Manas Kumar Mishra.
Roll No - 18B078 (ESD18I011)

Q1):- According to Divergence Theorem :-

Flux through the closed surface is equal the volume intergal of Divergence of that field through the ~~order~~ whole volume.

$$\therefore \iint_S \bar{F}\cdot\hat{n}\, ds = \iiint_V \nabla\cdot\bar{F}\, dv$$

$$\therefore \nabla\cdot\bar{F} = \frac{\partial}{\partial x}(3z^2y^2) + \frac{\partial}{\partial y}(9x^2y^2z) + \frac{\partial}{\partial z}(z - 4xz^2).$$

$$= 0 + 18x^2zy + 1 - 8xz.$$

$$\therefore \iiint_V \nabla\cdot\bar{F}\, dv = \iiint (18x^2zy + 1 - 8xz)\, dx\, dy\, dz.$$

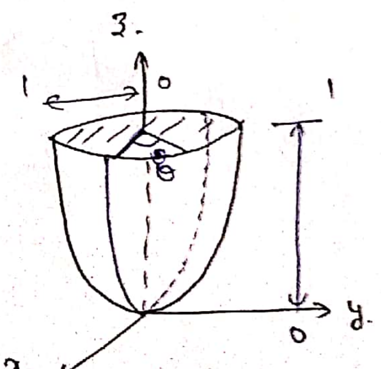
$\Rightarrow$ Now solving these integral use polar coordinates.

$x = r\cos\theta$, $y = r\sin\theta$ $\quad z = r^2$ $\left(\because z = x^2 + y^2\right)$.

where z changing $0 \to 1$ or r^2

r " $0 \to 1$ because of ($x^2+y^2=1$ surface)

θ " $0 \to 2\pi$



$$\therefore \iiint_0^{r^2} (18x^2y z + 1 - 8xz)\, dz\, dx\, dy$$

$$= \left. 9x^2yz^2 + z - 4xz^2 \right|_0^{r^2}$$

$$\Rightarrow 9r^4x^2y + r - 4xr^4$$

Now put ~~and~~ polar coordinates.

$$\iint_0^1 \left(9r^4\cos^2\theta\, r\sin\theta\, r^2 + r^2 - 4r\cos\theta\, r^4\right)(r\,dr\,d\theta)$$

$$(\because dx\,dy = r\,dr\,d\theta)$$

$$\Rightarrow \int_0^{2\pi}\left[\int_0^1 9r^8\cos^2\theta\sin\theta\, dr + \int_0^1 r^3\,dr - 4\int r^6\cos\theta\, dr\right] d\theta$$

$$\Rightarrow \int_0^{2\pi} \left. r^9\cos^2\theta\sin\theta \right|_0^1 + \left. \frac{r^4}{4}\right|_0^1 - \left. \frac{4r^7}{7}\cos\theta\right|_0^1 \, d\theta$$

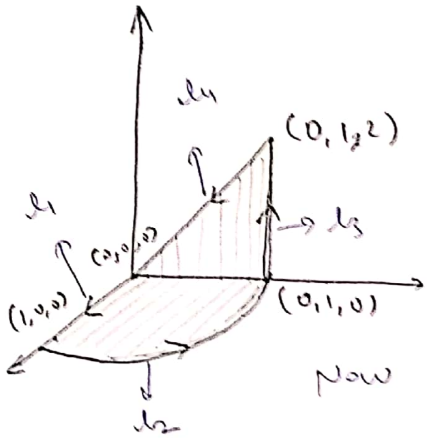
$$= \int_0^{2\pi} \cos^2\theta\sin\theta\, d\theta + \int_0^{2\pi} \frac{1}{4}\, d\theta - \frac{4}{7}\int_0^{2\pi}\cos\theta\, d\theta$$

$$\Rightarrow \cancel{-\frac{1}{3}\cos^3\theta \Big|_0^{2\pi}}^{\,0} + \frac{\theta}{4}\Big|_0^{2\pi} - \cancel{\frac{4}{7}\sin\theta\Big|_0^{2\pi}}^{\,0}$$

$$\Rightarrow \frac{2\pi}{4} = \boxed{\frac{\pi}{2}\ \text{Ans.}}$$

$$\therefore \iint_S \bar{F}\cdot\hat{n}\, dS = \frac{\pi}{2} A_z .$$

Q2:-) $\bar{V} = r\cos^2\theta\,\hat{r} - r\cos\theta\sin\theta\,\hat{\theta} + 3r\,\hat{\varphi}.$



We know that

$$d\bar{l} = \hat{r}\,dr + \hat{\theta}\,r\,d\theta + \hat{\phi}\,r\sin\theta\,d\varphi.$$

Now $\bar{V}\cdot d\bar{l} =$

$$= r\cos^2\theta\,dr - r^2\cos\theta\sin\theta\,d\theta + 3r^2\sin\theta\,d\varphi$$

Now for l_1 case

r is changing $0 \to 1$

φ is const $\varphi = 0$ $\therefore d\varphi = 0$

θ is const $\theta = \pi/2$. $\therefore d\theta = 0$

$$\therefore \int_{l_1} \bar{V}\cdot d\bar{l} = \int_0^1 r\cos^2\left(\frac{\pi}{2}\right) dr + 0 + 0 = 0.$$

0

Now for l_2 Case

r is constant $r = 1$ $\therefore dr = 0$

φ is changing $\phi = 0 \to \pi/2$

θ is constant $\theta = \frac{\pi}{2}$ $d\theta = 0$

$$\int_{l_2} \bar{V}\cdot d\bar{l} = \int_0^{\pi/2} 3(1)^2 \sin\left(\frac{\pi}{2}\right) d\varphi = \frac{3\pi}{2}.$$

For l_3 case :-

$\varphi = \frac{\pi}{2}$, But r and θ Both changing.

$\therefore$ $r\sin\theta = y = 1$

Then $r = \frac{1}{\sin\theta}$

$$dr = -\frac{1}{\sin^2\theta}\cos\theta\, d\theta.$$

~~and~~ θ $\frac{\pi}{2} \to \pi/4$.

$$\therefore \int_{l_3} \bar{V}\cdot d\bar{l} = \int_{\pi/2}^{\pi/4} r\cos^2\theta\, dr - (r\cos\theta\sin\theta)(r\, d\theta)$$

$$= \int_{\pi/2}^{\pi/4} \frac{-\cos^3\theta}{\sin^3\theta}\, d\theta - \frac{\cos\theta\,\cancel{\sin\theta}}{\sin^{\cancel{2}}\theta}\, d\theta.$$

$$= \int_{\pi/2}^{\pi/4} \frac{\cos\theta}{\sin\theta}\left[\frac{\cos^2\theta}{(-\sin^2\theta)} - 1\right] d\theta.$$

$$= \int_{\pi/2}^{\pi/4} -\frac{\cos\theta}{\sin^3\theta}\, d\theta$$

$$= \left.\frac{1}{2\sin^2\theta}\right|_{\pi/2}^{\pi/4} = \frac{1}{2}.$$

Similarly :-

For l_4 Case :-

$\theta = \frac{\pi}{4}$, $\varphi = \frac{\pi}{2}$, $r : \sqrt{2} \to 0$

$$\int_{l_4} \bar{v}\cdot d\bar{l} = \int_{\sqrt{2}}^{0} (r\cos^2\theta)\, dr = -\frac{1}{4}\cdot r^2\Big|_0^{\sqrt{2}} = -\frac{1}{2}.$$

$\Rightarrow \oint \bar{v}\cdot d\bar{l} = 0 + \frac{3\pi}{2} + \frac{1}{2} - \frac{1}{2} = \frac{3\pi}{2}.$

Now By Stokes' theorem

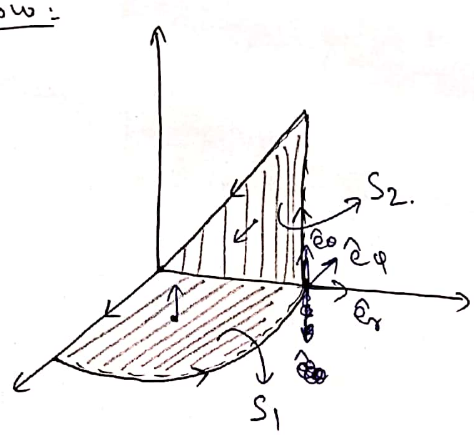
$$\iint_S (\nabla \times \bar{v})\, d\bar{a} = \oint \bar{v}\cdot d\bar{l}.$$

$\therefore$ Curl in spherical coordinate system.

$$= \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\,\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ V_r & rV_\theta & r\sin\theta\, V_\phi \end{vmatrix}.$$

$$\Rightarrow (\nabla \times \bar{v}) = 3\cot\theta\,\hat{r} - 6\hat{\theta}.$$

$\Rightarrow$ Now :-



For surface S_2 :-

$$\iint_{S_2} (\nabla\times\bar{v})\, d\bar{a} \quad \text{where} \quad d\bar{a} = -r\,dr\,d\phi\,\hat{\phi}.$$

$$\Rightarrow -\iint \left(3\cos\theta\,\hat{r} - 6\hat{\theta}\right) r\,dr\,d\phi\,\hat{\phi}$$

$$= 0$$

For surface S_1 :-

$$\iint_{S_1} (\nabla\times\bar{v})\, d\bar{a} = \iint \left(3\cos\theta\,\hat{r} - 6\hat{\theta}\right) r\sin\theta\, dr\, d\phi\, \hat{\theta}$$

$$\int_0^1 \int_0^{\pi/2} 6r\sin\left(\frac{\pi}{2}\right) dr\, d\phi \qquad \left(\because, \theta = \frac{\pi}{2}\right). \text{ in } S_1 \text{ surface}$$

$$\Rightarrow \frac{3\pi}{2}.$$

$\therefore \int (\nabla \times \bar{V}) \, d\bar{a} = \frac{3\pi}{2}$.

$\therefore \iint_S (\nabla \times \bar{V}) \, d\bar{a} = \oint \bar{V} \cdot d\bar{l}$ are same.

(3) :- Sol:-

$$\nabla \times (\nabla \times \bar{A}) = \nabla(\nabla \cdot \bar{A}) - \nabla^2 \bar{A}$$

Now:- assume:-

$$\bar{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}.$$

$$\nabla \times \bar{A} = i\left(\frac{\partial}{\partial y} A_z - \frac{\partial}{\partial z} A_y\right) - \left(\frac{\partial}{\partial x} A_z - \frac{\partial}{\partial z} A_x\right)\hat{j} + \hat{k}\left(\frac{\partial}{\partial x} A_y - \frac{\partial}{\partial y} A_x\right)$$

$$\Rightarrow \nabla \times (\nabla \times \bar{A}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \left(\frac{\partial}{\partial y} A_z - \frac{\partial A_y}{\partial z}\right) & \left(\frac{\partial}{\partial x} A_z - \frac{\partial A_x}{\partial z}\right) & \left(\frac{\partial}{\partial x} A_y - \frac{\partial}{\partial y} A_x\right) \end{vmatrix}$$

$$\Rightarrow \hat{i}\left[\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} A_y - \frac{\partial}{\partial y} A_x\right) - \frac{\partial}{\partial z}\left(\frac{\partial}{\partial x} A_z - \frac{\partial}{\partial z} A_x\right)\right]$$

$$- \hat{j}\left[\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} A_y - \frac{\partial}{\partial y} A_x\right) - \frac{\partial}{\partial z}\left(\frac{\partial}{\partial y} A_z - \frac{\partial}{\partial z} A_y\right)\right].$$

$$+ \hat{k}\left[\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} A_z - \frac{\partial}{\partial z} A_x\right) - \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y} A_z - \frac{\partial}{\partial z} A_y\right)\right].$$

$\nabla(\nabla\cdot\bar{A})$ Now.

$$\nabla\cdot\bar{A} = \frac{\partial A_x}{\partial x} + \frac{\partial}{\partial y}A_y + \frac{\partial}{\partial z}A_z.$$

$$\nabla(\nabla\cdot\bar{A}) = \hat{i}\,\frac{\partial}{\partial x}\left[\frac{\partial A_x}{\partial x} + \frac{\partial}{\partial y}A_y + \frac{\partial}{\partial z}A_z\right] + \hat{j}\,\frac{\partial}{\partial y}\left[\frac{\partial A_x}{\partial x} + \frac{\partial}{\partial y}A_y + \frac{\partial}{\partial z}A_z\right] + \hat{k}\,\frac{\partial}{\partial z}\left[\frac{\partial A_x}{\partial x} + \frac{\partial}{\partial y}A_y + \frac{\partial}{\partial z}A_z\right].$$

$$= \hat{i}\left[\frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2}{\partial x\partial y}A_y + \frac{\partial^2}{\partial x\partial z}A_z\right] + \hat{j}\left[\frac{\partial^2 A_x}{\partial y\partial x} + \frac{\partial^2}{\partial y^2}A_y + \frac{\partial^2}{\partial z^2}A_z\right] + \hat{k}\left[\frac{\partial^2 A_x}{\partial z\partial x} + \frac{\partial^2}{\partial z\partial y}A_y + \frac{\partial^2}{\partial z^2}A_z\right].$$

Now $(\nabla\times(\nabla\times\bar{A}) - \nabla(\nabla\cdot\bar{A})$

$$\Rightarrow -\hat{i}\left[\frac{\partial^2}{\partial x^2}A_x + \frac{\partial^2}{\partial y^2}A_x + \frac{\partial^2 A_x}{\partial z^2}\right] - \hat{j}\left[\frac{\partial^2}{\partial x^2}A_y + \frac{\partial^2}{\partial y^2}A_y + \frac{\partial^2}{\partial z^2}A_y\right] - \hat{k}\left[\frac{\partial^2}{\partial x^2}A_z + \frac{\partial^2}{\partial y^2}A_z + \frac{\partial^2}{\partial z^2}A_z\right].$$

which is Nothing but $-(\nabla^2\cdot\bar{A})$. $\left[\because \nabla^2 V = \nabla^2 V_x + \nabla^2 V_y + \nabla^2 V_z\right]$.

$$\therefore \quad \nabla\times(\nabla\times\bar{A}) = \nabla(\nabla\cdot\bar{A}) - \nabla^2\bar{A}$$

Hence proof.

Q. 4) :-

$$\int_S \nabla \phi \times ds = \oint_C \varphi \, d\bar{l}$$

Now stoke's Theorem :-

$$\int_S (\nabla \times \bar{v}) . d\bar{a} = \int \bar{v} \, d\bar{l}$$

Now assume :- $\bar{V} = \bar{C} \varphi$

where $\varphi \rightarrow$ any scalar function.

and $\bar{C} \rightarrow$ constant vector [Direction and magnitude are same].

$$\therefore \quad \nabla \times \bar{C}\varphi = \varphi(\cancel{\nabla \times \bar{C}}) - C \times (\nabla \varphi)$$

0

$$= -\bar{C} \times (\nabla \varphi) \quad \text{(Because } \bar{C} \text{ is constant vector)}$$

$$\therefore \int_S (\nabla \times \bar{v}) . d\bar{a} = \oint_C \bar{v} . d\bar{l}$$

$$-\int_S \underbrace{(\bar{C} \times \nabla \phi)}_{\text{Scalar Triple Product}} . d\bar{a} = \oint_C \bar{C} . \phi . d\bar{l}$$

$$\therefore \; -[C \times \nabla \phi] . d\bar{a} = C . [\nabla \phi \times d\bar{a}]$$

$$\therefore \int_S C . [\nabla \phi \times d\bar{a}] = \oint_C \bar{C} . \phi \, d\bar{l}$$

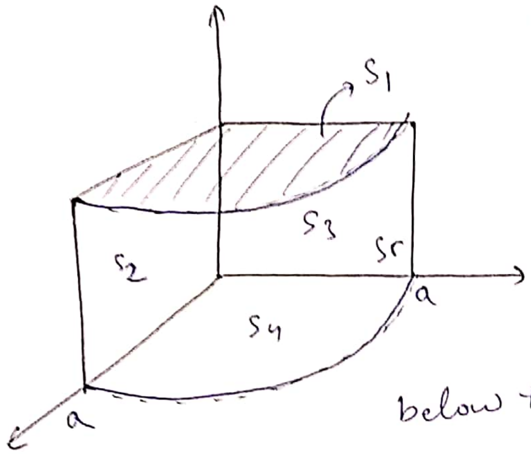
Put C from Both side.

$$\boxed{\int_S \nabla \phi \times d\bar{a} = \oint_C \bar{\phi} \varphi \, d\bar{l}}$$

Hence Proof.

Q5) $\bar{F} = z\hat{i} + x\hat{j} + y\hat{k}$ $\quad x^2+y^2=a^2$, $z=h$. 5

<u>Now find :-</u>



$\nabla \cdot \bar{F} = 0.$

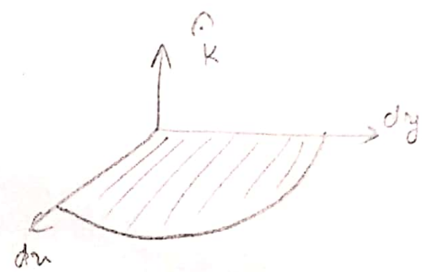
∴ it means the flux through the ~~cy~~ cylinder is zero

According to question flux ~~below~~ below the S_1 surface.

$$S_1 + S_2 + S_3 + S_4 + S_5 = 0$$

$$\therefore S_1 = -\sum_{i=2}^{5} S_i$$

∴ Now flux through the S_1



$$\Rightarrow \int_{S_1} \bar{F}\cdot\hat{n}\,d\bar{s}$$

$$\Rightarrow \int (z\hat{i} + x\hat{j} + y\hat{k})\cdot\hat{k}\,dx\,dy.$$

$$\iint y\,dx\,dy$$

Now use Polar coordinates $y = r\sin\theta$
$dx\,dy = r\,dr\,d\theta.$

$$\Rightarrow \int_0^{\pi/2}\int_0^{a} r^2 \sin\theta\,dr\,d\theta$$

$$\Rightarrow \int_0^{\pi/2} \frac{r^3}{3}\Big|_0^{a} \sin\theta\,d\theta = \int_0^{\pi/2} \frac{a^3}{3}\sin\theta\,d\theta$$

$$= \frac{a^3}{3}\left(-\cos\theta\right)_0^{\pi/2}$$

$$= \frac{a^3}{3}\left[1 - 0\right]$$

$$= \frac{a^3}{3}$$

$\therefore$ $\mathfrak{g}_o$ Flux $= \frac{a^3}{3}$. Ans.