Problem Set 3

Katherine Cheng, Richard Davis, Marty Keil

April 23, 2015

Problem Three: First-Order Negations (definitely needs to be checked/edited)

i. Negating the statement, "For all x in the set of real numbers, and for all y in the set of real numbers, if x is less than y then there exists q in the set of rational numbers such that x is less than q, and q is less than y."

$$\neg(\forall x \in \mathbb{R}.\ \forall y \in \mathbb{R}.(x < y \to \exists q \in \mathbb{Q}.(x < q \land q < y))) \tag{1}$$
$$\exists x \in \mathbb{R}.\neg(\forall y \in \mathbb{R}.(x < y \to \exists q \in \mathbb{Q}.(x < q \land q < y))) \tag{2}$$
$$\exists x \in \mathbb{R}.\ \exists y \in \mathbb{R}.\neg(x < y \to \exists q \in \mathbb{Q}.(x < q \land q < y)) \tag{3}$$
$$\exists x \in \mathbb{R}.\ \exists y \in \mathbb{R}.(x < y) \land \neg(\exists q \in \mathbb{Q}.(x < q \land q < y)) \tag{4}$$
$$\exists x \in \mathbb{R}.\ \exists y \in \mathbb{R}.(x < y) \land \neg(\exists q \in \mathbb{Q}.(x < q \land q < y)) \tag{5}$$
$$\exists x \in \mathbb{R}.\ \exists y \in \mathbb{R}.(x < y) \land \forall q \in \mathbb{Q}.\neg(x < q \land q < y)) \tag{6}$$
$$\exists x \in \mathbb{R}.\ \exists y \in \mathbb{R}.(x < y) \land \forall q \in \mathbb{Q}.\neg(x < q) \lor \neg(q < y) \tag{7}$$
$$\exists x \in \mathbb{R}.\ \exists y \in \mathbb{R}.(x < y) \land \forall q \in \mathbb{Q}.(x \geq q) \lor (q \geq y) \tag{8}$$
$$\tag{9}$$

The negation is, "There exists x in the set of real numbers, and there exists y in the set of real numbers, where x is less than y and for all q in the set of rational numbers x is greater than or equal to q, and q is greater than or equal to y."

ii. Negating the statement, "For all x, y, and z, if (R(x, y) and R(y, z) then R(x, z)) then if for all x, y, and z R(y, z) and R(z, y) then R(z, x).”

$$\neg(\forall x.\forall y.\forall z.(R(x,y) \wedge R(y,z) \rightarrow R(x,z))) \rightarrow (\forall x.\forall y.\forall z.(R(y,x) \wedge R(z,y) \rightarrow R(z,x))) \tag{10}$$
$$\exists x.\exists y.\exists z.\neg((R(x,y) \wedge R(y,z) \rightarrow R(x,z))) \rightarrow (\forall x.\forall y.\forall z.(R(y,x) \wedge R(z,y) \rightarrow R(z,x))) \tag{11}$$
$$\exists x.\exists y.\exists z.((R(x,y) \wedge R(y,z) \rightarrow R(x,z))) \wedge \neg(\forall x.\forall y.\forall z.(R(y,x) \wedge R(z,y) \rightarrow R(z,x))) \tag{12}$$
$$\exists x.\exists y.\exists z.((R(x,y) \wedge R(y,z) \rightarrow R(x,z))) \wedge \exists x.\exists y.\exists z.\neg(R(y,x) \wedge R(z,y) \rightarrow R(z,x)) \tag{13}$$
$$\exists x.\exists y.\exists z.((R(x,y) \wedge R(y,z) \rightarrow R(x,z))) \wedge \exists x.\exists y.\exists z.\neg R(y,x) \vee \neg(R(z,y) \rightarrow R(z,x)) \tag{14}$$
$$\exists x.\exists y.\exists z.((R(x,y) \wedge R(y,z) \rightarrow R(x,z))) \wedge \exists x.\exists y.\exists z.\neg R(y,x) \vee R(z,y) \wedge \neg R(z,x)) \tag{15}$$
$$\tag{16}$$

The negation is, “There exists x, y, and z, where if R(x, y) and R(y, z) then R(x, z), and there exists x, y, and z such that not R(y, z) or R(z, y) and not R(z, x).”

iii. Negating the statement, “For all x, there exists S such that S is a set and for all z, if z is in set S then z=x, and if z=x then z is in set S.”

$$\neg(\forall x.\exists S.(Set(S) \wedge \forall z.(z \in S \leftrightarrow z = x))) \tag{17}$$
$$\exists x.\neg(\exists S.(Set(S) \wedge \forall z.(z \in S \leftrightarrow z = x))) \tag{18}$$
$$\exists x.\forall S.\neg(Set(S) \wedge \forall z.(z \in S \leftrightarrow z = x)) \tag{19}$$
$$\exists x.\forall S.\neg Set(S) \vee \neg(\forall z.(z \in S \leftrightarrow z = x)) \tag{20}$$
$$\exists x.\forall S.\neg Set(S) \vee \exists z.\neg(z \in S \leftrightarrow z = x) \tag{21}$$
$$\exists x.\forall S.\neg Set(S) \vee \exists z.\neg(z \in S \leftrightarrow z = x) \tag{22}$$
$$\tag{23}$$

How to distribute negation over bidirectional arrow?

Problem 4: ’Cause I’m Happy

i) Statment 2. The only way to make a conditional statement false is if the antecedent is true and the consequent is false. In this case, when the antecedent is true everyone who is a person is happy. If this is true, then it is impossible that there exists a person who is unhappy.

ii) Statement 1. The only way to make a conditional statement false is if the antecedent is true and the consequent is false. In this case, when the antecedent is true there is at least one person in the world who is happy. To make the consequent false, we say that not everyone in the world is happy.

iii) Statement 2. If we negate this statement we get the formula...

iv) Statement 2. Take the negation and show that it is always false.

v) Statement 2. The only way to make an implication false is...

vi) Statement 2.

Problem 5: Translating into Logic

i)

ii)

iii)

iv) $\forall Q Set(Q).\ \exists P Set(P).\ \forall S Set(S).\ (S \in P \cap \forall x.\ \forall y.\ (x \in S \to x \in Q \cap y \notin S \to y \notin Q))$

v)

Problem 6: Raven Paradox

The Raven Paradox involves a scientific statement. As we know from Popper, it is only possible to falsify scientific statements. Induction is a myth.

Problem 7: Graph Coloring

Marty?

Problem 8: Tournament Cycles

A tournament is a directed graph with n nodes where there is exactly one edge between any pair of distinct nodes and there are no self-loops. Prove that if a tournament graph contains a cycle of any length, then it contains a cycle of length three.

Theorem. *If a tournament graph contains a cycle of any length, then it contains a cycle of length three.*

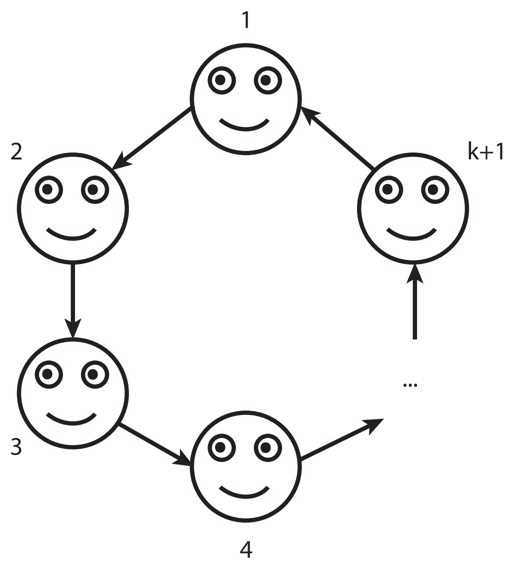

Figure 1: A figure

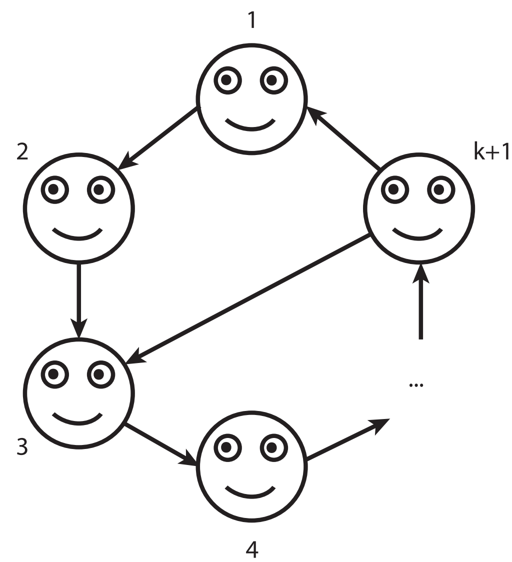

Figure 2: Another figure

Proof. By strong induction. Let $P(n)$ be the statement "if a tournament contains a cycle of length n, then it contains a cycle of length three." We will prove that $P(n)$ holds for $n \geq 3$.

For our base case, we show that $P(3)$ is true. $P(3)$ states that a tournament with a cycle of length 3 contains a cycle of length 3. This is a tautology.

For our inductive step, assume that for some $k \geq 3$ that $P(k)$ is true; that is, that a tournament containing a cycle of length k also contains a cycle of length three. We will prove that $P(k+1)$ is true, that if a tournament contains a cycle of length $k+1$, then it also contains a cycle of length 3.

Consider any tournament with a cycle of length $k+1$. Remove all the edges from this graph except for this cycle. A representation of this graph is shown in Figure 1. Because this is a tournament, we know that every node must be connected to every other node.

$\square$