Chapter 6
Graphs

2024 Spring

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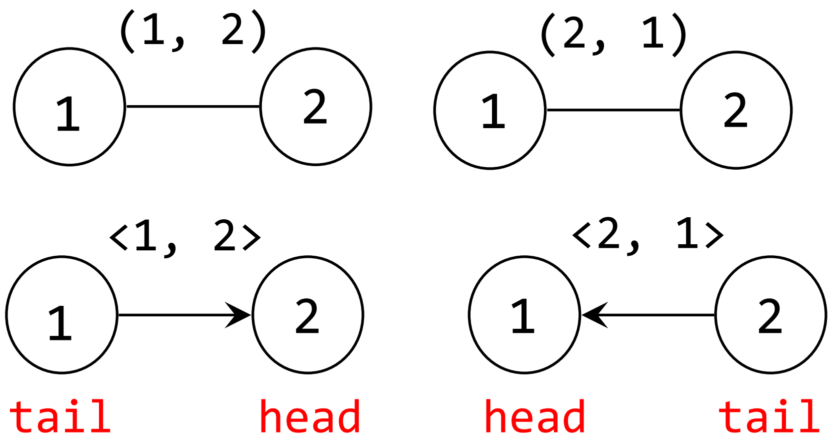
Contents

Graph Abstract Data Type
Elementary Operations
Spanning Trees
Shortest Paths

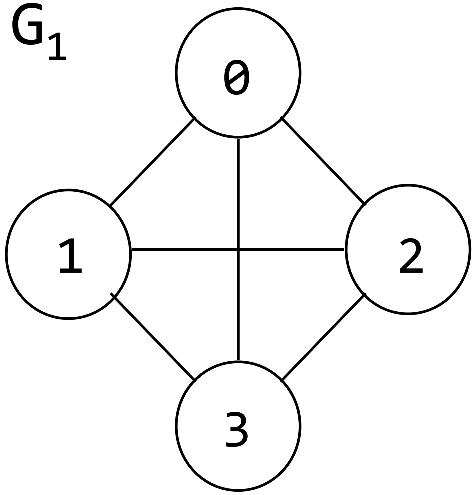
Definitions

- $G = (V, E)$, where
  - $V(G)$: set of vertices - finite and nonempty (정점)
  - $E(G)$: set of edges – finite and possibly empty (간선)
  - Restrictions
    - A graph may not have an edge from a vertex, I, back to itself
    - A graph may not have multiple occurrence of the same edge
- Undirected graph
  - unordered: $(u, v) = (v, u)$
- Directed graph (digraph)
  - ordered: $<u, v> \neq <v, u>$



Examples of Graph (1)

G_1



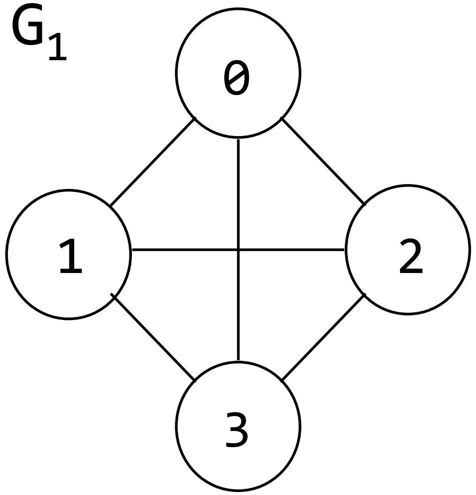
$V(G_1) = \{0, 1, 2, 3\}$
$E(G_1) = \{(0,1), (0,2), (0,3), (1,2), (1,3), (2,3)\}$

G_2

$V(G_2) = \{0, 1, 2, 3, 4, 5, 6\}$
$E(G_2) = \{(0,1), (0,2), (1,3), (1,4), (2,5), (2,6)\}$

Examples of Graph (1)

G_1



$V(G_1) = \{0, 1, 2, 3\}$

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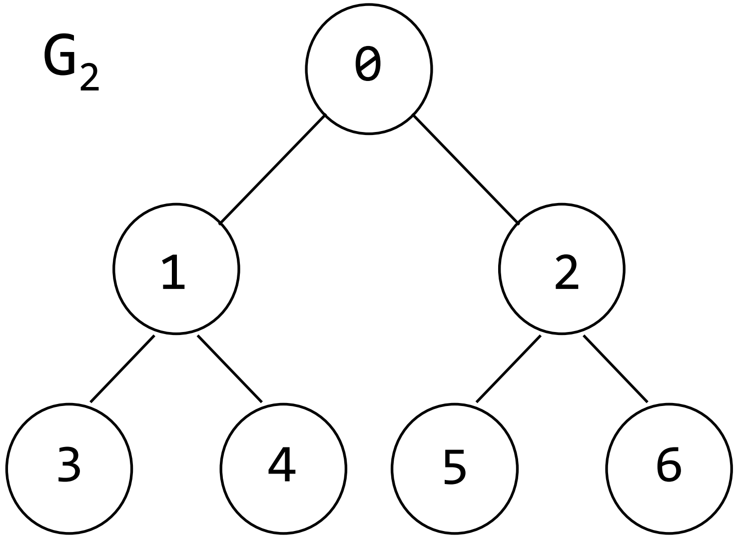
G_2

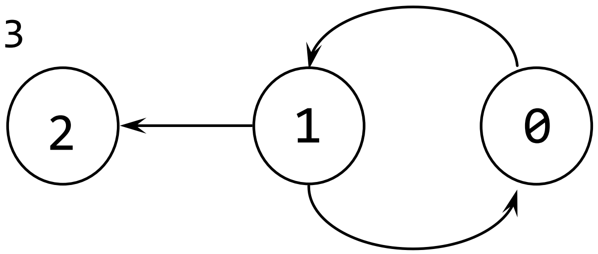


$V(G_2) = \{0, 1, 2, 3, 4, 5, 6\}$

$E(G_2) = \{(0,1), (0,2), (1,3), (1,4), (2,5), (2,6)\}$

Examples of Graph (2)

G_3



$V(G_3) = \{0, 1, 2\}$
$E(G_3) = \{\langle 0,1\rangle, \langle 1,0\rangle, \langle 1,2\rangle\}$

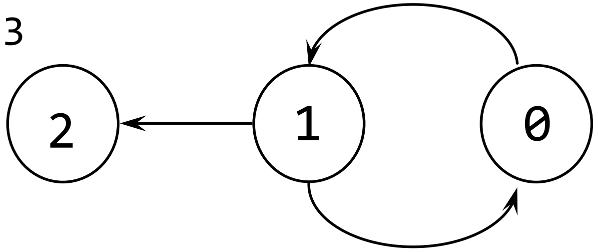
G_4

G_5 $V(G_5) = \{0, 1, 2, 3\}$
$E(G_5) = \{(0,1), (1,2), (1,3), (3,1), (3,2), (2,3), (3,2)\}$

$V(G_4) = \{0, 1, 2\}$
$E(G_4) = \{ \langle 0,0\rangle, \langle 0,2\rangle, \langle 1,0\rangle, \langle 2,1\rangle, \langle 2,0\rangle \}$

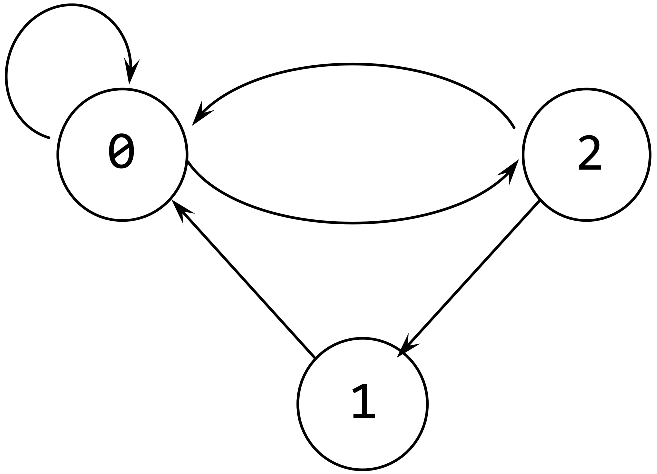
Examples of Graph (2)

G_3



$V(G_3) = \{0, 1, 2\}$
$E(G_3) = \{\langle 0,1\rangle, \langle 1,0\rangle, \langle 1,2\rangle\}$

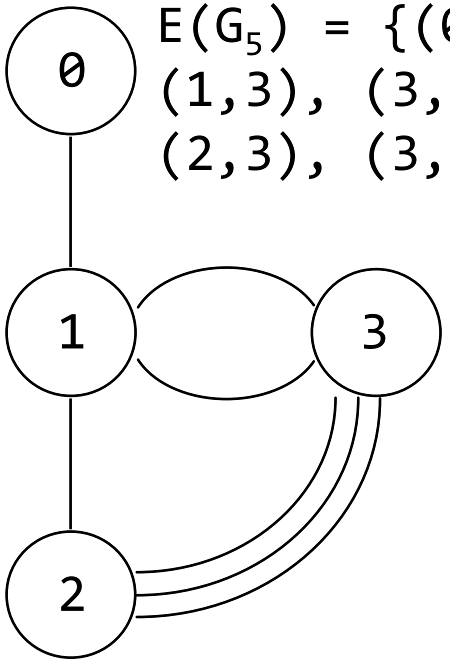
G_4



$V(G_4) = \{0, 1, 2\}$
$E(G_4) = \{ \langle 0,0\rangle, \langle 0,2\rangle, \langle 1,0\rangle, \langle 2,1\rangle, \langle 2,0\rangle \}$

G_5

$V(G_5) = \{0, 1, 2, 3\}$
$E(G_5) = \{(0,1), (1,2), (1,3), (3,1), (3,2), (2,3), (3,2)\}$



Terminology (1)

- Complete graph (완전 그래프)

  A graph that has the <u>maximum number of edges</u>

  →For an <u>undirected graph with *n* vertices</u>,

  the maximum number of the edges = $n(n\text{-}1)/2$

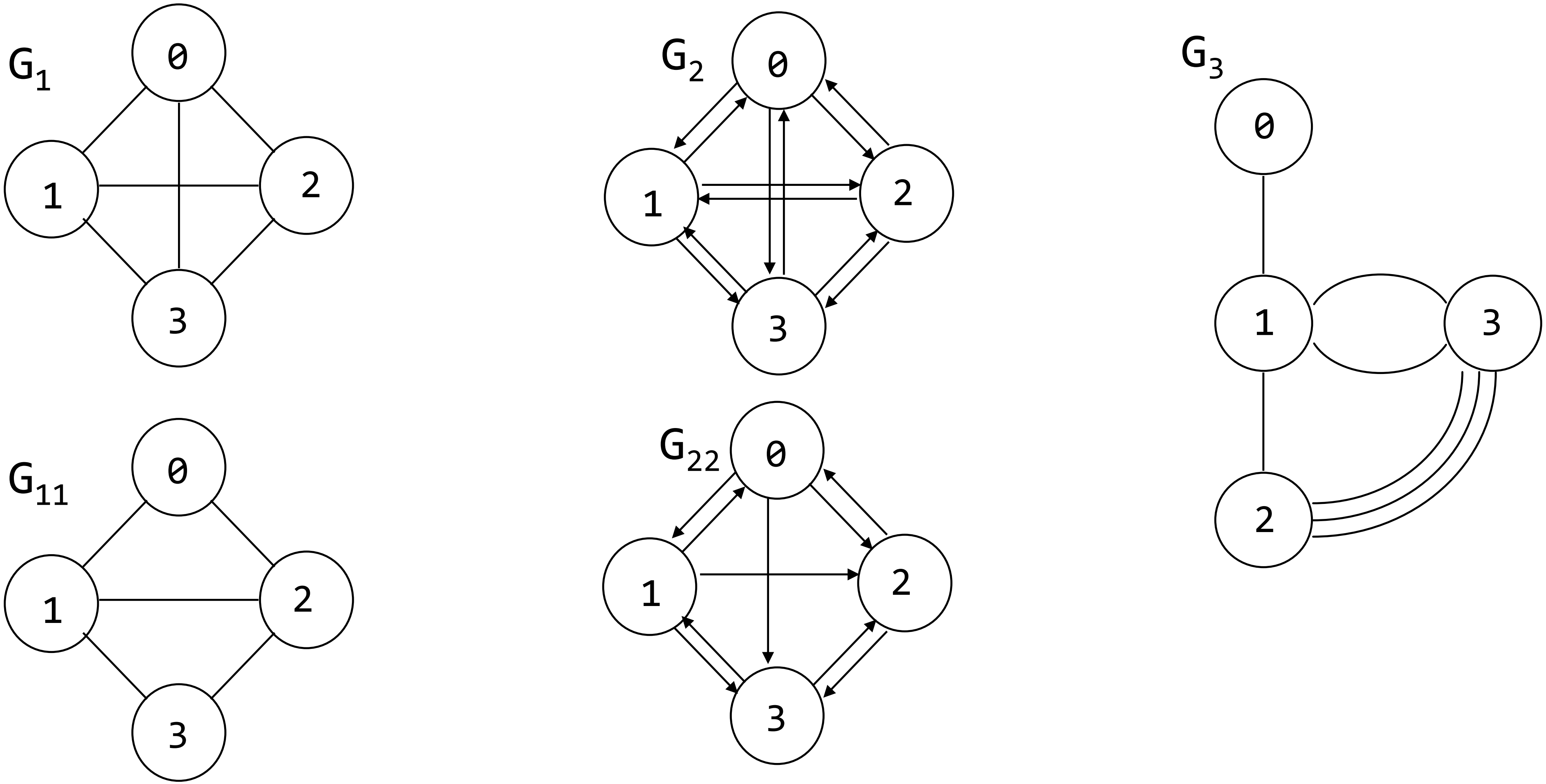
  →For a <u>directed graph with *n* vertices</u>,

  the maximum number of the edges = $n(n\text{-}1)$

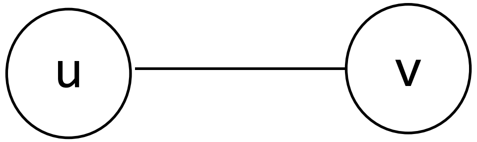
- Multigraph (다중그래프)

  A graph whose edges are unordered pairs of vertexes, and <u>the same pair of vertexes can be connected by multiple edges</u>

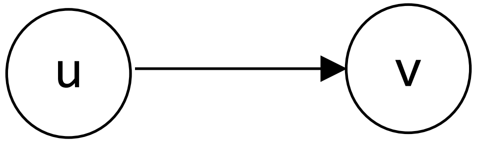
Terminology - Example



Terminology (2)



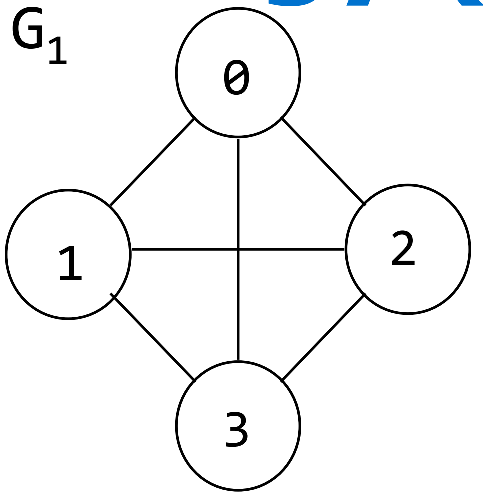
❖ If (u, v) is an edge of an undirected graph,
- The vertices u and v are adjacent (인접한)
- The edge (u, v) is incident (부속된) on u and v

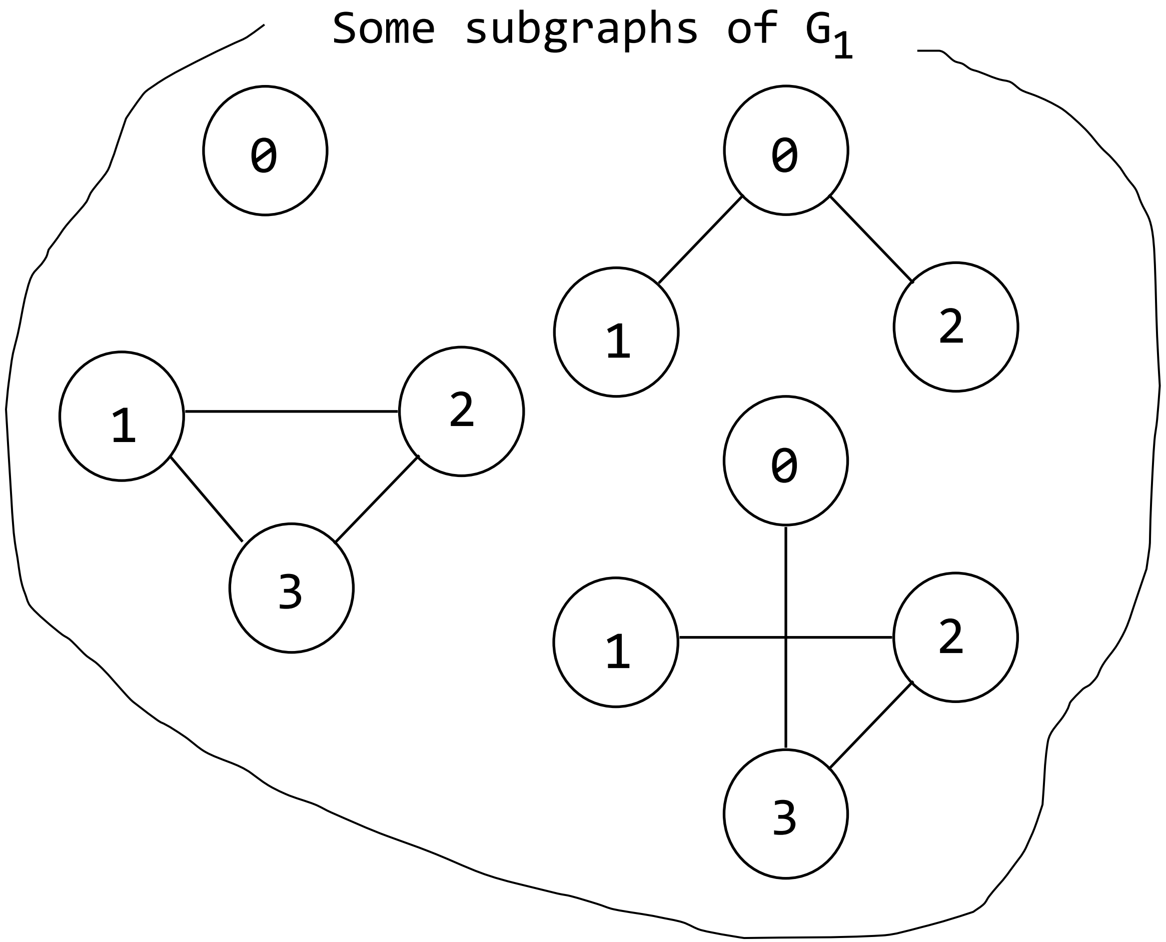


❖ If $<u, v>$ is a directed edge,
- The vertex u is adjacent to v
- The vertex v is adjacent from u
- The edge $<u, v>$ is incident on u and v

A subgraph of G is a graph G' such that
$V(G') \subseteq V(G)$ and $E(G') \subseteq E(G)$

Terminology (3)





Terminology (4)

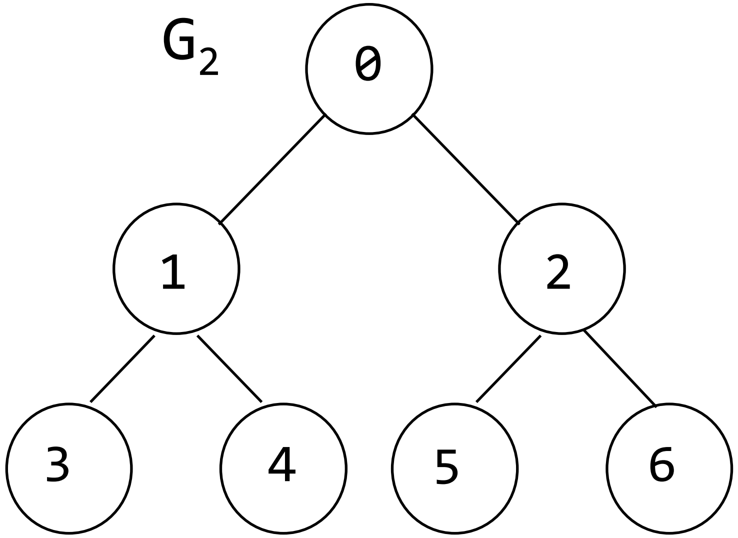
- A **path** from vertex u to v in graph G is a sequence of vertices, $\boldsymbol{u}$, i_1, i_2, …, i_k, $\boldsymbol{v}$ such that (u, i_1), (i_1, i_2), …, (i_k, v) are edges in an undirected graph
  - If G′ is a directed graph, the path consists of $<u, i_1>$, $< i_1, i_2 >$, …, $< i_k, v>$
  - The length of a path is the number of the edges on it

- A **simple path** is a path in which all vertices, except possibly the first and the last, are distinct

- A **cycle** is a simple path in which the first and the last vertices are the same

Terminology (5)

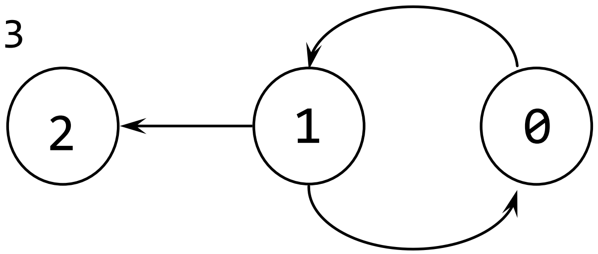
G_2



$V(G_2) = \{0, 1, 2, 3, 4, 5, 6\}$
$E(G_2) = \{(0,1), (0,2), (1,3), (1,4), (2,5), (2,6)\}$

Simple path: 0, 2, 6
0, 1, 4
2, 5
…
0, 2, 6, 2, 6 ??

G_3



$V(G_3) = \{0, 1, 2\}$
$E(G_3) = \{\langle 0,1\rangle, \langle 1,0\rangle, \langle 1,2\rangle\}$

Simple path: 0, 1, 2
1, 0
1, 2

Cycle: 0, 1, 0
0, 1, 0, 1, 2 ??

Terminology (6)

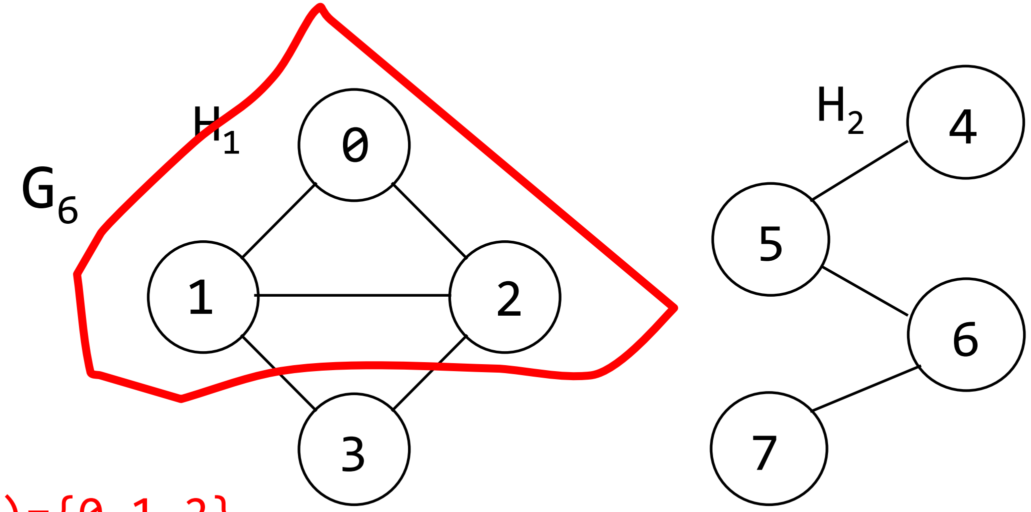
- In an undirected graph G, two vertices u and v are connected if there is a path in G from u and v

- A connected component or simply a component of an undirected graph is a maximal connected subgraph

- A tree is a graph that is connected and acyclic

Terminology (7)

$V(G_6) = \{0, 1, 2, 3, 4, 5, 6, 7\}$
$E(G_6) = \{(0,1), (0,2), (1,2), (1,3), (2,3), (4,5), (5,6), (6,7)\}$



Is 0 adjacent to 1?

Is 0 adjacent to 3?

Is 0 connected to 3?

Is 3 connected to 4?

$V(H_3)=\{0,1,2\}$
$E(H_3)=\{(0,1), (0,2), (1,2)\}$

Connected component?? No

Connected components

$V(H_1) = \{0, 1, 2, 3\}$
$E(H_1) = \{(0,1), (0,2), (1,2), (1,3), (2,3)\}$

$V(H_2) = \{4,5,6,7\}$
$E(H_2) = \{(4,5), (5,6), (6,7)\}$

Terminology (8)

- A directed graph is **strongly connected** if, for every pair of vertices, u and v in $V(G)$, there is a directed path from u and v and also from v to u

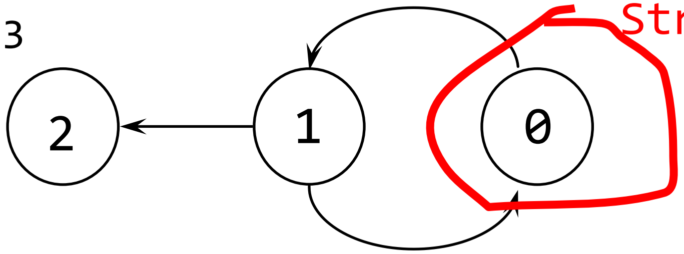
- A strongly connected component is a **maximal** subgraph that is strongly connected

Terminology (9)

$V(H_5)=\{0\}$, $E(H_5)=\{\}$

Strongly connected component?? No

G_3

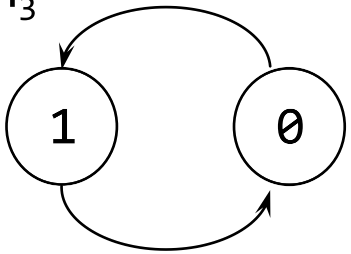


$V(G_3) = \{0, 1, 2\}$

$E(G_3) = \{\langle 0,1\rangle, \langle 1,0\rangle, \langle 1,2\rangle\}$

Strongly connected components

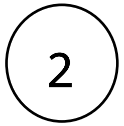
H_3



$V(H_3) = \{0, 1\}$

$E(H_3) = \{\langle 0,1\rangle, \langle 1,0\rangle\}$

H_4



$V(H_4) = \{2\}$

$E(H_4) = \{\}$

Terminology (10)

- The degree of a vertex is the number of edges incident to that vertex
- For a directed graph,

  the in-degree of a vertex v is defined as the number of edges that have v as the head

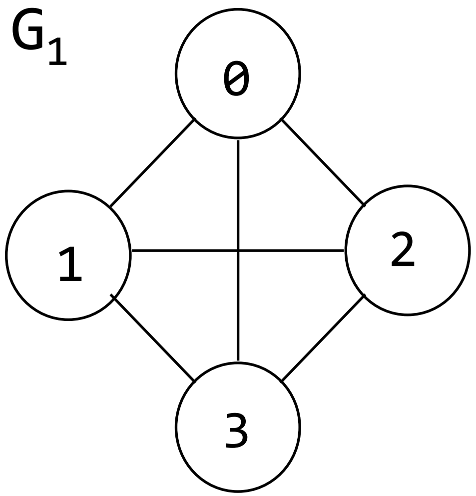
  the out-degree of a vertex v is defined as the number of edges that have v as the tail
- Property

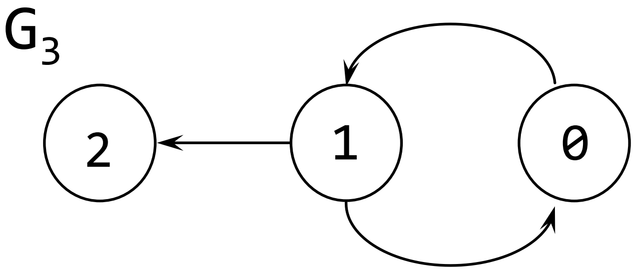
  e : the number of edges

  d_i : the degree of a vertex i in graph G

$$e = \left(\sum_{i=0}^{n-1} d_i\right) / 2$$

Terminology (11)





Degree of 0	
Degree of 1	
Degree of 3	

in-degree of 1	
out-degree of 1	
out-degree of 2	

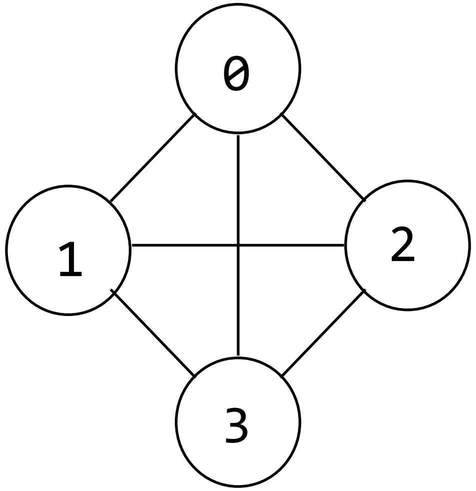
Graph representations

1. Adjacency Matrix
2. Adjacency Lists

Adjacency Matrix

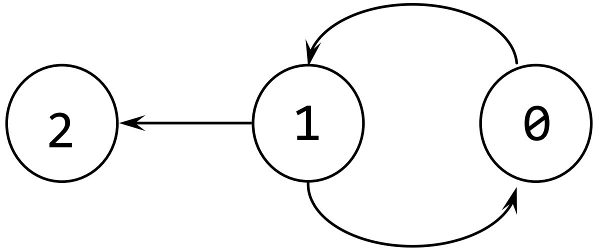
Two-dimensional $n \times n$ array, when the number of nodes is n

$$\texttt{a[i][j]} = \begin{cases} \texttt{1, if (i,j) is adjacent} \\ \texttt{0, otherwise} \end{cases}$$



$$\begin{array}{c|cccc} & [0] & [1] & [2] & [3] \\ \hline [0] & 0 & 1 & 1 & 1 \\ [1] & 1 & 0 & 1 & 1 \\ [2] & 1 & 1 & 0 & 1 \\ [3] & 1 & 1 & 1 & 0 \end{array}$$

G_1



$$\begin{array}{c|ccc} & [0] & [1] & [2] \\ \hline [0] & 0 & 1 & 0 \\ [1] & 1 & 0 & 1 \\ [2] & 0 & 0 & 0 \end{array}$$

G_3

Property of Adjacency Matrix

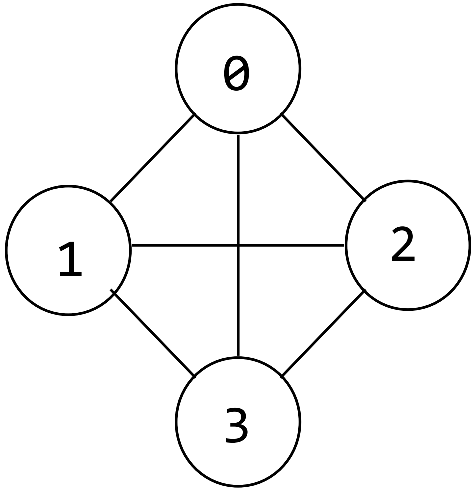
- The adjacency matrix for an undirected graph is symmetric
- The adjacency matrix for a digraph need not be symmetric
- For an undirected graph, the degree of any vertex *i* is its row sum
- For a directed graph,

  the row sum is the out-degree;

  the column sum is the in-degree

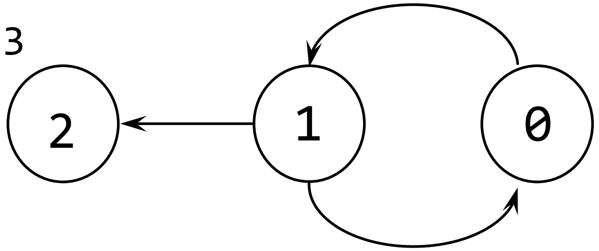
`<i,j> → row i, column j is set to 1`

G_1



$$
\begin{array}{c|cccc}
 & [0] & [1] & [2] & [3] \\
\hline
[0] & 0 & 1 & 1 & 1 \\
[1] & 1 & 0 & 1 & 1 \\
[2] & 1 & 1 & 0 & 1 \\
[3] & 1 & 1 & 1 & 0
\end{array}
$$

G_3



$$
\begin{array}{c|ccc}
 & [0] & [1] & [2] \\
\hline
[0] & 0 & 1 & 0 \\
[1] & 1 & 0 & 1 \\
[2] & 0 & 0 & 0
\end{array}
$$

In-degree of vertex 2 ?

out-degree of vertex 2 ?

0	사당
1	아주대
2	강남
3	의왕
4	평촌
5	수지
6	명동
7	서울역

(사당, 아주대)	(0, 1)
(사당, 강남)	(0, 2)
(강남, 의왕)	(2, 3)
(평촌, 의왕)	(4, 3)
(수지, 강남)	(5, 2)
(아주대, 강남)	(1, 2)
(명동, 서울역)	(6, 7)

	[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]
[0]								
[1]								
[2]								
[3]								
[4]								
[5]								
[6]								
[7]								

0	사당
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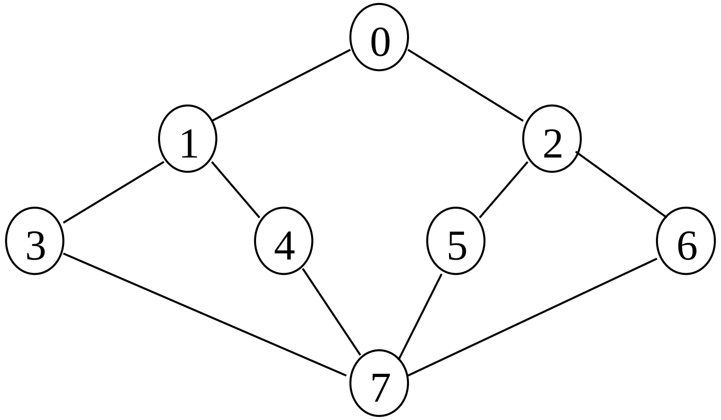
	[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]
[0]	0	1	1	0	0	0	0	0
[1]	1	0	1	0	0	0	0	0
[2]	1	1	0	1	0	1	0	0
[3]	0	0	1	0	1	0	0	0
[4]	0	0	0	1	0	0	0	0
[5]	0	0	1	0	0	0	0	0
[6]	0	0	0	0	0	0	0	1
[7]	0	0	0	0	0	0	1	0

Waste of memory



Utilization = 20/64 = 31(%)

0	1	1	0	0	0	0	0
1	0	0	1	1	0	0	0
1	0	0	0	0	1	1	0
0	1	0	0	0	0	0	1
0	1	0	0	0	0	0	1
0	0	1	0	0	0	0	1
0	0	1	0	0	0	0	1
0	0	0	1	1	1	1	0

Adjacency Lists

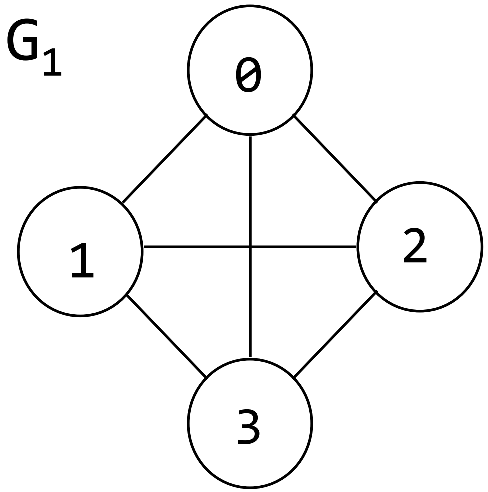
Replace n rows of adjacency matrix with n linked lists

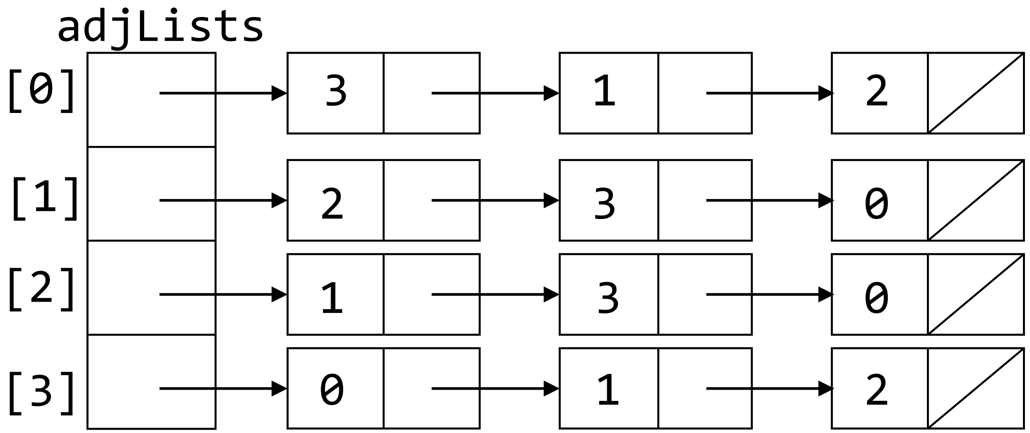
Every vertex i in G has one list

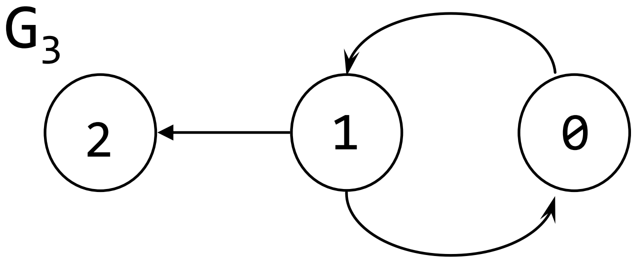
The nodes in chain i represent the vertices that are adjacent from i

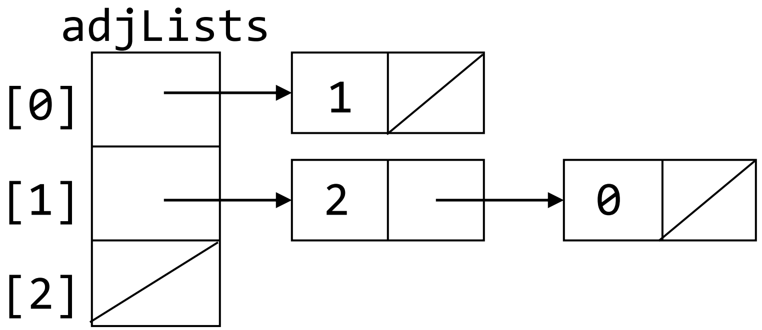
The vertices in each chain are not required to be ordered

Adjacency Lists - Examples









Adjacency Lists in C

```
typedef struct node *nodePointer;
typedef struct node {
    int vertex;
    nodePointer link;
};
nodePointer adjLists[MAX_NODES];
```

0	사당
1	아주대
2	강남
3	의왕
4	평촌
5	수지
6	명동
7	서울역

[0]	
[1]	
[2]	
[3]	
[4]	
[5]	
[6]	
[7]	

(사당, 아주대)	(0, 1)
(사당, 강남)	(0, 2)
(강남, 의왕)	(2, 3)
(평촌, 의왕)	(4, 3)
(수지, 강남)	(5, 2)
(아주대, 강남)	(1, 2)
(명동, 서울역)	(6, 7)

0	사당
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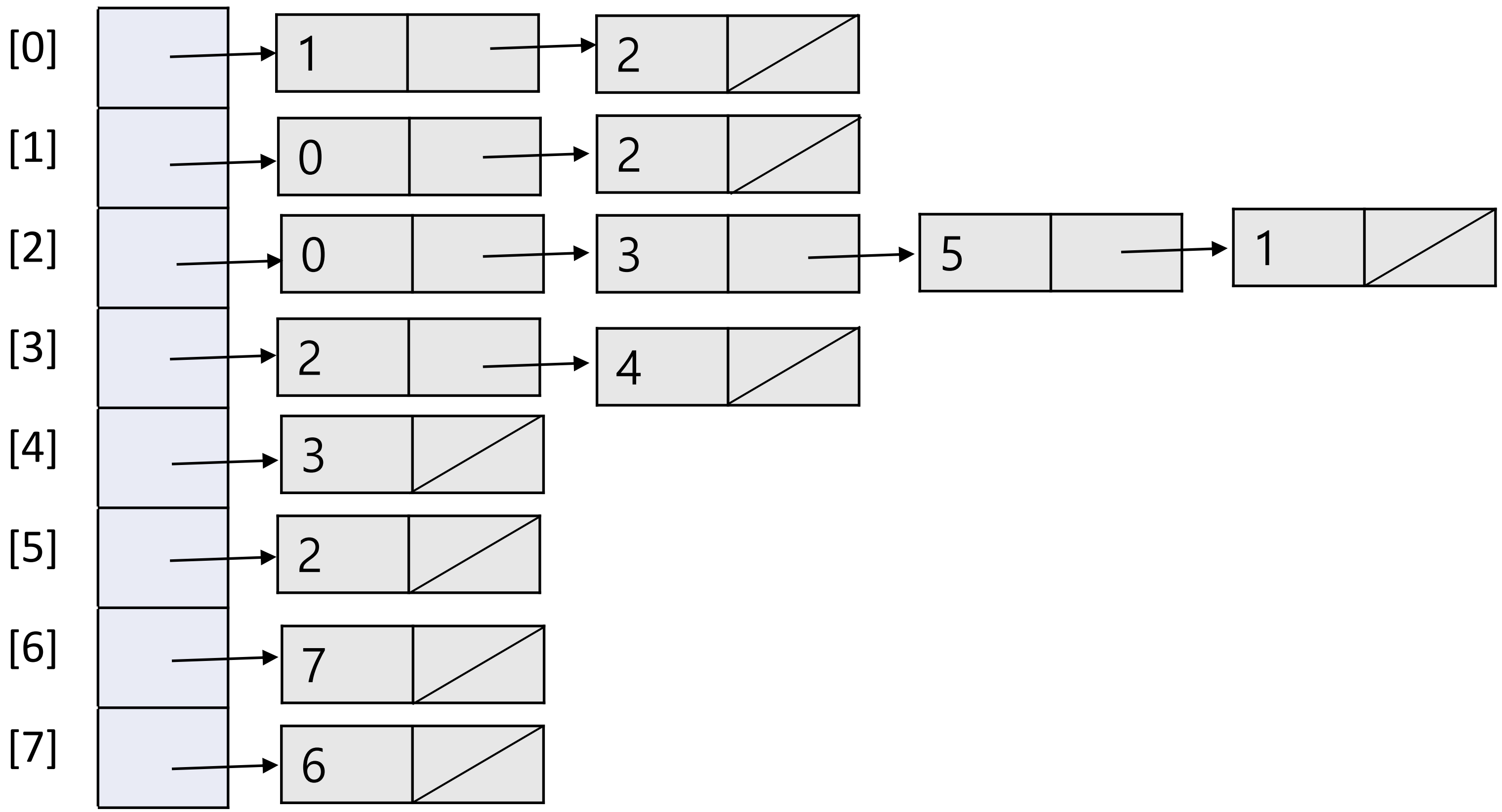


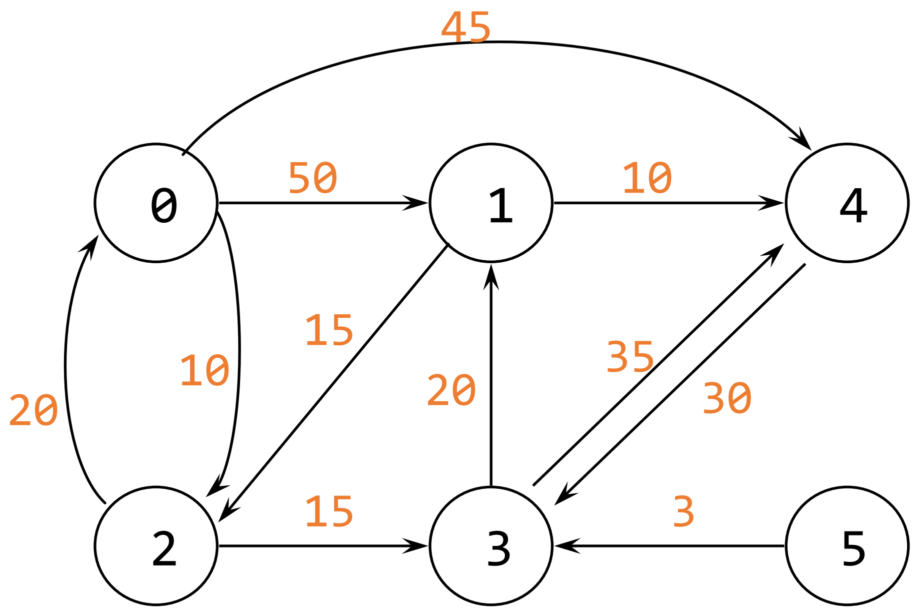
Weighted Edges (가중치 간선)

❖ Assign weights to edges of a graph

- Distance from one vertex to another, or
- Cost of going from one vertex to an adjacent vertex

❖ Modify representation to signify an edge with the weight of the edge

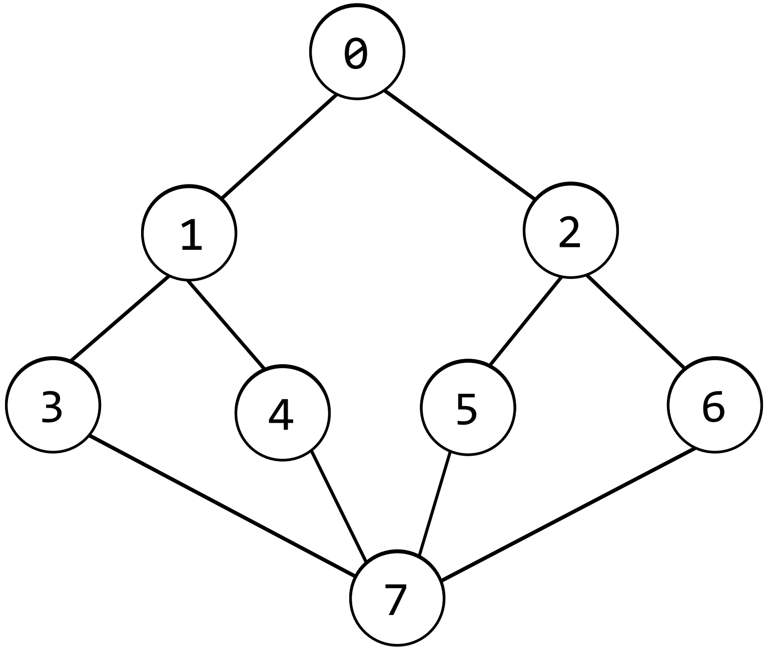
- for adjacency matrix : weight instead of 1
- for adjacency list : add weight field

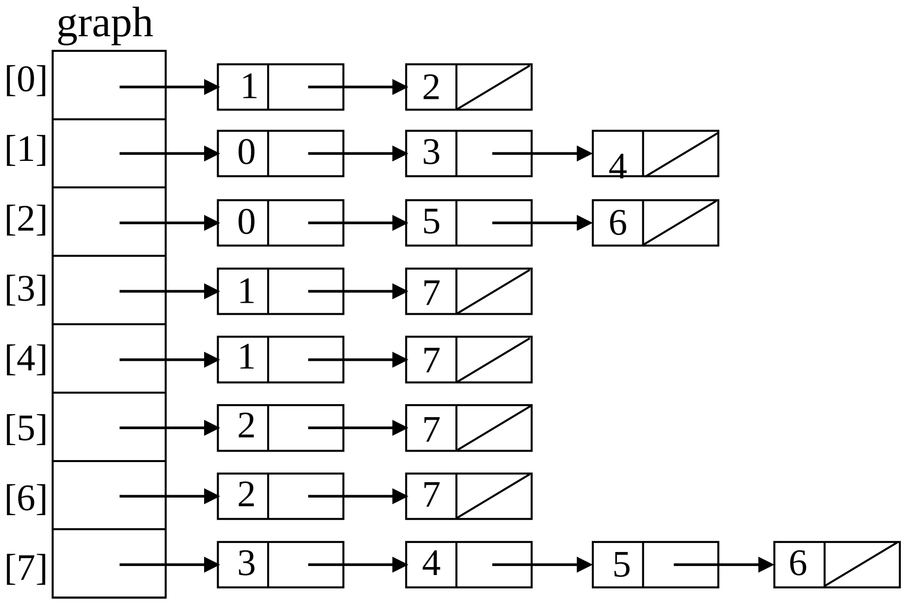


$$
\begin{array}{c|cccccc}
 & [0] & [1] & [2] & [3] & [4] & [5] \\
\hline
[0] & 0 & 50 & 10 & \infty & 45 & \infty \\
[1] & \infty & 0 & 15 & \infty & 10 & \infty \\
[2] & 20 & \infty & 0 & 15 & \infty & \infty \\
[3] & \infty & 20 & \infty & 0 & 35 & \infty \\
[4] & \infty & \infty & \infty & 30 & 0 & \infty \\
[5] & \infty & \infty & \infty & 3 & \infty & 0
\end{array}
$$

Graph Traversal

- ❖ Visit every vertex in a graph
  - DFS (Depth First Search) - similar to a preorder tree traversal
  - BFS (Breath First Search) - similar to a level-order tree traversal

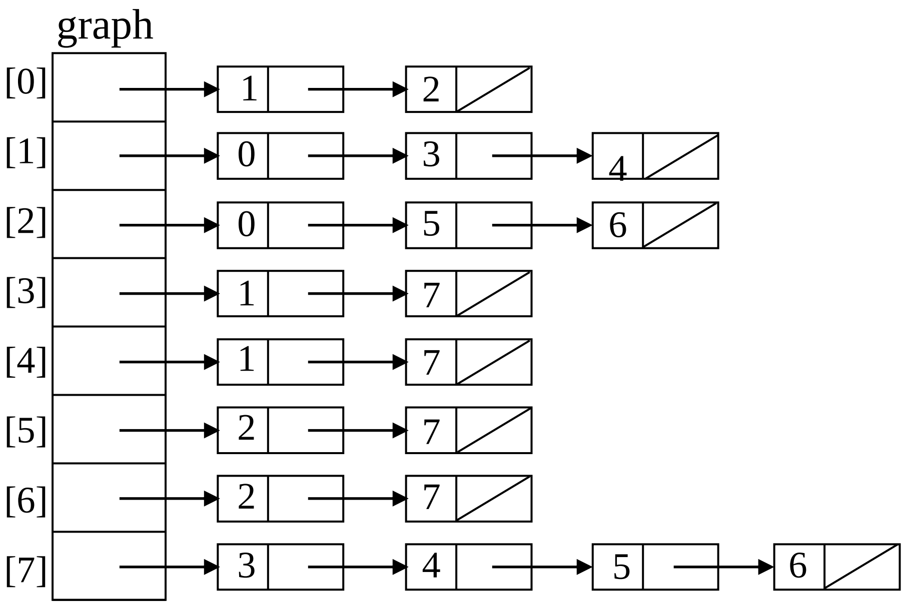




Depth First Search

```
typedef struct node *nodePointer;
typedef struct node {
    int vertex;
    nodePointer link;
};
nodePointer adjLists[MAX_NODES];
```

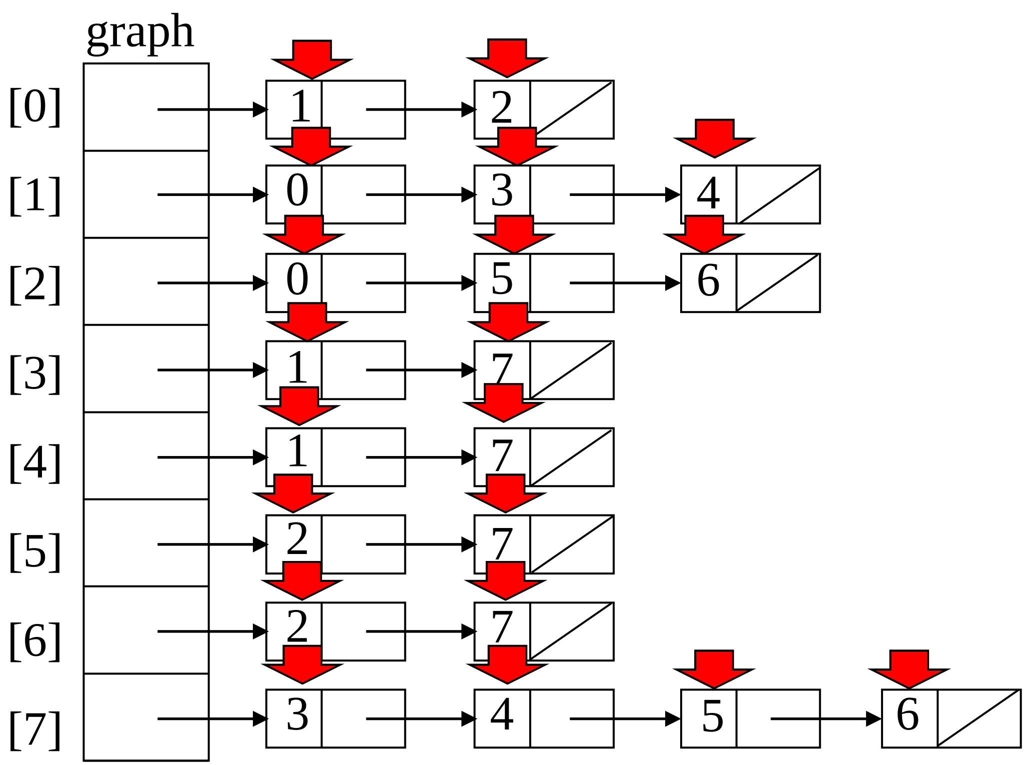
```
#define FALSE 0
#define TRUE 1
short int visited[MAX_VERTICES];
nodePointer graph[MAX_VERTICES];

void dfs(int v)
{
/* depth first search of a graph beginning at v */
   nodePointer w;
   visited[v] = TRUE;
   printf(“%5d”, v);
   for (w = graph[v]; w; w = w->link){
     if (!visited[w->vertex])
        dfs(w->vertex);
   }
}
```



Example of DFS

```
void dfs(int v)
{
   nodePointer w;
   visited[v] = TRUE;
   printf("%5d", v);
   for (w = graph[v]; w; w = w->link){
     if (!visited[w->vertex])
         dfs(w->vertex);
   }
}
```

`dfs(0);`



0 1 3 7 4 5 2 6

dfs(6)
dfs(2)
dfs(5)
dfs(7)
dfs(3)
dfs(1)
dfs(0)

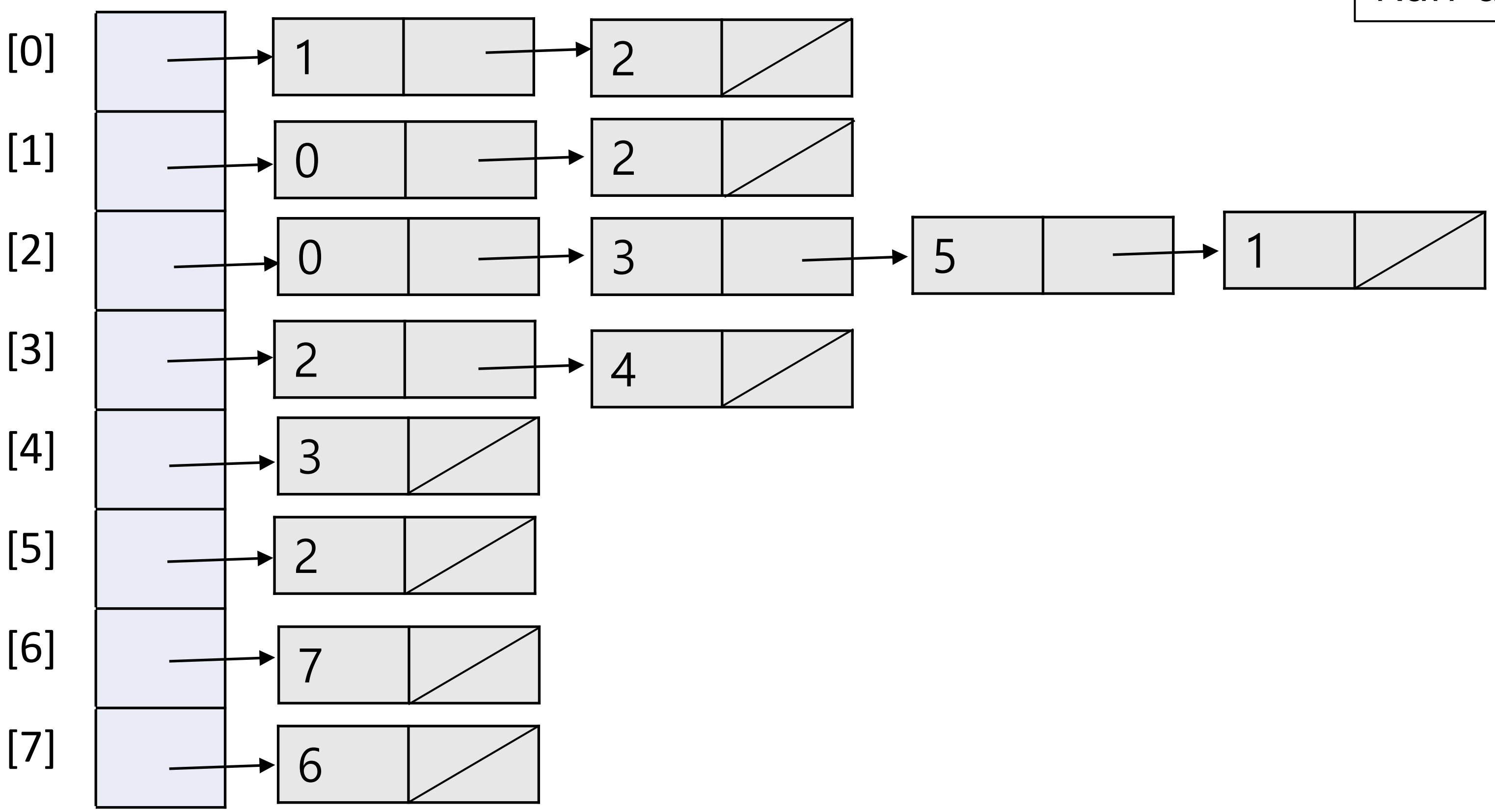
Function call stack

visited

[0]	1
[1]	1
[2]	1
[3]	1
[4]	1
[5]	1
[6]	1
[7]	1

0	사당
1	아주대
2	강남
3	의왕
4	평촌
5	수지
6	명동
7	서울역

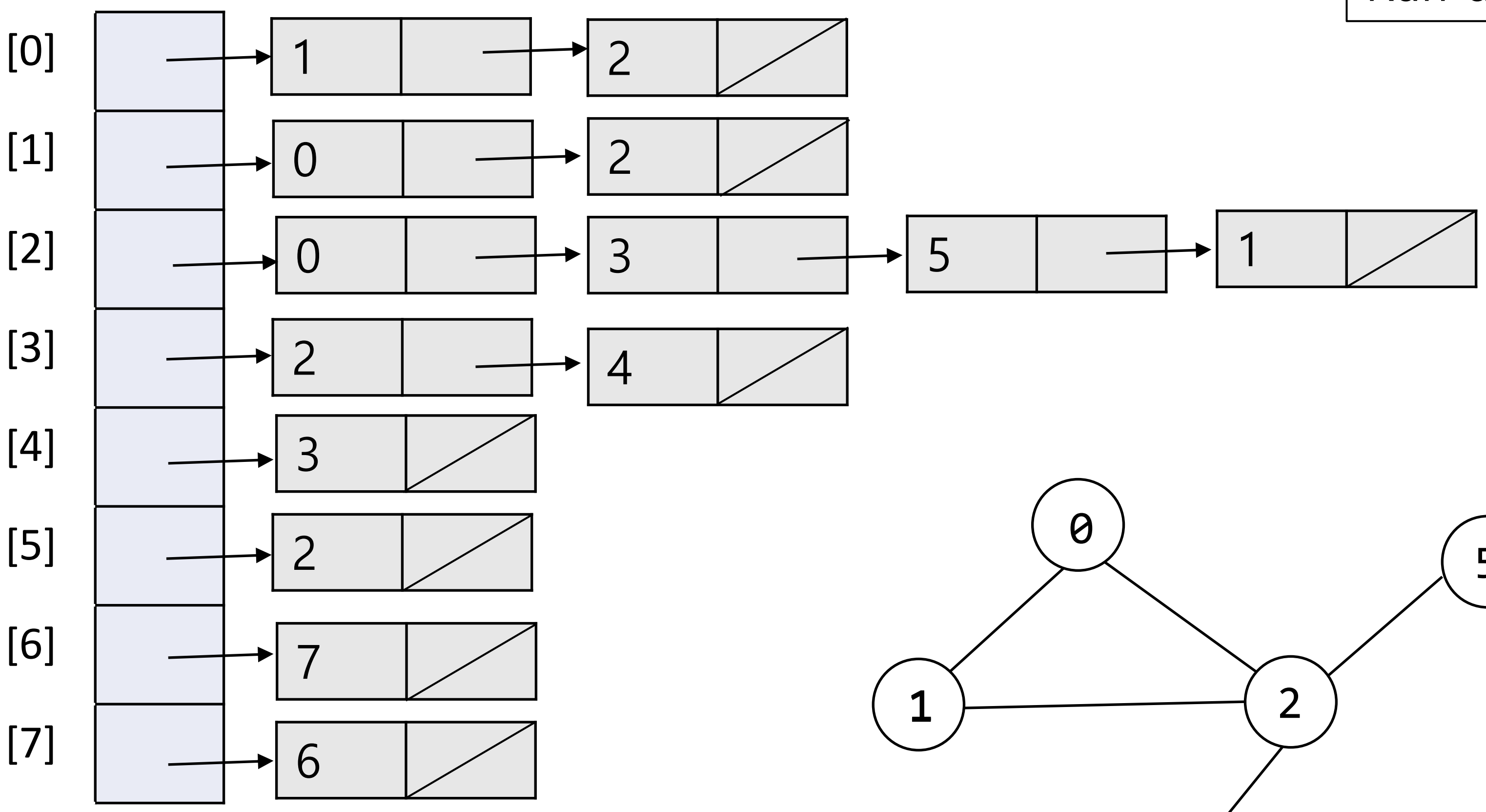
Run dfs(0) and show the result.

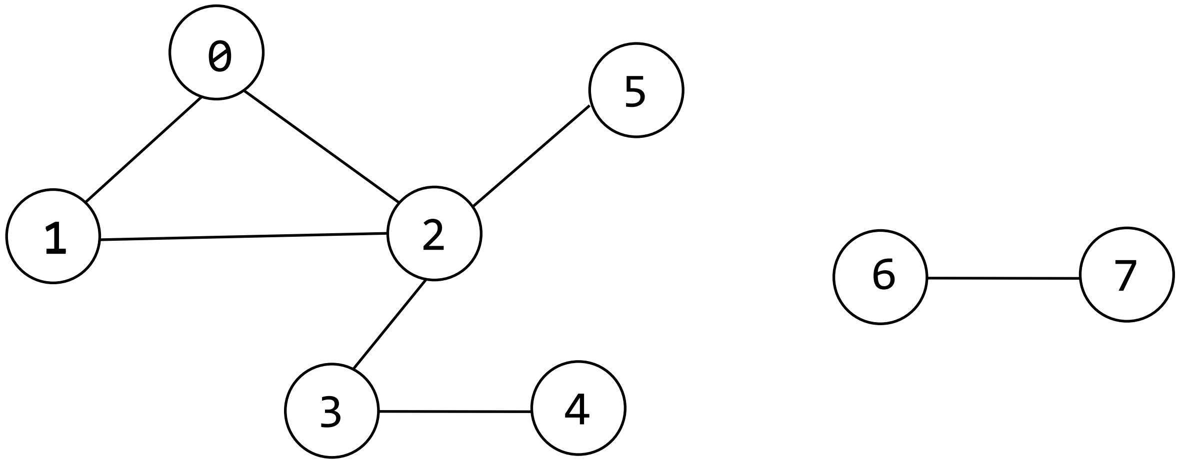



```
void dfs(int v)
{
   nodePointer w;
   visited[v] = TRUE;
   printf("%5d", v);
   for (w = graph[v]; w; w = w->link){
     if (!visited[w->vertex])
        dfs(w->vertex);
   }
}
```

0	사당
1	아주대
2	강남
3	의왕
4	평촌
5	수지
6	명동
7	서울역

Run dfs(0) and show the result.





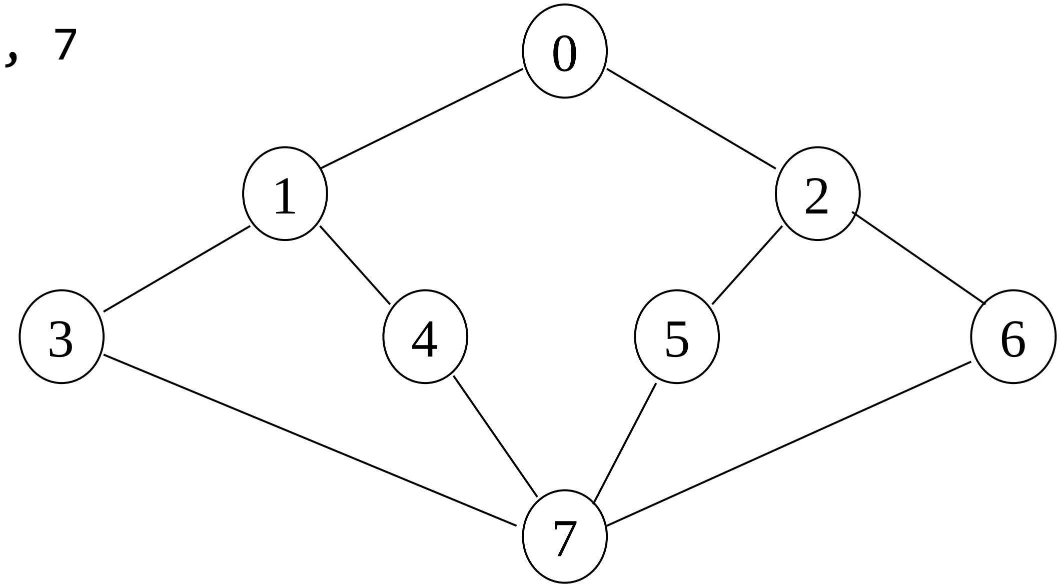

```
void dfs(int v)
{
   nodePointer w;
   visited[v] = TRUE;
   printf("%5d", v);
   for (w = graph[v]; w; w = w->link){
     if (!visited[w->vertex])
        dfs(w->vertex);
   }
}
```

0 1 2 3 4 5

Breadth First Search

- BFS starts at vertex *v* and marks it as visited
- It then visits each of vertices on *v*'s adjacency list
- When all the vertices on *v*'s adjacency list are visited, all the unvisited vertices are visited that are adjacent to the first vertex on *v*'s adjacency list

```
0, 1, 2, 3, 4, 5, 6, 7
```

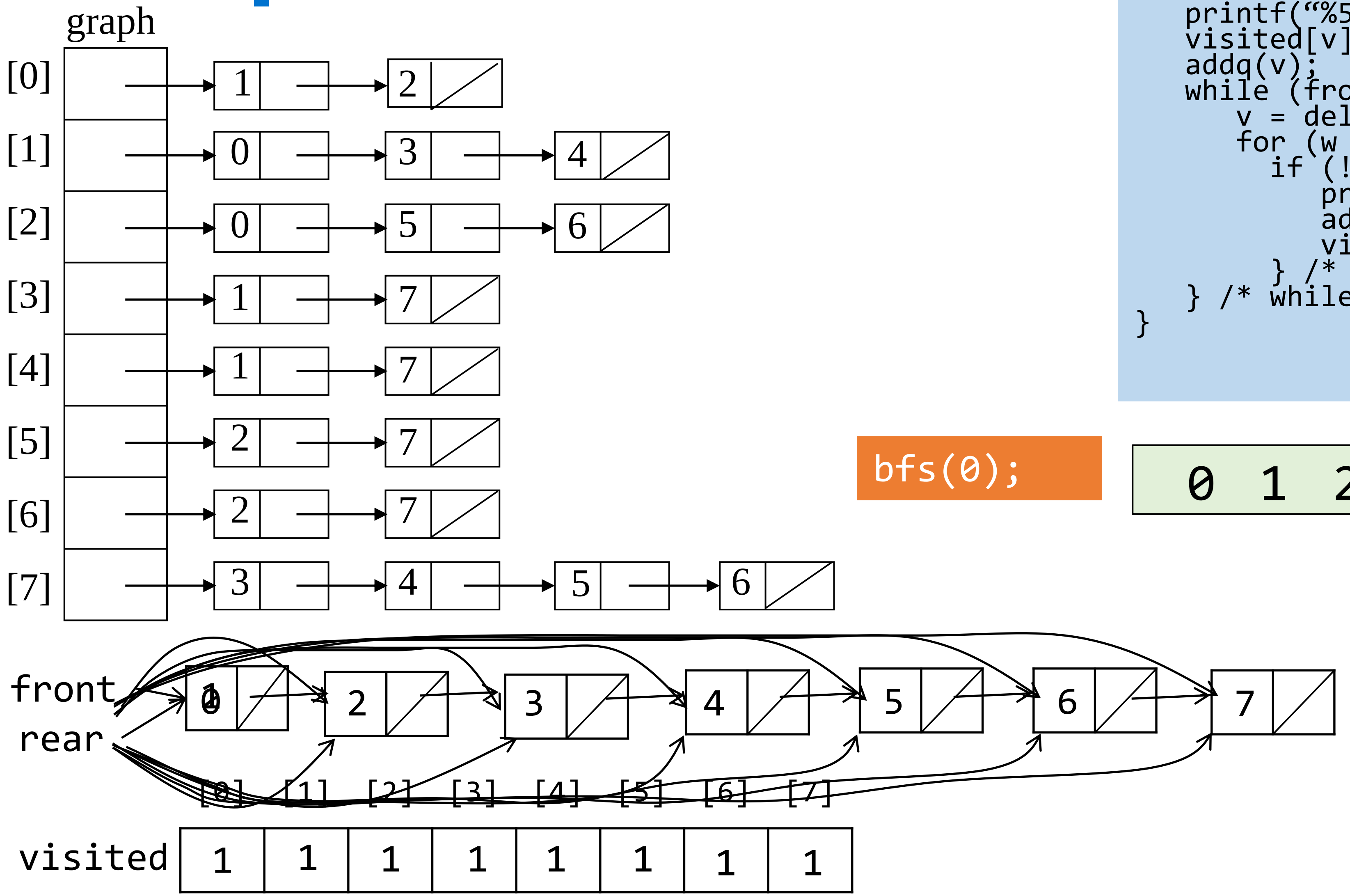


Breadth First Search

```
void bfs(int v)
{
   nodePointer w;
   front = rear = NULL;/* initialize queue */
   printf("%5d", v);
   visited[v] = TRUE;
   addq(v); /* p.159, Chapter 4 */
   while (front) {
      v = deleteq(); /* p.160, Chapter 4 */
      for (w = graph[v]; w; w = w->link)
        if (!visited[w->vertex]) {
           printf("%5d", w->vertex);
           addq(w->vertex);
           visited[w->vertex] = TRUE;
        } /* if */
   } /* while */
}
```

```
typedef struct node *queuePointer;
typedef struct node {
         int vertex;
         queuePointer link;
         };
queuePointer front, rear;
```

Example of BFS



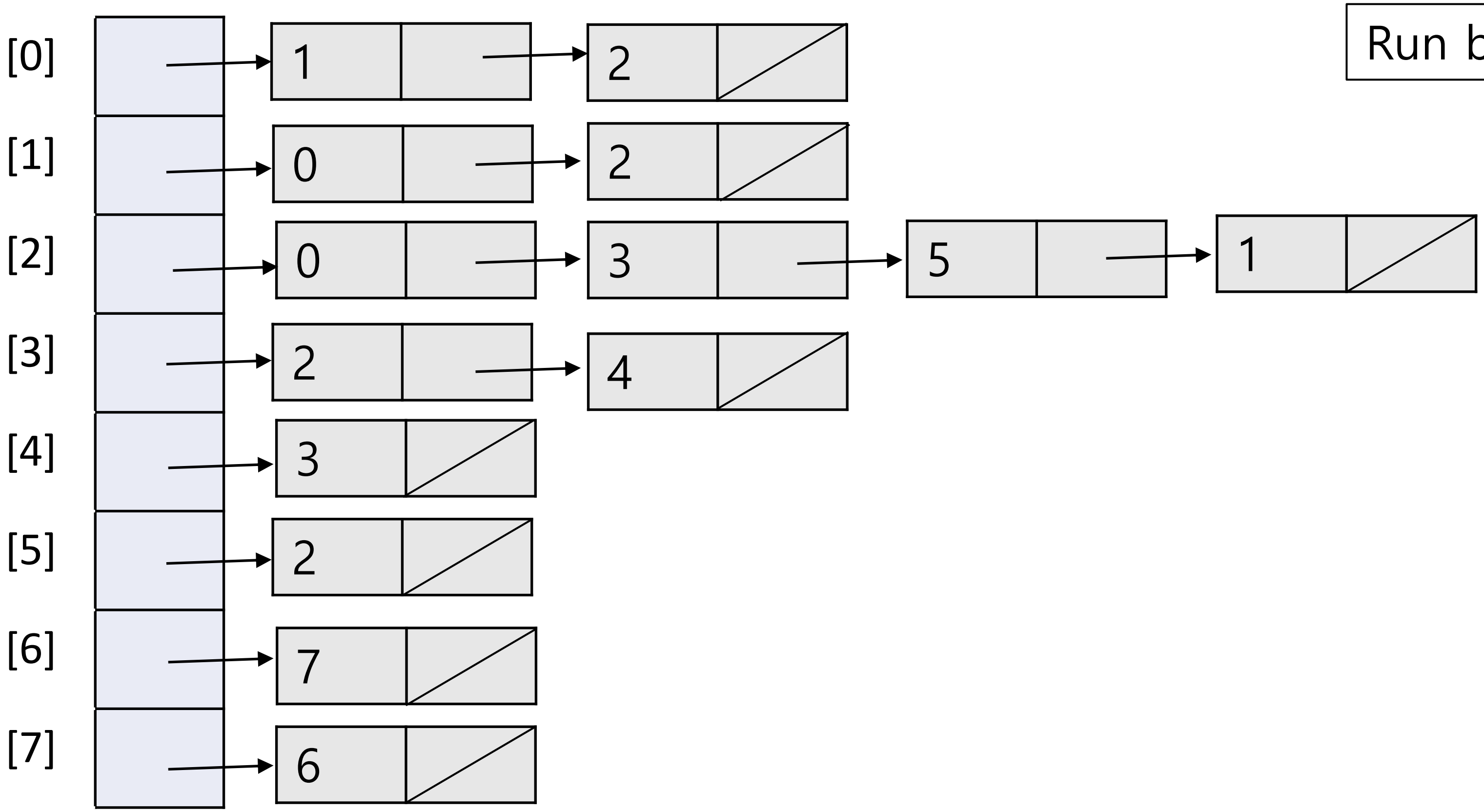

```
void bfs(int v)
{
    nodePointer w;
    front = rear = NULL;
    printf("%5d", v);
    visited[v] = TRUE;
    addq(v);
    while (front) {
        v = deleteq();
        for (w = graph[v]; w; w = w->link)
          if (!visited[w->vertex]) {
              printf("%5d", w->vertex);
              addq(w->vertex);
              visited[w->vertex] = TRUE;
          } /* if */
    } /* while */
}
```

bfs(0);

0 1 2 3 4 5 6 7

Run bfs(0) and show the result.

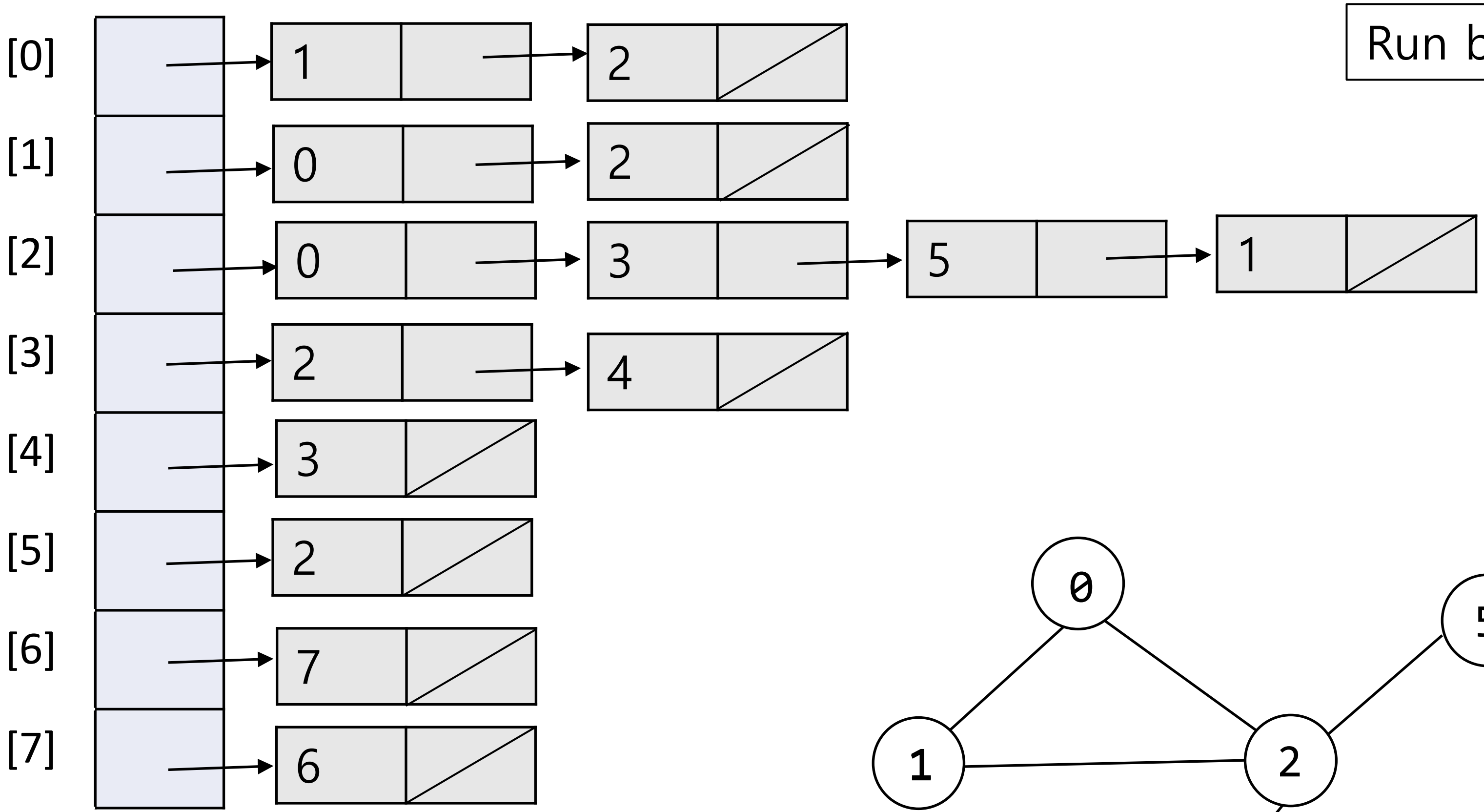
0	사당
1	아주대
2	강남
3	의왕
4	평촌
5	수지
6	명동
7	서울역



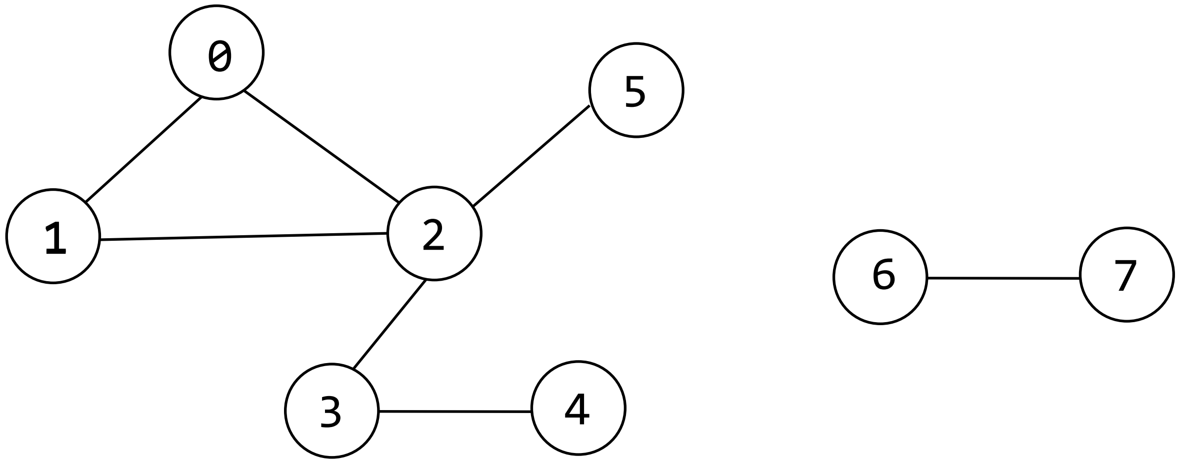

```
void bfs(int v)
{
   nodePointer w;
   front = rear = NULL;
   printf("%5d", v);
   visited[v] = TRUE;
   addq(v);
   while (front) {
      v = deleteq();
      for (w = graph[v]; w; w = w->link)
        if (!visited[w->vertex]) {
           printf("%5d", w->vertex);
           addq(w->vertex);
           visited[w->vertex] = TRUE;
        } /* if */
   } /* while */
}
```

Run bfs(0) and show the result.

0	사당
1	아주대
2	강남
3	의왕
4	평촌
5	수지
6	명동
7	서울역




```
void bfs(int v)
{
   nodePointer w;
   front = rear = NULL;
   printf(“%5d”, v);
   visited[v] = TRUE;
   addq(v);
   while (front) {
      v = deleteq();
      for (w = graph[v]; w; w = w->link)
        if (!visited[w->vertex]) {
           printf(“%5d”, w->vertex);
           addq(w->vertex);
           visited[w->vertex] = TRUE;
        } /* if */
   } /* while */
}
```



0 1 2 3 5 4

Connected Components (연결요소)

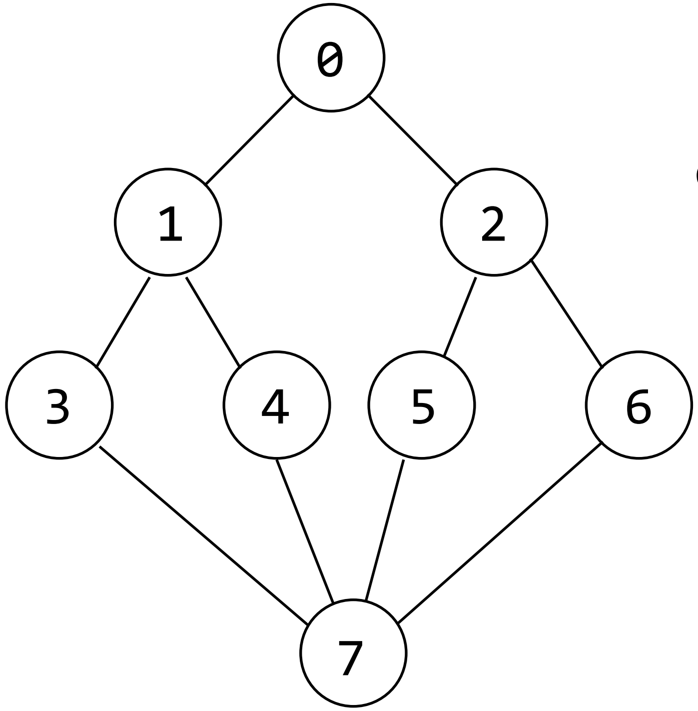
: a component of an undirected graph is a <u>maximal connected subgraph</u>

❖ To determine whether or not an undirected graph is connected
  – simply calling *dfs*(0) or *bfs*(0) and then determine if there are unvisited vertices

❖ To list the connected components of a graph
  – make repeated calls to either *dfs*(*v*) or *bfs*(*v*) where *v* is an unvisited vertex

```
void connected(void)
{ /* determine the connected components of a graph */
   int i;
   for (i = 0; i < n; i++) {
     if (!visited[i])
        dfs(i);
     printf("\n");
   }
}
```

Example of Connected Components



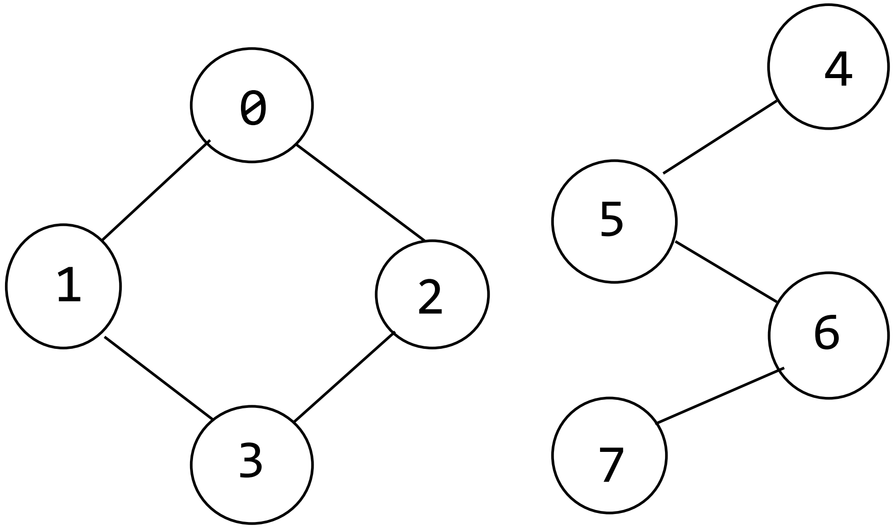
dfs(0);

0 1 3 7 4 5 2 6

dfs(0);
dfs(4);

0 1 3 2

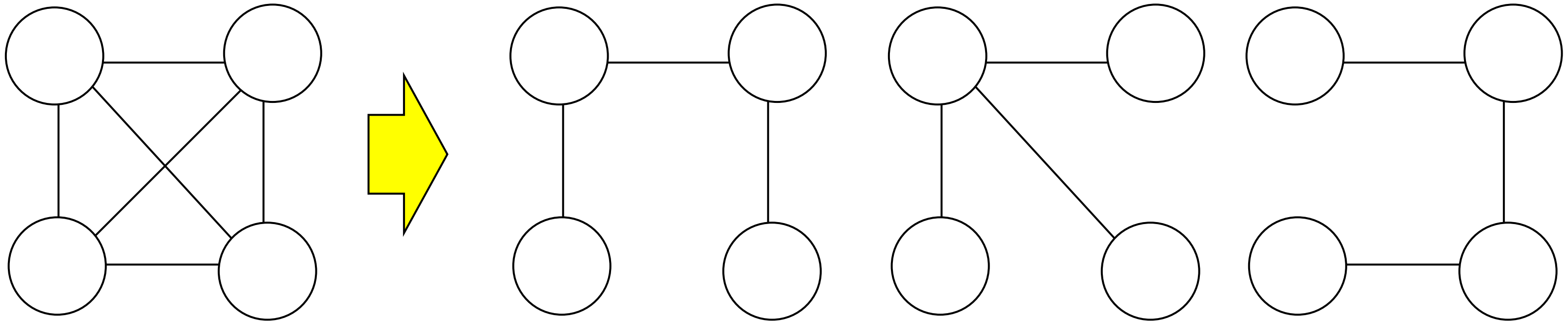
4 5 6 7



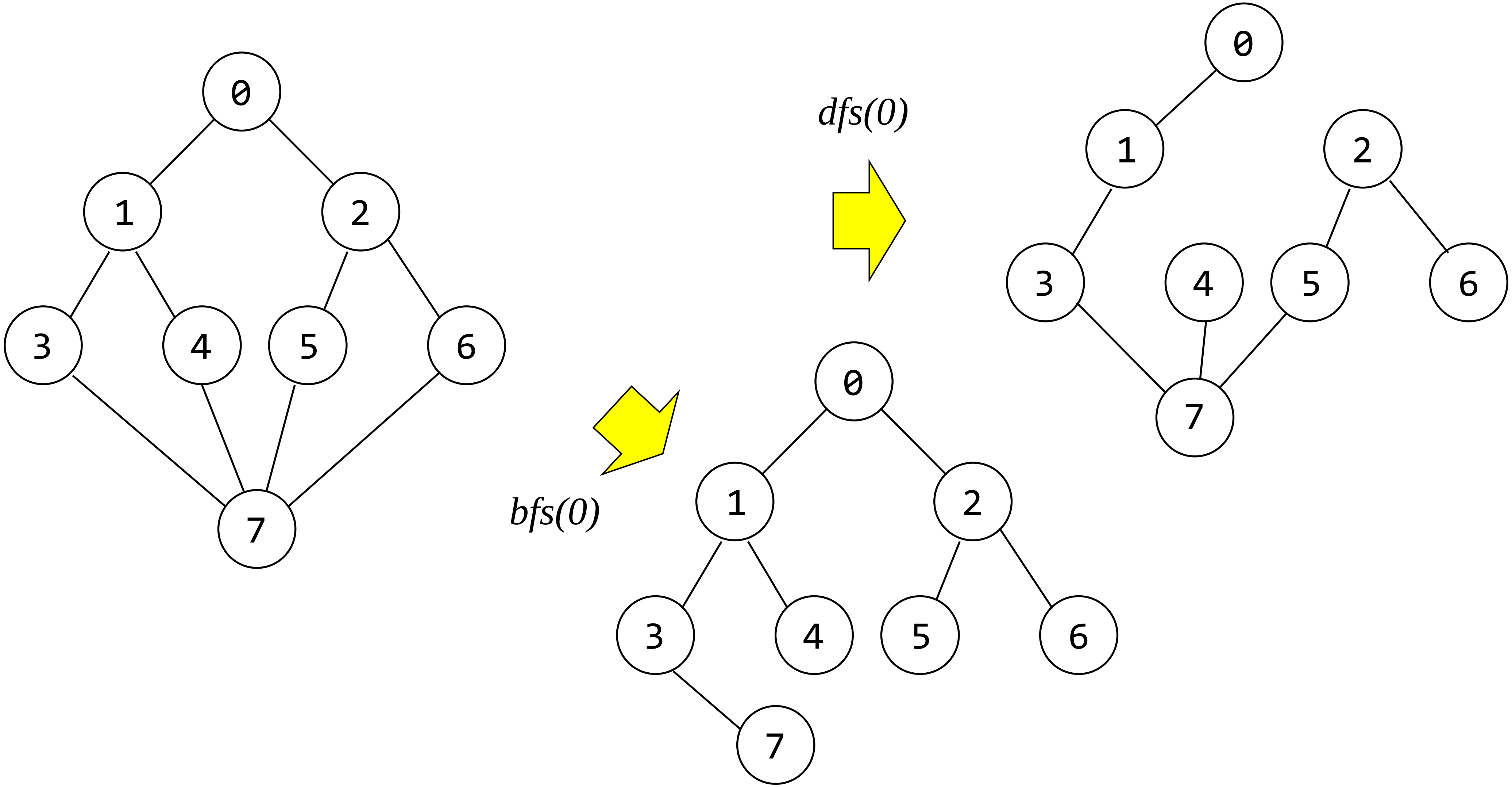
Spanning Trees (신장트리)

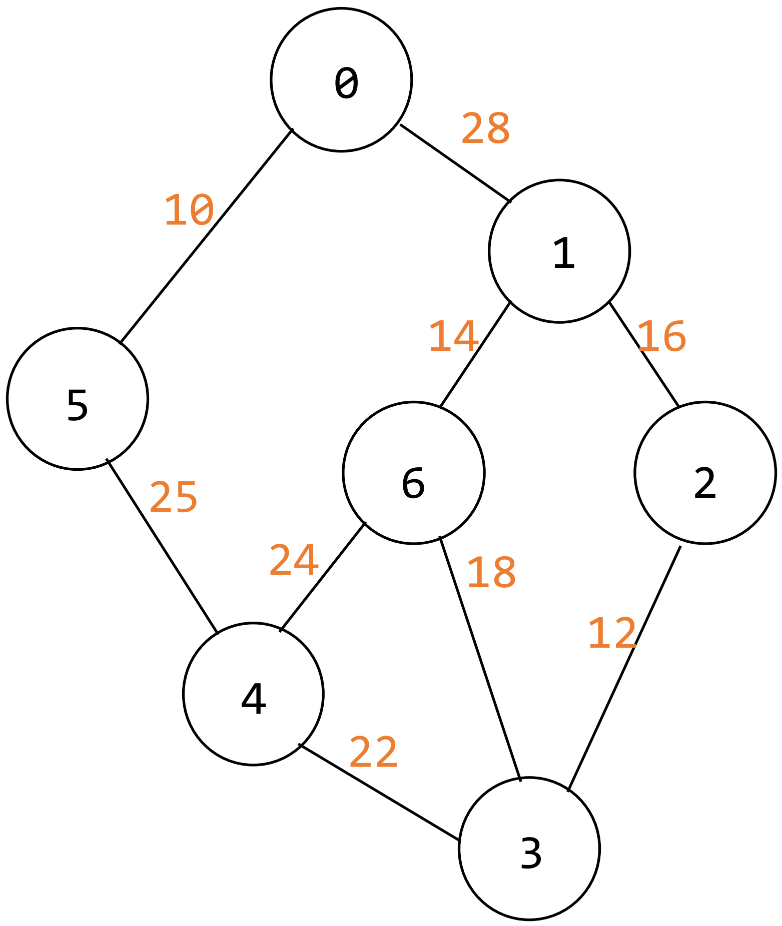
❖ If graph *G* is connected, *dfs()* or *bfs* implicitly partitions the edges in *G* into two sets:
- T : (for tree edges) set of edges used or traversed during the search
- N : (for nontree edges) set of remaining edges

❖ A spanning tree is any tree that consists solely of edges in *G* and that include all the vertices in *G*
1) if we add a nontree edge into a spanning tree → cycle
2) spanning tree is a minimal subgraph, G′, of G such that V(G)=V(G′) and G′ is connected



Example of DFS/BFS Spanning Trees




0
1
2
3
4
5
6
28
10
14
16
25
24
18
12
22

Minimum Cost Spanning Tree

- A spanning tree of least cost
  - Kruskal's algorithm
  - Prim's algorithm
  - Sollin's algorithm

- Applications
  - Network design: telephone, electrical, water, road, …

Minimum Cost Spanning Tree

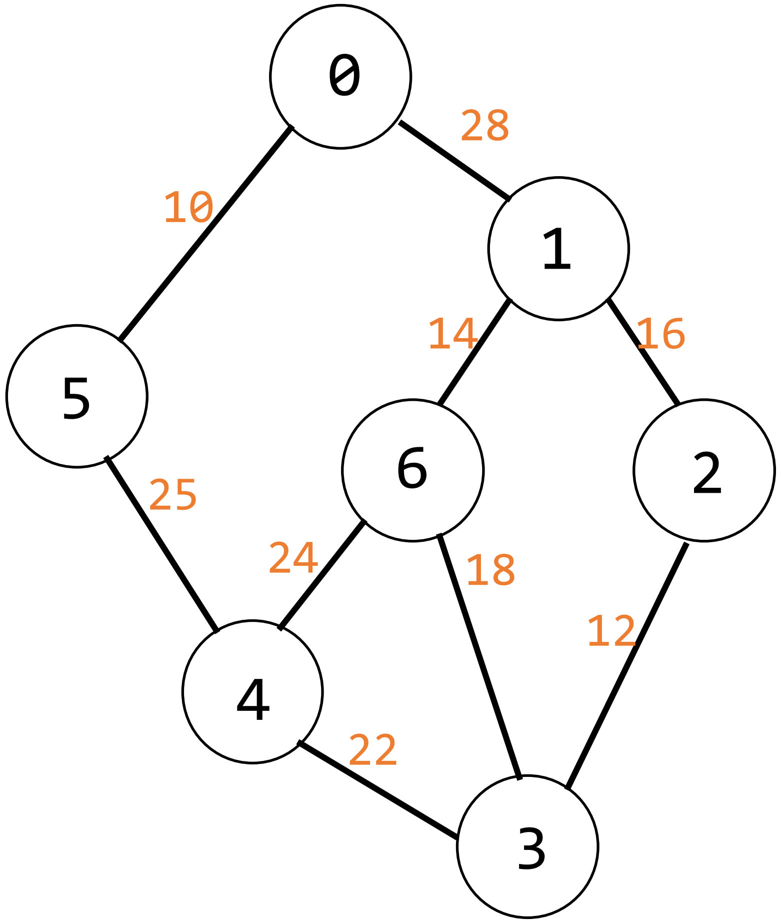
- Greedy method
  - Make the best decision, local optimum, at each stage using some criterion
  - When the algorithm terminates, we hope that the local optimum is equal to the global optimum.
- Spanning tree construction constraints
  - use only edges within the graph
  - use exactly *n*-1 edges
  - may not use edges that would produce a cycle

Kruskal’s Algorithm

```
T = {};
E = a set of every edges in the input graph
while (T contains less than (n-1) edges && E is not empty)
{
   choose a least cost edge (v,w) from E;
   delete (v,w) from E;
   if ((v,w) does not create a cycle in T)
      add(v,w) to T;
   else
      discard (v,w);
}
if (T contains fewer than (n-1) edges)
   printf(“no spanning tree\n”);
```

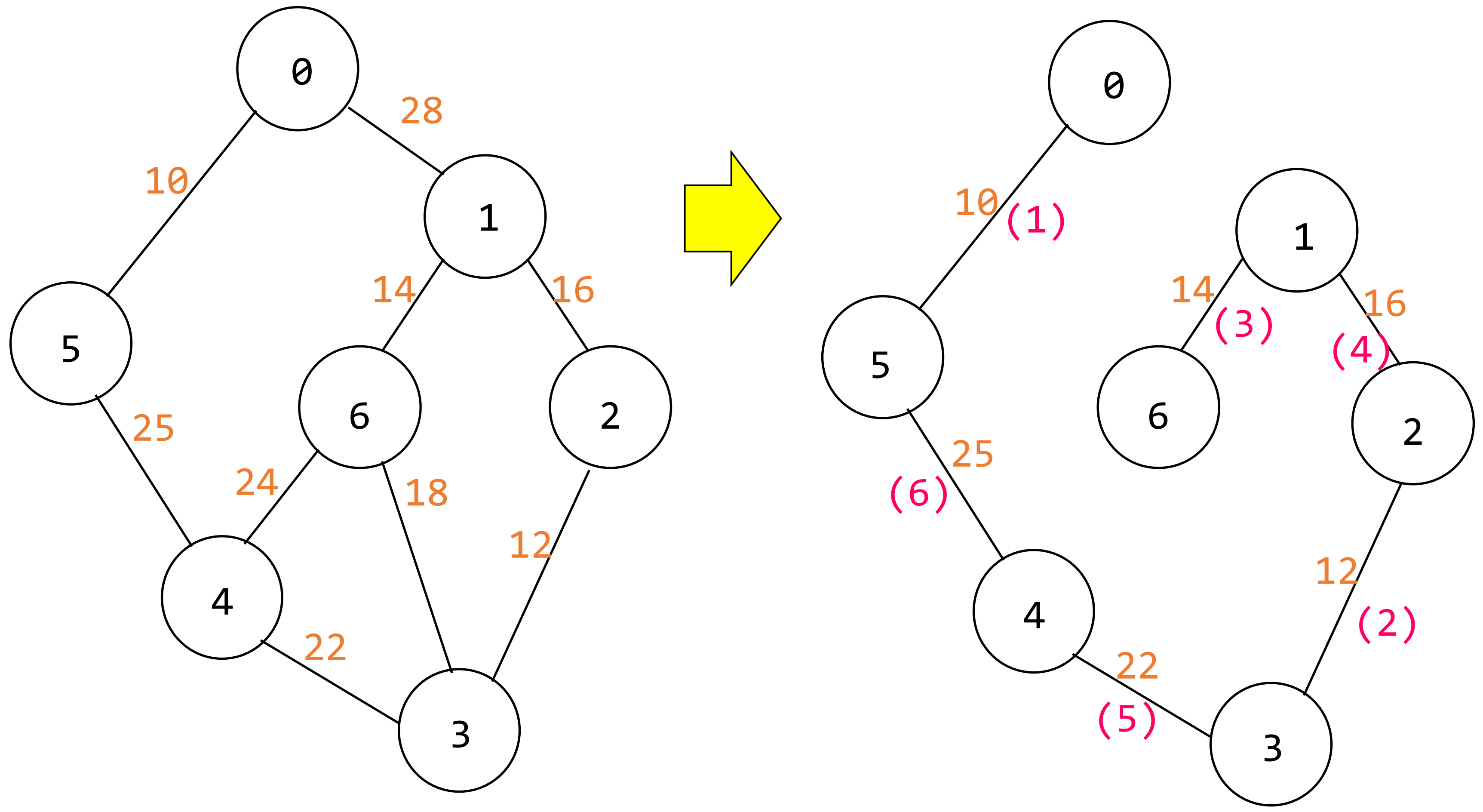
An Example of Kruskal’s Algorithm

```
T = {};
while (T contains less than (n-1) edges &&
    E is not empty)
{
   choose a least cost edge (v,w) from E;
   delete (v,w) from E;
   if ((v,w) does not create a cycle in T)
      add(v,w) to T;
   else
      discard (v,w);
}
if (T contains fewer than (n-1) edges)
   printf(“no spanning tree\n”);
```



T = {(0,5)(2,3)(1,6)(1,2)(3,4) (5,4)}

An Example of Kruskal's Algorithm

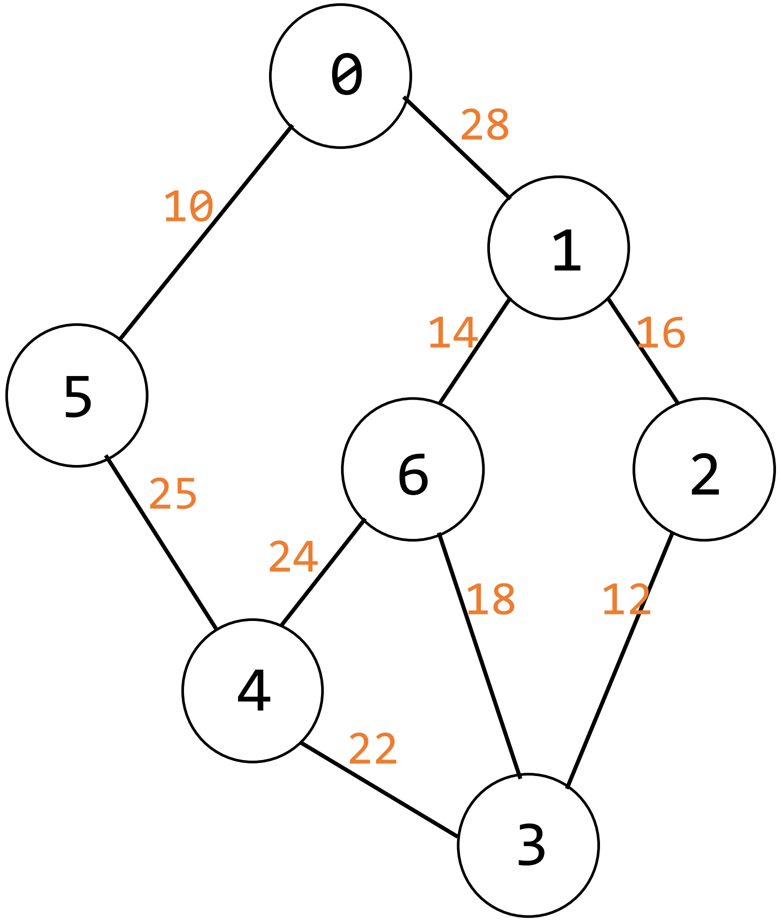


Prim’s Algorithm

```
T = {};
TV = {0}; /* start with vertex 0 and no edge*/
while (T contains fewer than n-1 edges)
{
   let (u,v) be a least cost edge such that u ∈ TV and v ∉ TV;
   if (there is no such edge) break;
   add v to TV;
   add (u,v) to T;
}
if (T contains fewer than n-1 edges)
printf(“no spanning tree\n”);
```

An Example of Prim's Algorithm

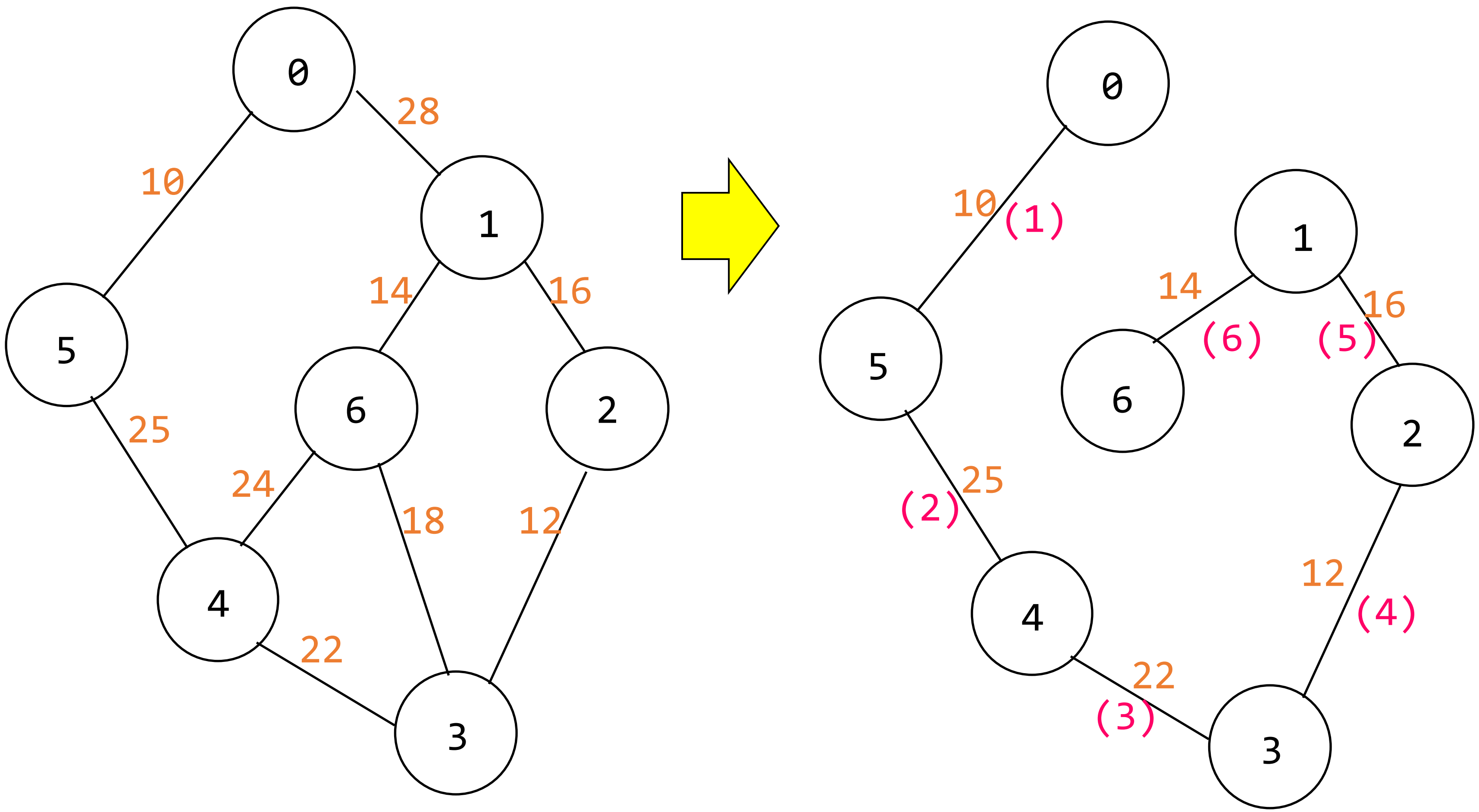
```
T = {};
TV = {0};
while (T contains fewer than n-1 edges)
{
   let (u,v) be a least cost edge such that u ∈ TV and v ∉
  TV;
   if (there is no such edge)
         break;
   add v to TV;
   add (u,v) to T;
}
if (T contains fewer than n-1 edges)
  printf("no spanning tree\n");
```



TV={ 0 5 4 3 2 1 6 }

T={(0,5) (5,4) (4,3) (3,2) (2,1) (1,6) }

An Example of Prim's Algorithm



Sollin's Algorithm

Start with a forest that has no edges

Select **several edges** for inclusion in T at each stage

1) Each component selects a least cost edge with which to connect to another component

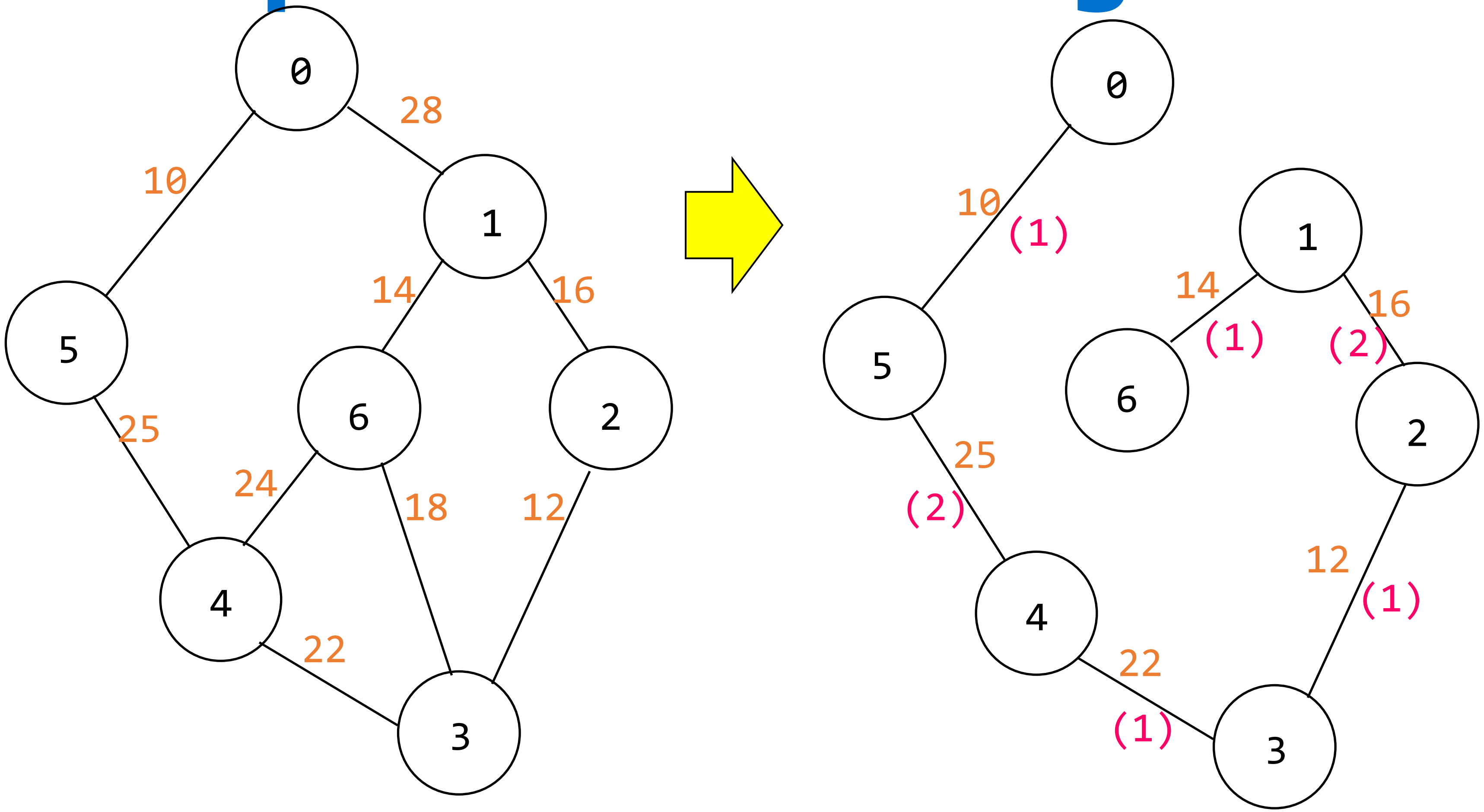
2) Duplicate selections are eliminated

- Cycles are possible when the graph has some edges that have the same cost

Terminate when

- There is only one tree at the end of a stage, or
- No edges remain for selection

An Example of Sollin's Algorithm



First stage: (0,5), (1,6), (2,3), (3,2), (4,3), (5,0), (6,1)

Second stage: (5,4), (1,2)

An Example of Sollin's Algorithm

Start with a forest that has no edges
Select several edges for inclusion in T at each stage

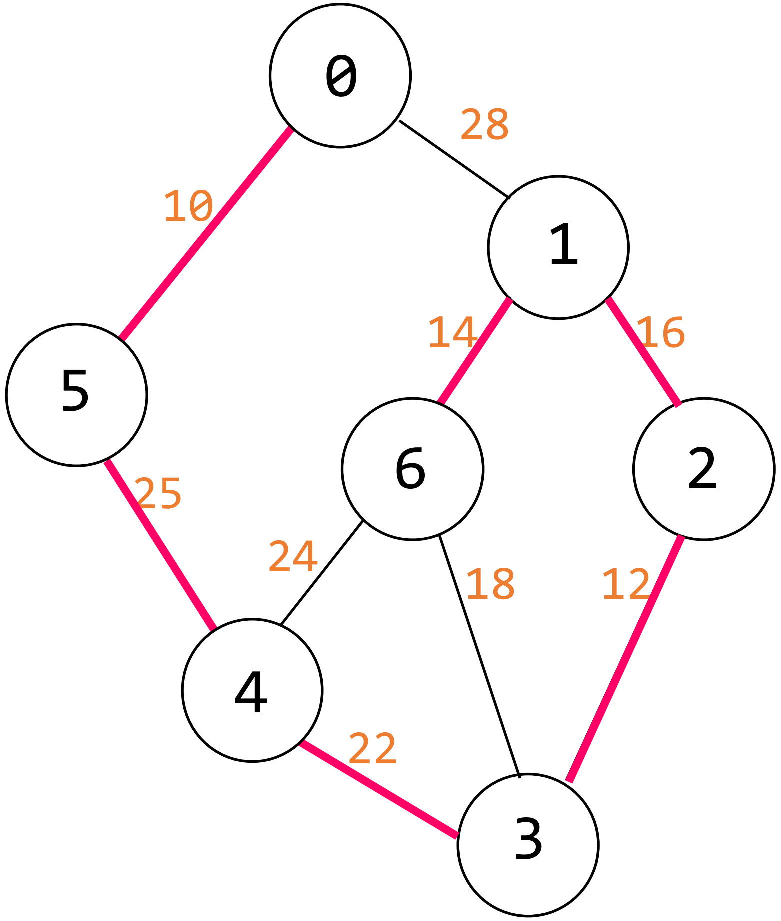
1) Each component selects a least cost edge with which to connect to another component
2) Duplicate selections are eliminated

First stage:

T={(0,5)(1,6)(2,3)(4,3)(6,1)}

Second stage:

T = { (0,5) (1,6) (2,3) (4,3) (5,4) (1,2) }



Shortest Paths

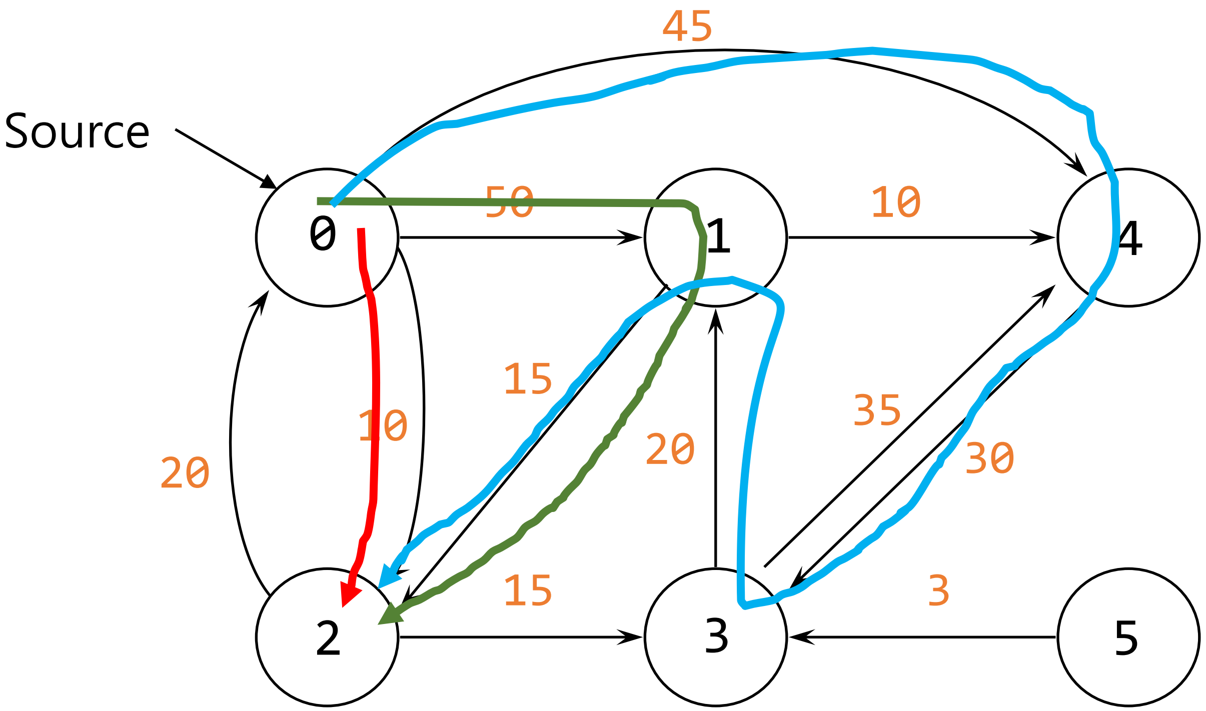
Single source -> all destinations

- Dijkstra algorithm
- Bellman-Ford algorithm

What is the shortest path?

Let's find all paths from 0 to 2.

The length of a path is now defined to be the **sum of the weights** of the edges on that path, rather than the number of edges.



Path	Length
1) 0 2	10
2) 0 2 3	25
3) 0 2 3 1	45
4) 0 4	45

Shortest paths from 0

Shortest Path Observations

v_0 : source vertex

S : set of vertices, including v_0, whose shortest paths have been found

w : any vertex ($\notin S$)

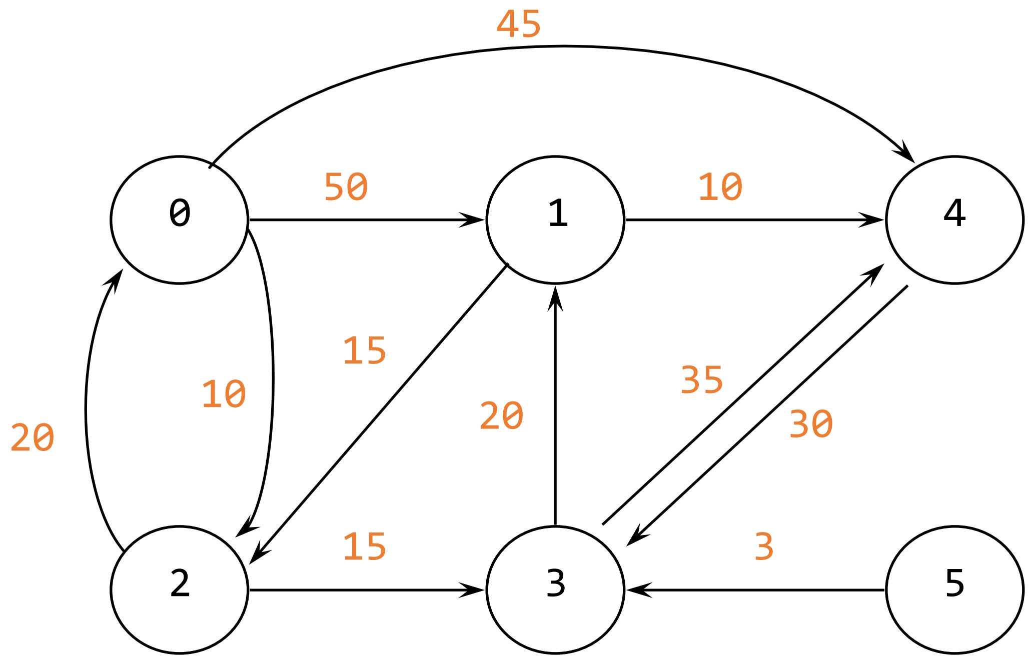
distance[w] : the length of the shortest path starting from v_0, going through vertices only in S, and ending in w

1) If the next shortest path from v_0 is to u, the path from v_0 to u goes through only those vertices that are in S
2) Vertex u is chosen so that it has the minimum distance, distance[u], among all the vertices not in S
3) Shortest path from v_0 to w can be changed by adding u to S

$\rightarrow$ distance [w] = distance[u] + cost(<u, w>)

Implementation of Dijkstra's Algorithm in C (1)

```
#define MAX_VERTICES 6
int cost[][MAX_VERTICES] =
 {{   0,   50,   10, 1000,   45, 1000},
  {1000,    0,   15, 1000,   10, 1000},
  {  20, 1000,    0,   15, 1000, 1000},
  {1000,   20, 1000,    0,   35, 1000},
  {1000, 1000,   30, 1000,    0, 1000},
  {1000, 1000, 1000,    3, 1000,    0}};
int distance[MAX_VERTICES];
short int found[MAX_VERTICES];
int n = MAX_VERTICES;
```



Implementation of Dijkstra’s Algorithm in C (2)

```
void shortestPath(int v, int cost[][MAX_VERTICES],
     int distance[], int n, short int found[])
{
   int i, u, w;
   for (i = 0; i < n; i++) {
       found[i] = FALSE;
       distance[i] = cost[v][i];
   }
   found[v] = TRUE;   distance[v] = 0;
   for (i = 0; i < n-2; i++) {
       u = choose(distance, n, found);
       found[u] = TRUE;
       for (w = 0; w < n; w++)
           if (!found[w])
              if (distance[u]+cost[u][w] < distance[w])
                    distance[w] = distance[u]+cost[u][w];
   } /* for i */
}
```

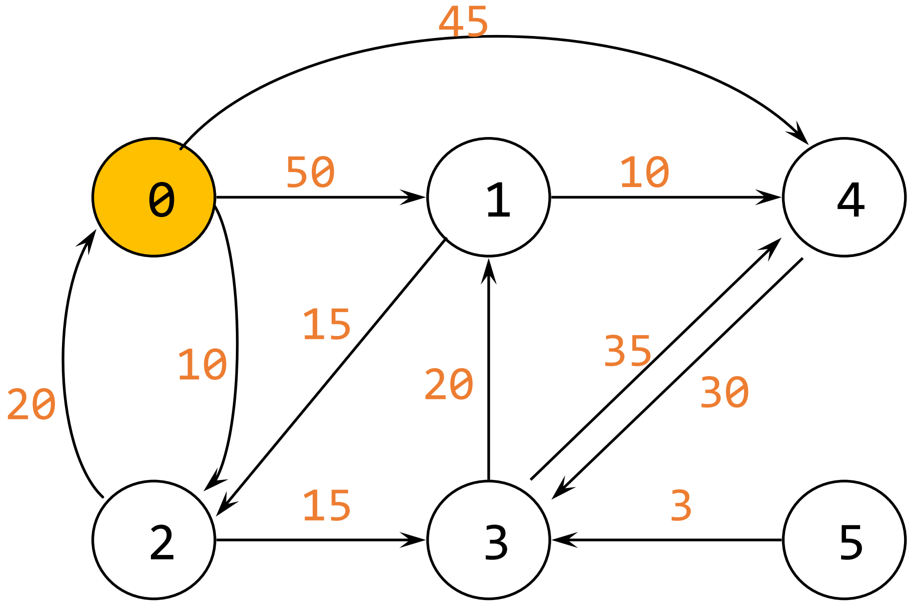
- v : source vertex
- cost[][] : adjacency matrix of a digraph
- distance[i] : the shortest path from vertex v to i
- found[i] : 0 if the shortest path from i has not been found
  1 otherwise ($i \in S$)

Implementation of Dijkstra’s Algorithm in C (3)

```
int choose(int distance[], int n, int found[])
{
  /* find smallest distance not yet checked */
   int i, min, minpos;
   min = INT_MAX;
   minpos = -1;
   for (i = 0; i < n; i++)
       if (distance[i] < min && !found[i]) {
          min = distance[i];
          minpos = i;
       }
   return minpos;
}
```

Dijkstra’s Algorithm Example 1

Initial condition



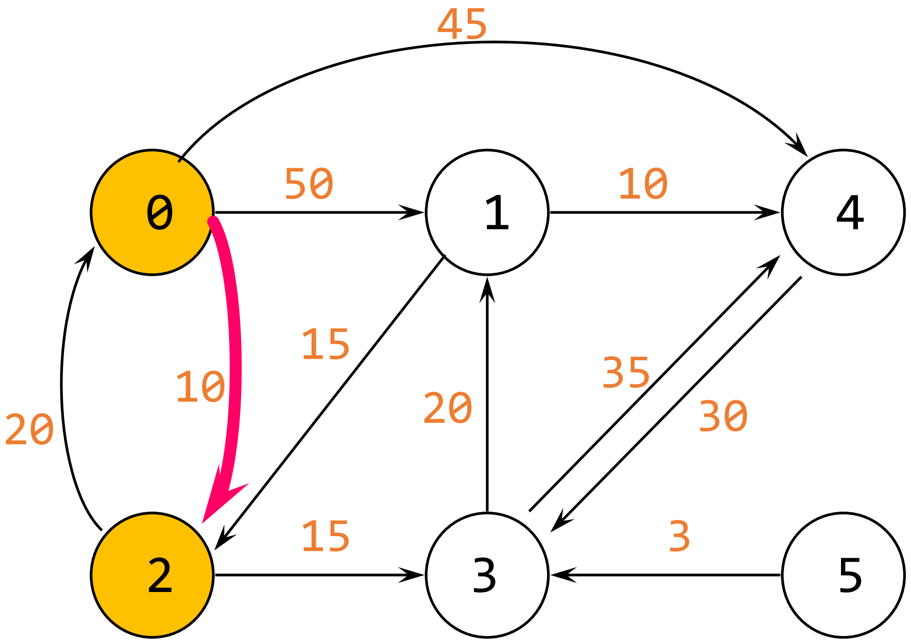
S = {0}

Cost adjacency matrix

$$
\begin{array}{c|cccccc}
 & [0] & [1] & [2] & [3] & [4] & [5] \\
\hline
[0] & 0 & 50 & 10 & \infty & 45 & \infty \\
[1] & \infty & 0 & 15 & \infty & 10 & \infty \\
[2] & 20 & \infty & 0 & 15 & \infty & \infty \\
[3] & \infty & 20 & \infty & 0 & 35 & \infty \\
[4] & \infty & \infty & \infty & 30 & 0 & \infty \\
[5] & \infty & \infty & \infty & 3 & \infty & 0
\end{array}
$$

distance

Dijkstra’s Algorithm Example 1 – 1st Step

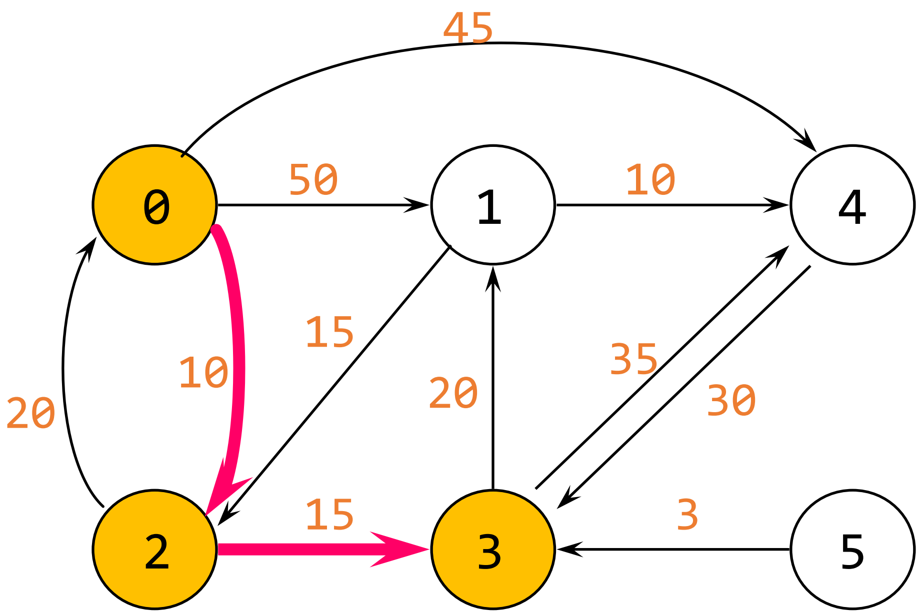


$$
\begin{array}{c|cccccc}
 & [0] & [1] & [2] & [3] & [4] & [5] \\
\hline
[0] & 0 & 50 & 10 & \infty & 45 & \infty \\
[1] & \infty & 0 & 15 & \infty & 10 & \infty \\
[2] & 20 & \infty & 0 & 15 & \infty & \infty \\
[3] & \infty & 20 & \infty & 0 & 35 & \infty \\
[4] & \infty & \infty & \infty & 30 & 0 & \infty \\
[5] & \infty & \infty & \infty & 3 & \infty & 0
\end{array}
$$

S = {0, 2}

0→2: (0, 2), 10

Dijkstra's Algorithm Example 1 – 2nd Step



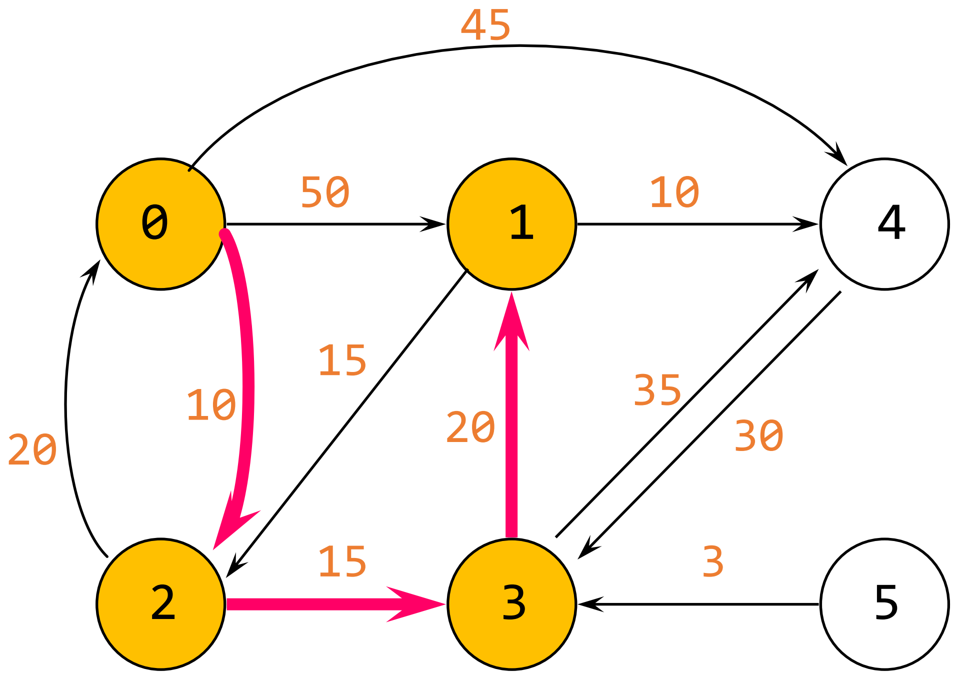
$$
\begin{array}{c|cccccc}
 & [0] & [1] & [2] & [3] & [4] & [5] \\
\hline
[0] & 0 & 45 & 10 & 25 & 45 & \infty \\
[1] & \infty & 0 & 15 & \infty & 10 & \infty \\
[2] & 20 & \infty & 0 & 15 & \infty & \infty \\
[3] & \infty & 20 & \infty & 0 & 35 & \infty \\
[4] & \infty & \infty & \infty & 30 & 0 & \infty \\
[5] & \infty & \infty & \infty & 3 & \infty & 0
\end{array}
$$

S = {0, 2, 3}

0→2: (0, 2), 10

0→3: (0, 2, 3), 25

Dijkstra's Algorithm Example 1 – 3rd Step



	[0]	[1]	[2]	[3]	[4]	[5]
[0]	0	45	10	25	45	∞
[1]	∞	0	15	∞	10	∞
[2]	20	∞	0	15	∞	∞
[3]	∞	20	∞	0	35	∞
[4]	∞	∞	∞	30	0	∞
[5]	∞	∞	∞	3	∞	0

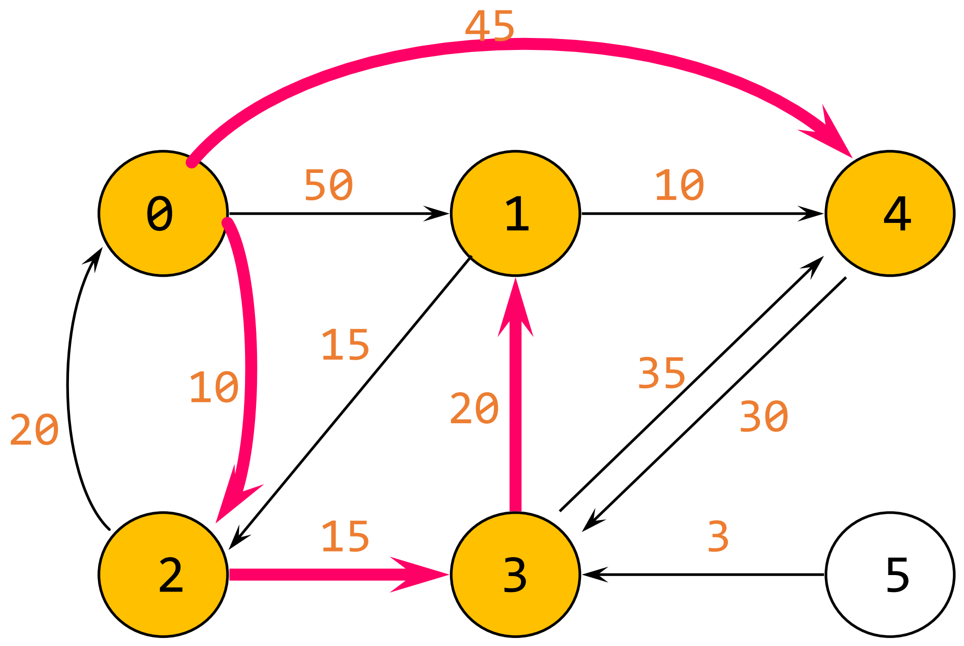
S = {0, 2, 3, 1}

0→2: (0, 2), 10

0→3: (0, 2, 3), 25

0→1: (0, 2, 3, 1), 45

Dijkstra’s Algorithm Example 1 – 4th Step



$$
\begin{array}{c|c}
 & \begin{matrix} [0] & [1] & [2] & [3] & [4] & [5] \end{matrix} \\
\begin{matrix} [0] \\ [1] \\ [2] \\ [3] \\ [4] \\ [5] \end{matrix} &
\begin{bmatrix}
0 & 45 & 10 & 25 & 45 & \infty \\
\infty & 0 & 15 & \infty & 10 & \infty \\
20 & \infty & 0 & 15 & \infty & \infty \\
\infty & 20 & \infty & 0 & 35 & \infty \\
\infty & \infty & \infty & 30 & 0 & \infty \\
\infty & \infty & \infty & 3 & \infty & 0
\end{bmatrix}
\end{array}
$$

S = {0, 2, 3, 1, <u>4</u>}

0→2: (0, 2), 10

0→3: (0, 2, 3), 25

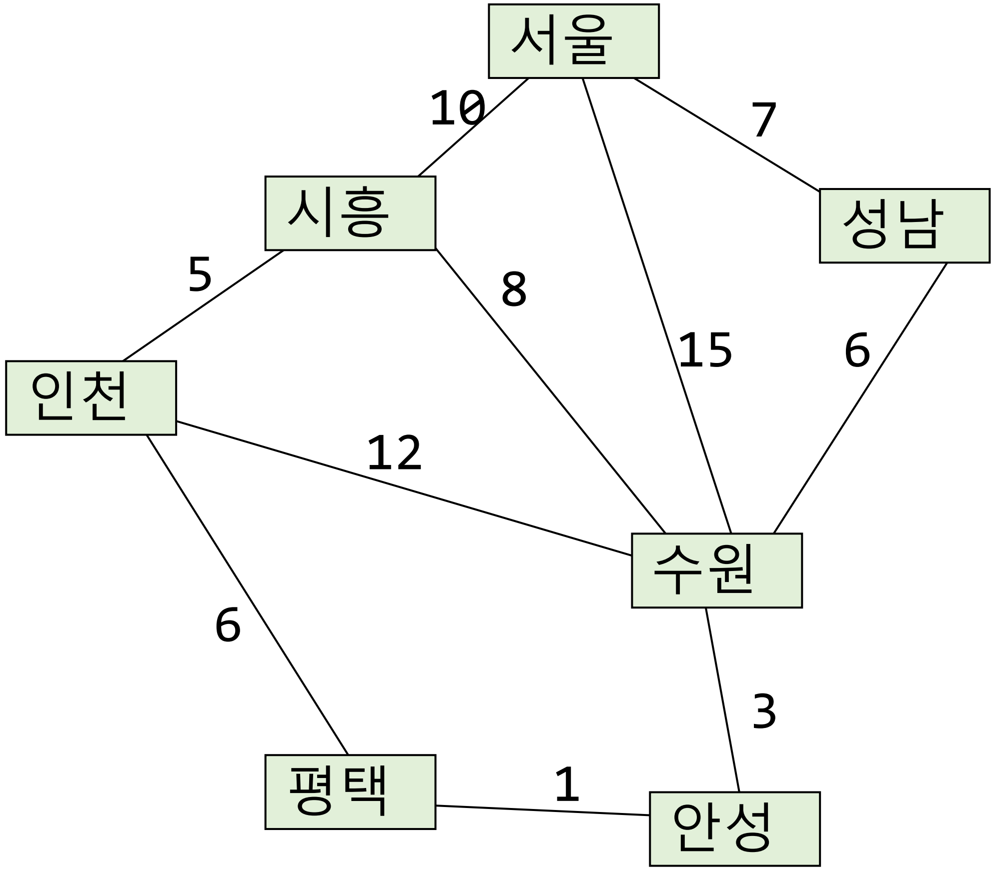
0→1: (0, 2, 3, 1), 45

$\boxed{0 \to 4: (0, 4), 45}$

Implementation of Dijkstra's Algorithm in C (4)

```
void shortestPath(int v, int cost[][MAX_VERTICES],
     int distance[], int n, short int found[], int pi[])     /* pi: an array of predecessors for each vertex */
{
   int i, u, w;
   for (i = 0; i < n; i++) {
       found[i] = FALSE;
       distance[i] = cost[v][i];
       if (distance[i] < MAX_DISTANCE) pi[i]=v; else pi[i]=-1;
   }
   found[v] = TRUE;   distance[v] = 0;
   for (i = 0; i < n-2; i++) {
       u = choose(distance, n, found);
       found[u] = TRUE;
       for (w = 0; w < n; w++)
           if (!found[w])
              if (distance[u]+cost[u][w] < distance[w]){
                     distance[w] = distance[u]+cost[u][w]; pi[w]=u;
              }
   } /* for i */
}
```

Dijkstra’s Algorithm: Example 2

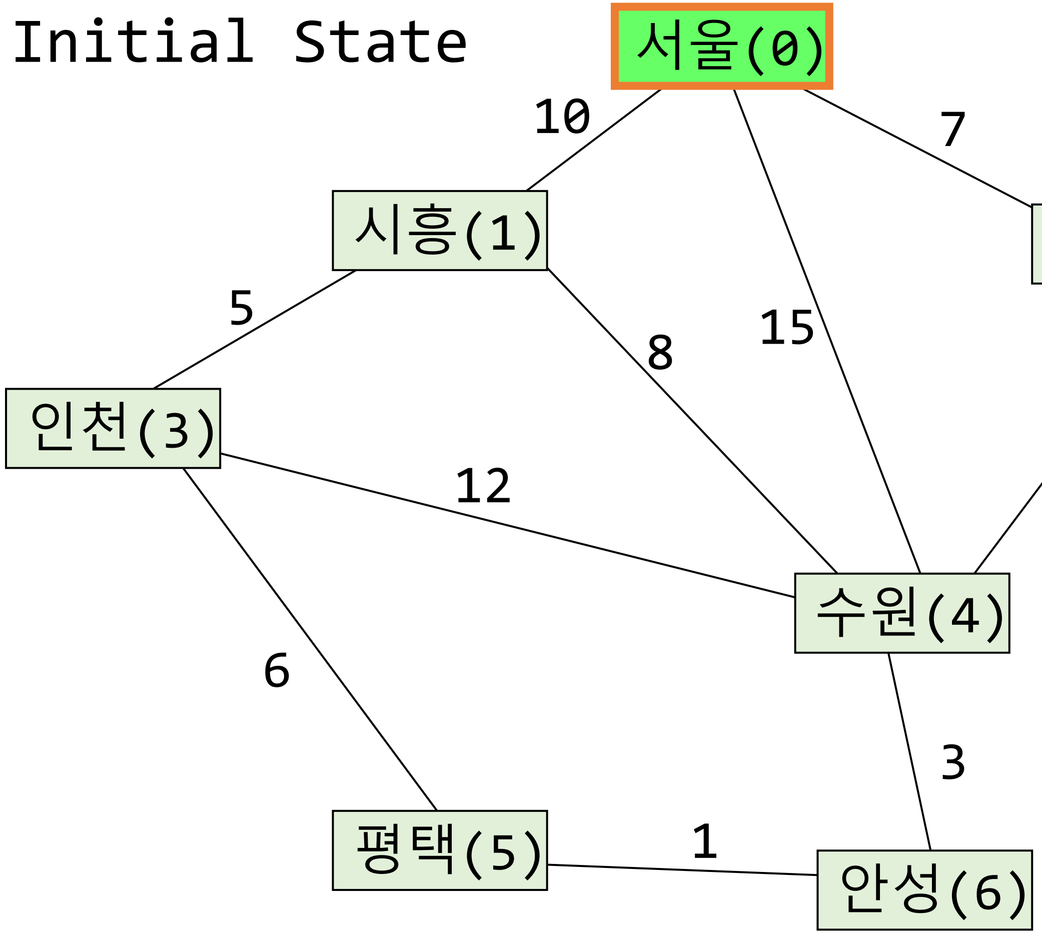


서울:0, 시흥:1, 성남:2, 인천:3, 수원:4, 평택:5, 안성:6

Cost adjacency matrix

$$\begin{bmatrix} 0 & 10 & 7 & 1000 & 15 & 1000 & 1000 \\ 10 & 0 & 1000 & 5 & 8 & 1000 & 1000 \\ 7 & 1000 & 0 & 1000 & 6 & 1000 & 1000 \\ 1000 & 5 & 1000 & 0 & 12 & 6 & 1000 \\ 15 & 8 & 6 & 12 & 0 & 1000 & 3 \\ 1000 & 1000 & 1000 & 6 & 1000 & 0 & 1 \\ 1000 & 1000 & 1000 & 1000 & 3 & 1 & 0 \end{bmatrix}$$

Initial State



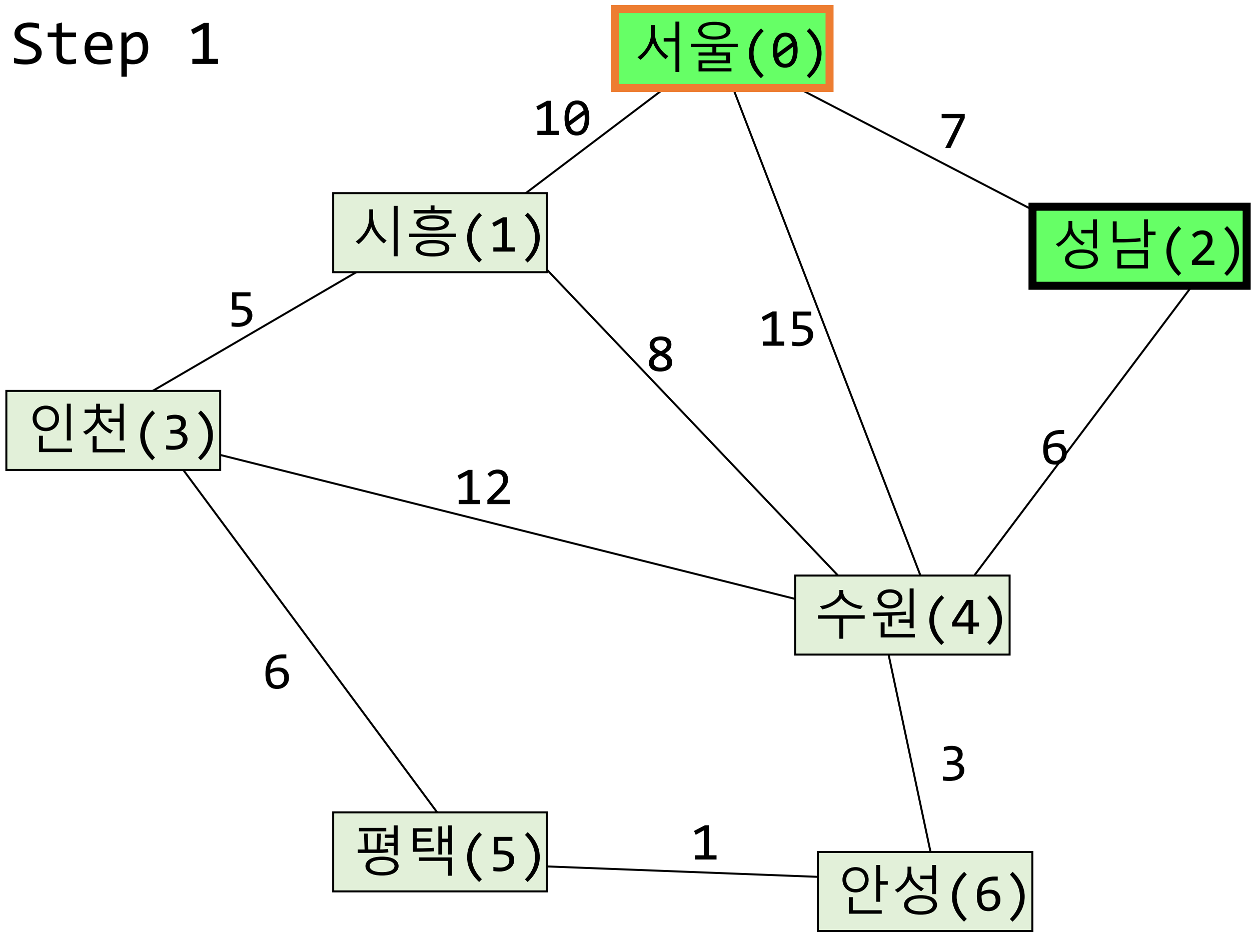
	distance	found
[0]	0	1
[1]	10	0
[2]	7	0
[3]	1000	0
[4]	15	0
[5]	1000	0
[6]	1000	0

S={서울(0)}

pi

[0]	[1]	[2]	[3]	[4]	[5]	[6]
0	0	0	-1	0	-1	-1

Step 1



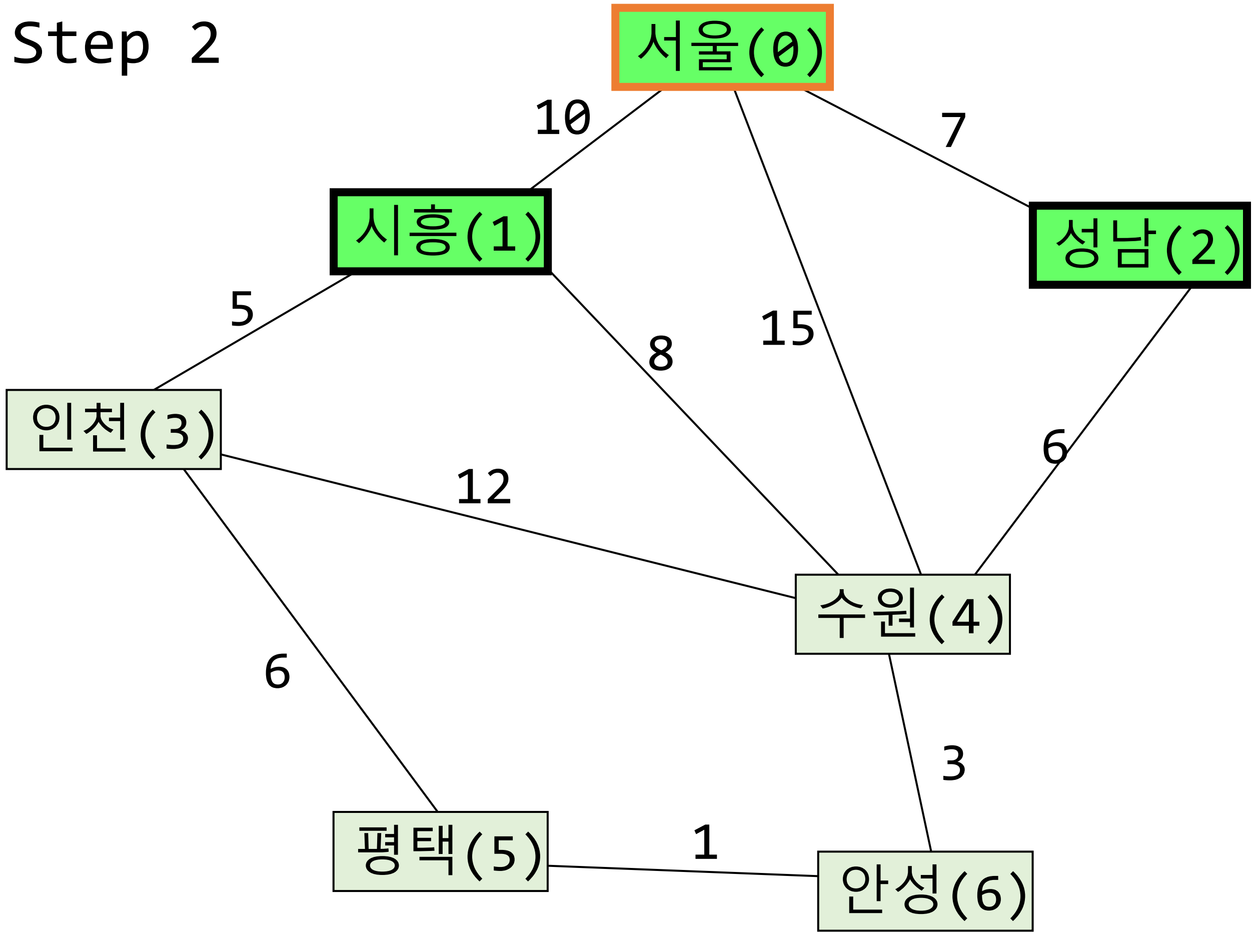
	distance	found
[0]	0	1
[1]	10	0
[2]	7	1
[3]	1000	0
[4]	15 13	0
[5]	1000	0
[6]	1000	0

(서울, 성남)+(성남, 수원)

S={서울(0), **성남(2)**}

	[0]	[1]	[2]	[3]	[4]	[5]	[6]
pi	0	0	0	-1	2	-1	-1

Step 2



	distance
[0]	0
[1]	10
[2]	7
[3]	1000 15
[4]	13
[5]	1000
[6]	1000

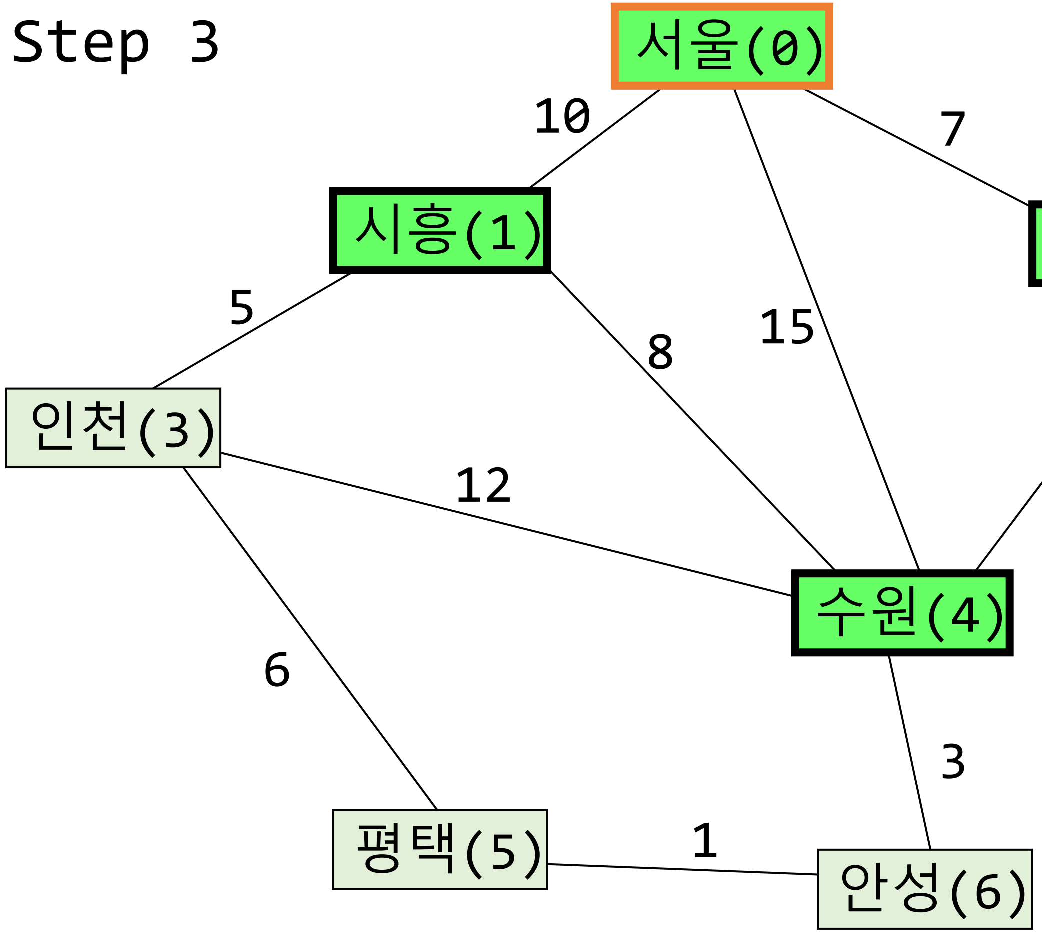
	found
[0]	1
[1]	1
[2]	1
[3]	0
[4]	0
[5]	0
[6]	0

(서울, 시흥)+(시흥,인천)

S={서울(0), 성남(2), **시흥(1)**}

	[0]	[1]	[2]	[3]	[4]	[5]	[6]
pi	0	0	0	1	2	-1	-1

Step 3



	distance
[0]	0
[1]	10
[2]	7
[3]	15
[4]	13
[5]	1000
[6]	1000 16

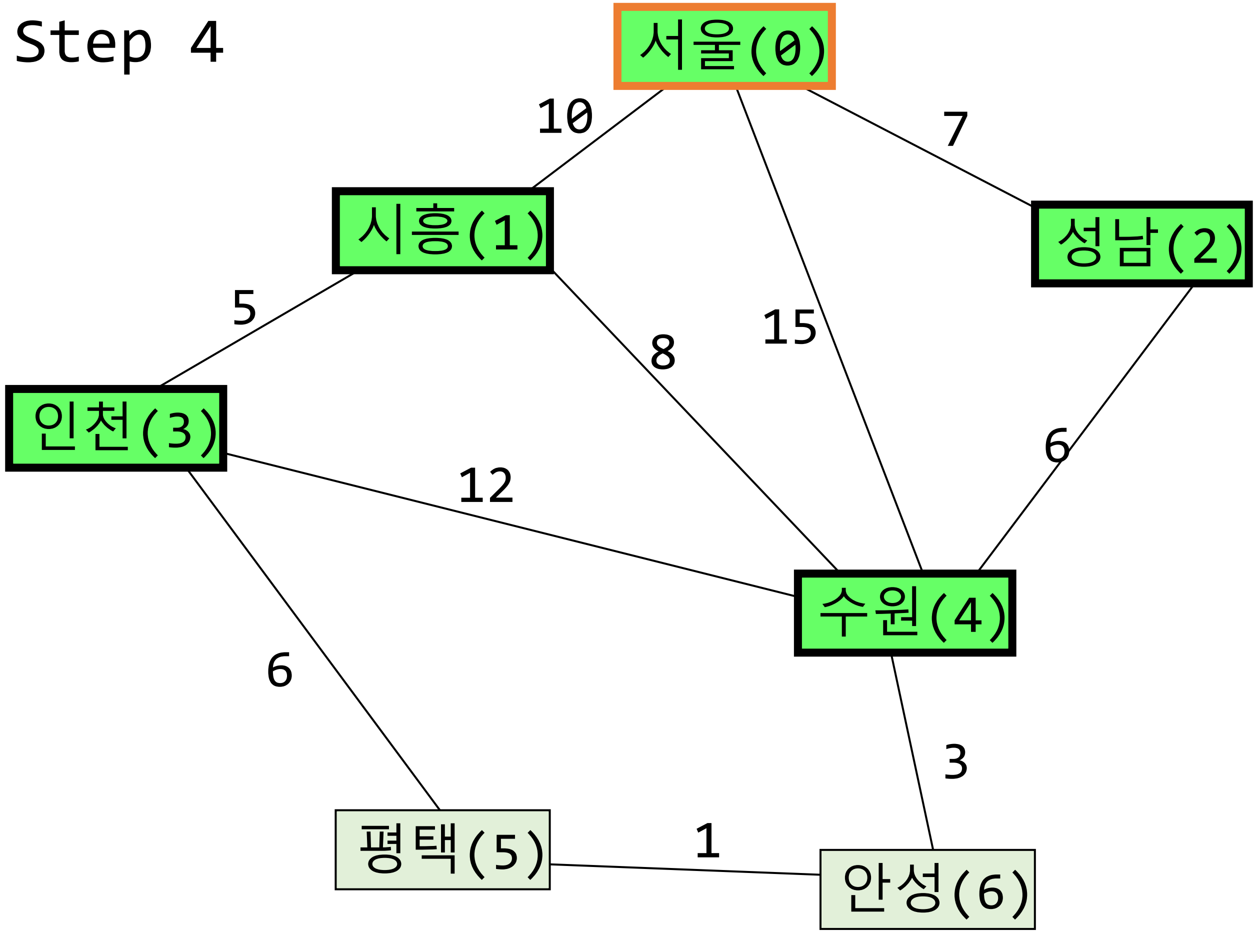
	found
[0]	1
[1]	1
[2]	1
[3]	0
[4]	1
[5]	0
[6]	0

(서울, 수원)+(수원,안성)

S={서울(0), 성남(2), 시흥(1), 수원(4)}

	[0]	[1]	[2]	[3]	[4]	[5]	[6]
pi	0	0	0	1	2	-1	4

Step 4



	distance
[0]	0
[1]	10
[2]	7
[3]	15
[4]	13
[5]	~~1000~~ 21
[6]	16

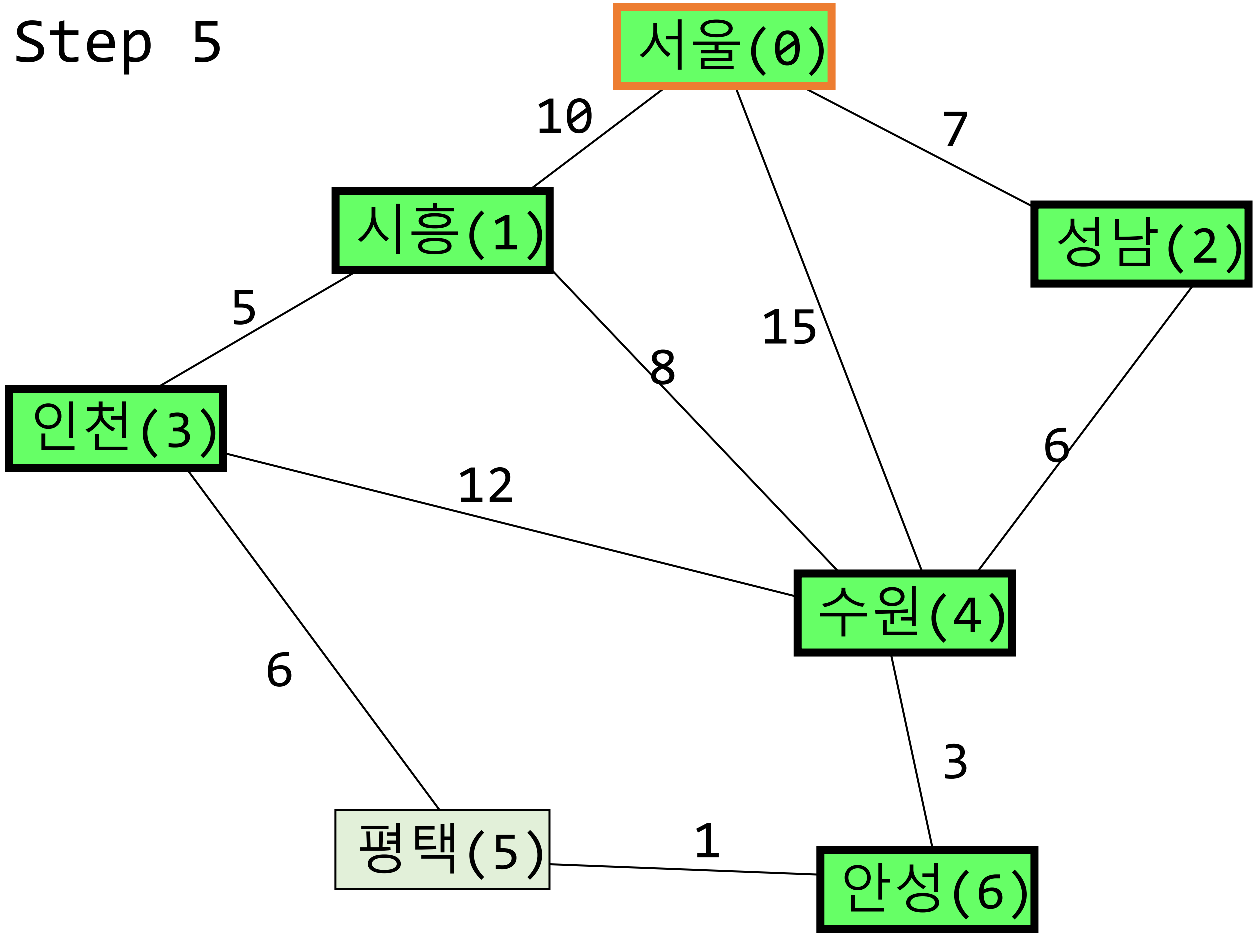
	found
[0]	1
[1]	1
[2]	1
[3]	1
[4]	1
[5]	0
[6]	0

(서울, 인천)+(인천,평택)

S={서울(0), 성남(2), 시흥(1), 수원(4), **인천(3)** }

	[0]	[1]	[2]	[3]	[4]	[5]	[6]
pi	0	0	0	1	2	3	4

Step 5



	distance	found
[0]	0	1
[1]	10	1
[2]	7	1
[3]	15	1
[4]	13	1
[5]	21 17	0
[6]	16	1

(서울, 안성)+(안성,평택)

S={서울(0), 성남(2), 시흥(1), 수원(4), 인천(3), **안성(6)** }

	[0]	[1]	[2]	[3]	[4]	[5]	[6]
pi	0	0	0	1	2	**6**	4

How to determine the path

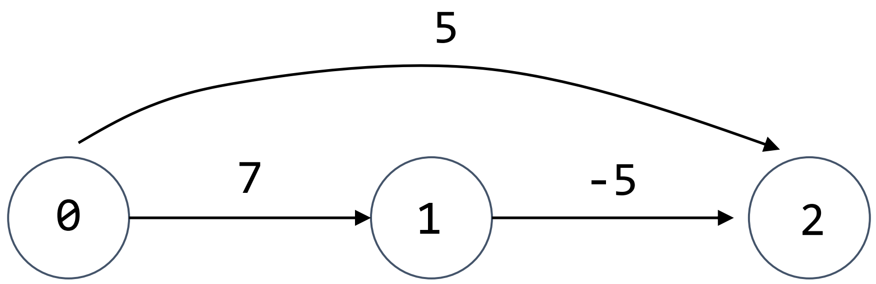
S={서울(0), 성남(2), 시흥(1), 수원(4), 인천(3), 안성(6), 평택(5) }

	[0]	[1]	[2]	[3]	[4]	[5]	[6]
pi	0	0	0	1	2	6	4

서울 → 평택 서울(0) 성남 (2) 수원 (4) 안성(6) 평택(5)

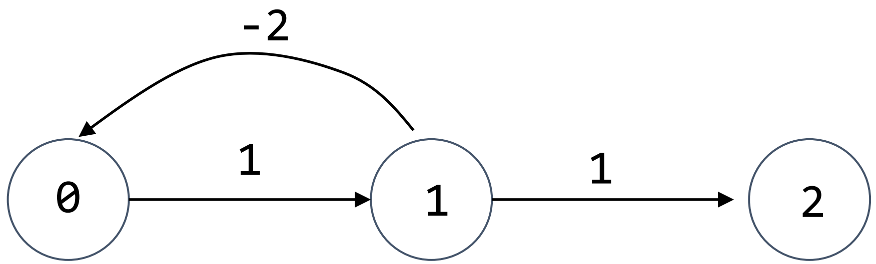
서울 → 인천 ?

Negative Edge Cases



Shortest path from 0 to 2 ? 0,1,2

Dikstra's algorithm cannot find this shortest path



Cost of the shortest path between 0 and 2 ?

0,1,0,1,0,1,0,1,…,2 → -∞

There is no shortest path!!

Bellman-Ford's algorithm: Overview

Allows negative weights

If there is a negative cycle, returns "a negative cycle exists"

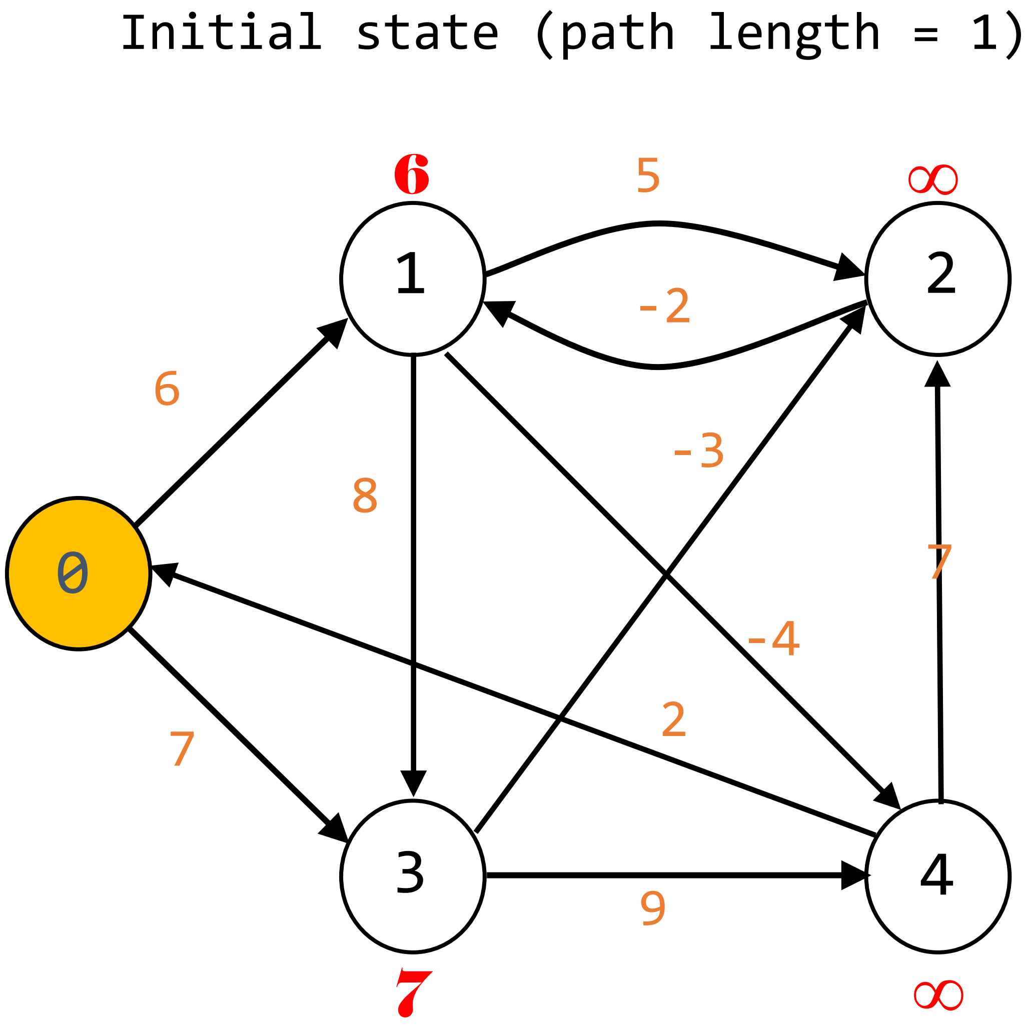
The idea:

- There is a shortest path from s to any other vertex that does not contain a negative cycle
- The maximal number of edges in such a path with no cycles is $|V|-1$, because it can have at most $|V|$ nodes on the path if there is no cycle
  $\Rightarrow$ it is enough to check paths of up to $|V|-1$ edges.

Bellman-Ford Algorithm

```
void BellmanFord(int n, int v)
{ /* single source all destination shortest paths with negative edge lengths. */
  for (int i=0; i<n; i++)
    dist[i] = length[v][i];  /* initialize dist */

  for (int k =2; k <= n-1; k++)
     for (each u such that u != v and u has at least one incoming edge)
        for (each <i, u> in the edge)
            if (dist[u] > dist[i] + length[i][u])
                 dist[u] = dist[i] + length[i][u];
  for each edge (u,w) in E
     if (dist(w) > dist(u)+length[u][w])
          return FALSE;                        /* negative cycle detection */
}
```

Bellman-Ford's algorithm: Example

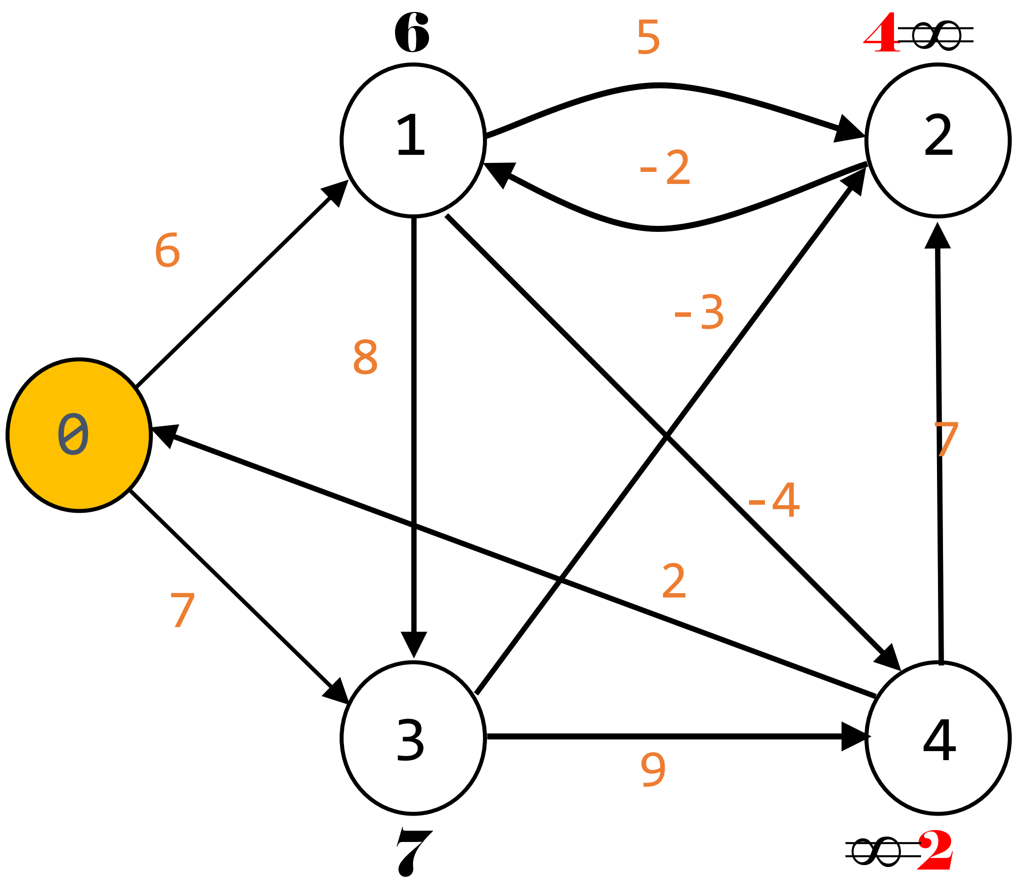


dist[1]= 6 (0,1)
dist[2]= ∞
dist[3]= 7 (0,3)
dist[4]= ∞

	[0]	[1]	[2]	[3]	[4]
[0]	0	6	∞	7	∞
[1]	∞	0	5	8	−4
[2]	∞	−2	0	∞	∞
[3]	∞	∞	−3	0	9
[4]	2	∞	7	∞	0

Path length = 2

```
for each <i,u>,
  min{dist[u], dist[i] + length[i][u]}
```



dist[1]= 6 (0,1)
  min{6, ∞+(-2)}

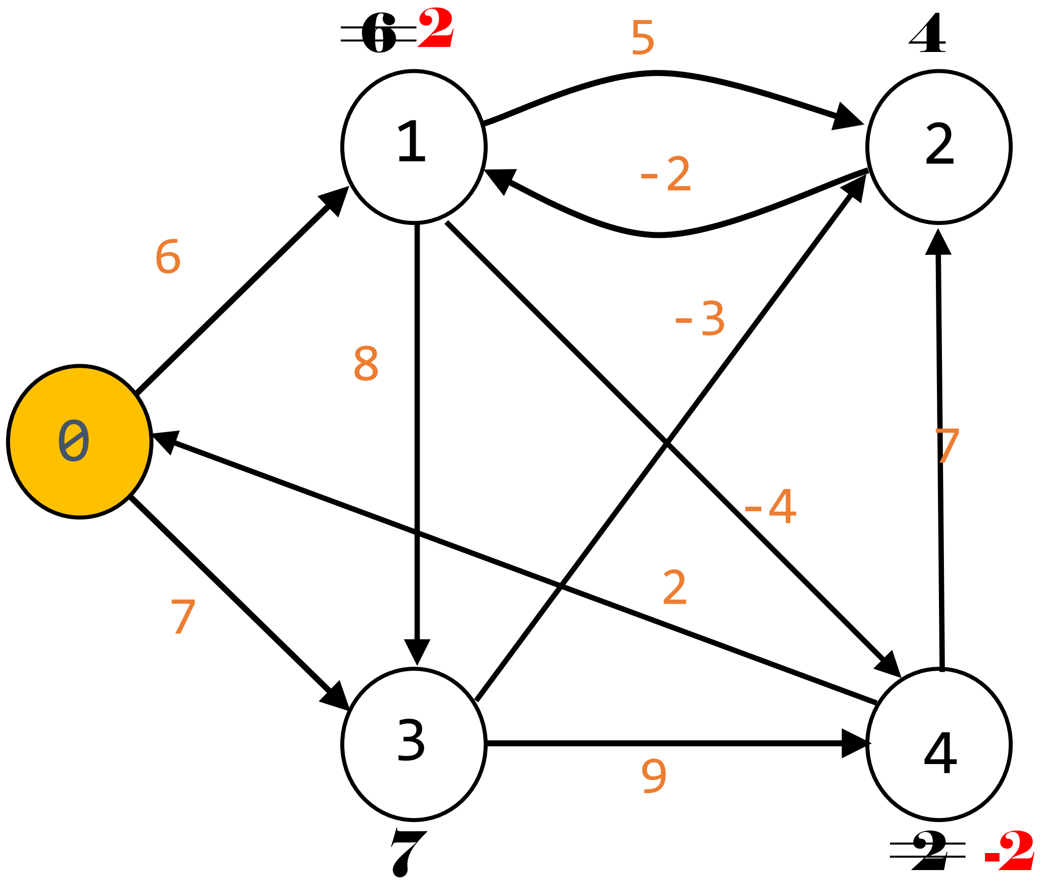
dist[2]= ∞ (0,3,2)
  min{∞, 6+5, 7+(-3), ∞+7}

dist[3]= 7 (0,3)
  min{7, 6+8}

dist[4]= ∞ (0,1,4)
  min{∞, 6+(-4), 7+9}

Path length = 3

```
for each <i,u>,
  min{dist[u], dist[i] + length[i][u]}
```



dist[1]= 6 (0,3,2,1)
  min{6, 4+(-2)}

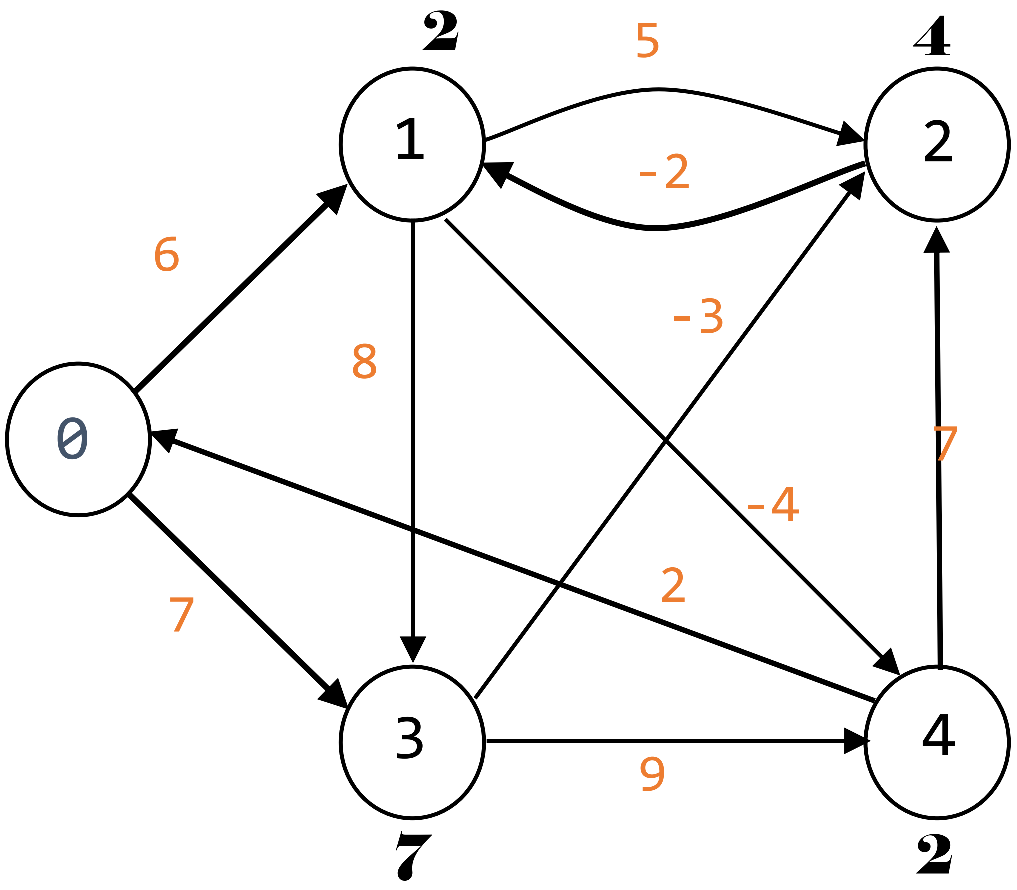
dist[2]= 4 (0,3,2)
  min{4, 6+5, 7+(-3), 2+7}

dist[3]= 7 (0,3)
  min{7, 2+8}

dist[4]= -2 (0,3,2,1,4)
  min{2, 2+(-4), 7+9}

Path length = 4

for each <i,u>,
min{dist[u], dist[i] + length[i][u]}



dist[1]= 2 (0,3,2,1)
min{2, 4+(-2)}

dist[2]= 4 (0,3,2)
min{4, 2+5, 7+(-3), 2+7}

dist[3]= 7 (0,3)
min{7, 2+8}

dist[4]= -2 (0,3,2,1,4)
min{-2, 2+(-4), 7+9}

Next Topic

Sorting