Week 3 Lecture - Multiple Linear Regression

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This lecture introduces basic concepts and presents examples of various regression techniques.

- Multiple linear regression
- Non-linear regression
    - Interaction Terms of X Variables
    - Polynomial Regressions
    - Transformations of the response and explanatory variables
- A collection of helpful R functions for regression analysis

Multiple Linear Regression

In multiple linear regression, the expected value of Y_i given $X_{1i}, X_{2i}, ..., X_{pi}$ is:

$$E(Y_i) = \alpha + \beta_1 X_{1i} + \beta_2 X_{2i} + ... + \beta_p X_{pi}$$

for $i = 1, 2, 3, ..., n$

Y_i has a ***normal distribution*** with ***standard deviation*** σ. It is the random component of the model, which has a normal distribution.

The response variable is Y, and Xs are continuous explanatory variables. The parameters are $\alpha, \beta_1, \beta_2, ..., \beta_p$:

- The ***intercept*** is α: The value of Y when $X_1 = X_2 = ... = X_p = 0$
- We interpret β_j as the average effect on Y of a one unit increase in X_j , holding all other predictors fixed.

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Multiple Linear Regression. . .

Here the ***estimated model*** is:

The expected value of Y given X is

$$E(\hat{Y}) = \hat{\alpha} + \hat{\beta}_1 X_1 + \hat{\beta}_2 X_2 + ... + \hat{\beta}_p X_p$$

(OR)

$$\hat{Y} = \hat{\alpha} + \hat{\beta}_1 X_1 + \hat{\beta}_2 X_2 + ... + \hat{\beta}_p X_p$$

In the advertising example, the estimated model becomes

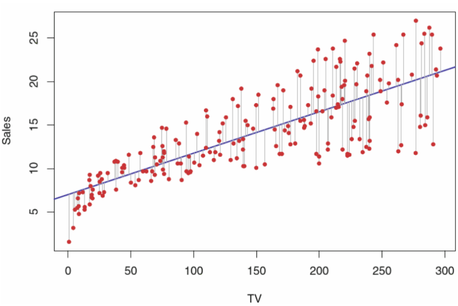
$$E(\hat{Sales}) = \hat{\alpha} + \hat{\beta}_1 TV + \hat{\beta}_2 Radio + \hat{\beta}_3 Newspaper$$

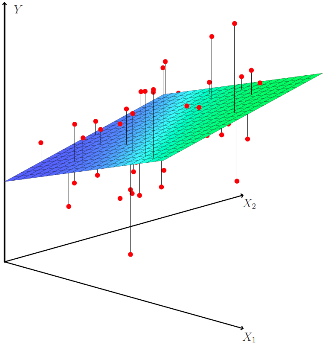
(OR)

$$\hat{Sales} = \hat{\alpha} + \hat{\beta}_1 TV + \hat{\beta}_2 Radio + \hat{\beta}_3 Newspaper$$

Parameter Estimation

Recall the parameter estimation in simple linear regression. Here also parameters are estimted by minimizing the sum of squared residuals.





(source: An Introdution to Statistical Learning-prescribed book)

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Import the Data Set "Advertising"

```r
Advertising <- read.csv("Advertising.csv")
attach(Advertising)
names(Advertising)
```

```
## [1] "TV"        "Radio"     "Newspaper" "Sales"
```

```r
head(Advertising)
```

```
##      TV Radio Newspaper Sales
## 1 230.1  37.8      69.2  22.1
## 2  44.5  39.3      45.1  10.4
## 3  17.2  45.9      69.3   9.3
## 4 151.5  41.3      58.5  18.5
## 5 180.8  10.8      58.4  12.9
## 6   8.7  48.9      75.0   7.2
```

Multiple Linear Regression

R code

```r
model=lm(Sales~TV+Radio+Newspaper)
summary(model)
```

```
##
## Call:
## lm(formula = Sales ~ TV + Radio + Newspaper)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -8.8277 -0.8908  0.2418  1.1893  2.8292
##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept)  2.938889   0.311908   9.422   <2e-16 ***
## TV           0.045765   0.001395  32.809   <2e-16 ***
## Radio        0.188530   0.008611  21.893   <2e-16 ***
## Newspaper   -0.001037   0.005871  -0.177     0.86
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 1.686 on 196 degrees of freedom
## Multiple R-squared:  0.8972, Adjusted R-squared:  0.8956
## F-statistic: 570.3 on 3 and 196 DF,  p-value: < 2.2e-16
```

Degree of scatter

- Residual standard error: 1.686 on 196 degrees of freedom
- Multiple R-squared: 0.8972
- Adjusted R-squared: 0.8956
- F-statistic: 570.3 on 3 and 196 DF
- p-value: $< 2.2\text{e-}16$ (2.2×10^{-16})

Linear relationship between Sales and TV, and Sales and Radio are ***significant***

Linear relationship between Sales and Newspaper are **NOT significant**.

*Why?

ANOVA Table and critical value of F:

```
anova(model)
```

```
## Analysis of Variance Table
##
## Response: Sales
##            Df Sum Sq Mean Sq   F value Pr(>F)
## TV          1 3314.6  3314.6 1166.7308 <2e-16 ***
## Radio       1 1545.6  1545.6  544.0501 <2e-16 ***
## Newspaper   1    0.1     0.1    0.0312 0.8599
## Residuals 196  556.8     2.8
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

```
qf(0.95,3,196)
```

```
## [1] 2.650677
```

Better model

```r
modelB=lm(Sales~TV+Radio)
summary(modelB)
```

```
##
## Call:
## lm(formula = Sales ~ TV + Radio)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -8.7977 -0.8752  0.2422  1.1708  2.8328
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept)  2.92110    0.29449   9.919   <2e-16 ***
## TV           0.04575    0.00139  32.909   <2e-16 ***
## Radio        0.18799    0.00804  23.382   <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 1.681 on 197 degrees of freedom
## Multiple R-squared:  0.8972, Adjusted R-squared:  0.8962
## F-statistic: 859.6 on 2 and 197 DF,  p-value: < 2.2e-16
```

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95% confidence intervals for the estimated parameters

```r
confint(modelB)
```

```
##                   2.5 %     97.5 %
## (Intercept) 2.34034299 3.50185683
## TV          0.04301292 0.04849671
## Radio       0.17213877 0.20384969
```

ANOVA Table and critical value of F:

```r
anova(modelB)
```

```
## Analysis of Variance Table
##
## Response: Sales
##            Df Sum Sq Mean Sq F value    Pr(>F)
## TV          1 3314.6  3314.6 1172.50 < 2.2e-16 ***
## Radio       1 1545.6  1545.6  546.74 < 2.2e-16 ***
## Residuals 197  556.9     2.8
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Recall, in simple linear regression, we checked the hypotheses $\beta_1 = 0$ vs $\beta_1 \neq 0$ to determine whether there is a significant relationship between the response and the predictor.

In multiple linear regression, by using $F - statistic$, we can the test the following;

$H_0 : \beta_1 = \beta_2 = ... = \beta_p = 0$

H_1 : at least one $\beta_j \neq 0$

(p is the number of predictors)

```
qf(0.95,2,197)
```

```
## [1] 3.041753
```

Accepted Model

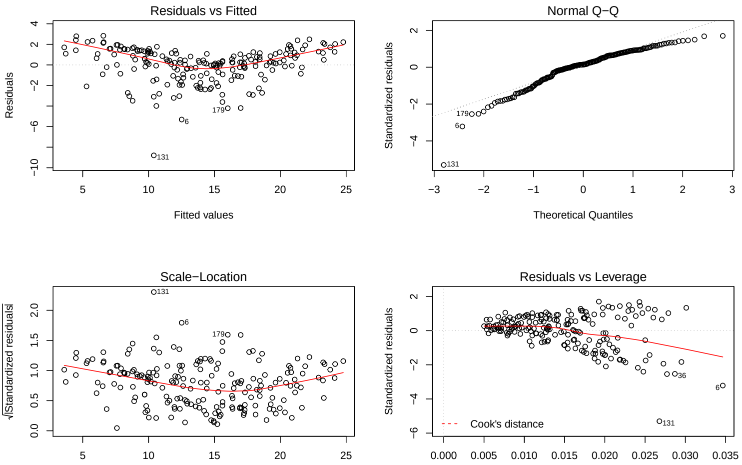
$$E(\hat{Sales}) = 2.92110 + 0.04575TV + 0.18799Radio$$

(OR)

$$\hat{Sales} = 2.92110 + 0.04575TV + 0.18799Radio$$

Assumption checking

```
par(mfrow=c(2,2))
plot(modelB)
```



Variable selection

- The direct approach is to compute the least squares fit for all possible subsets and then choose between them based on some criterion.
- However we often cannot examine all possible methods. There are 2^p of them; p= number of predictors.
- Commonly used approches;
    - Forward Selection
    - Backward Selection
    - Mixed (Stepwise) selection

Foward Selection

- Begin with the **null model** (model with an intercept, no predictors)
- Fit p simple linear regressions and add to the null model the variable that results in the lowest residial/error sum of squares
- Add to that model the variable that results in the lowest residial/error sum of squares amongst all two-varible models
- Continue until some stopping rule is satisfied (when all remaining variables have p-value above some threshold)

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Foward Selection ctd...

Step1:

	Model	Residual Sum of Squares
fit1	lm(Sales~TV)	**2102.5**
fit2	lm(Sales~Radio)	3618.5
fit3	lm(Sales~Newspaper)	5134.8

Step 2:

	Model	Residual Sum of Squares
fit4	lm(Sales~TV+Radio)	**556.9**
fit5	lm(Sales~TV+Newspaper)	1918.6

Step 3:

	Model	Residual Sum of Squares
fit6	lm(Sales~TV+Radio+Newspaper)	556.8

Foward Selection ctd...

However, in **fit6**, the Newspaper variable is **not significant**. Therrefore we may select the **fit4** as the best fit.

codes for above analysis;

```
fit1 <- lm(Sales~TV)
anova(fit1)
fit2 <- lm(Sales~Radio)
anova(fit2)
fit3 <- lm(Sales~Newspaper)
anova(fit3)

fit4 <- lm(Sales~TV+Radio)
anova(fit4)
fit5 <- lm(Sales~TV+Newspaper)
anova(fit5)

fit6 <- lm(Sales~TV+Radio+Newspaper)
anova(fit6)
```

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Backward Selection

- Start with all variables in the model
- Remove the variable with the largest p-value (the variable which is the least significant)
- Consider the new $(p-1)-variable$ model, and remove the variable with the largest p-value
- Continue until some stopping rule is satisfied (we may stop when all remaining variables have a significant p value)

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Backward Selection ctd...

Step 1:

```
m1 <- lm(Sales~TV+Radio+Newspaper)
summary(m1)
```

```
##
## Call:
## lm(formula = Sales ~ TV + Radio + Newspaper)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -8.8277 -0.8908  0.2418  1.1893  2.8292
##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept)  2.938889   0.311908   9.422   <2e-16 ***
## TV           0.045765   0.001395  32.809   <2e-16 ***
## Radio        0.188530   0.008611  21.893   <2e-16 ***
## Newspaper   -0.001037   0.005871  -0.177     0.86
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 1.686 on 196 degrees of freedom
## Multiple R-squared:  0.8972, Adjusted R-squared:  0.8956
## F-statistic: 570.3 on 3 and 196 DF,  p-value: < 2.2e-16
```

Remove **Newspaper** variable

Step 2:

```r
m2 <- lm(Sales~TV+Radio)
summary(m2)
```

```
##
## Call:
## lm(formula = Sales ~ TV + Radio)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -8.7977 -0.8752  0.2422  1.1708  2.8328
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept)  2.92110    0.29449   9.919   <2e-16 ***
## TV           0.04575    0.00139  32.909   <2e-16 ***
## Radio        0.18799    0.00804  23.382   <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 1.681 on 197 degrees of freedom
## Multiple R-squared:  0.8972, Adjusted R-squared:  0.8962
## F-statistic: 859.6 on 2 and 197 DF,  p-value: < 2.2e-16
```

All the variables are significant. We may select this model.

Mixed (Stepwise) Selection

- This is a combination of forward and backward selection.
- Start with forward selection and after each step in which a variable was added, all variables in the model are checked to see if their significance has been reduced below the specified tolerance level.
- If a nonsignificant variable is found, it is removed from the model.
- Continue these foward and backward steps untill all variables in the model have a sufficiently low p-value.

Model Selection

Various criteria can be used to judge the quality of a model. These incluses;

- Akaike Information Criterian (AIC)
- Bayasian Information Criterian (BIC)
- Mallow's C_p
- Adjusted R^2

etc...

Quick codes for variable selection

Forward Selection

```
step(lm(Sales~1), direction = "forward", scope = ~TV+Radio+Newspaper)
```

Backward Selection

```
step(lm(Sales~TV+Radio+Newspaper), direction = "backward")
```

Mixed Selection

```
step(lm(Sales~TV+Radio+Newspaper), direction = "both")
```

Extensions of the Linear Model

Removing the additive assumption: Interactions and Nonlinearity

Interactions:

- In our previous analysis of the Advertising data, we assumed that the effect on sales of increasing one advertising medium is independent of the amount spent on the other media.
- For example, the linear model

$$E(Sales) = \beta_0 + \beta_1 \text{TV} + \beta_2 \text{Radio} + \beta_3 \text{Newspaper}$$

states that the average effect on sales of a one-unit increase in TV is always β_1, regardless of the amount spent on radio.

Interactions - continued

- But suppose that spending money on radio advertising actually increases the effectiveness of TV advertising, so that the slope term for TV should increase as radio increases.
- In this situation, given a fixed budget of $100,000, spending half on radio and half on TV may increase sales more than allocating the entire amount to either TV or to radio.
- In marketing, this is known as a synergy effect, and in statistics it is referred to as an interaction effect.

Modeling interactions - Advertising data

Model with Interactions takes the form

$$E(Sales) = \alpha + \beta_1 TV + \beta_2 Radio + \beta_3 TV.Radio$$

and the ***estimated line*** takes the form

$$E(\hat{Sales}) = \hat{\alpha} + \hat{\beta}_1 TV + \hat{\beta}_2 Radio + \hat{\beta}_3 TV.Radio$$

(OR)

$$\hat{Sales} = \hat{\alpha} + \hat{\beta}_1 TV + \hat{\beta}_2 Radio + \hat{\beta}_3 TV.Radio$$

```
model4=lm(Sales~TV+Radio+TV*Radio)
summary(model4)
```

```
##
## Call:
## lm(formula = Sales ~ TV + Radio + TV * Radio)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -6.3366 -0.4028  0.1831  0.5948  1.5246
##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept) 6.750e+00  2.479e-01  27.233   <2e-16 ***
## TV          1.910e-02  1.504e-03  12.699   <2e-16 ***
## Radio       2.886e-02  8.905e-03   3.241   0.0014 **
## TV:Radio    1.086e-03  5.242e-05  20.727   <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.9435 on 196 degrees of freedom
## Multiple R-squared:  0.9678, Adjusted R-squared:  0.9673
## F-statistic:  1963 on 3 and 196 DF,  p-value: < 2.2e-16
```

ANOVA

```r
anova(model4)
```

```
## Analysis of Variance Table
##
## Response: Sales
##            Df Sum Sq Mean Sq F value    Pr(>F)
## TV          1 3314.6  3314.6 3723.36 < 2.2e-16 ***
## Radio       1 1545.6  1545.6 1736.22 < 2.2e-16 ***
## TV:Radio    1  382.4   382.4  429.59 < 2.2e-16 ***
## Residuals 196  174.5     0.9
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 '
```

Modeling interactions - Summary

Model 4

$$E(\hat{Sales}) = 6.750 + (1.910 \times 10^{-2})TV + (2.886 \times 10^{-2})Radio + (1.0868 \times 10^{-3})TV.Radio$$

(OR)

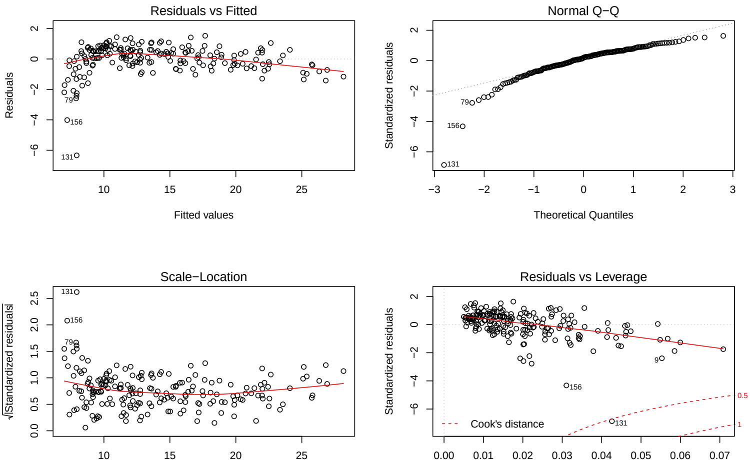
$$\hat{Sales} = 6.750 + (1.910 \times 10^{-2})TV + (2.886 \times 10^{-2})Radio + (1.0868 \times 10^{-3})TV.Radio$$

- Multiple R-squared: 0.9678
- Residual standard error: 0.9435 on 196 degrees of freedom
- F-statistic: 1963 on 3 and 196 DF
- p-value: $< 2.2e\text{-}16$

Significant Interation term

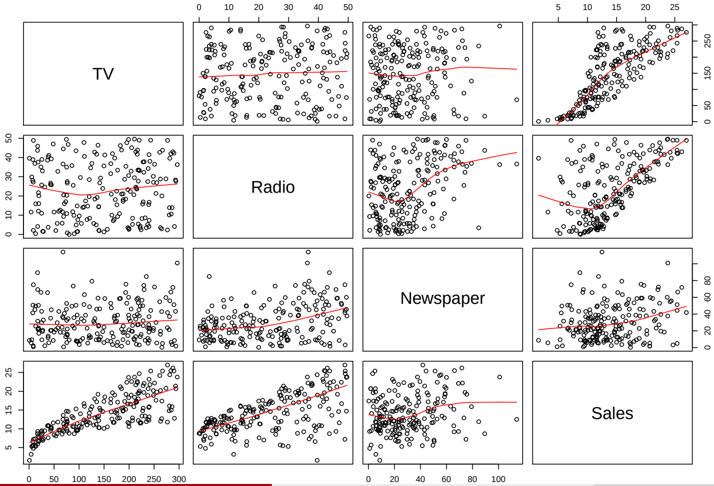
Model Checking

```
par(mfrow=c(2,2))
plot(model4)
```



Covariance and Correlations

```
pairs(Advertising,panel=panel.smooth)
```



Covariance and Correlations

```r
cov(Advertising,method="pearson") #covariance
```

```
##                   TV    Radio Newspaper     Sales
## TV        7370.94989 69.86249 105.91945 350.39019
## Radio       69.86249 220.42774 114.49698  44.63569
## Newspaper  105.91945 114.49698 474.30833  25.94139
## Sales      350.39019  44.63569  25.94139  27.22185
```

Covariance and Correlations

```r
cor(Advertising,method="pearson") #correlation
```

```
##                   TV      Radio  Newspaper     Sales
## TV        1.00000000 0.05480866 0.05664787 0.7822244
## Radio     0.05480866 1.00000000 0.35410375 0.5762226
## Newspaper 0.05664787 0.35410375 1.00000000 0.2282990
## Sales     0.78222442 0.57622257 0.22829903 1.0000000
```

```r
cor(TV,Sales)
```

```
## [1] 0.7822244
```

NOTE: Correlation is calculated only for ***Continuous variables.***

Code

```
model2$residuals
plot(predict(model2),model2$residuals)
hist(model2$residuals)
predict(model2)
predict(model2,interval='confidence')
predict(model2,interval='confidence')
predict(model2,as.data.frame(cbind
  (TV=50,Radio=50,Newspaper=50)))
```

Non-linear effects of predictors - Polynomial Regression

The relationship between Y and X often turns out not to be a straight line.

How do we assess the significance of departures from linearity?

One of the simplest ways is to use ***polynomial regression***.

As before, we have just ***one continuous explanatory variable***, X, but we can fit ***higher powers*** of X, such as X^2 and X^3, to the model in addition to X to explain curvature in the relationship between Y and X.

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Non-linear effects of predictors - Polynomial Regression

Consider the model

$$E(Sales) = \beta_0 + \beta_1 TV + \beta_2 TV^2$$

Will this model provide a better fit?

```
model5=lm(Sales~TV+I(TV*TV))
summary(model5)
```

```
##
## Call:
## lm(formula = Sales ~ TV + I(TV * TV))
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -7.6844 -1.7843 -0.1562  2.0088  7.5097
##
## Coefficients:
##               Estimate Std. Error t value Pr(>|t|)
## (Intercept)  6.114e+00  6.592e-01   9.275  < 2e-16 ***
## TV           6.727e-02  1.059e-02   6.349 1.46e-09 ***
## I(TV * TV)  -6.847e-05  3.558e-05  -1.924   0.0557 .
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 3.237 on 197 degrees of freedom
## Multiple R-squared:  0.619,  Adjusted R-squared:  0.6152
## F-statistic: 160.1 on 2 and 197 DF,  p-value: < 2.2e-16
```

Anova

```r
anova(model5)
```

```
## Analysis of Variance Table
##
## Response: Sales
##              Df Sum Sq Mean Sq  F value  Pr(>F)
## TV            1 3314.6  3314.6 316.4072 < 2e-16 ***
## I(TV * TV)    1   38.8    38.8   3.7036 0.05574 .
## Residuals   197 2063.7    10.5
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 '
```

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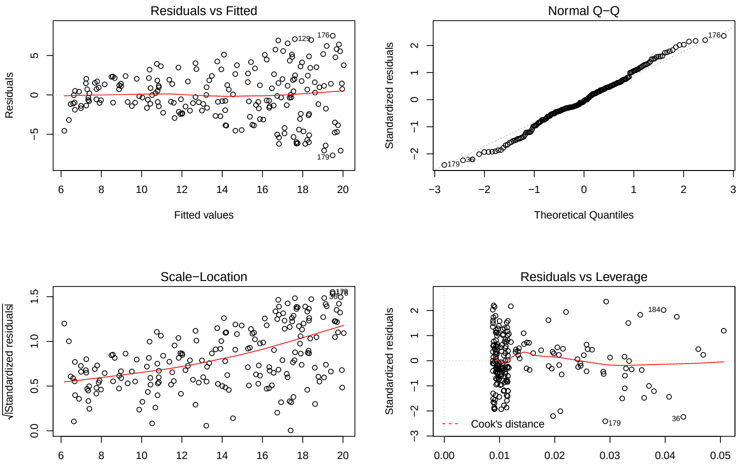
Model Checking

```
par(mfrow=c(2,2))
plot(model5)
```



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Transformations of the response and explanatory variables

The use of ***transformation to linearize the relationship*** between the response and the explanatory variables:

- ***log y*** against x for ***exponential relationships***
- ***log y*** against ***log x*** for ***power functions***
- ***exp y*** against x for ***logarithmic relationships***
- $1/y$ against $1/x$ for ***asymptotic relationships***
- ***log p/1-p*** against x for ***proportion data***

Transformations of the response and explanatory variables. . .

Other transformations are useful for ***variance stabilization***:

- ***sqrt(y)*** to stabilize the variance for ***count data***
- ***arcsin(y)*** to stabilize the variance of ***percentage data***

TEXT BOOK

Lecture notes are based on the textbook.

For further reference refer;

Prescribed Textbook - **Chapter 3**

- James, G., Witten, D., Hastie, T., & Tibshirani, R. (2013). An Introduction to Statistical Learning: with Applications in R Springer.