. This implies that every 3- $st$ -path of  $F$  is of the form

- (i)  $(su_1, u_1u_2, u_2t), (su_2, u_2u_3, u_3t), (su_1, u_1t), (su_3, u_3t), (st)$  or
- (ii)  $(su_1, u_1u_3, u_3t)$ .

If  $P$  is one of these  $st$ -paths, then  $|P \cap A| = 1$  (resp.  $|P \cap A| = 2$ ) if  $P$  is of type (i) (resp. (ii)). As  $F \cap [V_1, V_3] \neq \emptyset$ , it follows that  $F$  contains at least one path of type (ii) and therefore  $|F \cap A| \geq k + 1$ .

*Sufficiency.* Suppose that conditions (1) and (2) hold. First we show that  $\mathcal{F} \neq \emptyset$ . As  $||[s, V_1]| + |[V_3, t]| + |[s, t]| \geq k + 1$ , there exist node sets  $U_1 \subseteq V_1$ ,  $U_3 \subseteq V_3$ , and an edge set  $E_0 \subseteq [s, t] \setminus \{st\}$  such that  $|U_1| + |U_3| + |E_0| = k$ . We suppose, w.l.o.g., that  $u_1 \in U_1$  and  $u_3 \in U_3$ . Consider the  $st$ -paths  $(su, ut)$ ,  $u \in U_1 \cup U_3$  and  $(e)$ ,  $e \in E_0$ . Clearly, these  $st$ -paths form a set of  $k$  edge-disjoint 3- $st$ -paths. Moreover, each of these paths intersects  $T$  only once. Thus they induce a solution, say  $E_1$ , of  $k$ HPP whose incidence vector belongs to  $\mathcal{F}$ . Therefore  $\mathcal{F} \neq \emptyset$ .

Now suppose that there exists a facet defining inequality  $b^T x \geq \beta$  such that  $\mathcal{F} \subseteq \{x \in k$ HPP( $G$ ) |  $b^T x = \beta\}$ . As before, we will show that there exists a scalar  $\rho \neq 0$  such that  $b = \rho a$ .

Let  $e \in E_1 \cap T$  (where  $E_1$  is the solution introduced above). Let  $E_2 = (E_1 \setminus \{e\}) \cup \{st\}$ . Since  $E_2$  is a solution of the  $k$ HPP whose incidence vector belongs to  $\mathcal{F}$ , we have  $b^T x^{E_2} = b^T x^{E_1} = \beta$ , implying that  $b(e) = b(st)$ . As  $e$  is an arbitrary edge, we then obtain that

$$b(e) = \rho \quad \text{for all } e \in (E_1 \cap T) \cup \{st\}, \quad \text{for some } \rho \in \mathbb{R}. \quad (3.2)$$

Now let  $e \in E \setminus T$ . If  $e \notin E_1$ , then let  $E_3 = E_1 \cup \{e\}$  is a solution of  $k$ HPP. Moreover, its incidence vector belongs to  $\mathcal{F}$ . Hence,  $b(e) = b^T x^{E_3} - b^T x^{E_1} = 0$ . If  $e \in E_1 \setminus T$ , then  $e$  is either of the form  $su$ ,  $u \in U_1$ , or  $vt$ ,  $v \in V_3$ . Suppose, w.l.o.g., that  $e = su$ , the case where  $e = vt$  is similar. Note that, by the definition of  $E_1$ ,  $ut$  also belongs to  $E_1$ . Let  $E'_3 = (E_1 \setminus \{su, ut\}) \cup \{st\}$ . We have that  $E'_3$  induces the  $k$ HPP and  $x^{E'_3} \in \mathcal{F}$ . Hence,  $b^T x^{E'_3} = b^T x^{E_1} = \beta$  and, in consequence,  $b(su) + b(ut) = b(st)$ . As, by (3.2),  $b(ut) = b(st)$ , we have that  $b(su) = 0$ . Thus, we obtain that

$$b(e) = 0 \quad \text{for all } e \in E \setminus T. \quad (3.3)$$

Consider now an edge  $e \in T \setminus E_1$ . If  $e \in [s, t] \setminus \{st\}$ , then clearly, the edge set  $(E_1 \setminus \{g\}) \cup \{e\}$  induces a solution of the  $k$ HPP and its incidence vector belongs to  $\mathcal{F}$  where  $g$  is an edge of  $E_1$ . Hence, as before,  $b(e) = b(g) = \rho$ .

Now if  $e = sv$  (resp.  $e = vt$ ) with  $v \in V_2$ , then the edge set  $E_4 = (E_1 \setminus \{su_3\}) \cup \{e, vu_3\}$  (resp.  $E_4 = (E_1 \setminus \{u_1t\}) \cup \{u_1v, e\}$ ) induces a solution of the kHPP. Moreover, its incidence vector belongs to  $\mathcal{F}$ . Thus,  $b^T x^{E_4} - b^T x^{E_1} = b(e) + b(vu_3) - b(su_3) = 0$  (resp.  $b^T x^{E_4} - b^T x^{E_1} = b(u_1v) + b(e) - b(u_1t) = 0$ ). From (3.2) and (3.3) we get  $b(e) = \rho$ .

Let  $e = sv$  with  $v \in V_3$ . The case where  $e \in [V_1, t]$  is similar. If  $v \in U_3$ , then the edge set  $E_5 = (E_1 \setminus \{f\}) \cup \{e\}$ , where  $f$  is the edge of  $E_1$  between  $s$  and  $v$ , induces a solution of the kHPP whose incidence vector belongs to  $\mathcal{F}$ . Hence  $b^T x^{E_5} - b^T x^{E_1} = b(e) - b(su_3) = 0$ . By (3.2), we get  $b(e) = \rho$ . If  $v \notin U_3$ , then we have that  $E'_5 = (E_1 \setminus \{f'\}) \cup \{e, vt\}$ , where  $f' \in E_1 \cap [s, U_3]$ , also induces a solution of the kHPP and its incidence vector belongs to  $\mathcal{F}$ . Thus,  $b^T x^{E'_5} - b^T x^{E_1} = b(e) + b(u_3t) - b(f) = 0$ . By (3.2) and (3.3), we get  $b(e) = \rho$ .

Now suppose that  $e = uv \in [V_1, V_3]$ . If  $u \in U_1$  and  $v \in U_3$ , then by considering the edge set  $E_6 = (E_1 \setminus \{ut, sv\}) \cup \{e, st\}$ , we get  $b(e) + b(st) = b(sv) + b(ut)$ . From (3.2) and (3.3), we have that  $b(e) = \rho$ . If  $u \notin U_1$  and  $v \in U_3$ , then by considering the edge set  $E_7 = (E_1 \setminus \{g\}) \cup \{su, e\}$ , where  $g$  is the edge of  $E_1$  between  $s$  and  $v$ , we get  $b(e) + b(su) = b(g)$ . By (3.2) and (3.3), we have  $b(e) = \rho$ . If  $u \in U_1$  and  $v \notin U_3$ , then we show in a similar way that  $b(e) = \rho$ . If  $u \notin U_1$  and  $v \notin U_3$ , then by considering the edge set  $E_8 = (E_1 \setminus \{st\}) \cup \{su, e, vt\}$ , we get  $b(e) = \rho$ . Thus, we obtain

$$b(e) = \rho \quad \text{for all } e \in T \setminus (E_1 \cup \{st\}). \quad (3.4)$$

From (3.2)–(3.4), we have

$$b(e) = \begin{cases} \rho & \text{for all } e \in T, \\ 0 & \text{if not.} \end{cases}$$

Therefore,  $b = \rho a$ . Moreover  $\rho \neq 0$  since  $bx \geq \beta$  defines a facet which ends the proof of the theorem.  $\square$

As it will turn out in the next section, the conditions given for inequalities (2.1)–(2.3) to define facets will be useful for characterizing the kHPP polytope.

#### 4. Complete description of kHPP(G)

In this section, we will present our main result, that is the polytope  $P_k(G)$ , given by the  $st$ -cut, the 3-path-cut and the trivial inequalities, is integral, which implies that kHPP(G) is completely described by these inequalities. We will first give some known results related to disjoint paths in directed graphs that will be useful to prove our result. Then we will show that the kHPP can be transformed to finding  $k$  disjoint paths in an appropriate directed graph, which will be used for proving our main result.

##### 4.1. Disjoint $st$ -paths in directed graphs

Given a directed graph  $D = (V, A)$ , two nodes  $s, t \in V$ , an integer  $k \geq 2$  and a weight function  $c(\cdot)$  on the arcs of  $D$ , the  $k$  arc-disjoint  $st$ -paths problem (kADPP for short) consists in finding a minimum weight subgraph of  $D$  which contains at least  $k$  arc-disjoint paths from  $s$  to  $t$ . Let  $\text{kADPP}(D)$  be the convex hull of the solutions of the kADPP on  $D$ .

If  $B$  is an arc subset of  $A$  which induces a solution of the kADPP, then its incidence vector  $x^B$  satisfies the following inequalities:

$$x(\delta^{\text{out}}(W)) \geq k, \quad \text{for all } W \subseteq V, s \in W \text{ and } t \in \overline{W}, \quad (4.1)$$

$$0 \leq x(a) \leq 1, \quad \text{for all } a \in A. \quad (4.2)$$

Conversely, by Menger's theorem [32], any integral solution of the system given by inequalities (4.1) and (4.2) induces a solution of the kADPP. Inequalities (4.1) are called  $st$ -dicut inequalities and constraints (4.2) are called trivial inequalities. Thus, the kADPP is equivalent to

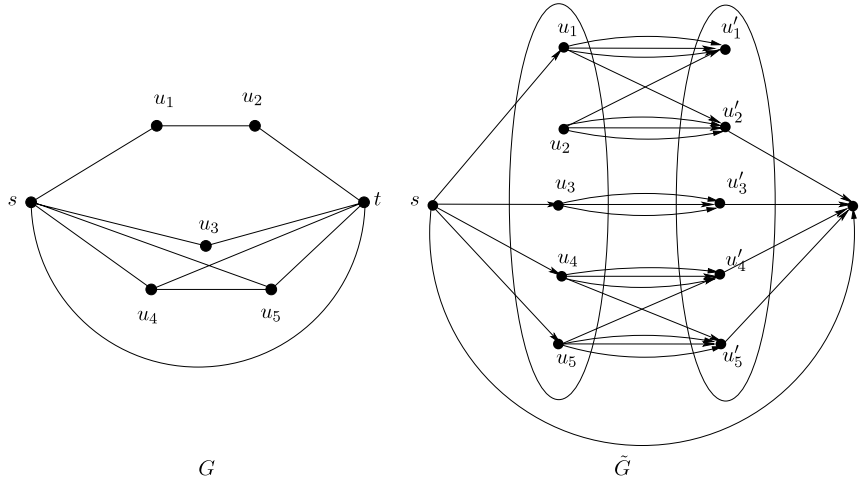
$$\min\{cx \mid x \text{ satisfies (4.1), (4.2), } x \in \{0, 1\}^A\}.$$

The following theorem shows that the  $st$ -dicut and the trivial inequalities suffice to describe the polytope  $\text{kADPP}(G)$ .

**Theorem 4.1** ([33]). *The polytope  $\text{kADPP}(G)$  is completely described by inequalities (4.1) and (4.2).*

The following theorem indicates that two node subsets  $W_1$  and  $W_2$  of  $V$  that induce tight  $st$ -dicut inequalities for a solution  $y \in \text{kADPP}(D)$ , can be seen as embedded node sets. This comes from the fact that the sets inducing  $st$ -dicuts in a graph form a laminar family. A family  $\mathcal{F} \subseteq 2^V$  of node sets is said to be laminar if and only if for every  $V_i, V_j \in \mathcal{F}$ , either  $V_i \cap V_j = \emptyset$  or  $V_i \subset V_j$  or  $V_j \subset V_i$ . For more details on laminar families see [34,33].

**Theorem 4.2** ([33]). *Let  $W_1$  and  $W_2$  be two node subsets of  $V$  that induce  $st$ -dicuts of  $D$  such that  $W_1 \cap W_2 \neq \emptyset \neq (V \setminus W_1) \cap W_2$ . If the  $st$ -dicut inequalities, induced by  $W_1$  and  $W_2$ , are tight for a solution  $x$  of  $\text{kADPP}(G)$ , then there exists a node set different from  $W_1$  and  $W_2$  contained either in  $W_1$  or in  $W_1 \cup W_2$  which induces a tight  $st$ -dicut inequality for  $x$ .*

Fig. 2. Construction of  $\tilde{G}$ .

#### 4.2. Graph transformation

One of the approaches that may be considered when studying hop-constrained network design problems is to transform the problem into an unconstrained version in an appropriate graph. This idea has already been used by Gouveia et al. [13,14] who transform the original undirected graph into an extended undirected graph, and introduce a hop-indexed formulation for the problem. For our purpose, we will introduce a transformation of the initial graph into a directed layered graph.

Consider an undirected graph  $G = (V, E)$ . Let  $N = V \setminus \{s, t\}$ ,  $N'$  be a copy of  $N$  and  $\tilde{V} = N \cup N' \cup \{s, t\}$ . The copy in  $N'$  of a node  $u \in N$  will be denoted by  $u'$ . Let  $\tilde{G} = (\tilde{V}, \tilde{A})$  be the directed graph such that  $\tilde{V} = N \cup N' \cup \{s, t\}$  and  $\tilde{A}$  is obtained from  $G$  as follows. To each edge  $e \in [s, t]$ , we associate an arc from  $s$  to  $t$  in  $\tilde{G}$ . To each edge  $su \in E$  (resp.  $vt \in E$ ), we associate in  $\tilde{G}$  the arc  $(s, u)$ ,  $u \in N$  (resp.  $(v', t)$ ,  $v' \in N'$ ). To each edge  $uv \in E$ , with  $u, v \notin \{s, t\}$ , we associate two arcs  $(u, v')$  and  $(v, u')$ , with  $u, v \in N$  and  $u', v' \in N'$ . Finally, to each node  $u \in V \setminus \{s, t\}$ , we associate in  $\tilde{G}$   $k$  arcs  $(u, u')$  (see Fig. 2 for an illustration for  $k = 3$ ).

Note that any  $st$ -dipath in  $\tilde{G}$  is of length no more than 3. Also note that each 3- $st$ -path in  $G$  corresponds to an  $st$ -dipath in  $\tilde{G}$  and vice versa. In fact, a 3- $st$ -path  $\Gamma = (s, u, v, t)$ , with  $u \neq v$ ,  $u, v \notin \{s, t\}$ , corresponds to an  $st$ -path in  $\tilde{G}$  of the form  $(s, u, v', t)$  with  $u \in N$  and  $v' \in N'$ , and a 3- $st$ -path  $L = (s, u, t)$ ,  $u \notin \{s, t\}$  corresponds to an  $st$ -path in  $\tilde{G}$  of the form  $(s, u, u', t)$ .

The main idea of the proof is to show that each solution of  $P_k(G)$  corresponds to a solution of  $kADPP(\tilde{G})$  and vice versa. We will use this correspondence together with Theorem 4.1 to obtain the proof.

Given a solution  $\bar{x}$  of  $\mathbb{R}^E$ , we let  $\bar{y}$  be the solution of  $\mathbb{R}^{\tilde{A}}$  obtained from  $\bar{x}$  as follows.

$$\bar{y}(a) = \begin{cases} \bar{x}(su) & \text{if } a = (s, u), u \in N, \\ \bar{x}(vt) & \text{if } a = (v', t), v' \in N', \\ \bar{x}(uv) & \text{if } a \in \{(u, v'), (v, u')\}, u, v \in N, u' \neq v, u' \neq v', \\ \bar{x}(st) & \text{if } a = (s, t), \\ 1 & \text{if } a = (u, u'), u \in N, u' \in N'. \end{cases}$$

We will say that the solutions  $\bar{x}$  and  $\bar{y}$  are associated.

In what follows we will show that any  $st$ -cut and 3-path-cut of  $G$  corresponds to an  $st$ -dicut in  $\tilde{G}$ . Indeed, let us consider an edge set  $C \subseteq E$  and an arc set  $\tilde{C} \subseteq \tilde{A}$  obtained from  $C$  as follows.

- (i) For an edge  $st \in C$ , add  $(s, t)$  in  $\tilde{C}$ ;
- (ii) for an edge  $su \in C$ , add  $(s, u)$  in  $\tilde{C}$ ,  $u \in N$ ;
- (iii) for an edge  $vt \in C$ , add  $(v', t)$  in  $\tilde{C}$ ,  $v' \in N'$ ;
- (iv) for an edge  $uv \in C$ ,  $u \neq v$ ,  $u, v \in N$ ,
  - (iv.a) if  $su \in C$  or  $vt \in C$ , then add  $(v, u')$  in  $\tilde{C}$ , with  $v \in N$  and  $u' \in N'$ ;
  - (iv.b) if  $su \notin C$  and  $vt \notin C$ , then add  $(u, v')$  in  $\tilde{C}$ .

Observe that  $\tilde{C}$  does not contain any arc of the form  $(u, u')$  with  $u \in N$  and  $u' \in N'$ . Also note that  $\tilde{C}$  does not contain at the same time two arcs  $(u, v')$  and  $(v, u')$ , for an edge  $uv \in E$  with  $u, v \in V \setminus \{s, t\}$ .

Conversely, an arc subset  $\tilde{C}$  of  $\tilde{A}$  can be obtained from an edge set  $C \subseteq E$  if  $\tilde{C}$  does not contain simultaneously two arcs  $(u, v')$  and  $(v, u')$ ,  $u, v \in N$ ,  $u', v' \in N'$ , and does not contain any arc of the form  $(u, u')$  with  $u \in N$ ,  $u' \in N'$ .

As each arc of  $C$  corresponds to a single arc of  $\tilde{C}$  and vice versa, both sets have the same weight, that is  $\bar{x}(C) = \bar{y}(\tilde{C})$ .



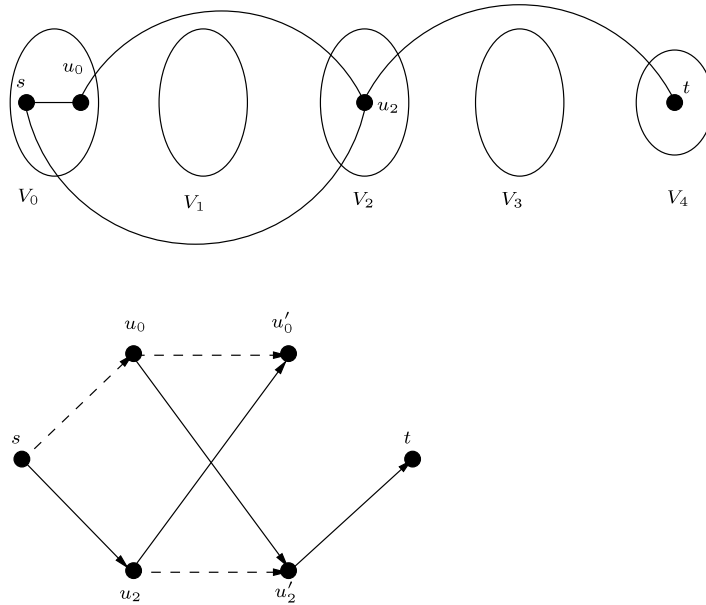


Fig. 3. A 3-path-cut in  $G$  which does not induce an  $st$ -cut in  $\tilde{G}$ .

**Lemma 4.1.** Let  $C \subseteq E$  be an edge set of  $G$  which is an  $st$ -cut or a 3-path-cut induced by a partition  $(V_0, \dots, V_4)$  such that  $|V_0| = |V_4| = 1$ . Then the arc set obtained from  $C$  by the procedure given above is an  $st$ -dicut of  $\tilde{G}$ . Moreover,  $\bar{x}(C) = \bar{y}(\tilde{C})$ .

**Proof.** Suppose first that  $C$  is an  $st$ -cut  $\delta(W)$  for some  $W \subseteq V$  with  $s \in W$  and  $t \in \bar{W}$ . Let  $\tilde{W} \subseteq \tilde{V}$  such that  $\tilde{W} = W \cup \{u' \mid u \in W \setminus \{s\}\}$ . We will show that  $\tilde{C} = \delta^{out}(\tilde{W})$ . We first show that  $\tilde{C} \subseteq \delta^{out}(\tilde{W})$ . Observe that any arc  $f$  of  $\tilde{C}$  is of the form  $(s, t)$ ,  $(s, u)$ ,  $u \neq t$ ,  $(v', t)$ ,  $(u, v')$  or  $(v, u')$ ,  $u, v \in N$ ,  $u', v' \in N'$ . If  $f = (s, u) \in \tilde{C}$ , with  $u \in N \cup \{t\}$ , then  $su \in C$ . Thus,  $u \in W$  and therefore,  $(s, u) \in \delta^{out}(\tilde{W})$ .

If  $f = (v', t)$  for  $v' \in N'$ , this implies that  $vt \in C$ . Thus,  $v \in W$  and hence  $(v', t) \in \delta^{out}(\tilde{W})$ .

If  $f = (v, u')$  for  $v \in N$ ,  $u' \in N'$ , then by step (iv.a) of the construction of  $\tilde{C}$ , we should have  $su$  and  $vt$  in  $C$ . Hence,  $v \in W$  and  $u \in \bar{W}$ . Therefore,  $v \in \tilde{W}$  and  $u' \in \tilde{V} \setminus \tilde{W}$ . Hence  $(v, u') \in \delta^{out}(\tilde{W})$ . If  $f = (u, v')$ , it similarly follows that  $f \in \delta^{out}(\tilde{W})$ . Consequently, we have that  $\tilde{C} \subseteq \delta^{out}(\tilde{W})$ .

Next, we show that  $\delta^{out}(\tilde{W}) \subseteq \tilde{C}$ . Let  $g$  be an arc of  $\delta^{out}(\tilde{W})$ . If  $g = (s, u)$  for  $u \in N$ , then  $u \in \tilde{V} \setminus \tilde{W}$  and hence  $su \in \delta(W) (= C)$ . This implies that  $(s, u) \in \tilde{C}$ .

If  $g = (v', t)$  for  $v' \in N'$ , then  $v'$  and hence  $v$  belong to  $\tilde{W}$ . Thus,  $vt \in \delta(W)$  and therefore  $(v', t) \in \tilde{C}$ . If  $g = (v, u')$  with  $v \in N$  and  $u' \in N'$ , then  $v \in \tilde{W}$ , and  $u, u' \in \tilde{V} \setminus \tilde{W}$ . This implies that  $v \in W$  and  $u \in \bar{W}$ . In consequence,  $su \in \delta(W)$  and  $vt \in \delta(W)$ , and thus  $(v, u') \in \tilde{C}$ .

If  $g = (u, v')$  with  $u \in N$  and  $v' \in N'$ , we similarly show that  $g \in \tilde{C}$ .

We thus obtain that  $\delta^{out}(\tilde{W}) \subseteq \tilde{C}$ , and hence  $\delta^{out}(\tilde{W}) = \tilde{C}$ .

Now suppose that  $C$  is a 3-path-cut induced by a partition  $(V_0, V_1, V_2, V_3, V_4)$  such that  $V_0 = \{s\}$  and  $V_4 = \{t\}$ . By considering  $\tilde{W} = V_1 \cup \{u' \mid u \in V_1 \cup V_2\}$ , we can show as before that  $\tilde{C} = \delta^{out}(\tilde{W})$ .  $\square$

Note that for an edge set  $C$  which is a 3-path-cut of  $G$ , induced by a partition  $(V_0, \dots, V_4)$  such that  $|V_0| \geq 2$  or  $|V_4| \geq 2$ , the corresponding arc set  $\tilde{C}$  may not be an  $st$ -dicut of  $\tilde{G}$ . In fact,  $\tilde{C}$  may simultaneously contain two arcs  $(s, u)$ ,  $(u, v')$  or  $(u, v')$ ,  $(v', t)$ . In the example of Fig. 3,  $\tilde{C}$  simultaneously contains the arcs  $(s, u_2)$  and  $(u_2, u'_0)$ . If there exists a node subset  $\tilde{W} \subseteq \tilde{V}$  such that  $\tilde{C} = \delta^{out}(\tilde{W})$ , we would have  $u_2 \in \tilde{W}$  and  $u_2 \in \tilde{V} \setminus \tilde{W}$ , a contradiction.

Also note that by Theorem 3.4, the  $L$ -path-cut inequalities induced by such partitions do not define facets of  $kHPP(G)$ .

The following lemma shows that an  $st$ -cut in  $\tilde{G}$  which does not contain any arc of the form  $(u, u')$ ,  $u \in V \setminus \{s, t\}$  corresponds to either an  $st$ -cut or a 3-path-cut in  $G$  with a lower weight.

**Lemma 4.2.** Let  $\tilde{C}$  be an  $st$ -dicut of  $\tilde{G}$  such that  $\tilde{C}$  does not contain an arc of the form  $(u, u')$ ,  $u \in V \setminus \{s, t\}$ . Then there exists an  $st$ -cut or a 3-path-cut  $C \subseteq E$  in  $G$  such that  $\bar{x}(C) \leq \bar{y}(\tilde{C})$ .

**Proof.** Let  $\tilde{C} = \delta^{out}(\tilde{W})$  with  $\tilde{W} \subset \tilde{V}$ . Since  $\tilde{C}$  does not contain any arc of the form  $(u, u')$ ,  $u \in N$ ,  $\tilde{C}$  may contain arcs of the form either  $(u, v')$  or  $(v, u')$  or none of them but not both.

If  $\tilde{C}$  contains an arc of the form  $(u, v')$  with  $u \in N$ ,  $v' \in N'$ , since  $\tilde{C}$  is an  $st$ -dicut in  $\tilde{G}$ , the arcs  $(s, u)$  and  $(v', t)$  are not in  $\tilde{C}$ . If  $\tilde{C}$  contains an arc  $(v, u')$ , as  $\tilde{C}$  does not contain arcs of the form  $(z, z')$ ,  $z \in N$ , we should have  $u \in \tilde{V} \setminus \tilde{W}$  and  $v' \in \tilde{W}$ . Hence  $(s, u)$  and  $(v', t)$  are in  $\tilde{C}$ . Therefore  $\tilde{C}$  can be obtained from an edge set  $C \subseteq E$  of  $G$ . Moreover  $\bar{x}(C) = \bar{y}(\tilde{C})$ .

Furthermore,  $C$  intersects all the 3- $st$ -paths of  $G$ . In fact, if there exists a 3- $st$ -path  $\Gamma = (su, uv, vt)$  which does not intersect  $C$ , then the arcs  $(s, u)$ ,  $(u, v')$ ,  $(v, u')$  and  $(v', t)$  of  $\tilde{G}$  are not in  $\tilde{C}$ . Thus, the  $st$ -path  $((s, u), (u, v'), (v', t))$  of  $\tilde{G}$  does not intersect  $\tilde{C}$ , contradicting the fact that  $\tilde{C}$  is an  $st$ -dicut of  $\tilde{G}$ . Thus  $C$  intersects all the 3- $st$ -paths of  $G$ .

If  $C$  is an  $st$ -cut then the result holds. If this is not the case, then we will show that there exists a 3-path-cut  $T$  such that  $T \subseteq C$ . Consider the graph  $G'$  obtained from  $G$  by deleting all the edges of  $C$ .  $G'$  does not contain any 3- $st$ -path since  $C$  intersects all these paths. Let  $\pi = (V_0, \dots, V_4)$  be a partition of  $V$  in  $G'$  such that  $V_0 = \{s\}$ ,  $V_i$ , for  $i = 1, 2, 3$ , is the set of nodes of  $G'$  at distance (in terms of edges)  $i$  from  $s$  and  $V_4 = V \setminus (\bigcup_{i=0}^3 V_i)$ . As  $C$  intersects all the 3- $st$ -paths of  $G$ , all the  $st$ -paths in  $G'$  are of length at least 4 and hence,  $t \in V_4$ . Moreover, the subgraph  $G'_\pi$  induced by  $\pi$  in  $G'$  does not contain any chord, that is an edge  $uv$  with  $u \in V_i$ ,  $v \in V_j$ , and  $|i - j| > 1$ . In fact, if  $uv$  is a chord, then  $v$  is at distance  $i + 1 < j$  of  $s$ , a contradiction. Therefore, if  $T$  is the 3-path-cut induced by  $\pi$ , we have that  $T \subseteq C$ . As  $\bar{x}(e) \geq 0$ , for all  $e \in E$ , this implies  $\bar{x}(T) \leq \bar{x}(C) = \bar{y}(\tilde{C})$ .  $\square$

#### 4.3. The polytope

In what follows, we will show that  $P_k(G)$  is integral. To this end, we first give some lemmas.

**Lemma 4.3.** Let  $\bar{x} \in P_k(G)$  and  $\bar{y}$  its associated solution. Then  $\bar{y} \in kADPP(\tilde{G})$ .

**Proof.** Clearly,  $\bar{y}$  satisfies inequalities  $0 \leq y(a) \leq 1$ , for all  $a \in \tilde{A}$ . Now suppose that there exists an  $st$ -dicut inequality, say  $y(\delta^{out}(\tilde{W})) \geq k$  with  $\tilde{W} \subseteq \tilde{V}$ , such that  $\bar{y}(\delta^{out}(\tilde{W})) < k$ .

First note that  $\delta^{out}(\tilde{W})$  does not contain any arc of the form  $(u, u')$ ,  $u \in N$ . In fact, if  $(u, u') \in \delta^{out}(\tilde{W})$ , for some  $u \in N$ , then one would have that  $[u, u'] \subseteq \delta^{out}(\tilde{W})$ . Since  $|[u, u']| = k$  and  $\bar{y}(a) = 1$  for all  $a \in [u, u']$ , one would have  $\bar{y}(\delta^{out}(\tilde{W})) \geq k$ , a contradiction. Hence, from Lemma 4.2, there exists either an  $st$ -cut or a 3-path-cut  $C \subseteq E$  of  $G$  such that  $\bar{x}(C) \leq \bar{y}(\delta^{out}(\tilde{W}))$  and therefore  $\bar{x}(C) < k$ . But this is impossible since  $\bar{x} \in P_k(G)$ .  $\square$

**Lemma 4.4.** Let  $e = uv$  be an edge of  $G$  such that  $u, v \in V \setminus \{s, t\}$ , and  $\bar{y} \in \mathbb{R}^{\tilde{A}}$  a solution of  $kADPP(\tilde{G})$ . If there exists an  $st$ -dicut  $\tilde{C}$  of  $\tilde{G}$  which does not contain any arc of the form  $(z, z')$ ,  $z \in V \setminus \{s, t\}$ , and such that  $(u, v') \in \tilde{C}$  and  $\bar{y}(\tilde{C}) = k$ , then  $\bar{y}(\tilde{C}') > k$  for all  $st$ -dicut  $\tilde{C}'$  of  $\tilde{G}$  containing the arc  $(v, u')$ .

**Proof.** Suppose that there exists an  $st$ -dicut  $\tilde{C} = \delta^{out}(\tilde{W})$  of  $\tilde{G}$  which does not contain arcs of the form  $(z, z')$ ,  $z \in V \setminus \{s, t\}$  and such that  $(u, v') \in \tilde{C}$  and  $\bar{y}(\tilde{C}) = k$ . Suppose also, on the contrary, that there exists an  $st$ -dicut  $\tilde{C}' = \delta^{out}(\tilde{W}')$  containing the arc  $(v, u')$  and such that  $\bar{y}(\tilde{C}') = k$ . From Theorem 4.2,  $\tilde{W}$  and  $\tilde{W}'$  can be chosen so that either  $\tilde{W}' \subseteq \tilde{W}$  or  $\tilde{W} \subseteq \tilde{W}'$ . As  $(u, v') \in \tilde{C}$ , we have that  $u \in \tilde{W}$  and  $v' \in \tilde{V} \setminus \tilde{W}$ . Since  $(z, z') \notin \tilde{C}$ , for all  $z \in V \setminus \{s, t\}$ , it follows that  $u, u' \in \tilde{W}$ , and  $v, v' \in \tilde{V} \setminus \tilde{W}$ . Similarly, as  $(v, u') \in \tilde{C}'$ , we have that  $v, v' \in \tilde{W}'$  and  $u, u' \in \tilde{V} \setminus \tilde{W}'$ .

If  $\tilde{W}' \subseteq \tilde{W}$ , then one would have  $v \in \tilde{W}$ . But this contradicts the fact that  $v \in \tilde{V} \setminus \tilde{W}$ . If  $\tilde{W} \subseteq \tilde{W}'$ , then we would obtain that  $u \in \tilde{W}'$ . As  $u \in \tilde{V} \setminus \tilde{W}'$ , this is a contradiction.  $\square$

Now we are ready to state our main result.

**Theorem 4.3.** The polytope  $kHPP(G)$  is completely described by inequalities (2.1)–(2.3).

**Proof.** We will show that the polytope  $P_k(G)$  is integral. For this, let us suppose, on the contrary, that there exists a fractional extreme point  $\bar{x}$  of  $P_k(G)$ . Then there exists a set of  $st$ -cuts  $C^*(\bar{x})$  and a set of 3-path-cuts  $T^*(\bar{x})$  such that  $\bar{x}$  is the unique solution of the system

$$S(\bar{x}) \begin{cases} x(e) = 0, & \text{for all } e \in E_0(\bar{x}), \\ x(e) = 1, & \text{for all } e \in E_1(\bar{x}), \\ x(C) = k, & \text{for all } C \in C^*(\bar{x}), \\ x(T) = k, & \text{for all } T \in T^*(\bar{x}), \end{cases}$$

where  $E_0(\bar{x})$  (resp.  $E_1(\bar{x})$ ) is the set of edges such that  $\bar{x}(e) = 0$  (resp.  $\bar{x}(e) = 1$ ) and  $|E_0(\bar{x})| + |E_1(\bar{x})| + |C^*(\bar{x})| + |T^*(\bar{x})| = |E|$ .

We will show that there exists a solution  $\bar{x}_1$  of  $P_k(G)$  different from  $\bar{x}$  which is also a solution of  $S(\bar{x})$ , yielding a contradiction.

Clearly, the solution  $\bar{y}$ , associated with  $\bar{x}$ , is fractional and, by Lemma 4.3, is a solution of  $kADPP(\tilde{G})$ . Let  $\tilde{A}_0(\bar{y}) = \{(u, v) \in \tilde{A} \mid \bar{x}(uv) = 0\}$  and  $\tilde{A}_1(\bar{y}) = \{(u, v) \in \tilde{A} \mid \bar{x}(uv) = 1\} \cup \{(u, u'), u \in N, u' \in N'\}$ . By Lemma 4.1, each  $st$ -cut  $C \in C^*(\bar{x})$  and 3-path-cut  $T \in T^*(\bar{x})$  corresponds to an  $st$ -dicut  $\tilde{C}$  of  $\tilde{G}$  having the same weight, that is  $\bar{y}(\tilde{C}) = k$ . We denote by  $C^*(\bar{y})$  the set of the corresponding  $st$ -dicuts. It then follows that  $\bar{y}$  is solution (not necessarily unique) of the system  $S(\bar{y})$  given by

$$S(\bar{y}) \begin{cases} y(a) = 0, & \text{for all } a \in \tilde{A}_0(\bar{y}), \\ y(a) = 1, & \text{for all } a \in \tilde{A}_1(\bar{y}), \\ y(\tilde{C}) = k, & \text{for all } \tilde{C} \in C^*(\bar{y}). \end{cases}$$

Since  $\bar{y}$  is fractional and hence, by Theorem 4.1, cannot be an extreme point of  $kADPP(\tilde{G})$ ,  $\bar{y}$  can be written as a convex combination of integral extreme points of  $kADPP(\tilde{G})$ . Let  $\bar{y}_1$  be one of these extreme points. Clearly,  $\bar{y}_1$  is also a solution of

$S(\bar{y})$ . In the following, we show that there exists an integer solution  $\bar{y}'_1$  of  $kADPP(\tilde{G})$  which is a solution of  $S(\bar{y})$  and such that  $\bar{y}'_1(u, v') = \bar{y}'_1(v, u')$  for all pair of arcs  $((u, v'), (v, u'))$  of  $\tilde{G}$ , corresponding to an edge  $uv \in E$  with  $u, v \in V \setminus \{s, t\}$  and  $u \neq v$ . If such a solution exists, then  $\bar{y}'_1$  can be associated with a solution  $\bar{x}'_1 \in P_k(G)$  satisfying  $S(\bar{x})$  and different from  $\bar{x}$ .

If for all pair of arcs  $((u, v'), (v, u'))$  of  $\tilde{G}$ , with  $u, v \in N, u', v' \in N', \bar{y}_1(u, v') = \bar{y}_1(v, u')$ , then we can take  $\bar{y}'_1 = \bar{y}_1$ . So suppose that there exist two nodes  $u, v \in V \setminus \{s, t\}$ , such that  $uv \in E$  and  $\bar{y}_1(u, v') \neq \bar{y}_1(v, u')$ . As  $\bar{y}_1$  is integral, we can suppose, w.l.o.g., that  $\bar{y}_1(u, v') = 1$  and  $\bar{y}_1(v, u') = 0$ . It follows that  $\bar{y}(u, v'), \bar{y}(v, u')$  and  $\bar{x}(uv)$  are fractional. Note that  $\bar{x}(uv) = \bar{y}(u, v') = \bar{y}(v, u')$ . Also note that any  $st$ -dicut of  $\tilde{G}$  inducing a tight  $st$ -dicut inequality for  $\bar{y}$  or  $\bar{y}_1$  does not contain arcs of the form  $(z, z'), z \in V \setminus \{s, t\}$ . If there is an  $st$ -dicut  $\tilde{C}$  of  $\tilde{G}$  which contains  $(u, v')$ , and such that  $\bar{y}_1(\tilde{C}) = k$ , then, by Lemma 4.4, every  $st$ -dicut containing  $(v, u')$  is not tight for  $\bar{y}_1$ . Let  $\bar{y}'_1$  be the solution given by

$$\bar{y}'_1(a) = \begin{cases} \bar{y}_1(a), & \text{for all } a \in \tilde{A} \setminus \{(u, v')\}, \\ 1, & \text{for } a = (v, u'). \end{cases}$$

Clearly,  $\bar{y}'_1$  is a solution of  $kADPP(\tilde{G})$  with  $\bar{y}'_1(u, v') = \bar{y}'_1(v', u) = 1$ , and satisfies with equality every  $st$ -dicut inequality which is tight for  $\bar{y}_1$ . In particular, the  $st$ -dicuts inequalities of  $\tilde{C}^*(\bar{y})$  are also tight for  $\bar{y}'_1$ . Hence,  $\bar{y}'_1$  is a solution of  $S(\bar{y})$ .

If there is an  $st$ -dicut  $\tilde{C}$  which contains  $(v, u')$  and such that  $\bar{y}_1(\tilde{C}) = k$ , then, by Lemma 4.4, every  $st$ -dicut  $\tilde{R} \subseteq \tilde{A}$  containing  $(u, v')$  is such that  $\bar{y}_1(\tilde{R}) \geq k + 1$ . Hence, the solution  $\bar{y}'_1$  given by

$$\bar{y}'_1(a) = \begin{cases} \bar{y}_1(a), & \text{for all } a \in \tilde{A} \setminus \{(u, v')\}, \\ 0, & \text{for } a = (u, v'), \end{cases}$$

is a solution of  $kADPP(\tilde{G})$  such that  $\bar{y}'_1(u, v') = \bar{y}'_1(v', u) = 0$ , and every  $st$ -dicut inequality which is tight for  $\bar{y}_1$  is also tight for  $\bar{y}'_1$ . Thus  $\bar{y}'_1$  is also a solution of  $S(\bar{y})$ .

Consequently, there exists an integer solution  $\bar{y}'_1 \in kADPP(\tilde{G})$  which is a solution of  $S(\bar{y})$  and such that  $\bar{y}'_1(u, v') = \bar{y}'_1(u', v)$  for all arcs  $(u, v'), (v, u') \in \tilde{A}$  corresponding to an edge  $uv \in E$ . Thus,  $\bar{y}'_1$  can be associated with a solution  $\bar{x}'_1$  of  $P_k(G)$ . As  $\bar{y}'_1$  is integral,  $\bar{x}'_1$  is also integral. Moreover,  $\bar{x}'_1$  is a solution of  $S(\bar{x})$ . In fact, it is not hard to see that,  $\bar{y}'_1$  is a solution of  $S(\bar{y})$ ,  $\bar{y}'_1(a) = 0$  for all  $a \in \tilde{A}_0(\bar{y})$  and  $\bar{y}'_1(a) = 1$  for all  $a \in \tilde{A}_1(\bar{y})$ . Hence  $\bar{x}'_1(e) = 0$  for all  $e \in E_0(\bar{x})$  and  $\bar{x}'_1(e) = 1$  for all  $e \in E_1(\bar{x})$ . Suppose that there is an  $st$ -cut (resp. 3-path-cut) inequality in  $C^*(\bar{x})$  (resp.  $T^*(\bar{x})$ ) which is not tight for  $\bar{x}'_1$ , say  $\bar{x}'_1(C_0) > k$ . Then by Lemma 4.2, we have that  $\bar{x}'_1(C_0) \leq \bar{y}'_1(\tilde{C}_0)$ , where  $\tilde{C}_0$  is the  $st$ -dicut of  $\tilde{C}^*(\bar{y})$  corresponding to  $C_0$ . We thus obtain that  $\bar{y}'_1(\tilde{C}_0) > k$ . Hence  $\bar{y}'_1$  is not a solution of  $S(\bar{y})$ , a contradiction. Thus,  $\bar{x}'_1$  is a solution of  $S(\bar{x})$ . Since  $\bar{x}'_1$  is integral and  $\bar{x}$  is fractional,  $\bar{x}'_1 \neq \bar{x}$ . In consequence,  $\bar{x}$  is not the unique solution of  $S(\bar{x})$ , contradicting the fact that  $\bar{x}$  is an extreme point of  $P_k(G)$ . Therefore,  $\bar{x}$  cannot be fractional, which ends the proof of the theorem.  $\square$

A direct consequence of Theorems 3.2–3.4 and 4.3 is the following.

**Corollary 4.1.** *If  $G = (V, E)$  is a complete graph and  $|V| \geq k + 2$ , a minimal complete linear description of  $kHPP(G)$  is given by*

$$\begin{aligned} x(\delta(W)) &\geq k, & \text{for all } st\text{-cut } \delta(W), \\ x(T) &\geq k, & \text{for all 3-path-cut } T \text{ induced by a partition satisfying conditions (1) and (2) of Theorem 3.4,} \\ x(e) &\geq 0, & \text{for all } e \in E, \\ x(e) &\leq 1, & \text{for all } e \in E. \end{aligned}$$

As mentioned in Section 2.2, the separation problem for the  $st$ -cut and 3-path-cut inequalities can be solved in polynomial time. Thus, the  $kHPP$  can be solved in polynomial time using a cutting plane algorithm.

From the above results, the  $kHPP$  can also be seen as a minimum cost flow problem in the graph  $\tilde{G}$  by associating with its arcs unit capacities and appropriate weights. In fact, an arc of  $\tilde{G}$  which corresponds to an edge of  $G$  takes the same weight as this edge while the arcs of the form  $(u, u'), u \in V \setminus \{s, t\}$  (which do not correspond to any edge in  $G$ ) are given the weight 0. By the correspondence between the 3- $st$ -paths of  $G$  and the  $st$ -paths in  $\tilde{G}$ , a minimum weight subgraph of  $G$  which contains  $k$  edge-disjoint 3- $st$ -paths corresponds to a subgraph of  $\tilde{G}$  containing  $k$  arc-disjoint  $st$ -paths of the same weight. Moreover, the weight of this subgraph is minimum. The  $kHPP$  is thus equivalent to finding a minimum cost flow from  $s$  to  $t$  of value  $k$  in  $\tilde{G}$ . This implies that the problem can also be solved in polynomial time using any minimum cost flow algorithm.

## 5. Concluding remarks

In this paper we have given a complete description of the polytope associated with the  $k$  edge-disjoint  $L$ -hop-constrained paths problem when  $L = 3$  and  $k \geq 2$ . We have presented valid inequalities for the problem and given an integer programming formulation. We have also described necessary and sufficient conditions for the trivial inequalities, the  $st$ -cut and  $L$ -path-cut inequalities to define facets of the polytope. Using these results together with a transformation of the  $kHPP$  in  $G$  into the  $kADPP$  in a directed graph  $\tilde{G}$ , we have shown that the polytope  $kHPP(G)$  is completely described by the trivial,  $st$ -cut and 3-path cut inequalities. As the separation problem for these inequalities can be solved in polynomial time, this

yields a polynomial time cutting plane algorithm to solve the problem. We have also shown that the  $k$ HPP can be solved as a minimum cost flow problem in the graph  $\tilde{G}$ .

In the light of these results, we believe that an alternative proof of the main result would consist in considering an extended flow formulation of the  $k$ HPP on the auxiliary graph  $\tilde{G}$ , showing that this formulation is integer and then projecting the formulation on the design variables. It would be very interesting to have such a proof.

These results generalize those obtained by Huygens et al. [1] and Dahl et al. [2] for  $k = 2$  and  $L = 2, 3$  and for  $k \geq 2$  and  $L = 2$ , respectively. Unfortunately the linear description of the  $k$ HPP is no longer valid when  $L \geq 4$ . As shown by Huygens and Mahjoub [9], further inequalities are even needed for an integer programming formulation of the problem when  $k = 2$  and  $L = 4$ .

The integer programming formulation for the  $k$ HPP can be easily extended to the more general case where more than one pair of terminals are considered. However, as pointed out in [11], the cut inequalities together with the  $L$ -path-cut and trivial inequalities do not suffice to completely describe the  $k$ HPP polytope even when only two pairs of terminals are considered,  $L \geq 3$  and  $k = 2$ .

The results of this paper can be exploited for designing a Branch-and-Cut algorithm for that general case. Also the transformation of the  $k$ HPP to the  $k$ ADPP in an appropriate directed graph, introduced and exploited in this paper, can be used to give flow-based formulations. It would also be interesting to investigate this type of approach for  $L \geq 4$ . This is our aim for future work.

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## References

- [1] D. Huygens, A.R. Mahjoub, P. Pesneau, Two edge-disjoint hop-constrained paths and polyhedra, *SIAM Journal on Discrete Mathematics* 18 (2) (2004) 287–312.
- [2] G. Dahl, D. Huygens, A.R. Mahjoub, P. Pesneau, On the  $k$  edge-disjoint 2-hop-constrained paths polytope, *Operation Research Letters* 34 (5) (2006) 577–582.
- [3] C.-W. Ko, C.L. Monma, Heuristics for designing highly survivable communication networks, Technical Report, Bellcore, Morristown, New Jersey, 1989.
- [4] H. Kerivin, A.R. Mahjoub, Design of survivable networks: a survey, *Networks* 46 (2005) 1–21.
- [5] G. Dahl, Notes on polyhedra associated with hop-constrained paths, *Operations Research Letters* 25 (2) (1999) 97–100.
- [6] G. Dahl, L. Gouveia, On the directed hop-constrained shortest path problem, *Operations Research Letters* 32 (2004) 15–22.
- [7] C.R. Coullard, A.B. Gamble, J. Lui, The  $k$ -walk polyhedron, in: D.-Z. Du, J. Sun (Eds.), *Advances in Optimization and Approximation, Nonconvex Optimization Application 1*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1994, pp. 9–29.
- [8] G. Dahl, N. Foldnes, L. Gouveia, A note on hop-constrained walk polytopes, *Operations Research Letters* 32 (2004) 345–349.
- [9] D. Huygens, A.R. Mahjoub, Integer programming formulation for the two 4-hop-constrained paths problem, *Networks* 49 (2) (2007) 135–144.
- [10] G. Dahl, B. Johannessen, The 2-path network problem, *Networks* 43 (3) (2004) 190–199.
- [11] D. Huygens, M. Labbé, A.R. Mahjoub, P. Pesneau, The two edge-connected hop-constrained network design problem: valid inequalities and Branch-and-Cut, *Networks* 49 (1) (2007) 116–133.
- [12] L. Gouveia, P. Patricio, A.F. de Sousa, R. Valadas, MPLS over WDM Network Design with Packet Level QoS Constraints based on ILP Models, in: *Proceedings of INFOCOM*, 2003.
- [13] L. Gouveia, P. Patricio, A. de Sousa, Compact models for hop-constrained node survivable network design: an application to MPLS, in: G. Anandilagam, S. Raghavan (Eds.), *Telecommunications Network Planning: Innovations and Pricing*, Network Design and Management, Springer, 2005, pp. 167–180.
- [14] L. Gouveia, P. Patricio, A. de Sousa, Hop-constrained node survivable network design: an application to MPLS over WDM, *Networks and Spatial Economics* 8 (1) (2008) 3–21.
- [15] L. Gouveia, Multicommodity flow models for spanning trees with hop constraints, *Operations Research Letters* 25 (2) (1999) 97–100.
- [16] L. Gouveia, Using variable redefinition for computing lower bounds for minimum spanning, *INFORMS Journal on Computing* 10 (2) (1998) 180–188.
- [17] L. Gouveia, C. Requejo, A new Lagrangean relaxation approach for the hop-constrained minimum spanning tree problem, *European Journal of Operations Research* 25 (2) (2001) 539–552.
- [18] G. Dahl, The 2-hop spanning tree problem, *Operations Research Letters* 23 (1–2) (1999) 21–26.
- [19] B. Fortz, M. Labbe, F. Maffioli, Solving the two-connected network with bounded meshes problem, *Operations Research Letters* 48 (2000) 866–877.
- [20] B. Fortz, A.R. Mahjoub, S.T. McCormick, P. Pesneau, Two-edge connected subgraphs with bounded rings: polyhedral results and Branch-and-Cut, *Mathematical Programming* 105 (1) (2006) 85–111.
- [21] M. Grötschel, C.L. Monma, Integer polyhedra arising from certain network design problems with connectivity constraints, *SIAM Journal on Discrete Mathematics* 3 (1990) 502–523.
- [22] M. Grötschel, C.L. Monma, M. Stoer, Polyhedral approaches to network survivability, in: F. Roberts, F. Hwang, C.L. Monma (Eds.), *Reliability of Computer and Communication Networks*, in: *Series Discrete Mathematics and Computer Science*, vol. 5, AMS/ACM, 1991, pp. 121–141.
- [23] M. Grötschel, C.L. Monma, M. Stoer, Facets for polyhedra arising in the design of communications networks with low-connectivity constraints, *SIAM Journal on Optimization* 2 (1992) 474–504.
- [24] M. Didi Biha, A.R. Mahjoub,  $k$ -edge connected polyhedra on series-parallel graphs, *Operations Research Letters* 19 (1996) 71–78.
- [25] M. Didi Biha, A.R. Mahjoub, Steiner  $k$ -edge connected subgraph polyhedra, *Journal of Combinatorial Optimization* 4 (2000) 131–134.
- [26] S. Chopra, The  $k$ -edge connected spanning subgraph polyhedron, *SIAM Journal on Discrete Mathematics* 7 (1994) 245–259.
- [27] F. Barahona, A.R. Mahjoub, On two-connected subgraph polytopes, *Discrete Mathematics* 147 (1995) 19–34.
- [28] J. Fonlupt, A.R. Mahjoub, Critical extreme points of the 2-edge connected spanning subgraph polytope, *Mathematical Programming* 105 (2006) 289–310.
- [29] M. Didi Biha, A.R. Mahjoub, The  $k$ -edge connected subgraph problem I: polytopes and critical extreme points, *Linear Algebra and its Applications* 381 (2004) 117–139.
- [30] H. Kerivin, A.R. Mahjoub, C. Nocq, (1, 2)-survivable networks: facets and Branch-and-Cut, in: M. Grötschel (Ed.), *The Sharpest Cut*, MPS-SIAM Series in Optimization, 2004, pp. 121–152.
- [31] F. Bendali, I. Diarrassouba, M. Didi Biha, A.R. Mahjoub, J. Mailfert, A Branch-and-Cut algorithm for the  $k$ -edge connected subgraph problem, *Networks* 55 (1) (2010) 13–32.
- [32] K. Menger, Zur allgemeinen kurventheorie, *Fundamenta Mathematicae* 10 (1927) 96–115.
- [33] A. Schrijver, *Combinatorial Optimization — Polyhedra and Efficiency*, Algorithms and Combinatorics, vol. 24, Springer, Berlin, Heidelberg, 2003.
- [34] G. Cornuejols, D. Naddef, J. Fonlupt, The traveling salesman problem on a graph and some related integer polyhedra, *Mathematical Programming* 33 (1985) 1–27.