Lecture 3
Probability Models

Dr. Franco Ubaudi

The Nature of Data
Western Sydney University

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Outline

- Probability spaces
- Random variables (rv)
- Expected value & variance
- Poisson distribution
- Binomial & Poisson
- Normal rv
- Approximating a binomial rv using a Normal rv

Probability spaces

A probability space consists of three elements:

1. a sample space Ω: all possible outcomes
2. an event space $\mathcal{F}$: all possible events
3. and a probability function P: a mapping of an event to a value in $[0, 1]$

Probability spaces cont.

Two example sample spaces:

- $\Omega = \{H, T\}$
- $\Omega = \{1, 2, 3, 4, 5, 6\}$

Probability spaces cont.

Considering the sample space
$\Omega = \{1, 2, 3, 4, 5, 6\}$

Possible event spaces are:

- $\mathcal{F} = \{\{\}, \{\Omega\}\}$
- A more sophisticated event space could be whether the outcome is divisible by 3

  $\mathcal{F} = \{\{\}, \{3, 6\}, \{1, 2, 4, 5\}\}$

  or

  $\mathcal{F} = \{\{\}, \{\mathit{Yes}\}, \{\mathit{No}\}\}$

Probability spaces cont.

Examples of mapping events to a probability are:

- Considering the event space $\mathcal{F} = \{1, 2, 3, 4, 5, 6\}$

  $P(X = e) = 1/6$ for each event $e \in \mathcal{F}$
- For the event space $\mathcal{F} = \{\{\}, \{3, 6\}, \{1, 2, 4, 5\}\}$

  $P(X = e) = 2/6$ for each event $e \in \{3, 6\}$
  $P(X = e) = 4/6$ for each event $e \in \{1, 2, 4, 5\}$

Single fair coin toss

Bernoulli with $p = 1/2$

Sample space is heads and tails: $\Omega = \{H, T\}$ (it's either heads or tails).

Event space $\mathcal{F}$ includes H and T but also "heads or tails"

All together we have

$\mathcal{F} = \{\{H\}, \{T\}, \{H \cup T\}\}$.

Probability measure maps:
$P(H) = 0.5, P(T) = 0.5, P(H \cup T) = 1$.

2 coins

Outcomes $\Omega = \{HH, HT, TH, TT\}$.

Events: the combinations of all possible outcomes.

e.g. event "2 heads or 2 tails" is $HH \cup TT$.

The probability function is $P(HH \cup TT) = 1/2$.

Random variables again

A random variable X is the mapping from the sample space to a (real) number – $X : \Omega \mapsto \mathbb{R}$.

Has probability $P(X = e)$ for each event $e \in \mathcal{F}$.

Central value for X?

e.g. n coin tosses: what is the typical number of heads?

Expected value and variance

$X =$ number of times heads comes up.

Want *expected value* of X, $E[X]$.

For 3 coin tosses, X can take on values $0, 1, 2$ or 3.

$$E[X] = 0 * P(X = 0) + 1 * P(X = 1) + 2 * P(X = 2) + 3 * P(X = 3)$$

From the binomial distribution, we could calculate all of this.

Quicker: for any Binomial random variable for given p and n:

$$E[X] = np.$$

So for $n = 3$ and $p = 1/2$, we get $3/2$ "expected" value.

For a lot of experiments the average would converge to np.

Variance

How much does the rv deviate from this central expected value?

The *variance* of a random variable as $E[(X - \mu)^2]$, where $\mu = E[X]$.

The binomial variance is given by $np(1 - p)$.

Poisson distribution

The Binomial distribution has non-zero probabilities for the integer numbers 0 through n, for n trials.

It is useful for calculating the probability of a number of successes out of a given number of trials.

What if instead we want to count occurrences of an event

- number of cars that pass though an intersection
- number of goals scored in a football match

Death by horse kick

In 1898, von Bortkiewicz published the number of deaths from horse kicks in the Prussian army for 10 regiments over 20 years (200 observations).

Chance of being kicked by a horse

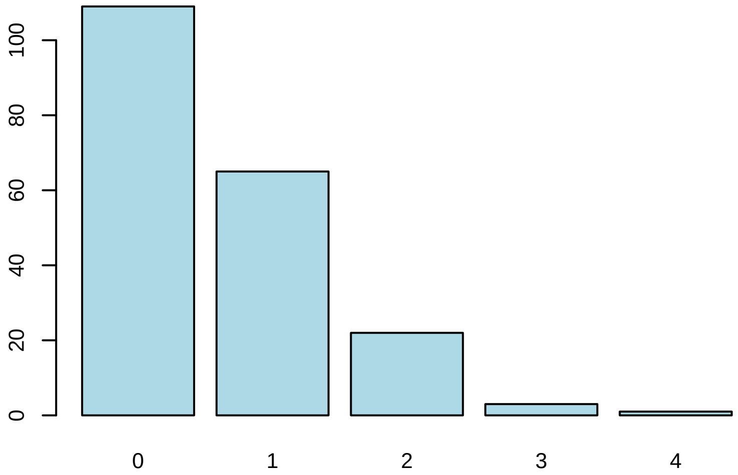


Figure: Number of deaths from horse kick for each regiment and year.

Being kicked by a horse

These are counts – not the result of a sequence of kicked/not kicked trials.

For this data, von Bortkiewicz proposed the Poisson distribution.

Let X be a Poisson random variable.

The probability of observing k kicks is

$$P(X = k) = \frac{\lambda^k}{k!} \exp(-\lambda)$$

where λ is the expected number of kicks, and $\exp(\cdot)$ is the exponential function.

Being kicked by a horse

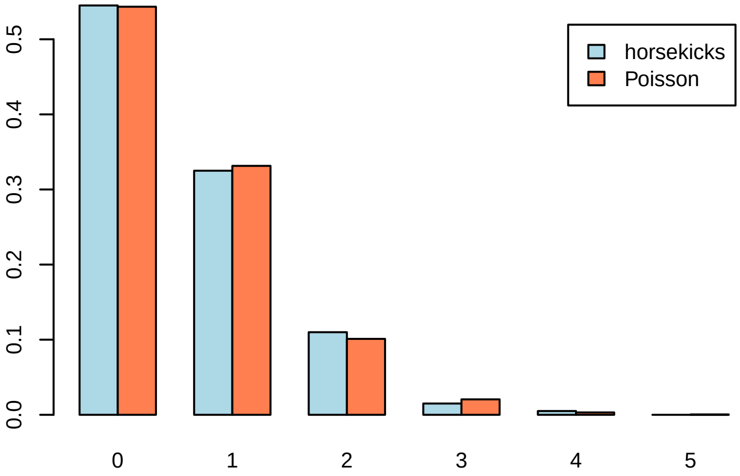


Figure: Number of deaths from horse kick compared to the Poisson distribution.

Poisson Distribution

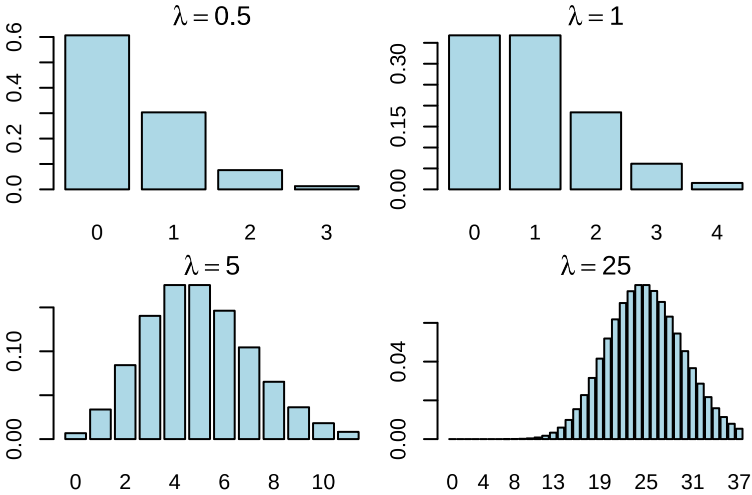


Figure: Examining the Poisson distribution.

Poisson Variance

A Poisson random variable has only one parameter λ

$$E[X] = \lambda$$

$$\text{variance} = E[(X - \lambda)^2] = \lambda$$

Horsekick data

For the horse kick data we observed 200 counts, tabulated

0	1	2	3	4
109	65	22	3	1

So the mean number of kicks per regiment per year:

$$E[X] = 0 * P(X = 0) + 1 * P(X = 1) + \cdots + 4 * P(X = 4)$$

$$\frac{(0 \times 109 + 1 \times 65 + 2 \times 22 + 3 \times 3 + 4 \times 1)}{200} = 0.61$$

Binomial and Poisson

Binomial Examples

- Number of geminating seeds out of a batch.
- Number of people with disease.
- Number of insects killed with certain pesticide dose.

Expected value: np
Variance: $np(1-p)$

Poisson Examples

- Number of car insurance claims.
- Number of plants in an area.
- Number of people waiting in a queue.

Expected value: λ
Variance: λ

Using the *R* functions

Take a quick look at using *R* function to calculate statistical data

- binomial distribution (discrete)
  - dbinom - probability of a single outcome
  - pbinom - probability of sequence of outcomes starting from the left
  - qbinom - reciprocal of pbinom
  - rbinom - randomly select
- poisson distribution (discrete)
  - dpois

    ⋮
- normal distribution (continuous)
  - dnorm - likelihood, not really used in this unit
  - pnorm - probability of outcomes starting from the left

    ⋮

Normal rv

Gauss proposed the Normal distribution – hence also known as the Gaussian distribution.

X is a Normal rv with mean μ and variance σ^2,

$$P(X = x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

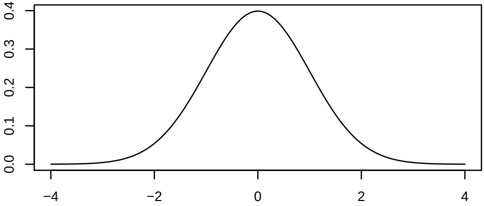


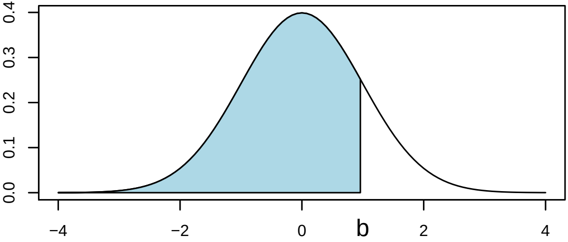
Figure: Mean 0, variance 1

Normal probabilities

Binomial rvs are discrete – to calc less than k heads, we add up the probabilities for all counts of heads less than k
e.g. *pbinom*(k - 1, n, p)

For a Normal distribution this translates to areas.

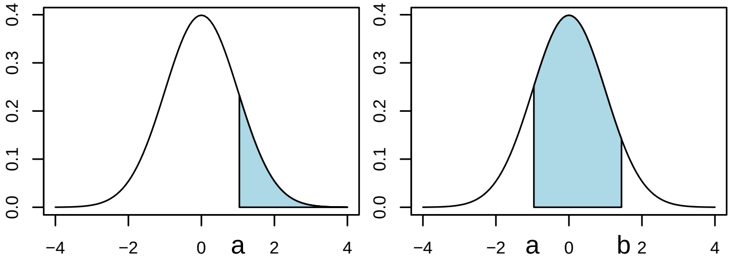
The probability that $X < b$ is the area under the normal density that is less than b.



pnorm(b) or *pnorm*(b, 0, 1)

Normal Probabilities

We can also compute probabilities like $P(X > a)$ and $P(a < X < b)$



- $P(X > a) =$ `1-pnorm(a)` or `pnorm(a, lower=FALSE)`
- $P(a < X < b) =$ `pnorm(b) - pnorm(a)`

Normal Probabilities

How does this help to approximate the probability of more than 60 heads in 100 tosses?

In the binomial case $1 - \mathit{pbinom}(60, 100, 0.5) = 0.0176$

Recall: the binomial rv has mean $np = 50$ and variance $np(1-p) = 25$.

A normal rv with mean μ and variance σ^2 has density:

$$\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

The binomial is approximated by a Normal with mean np and variance $np(1-p)$.

Approximate binomial by a normal

Let X be this approximation, we want to compute $P(X > 60)$.

- `pnorm(60, 100, 5, lower=FALSE)` $= 0.023$
  or
- `1 - pnorm(60, 100, 5)` $= 0.023$

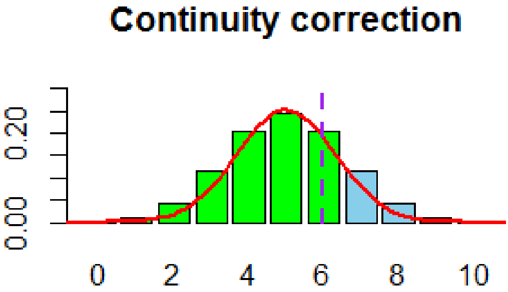
Continuity correction

Approximation can be improved by a continuity correction.

- Binomial is discrete, Normal is continuous.
- We approximate $P(B > k)$ with $P(X > k + 1/2)$, and $P(B < k)$ with $P(X < k - 1/2)$
- $P(B = k)$ with $P(k - 1/2 < X < k + 1/2)$

`R` can also compute Binomial probabilities with `pbinom`.

Continuity correction cont.



$pbinom(6, 10, p) = 0.828125$

$pnorm(6 + 0.5, mu, sigma) = 0.8286091$

Examples of normal approximation to binomial

```
n = c(5,10,100)
x = floor(n*0.6)
pb = 1-pbinom(x,size = n, prob = 0.5)
pn = 1-pnorm((x+0.5-n/2)/sqrt(n/4))
```

n	x	pb	pn
5	3	0.1875	0.1855
10	6	0.1719	0.1714
100	60	0.0176	0.0179